University of Durham

EXAMINATION PAPER

May/June 2014 Examination code: 042611/01

LEVEL 2 PHYSICS: MATHEMATICAL METHODS IN PHYSICS

SECTION A. Mathematical Methods part 1
SECTION B. Mathematical Methods part 2

Time allowed: 3 hours

Examination material provided: None

Calculators: The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron mass:	$m_e = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_p = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\,\mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_A = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\,\mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. MATHEMATICAL METHODS PART 1
Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Find a matrix S for which $S^{-1}AS$ is a diagonal matrix for the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -2 & -3 \end{pmatrix} .$$

Verify explicitly that this is the case. [4 marks]

(b) Given the cylindrical coordinates

$$\begin{aligned} x &= \rho \cos\phi \,, \\ y &= \rho \sin\phi \,, \\ z &= h \,, \end{aligned}$$

compute the basis vectors $\partial\underline{r}/\partial u_i$ and ∇u_j for this coordinate system. Give the infinitesimal surface element, $d\underline{S}$, for a surface of constant ϕ and find its components in both bases. [4 marks]

$$\left[\text{Hint: } \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.\right]$$

(c) Given the scalar field

$$\phi(x, y, z) = \frac{z - y^3}{x^2}$$

and the vector field $\underline{a}$

$$\underline{a}(x, y, z) = (y^2, y + z, xz^2) \,,$$

compute the quantities $\underline{\nabla}\phi$ and $\underline{\nabla} \times \underline{a}$. [4 marks]

(d) State Stokes' theorem. Draw a sketch to illustrate Stokes' theorem for the case of a half sphere and explain all symbols you are using. [4 marks]

(e) The Fourier series for a periodic function of period L is given by

$$f(x) = \frac{a_0}{2} + \sum_{p=0}^{\infty} a_p \cos\left(\frac{2\pi p x}{L}\right) + b_p \sin\left(\frac{2\pi p x}{L}\right) \,, \qquad a_0 = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\, dx,$$

$$a_p = \frac{2}{L}\int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi p x}{L}\right) dx \,, \qquad b_p = \frac{2}{L}\int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi p x}{L}\right) dx \,.$$

Compute the Fourier series of the following function,

$$f(x) = \begin{cases} -1 & \text{if } -1 \le x < 0 \,, \\ 1/2 & \text{if } 0 \le x < 1 \,, \end{cases}$$

periodically continued from the interval $[-1, 1]$ to the whole real axis. [4 marks]

(f) Given the definition of the Fourier transform

$$\mathcal{F}[f](w) = \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \ ,$$

compute the Fourier transform of the function

$$f(x) = \begin{cases} \cos(x)e^{-x} & \text{for} \quad x \geq 0 \\ 0 & \quad\quad\;\; x < 0 \,. \end{cases}$$

[4 marks]

(g) Given the Laplace transform

$$\bar{f}(s) \equiv \int\limits_{0}^{\infty} f(t)e^{-st}\, dt$$

and using a suitable variable transformation, show the following properties:

$$\bar{g}(s) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) \ , \qquad \text{for} \quad g(t) = f(at) \ .$$

and

$$\bar{h}(s) = e^{-as}\bar{f}\left(s\right) \ , \qquad \text{for} \quad h(t) = \begin{cases} f(t-a) & \text{for} \quad t \geq a \\ 0 & \quad\quad \text{otherwise} \end{cases} \ .$$

[4 marks]

(h) Compute the following integrals over the full three-dimensional space

$$I_1 = \int g(\underline{r} \cdot \underline{r})\delta(\underline{r} - \underline{r_0})d^3\underline{r} \ , \qquad r_0 = (2, -3, 4) \ ,$$

$$I_2 = \int \delta(\underline{r} \cdot \underline{r} - 4)\delta(\underline{r} \cdot \underline{\hat{e}_z})d^3\underline{r} \ .$$

[4 marks]

2. Consider the coordinate system defined by

$$\underline{r} = (x, y, z) = \begin{pmatrix} \lambda \sin\theta \cos\phi \\ \lambda \sin\theta \sin\phi \\ b\lambda \cos\theta \end{pmatrix}$$

and the surface S given by the parametric equations

$$\underline{r}(\theta, \phi) = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ b \cos\theta \end{pmatrix} \qquad 0 \le \theta \le \pi\ , \quad 0 \le \phi < 2\pi\ ,$$

which is the surface corresponding to a constant λ with $\lambda = 1$.

(a) Compute the infinitesimal surface element, $d\underline{S}$, for this surface. [3 marks]

(b) Using your result for (a) compute the area of the surface for the special case $b = 5/3$. [4 marks]

$$\left[\int_{-1}^{1} \sqrt{1 - a^2c^2}\, dc = \sqrt{1 - a^2} + \frac{\sin^{-1}(a)}{a}. \right]$$

(c) Compute the infinitesimal volume element, $d\underline{V}$, for the coordinate system (λ, θ, ϕ) and use it to compute the volume enclosed by the surface S for a general value of b. [3 marks]

(d) A body with shape given by the surface S is immersed in a fluid. The fluid exerts a force on the body normal to its surface and proportional to the pressure $P(z) = \rho g(z_0 - z)$, where ρ is the density of the fluid and z_0 is the height of the fluid's surface above the body. Compute the total force exerted by the fluid on the body. [4 marks]

(e) State the divergence theorem for a general volume, explain all symbols used and use it to prove Archimedes' law, namely that the upward force exerted on a body in a fluid is equal to the mass of the displaced fluid. [6 marks]

3. The definition of the Laplace transform $\bar{f}$ of a function f defined for positive values is given by:

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\, dt.$$

(a) Show that the Laplace transform of the convolution $f * g$ of two functions

$$f * g(t) = \int_0^\infty f(u)g(t-u)\, du,$$

is given by

$$\overline{f * g}(s) = \bar{f}(s)\bar{g}(s)\ .$$

[2 marks]

(b) Compute the Laplace transform of the function

$$f(x) = \begin{cases} h & \text{for} \quad x_0 \le x \le x_0 + \Delta x \\ 0 & \qquad\qquad \text{otherwise,} \end{cases}$$

for the case in which x_0 and Δx are both positive numbers. [2 marks]

(c) Using the result of (b) with an appropriate value of h and taking the limit $\Delta x \to 0$, compute the Laplace transform of a Dirac delta function $\delta(x - x_0)$. Verify your result with the explicit calculation of the Laplace transform of the Dirac delta function. [4 marks]

(d) Compute the Fourier series in terms of sines and cosines of the function

$$f(t) = \begin{cases} -1 & \text{for} \quad -a < t \le 0 \\ +1 & \text{for} \quad 0 < t \le a \\ 0 & \qquad \text{otherwise} \end{cases} \quad , \qquad a > 0$$

periodically repeated from the interval $[-L/2, +L/2]$ to the entire real axis, where $a < L/2$. [3 marks]

(e) Compute the Fourier transform of the function defined in (d) without periodic continuation. [3 marks]

(f) Using the identity

$$\lim_{\Delta\omega \to 0} \sum_{p=-\infty}^{p=\infty} f(p\Delta\omega)\Delta\omega = \int_{-\infty}^{\infty} f(\omega)\, d\omega\ ,$$

and by rewriting the Fourier series representation of f from (d) as a sum from $-\infty$ to ∞ instead of from 1 to ∞, and expressing the sine and cosine functions as the exponential function of complex arguments, show that the Fourier series representation of f becomes the Fourier transform representation in the limit of large L, that is:

$$\lim_{L\to\infty} \left(\frac{a_0}{2} + \sum_{p=1}^{\infty} a_p \cos\left(\frac{2\pi p x}{L}\right) + b_p \sin\left(\frac{2\pi p x}{L}\right) \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx\ .$$

Find the appropriate definition of $\Delta\omega$ in the formula above and make sure all occurences of the summation variable p appear with $\Delta\omega$ such that the small $\Delta\omega$ limit can be taken. [6 marks]

$$\left[\begin{array}{l}\text{Hints:}\\ \text{Some Taylor expansions:}\\ e^x = \sum\limits_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2} + \ldots \\ \sin x = \sum\limits_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \\ \cos x = \sum\limits_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots \end{array}\right]$$

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SECTION B. MATHEMATICAL METHODS PART 2
Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Solve the equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0,$$

showing all your steps. [4 marks]

(b) Solve the equation

$$2 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = 0.$$

[2 marks]
Is the behaviour of your solution $y(x)$ physical or not when $x \to \infty$? Explain why. [2 marks]

(c) Use the *Wronskian* method to solve the ordinary differential equation

$$6 \frac{d^2y}{dx^2} + \frac{dy}{dx} = 4 e^{-x/6},$$

given that $y(x) = k_1$ and $y(x) = k_2 e^{-x/6}$ are solutions to the homogeneous equation, where k_1 and k_2 are constants. [4 marks]

[Hint : Remember that if $y = k_1 y_1 + k_2 y_2$, then
$k_1' = -\frac{h(x)}{W(x)} y_2$ and $k_2' = \frac{h(x)}{W(x)} y_1$
where $W(x)$ is the Wronskian and $h(x)$ is the inhomogeneous term.]

(d) Using the method of *reduction of order*, solve

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y + 24x = 0.$$

[4 marks]

(e) State the type of this equation

$$(1 - x^2)\, y'' - 2xy' + 30\, y = 0.$$

Write down a solution to the equation and justify your answer.
[4 marks]

(f) Solve the partial differential equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + 4 \frac{\partial f(x,y)}{\partial y} = 0,$$

where x, y the Cartesian coordinates. [2 marks]
Determine whether or not your solution is physical if we impose the condition that $f(x,y)$ tends to zero when $y \to \infty$. [2 marks]

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(g) Draw the representation of the real part of the spherical harmonics Y_2^1 on a sphere of radius unity, given that

$$Y_l^m \;=\; A\, P_l^m(\cos\theta)\, e^{i\,m\,\phi},$$

where A is a constant and

$$P_2^1(\cos\theta) = \sin\theta \times \frac{d}{d\cos\theta}\left[P_2(\cos\theta)\right],$$

with

$$P_2(\cos\theta) = \frac{1}{2}\left(3(\cos\theta)^2 - 1\right).$$

[4 marks]

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5. Fission is an important process in nuclear physics. When a neutron hits a nucleus of Uranium, it can break the nucleus and generate more neutrons thus maintaining the fission reaction. In a nuclear reactor, the Uranium is enclosed in a cylinder of radius R and height h. The equation describing the evolution of the number of neutrons N with time in the cylinder accounts for both their diffusion in the material and their generation at a rate Γ

$$\frac{\partial N}{\partial t} = k\Delta N + \Gamma N,$$

where k is the diffusion coefficient and

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$$

is the Laplacian operator in cylindrical coordinates (assuming a homogeneous distribution of material inside the cylinder). Assume the Dirichlet conditions apply such that $N(R, z, t) = 0$ for $z \in [0, h]$ and $N(r, 0, t) = N(r, h, t) = 0$ for $r \in [0, R]$.

(a) Assuming that $\Gamma = 0$, find how the number of neutrons N evolves with time. Bear in mind that in absence of fission, the reaction cannot be sustained. [4 marks]

(b) Determine how the number of neutrons N depends on the position z within the cylinder given that the solution must vanish at $z = 0$ and $z = h$ so as to satisfy the imposed Dirichlet conditions. [6 marks]

(c) Determine how the number of neutrons N depends on the radius r within the cylinder. Show that the number of neutrons decreases as $r \to R$. [4 marks]

(d) State the implications of the condition $N(R, z, t) = 0$. [2 marks]

(e) The number of neutrons when there is both diffusion and fission evolves as

$$N \propto e^{(\Gamma - m)t},$$

where

$$m = \left(\frac{\lambda}{R}\right)^2 + \left(\frac{\pi n}{h}\right)^2,$$

where λ and n are constants.

Give the condition for fission to dominate over the diffusion. What is the interpretation of this condition? [4 marks]

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6. Solitons and kinks are phenomena which occur, for example, in solid state physics and superconductors. The equation of motion can be written as

$$\frac{\partial^2 u_1}{\partial t^2} - c_0^2 \frac{\partial^2 u_1}{\partial x^2} + a^2 u_2 = 0,$$

where u_1 and u_2 are functions which are considered in different regimes below and C_0 is a constant.

(a) Assume first that $u_1 = u$ and $u_2 = 0$. Is the above equation linear in this case? What is the name of such an equation? [2 marks]

(b) Determine the function $u(x, t)$ if there is no damping or enhancement of the solution with time. [4 marks]

(c) Express your solution in terms of a function $f(ax \pm bt)$ where a, b are constants. Give a physical interpretation of your findings. [2 marks]

(d) Explain what happens when $t = 0$. [2 marks]

(e) Now consider the case in which $u_1 = u_2 = u$. Determine the function u and show that the solution can be written as

$$u = e^{i(wt-kx)}.$$

[7 marks]

(f) Finally, consider the case where $u_1 = \theta$ and $u_2 = \sin\theta$. This is the Sine-Gordon equation. Is it linear or non-linear? [3 marks]

