(a) An ignorable coordinate is a generalised coordinate that does not appear explicitly in the Lagrangian of a system although its time derivative does. **[2 marks,B]**
Each ignorable coordinate has an associated constant of motion, the canonically conjugate momentum, that reduces by one the number of differential equations that need to be solved for the system. **[2 marks,B]**

(b) The auxiliary equation is the quadratic in λ for the assumed solution $x = e^{\lambda t}$. This is $\lambda^2 + b\lambda + k/m = 0$. **[2 marks,B]**
The critically damped solution has two equal roots, so $b = 2\sqrt{k/m}$. **[2 marks,B]**

(c) A Green's function describes the response of a system to the application of an impulsive force, i.e. the evolution of a dynamical system subsequent to the application of an impulsive force. **[2 marks,B]**
By considering the driving force to be made up of a sum of an infinite number of impulsive forces, the motion of a driven oscillator can be constructed using an integral of an infinite number of Green's functions. This can be written as a convolution. **[2 marks,B]**

(d) Using the fact that $p = \partial L/\partial \dot{q} = \alpha\beta\dot{q}^{\beta-1}$, and noting that $T = \alpha\dot{q}^{\beta}$, we have $H = \alpha\beta\dot{q}^{\beta} - T + V$. **[2 marks,U]**
By inspection, $E = T + V$. For $H = E$, we need $\alpha\beta\dot{q}^{\beta} - T = T$. Hence, $\alpha\beta\dot{q}^{\beta} = 2\alpha\dot{q}^{\beta}$ and $\beta = 2$ is a necessary condition for $H = E$. **[2 marks,U]**

(e) Noether's theorem: if the Lagrangian is invariant under a continuous symmetry transformation, then there are conserved quantities associated with that symmetry. **[2 marks,B]**
For example, if the Lagrangian is rotationally symmetric, then the angular momentum vector is conserved (or any other reasonable example). **[2 marks,B]**

(f) The Euler force is an inertial/fictitious/pseudo force. Such forces arise when considering dynamics in a non-inertial, i.e. accelerating, reference frame. **[2 marks,B]**
Its magnitude is proportional to the rate of change of the angular velocity of the rotating frame, which, for the Earth, is usually negligibly small. **[2 marks,B]**

(g) Using $\rho = M/(8a^3)$ to represent the uniform density of the cube, the required moment of inertia is

$$I = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} \rho(x^2 + y^2)dxdydz = \frac{2}{3}Ma^2.$$

[4 marks,U]

(h) The principal axis theorem states that it is always possible to rotate the coordinates such that the inertia tensor for a rigid body is diagonal. **[2 marks,B]**
An oblate symmetric ellipsoid has one short axis, with a large principal moment of inertia, and two long axes with smaller, equal, principal moments. Hence, $I_3 > I_1 = I_2$. ($I_1 > I_2 = I_3$ also fine.) **[2 marks,B]**

(a) [6 marks total] **(Unseen)**
The kinetic energy is $T = [m_1(a\dot{\theta})^2 + m_2(a\dot{\theta})^2]/2$.
The potential energy can be written as $V = m_1 ga(1-\sin\theta)+m_2 ga(1-\sin(\theta+\alpha))$.
Hence,

$$L = T - V = \frac{(m_1+m_2)}{2}(a\dot{\theta})^2 - ga[m_1+m_2-m_1\sin\theta - m_2\sin(\theta+\alpha)].$$

[4 marks]

Applying the Euler-Lagrange equation implies

$$(m_1+m_2)a^2\ddot{\theta} - ga[m_1\cos\theta + m_2\cos(\theta+\alpha)] = 0,$$

from which the required result follows. **[2 marks]**

(b) (i) [1 mark total] **(Unseen)**
The equilibrium configuration has $\theta_{\text{eq}} + \alpha/2 = \pi/2$, i.e. $\theta_{\text{eq}} = \pi/2 - \alpha/2$. **[1 mark]**

(ii) [7 marks total] **(Unseen)**
Putting $m_1 = m_2 = m$ into the expression for $\ddot{\theta}$ gives

$$\ddot{\theta} = \frac{g}{a}\cos\left(\frac{\alpha}{2}\right)\cos\left(\theta+\frac{\alpha}{2}\right).$$

[1 mark]

As $\phi = \theta - \theta_{\text{eq}}$, $\theta + \alpha/2 = \phi + \pi/2$. Therefore

$$\begin{aligned}\ddot{\theta} = \ddot{\phi} &= \frac{g}{a}\cos\left(\frac{\alpha}{2}\right)\cos\left(\phi+\frac{\pi}{2}\right)\\ &= -\frac{g}{a}\cos\left(\frac{\alpha}{2}\right)\sin\phi\\ &\approx -\omega^2\phi,\end{aligned}$$

where $\omega = \sqrt{g\cos(\alpha/2)/a}$ and the oscillations are assumed to be small. **[5 marks]**
Hence,

$$\phi(t) = \phi(0)\cos\omega t + \frac{\dot{\phi}(0)}{\omega}\sin\omega t.$$

[1 mark]

(iii) [3 marks total] **(Unseen)** When $\alpha = \pi$, the equal mass particles are on opposite sides of the diameter of the hoop and $\ddot{\phi} = 0$. Hence $\phi(t) = \phi(0) + \dot{\phi}(0)t$ and the rod rotates indefinitely with a constant angular velocity. **[3 marks]**

(c) [3 marks total] **(Unseen)** In this case, $L = (2m)(a\dot{\phi})^2/2 - 2mga(1-\cos\phi)$ and $\ddot{\phi} \approx -(g/a)\phi$. Hence $\omega_{2m} = \sqrt{g/a}$. (This is also the limiting case of $\alpha \to 0$ in the previous case.) **[3 marks]**

(a) (i) [4 marks total] **(Unseen)**
The gravitational acceleration is GM/R^2 towards the centre of the planet, so the centrifugal force needs to balance this. **[2 marks]**
The centrifugal acceleration points radially away from the axis of rotation and has size $\omega^2 R$. Hence $\omega = \sqrt{GM/R^3}$. **[2 marks]**

(ii) [3 marks total] **(Bookwork)**
The centrifugal force vanishes at the poles as $\omega \times r = 0$. **[1 mark]**
The Coriolis acceleration has size $|\underline{a}_{Cor}| = 2\omega v$ and it acts to the right of the direction of motion of the particle. **[2 marks]**

(b) (i) [5 marks total] **(Unseen)**
The time taken for the arrow to reach the target can be approximated as $T = l/v$. **[1 mark]**
During this period, an acceleration of $\ddot{x} = 2\omega v$ to the right is experienced by the arrow. **[1 mark]**
Thus, the arrow will be displaced by $x_{disp} = \ddot{x}T^2/2$ to the right at the distance of the target, which is an angle $\theta_{disp} = x_{disp}/l = \omega l/v$ to the right of the target. Thus, the north pole archer will aim this same angle to the left of the target centre in order to hit it. **[3 marks]**

(ii) [2 marks total] **(Unseen)**
The Coriolis force acts to the left at the south pole, so the south pole archer will aim an angle $\omega l/v$ to the right of the target centre and miss by a distance $2\omega l^2/v$ to the right. **[2 marks]**

(c) [6 marks total] **(Unseen)**
The north pole archer will again aim a distance $\omega l^2/v$ to the left of the target. However, this time the arrow will only take a time $T = fl/v$ to reach the target. Thus it will miss the centre by a distance

$$\Delta x_{NP} = -\frac{\omega l}{v}.v.\frac{fl}{v} + \frac{1}{2}(2\omega v)\left(\frac{fl}{v}\right)^2 = \frac{\omega l^2}{v}(-f + f^2),$$

i.e. to the left of the centre. **[2 marks]**
The south pole archer will aim a distance $\omega(fl)^2/v$ to the right of the target, taking into account that they know the new distance to the target. Thus their arrow will miss the centre by a distance

$$\Delta x_{SP} = \frac{\omega(fl)}{v}.v.\frac{fl}{v} + \frac{1}{2}(2\omega v)\left(\frac{fl}{v}\right)^2 = \frac{2\omega(fl)^2}{v},$$

to the right of the centre. **[2 marks]**
For the southerner to win the competition, we need $|\Delta x_{SP}| < |\Delta x_{NP}|$. Thus $2f^2 < f - f^2$, which is satisfied by $f < 1/3$. **[2 marks]**

All the short questions are bookwork.

(a) Linear independence of the set $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle$ means that there are **no non-zero complex numbers** b_i such that

$$\sum_i b_i|\phi_i\rangle = \underline{0}$$

where $\underline{0}$ is the null state vector. [2 marks]
An orthonormal basis satisfies the condition

$$\langle\phi_i|\phi_j\rangle = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$, 0 otherwise is the Kroenecker symbol. [2 marks]

(b) Hermitian operators $\hat{A}$ have the defining property that

$$\langle\alpha|\{\hat{A}|\beta\rangle\} = [\langle\beta|\{\hat{A}|\alpha\rangle\}]^*$$

where $*$ denotes the complex conjugate. [2 marks]

If $\hat{A}|\alpha_i\rangle = a_i|\alpha_i\rangle$ then

$$\langle\alpha_i|\hat{A}|\alpha_i\rangle = a_i\langle\alpha_i|\alpha_i\rangle$$

The left-hand side of this expression is real using the Hermiticity condition above. On the right-hand side $\langle\alpha_i|\alpha_i\rangle$ is real since it is an inner product of identical states, hence we conclude that the eigenvalues a_i are real. [2 marks]

(c) In Dirac notation the corresponding wavefunction is $\psi(x) = \langle x|\psi\rangle$. [1 mark]

Inserting a complete set of position eigenstates

$$\langle\phi|\psi\rangle = \int dx \langle\phi|x\rangle\langle x||\psi\rangle$$

[2 marks]

$$= \int dx \phi^*(x)\psi(x)$$

an overlap integral of the two wavefunctions. [2 marks]

(d) An operator $\hat{A}$ is unitary if $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} = \hat{1}$ where $\hat{A}^\dagger$ is the adjoint operator defined by

$$\langle\alpha|\hat{A}^\dagger|\beta\rangle = [\beta|\hat{A}|\alpha\rangle]^*$$

and $\hat{1}$ is the unit operator. [2 marks]

We have $\langle\phi'|\psi'\rangle = \langle\phi'|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi'\rangle^* = \langle\psi|\hat{A}^\dagger\hat{A}|\phi\rangle^* = \langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle$. as required. [2 marks]

(e) We have the anticommutation relation $\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = 1$ If we require that b annihilates the vacuum state then $b|0\rangle = 0$. Acting with the number operator $\hat{N} = \hat{b}\hat{b}^\dagger$ on $\hat{b}^\dagger|0\rangle$ yields

$$\hat{N}\hat{b}^\dagger|0\rangle = \hat{b}^\dagger\hat{b}\hat{b}^\dagger|0\rangle = \hat{b}^\dagger(1 - \hat{b}^\dagger\hat{b})|0\rangle = (\hat{b}^\dagger - \hat{b}^\dagger\hat{N})|0\rangle = \hat{b}^\dagger|0\rangle$$

So we see that $\hat{b}^\dagger|0\rangle = |1\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n = 1$. [1 mark]
If we attempt to create another quantum by acting again with $\hat{b}^\dagger$, however, we find

$$\hat{N}\hat{b}^\dagger|1\rangle = \hat{b}^\dagger(1 - \hat{b}^\dagger\hat{b})|1\rangle = \hat{b}^\dagger(1 - \hat{N})|1\rangle = 0$$

So the state $|1\rangle$ is annihilated by $\hat{b}^\dagger$, and the possible eigenvalues are $n = 0$ and $n = 1$. [2 marks]
This is exactly what is required by the Pauli exclusion principle for fermions. [1 mark]

(f) The Heisenberg picture evolution equation is

$$\frac{\partial\hat{0}}{\partial t} = \frac{i}{\hbar}[\hat{H}, \hat{O}]$$

[2 marks]
If the operator corresponds to a conserved quantity one then finds $[\hat{H}, \hat{O}] = 0$, and the operator commutes with the Hamiltonian. [2 marks]

(g) The general expression is

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$$

where ϵ_{ijk} is the Levi-Civita symbol with $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$ and $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$. Corresponding to odd and even signature permutations of $\{1, 2, 3\}$ respectively. [3 marks]

So we find $[\hat{L}_x, \hat{L}_z] = i\epsilon_{132}\hat{L}_y = -i\hat{L}_y$. [1 mark]

Parts (a)-(c) are bookwork, parts (d) and (e) are unseen.

(a) The creation operator in terms of $\hat{x}$ and $\hat{p}$ is

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

and the annihilation operator is

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

[2 marks]

(b) From the expressions in (a) we find directly

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2\hbar}(-i[\hat{x}, \hat{p}] + i[\hat{p}, \hat{x}]) = 1$$

as required, using $[\hat{x}, \hat{p}] = i\hbar$. [2 marks]

(c) We have

$$\hat{N}\hat{a}|n\rangle = \hat{a}^\dagger\hat{a}\hat{a}|n\rangle = (\hat{a}\hat{a}^\dagger - 1)\hat{a}|n\rangle = (\hat{a}\hat{N} - \hat{a})|n\rangle = (n-1)\hat{a}|n\rangle$$

So we see that $\hat{a}|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n-1$, i.e $\hat{a}|n\rangle = c_-|n-1\rangle$ where c_- is a normalization constant. [1 mark]
Similarly

$$\hat{N}\hat{a}^\dagger|n\rangle = \hat{a}^\dagger\hat{a}\hat{a}^\dagger|n\rangle = \hat{a}^\dagger(\hat{a}^\dagger\hat{a} + 1)|n\rangle = (\hat{a}^\dagger\hat{N} + \hat{a}\dagger)|n\rangle = (n+1)\hat{a}^\dagger|n\rangle$$

So we see that $\hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n+1$, i.e $\hat{a}|n\rangle = c_+|n-1\rangle$ where c_+ is a normalization constant. [1 mark]
To fix c_- and c_+ if we assume that $|n\rangle$ is normalized with $\langle n|n\rangle = 1$ then we have

$$n = \langle n|\hat{N}|n\rangle = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle \hat{a}n|\hat{a}n\rangle = c_-^2\langle n-1|n-1\rangle = c_-^2$$

So we conclude that up to a phase $c_- = \sqrt{n}$ as required. [1 mark]
Similarly one can write

$$\langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n|(\hat{a}^\dagger\hat{a} + 1)|n\rangle = \langle n|(\hat{N} + 1)|n\rangle = n + 1$$

This can also be written as

$$\langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle \hat{a}^\dagger n|\hat{a}^\dagger n\rangle = c_+^2$$

Comparing these expressions we find up to a phase $c_+ = \sqrt{n+1}$ as required. [1 mark]

(d) Given that $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ we can rewrite the result for $\hat{a}$ obtained in (a) as

$$\hat{a} = \sqrt{\frac{\alpha}{2}}\left(x + \frac{1}{\alpha}\frac{\partial}{\partial x}\right)$$

where $\alpha = \frac{m\omega}{\hbar}$. [3 marks]
Then $\hat{a}|0\rangle = 0$ yields the differential equation

$$\sqrt{\frac{\alpha}{2}}\left(x + \frac{1}{\alpha}\frac{d}{dx}\right)\psi_0(x) = 0$$

So

$$\frac{d\psi_0}{dx} = -\alpha x \psi_0$$

[2 marks]

with solution

$$\psi_0(x) = N \exp\left(-\alpha\frac{x^2}{2}\right)$$

[1 mark]

(e) We have

$$\begin{aligned}
\hat{T} &= \frac{1}{2m}\hat{p}^2 = -\frac{\hbar m\omega}{4m}\left[\hat{a}^{\dagger 2} + \hat{a}^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\right] \\
\hat{V} &= \frac{m\omega^2}{2}\hat{x}^2 = \frac{\hbar m\omega^2}{4m\omega}\left[\hat{a}^{\dagger 2} + \hat{a}^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right]
\end{aligned}$$

[2 marks]

When sandwiched between the same eigenstates $\langle n|$ and $|n\rangle$, the quadratic terms $\hat{a}^{\dagger 2}$ and $\hat{a}^2$ must vanish, leading to

$$\begin{aligned}\langle n|\hat{T}|n\rangle &= \langle n|\hat{V}|n\rangle = \frac{\hbar\omega}{4}\left\langle n\left|\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)\right|n\right\rangle \\ &= \frac{\hbar\omega}{2}\left\langle n\left|\left(\hat{N} + \frac{1}{2}\right)\right|n\right\rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) .\end{aligned}$$

[4 marks]

Parts (a)-(d) are bookwork, parts (e) and (f) are unseen.

(a)

$$\begin{aligned}\left[\hat{L}_i,\,\hat{L}_j\right] &= i\hbar\epsilon_{ijk}\hat{L}_k \\ \left[\hat{L}_i,\,\hat{L}^2\right] &= 0\,.\end{aligned}$$

[4 marks]

(b) They are the spherical harmonics:

$$\begin{aligned}\hat{L}^2 Y_{lm} &= l(l+1)\hbar^2 Y_{lm} \\ \hat{L}_z Y_{lm} &= m\hbar Y_{lm}\,.\end{aligned}$$

[2 marks]

(c) The eigenfunctions are the spherical harmonics, and

$$\hat{H}Y_{lm} = \frac{l(l+1)\hbar^2}{2J}Y_{lm}$$

[1 mark]

The energy-eigenvalues $l(l+1)\hbar^2/(2J)$ are $(2l+1)$-fold degenerate, since for each given l there are $(2l+1)$ eigenfunctions, corresponding to $m = -l, -(l-1), -(l-2), \ldots, 0, 1, 2, 3\ldots, l$. [1 mark]

(d) If we choose the $\underline{B}$ field in the z-direction the Hamiltonian in the presence of the field will become

$$\hat{H}' = \frac{\hat{L}^2}{2}2J - \mu|\underline{B}|\hat{L}_z$$

So the energy eigenvalues become

$$\hat{H}'|l,m\rangle = \left(\frac{\hbar^2 l(l+1)}{2J} - \mu|\underline{B}|m\hbar\right)|l,m\rangle$$

.

So each original energy of $\hbar^2 l(l+1)/2J$ splits into $2l+1$ distinct energies, distributed symmetrically above and below the original energy eigenvalues and spaced by an energy of $\mu|\underline{B}|\hbar$. [2 marks]

(e) With the definitions in the hint

$$\begin{aligned}\cos^2\theta &= \frac{\sqrt{4\pi}}{3}\left[\sqrt{\frac{4}{5}}Y_{20}(\theta,\,\phi)+Y_{00}(\theta,\,\phi)\right]\\ \sin^2\theta\cos(2\phi) &= \sqrt{\frac{8\pi}{15}}\left[Y_{22}(\theta,\,\phi)+Y_{2-2}(\theta,\,\phi)\right]\end{aligned}$$

[2 marks]

After identifying the wavefunction as

$$\begin{aligned}|\psi\rangle &= N\left\{\sqrt{\frac{16\pi}{45}}Y_{20}(\theta,\,\phi)+\sqrt{\frac{4\pi}{9}}Y_{00}(\theta,\,\phi)\right.\\ &\qquad\left.+\sqrt{\frac{8\pi}{15}}\left[Y_{22}(\theta,\,\phi)+Y_{2-2}(\theta,\,\phi)\right]\right\},\end{aligned}$$

the normalisation follows from the orthonormality of the spherical harmonics, yielding

$$\frac{1}{N^2}=\frac{16\pi}{45}+\frac{4\pi}{9}+\frac{8\pi}{15}+\frac{8\pi}{15}=\frac{(16+20+24+24)\pi}{45}=\frac{28\pi}{15},$$

Therefore

$$\begin{aligned}|\psi\rangle &= \sqrt{\frac{4}{21}}|20\rangle+\sqrt{\frac{5}{21}}|00\rangle+\sqrt{\frac{2}{7}}\left[|22\rangle+|2-2\rangle\right]\\ &= \sqrt{\frac{12}{63}}Y_{20}(\theta,\,\phi)+\sqrt{\frac{15}{63}}Y_{00}(\theta,\,\phi)\\ &\qquad+\sqrt{\frac{2}{7}}\left[Y_{22}(\theta,\,\phi)+Y_{2-2}(\theta,\,\phi)\right].\end{aligned}$$

[4 marks]

(f) The energy eigenvalues are $l(l+1)\hbar^2/2J$ so energies of $3\hbar^2/J$, $\hbar^2/J$ and zero correspond to $l=2,1,0$, respectively. The probabilities for observing these energies in the state $|\psi\rangle$ are

$$P(l(l+1)\hbar^2/2J)=\sum_{m=-l}^{l}|\langle lm|\psi\rangle|^2$$

So from the expression for $|\psi\rangle$ above we can obtain $P(3\hbar^2/J) = \frac{16}{21}$, $P(\hbar^2/J) = 0$, and $P(0) = \frac{5}{21}$. [4 marks]