Foundations 3A - QM
Worksheet 2

Problem 1

(a) Show that

$$\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix} = \begin{pmatrix} 4\sqrt{2} \\ -8i \end{pmatrix}.$$

(b) Is it the case that the column vectors

$$\begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{2} \\ i \end{pmatrix}$$

are eigenvectors of this matrix? If they are, what are the corresponding eigenvalues?

(c) Show that these two column vectors are orthogonal (i.e., show that their inner product is zero).

Problem 2

(a) Show that the states $\chi_{n\uparrow}$ and $\chi_{n\downarrow}$ defined by the equations

$$\chi_{n,\uparrow} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(\mathrm{i}\phi) \end{pmatrix}$$

and

$$\chi_{n,\downarrow} = \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2)\exp(\mathrm{i}\phi) \end{pmatrix}$$

are eigenstates of S_n and find the corresponding eigenvalues. Further, show that $\chi_{n,\uparrow}$ and $\chi_{n,\downarrow}$ are orthogonal to each other and that both are normalized to unity. Here $S_n = \hat{\mathbf{n}} \cdot \mathbf{S}$, where $\hat{\mathbf{n}}$ is a unit vector in the θ, ϕ direction: $\hat{\mathbf{n}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$.

(b) Use the above to find an eigenstate of S_x with eigenvalue $\hbar/2$ and an eigenstate of S_x with eigenvalue $-\hbar/2$. Show that these eigenstates are **not** orthogonal to the eigenstates of S_z.

(c) Using 2-component column and row vectors, show that the expectation value of S_y for an eigenstate of S_x with eigenvalue $-\hbar/2$ is zero. (Note the indexes: the question is about the expectation value of the y-component of $\mathbf{S}$ for a system prepared in a certain eigenstate of the x-component of $\mathbf{S}$.) Why does this result imply that if a spin-1/2 particle is prepared in that state and then the y-component of its spin is measured, there is a probability of 1/2 that a value $\hbar/2$ is found?

(d) Suppose that a spin-1/2 particle is prepared in the state of spin up, and then that it is passed through an analyzer which would tell whether it is in the state $\chi_{n\uparrow}$ or in the state $\chi_{n\downarrow}$. What would be the probability to find $\chi_{n\uparrow}$ and what would be the probabilty to find $\chi_{n\downarrow}$?

Note: By analogy with the words spin up and spin down in the z-direction to refer to the eigenstates of S_z, the projection of $\mathbf{S}$ on the unit vector $\hat{\mathbf{z}}$, one may say that in the states $\chi_{n\uparrow}$ and $\chi_{n\downarrow}$ the spin "points" in the positive or the negative $\hat{\mathbf{n}}$ direction. However, spin states are not little arrows pointing in a certain direction or another: a measurement of the z-component of $\mathbf{S}$ on a state "pointing" in the x-direction will find a non-zero result (either $+\hbar/2$ or $-\hbar/2$ with equal probability).

Problem 3

As is explained in Part 5 of the course notes, electrons, protons and neutrons have a non-zero magnetic dipole moment. The corresponding quantum mechanical operator, $\boldsymbol{\mathcal{M}}_S$, is proportional to the spin operator $\mathbf{S}$. We will express $\mathbf{S}$ and $\boldsymbol{\mathcal{M}}_S$ in terms of the vector $\boldsymbol{\sigma}$ whose x-, y- and z-components are the Pauli spin matrices. Namely, $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ and $\boldsymbol{\mathcal{M}}_S = \hbar\gamma\,\boldsymbol{\sigma}/2$, where γ is a constant. (The value of γ depends on the nature of the particle: $\gamma < 0$ for electrons and neutrons, while $\gamma > 0$ for protons; compared to electrons, $|\gamma|$ is about a thousand times smaller for protons and neutrons.)

Let us consider an electron (or proton or neutron) at rest in a static, uniform magnetic field $\boldsymbol{\mathcal{B}}$. Since this particle is at rest, it has no kinetic energy. It has a potential energy, though, because of its magnetic dipole moment: The classical potential energy of a magnetic dipole moment $\boldsymbol{\mathcal{M}}$ in a static, uniform magnetic field $\boldsymbol{\mathcal{B}}$ is $-\boldsymbol{\mathcal{M}} \cdot \boldsymbol{\mathcal{B}}$. Therefore we take the quantum mechanics Hamiltonian of the system to be $-\boldsymbol{\mathcal{M}}_S \cdot \boldsymbol{\mathcal{B}}$, that is $H = -\hbar\gamma\,\boldsymbol{\mathcal{B}} \cdot \boldsymbol{\sigma}/2$. We choose the z-direction of the system of coordinates to be in the direction of $\boldsymbol{\mathcal{B}}$. Thus $\boldsymbol{\mathcal{B}} = \mathcal{B}\hat{\mathbf{z}}$ and $H = -\hbar\gamma\mathcal{B}\sigma_z/2$.

Show that if initially (at $t = 0$) the particle is in the state $\chi_{n,\uparrow}$ defined in Problem 2, then at later times it will be in the state

$$\chi_\uparrow(t) = \exp(-\mathrm{i}\omega t/2) \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\ \exp[\mathrm{i}(\phi + \omega t)] \end{pmatrix},$$

where $\omega = -\gamma\mathcal{B}$. You can use that $\chi_\uparrow(t)$ must be a solution of the time-dependent Schrödinger equation,

$$\mathrm{i}\hbar\frac{\partial\chi_\uparrow}{\partial t} = H\chi_\uparrow(t).$$

Note: From Problem 2 above, we see that the state $\chi_\uparrow(t)$ is an eigenstate of the operator $\hat{\mathbf{n}}(t) \cdot \mathbf{S}$, where $\hat{\mathbf{n}}(t)$ is a unit vector in the direction $(\theta, \phi + \omega t)$. This result shows that in the presence of a uniform magnetic field, the direction in which the spin of the particle "points" rotates about the field direction with the angular frequency ω. One says that the spin precesses about that direction.

Problem for extra practice

Problem 4

Show that for a spin-1/2 particle,

$$\mathbf{S}^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the usual representation of spin operators by square matrices.