University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 043621/01

LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3A

SECTION A. QUANTUM MECHANICS 3
SECTION B. NUCLEAR AND PARTICLE PHYSICS

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU = 1.50×10^{11} m
Parsec:	pc = 3.09×10^{16} m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. QUANTUM MECHANICS 3

Answer Question 1 and **at least one** of Questions 2 and 3.

1. (a) What is the dipole approximation? What is the sudden approximation? [4 marks]

   (b) The lifetime of the $2\mathrm{p}_{m=0}$ state of hydrogen is 1.6 ns. This state has a 100% probability to decay to the ground state. Calculate the corresponding A-coefficient and the natural width in frequency of the corresponding spectral line. The lifetime of the $3\mathrm{p}_{m=0}$ state, which has a 88% probability to decay to the ground state and a 12% probability to decay to another excited state, is 5.4 ns. What makes the $3\mathrm{p}_{m=0}$ state longer lived than the $2\mathrm{p}_{m=0}$ state? [4 marks]

   (c) Give an example of a radiative transition between states of atomic hydrogen forbidden by the electric dipole selection rules on the magnetic quantum number, and give a mathematical proof of the selection rule forbidding that particular transition. [4 marks]

   (d) What are the possible values of J, the total angular momentum quantum number, for a state with an orbital quantum number L and a spin quantum number S? Give the possible values of J for the following spectral terms: (i) ^{1}S, (ii) ^{2}D, (iii) ^{4}P. [4 marks]

   (e) Neglecting spin-orbit coupling, singlet and triplet states of helium are eigenstates of the operator $\underline{S}^2$, where $\underline{S} = \underline{S}_1 + \underline{S}_2$ and $\underline{S}_1$ and $\underline{S}_2$ are the spin operators for electron 1 and for electron 2, respectively. What are the corresponding eigenvalues? Hence, show that these states are also eigenstates of the operator

   $$\mathcal{J} - \frac{\mathcal{K}}{2}\left[1 + \frac{2}{\hbar^2}\left(\underline{S}^2 - \underline{S}_1^2 - \underline{S}_2^2\right)\right],$$

   where $\mathcal{J}$ and $\mathcal{K}$ are two constants, and find the corresponding eigenvalues. [4 marks]

   (f) State what is meant by the terms gerade and ungerade as applied to the molecular orbitals of homonuclear diatomic molecules. Briefly explain, through a simple example or otherwise, why gerade orbitals are bonding and ungerade orbitals antibonding. [4 marks]

2. A time-independent Hamiltonian, H_0, is perturbed by a time-dependent perturbation, $H'(t)$. The perturbation Hamiltonian, $H'(t)$, is negligible for $t \rightarrow \pm\infty$ but not for $t \approx 0$. The system for which $H_0 + H'(t)$ is the total Hamiltonian is in the eigenstate a of H_0 before $H'(t)$ becomes non-negligible. The probability $P(a \rightarrow b)$ that it is in the eigenstate b after $H'(t)$ has become negligible again is given by the following equation, in first order perturbation theory and assuming that $\omega_{ba} \neq 0$:

$$P(a \rightarrow b) = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} H'_{ba}(t) \exp(i\omega_{ba} t) dt \right|^2 .$$

Define the quantities ω_{ba} and $H'_{ba}(t)$ appearing in this equation in terms of eigenenergies and eigenfunctions of H_0. Briefly explain how this equation is obtained, starting from the time-dependent Schrödinger equation. [8 marks]

An atom of hydrogen, initially in the ground state, is perturbed by a negatively charged muon passing in its vicinity. The atomic nucleus remains at rest at the origin of the system of coordinates. The muon follows a straight line trajectory with the equation $X = X_0, Y = Y_0, Z = vt$, where X, Y and Z are the muon's x-, y- and z-coordinates and X_0, Y_0 and v are three constants.

(i) Assuming that the distances X_0 and Y_0 are much larger than the Bohr radius, one can make the approximation that

$$H'(t) = \frac{e^2}{4\pi\epsilon_0} \left[\frac{1}{R(t)} + \frac{xX_0 + yY_0 + vtz}{R^3(t)} \right],$$

where x, y and z are the x-, y- and z-coordinates of the atomic electron and $R(t) = (X_0^2 + Y_0^2 + v^2t^2)^{1/2}$. Show that within this approximation $P(1\text{s} \rightarrow 2\text{p}_{m=0})$ does not depend on the sign of v and that $P(1\text{s} \rightarrow 2\text{p}_{m=1}) = P(1\text{s} \rightarrow 2\text{p}_{m=-1})$. [9 marks]

(ii) What would you take for $H'(t)$ if X_0 and Y_0 could not be assumed to be much larger than the Bohr radius? [3 marks]

$$\left[\begin{array}{c} Y_{l=0,m=0}(\theta,\phi) = \left(\frac{1}{4\pi}\right)^{1/2}, \quad Y_{l=1,m=0}(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta, \\ Y_{l=1,m=\pm1}(\theta,\phi) = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\,(\cos\phi \pm i\sin\phi), \\ \text{where } r,\theta,\phi \text{ are such that } x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta. \end{array} \right]$$

3. Outline the principles of the Rayleigh-Ritz variational method for the calculation of the ground state energy of a quantum system. [5 marks]

   Let $\phi(\underline{r}_1, \underline{r}_2; \lambda) = (\lambda/a_0)^3(1/\pi)\exp(-\lambda|\underline{r}_1|/a_0)\exp(-\lambda|\underline{r}_2|/a_0)$, where λ is a parameter ($\lambda > 0$), a_0 is the Bohr radius (approximately 0.529 Å), and $\underline{r}_1$ and $\underline{r}_2$ are the position vectors of the two electrons of a helium atom with respect to the nucleus.

   (i) Show that $\int |\phi(\underline{r}_1, \underline{r}_2; \lambda)|^2 \, d\underline{r}_1 d\underline{r}_2 = 1$. [5 marks]

   $$\left[\text{Hint: } \int_0^\infty r^n \exp(-ar) dr = n!/a^{n+1}.\right]$$

   (ii) Given that the expectation value of the Hamiltonian of helium in the state with the wave function $\phi(\underline{r}_1, \underline{r}_2; \lambda)$ is $(\lambda^2 - 27\lambda/8)\, e^2/(4\pi\epsilon_0 a_0)$, show that the best estimate of the ground state energy of this atom one can obtain by using $\phi(\underline{r}_1, \underline{r}_2; \lambda)$ as a trial function is approximately -77.4 eV. [5 marks]

   (iii) Why is $\phi(\underline{r}_1, \underline{r}_2; \lambda)$ a suitable trial function for calculating the ground state energy although this function has no spin dependence? [2 marks]

   In contrast to the case of helium above, why would a variational calculation of the ground state energy of lithium ($Z = 3$) based on the normalized trial function

   $$(\lambda/a_0)^{9/2}(1/\pi)^{3/2}\exp(-\lambda|\underline{r}_1|/a_0)\exp(-\lambda|\underline{r}_2|/a_0)\exp(-\lambda|\underline{r}_3|/a_0)$$

   give no information on the true value of this energy? [3 marks]

SECTION B. NUCLEAR AND PARTICLE PHYSICS

Answer Question 4 and **at least one** of Questions 5, 6, 7 and 8.

4. (a) Sketch the distribution of stable nuclei on a plot of atomic mass number Z versus neutron number N including the regions occupied by unstable nuclei with different decay modes. Briefly explain the main features of this plot. [4 marks]

(b) Explain, with the aid of an appropriate diagram, how there can be two nuclei with the same, even, atomic mass number which cannot undergo beta decay. [4 marks]

(c) A beam of positively charged pions, π^+, collides with a stationary proton target. What is the minimum energy of the pion beam required to produce the Δ^{++} resonance given that the masses of the Δ^{++}, π^+ and proton are $1232\,\mathrm{MeV}/c^2$, $139.6\,\mathrm{MeV}/c^2$ and $938.3\,\mathrm{MeV}/c^2$, respectively? [4 marks]

(d) What key property of the nuclear force is explained by the concept of isospin? Give the isospin quantum numbers of the proton and neutron. Explain how isospin is related to the nuclear energy levels of ${}^{10}_{4}\mathrm{Be}_6$, ${}^{10}_{5}\mathrm{B}_5$ and ${}^{10}_{6}\mathrm{C}_4$ [4 marks]

(e) In the independent particle shell model of the nucleus the first five energy levels in order of increasing energy are $1\mathrm{s}_{\frac{1}{2}}$, $1\mathrm{p}_{\frac{3}{2}}$, $1\mathrm{p}_{\frac{1}{2}}$, $1\mathrm{d}_{\frac{5}{2}}$ and $2\mathrm{s}_{\frac{1}{2}}$. What values are predicted by the independent particle shell model for the spin, parity, and magnetic dipole moment of the ${}^{30}_{14}\mathrm{Si}$ nucleus. [4 marks]

(f) What is the quark content and isospin of the Δ^{++} baryon? Use the symmetry of the Δ^{++} wavefunction to explain why the colour quantum number is required. [4 marks]

(g) Calculate the ratio of the hadronic to muonic e^+e^- cross section

$$R = \frac{\sigma(e^+e^- \rightarrow \ \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)},$$

when the centre-of-mass energy is between twice the mass of the charm and bottom quarks. [4 marks]

(h) For each of the following reactions state whether or not it is allowed in the Standard Model:

i) $d\bar{d} \rightarrow t\bar{t}$;
ii) $W^+\gamma \rightarrow e^+\gamma$;
iii) $\nu_\mu e^- \rightarrow \nu_\mu e^-$;
iv) $\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu$.

If the reaction is allowed draw at least one Feynman diagram responsible for it. If it is forbidden, explain why. [4 marks]

(i) List the Standard Model fermions arranging them in families (generations) and indicate which interactions they experience. [4 marks]

5. a) Show that if the nucleus is assumed to be a uniformly charged sphere of radius R the Coulomb energy for a nucleus with atomic number Z is

$$E_C = \frac{3}{5}\frac{\alpha}{R}Z(Z-1),$$

where the fine-structure constant $\alpha = 1/137$. [8 marks]

The semi-empirical mass formula predicts that the nuclear binding energy for a nucleus with mass number A and neutron number N is

$$B = a_V A - a_S A^{\frac{2}{3}} - a_c \frac{Z^2}{A^{\frac{1}{3}}} - a_a \frac{(N-Z)^2}{4A} - \frac{\delta}{A^{\frac{1}{2}}}$$

where $a_V = 15.67\,\mathrm{MeV}$, $a_S = 17.23\,\mathrm{MeV}$, $a_c = 0.714\,\mathrm{MeV}$, $a_a = 93.15\,\mathrm{MeV}$ and

$$\delta = \begin{cases} -11.2\,\mathrm{MeV} & \text{for even } Z \text{ and } N \text{ (even} - \text{even nuclei)}, \\ 0\,\mathrm{MeV} & \text{for odd } A \text{ (odd} - \text{even nuclei)}, \\ 11.2\,\mathrm{MeV} & \text{for odd } Z \text{ and } N \text{ (odd} - \text{odd nuclei)}. \end{cases}$$

b) The nuclear radius $R \approx r_0 A^{\frac{1}{3}}$, where $r_0 = 1.2\,\mathrm{fm} = 0.006\,\mathrm{MeV}^{-1}$. Calculate a theoretical value for a_c using the result of part a). [2 marks]

[Hint: $\hbar c = 197\,\mathrm{MeVfm}$.]

c) A neutron star consists solely of neutrons and is bound together by the gravitational force. Use the semi-empirical mass formula, together with the contribution from the gravitational binding energy, to estimate the minimum radius of a neutron star. [10 marks]

[Hint: The surface and pairing terms can be neglected, the neutron mass $m_{\mathrm{n}} = 939.6\,\mathrm{MeV}/c^2$ and $G = 6.708 \times 10^{-45}\hbar c\,(\mathrm{MeV}/c^2)^{-2}$]

6. The differential cross section for the scattering of an electron from an extended charge distribution is

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{exp}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{*}_{\mathrm{Mott}} \left|F(\underline{q}^2)\right|^2,$$

where $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{*}_{\mathrm{Mott}}$ is the Mott scattering cross section and the form factor

$$F(\underline{q}^2) = \int \exp\left(i\underline{q}\cdot\underline{x}\right) f(\underline{x})\mathrm{d}^3x,$$

where $\underline{q} = \underline{k} - \underline{k}'$ is the change in the momentum of the electron, $\underline{k}$ is the momentum of the incoming electron, $\underline{k}'$ is the momentum of the scattered electron and $f(\underline{x})$ is the charge distribution.

a) Briefly discuss how the form of the charge distribution affects the form factor. [3 marks]

b) Show that for a spherically symmetric charge distribution

$$F(\underline{q}^2) = \frac{4\pi}{|\underline{q}|}\int f(r)r\sin\left(|\underline{q}|r\right)\mathrm{d}r.$$

[3 marks]

c) For a homogeneous charge distribution with radius R

$$f(r) = \begin{cases} \frac{3}{4\pi R^3} & r \leq R, \\ 0 & r > R, \end{cases}$$

show that

$$F(\underline{q}^2) = \frac{3}{(|\underline{q}|R)^3}\left[\sin\left(|\underline{q}|R\right) - |\underline{q}|R\cos\left(|\underline{q}|R\right)\right].$$

[6 marks]

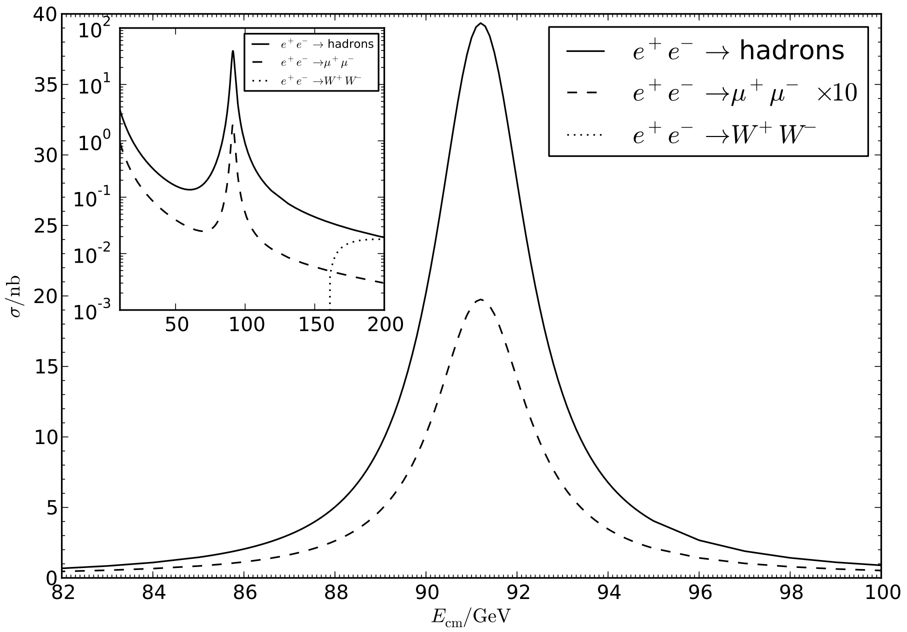
d) The second minimum of the cross section for the scattering of electrons with an energy of 302 MeV from $^{90}_{30}\mathrm{Zr}$ occurs at an angle of 60^0. Calculate the radius of the Zirconium nucleus. [3 marks]

[Hint: The solutions of $\tan x = x$ are $x = 0, 4.493, 7.725, \ldots$ and $\hbar c = 197\,\mathrm{MeVfm}$.]

e) The gradient $\left.\frac{\mathrm{d}F(|\underline{q}|^2)}{\mathrm{d}|\underline{q}|^2}\right|_{|\underline{q}|^2\to 0} = -0.532\,\mathrm{fm}^2$ for the scattering of 198.5 MeV electrons from $^{11}_{5}\mathrm{B}$. By expanding the form factor for small values of $|\underline{q}|$, or otherwise, calculate the radius of the Boron nucleus. [5 marks]

[Hint: $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$, $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$]

7. The cross sections for $e^+e^- \to$ hadrons, $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to W^+W^-$ are shown below as a function of the centre-of-mass energy of the electron-positron collision. The main figure shows the cross section near the resonance peak, while the inset frame shows the cross section over a wider energy range.



a) Draw the leading-order Feynman diagrams for all the processes and use them to explain the key features of the dependence of the cross section on the centre-of-mass energy of the collisions shown in the inset frame. [10 marks]

The Breit-Wigner cross section for a resonance R is

$$\sigma_{i\to f}(s) = 12\pi \frac{\Gamma_{R\to i}\Gamma_{R\to f}}{\left(s - M_R^2\right)^2 + M_R^2\Gamma_{R\ \rm total}^2},$$

where s is the centre-of-mass energy squared, M_R is the mass of the resonance, $\Gamma_{R\ \rm total}$ is the total width of the particle, and $\Gamma_{R\to i,f}$ is the partial width for the decay of the particle into the initial- and final-state particles, respectively.

b) Assuming that $\Gamma_{R\to e^+e^-} = \Gamma_{R\to\mu^+\mu^-} = \Gamma_{R\to\tau^+\tau^-}$ use the plot of the cross section near the peak to obtain $\Gamma_{R\ \rm total}$, $\Gamma_{R\to\mu^+\mu^-}$, $\Gamma_{R\to\rm hadrons}$ and the mass of the resonance. [6 marks]

[Hint: $(\hbar c)^2 = 0.389\,\mathrm{GeV^2 mb}$ and the cross section for $e^+e^- \to \mu^+\mu^-$ has been multiplied by ten in the main figure.]

c) Calculate the partial width for the decay of the resonance into invisible particles and explain the significance of this result. [4 marks]

8. The quark-antiquark spin-spin interaction potential is

$$V_{\mathrm{ss}}(q\bar{q}) = \frac{32\pi\alpha_S}{9}\frac{\underline{s}_q \cdot \underline{s}_{\bar{q}}}{m_q m_{\bar{q}}}\delta(\underline{x}),$$

where α_S is the strong coupling constant and $\underline{s}_{q,\bar{q}}$ and $m_{q,\bar{q}}$ are respectively, the spins and constituent masses of the quark and antiquark.

a) Show that this interaction gives a contribution

$$\Delta M_{\mathrm{ss}} = g_{P,V}\frac{8\pi}{9m_q m_{\bar{q}}}\alpha_S\left|\psi(0)\right|^2,$$

to the masses of pseudoscalar and vector mesons where $\psi(0)$ is the meson wavefunction at the origin and $g_{P,V}$ are constants for the pseudoscalar and vector mesons, respectively. Determine the constants $g_{P,V}$. [8 marks]

b) The mass of the pseudoscalar π^+ meson ($|\pi^+\rangle = |u\bar{d}\rangle$) is $139.6\,\mathrm{MeV}/c^2$ while the mass of the vector ρ^+ meson ($|\rho^+\rangle = |u\bar{d}\rangle$) is $775.5\,\mathrm{MeV}/c^2$. Assuming that the constituent masses of the up and down quarks are equal and that $\alpha_S\left|\psi(0)\right|^2$ is the same for all pseudoscalar and vector mesons calculate the constituent masses of the up and down quarks and extract $\alpha_S\left|\psi(0)\right|^2$. [4 marks]

c) The constituent mass of the strange quark is $483\,\mathrm{MeV}/c^2$. Use the results from part b) to calculate the masses of the pseudoscalar K^+ ($|K^+\rangle = |u\bar{s}\rangle$) and vector ϕ ($|\phi\rangle = |s\bar{s}\rangle$) mesons. [4 marks]

d) Using your results how do you expect the ϕ meson to decay? Draw the Feynman diagram for the dominant decay mode. [4 marks]