# QM3 2020/21 — Problem 1

Consider two spin–1/2 particles, particle 1 and particle 2. We denote by $\mathbf{S}_1$ and $\mathbf{S}_2$ the respective spin operators for particle 1 and particle 2 and by $S_{1z}$, $S_{2z}$ their projection on the $z$-axis: $S_{1z} = \hat{\mathbf{z}} \cdot \mathbf{S}_1$ and $S_{2z} = \hat{\mathbf{z}} \cdot \mathbf{S}_2$ where $\hat{\mathbf{z}}$ is the unit vector in the $z$-direction.

$\mathbf{S}$ is the total spin operator, $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. $\alpha(1)$ and $\beta(1)$ the spin-up and spin-down eigenstates of $S_{1z}$ and $\alpha(2)$, $\beta(2)$ the corresponding eigenstates of $S_{2z}$.

The operator $\mathbf{S}_1$ does not act on $\alpha(2)$ and $\beta(2)$, and the operator $\mathbf{S}_2$ does not act on $\alpha(1)$ and $\beta(1)$.

Consider now the following joint spin states (states in the Hilbert space of two spin-1/2 particles), the "singlet" state $\chi_{0,0}$ and the "triplet" state $\chi_{1,0}$

$$\chi_{0,0} = \frac{\alpha(1)\beta(2) - \beta(1)\alpha(2)}{\sqrt{2}}, \quad \chi_{1,0} = \frac{\alpha(1)\beta(2) + \beta(1)\alpha(2)}{\sqrt{2}}$$

(a) Show that these two spin states are eigenstates of the $z$ projection of the total spin, $S_z = S_{1z} + S_{2z}$ with eigenvalue $M = 0$.

(b) Let $\alpha_x(1) = [\alpha(1) + \beta(1)]/\sqrt{2}$ and $\beta_x(1) = [-\alpha(1) + \beta(1)]/\sqrt{2}$. Show that $\alpha_x(1)$ and $\beta_x(1)$ are eigenstates of $S_{1x}$, the projection of $\mathbf{S}_1$ on the $x$-axis, and that the corresponding eigenvalues are, respectively, $\hbar/2$ and $-\hbar/2$. Show also that $\alpha_x(1)$ is orthogonal to $\beta_x(1)$.

(c) Show that

$$\alpha(1)\beta(2) - \beta(1)\alpha(2) = \alpha_x(1)\beta_x(2) - \beta_x(1)\alpha_x(2). \tag{1}$$

Is $\chi_{0,0}$ an eigenstate of $S_x$? Is $\chi_{1,0}$ an eigenstate of $S_x$? ($S_x = S_{1x} + S_{2x}$.)

(d) Why does Eq. (1) imply that if these two particles are in the joint spin state $\chi_{0,0}$, then a measurement of their spin in the $x$-direction must return $\hbar/2$ for one of the particles and $-\hbar/2$ for the other one?

[Hint: There are several ways you can show this. One way is to show that the probability of getting either $\hbar/2$ for both spins or $-\hbar/2$ for both spins is zero. ]

Self assessment: Consider yourself to have been successful if you completed all the four parts reasonably well, and partially successful if you could complete at least two or three of the four parts (except perhaps for minor errors) but not the other one(s).