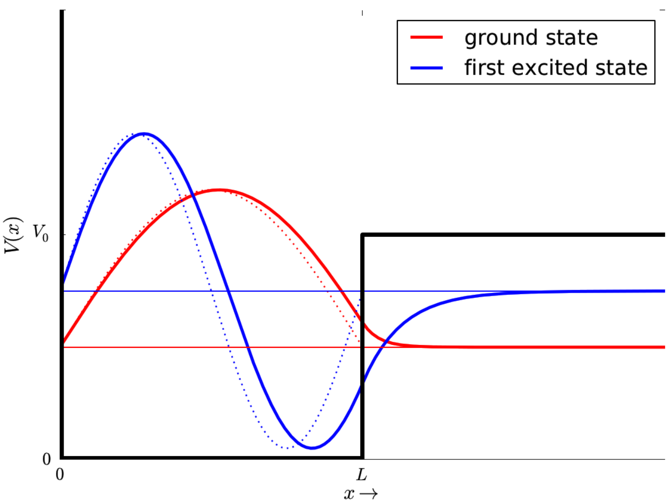


(a) The sketch does not have to be as detailed as the following but should clearly show the following features:

Ground has one turning point and first excited state has two. [1 mark,unseen]

The wavefunctions go to zero at $x = 0$ and have a continuous gradient at $x = L$ [1 mark,unseen]

The peaks (and trough) are shifted to the right relative to those of the infinite square well [1 mark,unseen]

The exponential decay of the excited state is slower (longer length scale) than that of the ground state. [1 mark,unseen]

(b) As the Hermitian property for an operator H is $\langle\psi_1|H\psi_2\rangle = \langle H\psi_1|\psi_2\rangle$ then linear combinations with real coefficients of Hermitian operators are Hermitian.

Hence i) $A + B$ and ii) cA must be Hermitian. [2 marks,bookwork]

iii) The following two equalities follow respectively as A and B are Hermitian

$\langle\psi|AB\psi\rangle = \langle A\psi|B\psi\rangle = \langle BA\psi|B\psi\rangle \neq \langle AB\psi|\psi\rangle$ but the last relation is not true unless $[A, B] = 0$

AB is not necessarily Hermitian. (e.g. $\hat{x}\hat{p}$ not Hermitian as they do not commute) [1 mark,unseen]

iv) $AB + BA$ is Hermitian as $[AB + BA, AB + BA] = 0$ [1 mark,unseen]

(Explanations are not required to get full marks)

(c) To determine $\langle A(t)\rangle$ we need to express u_1 and u_2 in terms of ψ_1 and ψ_2

$u_1 = (\psi_1 + \psi_2)/\sqrt{2} \qquad \& \qquad u_2 = (\psi_1 - \psi_2)/\sqrt{2}$ [1 mark,unseen]

Hence

$$\Psi = \frac{\psi_1 + \psi_2}{2}e^{-iE_1t/\hbar} + \frac{\psi_1 - \psi_2}{2}e^{-iE_2t/\hbar} = \frac{\psi_1}{2}\left[e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right] + \frac{\psi_2}{2}\left[e^{-iE_1t/\hbar} - e^{-iE_2t/\hbar}\right]$$ [1 mark,unseen]

So the probability of being in state ψ_1 at time t is

$$P_1 = \frac{1}{4}\left[e^{iE_1t/\hbar} + e^{iE_2t/\hbar}\right]\left[e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right] = \frac{1}{4}\left[2 + e^{i(E_1-E_2)t/\hbar} + e^{-i(E_1-E_2)t/\hbar}\right] = \frac{1}{2}\left[1 + \cos\omega t\right]$$

where the (angular) frequency of the oscillation between the 2 states is $\omega = (E_2 - E_1)/\hbar$.

$\langle A(t)\rangle = a_1P_1 + a_2(1 - P_1) = a_1\left[1 + \cos\omega t\right]/2 + a_2\left[1 - \cos\omega t\right]/2$ [2 marks,unseen]

(Can also be written as $\langle A(t)\rangle = a_1\cos^2(\omega t/2) + a_2\sin^2(\omega t/2)$)

(d) As the hydrogen wavefunctions are orthonormal

$\langle\Psi|\Psi\rangle = \frac{16}{N^2} + \frac{4}{N^2} + \frac{5}{N^2} = 1 \Rightarrow N^2 = 16 + 4 + 5 = 25$ i.e. $N = 5$ [2 marks,unseen]

The expectation value $\langle E\rangle = \frac{16}{25}E_1 + \frac{9}{25}E_2$ $\langle E\rangle = -\frac{16}{25}13.6 - \frac{9}{25}\frac{13.6}{4} = -9.9\cancel{28}\text{eV}$ [2 marks,unseen]

L2 Quantum Mechanics 2A Question 1 Solution 2016/2017

(e) **(Near bookwork).** We need to integrate $|\psi|^2$ (multiplied by the volume element) over the angle coordinates θ and ϕ. **[2 marks]**
Hence $\rho(r)\, dr = \iint |\psi|^2 r^2 \sin(\theta)\, d\theta\, d\phi\, dr \Rightarrow \rho(r) = \iint |\psi|^2 r^2 \sin(\theta)\, d\theta\, d\phi$. **[2 marks]**

(f) **(Unseen).** As $2\hbar$ must be the maximal value of L_z, the eigenvalue of L^2 must be $\hbar^2 \times 2 \times 3 = \hbar^2 6$ **[2 marks]**
and there are 5 possible values of L_z $(-2\hbar, -\hbar, 0, \hbar, 2\hbar)$. **[2 marks]**

(g) **(Unseen).** This would tells us that there would be three possible values for the spin of the electron, **[2 marks]**
and that these would be $-1, 0, +1$. **[2 marks]**

(h) **(Near bookwork).** The odd excited states are $= 0$ at the centre of the well. **[2 marks]**
Hence, the δ-function perturbation (which is located at the centre of the well, and which has no width) cannot perturb these wavefunctions. **[2 marks]**

(a) $[a_-, a_+] = \frac{1}{2}[\alpha x + ip/\alpha\hbar, \alpha x - ip/\alpha\hbar]$

$[a_-, a_+] = \frac{1}{2}\cancel{[\alpha x, \alpha x]} - \frac{1}{2}[\alpha x, ip/\alpha\hbar] + \frac{1}{2}[ip/\alpha\hbar, \alpha x] - \frac{1}{2}\cancel{[ip/\alpha\hbar, ip/\alpha\hbar]}$ [1 mark,unseen]

$[a_-, a_+] = -\frac{i}{2\hbar}[x,p] + \frac{i}{2\hbar}[p,x]$

$[a_-, a_+] = -\frac{i}{2\hbar}[x,p] - \frac{i}{2\hbar}[x,p] = \frac{-i}{2\hbar}2i\hbar = 1$ [2 marks,unseen]

(b) The ground state is the lowest state and so the lowering operator applied to it must produce zero

$a_-\psi_0 = 0$ [1 mark,bookwork]

$\alpha x\psi_0 + \frac{i}{\alpha\hbar}(-i\hbar)\frac{d\psi_0}{dx} = 0$ [1 mark,bookwork]

$\alpha x\psi_0 + \frac{1}{\alpha}\frac{d\psi_0}{dx} = 0$

$\frac{1}{\psi_0}\frac{d\psi_0}{dx} = -\alpha^2 x$

Integrating gives

$\ln\psi_0 = -\alpha^2x^2/2 + \text{constant}$

$\Rightarrow \psi_0 \propto \exp(-\alpha^2x^2/2) = A\exp(-\alpha^2x^2/2)$ [2 marks,bookwork]

Normalization requires

$\int_0^\infty \psi_0^*\psi_0\,dx = A^2\int_0^\infty e^{-\alpha^2x^2}\,dx$ [2 marks,unseen]

Using the hint for the standard integral of a Gaussian $A^2\left(\frac{\sqrt{\pi}}{\alpha}\right) = 1$

$\Rightarrow A = \left(\frac{\alpha^2}{\pi}\right)^{1/4}$ and so the normalized ground state wavefunction is $\psi_0 = \left(\frac{\alpha^2}{\pi}\right)^{1/4}\exp(-\alpha^2x^2/2)$ [2 marks,unseen]

(c) We can use the raising operator to generate the first excited state

$\psi_1 \propto a_+\psi_0$ [1 mark,bookwork]

$\psi_1 \propto \alpha x\psi_0 - \frac{i}{\alpha\hbar}(-i\hbar)\frac{d\psi_0}{dx}$

$\psi_1 \propto \alpha x\psi_0 - \alpha\frac{d\psi_0}{dx}$ [1 mark,bookwork]

$\psi_1 \propto xe^{-\alpha^2x^2/2} - \frac{d}{dx}e^{-\alpha^2x^2/2}$

$\psi_1 \propto xe^{-\alpha^2x^2/2} + \alpha^2xe^{-\alpha^2x^2/2}$

$\psi_1 \propto xe^{-\alpha^2x^2/2}$ [2 marks,bookwork]

(d) Sub. into the Schrödinger equation

$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V(x)\psi_0 = E_0\,\psi_0$ [1 mark,unseen]

$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}e^{-\alpha^2x^2/2} + \frac{1}{2}mw^2x^2e^{-\alpha^2x^2/2} = E_0\,e^{-\alpha^2x^2/2}$

$\frac{\hbar^2}{2m}\frac{d}{dx}\alpha^2xe^{-\alpha^2x^2/2} + \frac{1}{2}mw^2x^2e^{-\alpha^2x^2/2} = E_0\,e^{-\alpha^2x^2/2}$ [1 mark,unseen]

$\frac{\hbar^2}{2m}\left(\alpha^2 - \alpha^4x^2\right)e^{-\alpha^2x^2/2} + \frac{1}{2}mw^2x^2e^{-\alpha^2x^2/2} = E_0\,e^{-\alpha^2x^2/2}$ [1 mark,unseen]

$\frac{\hbar^2}{2m}\left(\frac{m\omega}{\hbar} - \frac{m^2\omega^2}{\hbar^2}x^2\right)e^{-\alpha^2x^2/2} + \frac{1}{2}mw^2x^2e^{-\alpha^2x^2/2} = E_0\,e^{-\alpha^2x^2/2}$

$$\left(\frac{\hbar\omega}{2} - \frac{m\omega^2}{2}x^2\right) e^{-\alpha^2 x^2/2} + \frac{1}{2}mw^2x^2e^{-\alpha^2x^2/2} = E_0\, e^{-\alpha^2x^2/2}$$

LHS = RHS if $E_0 = \hbar\omega/2$ as expected $[E_n = (1/2+n)\hbar\omega\ ]$ [2 marks,unseen]

L2 Quantum Mechanics 2A Question 3 Solution 2016/2017

(a) **(Near bookwork, although in a new context).** Considering $\psi_{n-}(\phi)$,

$$-\frac{\hbar^2}{2mR^2}\frac{d^2}{d\phi^2}\left[\frac{1}{\sqrt{\pi}}\sin(n\phi)\right] = -\frac{\hbar^2}{2mR^2}\frac{n}{\sqrt{\pi}}\frac{d}{d\phi}\cos(n\phi) = \frac{\hbar^2 n^2}{2mR^2}\left[\frac{1}{\sqrt{\pi}}\sin(n\phi)\right]$$

shows it is an eigenfunction, **[2 marks]**
and considering $\psi_{n+}(\phi)$,

$$-\frac{\hbar^2}{2mR^2}\frac{d^2}{d\phi^2}\left[\frac{1}{\sqrt{\pi}}\cos(n\phi)\right] = \frac{\hbar^2}{2mR^2}\frac{n}{\sqrt{\pi}}\frac{d}{d\phi}\sin(n\phi) = \frac{\hbar^2 n^2}{2mR^2}\left[\frac{1}{\sqrt{\pi}}\cos(n\phi)\right]$$

shows it is also an eigenfunction. **[2 marks]**
The eigenvalues $\hbar^2 n^2/2mR^2$ are the same for the same n, hence we have degenerate pairs. **[2 marks]**

(b) **(Unseen).**

(i) Noting that $\delta(-\phi) = \delta(\phi)$ (the delta function is centred on zero), **[2 marks]**

$$[H', M]\psi(\phi) = g[\delta(\phi)M\psi(\phi) - M\delta(\phi)\psi(\phi)] = g[\delta(\phi)\psi(-\phi) - \delta(-\phi)\psi(-\phi)] = g[\delta(\phi) - \delta(\phi)]\psi(-\phi) = 0.$$

[2 marks]

(ii) Considering $\psi_{n-}(\phi)$,

$$M\left[\frac{1}{\sqrt{\pi}}\sin(n\phi)\right] = \frac{1}{\sqrt{\pi}}\sin(-n\phi) = -\frac{1}{\sqrt{\pi}}\sin(n\phi),$$

hence is an eigenfunction with eigenvalue $= -1$, **[2 marks]**
and considering $\psi_{n+}(\phi)$,

$$M\left[\frac{1}{\sqrt{\pi}}\cos(n\phi)\right] = \frac{1}{\sqrt{\pi}}\cos(-n\phi) = \frac{1}{\sqrt{\pi}}\cos(n\phi),$$

hence is an eigenfunction with eigenvalue $= +1$. **[2 marks]**

(c) **(Unseen).** The corrections are given by

$$\langle\psi_{n-}|H'|\psi_{n-}\rangle = \frac{g}{\pi}\int_{-\pi}^{\pi}\sin^2(n\phi)\delta(\phi)\,d\phi = \frac{g}{\pi}\sin^2(0) = 0,$$

[2 marks]

and

$$\langle\psi_{n+}|H'|\psi_{n+}\rangle = \frac{g}{\pi}\int_{-\pi}^{\pi}\cos^2(n\phi)\delta(\phi)\,d\phi = \frac{g}{\pi}\cos^2(0) = \frac{g}{\pi}.$$

[2 marks]

Because the wavefunctions ψ_{n-} and ψ_{n+} (together with ψ_0) are eigenfunctions of an operator (M), which commutes with H', and which also commutes with the unperturbed Hamiltonian (which we know because we know that $\{\psi_0, \psi_{n-}, \psi_{n+}\}$ are also solutions of the original time-independent Schrödinger equation and thus form a common eigenbasis), we know that ψ_{n-} and ψ_{n+} are the "right" pair of degenerate states that "follow the perturbation." **[2 marks]**
(Hence the above terms give the correct energy shift, without having to go through the matrix diagonalisation process associated with degenerate perturbation theory).

Electromagnetism

Prof. Hampshire
June 2017 Qn 1.

(a). 4 Marks, Seen/Unseen

$\nabla \times \underline{E} = -\partial \underline{B}/\partial t$ – equivalent to $V = -\partial \phi/\partial t$

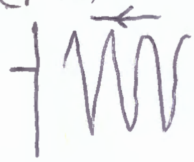
– follow Faraday. Ramp the magnetic field inside a single loop and measure the voltages produced (or equivalent).

(b) 4 Marks, Seen/Unseen

$\underline{F} = q_1 q_2 \hat{r} / 4\pi\varepsilon_0 (\underline{r}_1 - \underline{r}_2)^2 \quad \underline{r}_1 - \underline{r}_2 = (5\hat{i} + 3\hat{j}) - (2\hat{i} + \hat{j}) = 3\hat{i} + 2\hat{j}$

$(\underline{r}_1 - \underline{r}_2)^2 = 9 + 4 = 13. \Rightarrow F = 3\times10^{-12}/4\pi . 8{\cdot}85\times10^{-12} . 13 = 2 \times 10^{-3}\,N.$

(c). 4 Marks, Seen/Unseen



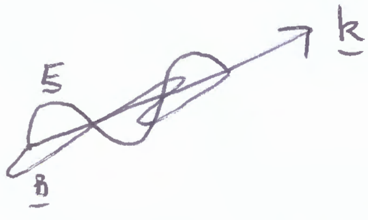
An aerial (or receiver) is an arrangement of conductors that detect a (modulated) electromagnetic wave by coupling to the $\underline{E}$-field or $\underline{B}$-field to produce an ac. voltage.

(d) 4 Marks, Seen/Unseen

A magnetic dipole is a current loop that maps out an area. It has a vectorial character that by convention is orthogonal to the surface area. $\underline{m}$: moment

$\underline{m} = IA\,\hat{n}$, I: current, A: area, $\hat{n}$: unit vector

(e). 4 Marks, Seen/Unseen



$\underline{E} \perp \underline{B} \perp \underline{k}$

$\underline{E} \times \underline{B} \,//\, \underline{k}.$

(f). 4 Marks, Seen/Unseen

An intensive property (bulk or physical property) does not depend on size of the material (eg resistivity). An extensive property depends on the amount or shape of the material (eg resistance).

(g). 4 Marks, Seen/Unseen

Lorentz force + circular motion $\quad \underline{F} = q(\underline{v} \times \underline{B}) = mv^2/r$

$v \perp B \Rightarrow v = qBr/m = \dfrac{1{\cdot}6\times10^{-19} . 5\times10^{-6} . 5\times10^{-3}}{9{\cdot}11\times10^{-31}} = 4 \times 10^{3}\,ms^{-1}$

Electromagnetism

Prof Hampshire June 2017, Qn 2.

(a) 2 Marks Seen/Unseen

Drude Model.
The charge carriers accelerate for time τ. The carriers then scatter and instantaneously stop.
Use: $\sigma_N = ne^2\tau/m$ where τ is the time between scattering events.

(b) 6 Marks Unseen

$$k^2 = \mu_0 \varepsilon_0 \omega^2 - ne^2\mu_0/m_e.$$

$$2k\partial k/\partial\omega = 2\omega\mu_0\varepsilon_0, \Rightarrow \partial\omega/\partial k = k/\omega \cdot 1/\mu_0\varepsilon_0.$$

$$k = (\mu_0\varepsilon_0)^{\frac{1}{2}}\omega\left[1 - ne^2\mu_0/m\mu_0\varepsilon_0\omega^2\right]^{1/2} = (\mu_0\varepsilon_0)^{\frac{1}{2}}\omega\left[1 - ne^2/2m\varepsilon_0\omega^2 + \ldots\right]$$

$$V_g = \partial\omega/\partial k = 1/(\mu_0\varepsilon_0)^{1/2}\left(1 - \omega_p^2/2\omega^2\right) \text{ where } \omega_p^2 = ne^2/m\varepsilon_0$$
$$c = 1/\sqrt{\mu_0\varepsilon_0}$$

(c) 8 Marks Unseen

Estimate (1).
At the speed of light in vacuum $t = L/V = \frac{1}{3}\times 10^8 = 3\,ns$.
The speed of the slower pulse is at least $\times 300$ slower than c.
$\Rightarrow \omega_p = \omega_L = 2\pi\times 10^{10}\,Hz \approx 6\times 10^{10}\,Hz$.

or (c) 8 Marks Unseen.

Estimate (2).

$$\Delta T = T_L - T_s = L\left(\frac{1}{V_{gL}} - \frac{1}{V_{gS}}\right) = L\left(\frac{1}{1-\omega_p^2/2\omega_L} - \frac{1}{1-\omega_p^2/2\omega_S^2}\right)$$

$$\Delta\tau = \frac{L}{c}\left(\frac{\omega_p^2}{2\omega_L^2} - \frac{\omega_p^2}{2\omega_S^2}\right) = \frac{L\omega_p^2}{2c}\left(\frac{1}{\omega_L^2} - \frac{1}{\omega_S^2}\right)$$

ω_L : 10GHz
ω_S : 20GHz
$\omega_S = 2\omega_L$

$$\omega_p = \frac{2\Delta T c}{L}\left(\frac{1}{\omega_L^2} - \frac{1}{4\omega_L^2}\right)^{-1} = \frac{\Delta T c}{L}\cdot\frac{8\omega_L^2}{3} = 800\,\omega_L^2$$

$$\Rightarrow \omega_p \approx \sqrt{800}\,\omega_L = 1.7\times 10^{12}\,Hz.$$ Estimate $\omega_p > \omega_L$

(d) 4 Marks Unseen.

The plasma cannot be isotropic.
In the direction of the polarised E-field, if the plasma frequency is higher, the density of the electrons must be higher. The anisotropy of the plasma can be measured using two polarised waves.

Electromagnetism

Prof Hampshire
June 2017 Qn 3.

(a) 2 Marks Seen/Unseen

$\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \partial^2 \underline{E}/\partial t^2$ Let $\underline{E} = \underline{E}_0 \exp i(kx - \omega t)$.

$\Rightarrow k^2 = \mu_0 \varepsilon_0 \omega^2$, $v_g = \partial \omega / \partial k = 1/\sqrt{\mu_0 \varepsilon_0}$.

(b) 6 Marks Unseen.

$$\underline{E} = \underline{E}_0 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi z}{a}\right) \exp i\left(ky - \omega t\right).$$

(3)

$$\left(-4\pi^2/a^2 - 4\pi^2/a^2 - k^2\right)\underline{E} = -\mu_0 \varepsilon_0 \omega^2 \underline{E}$$ – Dispersion relation

$$k^2 = \mu_0 \varepsilon_0 \omega^2 - 8\pi^2/a^2.$$

For propagation $k^2 > 0$,

$$\Rightarrow \omega^2 = 8\pi^2/a^2 . \mu_0 \varepsilon_0 . = \frac{8\pi^2}{(0.1)^2} (3\times10^8)^2 = 7 \times 10^{20}$$

(3)

$$f_{min} = \omega/2\pi = (7.1\times10^{20})^{1/2}/2\pi = 4 \times 10^9 \text{ Hz}.$$

(c) 2 Marks Unseen.

Dispersion relation $k^2 = \mu_0 \varepsilon_0 \omega^2 - 8\pi^2/a^2$

$\Rightarrow 2k = \mu_0 \varepsilon_0 2\omega \, \partial\omega/\partial k$, $2 = \mu_0 \varepsilon_0 \left(2\left(\frac{\partial \omega}{\partial k}\right)^2 + 2\omega \frac{\partial^2 \omega}{\partial k^2}\right)$

$\Rightarrow \partial^2 \omega/\partial k^2 \neq 0 \Rightarrow$ Dispersion, so the shape changes.

(d) 6 Marks Unseen.

(2) Using $\nabla \times \underline{E} = -\partial \underline{B}/\partial t$.

$$\nabla \times \underline{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & E_z \end{vmatrix} = \left[ik\left(\hat{i}\,|E_0| \sin\frac{2\pi x}{a} \sin\frac{2\pi z}{a}\right) - \frac{2\pi}{a}\hat{j}\,|E_0| \cos\frac{2\pi x}{a} \sin\frac{2\pi z}{a}\right] \times \exp i(ky - \omega t).$$

(2)

$$\underline{B} = \frac{1}{-i\omega} \nabla \times \underline{E} = i \nabla \times \underline{E} = \left(-k\hat{i}\,|E_0| \sin\frac{2\pi x}{a} . \sin\frac{2\pi z}{a} - \frac{2\pi i}{a}\hat{j}\,|E_0| \cos\frac{2\pi x}{a} . \sin\frac{2\pi z}{a}\right) \times \exp i(ky - \omega t).$$

(2).

(e) 4 Marks Unseen.

$v_g = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0}}$ so the velocity decreases. The frequency of the wave is unchanged so hf is unchanged.