QUANTUM MECHANICS 2 - WORKSHOP 3

Q1) The 3 dimensional harmonic potential has energies

$$E_{n_x,n_y,n_z} = (n_x + n_y + n_z + 3/2)\,\hbar\omega.$$

Write down the energies and degeneracies of the first 4 energy levels.

Q2) The 1D harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. The ground state ($n = 0$) energy eigenfunction is $\psi_0(x) = Ne^{-m\omega x^2/2\hbar}$. The energy of the n^{th} state is $E_n = (n + 1/2)\hbar\omega$.

[In sections a) and b) we are not concerned with normalisation.]

a) Use the upwards ladder operator $a_+ = A(-ip + m\omega x)$, where $A^2 = 1/(2\hbar\omega m)$, to show that

$$a_+\psi_0(x) \propto xe^{-m\omega x^2/2\hbar}$$

This is also $\propto \psi_1(x)$. The upward ladder operator enables the next energy eigenfunction to be calculated so $\boxed{a_+\psi_n \propto \psi_{n+1}.}$

b) The downwards ladder operator $a_- = A(ip + m\omega x)$. Use this together with a_+ above and the fundamental commutator $[x, p] = i\hbar$ to show that $[a_-, a_+] = 1$.

c) Write x in terms of a_- and a_+ only (i.e. no p). Hence show that

$$x^2 = \frac{\hbar}{2m\omega}\Big((a_+)^2 + a_+a_- + a_-a_+ + (a_-)^2\Big)$$

Hence write $V(x) = \frac{1}{2}m\omega^2x^2$ as a function of $a_\pm$.

d) For any energy eigenstate ψ_n show that $\langle V_n\rangle = \int \psi_n^* V(x)\psi_n\,dx = 1/2m\omega^2\langle x^2\rangle = E_n/2$ given that $a_+a_-\psi_n = n\psi_n$ and $a_-a_+\psi_n = (n + 1)\psi_n$, and that energy eigenfunctions of differing n are orthogonal.