Mathematical Methods in Physics

Workshop 9

9.1

Verify the divergence theorem for the vector field $\mathbf{a} = xy\,\mathbf{i}$ and volume V, where V is the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, that is calculate both sides of the following expression

$$\int_V (\nabla \cdot \mathbf{a})\, dV = \int_S \mathbf{a} \cdot d\mathbf{S}.$$

[*Hint: Use Cartesian coordinates in order to parametrise the surface S. Remember that $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ where $\hat{\mathbf{n}}$ is normal to each side of the cube and dS represent the infinitesimal scalar element of each side of the cube. For instance, consider the side of the cube in the xy plane. Its $d\mathbf{S}$ is $-\mathbf{k}\, dxdy$.*]

9.2

Verify Stokes' theorem for the vector field $\mathbf{a} = \mathbf{k} \times \mathbf{r}$ and the surface S, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, $\mathbf{r}$ the position vector and $\mathbf{k}$ the constant unit vector along the z-axis. In other words calculate both sides of the equation

$$\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \int_C \mathbf{a} \cdot d\mathbf{r}.$$

[*Hint: In order to calculate curl $\mathbf{a}$ use the following formula*

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} - (\nabla \cdot \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a},$$

where $\mathbf{a}$ and $\mathbf{b}$ are two vector fields. You may wish to use spherical polar coordinates, i.e. $x = r\,\sin\theta\cos\phi$, $y = r\,\sin\theta\sin\phi$, $z = r\,\cos\theta$, to calculate the surface integral.]

9.3

Consider the vector function

$$\mathbf{F} = F_0 \left(\frac{x\cos(\lambda z)}{a}\,\mathbf{i} + \frac{y\cos(\lambda z)}{a}\,\mathbf{j} + \sin(\lambda z)\,\mathbf{k} \right).$$

Use the divergence theorem to calculate the flux of $\mathbf{F}$ through the closed surface bounded by the cylinders $\rho = a$, $\rho = 2a$ and the planes $z = \pm a\pi/2$, that is

$$\int_{S_1 \cup S_2} \mathbf{F} \cdot d\mathbf{S},$$

where S_1 represents the cylindrical surfaces and S_2 the planes.

[*Hint: Use cylindrical polar coordinates:* $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.]

9.4

The spherical polar coordinates are expressed by the following equations

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

where the range of the variables r, θ, ϕ are not relevant for this exercise.
A rigid body is rotating about a fixed axis with a constant angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{k}$. Using spherical polar coordinates calculate

a) $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$,

b) $\nabla \times \mathbf{v}$, where the curl of a vector $\mathbf{a}$ in orthogonal curvilinear coordinates is:

$$\nabla \times \mathbf{a} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u\,\hat{\mathbf{e}}_u & h_v\,\hat{\mathbf{e}}_v & h_w\,\hat{\mathbf{e}}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u\,a_u & h_v\,a_v & h_w\,a_w. \end{vmatrix}$$

[*Hint: The starting point is the position vector in spherical polar coordinates.*]

9.5

Consider the coordinate ρ, ϕ and a defined in terms of Cartesian coordinates as follows

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = a(1+\rho^2),$$

for $x > 0\,, y > 0\,, z > 0$. Compute the cartesian components of the vectors

$$\mathbf{e}_\rho = \frac{\partial\mathbf{r}}{\partial\rho}\,,\ \mathbf{e}_\phi = \frac{\partial\mathbf{r}}{\partial\phi}\,,\ \mathbf{e}_a = \frac{\partial\mathbf{r}}{\partial a} \quad \text{and} \quad \mathbf{f}_\rho = \nabla\rho\,,\ \mathbf{f}_\phi = \nabla\phi\,,\ \mathbf{f}_a = \nabla a$$

and verify that they are reciprocal bases that is $(\mathbf{e}_l \cdot \mathbf{f}_m) = \delta_{lm}$ with $l, m = \rho,\ \phi,\ a$. Note that the ρ, ϕ and a are not orthogonal curvilinear coordinates (you can verify that the vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_a$ are not orthogonal), hence you cannot use the expression for the gradient in orthogonal curvilinear coordinates. First you need to write ρ, ϕ and a in cartesian coordinates and use the expression for the gradient in Cartesian coordinates. [*Hint:* $\frac{d}{dt}\arctan(t) = \frac{1}{1+t^2}$].