Foundations of Physics 3A 2020/21 — Problem 4

Preparation and background reading: Notes Lecture 02, Larmor precession, Time dependent perturbation theory

Suppose that the Hamiltonian of a certain spin-1/2 particle submitted to a magnetic field $\mathcal{B}_x\hat{\mathbf{x}} + \mathcal{B}_y\hat{\mathbf{y}} + \mathcal{B}_z\hat{\mathbf{z}}$ is given by the equation

$$H = \tilde{\gamma}\left[\mathcal{B}_x\sigma_x + \mathcal{B}_y\sigma_y + \mathcal{B}_z\sigma_z\right],$$

where $\tilde{\gamma}$ is a constant and σ_x, σ_y and σ_z are the Pauli spin matrices. (The vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are unit vectors in the x-, y- and z-directions, respectively.)

(a) What is the constant $\tilde{\gamma}$ if the particle is an electron?

(b) Suppose that $\mathcal{B}_x = \mathcal{B}_y = 0$, so that $H = \tilde{\gamma}\mathcal{B}_z\sigma_z$. Show that the spin states

$$\alpha \equiv \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad \beta \equiv \begin{pmatrix}0\\1\end{pmatrix} \tag{1}$$

are eigenstates of that Hamiltonian, the corresponding eigenenergies being, respectively, $\tilde{\gamma}\mathcal{B}_z$ and $-\tilde{\gamma}\mathcal{B}_z$.

(c) Suppose that initially ($t \leq 0$), the particle is in the spin state β and that the field is constant and oriented in the z-direction. Suppose, further, that at $t = 0$ the field acquires a constant non-zero x-component so that for $t > 0$ the Hamiltonian of the particle is $\tilde{\gamma}[\mathcal{B}_x\sigma_x + \mathcal{B}_z\sigma_z]$, where $\mathcal{B}_z$ is constant and the same as for $t < 0$. Let $P_{\alpha\beta}(t)$ be the probability that at a time $t > 0$ the particle is in state α in Eq. 1. Show that in first order perturbation theory $P_{\alpha\beta}(t)$ oscillates between 0 and $\mathcal{B}_x^2/\mathcal{B}_z^2$. [Hints: What do you take for unperturbed Hamiltonian H_0? What do you take for the perturbation Hamiltonian H'? What is the Bohr transition frequency?]

(d) At what angular frequency does $P_{\alpha\beta}(t)$ oscillate, as predicted by first order perturbation theory?

(e) [This question is more difficult than the other parts of this homework.] Why can one expect that the angular frequency predicted by first order perturbation theory differs from the exact angular frequency by a factor $[\mathcal{B}_z^2/(\mathcal{B}_x^2 + \mathcal{B}_z^2)]^{1/2}$?

Self-assessing your work: The model solution has no numerical marking scheme. It is suggested that you base your assessment only on parts (a) to (d), and consider yourself to have been "successful" if you could do parts (a) to (d), apart perhaps for minor errors in the maths, "partially successful" if you could do much of part (c) but did not go far in parts (a), (b) and (d), and "unsuccessful" if you did not know how to handle the calculations involved in part (c).