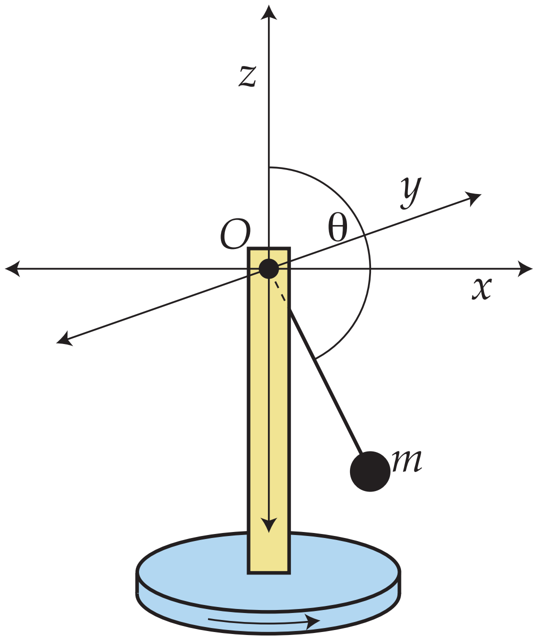
CM2: Lagrangian of a Spinning Plane Pendulum



Consider the situation of a plane pendulum, consisting of a pendulum bob of mass m connected by a rigid massless rod of length R to a pivot at a point in space O. Let us set the origin of our coordinate system at O. An upright connects the pivot, rigidly, to the centre of a disc that is driven by a motor to spin around the z (vertical) axis with a constant angular velocity ω, rotating the plane pendulum with it.

1. Using spherical coordinates, with r being the distance of the pendulum bob from the pivot, θ, the angle away from straight up of the pendulum, and ϕ, the azimuthal angle describing how far the whole apparatus is rotated about the z axis ($\phi = 0$ along the x axis), write down expressions for x, y, z, in terms of r, θ, ϕ, and hence determine expressions for $\dot{x}$, $\dot{y}$, $\dot{z}$ in terms of r, $\dot{r}$, θ, $\dot{\theta}$, ϕ, $\dot{\phi}$.

2. Consider a point mass of mass m moving in 3 dimensions. Using the expressions you have derived for $\dot{x}$, $\dot{y}$, $\dot{z}$, rewrite the standard expression for the kinetic energy T of the point mass in spherical polar form (i.e., in terms of r, $\dot{r}$, θ, $\dot{\theta}$, $\dot{\phi}$). Rewrite the expression for the potential energy $V = mgz$ in spherical polar form.

3. Write down equations that express the two constraints (that the point mass is always a fixed distance R from the origin, and that the azimuthal angle describing the location of the point mass changes in time at a constant rate ω) in mathematical form. Use these constraint equations to simplify the kinetic energy T and the potential energy V, and hence determine the Lagrangian L for the spinning plane pendulum, where θ is the only dynamical variable.

4. Use the Euler-Lagrange equation to determine an expression for $\ddot{\theta}$. From this, determine a condition for the existence of horizontal circular orbits, i.e., motion of the pendulum bob such that the angle of elevation never changes. For what range of angles are such orbits possible? (Do not consider $\theta = 0$ or π).