ELECTROMAGNETISM

Workshops 1st Set (Qns) - Vectors and vector calculus

Professor D P Hampshire – 2nd Year Physics Lecture Course

The material for this workshop is split into three parts. Part I: A review of vectors and vector calculus. Part II contains worked examples. Part III gives some additional unseen questions. Please make sure you can answer worked examples in Part II from scratch without reference to the worked solutions.

Workshops/Examples classes: - there are 3 functions for the examples classes:

i) Lectures – You are welcome to ask any questions about any of the lectures.

ii) Homework – You are welcome to bring a copy of your homework problem sheet and your marked homework scripts with you to the workshops/examples classes. We can go through the model solutions.

iii) Workshop/examples class problems – A dedicated sheet of problems will be provided for you to attempt during each workshop.

Office hours: This lecture course is for about 250 students. In first instance please use the workshop/examples class as my office hour – be brave, put your hand up and I can come over and chat with you about electromagnetism/homework/Judo/Waterpolo/life/... Please ask the workshop support staff and your friends about difficult parts of the course. If all else fails – mention to me at a workshop that you have exhausted all reasonable resources/usual suspects and would like to (bring some friends who are also stuck and) chat. (By appointment Friday 5.00 p.m. – Rm. 143).

Errors: If you find any typos or errors in the workshop materials, lectures notes or homeworks, please note the correction on a hard copy and give the hard copy to Professor Hampshire.

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1 Vectors and vector calculus

1.1 Definitions

1.1.1 Vector field notation

A vector field is fully characterised by knowing its magnitude and direction at every point in space. Physical parameters such as current density, force, electric field and magnetic field are all important vector fields.

Figure 1 : In order to describe vectors that come into and out of the paper, we consider the tip and tail of a medieval arrow. LHS: A dot demotes the tip of the arrow so the vector $\underline{\boldsymbol{J}}$ is coming out of the board. RHS: A cross demotes the tail of the arrow and so denotes the vector $\underline{\boldsymbol{K}}$ going into the paper.

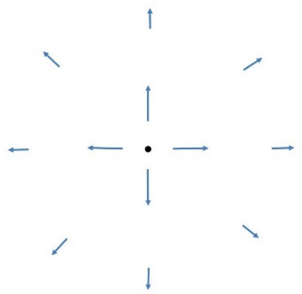

Figure 2 : A vector field in a plane represented by arrows where the directions of the vector field throughout space are represented by the directions in which the arrow points and the magnitudes of the vector field are represented by the lengths of the arrows. Shown is the vector field, which we know as electric field, close to a point charge.

1.1.2 The Del operator

The symbol $\underline{\boldsymbol{\nabla}}$ denotes the vector operator 'del' where,

$$\underline{\boldsymbol{\nabla}} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}. \qquad 1\text{-}1$$

1.1.3 The common three-dimensional coordinate systems

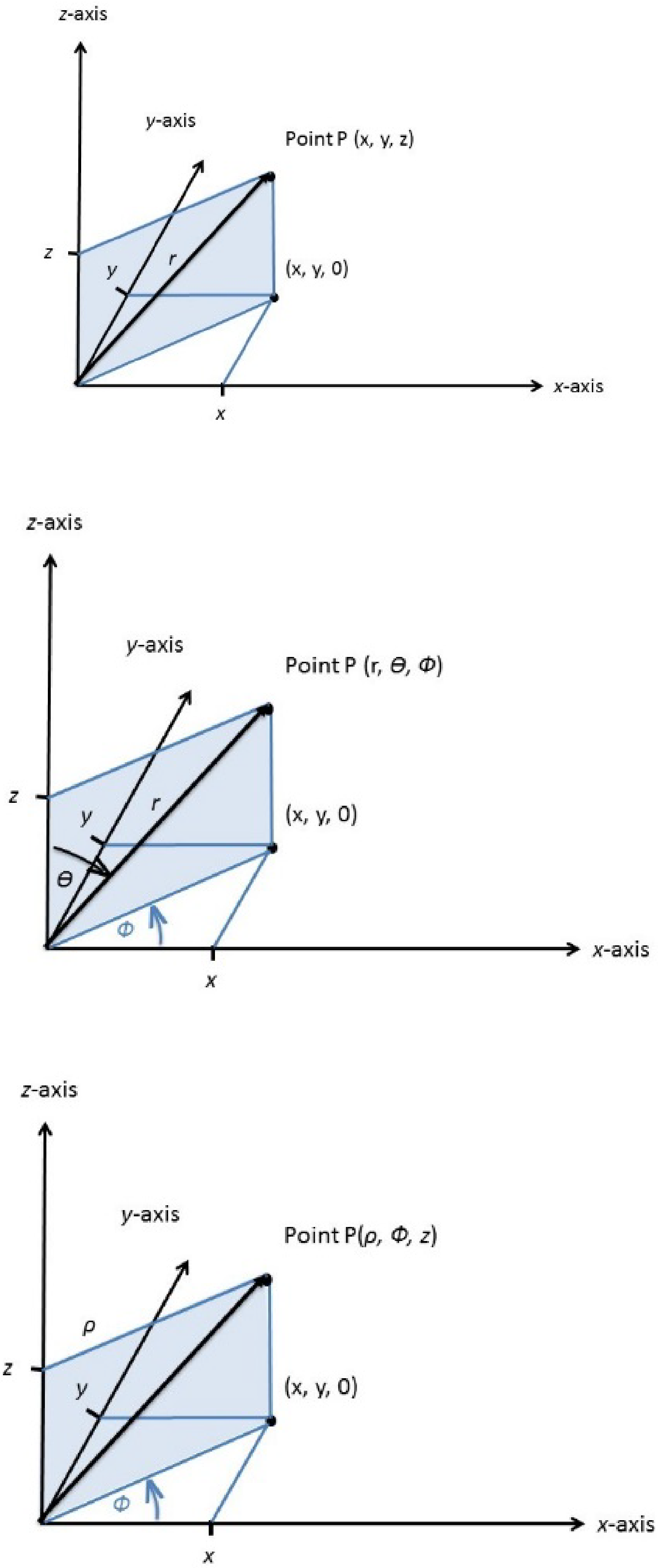

Figure 3 : The variables used in the three commonly used coordinate systems - (Left) the rectangular, (Middle) the spherical polar and (Right) the cylindrical polar systems.

1.1.4 The Gradient of a Scalar Field

When operating on a scalar function (a function which has a sign and magnitude but no directional properties) e.g. temperature, electrostatic potential , $V(x, y, z)$ say, this gives the gradient of V:

$$\text{grad } V = \underline{\boldsymbol{\nabla}}V = \hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z} \qquad 1\text{-}2$$

Note that the gradient of a scalar function is vector.

1.1.5 The Divergence and Curl of a Vector Field

The del operator can operate on a vector field, say $\underline{\boldsymbol{A}}$ where:

$$\underline{\boldsymbol{A}}(x, y, z) = \hat{\mathbf{i}}A_x(x, y, z) + \hat{\mathbf{j}}\mathrm{A}_{\mathrm{y}}(x, y, z) + \hat{\mathbf{k}}A_z(x, y, z) \qquad 1\text{-}3$$

In Cartesian coordinates, the divergence of $\underline{\boldsymbol{A}}$(x,y,z) is:

$$\text{div } \underline{\boldsymbol{A}} = \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \qquad 1\text{-}4$$

Note that the divergence of a vector function is a scalar function.

The curl (or circulation) of $\underline{\boldsymbol{A}}(x,y,z)$ is:

$$\text{curl } \underline{\boldsymbol{A}} = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}} = \hat{\mathbf{i}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{\mathbf{j}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \qquad 1\text{-}5$$

The circulation of $\underline{\boldsymbol{A}}$ can be written in determinant form as;

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad 1\text{-}6$$

Note: (i) the curl of a vector function is a vector function; (ii) the similarities with standard vector algebra (i.e. cross-products are vectors, dot products are scalars).

1.1.6 Laplacian

The Laplacian operator is denoted by “del-squared” or ∇^2 operator, where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \qquad 1\text{-}7$$

(Note ∇^2 is not in bold because it is a scalar operator). The Laplacian can operate on a scalar or vector function. When operating on a scalar function (e.g. potential) the result is a scalar:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \qquad 1\text{-}8$$

When operating on a vector function it gives a vector where:

$$\nabla^2 \underline{\boldsymbol{A}} = \hat{\mathbf{i}}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right) + \hat{\mathbf{j}}\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right). \qquad 1\text{-}9$$

1.2 Vector calculus proofs and identities

1.2.1 The Divergence (or Gauss') theorem

The divergence theorem can be stated as

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}\, \mathrm{d}V \;-\; \text{Gauss'theorem} \qquad 1\text{-}10$$

where $\underline{\boldsymbol{E}}$ is any arbitrary vector field and V is the volume of any arbitrarily shaped volume.

Proof of divergence theorem

The standard convention is the unit vector associated with the area surrounding a volume is in a direction outward from the volume at the surface. Hence the net flux out of the box is positive.

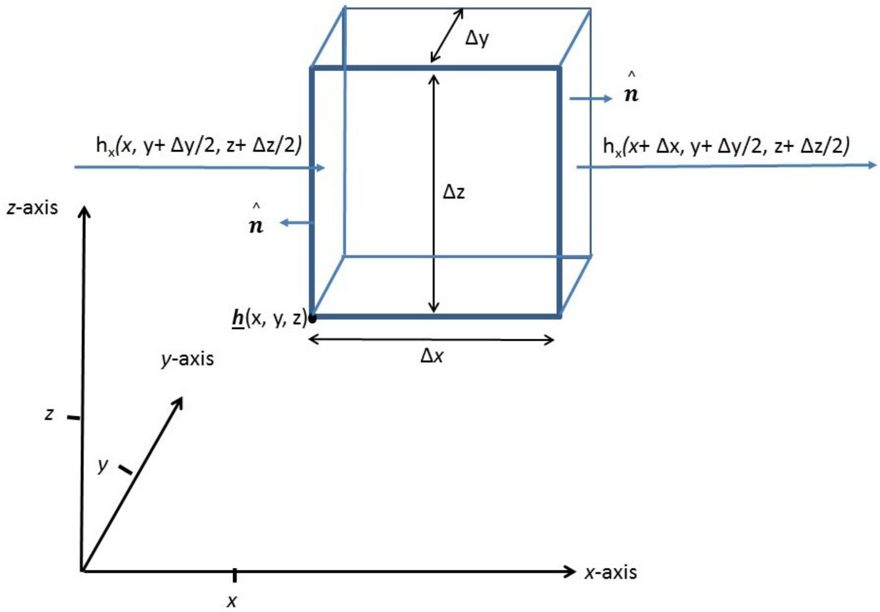

Figure 4 : A small elemental cube with sides Δx, Δy and Δz. The figure shows explicitly the change in flux along the x-axis.

For the elemental cube shown in Figure 4, the flux of the vector field in the x-direction is determined by the contribution from just by two surfaces – those at the extreme left and right of the cube. The vector field is not constant across these surfaces. We can approximate the net flux in the x-direction, by taking an average of the x-component of vector field across each surface multiplied by the area of each surface and adding them together. Hence:

Coordinate system (coordinates)	$\underline{\nabla} V$	$\underline{\nabla} \cdot \underline{\boldsymbol{A}}$	$\underline{\nabla} \times \underline{\boldsymbol{A}}$	$\nabla^2 V$
Rectangular or Cartesian (x, y, z)	$\hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{A_x} & \mathrm{A_y} & \mathrm{A_z} \end{vmatrix}$	$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$
Spherical polar (r, θ, ϕ)	$\frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$	$\begin{vmatrix} \frac{\hat{\mathbf{r}}}{r^2\sin\theta} & \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta} & \frac{\hat{\boldsymbol{\phi}}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \mathrm{A}_r & r\mathrm{A}_\theta & r\mathrm{A}_\phi \sin\theta \end{vmatrix}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \mathrm{V}}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial \mathrm{V}}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$
Cylindrical polar (ρ, ϕ, z)	$\frac{\partial V}{\partial \rho}\hat{\mathbf{r}} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$	$\begin{vmatrix} \frac{\hat{\boldsymbol{\rho}}}{\rho} & \hat{\boldsymbol{\phi}} & \frac{\hat{\mathbf{z}}}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \mathrm{A}_\rho & \rho\mathrm{A}_\phi & \mathrm{A}_z \end{vmatrix}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \mathrm{V}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

Table 1 : In the three common coordinate systems : The 'Grad' of a scalar field V ($\underline{\nabla} V$). The divergence ($\underline{\nabla} \cdot \underline{\boldsymbol{A}}$) and curl ($\underline{\nabla} \times \underline{\boldsymbol{A}}$) of a vector field $\underline{\boldsymbol{A}}$, as well as the Laplacian for the scalar field V.

$$(Net\ outward\ flux)_x \approx \{\bar{h}_x(\mathrm{RHS}) - \bar{h}_x(\mathrm{LHS})\}\Delta y \cdot \Delta\delta z. \qquad 1\text{-}11$$

where for example $\bar{h}_x$(LHS) is the average vector field in the x-direction across the LHS surface. We assume the flux changes smoothly throughout all space so can find the average from say the average of the values at the four corners or (as used here) the value of $\bar{h}_x$ at the centre of the surface:

For the LHS surface, we use a standard Taylor series to write $\bar{h}_x$ in terms of the vector field h_x, and its derivatives, at point (x,y,z):

$$\bar{h}_x(\mathrm{LHS}) \approx h_x\left(x, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2}\right) \approx h_x(x,y,z) + \frac{1}{2}\frac{\partial h_x(x,y,z)}{\partial y}\Delta y + \frac{1}{2}\frac{\partial h_x(x,y,z)}{\partial z}\Delta z. \qquad 1\text{-}2$$

Equally for the RHS surface we have:

$$\bar{h}_x(\mathrm{RHS}) \approx h_x(x + \Delta x, y + \Delta y/2, z + \Delta z/2). \qquad 1\text{-}13$$

So again using a Taylor series:

$$\bar{h}_x(\mathrm{RHS}) \approx h_x(x,y,z) + \frac{\partial h_x(x,y,z)}{\partial x}\Delta x + \frac{1}{2}\frac{\partial h_x(x,y,z)}{\partial y}\Delta y + \frac{1}{2}\frac{\partial h_x(x,y,z)}{\partial z}\Delta z. \qquad 1\text{-}14$$

Hence:

$$(Net\ outward\ flux)_x \approx \left\{\frac{\partial h_x(x,y,z)}{\partial x}\Delta x\right\}\Delta y \Delta z \approx \frac{\partial h_x}{\partial x}\Delta V. \qquad 1\text{-}15$$

Where the volume of the small cube is $\Delta V = \Delta x \Delta y \Delta z$.

There are similar contributions for the sides with area normal vectors parallel to the y and z axes. Hence:

$$Net\ outward\ \mathrm{flux}\ for\ the\ entire\ cube \approx \left(\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z}\right)\Delta V. \qquad 1\text{-}16$$

If the volume of the box is very small i.e. the limit of δV goes to dV. The term in the bracket is the divergence of $\underline{\boldsymbol{h}}$, hence:

$$Net\ outward\ \mathrm{flux}\ for\ the\ entire\ cube = \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{h}}\, \mathrm{d}V. \qquad 1\text{-}17$$

Therefore, the total outward flux (ϕ) over an arbitrary volume is the volume integral of the $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{h}}\, \mathrm{d}V$:

$$\phi = \oint \underline{\boldsymbol{h}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{h}}\, \mathrm{d}V \ - \mathrm{Gauss's\ theorem.} \qquad 1\text{-}18$$

Equation 1- is known as Gauss' theorem or the Divergence Theorem.

1.2.2 Stoke's theorem

Stokes Theorem can be stated as:

$$\oint \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \int \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}\right) \cdot \mathrm{d}\underline{\boldsymbol{S}}. \qquad \text{1-19}$$

Where $\underline{\boldsymbol{A}}$ is any arbitrary vector field.

<u>Proof of Stoke's theorem</u>

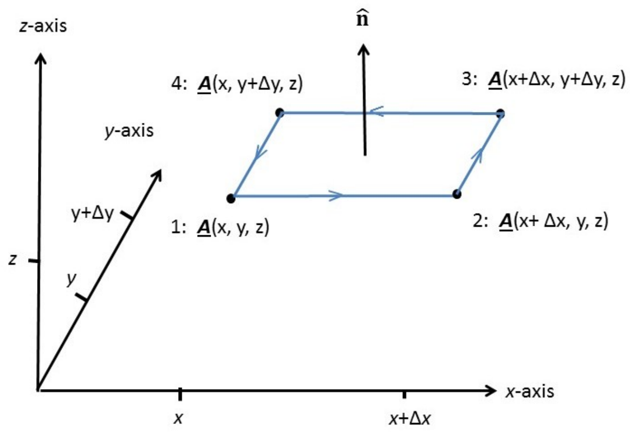

Figure 5 : An elemental path integral in the x-y plane that can be used to prove Stoke's theorem.

Consider the line integral of a vector field $\underline{\boldsymbol{A}}$ around the infinitesimal closed loop in the x-y plane shown in Figure 5. The standard convention used is the line integral in anticlockwise direction is positive. Only the x-component of $\underline{\boldsymbol{A}}$ contributes to the line integral from point 1 to point 2. We assume that the vector field varies smoothly throughout space so can find an average of the vector field from the value of the vector field at the mid-point of any straight path. Using a Taylor series, we find :

$$\oint_{Point\ 1 \rightarrow\ Point\ 2} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \approx A_x(x + \Delta x/2, y, z)\Delta x \approx A_x(x, y, z)\Delta x + \frac{1}{2}\frac{\partial A_x(x,y,z)}{\partial x}\{\Delta x\}^2. \qquad \text{1-8}$$

Equally

$$\oint_{Point\ 3 \rightarrow\ Point\ 4} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \approx -A_x\left(x + \frac{\Delta x}{2}, y + \Delta y, z\right)\Delta x \approx -\left\{A_x(x, y, z)\Delta x + \frac{1}{2}\frac{\partial A_x(x,y,z)}{\partial x}\{\Delta x\}^2 + \frac{\partial A_x(x,y,z)}{\partial y}\Delta y \Delta x\right\}. \qquad \text{1-9}$$

Therefore:

$$\oint_{Point\ 1 \rightarrow\ Point\ 2} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} + \oint_{Point\ 3 \rightarrow\ Point\ 4} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = -\frac{\partial A_x(x,y,z)}{\partial y}\Delta y \Delta x. \qquad \text{1-10}$$

If we consider the whole closed loop in Figure 5 (i.e. adding contributions for paths: Point 2→Point 3 and Point 4→Point 1), we have:

$$\oint_{\substack{Closed \\ path\ in \\ x-y\ plane}} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \approx \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \Delta x \Delta y. \qquad \text{1-11}$$

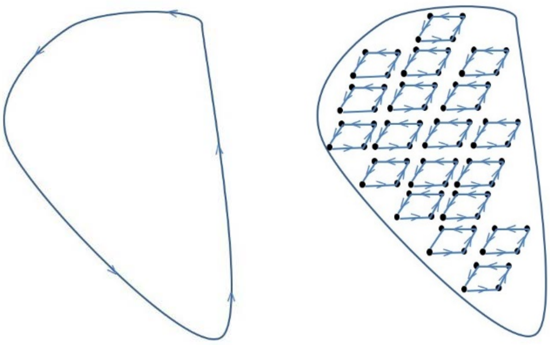

Figure 6 : Superposition of path integrals: The line integral on the LHS is equivalent to the sum of the all the path integrals on the RHS.

Now remember superposition - there are similar contributions for the y-z and z-x planes:

$$\oint_{\substack{Closed \\ path\ in \\ y-z\ plane}} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \approx \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \Delta z \Delta y. \qquad \text{1-12}$$

$$\oint_{\substack{Closed \\ path\ in \\ z-x\ plane}} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \approx \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \Delta z \Delta x. \qquad \text{1-13}$$

Therefore, the total line integral in 3 dimensions around an arbitrary path is:

$$\oint_{\substack{Closed \\ path}} \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \approx \left(\hat{\mathbf{i}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{\mathbf{j}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right) \cdot \left(\hat{\mathbf{i}}\Delta z \Delta y + \hat{\mathbf{j}}\Delta z \Delta x + \hat{\mathbf{k}}\Delta x \Delta y\right). \qquad \text{1-14}$$

If the small surface area is considered in the limit that $\Delta\underline{\boldsymbol{S}}$ goes to zero and we remember that the term in the bracket is the circulation of $\underline{\boldsymbol{A}}$, hence:

$$\oint \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \int \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \quad \text{Stoke's theorem} \qquad \text{1-15}$$

where, $\mathrm{d}\underline{\boldsymbol{S}}$ is the vector area defined as:

$$\mathrm{d}\underline{\boldsymbol{S}} = \hat{\mathbf{i}} dz dy + \hat{\mathbf{j}} dz dx + \hat{\mathbf{k}} dx dy. \qquad \text{1-16}$$

1.3 Worked examples

1.3.1 Questions

1. If $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} = r$ show that $\underline{\nabla} f = \hat{\mathbf{r}}$.

2. Sketch the vector function $\underline{\boldsymbol{h}} = \frac{\hat{\mathbf{r}}}{r^2}$, and compute its divergence.
[Hint: Use spherical polar co-ordinates $\underline{\nabla} \cdot \underline{\boldsymbol{F}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta F_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}F_\phi$]

3. Calculate the Laplacian of the following,
(i) $\underline{\boldsymbol{F}} = x^2\hat{\mathbf{i}} - 3xy\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$ (i.e. $\nabla^2\underline{\boldsymbol{F}}$)
(ii) $f = x^2 + 2xy + 3z + 4$.

4. Given that $T = xy^2$, find the path integral of $\underline{\nabla} T$ along the straight line joining the origin and point (3,4,0). Is it path independent?

5. Given that the potential function is $V = \frac{1}{\sqrt{x^2+y^2+z^2}}$, find the electric field at point (1,1,1).

6. State the divergence theorem.

Given the vector function $\boldsymbol{F} = y^2\hat{\mathbf{i}} + (2xy + z^2)\hat{\mathbf{j}} + 2yz\hat{\mathbf{k}}$.
(a) By solving a volume integral and using the divergence theorem, find the total flux out of the unit cube located at the origin.
(b) Do the surface integrals to find the total flux out of the cube. Is the value the same as that obtained by Divergence Theorem?

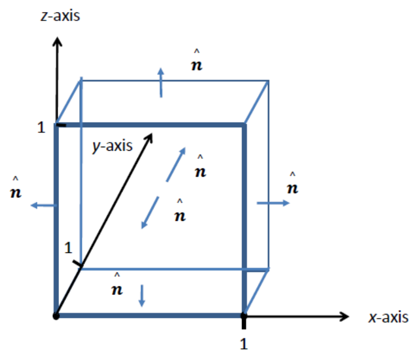

Figure 7 : The six faces of a cube with flux coming out of each of them.

1.3.2 Answers

1.

$$\underline{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

$$= \frac{1}{2r}2x\hat{\mathbf{i}} + \frac{1}{2r}2y\hat{\mathbf{j}} + \frac{1}{2r}2z\hat{\mathbf{k}}$$

$$= \frac{1}{r}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) = \hat{\mathbf{r}}$$

2.

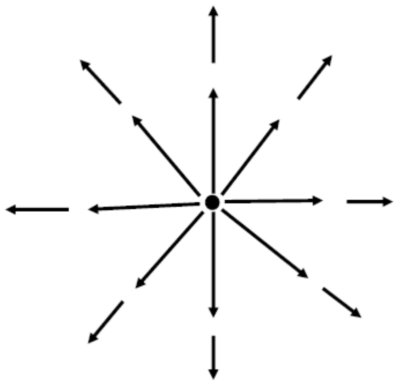

Figure 8 : The vector function $\underline{\boldsymbol{h}} = \frac{\hat{\mathbf{r}}}{r^2}$ shown graphically.

Use spherical polar co-ordinates

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{F}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta F_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}F_\phi$$

Therefore, for $\underline{\boldsymbol{h}} = \frac{1}{r^2}\hat{\mathbf{r}}$:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{h}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}r^2\right) = 0$$

Although at r = 0, we can write $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{h}}$ in terms of a delta function.

3.

$$\nabla^2 \underline{\boldsymbol{F}} = \frac{\partial^2}{\partial x^2}\underline{\boldsymbol{F}} + \frac{\partial^2}{\partial y^2}\underline{\boldsymbol{F}} + \frac{\partial^2}{\partial z^2}\underline{\boldsymbol{F}}$$

$$\frac{\partial^2}{\partial x^2}\underline{\boldsymbol{F}} = 2\hat{\mathbf{i}}, \qquad \frac{\partial^2}{\partial y^2}\underline{\boldsymbol{F}} = 0, \qquad \frac{\partial^2}{\partial z^2}\underline{\boldsymbol{F}} = 0$$

$$=> \nabla^2 \underline{\boldsymbol{F}} = 2\hat{\mathbf{i}}$$

$$\nabla^2 f = \frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f + \frac{\partial^2}{\partial z^2}f = 2 + 0 + 0 = 2$$

4.

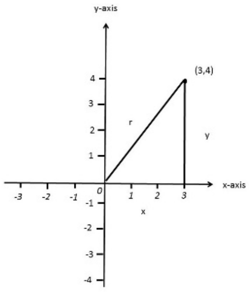

Figure 9 : The path integral of $\underline{\nabla} T$ along the straight line joining the origin and point (3,4)

$$\underline{\nabla} T = y^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}}$$

Given $\underline{\boldsymbol{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, we know $d\underline{\boldsymbol{r}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$. Therefore the path integral is defined by:

$$\int \underline{\nabla} T \cdot d\underline{\boldsymbol{r}} = \int y^2 dx + \int 2xydy$$

In order to solve the path integral, we need to know the values of y at any given value of x. This is of course specified by the path itself for which $y = \frac{4x}{3}$. This gives

$$\int \underline{\nabla} T \cdot d\underline{\boldsymbol{r}} = \int_0^3 \frac{16}{9} x^2 dx + \int_0^4 \frac{6}{4} y^2 dy = 48$$

We can also show that the path integral is independent of the particular path chosen, by noting:

$$\underline{\nabla} T \cdot d\underline{\boldsymbol{r}} = y^2 dx + 2xydy = d(xy^2)$$

Path integral $= \int \underline{\nabla} T \cdot d\underline{\boldsymbol{r}} = \int_{(0,0)}^{(3,4)} d(xy^2) = 3(4)^2 - 0 = 48$

The integral equals $[xy^2]_{(0,0)}^{(3,4)}$and so is <u>independent</u> of path

5.

$$\underline{\boldsymbol{E}} = -\underline{\nabla} V = -\left[\frac{\partial}{\partial r} V\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta} V\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} V\hat{\boldsymbol{\phi}}\right]$$

Use gradient in spherical form.

$$\underline{\nabla} V = \underline{\nabla}\frac{1}{\sqrt{x^2+y^2+z^2}} = \underline{\nabla}\frac{1}{r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = -\frac{1}{r^2}\hat{\mathbf{r}}$$

Therefore,

$$\underline{\boldsymbol{E}} = \frac{1}{r^2}\hat{\mathbf{r}} = \frac{1}{3}\hat{\mathbf{r}}$$

6. a)

$$Flux = \oint_{volume} \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{F}}\, dV$$

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{F}} = 2x + 2y$$

$$\text{Flux} = \oint (2x + 2y) dxdydz = 2\int_0^1 \int_0^1 \int_0^1 (x + y)\, dzdxdy$$

$$= 2(1-0)\int_0^1 \int_0^1 (x + y)\, dx\, dy = 2\int_0^1 \left(\left[\frac{x^2}{2}\right]_0^1 + [x]_0^1\, y \right) dy$$

$$= 2\int_0^1 \left({}^1\!/_2 + y \right) dy = 2\left[{}^1\!/_2 + {}^1\!/_2\right] = 2$$

b) We need to consider the 6 faces one by one.

i) On this plane, x = 1, dx = 0 and $\hat{\mathbf{n}} = \hat{\mathbf{i}}$

$$=> \underline{\boldsymbol{F}} = y^2\hat{\mathbf{i}} + (2y + z^2)\hat{\mathbf{j}} + 2yz\hat{\mathbf{k}}$$

So

$$\underline{\boldsymbol{F}} \cdot \hat{\mathbf{n}} = y^2$$

$$(flux)_i = \int_0^1 \int_0^1 y^2 dydz = \frac{1}{3}[y^3]_0^1 \int_0^1 dz = {}^1\!/_3$$

ii) On this plane, x = 0, dx = 0 and

$$\underline{\boldsymbol{F}} \cdot \hat{\mathbf{n}} = -y^2$$

$$(flux)_{ii} = -\int_0^1 \int_0^1 y^2\, dydz = -{}^1\!/_3$$

iii) On this plane, y = 1, dy = 0 and $\hat{\boldsymbol{n}} = \hat{\mathbf{j}}$

$$\underline{\boldsymbol{F}} \cdot \hat{\mathbf{n}} = (2xy + z^2) = 2x + z^2$$

$$(flux)_{iii} = \int_0^1 \int_0^1 (2x + z^2)\, dxdz = \int_0^1 (1 + z^2)\, dz = {}^4\!/_3$$

iv) On this plane,, y = 0, dy = 0, $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$

$$=> \underline{\boldsymbol{F}} \cdot \hat{\mathbf{n}} = -(2xy + z^2) = -z^2$$

$$(flux)_{iv} = -\int_0^1 \int_0^1 z^2 dxdz = -\int_0^1 z^2 dz = -{}^1\!/_3$$

v) On this plane, z = 1, dz = 0, and $\hat{\mathbf{n}} = \hat{\mathbf{k}}$

$$=> \underline{\boldsymbol{F}} \cdot \hat{\mathbf{n}} = 2yz = 2y$$

$$(flux)_v = \int_0^1 \int_0^1 2ydxdy = 2\int_0^1 ydy = 1$$

vi) On this plane, z = 0, dz = 0, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$

$$=> \underline{\boldsymbol{F}} \cdot \hat{\mathbf{n}} = -2yz = 0$$

$$(flux)_{vi} = 0$$

Total flux = $^1/_3 - {}^1/_3 + {}^4/_3 - {}^1/_3 + 1 + 0 = 2$

1.4 Unseen problems

1.4.1 Dot products and cross products

When using scalar quantities or numbers, there are four core operations: addition, subtraction, division and multiplication. When using vectors there four core operations: addition and subtraction as well as dot product and cross product.

1. Given that $\underline{\mathbf{a}} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\underline{\mathbf{b}} = 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$. Find
(a) $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$ (b) $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ (c) $\hat{\mathbf{a}}$

2. If $\underline{\mathbf{a}} = 4\hat{\mathbf{i}} - 8\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\underline{\mathbf{b}} = 3\hat{\mathbf{j}} + b\hat{\mathbf{k}}$ are orthogonal to each other, find the value of b.

3. Find a unit vector orthogonal to $10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 16\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - \hat{\mathbf{j}}$.

4. Given that $\underline{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\underline{\mathbf{b}} = 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\underline{\mathbf{c}} = -\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, find
(a) $\underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$ (b) $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$.
Can you calculate $\underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \cdot \underline{\mathbf{c}})$? Give the explanation.

1.4.2 Vector addition

When considering adding or subtracting important scalar quantities such as mass or charge, one can simply add or subtract numbers to find resultant masses or charges. However for important vector quantities such as force or velocity or electric field, we must use vector manipulation to calculate resultant forces or electric fields.

5. A charge $q_1 = 4.0$ μC is at the origin, and a charge $q_2 = 6.0$ μC is on the x axis at $x =$ 3.0 m.
(a) Find the force on q_1
(b) Find the force on q_2
(c) How would your answers for part (a) and (b) differ if q_2 were -6.0 μC?

6. A charge $q = -3.64$ nC moves with a velocity of $2.75 \times 10^6\hat{\mathbf{i}}$ ms^{-1} Find the force on the charge if the magnetic field is $(0.75\hat{\mathbf{i}} + 0.75\hat{\mathbf{j}})$ T.

7. Prove that for a charge moving in a constant magnetic field $\underline{\boldsymbol{B}}$, there is no work done by magnetic force.

8. A current wire is bent into a semicircular loop of radius R that lies in the x-y plane. There is a current flowing through the wire and uniform magnetic field $\underline{\mathbf{B}} = B_0\hat{\mathbf{k}}$ perpendicular to the plane of the loop as shown in the figure. Verify that the force acting on the loop is zero.

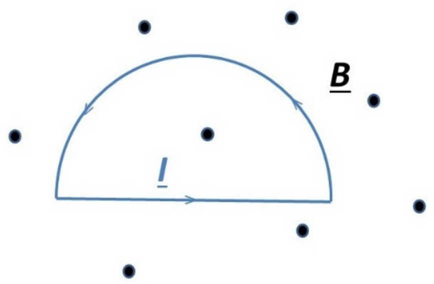

Figure 10 : A wire bent into a semi-circular loop of radius R carrying a current $\underline{\boldsymbol{I}}$ in a magnetic field $\underline{\boldsymbol{B}}$.

1.4.3 Calculations using the divergence theorem and Stoke's theorem

Important concepts in electromagnetism are encapsulated by the divergence theorem and Stokes theorem. These equations include calculations about vector fields.

9. A charge q sits at the middle of a cubic box. What is the flux of $\underline{\boldsymbol{E}}$ through the cubic box (Be careful this is a tricky question that can be either solved directly with a great deal of algebra or very easily with a little deep thought) ?

10. Let $\underline{\boldsymbol{E}}$ be the electric field $\underline{\boldsymbol{E}} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$.
(i) Calculate $\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}$.
(ii) Find a potential function (V) for $\underline{\boldsymbol{E}}$ where $\underline{\boldsymbol{E}} = -\underline{\boldsymbol{\nabla}}V$.

11.
(a) What is the gradient of $f(x, y, z) = 2xz + 4xy^2 + 16z + 3yz^2$?
(b) What is the curl and divergence of the vector field,

$$\underline{\boldsymbol{F}} = (2z + 4y^2)\hat{\mathbf{i}} + (8xy + 3z^2)\hat{\mathbf{j}} + (2x + 16 + 6yz)\hat{\mathbf{k}}$$

12. Calculate the total flux of the vector field,

$$\underline{\boldsymbol{E}}(x, y, z) = (x^3 - y^3)\hat{\mathbf{i}} - xyz^2\hat{\mathbf{j}} + (x^2z - 2x^2)\hat{\mathbf{k}},$$

out of a box given by $0 \le x \le 1$, $0 \le y \le 3$ and $0 \le z \le 2$.

First use surface integrals to calculate the flux directly, then use volume integrals to explicitly show that the divergence theorem is correct in this particular case,

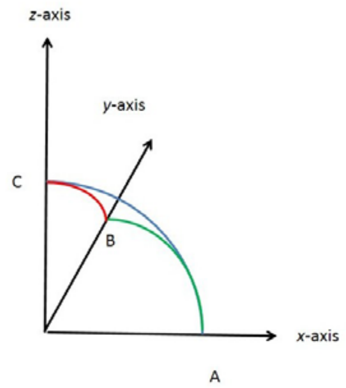

Figure 11 : A three component path integral along a sphere and bounded by the Cartesian coordinates.

13. Given that $\underline{\boldsymbol{F}} = xz\hat{\mathbf{i}}$, show by direct evaluation of the line integral that $I = \oint_{ABCA} \underline{\boldsymbol{F}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \frac{a^3}{3}$ around the closed path shown defined by the edge of the section of the spherical surface satisfying $x^2 + y^2 + z^2 = a^2$ where $x \geq 0$, $y \geq 0$, $z \geq 0$.

Then use Stokes' theorem to obtain the same result by considering either of two surfaces:
a) The open surface defined by the curved section of the spherical surface bounded by the above edge (you will eventually need to employ a change to spherical polar coordinates to evaluate this integral)
b) An alternative surface consisting of three component planar parts in the *xy*, *yz* and *zx* planes, again, of course, bounded by the same outside edge. (This time make use of a cylindrical coordinate system to evaluate the surface integral.)