# University of Durham

## EXAMINATION PAPER

May/June 2017 Examination code: PHYS2631-WE01

**THEORETICAL PHYSICS 2**

**SECTION A**. Classical Mechanics
**SECTION B**. Quantum Theory 2

**Time allowed:** 3 hours

**Additional material provided:** None

**Materials permitted:** None

**Calculators permitted:** Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

**Visiting students may use dictionaries:** No

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**Instructions to candidates:**

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

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**Information**

A list of physical constants is provided on the next page.

 

## Information

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|---|---|
| Elementary charge: | $e = 1.60 \times 10^{-19}$ C |
| Speed of light: | $c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$ |
| Boltzmann constant: | $k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$ |
| Bohr magneton: | $\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$ |
| Electron mass: | $m_\mathrm{e} = 9.11 \times 10^{-31}$ kg |
| Gravitational constant: | $G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$ |
| Proton mass: | $m_\mathrm{p} = 1.67 \times 10^{-27}$ kg |
| Planck constant: | $h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$ |
| Permittivity of free space: | $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$ |
| Magnetic constant: | $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$ |
| Molar gas constant: | $R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$ |
| Avogadro's constant: | $N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$ |
| Gravitational acceleration at Earth's surface: | $g = 9.81\ \mathrm{m\,s^{-2}}$ |
| Stefan-Boltzmann constant: | $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$ |
| Astronomical Unit: | $\mathrm{AU} = 1.50\ \times\ 10^{11}$ m |
| Parsec: | $\mathrm{pc} = 3.09\ \times\ 10^{16}$ m |
| Solar Mass: | $M_\odot = 1.99\ \times\ 10^{30}$ kg |
| Solar Luminosity: | $L_\odot = 3.84\ \times\ 10^{26}$ W |

**SECTION A**. CLASSICAL MECHANICS

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) A plane pendulum consists of a point mass attached to a massless, rigid rod. The rod is fixed at one end and confined to move in a plane. Write down 2 constraint equations for the system, specifying explicitly the variables being used. [4 marks]

   (b) Using $q$ as the displacement from equilibrium, write down the Lagrangian for a mechanical system with 1 degree of freedom in the vicinity of an equilibrium configuration. How does the Lagrangian differ between stable and unstable equilibrium configurations? [4 marks]

   (c) What does a Green's function represent? Explain, briefly, how they can be used to find the motion of a driven linear oscillator. [4 marks]

   (d) A mechanical system consists of two equal point masses connected by a massless spring, constrained to move along the line connecting them. How many degrees of freedom does this system have? Describe the normal modes for small oscillations about the equilibrium extension of the spring. [4 marks]

   (e) What is a central force? For a system of two point masses interacting only via a central force, the Lagrangian is translationally invariant. What consequence does this have for the motion of the centre of mass? [4 marks]

   (f) The Poisson bracket of two arbitrary functions $J$ and $K$ is given by

$$\{J, K\} = \frac{\partial J}{\partial q}\frac{\partial K}{\partial p} - \frac{\partial J}{\partial p}\frac{\partial K}{\partial q}.$$

   Using the implicit transformation equations $p = \partial F/\partial q$ and $P = -\partial F/\partial Q$, determine whether or not the generating function $F = Qe^q$ produces a canonical transformation. [4 marks]

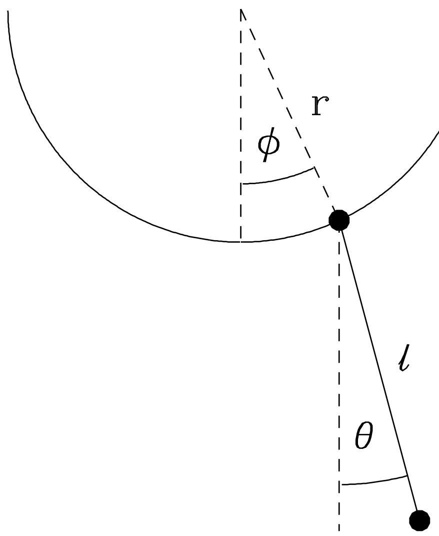
   (g) The Coriolis force on a mass $m$ is $\underline{F} = -2m\underline{\omega} \times \dot{\underline{r}}$. What gives rise to this force and what are $\underline{\omega}$ and $\dot{\underline{r}}$? In which direction does the Coriolis force act on the Newcastle to Carlisle train as it heads due west? [4 marks]

   (h) Euler's equations of motion for a rigid body are

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= N_1, \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= N_2, \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= N_3. \end{aligned}$$

   Explain, briefly, which quantities the symbols in these equations represent, making clear what coordinate system is being used. [4 marks]

2. Two beads of mass $m$ are connected by a massless rigid rod of length $l$. One of the beads is constrained to move on a circular wire of radius $r$. There is a uniform gravitational field of strength $g$. The displacements from vertical of the two beads are given by angles $\phi$ and $\theta$, as shown in the diagram.




(a) Show that the kinetic and potential energies of the system can be written as

$$T = \frac{ml^2}{2}\left[2(\lambda\dot{\phi})^2 + \dot{\theta}^2 + 2\lambda\dot{\phi}\dot{\theta}(\cos\phi\cos\theta + \sin\phi\sin\theta)\right]$$

and

$$V = 2mg\lambda l(1-\cos\phi) + mgl(1-\cos\theta)$$

respectively, where $\lambda = r/l$. [6 marks]

(b) Consider the case of small oscillations around the stable equilibrium configuration. Use the matrix formulation of the Euler-Lagrange equations, $\hat{\tau}\underline{\ddot{q}} + \hat{\upsilon}\underline{q} = 0$, where the elements of the $2\times 2$ matrices are given by

$$\tau_{jk} = \frac{1}{2}\frac{\partial^2 T}{\partial\dot{q}_j\partial\dot{q}_k} \qquad \text{and} \qquad \upsilon_{jk} = \frac{1}{2}\frac{\partial^2 V}{\partial q_j \partial q_k},$$

and $\underline{q} = (\phi, \theta)$, to show that the normal modes satisfy

$$\frac{ml^2}{2}\begin{pmatrix} 2\lambda\omega_0^2 - 2\lambda^2\omega^2 & -\lambda\omega^2 \\ -\lambda\omega^2 & \omega_0^2 - \omega^2 \end{pmatrix}\underline{b} = 0,$$

where $\omega_0^2 = g/l$, $\omega$ is the normal mode frequency and $\underline{b}$ is the mode vector. [7 marks]

(c) Find the normal mode frequencies as a function of $\lambda$ and, for the case $\lambda = 3/4$, find the relative amplitudes of the oscillations of the two generalised coordinates for the in-phase and antiphase modes. [7 marks]

3. For continuous rigid bodies, the elements of the inertia tensor are defined as

$$I_{\alpha\beta} = \int_{\text{volume}} dxdydz\rho(x,y,z)(r^2\delta_{\alpha\beta} - r_\alpha r_\beta),$$

where $\rho$ is the mass density, $\delta_{\alpha\beta}$ is the Kronecker delta and $\underline{r} = (x, y, z)$ is the displacement from the point with respect to which the inertia tensor is being determined. $\alpha$ and $\beta$ run from 1 to 3, such that $r_1 \equiv x$, $r_2 \equiv y$ and $r_3 \equiv z$.

(a) A uniform density cuboid of mass $M$ has sides of length $2a$, $2b$ and $2c$.

(i) Show that the inertia tensor with respect to the centre of mass is

$$\hat{I}_{\mathrm{C}} = \frac{M}{3}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & c^2+a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}.$$

[6 marks]

(ii) The displaced axis theorem relates the inertia tensor with respect to the centre of mass, $\hat{I}_{\mathrm{C}}$, to that with respect to another origin, $\hat{I}$, via

$$\hat{I} = \hat{I}_{\mathrm{C}} + M\hat{A},$$

where the elements of matrix $\hat{A}$ are determined by the elements of the centre of mass position vector, $\underline{R}_{\mathrm{C}}$, through

$$A_{\alpha\beta} = R_C^2\delta_{\alpha\beta} - R_{C,\alpha}R_{C,\beta}.$$

Using the displaced axis theorem, or otherwise, find the inertia tensor for the cuboid with respect to a point in the centre of a face with sides $2b$ and $2c$. [3 marks]

A car with an open door accelerates forwards from rest with constant acceleration $f$. The door swings freely from its hinge at the front and is hence closed as a result of the acceleration. You should approximate the door as a plane with height $2c$ and width $2a$.

(b) By splitting the kinetic energy of the door into components due to the translation of the centre of mass and rotation about the centre of mass, or otherwise, show that the Lagrangian can be written as

$$L = \frac{M}{2}\left(\frac{4}{3}a^2\dot{\theta}^2 + f^2t^2 + 2aft\dot{\theta}\sin\theta\right),$$

where $\theta$ is the opening angle of the door and $t$ is the time since the car started accelerating. [6 marks]

(c) Using the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

find an expression for the angular acceleration of the door, $\ddot{\theta}$. Use this to determine how long the door takes to close for the case when it is initially open by a very small amount, i.e. $\theta(t = 0) \ll 1$. [5 marks]

**SECTION B**. QUANTUM THEORY 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) What condition is required for a set of state vectors $|\phi_1\rangle,\ |\phi_2\rangle\ \ldots,\ |\phi_n\rangle$ to be linearly independent? What property is satisfied if they also form an orthonormal set? [4 marks]

(b) What is the defining property of a Hermitian operator? Show that such operators have real eigenvalues. [4 marks]

(c) Given a state vector $|\psi\rangle$, write down the corresponding wavefunction $\psi(x)$ in Dirac notation. By inserting a complete set of position eigenstates $|x\rangle$, obtain the inner product of state vectors $\langle\phi|\psi\rangle$ as an overlap integral of wavefunctions. [4 marks]

(d) What is the defining property of a unitary operator? Show that if $\hat{A}$ is a unitary operator and if $\hat{A}|\phi\rangle = |\phi'\rangle$, $\hat{A}|\psi\rangle = |\psi'\rangle$, then $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$, so that inner products of states are preserved under a unitary operation. [4 marks]

(e) Fermionic annihilation and creation operators satisfy the algebra

$$\{\hat{b},\hat{b}^\dagger\} = \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = \hat{1}\ .$$

Show that the possible eigenvalues of the number operator $\hat{N} = \hat{b}^\dagger\hat{b}$ are $n = 0, 1$. Comment on the physical significance of this result. [4 marks]

(f) Write down the time evolution equation for an operator $\hat{O}$ in the Heisenberg picture with Hamiltonian $\hat{H}$. If the operator corresponds to a conserved physical quantity what condition is satisfied? [4 marks]

(g) Give the general expression for the commutator of angular momenta $[\hat{L}_i,\hat{L}_j]$, where $\{i,j,k\} = \{x,y,z\} = \{1,2,3\}$. Evaluate $[\hat{L}_x,\hat{L}_z]$. [4 marks]

5. Consider a particle in a state with the wavefunction

$$\psi(x, y, z) = N(x + y + z) \exp\left(-\frac{(x^2 + y^2 + z^2)}{\alpha^2}\right) ,$$

where $N$ is a normalization constant and $\alpha$ a parameter. We measure the values of $\hat{L}^2$ and $\hat{L}_z$, where $\underline{\hat{L}}$ is the angular momentum. Write the wavefunction in spherical polar coordinates and decompose as a sum of spherical harmonics. [14 marks]

Hence, find the probabilities that the measurements yield:

(a) $\hat{L}^2 = 2\hbar^2$, $\hat{L}_z = 0$,
(b) $\hat{L}^2 = 2\hbar^2$, $\hat{L}_z = \hbar$,
(c) $\hat{L}^2 = 2\hbar^2$, $\hat{L}_z = -\hbar$.

[6 marks]

$$\left[\begin{array}{c} \text{Hint: } x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r \cos\theta\ . \\ Y_{11}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}\ \ ,\ \ Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta\ \ ,\ \ Y_{1-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}\ . \end{array}\right]$$

6. Consider the Hamiltonian, $\hat{H}$, for an electron with magnetic moment of magnitude $\mu$, at rest in a magnetic field $\underline{B} = (B_x, 0, B_z)$. The matrix elements with respect to the orthonormal basis states $|\pm\rangle$ of electron spin $\pm\frac{1}{2}\hbar$ along the $z$-axis will be

$$\langle +|\hat{H}|+\rangle = -\mu B_z \quad , \quad \langle +|\hat{H}|-\rangle = -\mu B_x$$
$$\langle -|\hat{H}|+\rangle = -\mu B_x \quad , \quad \langle -|\hat{H}|-\rangle = +\mu B_z \; .$$

The Hamiltonian matrix is then

$$\hat{H} = \begin{pmatrix} \langle +|\hat{H}|+\rangle & \langle +|\hat{H}|-\rangle \\ \langle -|\hat{H}|+\rangle & \langle -|\hat{H}|-\rangle \end{pmatrix} \; .$$

(a) Compute the eigenvalues and eigenkets of the Hamiltonian matrix in terms of the orthonormal states $|\pm\rangle$. [12 marks]

(b) Show that an orthonormal pair of eigenkets can be written in the form

$$|\psi_+\rangle = \frac{1}{\sqrt{1+k^2}}(k|+\rangle + |-\rangle) \; , \; |\psi_-\rangle = \frac{1}{\sqrt{1+k^2}}(|+\rangle - k|-\rangle) \; ,$$

and give an expression for $k$ in terms of the components of $\underline{B}$. [4 marks]

(c) If the system is prepared in the state $|\psi(0)\rangle = |+\rangle$ at time $t = 0$ determine the time evolution of the state at later times, $|\psi(t)\rangle$. [4 marks]