Quantum Theory - Worksheet 2

Problem 1

Consider an electron in a spin state $|\chi\rangle$ given by the column vector

$$\begin{pmatrix} \sqrt{3}/2 \\ i/2 \end{pmatrix}.$$

Show that $|\chi\rangle$ is normalized.

Problem 2

(a) Are the functions $\exp(in\phi)$, $n = 0, \pm 1, \pm 2, \ldots$, orthogonal on $[0, 2\pi]$? (A yes/no answer is OK here, as long as you can justify your answer. We have passed through the relevant calculations in a lecture, there is no need to repeat them here.)

(b) Are these functions linearly independent? (Justify your answer.)

(c) Consider the vector space that consists of all possible linear combinations of the following functions: 1, $\sin x$, $\cos x$, $(\sin x)^2$, $(\cos x)^2$, $\sin(2x)$ and $\cos(2x)$. What is the dimension of this space? Exhibit a possible set of basis vectors, and demonstrate that it is complete. [Note: That a set of vectors is complete means that it spans the whole vector space — i.e., that any vector in that space can be written as a linear combination of the vectors in the set. A set of vectors must be complete to form a basis, by definition of a basis.]

Problem 3

In this problem, a symbol such as $1/O$ where O is an operator represents the inverse of this operator (thus $1/O$ is just another notation for O^{-1}). Recall that an operator is said to be invertible if it has an inverse.

(a) "By definition of the inverse, an invertible operator always commutes with its inverse." True or false?

(b) As different operators may not commute, one should be careful with the order of the factors in expressions involving products of operators and the inverse of operators. Take, for example, two invertible operators, A and B. If these operators do not commute, it is not true, in general, that

$$\frac{1}{A} - \frac{1}{B} = \frac{B - A}{AB}.$$

However, one always has that

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{A}(B - A)\frac{1}{B} = \frac{1}{B}(B - A)\frac{1}{A}. \tag{1}$$

Prove this double identity.

(c) Eq. (1) is used in the quantum theory of collisions to obtain an important equation relating two operators G and G_0, called Green's operators and defined in the following way:

$$G = \frac{1}{E - H + i\epsilon}, \qquad G_0 = \frac{1}{E - H_0 + i\epsilon},$$

where E (the collision energy) and ϵ are real numbers and H and H_0 (two Hamiltonians) are operators. (Since one cannot add numbers to operators, one should really write the denominators as $E\,I - H - i\epsilon\,I$ and $E\,I - H_0 - i\epsilon\,I$, where I is the identity operator; however, doing so would be excessively pedantic, even by the standards of this course.) Suppose that $H = H_0 + V$. Show that

$$G = G_0 + G_0\,V\,G.$$

[You may note that this last equation can be iterated by replacing G in the right-hand side by $G_0 + G_0\,V\,G$. Further iterations yield a formal expansion of G in powers of V:

$$G = G_0 + G_0\,V\,G_0 + G_0\,V\,G_0\,V\,G_0 + \cdots$$

In Particle Physics, each of these terms would correspond to a set of Feynman diagrams.]

Problem 4

The exponential of an operator A is defined as follows:

$$\exp(A) = 1 + A + \frac{1}{2\,!}A^2 + \frac{1}{3\,!}A^3 + \cdots$$

Suppose that the quantum state of a one-dimensional system, such as a linear harmonic oscillator, is described by a wave function $\psi(x)$. Suppose, further, that this wave function can be expanded in a Taylor series about any value of the position variable x. Show that

$$\exp(-ix_0P/\hbar)\psi(x) = \psi(x - x_0),$$

where x_0 is a length and P is the momentum operator,

$$P = -i\hbar\,\frac{\mathrm{d}}{\mathrm{d}x}.$$

(In view of this result, the operator $\exp(-ix_0P/\hbar)\psi(x)$ is called the displacement operator.)