Mathematical Methods II
PDF 9

Craig Testrow

Key Points

- Revision Lecture

Revision

- **Common terms in y_c and y_p:** Let's say you have the following solution to a homogeneous ODE $y_c = c_1 x + c_2$. Let's also say the inhomogeneous RHS is x^2. So you decide your $y_p = ax^2 + bx + c$. But as you want to avoid terms already found in y_c you multiply by x^2. Now $y_p = ax^4 + bx^3 + cx^2$, *not* $y_p = ax^4 + bx^3 + cx^2 + dx + e$. This is the equation you would use if you started with a quartic RHS. The d and e terms won't invalidate the solution, they will just add to the c_1 and c_2 terms in y_c. These terms are unecessary and just cause more work if included.

- **Legendre:** Please bare in mind Adrien-Marie Legendre, like many famous mathematicians, lent his name to more than one equation or technique. Legendre linear equations are not the same as the Legendre differential equation.

  2^{nd} order Legendre linear equation (from L4),

  $$a_2(\alpha x + \beta)^2 y'' + a_1(\alpha x + \beta)y' + a_0 y = f(x).$$

  Legendre's differential equation (from L8),

  $$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0.$$

- **Singular points at ∞:** Show that Legendre's equation has a regular singular point at $|z| \to \infty$.

  $$(1 - z^2)y'' - 2zy' + \ell(\ell + 1)y = 0$$

  Let $w = 1/z$. We need to eliminate z from the derivatives, expressing them in terms of w

  $$\frac{dy}{dz} = \frac{dw}{dz}\frac{dy}{dw} = \frac{d}{dz}\left(\frac{1}{z}\right).\frac{dy}{dw} = -\frac{1}{z^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

$$
\begin{aligned}
\frac{d^2y}{dz^2} &= \frac{dw}{dz}\frac{d}{dw}\left(\frac{dy}{dz}\right), \\
&= \frac{-1}{z^2}\frac{d}{dw}\left(-w^2\frac{dy}{dw}\right), \\
&= -w^2\left(-2w\frac{dy}{dw} - w^2\frac{d^2y}{dw^2}\right), \\
&= 2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}.
\end{aligned}
$$

Sub into the ODE,

$$
\left(1-\frac{1}{w^2}\right)\left(2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}\right) + 2\left(\frac{1}{w}\right)\left(w^2\frac{dy}{dw}\right) + \ell(\ell+1)y = 0.
$$

Expanding and simplifying,

$$
\left(2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}\right) - \left(2w\frac{dy}{dw} + w^2\frac{d^2y}{dw^2}\right) + 2w\frac{dy}{dw} + \ell(\ell+1)y = 0,
$$

which leaves us with a 2^{nd} order ODE in terms of w,

$$
(w^4 - w^2)\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw} + \ell(\ell+1)y = 0.
$$

Dividing by $(w^4 - w^2)$ we find

$$
p(w) = \frac{2w}{w^2-1}, \qquad q(w) = \frac{\ell(\ell+1)}{w^4-w^2}
$$

Examining $p(0)$ and $q(0)$ we find,

$$
\lim_{w\to 0} p(w) = \frac{2w}{w^2-1} \to 0, \qquad \lim_{w\to 0} q(w) = \frac{\ell(\ell+1)}{w^4-w^2} \to \infty,
$$

$p(0) = 0$ but $q(0)$ diverges, so $|z| \to \infty$ is a singular point.
Testing $(w-w_0)p = wp$ and $(w-w_0)^2 q = w^2 q$ we find,

$$
\lim_{w\to 0} wp(w) = \frac{2w^3}{w^2-1} \to 0, \qquad \lim_{w\to 0} w^2 q(w) = \frac{\ell(\ell+1)w^2}{w^4-w^2} = \frac{\ell(\ell+1)}{w^2-1} \to -\ell(\ell+1),
$$

both converge at $w = 0$, so $|z| \to \infty$ is a regular singular point.

Name of ODE / method	Form/Condition	Order	Coeff.	Notes
Separable	$dy/dx = u(x)v(y)$	1	Var	Integrate independently
Exact	$du = A(x, y)dx$ $+B(x, y)dy = 0$ Test if $\partial A/\partial y = \partial B/\partial x$ $\partial u/\partial x = A, \partial u/\partial y = B$	1	Var	Find $u(x, y) = C$ by integrating A or B, use other to find F(x or y) from integral.
Integrating factor	$\mu(x, y)A(x, y)dx+$ $\mu(x, y)B(x, y)dy = 0$	1	Var	For inexact eqns
Homogeneous	$A(x, y)dx = B(x, y)dy$ $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. Sub $y = vx$	1	Var	
Isobaric	$A(x, y)dx = B(x, y)dy$ $f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x, y)$. Sub $y = vx^m$	1	Var	Set powers of: $x, dx = 1, y, dy = m$
Linear 1st order	$dy/dx + p(x)y = q(x)$ $y = 1/\mu(x) \int \mu(x)q(x)dx$ $\mu(x) = e^{\int p(x)dx}$	1	Var	
Bernoulli	$dy/dx + b(x)y = c(x)y^n$ $z = y^{1-n}$	1	Var	Solve as linear 1st order
Linear nth order	$a_n(x)d^ny/dx^n$ $+a_{n-1}(x)d^{n-1}y/dx^{n-1} + ...$ $+a_1(x)dy/dx + a_0(x)y = f(x)$	n	Var	
Linear 2nd order	$y'' + p(z)y' + q(z)y = f(z)$	2	Var	
Complementary function (linear superposition)	$y_c = c_1y_1(x) + c_2y_2(x)$	2+	Const	Solve as RHS=0. y_1 and y_2 must be linearly independent
Auxiliary equation	Sub $y = Ae^{\lambda x}$ Real: $c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ Repeat: $(c_1 + c_2x)e^{\lambda_1 x}$ Complex: $c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}$	2+	Const	Identify roots
Particular integral / trial functions	$y_p = be^{rx}$ or $b_1 sinrx + b_2 cosrx$ or $b_0 + b_1x + ... + b_Nx^N$	2+	Const	To find $RHS \neq 0$
General solution	$y = y_c + y_p$	2+	Const	
Laplace transform	$\bar{f}(s) \equiv \int_0^\infty e^{-sx} f(x)dx$ $\bar{f^n}(s) = s^n\bar{f}(s) - s^{n-1}f(0)$ $-s^{n-2}f'(0) - ...$ $-sf^{(n-2)}(0) - f^{(n-1)}(0)$	2+	Const	

Name of ODE / method	Form/Condition	Order	Coeff.	Notes
Legendre linear eqns	$a_n(\alpha x+\beta)^n \frac{d^n y}{dx^n} + \ldots$ $+a_1(\alpha x+\beta)\frac{dy}{dx} + a_0 y = f(x)$ Sub $\alpha x+\beta = e^t$	n	Var	Make coeffs. const. with sub.
Euler linear eqns	$a_n x^n \frac{d^n y}{dx^n} + \ldots$ $+a_1 x\frac{dy}{dx} + a_0 y = f(x)$ Sub $x = e^t$	n	Var	Make coeffs. const. with sub.
Wronskian	$W = y_1 y_2' - y_1' y_2$	2+	Var	Check for linear independence
Wronskian method / variation of parameters	$y_p(x) =$ $k_1(x)y_1(x) + k_2(x)y_2(x)$ $k_1' = \frac{-f(x)}{W(x)} y_2$ $k_2' = \frac{f(x)}{W(x)} y_1$	2+	Var	Find y_c as usual. $y = y_p$ as y_c is implicit in y_p
Ordinary and singular points	p and q finite $\to$ ordinary p or q infinite $\to$ singular $(z-z_0)p$ and $(z-z_0)^2 q$ finite $\to$ regular singular $(z-z_0)p$ or $(z-z_0)^2 q$ infinite $\to$ irregular singular	2+	Var	
Taylor series	$y(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ $= \sum_{n=0}^{\infty} a_n (z-z_0)^n$ $y' = \sum_{n=0}^{\infty} n a_n z^{n-1}$ $y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}$	2+	Var	Requires ordinary point. Shift index by adding to n terms. Determine recurrance relation(s) for a_n.
Legendre's DE	$(1-x^2)y'' - 2xy'$ $+\ell(\ell+1)y = 0$ $P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$	2	Var	Determine ℓ, solve with Rodrigues' formula