L2 Foundation of Physics 2B Optics 2019-20

O.9 Summary

February 18, 2020

Learning outcomes:

1. To understand Fraunhofer diffraction as a limiting case of Fresnel diffraction that applies (i) exactly in the focal place of a lens (ii) approximately in the far field. [Optics $f2f$ Sec. 5.7]
2. To understand the property of Cartesian separability in diffraction problems. [Optics $f2f$ Sec. 5.6]
3. To apply the Fraunhofer diffraction formula to the case of a single slit [Optics $f2f$ Sec. 5.8].

Key equations: The Fresnel diffraction integral is

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\mathrm{i}\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(x',y')\mathrm{e}^{\mathrm{i}kr_\mathrm{p}}\mathrm{d}x'\mathrm{d}y' \ , \quad \text{where} \quad r_\mathrm{p} = z + \frac{(x-x')^2+(y-y')^2}{2z} \ . \tag{1}$$

In the **Fraunhofer regime**, terms which depend quadratically on the *input* coordinates (i.e. terms that contain $x'^2/z, y'^2/z$) are either (i) cancelled by using a lens and setting $z = f$ or (ii) neglected in the far field $z \gg \rho'$. The latter is known as the Fraunhofer approximation.

The result is the Fraunhofer diffraction formula:

$$I^{(z)} = \frac{I_0}{\lambda^2 z^2}\left|\int\int_{-\infty}^{\infty} f(x',y')\mathrm{e}^{-ik(xx'+yy')/z}\mathrm{d}x'\mathrm{d}y'\right|^2 \ .$$

Note that this has the form of a **Fourier transform** with Fourier variables $u = x/(\lambda z)$ and $v = y/(\lambda z)$. For case (i) in the focal plane of a lens the same formula applies with $z = f$. If the aperture function can be written in the form $f(x',y') = g(x')h(y')$ then the problem is said to be **cartesian separable**, and the integrals over x' and y' can be carried out independently (see equation 5.21). For a long slit along the y axis located at $x = 0$ the aperture function is $g(x') = 1$ for $|x| \leq a/2$, see [Optics $f2f$ Ex. 5.2 on p. 79] so

$$I^{(z)} = \frac{\mathcal{I}_0}{\lambda z}\left|\int_{-a/2}^{a/2}\mathrm{e}^{-\mathrm{i}kxx'/z}\mathrm{d}x'\right|^2 = \frac{I_0 a^2}{\lambda z}\mathrm{sinc}^2\left(\frac{\pi a x}{\lambda z}\right) \ . \tag{2}$$

The sinc-function is unity at $x = 0$ and zero when $x = \pm(\lambda/a)z$. We define the **angular width** of the diffraction pattern as $\Delta\theta = \lambda/a$.

Outlook: In the next lecture, we shall look at diffraction by a double slit [Optics $f2f$ Ex. 5.3 and 5.4 on p. 80-81].