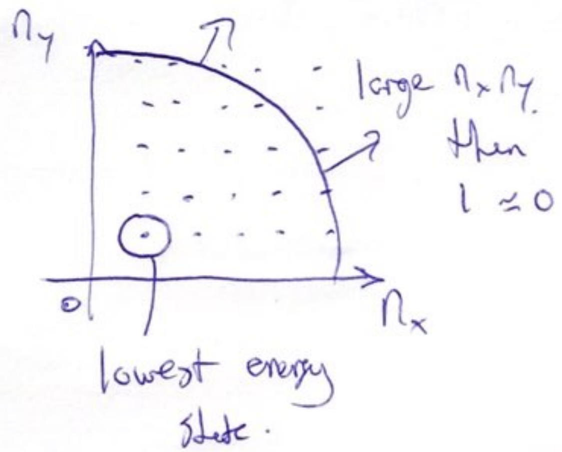
B.E. Density of States

Energetically favourable for all particles to occupy $\varepsilon = 0$ as $T \to 0$.

i.e. $\lim_{T \to 0} f_{BE}(0) = N$.

We have
$$N = \int_0^{\delta\varepsilon} g(\varepsilon) f_{BE}(\varepsilon)\, d\varepsilon = C \int_0^{\delta\varepsilon} \frac{\varepsilon^{1/2}\, d\varepsilon}{e^{\beta\varepsilon} - 1} \simeq 0$$



The "error" is in $g(\varepsilon)$ so we had

$g(\varepsilon) \sim \varepsilon^{1/2}$ but we require

one state kept at $\varepsilon = 0$ not 0 states.

quantized.

Rewrite adding a δ-function at this state:

$$g(\varepsilon) \rightarrow g(\varepsilon) + \delta(\varepsilon).$$

giving $$N = \int_0^{\delta\varepsilon} \delta(\varepsilon)\, f_{BE}(\varepsilon)\, d\varepsilon + \underbrace{\int_0^{\delta\varepsilon} g(\varepsilon)\, f_{BE}(\varepsilon)\, d\varepsilon}_{0} = \frac{1}{e^{-\beta\mu}-1} \xrightarrow[T\to 0]{} N.$$

Let's attempt to calculate number of bosons $n_0(T)$ in the single particle ground state.

$$N = \underbrace{\frac{1}{e^{-\beta\mu}-1}}_{n_0(T)} + \underbrace{\int_0^{\infty} g(\varepsilon)\, f_{BE}(\varepsilon)\, d\varepsilon}_{N(T)} = n_0(T) + N(T)$$

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Examine the integral:

$$N(T) = \frac{2\pi}{h^3} V (2m)^{3/2} \int_0^\infty \frac{\sqrt{\varepsilon}}{e^{-\beta\mu} e^{\beta\varepsilon} - 1} \, d\varepsilon.$$

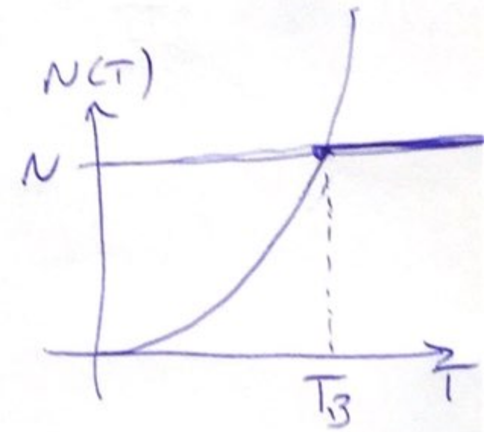
$$\simeq \underbrace{2.612 \frac{V}{h^3} (2\pi M k_B)^{3/2}}_{B} T^{3/2}.$$

Note $\int_0^\infty \frac{\sqrt{x}}{e^x - 1} dx = \frac{\sqrt{\pi}}{2} \left(\frac{3}{2}\right)!$

$\simeq 2.612$

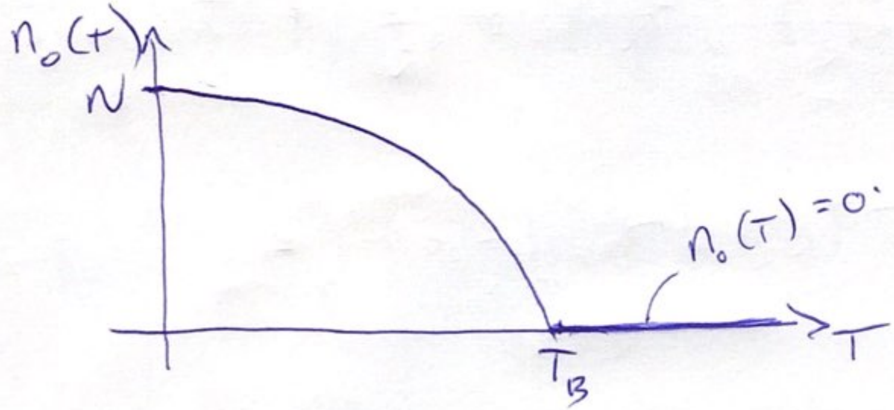
We have

$$N(T) = \begin{cases} B T^{3/2}, & T \leq T_B \\ N, & T \geq T_B \end{cases}$$



Therefore we can also examine the number of particles in the ground state, $n_0(T)$.

Rearranging we get $n_0(T) = N\left(1 - \left(T/T_B\right)^{3/2}\right)$.

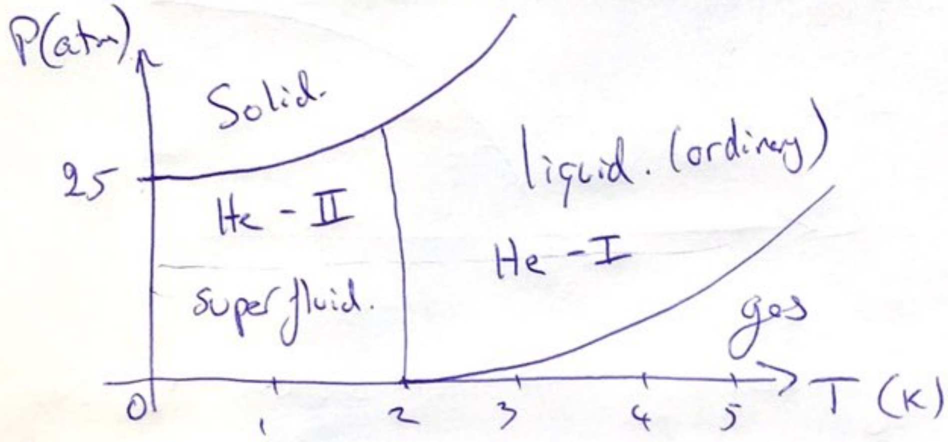


As we decrease T from a high temperature as T drops below T_B there is a sudden and rapid ~~increase~~ increase in the population of the ground state.

This is known as Bose-Einstein Condensation.

Example: ^4He is 2 protons + 2 neutrons + 2 electrons - compound boson.

There is liquid - liquid phase transition at 2K.



Look up properties of superfluids.

{ Also investigate superconductivity }.

Phonons and Photons - These particles obey BE statistics, it costs no energy to remove or add particles. - the chemical potential is zero.

(e.g. "photon gas" - Black body radiation,
" phonon gas " - vibrations in a crystal.).

The average number of photons/phonons per single particle state is

$$f_{BE}(\varepsilon) = \frac{1}{e^{\beta\varepsilon} - 1}, \quad \varepsilon = \hbar\omega$$

Average energy is :

average number of photons/phonons per state $f(\hbar\omega) \sim f(\omega)$

$\times$ energy per photon/phonon $\hbar\omega$

$$= \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} = U/N \quad \text{(average energy per oscillator)}$$

So one oscillator with energy $\varepsilon_n = n\hbar\omega$ ← no zero point motion., $n = 0, 1, 2, 3, \ldots$

Consider an oscillator in ground state $\varepsilon_0 = 0$ and first excited state $\varepsilon_1 = \hbar\omega$, if there are n Bosons in the excited state it has energy $n\hbar\omega$, note this is equivalent to having one particle in the n^{th} state (i.e. $\varepsilon_n = n\hbar\omega$). The average number of bosons in ε_1 is $f_1 = \dfrac{1}{(e^{\beta\hbar\omega}-1)}$ and average energy $\dfrac{\hbar\omega}{(e^{\beta\hbar\omega}-1)}$

<u>Spectral Density</u> It's convenient to express quantities in terms of ω (rad/sec.) or sometimes ν (Hz.) ($\omega \sim \nu$, factor of 2π).

<u>Spectral density</u>: $u(\omega)\,d\omega = \underbrace{g(\omega)}_{\text{DoS}}\ \underbrace{f(\omega)}_{\text{distribution function}}\ d\omega \cdot \underbrace{\hbar\omega}_{\text{energy}}$

Density of states in k-space is

$$g(k)\,dk = \frac{V}{(2\pi)^3} 4\pi k^2 \, dk \times 2$$ ← photons have 2 polarisations.

Also $c = \omega/k$ we have $k = \omega/c$ hence.

we get spectral ~~function~~ density is $\frac{\hbar V}{\pi^2 c^3} \omega^3 d\omega \frac{1}{e^{\beta\hbar\omega}-1}$

or $\frac{8\pi h V}{c^3} \nu^3 d\nu \frac{1}{e^{\beta\cancel{\hbar}\nu/h}-1}$

Planck's radiation formula.

From which relations such as Wien's displacement law, etc were obtained.

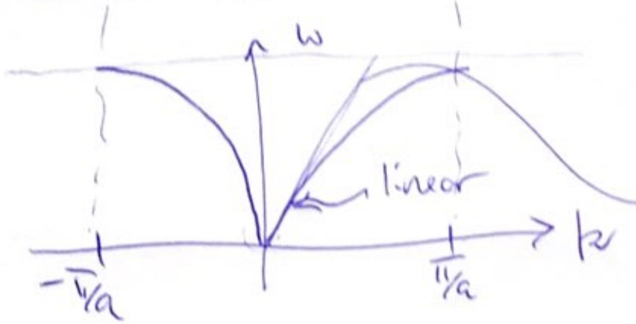
Total energy

all the constants

$$U = \frac{8\pi V h}{c^3} \int_0^\infty \frac{\nu^3}{e^{\beta h \nu} - 1} = C \underbrace{\int_0^\infty \frac{t^3}{e^t - 1}\, dt}_{\pi^4/15.}$$

$$U/V = \frac{8\pi^5 k_B^4}{15 c^3 h^3} T^4 \quad \text{(Stefan-Boltzmann law).}$$

Phonons Zero mass, zero chemical potential. Main difference is the number of polarisations which is 3. (two transverse, one longitudinal).



Approximately linear near $k \sim 0$.

this gradient is speed of sound.

For phonons $k_n = \frac{2\pi n}{Na}$, $n = 1, 2, 3, \ldots, N$. so

wavevectors take on a maximum value, k_N, because

$$e^{i(k_N + k_n)x} = e^{ik_n x} \quad \text{(periodicity).}$$

The maximum wavevector, denoted k_D (Debye wavevector) is associated

with maximum frequency ω_D. Let's approximate the dispersion to

be linear $\quad \varepsilon = h\nu = \frac{hk}{2\pi} c_s$ (subscript s for sound)

3N modes hence $3N = \int_0^{\nu_D} g(\nu) f_{BE}(\nu)\, d\nu = C \int_0^{\nu_D} \frac{\nu^2}{e^{\beta h\nu} - 1}\, d\nu$, etc.

Also for internal energy :

$$U = \frac{12\pi V h}{c_s^3}\int_0^{\nu_D}\frac{\nu^3}{e^{\beta h\nu}-1}\,d\nu = \dots = \frac{4\pi^3 V k_B^4}{5\, c_s^3 h^3}\, T^4 .$$

and so $C_V = \partial U / \partial T$ gives $C_V \propto T^3$. etc.

Done