University of Durham
EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2018	PHYS3621-WE01

Title:
Foundations of Physics 3A

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper.
- Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** the answers that are not to be marked.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Quantum Mechanics 3
Section B: Nuclear and Particle Physics

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50\, \times\, 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09\, \times\, 10^{16}$ m
Solar Mass:	$M_\odot = 1.99\, \times\, 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84\, \times\, 10^{26}$ W

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SECTION A: QUANTUM MECHANICS 3

Answer Question 1 and **at least one** of Questions 2 and 3.

1. (a) Describe the electric dipole approximation for the case of an atom interacting with an electromagnetic field. [4 marks]

   (b) (i) Why should one expect that the $(n = 2, l = 0, m = 0)$ state of atomic hydrogen is much longer lived than the $(n = 2, l = 1, m = 0)$ state in the absence of any external perturbation? [3 marks]
   (ii) What is the natural width of a spectral line? [1 mark]

   (c) State how one can approximately calculate eigenenergies of a Hamiltonian by using a finite basis formed by orthonormal eigenfunctions of another Hamiltonian. [4 marks]

   (d) Write down the Hamiltonian of Be^{2+}, a helium-like ion formed by a nucleus of beryllium ($Z = 4$) and two electrons. Briefly explain the physical origin of the potential energy term(s) it contains. (Assume no external electric or magnetic field. Neglect spin-orbit coupling and other relativistic effects. Also, assume that the nucleus is a point charge fixed in space rather than an extended distribution of charges of finite mass.) [4 marks]

   (e) Let S_z be the z-component of the total spin operator of a system of two electrons: $S_z = S_{1z}+S_{2z}$, where S_{1z} is the z-component of the spin operator for electron 1 and S_{2z} the z-component of the spin operator for electron 2.
   (i) Show that $S_z\alpha(1)\beta(2) = 0$, where $\alpha(1)$ denotes the spin-up eigenstate of S_{1z} and $\beta(2)$ the spin-down eigenstate of S_{2z}. [2 marks]
   (ii) Write down an eigenstate of S_z with eigenvalue $-\hbar$. Justify your answer by a calculation of this eigenvalue. [2 marks]

   (f) How many levels does the $2\,^2P_{3/2}$ state of He^+ split into under the effect of a weak magnetic field? (Assume you can ignore any hyperfine structure.) [4 marks]

2. Consider a 1D system whose Hamiltonian is H_0 for $t < 0$ and $H_0 + A\exp(i\omega t) + A^\dagger \exp(-i\omega t)$ for $t > 0$, where the initial Hamiltonian H_0, the operators A and $A^\dagger$ and the angular frequency ω are time-independent. This system is initially in state a, a bound eigenstate of H_0 of energy $E_a < 0$. Suppose that it may make a transition to a continuum state of energy $E_\mathrm{f} > 0$ when $t > 0$ and that the transition rate, $R(a \to k_\mathrm{f})$, is given by Fermi's Golden Rule. Thus $R(a \to k_\mathrm{f}) = (2\pi/\hbar)\,\rho_\mathrm{1D}(E_\mathrm{f})\,|A^\dagger_{k_\mathrm{f}a}|^2$, where the energy E_f is related to the wave number k_f labelling the continuum state by the equation $E_\mathrm{f} = \hbar^2 k_\mathrm{f}^2/(2m)$ (m is a mass).

   (a) Define the matrix element $A^\dagger_{k_\mathrm{f}a}$ in terms of the wave functions of the initial and final states, $\psi_a(x)$ and $\psi_{k_\mathrm{f}}(x)$ respectively (x varies from $-\infty$ to $+\infty$ in this problem). [4 marks]

   (b) By what name is the function $\rho_\mathrm{1D}(E)$ often referred to? [1 mark]

   (c) Suppose that $\psi_a(x) = \kappa^{1/2}\exp(-\kappa|x|)$, that $\psi_{k_\mathrm{f}}(x) = \pi^{-1/2}\sin(k_\mathrm{f}x)$ and that $A^\dagger = qFx$, where q and F are two constants and κ is related to the energy of state a by the equation $E_a = -\hbar^2\kappa^2/(2m)$.

      (i) Show that $R(a \to k_\mathrm{f}) \propto (\hbar\omega - |E_a|)^{1/2}/\omega^4$ given that $\rho_\mathrm{1D}(E_\mathrm{f}) = [m/(2\hbar^2 E_\mathrm{f})]^{1/2}$. [8 marks]

$$\left[\text{Hint:} \qquad \int_{-\infty}^{\infty} x\sin(kx)\exp(-\kappa|x|)\,dx = \frac{4k\kappa}{(k^2+\kappa^2)^2}.\right]$$

      (ii) Due to the perturbation represented by the operators A and $A^\dagger$, state a decays to the continuum when $t > 0$. Suppose that this state is stable in the absence of this perturbation and that its lifetime is 10 ns for $F = 1 \times 10^{-10}$ J m^{-1}. What is its lifetime for $F = 2 \times 10^{-10}$ J m^{-1}? Justify your answer. [7 marks]

3. Suppose that besides the electron there would also exist another kind of spin-1/2 particle, the "pseudo-electron", with exactly the same mass and same electric charge as the electron but different in some other respects. Consider the case of an atom of lithium containing two electrons and a pseudo-electron. Make the approximation that the ground state wave function of this system is $\Phi(\underline{r}_1, \underline{r}_2, \underline{r}_\mathrm{p})\chi(1,2)\alpha(\mathrm{p})$ where $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_\mathrm{p}$ are the position vectors of, respectively, electron 1, electron 2 and the pseudo-electron, $\chi(1,2)$ is the joint spin state of the two electrons, $\alpha(\mathrm{p})$ is the spin state of the pseudo-electron, and $\Phi(\underline{r}_1, \underline{r}_2, \underline{r}_\mathrm{p}) = \psi(\underline{r}_1)\psi(\underline{r}_2)\psi(\underline{r}_\mathrm{p})$ with $\psi(\underline{r}) = N\exp(-3\,r/a_0)Y_{00}(\theta,\phi)$. In the last equation, a_0 is the Bohr radius, θ and ϕ are the polar angles of the point of position vector $\underline{r}$, r is the distance of this point to the origin, $Y_{00}(\theta,\phi)$ is the $l = 0$ spherical harmonic, and N is a normalization constant such that

$$\int |\Phi(\underline{r}_1, \underline{r}_2, \underline{r}_\mathrm{p})|^2 \, d^3r_1 \, d^3r_2 \, d^3r_\mathrm{p} = 1.$$

(a) Find the normalization constant N. [7 marks]

$$\left[\text{Hint: } \int_0^\infty r^n \exp(-ar)\, dr = n!/a^{n+1}, \qquad Y_{00}(\theta,\phi) = (4\pi)^{-1/2}.\right]$$

(b) (i) Why are the two electrons necessarily in a singlet spin state? [4 marks]

(ii) The product $\chi(1,2)\alpha(\mathrm{p})$ is an eigenstate of $\underline{S}^2$, where $\underline{S} = \underline{S}_1 + \underline{S}_2 + \underline{S}_\mathrm{p}$ with $\underline{S}_i$ ($i = 1,2$) the spin operator for electron i and $\underline{S}_\mathrm{p}$ the spin operator for the pseudo-electron. What is the corresponding eigenvalue? [3 marks]

(c) Given that

$$E[\Phi] = \int \Phi^*(\underline{r}_1, \underline{r}_2, \underline{r}_\mathrm{p})\, H\, \Phi(\underline{r}_1, \underline{r}_2, \underline{r}_\mathrm{p})\, d^3r_1\, d^3r_2\, d^3r_\mathrm{p} = -214 \text{ eV},$$

where H is the Hamiltonian, and given that the energy of the $2\mathrm{P}_{1/2}$ state is -202 eV, what can you say about the wavelength of the light emitted in a transition from this excited state to the ground state? [6 marks]

SECTION B: NUCLEAR AND PARTICLE PHYSICS
Answer Question 4 and **at least one** of Questions 5, 6, 7 and 8.

4. (a) The semi-empirical mass formula for the mass of a nucleus with N neutrons, Z protons and atomic number $A = Z + N$ contains a volume term, a surface term and an asymmetry term.

$$M(Z, N) = \cdots + a_V A + a_S A^{2/3} + a_A \frac{(N-Z)^2}{4A} + \ldots$$

How would these three terms change if the neutron and proton were bosons and the nucleons arrange themselves as a 2D disk rather than a sphere? [4 marks]

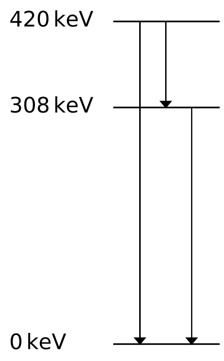
(b) An electron of mass m collides with a proton of mass M. The electron energy in the frame where the proton is at rest is E_{lab}. Calculate the sum of the energies of the proton and the electron in the centre of mass frame of the electron-proton system (i.e. the frame where the total momentum is 0). [4 marks]

(c) The order of the four lowest shells in the nuclear shell model for protons and neutrons is $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$. The next shell is either $2s_{1/2}$ or $1d_{3/2}$. Explain why knowing the spin of $^{33}_{16}$S does not help determining which of the two is next. Which S isotope would be more useful? [4 marks]

(d) A uniform beam of particles A of density 10^6 particles/mm^3 travelling at the speed of light collides with a stationary spherical target of radius 1 mm of particles B with density 9×10^{19} particles/mm^3. The beam is run for 10 seconds and 2×10^6 collisions between a particle A and a particle B are observed. What is the cross section for the interaction between particle A and B? [4 marks]

(e) $^{48}_{24}$Cr decays through β^+-decay to the two first excited states of $^{48}_{23}$V. $^{48}_{24}$Cr has $J^P = 0^+$ and the ground state of $^{48}_{23}$V has $J^P = 4^+$.
The following gamma emission energies are observed:

Energy [keV]	multipolarity
420	M3/E4
308	E2
112	?



No other gamma energies have been seen. Given that the probability for the β-decay into the two excited states is much larger than the probability to decay into the ground state, give the spin and parity of the excited states, explaining your reasoning. What is the multipolarity of the 112 keV transition? [4 marks]

(f) What are all the initial angular momentum and parity combinations J^P that allow the following decays if the interaction conserves parity?

(i) $J^P \rightarrow 0^+0^-$

(ii) $J^P \rightarrow 1^+1^-$

N^P in the reactions above represents a particle with spin N and intrinsic parity P. [4 marks]

(g) Assuming the colour wave function is the same for all baryons and that the spatial wave-function is symmetric, explain why there are no spin $J = 1/2$ baryons with quark content sss. [4 marks]

(h) In an experiment, a beam of electrons collides with a beam of electron anti-neutrinos. They combine through the weak interaction into a W boson, which decays into a muon and a muon anti-neutrino. Are we more likely to see the muon flying in the same direction as the electron or opposite to it? Explain your reasoning. [4 marks]

(i) For each of the following reactions give all single particles X that are allowed in the reaction in the Standard Model. If there is more than one possibility explain which is most likely. [4 marks]

- $\tau^+ \rightarrow \mu^+ \nu_\mu X$
- $\bar{u}d \rightarrow \mu^- X$
- $c \rightarrow \nu_e e^+ X$

5. The form factor for a charge distribution $f(\underline{r})$ is given by:

$$F(\underline{q}) = \int d^3\underline{r} f(\underline{r}) e^{i\underline{q}\cdot\underline{r}} ,$$

where $f(\underline{r})$ is normalised such that

$$\int d^3\underline{r} f(\underline{r}) = 1 .$$

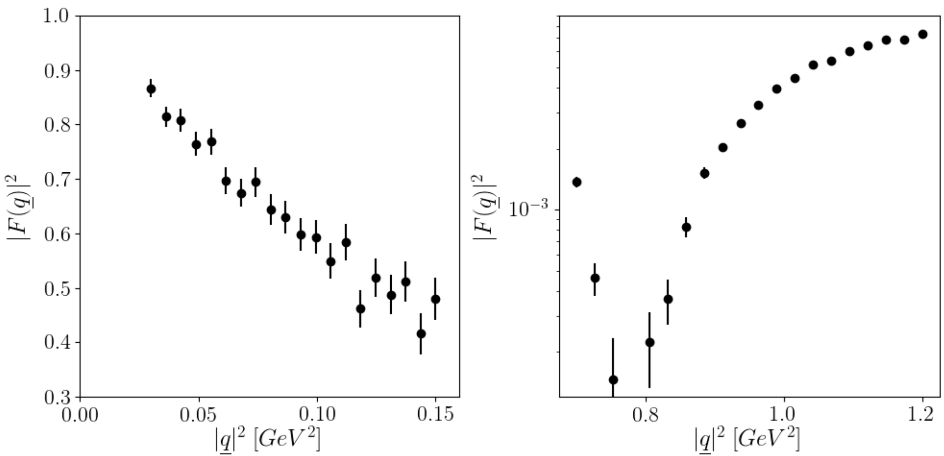
(a) Show that for a homogeneous sphere of radius R the form factor is given by

$$F(\underline{q}) = \frac{3}{a^3}(\sin a - a\cos a) \qquad \text{with} \qquad a = |\underline{q}|R . \qquad [6 \text{ marks}]$$

(b) Explain why the form factor should be 1 in the limit $|q| \to 0$. Show that this is the case by expanding the result of (a) for small values of $|q|$. [4 marks]

$$\left[\text{Hint: } \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \mathcal{O}(x^7) , \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \mathcal{O}(x^6)\right]$$

(c) Could these two measurements of the form factor come from the same spherical charge density ? [5 marks]



[Hint: The zeros of $\sin x - x\cos x$ are located at $x = 0, 4.49341, 7.72525, ...$]

(d) What is the form factor if the charge density is homogeneous in an outer layer of width νR, $0 < \nu < 1$ of the sphere of radius R and zero inside that outer layer? [5 marks]

[Hint: Reuse earlier results to save some integrations.]

6. In this question we consider the parton distribution functions (PDFs) of the proton. We neglect the sea-quark distribution. We will use the functions

$$I_{\alpha,\beta}(x) = x^\alpha(1-x)^\beta$$

to parametrise the PDFs. The following table displays the value of

$$\left(\int_0^1 I_{\alpha,\beta}\, dx\right)^{-1} ,$$

so for example the integral of $I_{3,8}(x)$ from 0 to 1 is 1/1980.

$\alpha\backslash\beta$	0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8	9
1	2	6	12	20	30	42	56	72	90
2	3	12	30	60	105	168	252	360	495
3	4	20	60	140	280	504	840	1320	1980

(a) Assuming $u(x) = u_0 I_{1,4}(x)$ and $d(x) = d_0 I_{1,3}(x)$, calculate the normalisation constants u_0 and d_0. [3 marks]

(b) Assuming the entire momentum of the proton is carried by the valence quarks and the gluons only, and assuming that the gluon PDF is given by

$$g(x) = g_0 I_{1,6}(x) ,$$

determine the value of g_0. [6 marks]

(c) The cross section for low-energy scattering of an electron neutrino with momentum k with a d quark with a fraction x of the proton momentum P is given by

$$\sigma(\nu_e(k_1) + d(xP_1) \to e^-(k_2)u(P_2)) = \frac{xG_F^2 s}{\pi} ,$$

where $s = (k_1 + P_1)^2$ is the invariant mass squared of the initial state and G_F is the Fermi constant. Draw the Feynman diagram for the reaction in the high energy regime and use it to explain what happens in the low energy regime. [3 marks]

(d) A beam of neutrinos of energy 10 GeV collides with a fixed proton target, a cross section of 93 fb is measured for the reaction $\nu p \to e^- X$, where X represents any hadronic final state. Calculate the value of the Fermi constant using the PDFs you determined above. [6 marks]

$$\left[\text{Hints: } m_p = 0.9383\,\text{GeV}, \quad 0.389\,\text{mb}\,\text{GeV}^2 = 1, \quad 1\text{fb} = 10^{-12}\text{mb}\right]$$

(e) What part of the sea quark distribution is the experiment in (d) also sensitive to? [2 marks]

7. The baryons which contain a single bottom quark are:

$$\Lambda_b^0(bud, I=0); \quad \Sigma_b^-, \Sigma_b^0, \Sigma_b^+(bdd, bud, buu, I=1); \quad \Xi_b^-, \Xi_b^0(bds, bus); \quad \Omega_b^-(bss),$$

where I is the isospin of the baryon.

(a) What are the possible spins S_{qq} of the pair of light quarks in each of these baryons? Use the spins of the light quark pairs to obtain the possible spins of the baryons and hence show that there are nine $J=\frac{1}{2}$ and six $J=\frac{3}{2}$ baryons containing a single bottom quark. [6 marks]

[Hint: If there are baryons, X, with the same quark content it is conventional to call the $J=\frac{3}{2}$ states X^* and the $J=\frac{1}{2}$ state X. If there are two $J=\frac{1}{2}$ states the one with a $S=1$ light quark pair is called X' and the one with a $S=0$ quark pair X.]

The hyperfine mass splitting for baryons is

$$\Delta M_{ss} = \frac{16}{9}\pi\alpha_S\,|\psi(0)|^2 \sum_{\substack{i,j=1\\ i<j}}^{3} \frac{\vec{s}_i\cdot\vec{s}_j}{m_i m_j},$$

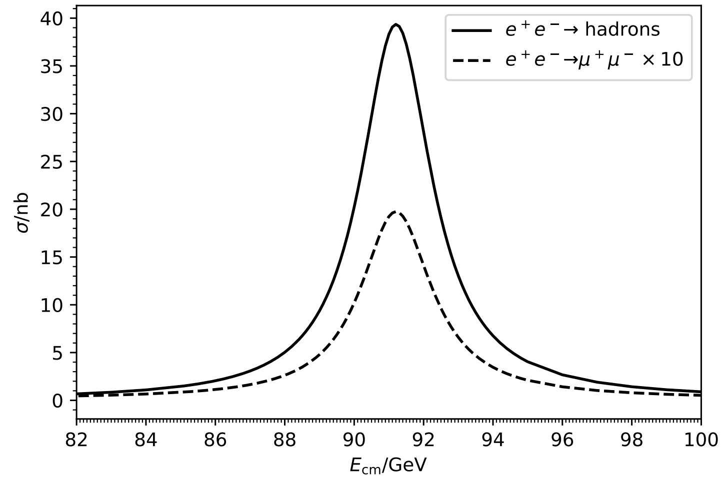
where the sum is over the three quarks in the baryon, and $\vec{s}_i$ and m_i are the spin and constituent masses of the quark respectively. Here $|\psi(0)|^2$ is the probability that the two quarks are at the same point in space.

(b) Show that for the spin$-\frac{1}{2}$ baryons the spins of the quarks satisfy the following equation [2 marks]

$$\sum_{\substack{i,j=1\\ i<j}}^{3} \vec{s}_i\cdot\vec{s}_j = -\frac{3}{4}.$$

(c) Assuming that the up and down quark constituent masses are equal calculate the masses of the $J=\frac{1}{2}$ Λ_b^0 and Ω_b^- as a function of the quark masses and $\alpha_S\,|\psi(0)|^2$. [12 marks]

8. The cross sections for $e^+e^- \to$ hadrons and $e^+e^- \to \mu^+\mu^-$ are shown below as a function of the centre-of-mass energy of the electron-positron collision, $E_{cm} = \sqrt{s}$, where $s = (p_1 + p_2)^2$, and p_1, p_2 are the electron and positron four-momenta.



(a) Draw the leading-order Feynman diagrams for the processes $e^+e^- \to \mu^+\mu^-$, $e^+e^- \to q\bar{q}$. Explain which of the diagrams represents the dominant process for the regions (i) $s \ll M_Z^2$ and (ii) $s \simeq M_Z^2$. [6 marks]

(b) Assuming the masses of the electron and muons can be neglected, use a dimensional argument to deduce the scaling of the cross section in the region $s \ll M_Z^2$ as a function of s. [2 marks]

(c) If the coupling of a quark to the Z boson was the same as the coupling of a muon to the Z boson, what would you expect the following ratio to be? [2 marks]

$$\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

The Breit-Wigner cross section for a resonance R is

$$\sigma_{i\to f}(s) = 12\pi \frac{\Gamma_{R\to i}\Gamma_{R\to f}}{(s - M^2)^2 + M_R^2\Gamma_{R\ \text{total}}^2},$$

where M_R is the mass of the resonance, $\Gamma_{R\ \text{total}}$ is the total width of the particle, and $\Gamma_{R\to i,f}$ is the partial width for the decay of the particle into the initial- and final-state particles, respectively.

(d) Assuming that $\Gamma_{R\to e^+e^-} = \Gamma_{R\to\mu^+\mu^-} = \Gamma_{R\to\tau^+\tau^-}$ use the plot of the cross section near the peak to obtain $\Gamma_{R\ \text{total}}$, $\Gamma_{R\to\mu^+\mu^-}$, $\Gamma_{R\to\text{hadrons}}$ and the mass of the resonance. [6 marks]

$$\left[\begin{array}{l}\text{Hint: } (\hbar c)^2 = 0.389\,\text{GeV}^2\text{mb and the cross section for } e^+e^- \to \\ \mu^+\mu^- \text{ has been multiplied by ten in the main figure.}\end{array}\right]$$

(e) Assuming there are no other observed decay channels than into hadrons or leptons, calculate the partial width for the decay of the resonance into invisible particles and explain the significance of this result. [4 marks]

