Mathematical Methods II
Lecture 5

Craig Testrow

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Key Points

- Calculating the Wronskian
- Method of variation of parameters/Wronskian method

Solving linear ODEs with variable coefficients

- **Wronskian:** The Wronskian of a set of functions is the determinant of a square matrix of those functions and their derivatives. It can be used to determine if the functions are linearly independent. The Wronskian for an n^{th} order ODE has the following form

$$W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & \dots \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix}$$

$W \neq 0$ between specified limits when all of the functions are linearly independent and $W = 0$ when they are not. Recall that the general solutions to an n^{th} order ODE must be constructed from n linearly independent solutions. A zero W would indicate that you need to seek further independent solutions.

The Wronskian for a 2^{nd} order ODE can be expressed as

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

e.g. 5.1 Check that the following solutions to a 2^{nd} order ODE are linearly independent.

$$y_1 = e^x$$

$$y_2 = e^{2x}$$

Calculate the Wronskian

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x.2e^{2x} - e^x.e^{2x} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

Hence y_1 and y_2 are linearly independent.

e.g. 5.2 Check that the following solutions to a 2$^{\text{nd}}$ order ODE are not linearly independent.

$$y_1 = x^2 + 1$$
$$y_2 = 2x^2 + 2$$

Calculate the Wronskian

$$W = \begin{vmatrix} x^2+1 & 2x^2+2 \\ 2x & 4x \end{vmatrix} = (x^2+1)4x - 2x(2x^2+2) = 4x^3 + 4x - 4x^3 - 4x = 0$$

Hence y_1 and y_2 are not linearly independent.

*** The following is for interest and not examinable ***

- **Wronskian method or method of variation of parameters (background):** This method can be used to determine the particular solution of an ODE and works with constant or variable coefficients.

Consider the general form of an n^{th} order linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

The complementary function of this ODE is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

which is the general solution to the homogeneous part $[f(x) = 0]$. Let's assume that the particular integral y_p can be expressed in a similar form to y_c, but with the constant coefficients replaced with functions of x to ensure linear independence (recall that if y_p contains a term that already exists in y_c, we multiply it by powers of x until there is no longer a duplicated term).

$$y_p(x) = k_1(x)y_1(x) + k_2(x)y_2(x) + ... + k_n(x)y_n(x)$$

This no longer satisfies the complementary equation, but if $f(x) \neq 0$ and we pick suitable $k_i(x)$ terms it could be made equal to $f(x)$, providing us with a particular integral.

We have n arbitrary functions, meaning we need n constraints on our system to determine what they are. The simplest way to obtain them is to differentiate y_p $n-1$ times

$$y_p' = k_1 y_1' + k_2 y_2' + ... + k_n y_n' + [k_1' y_1 + k_2' y_2 + ... + k_n' y_n].$$

We are free to choose our constraints as we wish, so let's say the bracketed term is equal to zero, leaving us with

$$y_p' = k_1 y_1' + k_2 y_2' + ... + k_n y_n'.$$

Which we can differentiate and discard the bracketed term again

$$y_p'' = k_1 y_1'' + k_2 y_2'' + ... + k_n y_n''.$$

We repeat this process until we have $(n-1)$ equations. We still want one more, in which we will not set the bracketed term to zero. The m^{th} derivative is given by

$$y_p^{(m)} = k_1 y_1^{(m)} + k_2 y_2^{(m)} + ... + k_n y_n^{(m)}.$$

If we differentiate one more time to obtain the n^{th} derivative

$$y_p^{(n)} = k_1 y_1^{(n)} + k_2 y_2^{(n)} + ... + k_n y_n^{(n)} + \left[k_n y_n^{(n)} + k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + ... + k_n' y_n^{(n-1)} \right].$$

Now we have n equations, of which the first $n-1$ are of the same form. Let's substitute them back into our ODE

$$\sum_{j=1}^{n} k_j \left[a_n y_j^{(n)} + ... + a_1 y_j' + a_0 y_j \right] + a_n \left[k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + ... + k_n' y_n^{(n-1)} \right] = f(x).$$

But since the functions y_j are solutions of the complementary equation, we have (for all j)

$$a_n y_j^{(n)} + ... + a_1 y_1' + a_0 y_j = 0.$$

So the sum term disappears, leaving

$$a_n \left[k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + ... + k_n' y_n^{(n-1)} \right] = f(x).$$

So, we are left with a set of n linearly independent equations

$$\begin{aligned}
k_1'(x) y_1(x) + k_2'(x) y_2(x) + ... + k_n'(x) y_n(x) &= 0 \\
k_1'(x) y_1'(x) + k_2'(x) y_2'(x) + ... + k_n'(x) y_n'(x) &= 0 \\
&... \\
k_1'(x) y_1^{(n-2)}(x) + k_2'(x) y_2^{(n-2)}(x) + ... + k_n'(x) y_n^{(n-2)}(x) &= 0 \\
k_1'(x) y_1^{(n-1)}(x) + k_2'(x) y_2^{(n-1)}(x) + ... + k_n'(x) y_n^{(n-1)}(x) &= \frac{f(x)}{a_n(x)}
\end{aligned}$$

Since these equations are linearly independent the determinant of their coefficients must be equal to a non-zero Wronskian.

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Aside - Cramer's rule: In order to proceed we will need to make use of Cramer's rule for a 2x2 matrix. Given a system of linear equations this rule allows you to solve for just one of the variables without having to solve the whole system of equations. Take the following linear equations

$$ax + by = m$$

$$cx + dy = n$$

Find three determinants. D, D_x and D_y.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$D_x = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$$

$$D_y = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$$

The solutions are given by $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$

- **Wronskian method or method of variation of parameters (derivation for 2^{nd} order ODE):**

Consider a 2^{nd} order ODE

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$$

This equation has the homogeneous solution

$$y_c = c_1 y_1 + c_2 y_2.$$

We want to find two functions $k_1(x)$ and $k_2(x)$ such that

$$y_p(x) = k_1(x)y_1(x) + k_2(x)y_2(x) = f(x).$$

i.e. we want y_p to be constructed from the same solutions as y_c. This will only work if the coefficients k_1 and k_2 are functions of x, rather than constant, or we will not find linearly independent solutions.

We start by finding y_p' and y_p'' and making an important assumption,

$$k_1' y_1 + k_2' y_2 = 0.$$

This simplifies our derivatives and helps us find k_1 and k_2. Notice that this is the first of the set of n equations we derive in the background section above.

$$y_p' = k_1 y_1' + k_1' y_1 + k_2 y_2' + k_2' y_2 = k_1 y_1' + k_2 y_2'$$

$$y_p'' = k_1 y_1'' + k_1' y_1' + k_2 y_2'' + k_2' y_2'$$

Now sub these into our ODE

$$\left[k_1 y_1'' + k_1' y_1' + k_2 y_2'' + k_2' y_2'\right] + p(x)\left[k_1 y_1' + k_2 y_2'\right] + q(x)\left[k_1 y_1 + k_2 y_2\right] = f(x).$$

Rearranging

$$(k_1' y_1' + k_2' y_2') + k_1\left[y_1'' + p(x)y_1' + q(x)y_1\right] + k_2\left[y_2'' + p(x)y_2' + q(x)y_2\right] = f(x)$$

Now, since y_1 and y_2 are solutions to y_c, the homogeneous equation, the terms in square brackets must equal zero, and so

$$k_1'y_1' + k_2'y_2' = f(x)$$

This equation combined with the assumption we made can be used to solve the equation using Cramer's rule. Our determinants take the following forms

$$k_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-f(x)}{W(y_1, y_2)} y_2 \to k_1 = -\int \frac{y_2 f(x)}{W(y_1, y_2)} dx$$

$$k_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{f(x)}{W(y_1, y_2)} y_1 \to k_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

This process allows us to use the Wronskian to determine the coefficients of the solutions in y_p, allowing us to combine it with y_c and reach a general solution to an inhomogeneous linear ODE.

Note that this method still works if the coefficient of the 2^{nd} derivitive term is not equal to 1, with a small modification. Assume our 2^{nd} order ODE is

$$ay'' + by' + cy = f(x)$$

Then our k_1' and k_2' become

$$k_1' = \frac{-f(x)}{aW(x)} y_2$$

$$k_2' = \frac{f(x)}{aW(x)} y_1$$

Note that $W(y_1, y_2) \equiv W(x)$. This is not necessary if the equation is arranged into the standard form shown above where the 2^{nd} derivitive term has no coefficient.

- **Wronskian method or method of variation of parameters (example):**

  e.g. 5.3 Solve the following equation using the Wronskian method

$$y'' - 4y' + 4y = (x + 1)e^{2x}$$

First, identify $a = 1$ and $f(x) = (x+1)e^{2x}$. Next, find y_c. The auxiliary equation is given by

$$\lambda^2 - 4\lambda + 4 = 0 \to \lambda = 2, 2$$

So,

$$y_c = (c_1 + xc_2)e^{2x}$$

This allows us to identify $y_1 = e^{2x}$ and $y_2 = xe^{2x}$. Now let's find the Wronskian,

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{2x}(e^{2x} + 2xe^{2x}) - 2e^{2x}.xe^{2x} = e^{4x}$$

Now we have everything we need to determine the coefficients of y_p

$$k_1' = \frac{-f(x)}{aW(x)} y_2 = \frac{-(x+1)e^{2x}}{e^{4x}} x e^{2x} = -x^2 - x$$

$$k_2' = \frac{f(x)}{aW(x)} y_1 = \frac{(x+1)e^{2x}}{e^{4x}} e^{2x} = x + 1$$

Integrating w.r.t x

$$k_1 = -\frac{x^3}{3} - \frac{x^2}{2} + c_3$$

$$k_2 = \frac{x^2}{2} + x + c_4$$

Our particular solution is therefore

$$y_p = k_1 y_1 + k_2 y_2 = \left(-\frac{x^3}{3} - \frac{x^2}{2} + c_3\right) e^{2x} + \left(\frac{x^2}{2} + x + c_4\right) x e^{2x}$$

$$= e^{2x} \left(\frac{x^3}{2} - \frac{x^3}{3} + x^2 - \frac{x^2}{2} + c_4 x + c_3\right)$$

$$= e^{2x} \left(\frac{x^3}{6} + \frac{x^2}{2} + c_4 x + c_3\right)$$

Therefore our general solution is

$$y = (c_1 + x c_2) e^{2x} + y_p = e^{2x} \left(\frac{x^3}{6} + \frac{x^2}{2} + c_5 x + c_6\right)$$

Note: The above example is solved in the traditional way, by finding $y = y_c + y_p$, to match the logic of previous examples. This is fine, but a curious result arises: the particular solution contains two terms that were already in the complementary function. This is a natural result of determining y_p based on the solutions to y_c, i.e. y_1 and y_2. In fact using this method, the general solution $y = y_p$ is equivalent to $y = y_c + y_p$, because y_p will naturally contain y_c. There is not a contradiction here, but a subtle point.

My advice to you is to short-cut the last line of the solution, which yields the same result and do it this way in future problems to avoid confusion. Instead of $y = y_c + y_p$ just state $y = y_p$ when solving problems with this method. So, in the case of e.g. 5.4, we would state that

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$$y_p = k_1(x) e^{2x} + k_2(x) x e^{2x}$$

and instead of

$$y = y_c + y_p = e^{2x} \left(\frac{x^3}{6} + \frac{x^2}{2} + c_5 x + c_6\right)$$

we write

$$y = y_p = e^{2x} \left(\frac{x^3}{6} + \frac{x^2}{2} + c_4 x + c_3\right)$$

as the general solution. It should be clear that these are equivalent, given that the constant terms are abitrary.

Perhaps an example will clarify this. Consider a problem that has a homogeneous solution

$$y_c = Ae^x + Be^{2x}$$

Using the method of variation of parameters we are going to solve

$$y_p = C(x)e^x + D(x)e^{2x}$$

Now, imagine our solution to y_p yielded the results

$$C(x) = x + E$$

$$D(x) = x + F$$

Giving a general solution

$$y = (x + E)e^x + (x + F)e^{2x}$$

If we can expand this out we note that the constant coefficients give us an equivalent expression to the complementary function, with undetermined constant coefficients. Compare

$$Ee^x + Fe^{2x} \equiv Ae^x + Be^{2x}$$

If we desired to represent the solution in the traditional form we could deconstruct this solution so y_c and y_p are displayed separately, i.e.

$$y = \left[Ee^x + Fe^{2x}\right] + \left[xe^x + xe^{2x}\right] = [y_c] + [y_p]$$

which should look more familiar compared to earlier problems, but this is unecessary.