ELECTROMAGNETISM

Level 2 Physics problems – Foundations of physics 2

Question 7 Cycle 2 Version 1

Professor D P Hampshire – 2nd Year Physics Lecture Course

These problems are formatively self-assessed. Students who showed the chutzpah to volunteer for the peer-marking pilot scheme will also mark one of their peer's scripts.

Reading Material (Please note that questions may not be exclusively from these chapters): Please read chapters 9, 10, 11 and 12 of Griffiths.

1.

a) Write down an expression for the Poynting vector and explain its significance. [1 mark]

b) An electromagnetic wave is travelling in free space with magnetic and electric fields given by; [1 mark]

$$\underline{\boldsymbol{B}} = \underline{\mathbf{B}}_{\mathrm{o}}\mathrm{exp}i(\underline{\mathbf{k}}\cdot\underline{\boldsymbol{r}} - \omega t) \text{ and } \underline{\boldsymbol{E}} = \underline{\mathbf{E}}_{\mathrm{o}}\mathrm{exp}i(\underline{\mathbf{k}}\cdot\underline{\boldsymbol{r}} - \omega t)$$

where $\underline{\mathbf{k}}$, ω, $\underline{\mathbf{B}}_{\mathrm{o}}$ and $\underline{\mathbf{E}}_{\mathrm{o}}$ are constants. Show, using Maxwell's equations, that the direction of propagation of the wave, the magnetic field and the electric field are all orthogonal to each other.

c) The Poynting vector is in the $3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 1\widehat{\boldsymbol{k}}$ direction, $\underline{\mathbf{B}}_{\mathrm{o}}$ has a magnitude of 10 nT and points in the direction $3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$ and ω is 3×10^{15} radians s^{-1}. Calculate the vectors $\underline{\boldsymbol{k}}$ and $\underline{\mathbf{B}}_{\mathrm{o}}$ as well as the magnitude of $\underline{\mathbf{E}}_{\mathrm{o}}$ [1 mark]

2. Using Fresnel's equations;

a) Derive an expression for the ratio of the refractive indices of two media given that electromagnetic radiation at normal incidence to a planar interface between the two media has equal energy transmitted and reflected if both media are non-magnetic and non-conducting dielectrics. [1 mark]

b) If one of the media has a relative permittivity of 9 what is the refractive index of the other media? [1 mark]

3. An object radiates electromagnetic waves with a time-averaged power distribution per unit area, $\underline{\boldsymbol{N}}_{\text{timeaverage}}(r,\theta)$, given by:

$$\underline{\boldsymbol{N}}_{\text{timeaverage}}(r,\theta) = \frac{C_0 I_o^2}{r^2}\sin\theta\,\hat{\boldsymbol{r}}$$

where C_0 = 10 Ω.

a) What is the radiation resistance, R_r, for this object where; [1 mark]

$$P = \frac{1}{2} I_o^2 R_r$$

and P is the total power radiated?

b) What is the beam, width, W, for the object? The beam width is the angle subtended between the half-power directions. [1 mark]

c) What is the value of the directivity? The directivity, D, is given by, [1 mark]

$$D = \frac{\text{Power in the peak direction per unit area}}{\text{Total power radiated in all directions/Total area}}$$

4. The magnetic vector field for a Hertzian dipole is given by;

$$\underline{\tilde{\boldsymbol{A}}} = \frac{\mu_o I_0 \underline{\boldsymbol{\delta l}}}{4\pi} e^{i(kr-wt)}$$

Which is equivalent to;

$$\underline{\tilde{\boldsymbol{A}}}(r,t) = \frac{\mu_0 I_0}{4\pi}\frac{e^{i(kr-wt)}}{r}\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right)\delta l$$

By using the definition $\underline{\tilde{\boldsymbol{B}}} = \underline{\boldsymbol{\nabla}} \times \underline{\tilde{\boldsymbol{A}}}$ and spherical harmonics (or otherwise), prove the result given in lectures namely :

$$\underline{\tilde{\boldsymbol{B}}} = \frac{\mu_o I_0 \delta l}{4\pi r^2}(1 - ikr)\sin(\theta)e^{i(kr-\omega t)}\hat{\boldsymbol{\phi}}$$

Hint:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{f}}(r,\theta,\phi) = \begin{vmatrix} \dfrac{\hat{\mathbf{r}}}{r^2\sin\theta} & \dfrac{\hat{\boldsymbol{\theta}}}{r\sin\theta} & \dfrac{\hat{\boldsymbol{\phi}}}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f_r & r f_\theta & r f_\phi \sin\theta \end{vmatrix}$$

where,

$$\underline{\boldsymbol{f}}(r,\theta,\phi) = \left(f_r\hat{\mathbf{r}} + f_\theta\hat{\boldsymbol{\theta}} + f_\phi\hat{\boldsymbol{\phi}}\right)$$

and hence that the far-field B-field is given by :

$$\underline{\boldsymbol{B}}_{\text{Far}} = \frac{\mu_o I_0 \delta l}{4\pi r} k\sin(\theta)\sin(kr - \omega t)\hat{\boldsymbol{\phi}}$$ [1 mark]

And the far-field E-field is given by:

$$\underline{\boldsymbol{E}}_{\text{Far}} = \frac{\mu_0 I_0 \delta l}{4\pi}\frac{\omega}{r}\sin(\theta)\sin(kr - \omega t)\hat{\boldsymbol{\theta}} \qquad \text{[1mark]}$$

Reading Material (Please note that questions may not be exclusively from these chapters):
Please read chapter 11 of Griffiths. Please complete any remaining chapters from Griffiths.

Below are some additional straightforward revision questions (no marks):

5.

a) Write down Maxwell's four equations in differential form.

b) Explain why the relative permeability of a metallic wire that is magnetic is found experimentally to be frequency dependant.

c) Consider an infinite line of equidistant electrons where there is 1 nm between neighbouring electrons. Calculate an approximate value for the magnitude of the electric field at a distance of 2 m from the electrons.

d) Use a diagram to explain the relationships between the electric field, the magnetic field and the direction of propagation of a plane electromagnetic wave propagating in free space.

e) Provide a brief outline of Ampère's model, which describes how a magnetic material generates a magnetic field.

f) Explain what a plasma is and give two examples.

g) Explain the difference between polar and non-polar dielectrics.

6. The wave equation for an electromagnetic wave propagating in vacuum is given by: [1 marks]

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2}$$

What is the group velocity of an electromagnetic pulse obeying this wave equation?

7. It is known that one possible solution for the wave equation within the interior of an infinitely long hollow square metallic tube (known as a waveguide) of width a is:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \sin\left(\frac{5\pi x}{a}\right)\sin\left(\frac{5\pi z}{a}\right)\exp i(ky - \omega t)$$

where x, y and z are the conventional Cartesian coordinates, ω is the angular frequency, k is the wavevector, $\underline{\boldsymbol{E}}_0$ gives the magnitude and polarisation of the wave and the axis of the tube is parallel to the y-axis.

a) Given that a = 0.2 m, find the minimum frequency at which this particular wave can propagate along the waveguide.

[**Hint:** Derive the dispersion relation and find the condition that the wavevector k is real]

b) Above this minimum frequency, determine whether a pulsed wave propagates without changing shape. What is the name of this type of propagation?

c) Consider another solution to the wave equation which describes a different wave that can propagate through the waveguide with a higher minimum frequency. Which of the following statements is true when comparing the energy of photons associated with the wave you have proposed when compared to the original wave considered above? Show your reasoning.

I. **If the two waves have the same frequency their energies are the same.**

II. **The wave with the highest minimum frequency must have the highest energy photons.**

III. **The wave with the lowest minimum frequency must have the highest energy photons.**

IV. **You can't really compare the energy of photons in this way.**

V. **The two waves always have the same energy.**