University of Durham

EXAMINATION PAPER

May/June 2014 Examination code: 043631/01

LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3B

SECTION A. Statistical Physics
SECTION B. Condensed Matter Physics part 1
SECTION C. Condensed Matter Physics part 2

Time allowed: 3 hours

Examination material provided: None

Calculators: The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. STATISTICAL PHYSICS

Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) (i) Explain the concepts of thermodynamic macrostate of a system and a microstate of a system and explain the difference between them. [2 marks]

   (ii) For the *microcanonical*, or (N, U, V) macrostate describing a system in isolation, where the number of particles N, the internal energy U, and the volume V of the system are all fixed: Are the various microstates that are consistent with the (N, U, V) macrostate equally probable, or do they have different probabilities in general? [1 mark]

   (iii) Similarly, for the *canonical*, or (N, T, V) macrostate (ensemble), describing a system in thermodynamic equilibrium at fixed N, V and held at fixed temperature T through contact with a heat bath: Are the various accessible microstates equally probable, or do they have different probabilities in general? [1 mark]

   (b) Give the statistical definition for the entropy of an isolated system. Show that the total entropy of a composite system, made up of two independent subsystems, is equal to the sum of the entropies of the subsystems. [4 marks]

   (c) Give the expressions for the Fermi-Dirac (FD) and Bose-Einstein (BE) distributions. Explain the limit where the FD and BE distributions reduce to the classical Maxwell-Boltzmann (MB) distribution. [4 marks]

   (d) Derive the density of states in k-space, $g(k)\,\delta k$, for a free particle in three dimensions. Then, derive the density of states in energy $g(\epsilon)\,\delta\epsilon$. For the derivation, you are free to use either periodic boundary conditions, or to consider that the particle is in a potential well with infinite walls. [4 marks]

   (e) The two lowest-lying energy levels of a hydrogen atom are $E_0 = -13.6$ eV and $E_1 = -3.4$ eV. Ignoring degeneracies, at what temperature would we find one hundredth as many hydrogen atoms in the first excited state as in the ground state? ($k_{\rm B} = 8.617 \times 10^{-5}$ eV K^{-1}) [4 marks]

2. A paramagnetic solid consists of N ions with spin 1/2 and magnetic moment $\mu_{\rm B}$. The system lies in a magnetic field B and each magnetic moment is oriented either parallel to the field (up), with energy $\epsilon_\uparrow = -\mu_{\rm B}B$, or antiparallel (down) with energy $\epsilon_\downarrow = +\mu_{\rm B}B$. The system is in contact with a heat bath at temperature T.

   (a) Write down the single-particle partition function Z_1. Give the probability $p_\uparrow$ that a magnetic moment is up and the probability $p_\downarrow$ that it is down. [2 marks]

   (b) The magnetisation per ion is equal to the average magnetic moment: $M/N = \sum_i p_i m_i$, where $i = \uparrow, \downarrow$ and $m_\uparrow = \mu_{\rm B}$, $m_\downarrow = -\mu_{\rm B}$.
   Show that the magnetisation per ion, M/N is given by:

   $$\frac{M}{N} = \mu_{\rm B} \tanh\left(\frac{\mu_{\rm B}B}{k_{\rm B}T}\right)$$

   where $k_{\rm B}$ is Boltzmann's constant. What is the limit of M/N for low and for high temperatures? [2 marks]

   (c) Obtain the internal energy U of the system of ions, directly from the definition, $U = N\sum_i p_i \epsilon_i$, or from the partition function. Compare with the energy of N magnetic moments, each of magnitude M/N and oriented along B. [3 marks]

   (d) The system of ions is brought into a state where the internal energy U is positive.

      (i) Show that the temperature of the system is negative. Is a negative temperature "hotter" (of higher energy) or "colder" than infinite temperature? [4 marks]

      (ii) Suggest a way to bring the system into such a state of negative temperature. [1 mark]

      (iii) What is the internal energy U, entropy S and the temperature T of the system in the limit where all the magnetic moments tend to become antiparallel to B? [3 marks]

   (e) (i) Using Gibbs' definition ($S = -Nk_{\rm B}\sum_i p_i \ln p_i$), or otherwise, show that the entropy S of the system depends on the magnetic field B and on the temperature T through the ratio B/T. [1 mark]

      (ii) Sketch the graph of the entropy versus temperature for two different applied magnetic fields and then explain how a dilute paramagnetic solid can be cooled. [4 marks]

3. (a) A gas of weakly interacting fermions is in a box of volume V. The single-particle energy levels for free particles in the box are ϵ_i and each level has degeneracy g_i. Consider a distribution of the fermions $(n_0, n_1, \ldots)$ in the single-particle levels $(\epsilon_0, \epsilon_1, \ldots)$, where n_k fermions are spread among the g_k degenerate single-particle states with energy ϵ_k.
Show that the statistical weight (number of microstates) of the distribution $\{n_i\}$ is given by

$$\Omega = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}.$$

[4 marks]

(b) Derive the Fermi-Dirac (FD) distribution for the average number of fermions, $f_k = n_k/g_k$, in a single-particle state with energy ϵ_k. Assume that n_k, g_k are sufficiently large for Stirling's approximation to be valid. (For large N: $\ln N! \simeq N \ln N - N$.) [4 marks]

(c) A gas of N free electrons is in a box of volume V at temperature $T = 0$. The magnetic moment of the electrons is $\mu_{\rm B}$. After applying a magnetic field B, the single-particle energies for the spin-up and spin-down electrons are: $\epsilon_k^{\uparrow} = \hbar^2 k^2/(2m) - \mu_{\rm B} B$ and $\epsilon_{k'}^{\downarrow} = \hbar^2 k'^2/(2m) + \mu_{\rm B} B$.

[Hint: The Fermi level, or chemical potential, $\epsilon_{\rm F}$ (i.e., the highest occupied single-particle level) is the same for the spin-up and spin-down electrons: $-\mu_{\rm B} B \leq \epsilon_k^{\uparrow} \leq \epsilon_{\rm F}$ and $\mu_{\rm B} B \leq \epsilon_{k'}^{\downarrow} \leq \epsilon_{\rm F}$.
Note, when $B \neq 0$ the Fermi kinetic energies $\hbar^2 k_{\rm F}^2/(2m)$, $\hbar^2 k_{\rm F}'^2/(2m)$ for spin-up and spin-down electrons are not equal to $\epsilon_{\rm F}$.]

(i) When B is weak, only few spin-down electrons can flip spin and align with B. Explain qualitatively why it is not possible for all spin-down electrons to follow the magnetic field. [2 marks]

(ii) Describe what happens when B increases to a value B_0, where $\mu_{\rm B} B_0$ equals the chemical potential: $\mu_{\rm B} B_0 = \epsilon_{\rm F}$. [2 marks]

(iii) At $B = B_0$, show that the Fermi wavevector $k_{\rm F}$ for spin-up electrons is given by: $k_{\rm F}^3 = 6\pi^2 N/V$. [4 marks]

(iv) Obtain the value B_0. [4 marks]

SECTION B. CONDENSED MATTER PHYSICS part 1
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) Show that for a primitive square lattice in two dimensions the kinetic energy of a free electron at a corner of the first Brillouin zone is higher than that of an electron at the midpoint of a side face of the zone by a factor of 2. What is the corresponding factor for a primitive cubic lattice in three dimensions? [4 marks]

   (b) The periodicity $\Delta(1/B)$ of the de Haas-van Alphen oscillations measures the extremal cross-sectional area S in k-space of the Fermi surface. Calculate the period expected for potassium within the free electron model given that the Fermi wavevector is 0.75×10^{10} m^{-1}. [4 marks]

   (c) Give a brief explanation of the origin of diamagnetism in Langévin's model and state the Langévin equation for the susceptibility of a diamagnetic solid. Identify all the terms in the equation. [4 marks]

   (d) Use Hund's rules to find the values of the total spin, S, the total orbital angular momentum, L, and the total orbital angular momentum, J, of an isolated dysprosium ion, Dy^{3+}, which has 9 electrons in the $4f$ shell. [4 marks]

   (e) Sketch the typical hysteresis curve for a magnetically hard material and give one example of a technologically useful application of such a material. [4 marks]

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5. (a) Consider a one-dimensional chain of atoms of lattice constant a. Starting from the energy - wavevector $E(k)$ relationship for free electrons explain why the introduction of a periodic potential causes band gaps to open up at the wavevectors $k = \pm\pi/a$. [4 marks]

   (b) Show that the wavefunctions at $k = \pm\pi/a$ are not travelling waves of the form $\exp(\pm i\pi x/a)$ but are instead standing waves. [4 marks]

   (c) By considering the distribution of probability densities of these standing waves show that the energy gap, U, is equal to the Fourier component of the crystal potential. [4 marks]

   (d) A two-dimensional metal has a monovalent atom in a primitive rectangular crystal structure with lattice parameters $a = 0.2$ nm and $b = 0.4$ nm. Draw the reciprocal lattice and the first two Brillouin zones. Calculate the radius of the free electron Fermi circle and draw the Fermi circle within the first Brillouin zone. [4 marks]

   (e) Make another sketch to show the first few periods of the electron energy bands in the periodic zone scheme, for both the 1st and 2nd Brillouin zones, assuming a small energy gap at the zone boundary. [4 marks]

6. (a) What are ferromagnetic spin waves and ferromagnetic magnons? [4 marks]

(b) A three-dimensional ferromagnet with a primitive cubic crystal structure has a magnon dispersion relation given by

$$\omega = \left(\frac{2\mathrm{J}S}{\hbar}\right)\left[z - \sum_{\underline{r}} \cos\left(\underline{q}\cdot\underline{r}\right)\right],$$

where J is the exchange integral, S is the spin of the atom in the solid, $\underline{q}$ is the wavevector of the spin wave, z is the number of nearest neighbours an atom has in the solid, and $\underline{r}$ is the vector joining an atom to one of its neighbours and $|\underline{r}| = a$. Use this equation to obtain the form of the dispersion relation, in the long wavelength limit, for a ferromagnet with a primitive cubic crystal structure. State an expression for the average number of thermally excited magnons of angular frequency ω at a temperature T. [4 marks]

(c) Use this expression to show that in the long wavelength limit, the total number of all possible magnons excited in a solid of volume V at a temperature T is given by

$$\sum_{\underline{q}} n_{\underline{q}} = \frac{V}{4\pi^2}\left(\frac{\hbar}{2\mathrm{J}Sa^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{\omega^{\frac{1}{2}}}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1} d\omega.$$

Evaluate this integral to show that the total number of magnons varies with temperature as $T^{3/2}$. [7 marks]

$$[\text{Hint}: \int_0^\infty \frac{x^{1/2}}{\exp(x) - 1} dx = 0.0587 \times 4\pi^2].$$

(d) A new ferromagnetic material is found to have a primitive cubic crystal structure with a lattice constant of $a = 0.20$ nm. Each atom has a spin $S = 1/2$ and J $= 0.010$ eV. Calculate both the magnetisation of the solid at $T = 0$ K and the change in magnetisation when the solid is heated to 10 K. Take the Bohr magneton to be $\mu_{\mathrm{B}} = 9.27 \times 10^{-24}$ J T^{-1}. [5 marks]

SECTION C. CONDENSED MATTER PHYSICS part 2
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) The energy of an electron in the conduction band of a two-dimensional semiconductor is given by

$$E(\underline{k}) = Ak^2 + Bk_x^2,$$

where A and B are positive constants, and $\underline{k} = (k_x, k_y)$ is the electron wavevector. Compute the electron effective mass in the k_x and k_y directions. [4 marks]

(b) Sketch how the electric current depends on the applied voltage for an ideal p-n junction. Explain the use of a p-n junction as a rectifier. [4 marks]

(c) Starting from the London equation

$$\nabla \times \underline{j} = -\frac{nq^2}{m}\underline{B},$$

show that the magnetic field $\underline{B} = (0, 0, B_z(r))$ inside an infinitely long cylindrical superconducting material with radius R and centered on the z-axis obeys the following equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right) = \frac{B_z}{\lambda_L^2},$$

where $\lambda_L = (m/\mu_0 nq^2)^{1/2}$ is the London penetration depth, $\underline{j}$ is the current density, n, q, and m are the density, charge, and mass of the electrons. [2 marks]

When the radius of the superconducting cylinder is very large, $R \to \infty$, show that the magnetic field inside the cylinder can be approximated by

$$B_z(r) = B_0 \exp[(r - R)/\lambda_L].$$

Assume $\underline{B} = (0, 0, B_0)$ outside the superconducting material. [2 marks]

(d) The Landau free energy density G for a ferroelectric material can be expressed in the form

$$G(E, P, T) = -EP + a_0 + a_2(T - T_c)P^2 + a_4P^4$$

as a function of temperature T, polarization P and electric field E. a_0, a_2 and a_4 are positive constants, and T_c is the Curie temperature. For $T < T_c$ and zero applied electric field, show that the spontaneous polarization is given by

$$|P_s| = \left(\frac{a_2}{2a_4}\right)^{1/2}(T_c - T)^{1/2}.$$

[2 marks]

What is the maximum value of the spontaneous polarization in this model? Compute the temperature at which the spontaneous polarization is half of the maximum value. [2 marks]

(e) To describe the frequency-dependent relative permittivity $\epsilon(\omega)$, Debye proposed the use of the following equation

$$\frac{dP}{dt} = \frac{P_E - P}{\tau},$$

where τ is the relaxation time for molecular rotation, and $P_E = \chi(0)\epsilon_0 E$ is the equilibrium value of the polarization given an instantaneous applied field E. Assume the applied electric field has the form $E(\omega) = E_0 \exp(i\omega t)$, and $\chi(0)$ is the static susceptibility. Show that this model results in an expression for $\epsilon(\omega)$ given by

$$\epsilon(\omega) = \epsilon_r(\omega) + i\epsilon_i(\omega),$$

where

$$\epsilon_r(\omega) = \frac{1 + \omega^2\tau^2 + \chi(0)}{1 + \omega^2\tau^2},$$

$$\epsilon_i(\omega) = -\frac{\omega\tau\chi(0)}{1 + \omega^2\tau^2}.$$

[4 marks]

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8. A semiconductor has conduction and valence band effective densities of states N_C and N_V respectively. Write down expressions for the electron concentration in the conduction band and the hole concentration in the valence band in terms of the effective densities of states N_C and N_V, the energy band gap E_G, and the chemical potential μ. [2 marks]

   Using the same notation, write down and explain the law of mass action as used in semiconductor physics. [3 marks]

   A semiconducting material is found to have the following properties: (i) At very low temperature, electrons are the dominant charge carriers. Their concentration, n, is found to increase with temperature, T, as $n \propto T^{3/2}\exp(-\alpha/T)$, with the constant $\alpha \approx 160$ K; (ii) The electron concentration is approximately constant around room temperature, $n \approx 10^{22}\,\mathrm{m}^{-3}$; (iii) Finally, at high temperature, the hole concentration becomes comparable to the electron concentration and they increase with temperature as $n = p \propto T^{3/2}\exp(-\beta/T)$, with the constant $\beta \approx 3800$ K.

   Is the material an intrinsic or a doped semiconductor? If the latter, is it a p-type or n-type semiconductor? Explain your reasoning. [5 marks]

   From the measurements described above, state whether it is possible to estimate the energy band gap E_G, donor energy level E_D, acceptor energy level E_A, donor concentration N_D, and acceptor concentration N_A. Where possible, compute the values, and with the help of the table below, show which semiconducting material is consistent with the observations. [8 marks]

   Estimate the temperature at which the transition between regimes (ii) and (iii) occurs, assuming that $N_C = 1.04 \times 10^{25}\,\mathrm{m}^{-3}$ and $N_V = 0.60 \times 10^{25}\,\mathrm{m}^{-3}$. For simplicity, assume N_C and N_V to be constant. [2 marks]

	E_G(eV)	E_D(eV) As	E_D(eV) Sb	E_A(eV) B	E_A(eV) Al
Si	1.12	0.054	0.043	0.045	0.072
Ge	0.66	0.014	0.010	0.011	0.011

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9. In a magnetic field, the change in the Gibbs free energy per unit volume is given by

$$dG = -SdT - MdB,$$

where the symbols M and S correspond to the magnetization and entropy per unit volume.

Show that, for a superconductor with critical field B_c, the difference between the Gibbs functions for the normal and superconducting states in zero field is

$$G_n(0,T) - G_s(0,T) = \frac{B_c^2}{2\mu_0},$$

where $G_n(B,T)$ and $G_s(B,T)$ are the Gibbs functions for the normal and superconducting states in a field B at temperature T. [5 marks]

Experiments on Gallium yield values for the temperature dependence of the critical field B_c, which can be fitted to

$$B_c(T) = B_0\left[1 - \left(\frac{T}{T_c}\right)^2\right].$$

Using this form of $B_c(T)$, show that the latent heat, L, per unit volume at the field-induced normal-to-superconducting transition is given by

$$L = -\frac{2B_0^2T^2}{\mu_0 T_c^2}\left[1 - \left(\frac{T}{T_c}\right)^2\right],$$

and that the difference in heat capacity per unit volume is

$$\Delta C = C_s - C_n = -\frac{2B_0^2 T}{\mu_0 T_c^2} + \frac{6B_0^2T^3}{\mu_0 T_c^4}.$$

C_s and C_n are the heat capacities per unit volume of the superconducting and normal states respectively. [8 marks]

Find the temperature at which $C_s = C_n$. [2 marks]

Explain whether the normal-superconducting transition is first-order or second-order at $T = T_c$, where T_c is the critical temperature. [3 marks]

Obtain an estimate for the discontinuity in the heat capacity which would be observed in Ga at temperature $T \to T_c$, given that $T_c = 1.09\,\mathrm{K}$ and $B_0 = 5.10 \times 10^{-3}\,\mathrm{T}$. [2 marks]