CM3 Solutions: Oscillations about stable equilibrium

1. **(3 marks total)** The kinetic energy is $T = [m_1(a\dot{\theta})^2 + m_2(a\dot{\theta})^2]/2$.
   The potential energy can be written as $V = m_1ga(1-\sin\theta) + m_2ga(1-\sin(\theta+\alpha))$.
   Hence,
   $$L = T - V = \frac{(m_1+m_2)}{2}(a\dot{\theta})^2 - ga[m_1+m_2-m_1\sin\theta - m_2\sin(\theta+\alpha)].$$
   [2 marks]

   Applying the Euler-Lagrange equation implies
   $$(m_1+m_2)a^2\ddot{\theta} - ga[m_1\cos\theta + m_2\cos(\theta+\alpha)] = 0,$$
   from which the required result follows. **[1 mark]**

2. **(1 mark total)** The equilibrium configuration has $\theta_{\rm eq} + \alpha/2 = \pi/2$, i.e. $\theta_{\rm eq} = \pi/2 - \alpha/2$. **[1 mark]**

3. **(4 marks total)** Putting $m_1 = m_2 = m$ into the expression for $\ddot{\theta}$ gives
   $$\ddot{\theta} = \frac{g}{a}\cos\left(\frac{\alpha}{2}\right)\cos\left(\theta+\frac{\alpha}{2}\right).$$
   [1 mark]

   As $\phi = \theta - \theta_{\rm eq}$, $\theta + \alpha/2 = \phi + \pi/2$. Therefore
   $$\begin{aligned}\ddot{\theta} = \ddot{\phi} &= \frac{g}{a}\cos\left(\frac{\alpha}{2}\right)\cos\left(\phi+\frac{\pi}{2}\right)\\ &= -\frac{g}{a}\cos\left(\frac{\alpha}{2}\right)\sin\phi\\ &\approx -\omega^2\phi,\end{aligned}$$
   where $\omega = \sqrt{g\cos(\alpha/2)/a}$ and the oscillations are assumed to be small. **[2 marks]**
   Hence,
   $$\phi(t) = \phi(0)\cos\omega t + \frac{\dot{\phi}(0)}{\omega}\sin\omega t.$$
   [1 mark]

4. **(1 mark total)** When $\alpha = \pi$, the equal mass particles are on opposite sides of the diameter of the hoop and $\ddot{\phi} = 0$. Hence $\phi(t) = \phi(0) + \dot{\phi}(0)t$ and the rod rotates indefinitely with a constant angular velocity. **[1 mark]**

5. **(1 mark total)** In this case, $L = (2m)(a\dot{\phi})^2/2 - 2mga(1-\cos\phi)$ and $\ddot{\phi} \approx -(g/a)\phi$. Hence $\omega_{2m} = \sqrt{g/a}$. (This is also the limiting case of $\alpha \to 0$ in the previous case.) **[1 mark]**