University of Durham

EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2019	PHYS3711-WE01

Title:
Condensed Matter Physics 3

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Attempt **all** questions. The short-answer questions at the start of each section carry 50% of the total marks for the paper. The remaining 50% of the marks are carried by the longer questions, which are equally weighted.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklets for Sections B and C, in order, inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Symmetry, Structure and Excitations
Section B: Broken Symmetry
Section C: Introduction to Soft Matter Physics

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A: SYMMETRY, STRUCTURE AND EXCITATIONS

1. (a) The binding energy per formula unit of an ionic crystal of formula XY can be modelled as

$$U_{lattice} = -\alpha \frac{|Q_+ Q_-|}{4\pi\epsilon_0 r_0} + mB \exp\left(-r_0/\rho\right),$$

where all symbols have their usual meaning. Given that in LiF $B = 4.92 \times 10^{-17}$ J, $\rho = 0.291$ Å, $m = 6$ and $r_0 = 2.01$ Å, calculate the value of the Madelung constant α for this crystal structure. [4 marks]

(b) Sketch the pair distribution functions (pdfs) for a real gas and an ideal gas, labelling axes clearly. Explain the origin of the differences between the two pdfs. [4 marks]

(c) The structure factor of a lattice is given by

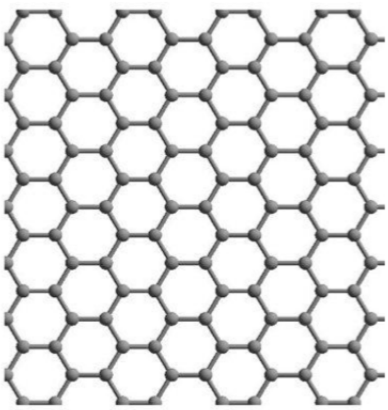
$$F(hkl) = \sum_i f_i \exp[2\pi i(hx_i + ky_i + lz_i)],$$

where all symbols have their usual meaning.
α-Uranium has an orthorhombic structure with a motif of four uranium atoms in positions:

$$\pm\left(0, y, \frac{1}{4}\right); \ \pm\left(\frac{1}{2}, \left[\frac{1}{2} + y\right], \frac{1}{4}\right).$$

Given that the structure factor for the (110) reflection is $|F(110)| = 258.9$ and the atomic scattering factor for uranium at the (110) reflection is $f_U(110) = 80$, find the value of y. [4 marks]

(d) The two-dimensional graphene lattice is shown below. Identify the point group symmetry operations of this lattice and hence write down the two-dimensional point group symbol. [4 marks]



(e) Sketch the dispersion relation for the phonons in a one-dimensional lattice with a motif of two atoms, labelling the acoustic and optic branches. How many optic branches would you expect in a solid in two dimensions with the same motif? Explain your answer. [4 marks]

2. Ge is a semiconductor which crystallises in a diamond lattice with point group $m3m$. Ge has an indirect band gap, with the lowest energy direct transition occurring at the Γ ($k = 0$) point in the Brillouin Zone.

   (a) Explain the differences between a direct and an indirect band gap in a semiconductor crystal. [6 marks]

   (b) At the Γ point the valence and conduction bands of unstrained Ge transform as the Γ_8^+ and Γ_7^- irreducible representations of the $m3m$ point group, respectively. Consider a uniaxial strain applied to the Ge crystal, which lowers the point group symmetry from $m3m$ to $4/mmm$. Using the character tables provided at the end of this examination paper and the decomposition formula for a reducible representation,

$$a_{\Gamma_j} = \frac{1}{h} \sum_k N_k \chi^{(\Gamma_j)}(C_k)^* \chi^{\text{reducible}}(C_k),$$

   where all symbols have their usual meaning, determine if the valence and conduction band states split upon application of the strain. If so, find the degeneracies of the symmetry-split states. [10 marks]

   (c) Given that the dipole operator belongs to a representation which is odd under inversion, justify why no optical transitions of the type $\Gamma_i^+ \rightarrow \Gamma_j^+$ are symmetry allowed in either the strained or the unstrained crystal. [4 marks]

[*Hint:* An even reducible representation cannot contain an odd irreducible representation and *vice versa*.]

SECTION B: BROKEN SYMMETRY

3. (a) The free energy of a system is given by

$$F(M) = a_0(T - T_c)M^2 + bM^4,$$

where M is the order parameter, T is temperature and a_0, b and T_c are positive constants. Identify the possible equilibrium states of this system, describing when they would be expected to be realised. [4 marks]

(b) Consider a ferromagnetic chain formed from N spins with $S = 1/2$, described by a Hamiltonian $\hat{H} = -J\sum_i^{N-1} \hat{S}_i^z \hat{S}_{i+1}^z$, where J is an exchange constant and $\hat{S}_i^z$ is the z-component spin operator for the ith spin in the chain. What is the energy cost of a single domain wall compared to the ferromagnetically aligned state? What is the entropy gain associated with the formation of such a wall? [4 marks]

(c) The energy E of an excitation in a superfluid is given in terms of momentum p by

$$E(p) = \sqrt{\left[\frac{p^2}{2m}\left(\frac{p^2}{2m} + ng\right)\right]},$$

where m is the mass of each particle in the fluid, n is a constant number density and g is a coupling constant. By considering the limits of this expression, describe the behaviour of the low- and high-energy excitations. [4 marks]

(d) Suggest how inelastic neutron scattering can be used to probe the excitations that emerge in a magnet upon symmetry breaking, and give a sketch of a suitable experimental setup. [4 marks]

(e) Consider a ferromagnetic, two-dimensional XY model. Sketch excitations with winding numbers $w = 1$ and $w = -1$ and state their type. [4 marks]

4. The critical exponents β, γ and δ are defined near the magnetic phase transition temperature $T_{\rm c}$ in terms of a reduced temperature $t = (T - T_{\rm c})/T_{\rm c}$ via

$$\begin{aligned} M(H = 0,\, T \to T_{\rm c}^{-}) &\propto (-t)^{\beta}, \\ M(H \to 0,\, T = T_{\rm c}) &\propto H^{\frac{1}{\delta}}, \\ \chi(H \to 0,\, T \to T_{\rm c}) &\propto |t|^{-\gamma}. \end{aligned}$$

(a) Explain what the symbols M, H and χ represent in these expressions, justifying the circumstances under which the three expressions are valid. [5 marks]

(b) Calculate the critical exponents β and δ for the free energy

$$F(M) = \frac{a(T)}{2} M^2 + \frac{c}{6} M^6 - \mu_0 M H,$$

where $a(T) = a_0(T - T_{\rm c})$, and a_0 and c are positive constants. [6 marks]

(c) Calculate the critical exponent γ for the free energy given above for both $T > T_{\rm c}$ and $T < T_{\rm c}$. [5 marks]

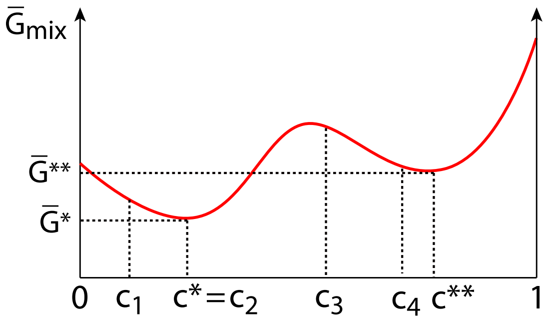
(d) If a term proportional to M^4 is added to $F(M)$, the critical exponents take on their usual mean field values. Justify why this is the case. [4 marks]

  Continued Next Page

SECTION C: INTRODUCTION TO SOFT MATTER PHYSICS

5. (a) The interaction between two adjacent molecules inside a solid can be described by a Lennard-Jones potential with a minimum of $-\varepsilon$ at a distance r_0. Knowing that each molecule has 4 nearest neighbours, provide an estimate for the melting temperature T_m of this solid and justify your reasoning. Explain why the vaporisation temperature of this material cannot be estimated from the given information. [4 marks]

(b) The molecular Gibbs free energy of mixing of a binary liquid mixture is plotted in the figure below as a function of the relative concentration c of liquid 2. The concentrations c^* and c^{**} indicate the local energy minima $\bar{G}^*$ and $\bar{G}^{**}$ respectively. The concentrations c_1, $c_2 = c^*$, c_3 and c_4 have respective energies $\bar{G}_1$, $\bar{G}_2 = \bar{G}^*$, $\bar{G}_3$ and $\bar{G}_4$ when the system is mixed homogeneously. Calculate the molecular Gibbs free energy of the system at equilibrium for c_1, c_2, c_3 and c_4, and comment on the process by which the system reaches its equilibrium. [4 marks]



(c) A homogenous metal sphere of radius R, density ρ and Young's modulus E_s is placed on a flat piece of rubber. The rubber has a Young's modulus of $E_r \ll E_s$ and is incompressible. Both materials deform elastically due to gravity. The reduced Young's modulus is $E^* = \left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)^{-1}$, and the adhesion force $F = \frac{4}{3}E^*\sqrt{R^*}\delta^{3/2} - \pi R^* w_{\mathrm{adh}}$, where all symbols have their usual meanings. Show that the deformation of the sphere is negligible compared to that of the rubber. Calculate the maximum indentation depth δ of the sphere into the rubber assuming no adhesion between the materials. [4 marks]

(d) The free energy $G(r)$ of a polymer is given by: $G(r) = \alpha r^2 + \beta/r^2$ where r is the polymer extension, and α and β are positive constants that depend on the polymer characteristics and temperature. Calculate r_0, the equilibrium end-to-end distance of the polymer as a function of α and β. Estimate how much force is needed to extend the polymer by $r_0/8$ from its equilibrium end-to-end distance. You may assume the potential to be locally parabolic around r_0. [4 marks]

(e) A material has a viscosity under shear given by $\eta(\dot{e}) = \dot{e}^2 - \sqrt{2\eta_0}\dot{e} + \eta_0$, where $\dot{e}$ is the shear rate and η_0 is a positive constant. Sketch the evolution of the viscosity as a function of $\dot{e}$. Comment on the behaviour of this material for $\dot{e} < \sqrt{\eta_0/2}$, $\dot{e} > \sqrt{\eta_0/2}$ and in the region where $\dot{e} \sim \sqrt{\eta_0/2}$. [4 marks]

6. A glass is filled with water, creating an air-water interface of area A. The water has a surface tension γ_w. A solid spherical particle of radius R and denser than water is placed at the surface of the water where it floats. The water makes an angle $\theta = 90°$ with the surface of the particle. The air-water interface surrounding the particle can be assumed undeformed due to the particle's small size.

   (a) Basing your explanation of Young's equation, $\gamma_{SG} - \gamma_{SL} = \gamma_{LG} \cos\theta$, where all symbols have their usual meaning, explain why the particle floats and calculate the work that would be needed to force the particle to sink. Clarify any approximation you make. [6 marks]

   (b) The particle can diffuse laterally at the surface of water. Assuming a diffusion coefficient $D = k_{\mathrm{B}}T/6\pi\eta R$, where all symbols have their usual meaning, calculate the distance d a particle travels over a period P. How would d vary for a particle made of identical material but with a diameter half as large? You may neglect the effect of air on the Stokes force experienced by the particle. [4 marks]

   (c) We progressively add more particles to the surface of water. The particles are identical and behave like a 2-dimensional gas with a configurational entropy given by $S = k_{\mathrm{B}}N \ln(aA/N)$ where a is a positive constant and N the number of particles. Given that hexagonally-packed discs cover a planar surface with a density of $\pi/(2\sqrt{3})$, determine the value a must take and justify your reasoning. [5 marks]

   (d) Comment on the effect of entropy on the spatial arrangement of the particles at the surface of water as N increases. Determine whether there exists a number of particles N_s beyond which the particles start to self-assemble. [5 marks]

$m3m$	1	$\mathcal{R}1$	3	3	2, $\mathcal{R}2$	4	$\mathcal{R}4$	2,$\mathcal{R}2$	$\bar{1}$	$\mathcal{R}\bar{1}$	$\bar{3}$	$\mathcal{R}\bar{3}$	m, $\mathcal{R}m$	$\bar{4}$	$\mathcal{R}\bar{4}$	m, $\mathcal{R}m$
Mult.	1	1	8	8	6	6	6	12	1	1	8	8	6	6	6	12
Γ_8^+	4	-4	-1	1	0	0	0	0	4	-4	-1	1	0	0	0	0
Γ_7^-	2	-2	1	-1	0	$-\sqrt{2}$	$\sqrt{2}$	0	-2	2	-1	1	0	$\sqrt{2}$	$-\sqrt{2}$	0

$4/mmm$	1	$\mathcal{R}1$	4	$\mathcal{R}4$	2,$\mathcal{R}2$	2,$\mathcal{R}2$	2,$\mathcal{R}2$	$\bar{1}$	$\mathcal{R}\bar{1}$	$\bar{4}$	$\mathcal{R}\bar{4}$	m,$\mathcal{R}m$	m,$\mathcal{R}m$	m,$\mathcal{R}m$
Mult.	1	1	2	2	2	4	4	1	1	2	2	2	4	4
Γ_1^+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2^+	1	1	1	1	1	-1	-1	1	1	1	1	1	-1	-1
Γ_3^+	1	1	-1	-1	1	1	-1	1	1	-1	-1	1	1	-1
Γ_4^+	1	1	-1	-1	1	-1	1	1	1	-1	-1	1	-1	1
Γ_5^+	2	2	0	0	-2	0	0	2	2	0	0	-2	0	0
Γ_1^-	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
Γ_2^-	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1
Γ_3^-	1	1	-1	-1	1	1	-1	-1	-1	1	1	-1	-1	1
Γ_4^-	1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	-1
Γ_5^-	2	2	0	0	-2	0	0	-2	-2	0	0	2	0	0
Γ_6^+	2	-2	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	2	-2	$\sqrt{2}$	$-\sqrt{2}$	0	0	0
Γ_7^+	2	-2	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	2	-2	$-\sqrt{2}$	$\sqrt{2}$	0	0	0
Γ_6^-	2	-2	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	-2	2	$-\sqrt{2}$	$\sqrt{2}$	0	0	0
Γ_7^-	2	-2	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	-2	2	$\sqrt{2}$	$-\sqrt{2}$	0	0	0

Character Tables for use with Question 2.