CM7 Solutions: Poisson Brackets

1. **(2 marks total)** For $i = x, y$ and z,

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i.$$

Hence, the canonical momenta are equal to the mechanical momenta. **[1 mark]**

$$\{p_x, p_y\} = \sum_{k=x,y,z} \left(\frac{\partial p_x}{\partial q_k}\frac{\partial p_y}{\partial p_k} - \frac{\partial p_x}{\partial p_k}\frac{\partial p_y}{\partial q_k} \right) = 0.$$

By symmetry,

$$\{p_y, p_z\} = \{p_z, p_x\} = 0.$$

[1 mark]

2. **(2 marks total)** From the definition,

$$\underline{J} = m\underline{q} \times \underline{\dot{q}} = \underline{q} \times \underline{p} = \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \begin{pmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{pmatrix}.$$

[2 marks]

3. **(4 marks total)** Using the definition of a Poisson bracket,

$$\begin{aligned} \{J_x, J_y\} &= \sum_{k=x,y,z} \left(\frac{\partial J_x}{\partial q_k}\frac{\partial J_y}{\partial p_k} - \frac{\partial J_x}{\partial p_k}\frac{\partial J_y}{\partial q_k} \right) \\ &= [-p_y. - x - y.p_x] = xp_y - yp_x = J_z. \end{aligned}$$

By symmetry,

$$\{J_y, J_z\} = J_x \quad \text{and} \quad \{J_z, J_x\} = J_y.$$

[2 marks]

Using $\{F, G\} = -\{G, F\}$,

$$\begin{aligned} \{J_y, \{J_y, \{J_y, J_x\}\}\} &= \{J_y, \{J_y, -J_z\}\} \\ &= \{J_y, -J_x\} \\ &= \{J_x, J_y\} = J_z. \end{aligned}$$

[2 marks]

4. **(2 marks total)** The canonically conjugate momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta.$$

[1 mark]

$\{r, p_r\} = \{\theta, p_\theta\} = \{\phi, p_\phi\} = 1$, because they are canonically conjugate pairs. **[1 mark]**