CM6: An application of Hamiltonian mechanics

A pendulum, of length l with a bob of mass m in a uniform gravitational field g, is driven by an external force such that its top has an x coordinate given by the function $x_0(t)$. These displacements are small, such that the oscillations of the pendulum are also small.

1. If the pendulum makes an angle θ with the downward vertical, then show that the Lagrangian can be written as
$$L = \frac{m}{2}(\dot{x}_0^2 + 2\dot{x}_0 l\dot{\theta}\cos\theta + l^2\dot{\theta}^2) + mgl\cos\theta.$$

2. Using the Legendre transformation $H(p_q, q) = p_q\dot{q} - L(q, \dot{q})$, determine the Hamiltonian of the system, $H(p_\theta, \theta)$, under the assumptions that $\theta, \dot{\theta} \ll 1$. Is $H(p_\theta, \theta) = E$, the total energy of the system?

3. Assume the small angle approximation from the start and repeat the steps above using the x coordinate of the pendulum bob, i.e. $x = x_0 + l\theta$, to find $L(x, \dot{x})$ and show that
$$H'(p_x, x) = \frac{p_x^2}{2m} - mgl + \frac{mg}{2l}(x - x_0)^2.$$
Is $H'(p_x, x) = E$?

4. Use Hamilton's equations of motion,
$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q},$$
to find a second order differential equation for x.

   Describe how Green's functions could be used to solve this equation and the main features of the motion in the case when $x_0 = A\sin(\omega_0 t)$, where A is a constant.