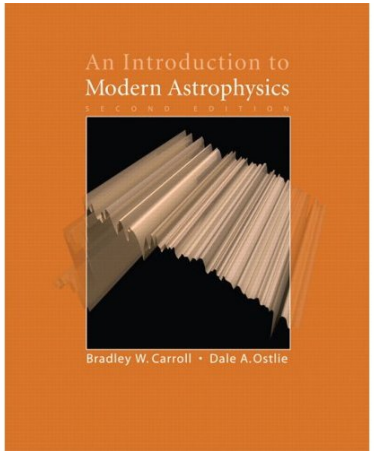
Lecture 6:
Stellar power source –
Nuclear fusion

Professor David Alexander
Ogden Centre West 119

Chapter 10 and 11 of Carroll and Ostlie



Aims of lecture

Key concept: nuclear fusion

Aims:

- Understand why energy is released by nuclear fusion processes and the factors behind the probability that a nuclear reaction will occur

- Know the difference between the classical and quantum temperature

- Know and be able to use:

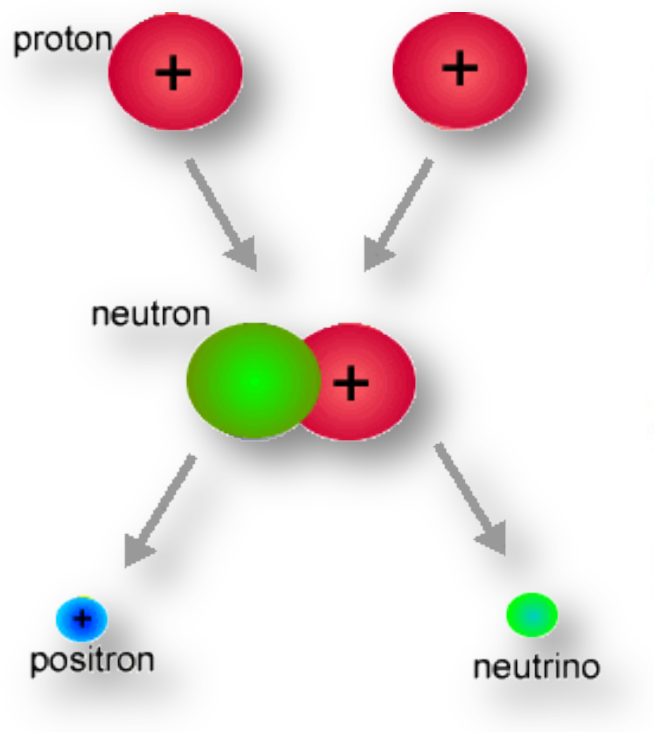
$$T_{classical} = \frac{Z_1 Z_2 e_c^2}{6\pi\varepsilon_0 kr}$$

Classical temperature for nuclear reaction

$$\varepsilon_{ix} = \varepsilon_0' X_i X_x \rho^\alpha T^\beta$$

Energy release from nuclear reactions

Nuclear Fusion



On the basis that 0.7% of the mass of Hydrogen is converted to energy when forming a Helium nucleus, the amount of energy available from the sun by converting 10% of its mass into Helium is:

$$E = (0.1\text{x}0.007) \text{ x } M c^2 = 1.3\text{x}10^{44} \text{ J}$$

This gives a nuclear timescale of $t \sim E/L$ or $\sim 10^{10}$ years

How is such a high efficiency (0.7%) achieved for nuclear fusion?

Binding energy of atomic nuclei: energy release

We know that the total mass of a nucleus is less than the mass of its constituent nucleons. The mass loss results in a release of energy, and this is known as the binding energy of the nucleus. The binding energy is the energy required to break the nucleus into its constituent parts - it gives atoms stability.

If a nucleus consists of Z protons and N neutrons, its binding energy ($E_b(Z,N)$) is:

$$E_b(Z,N) = \Delta mc^2 = \left[Zm_p + Nm_n - m(Z,N)\right]c^2$$

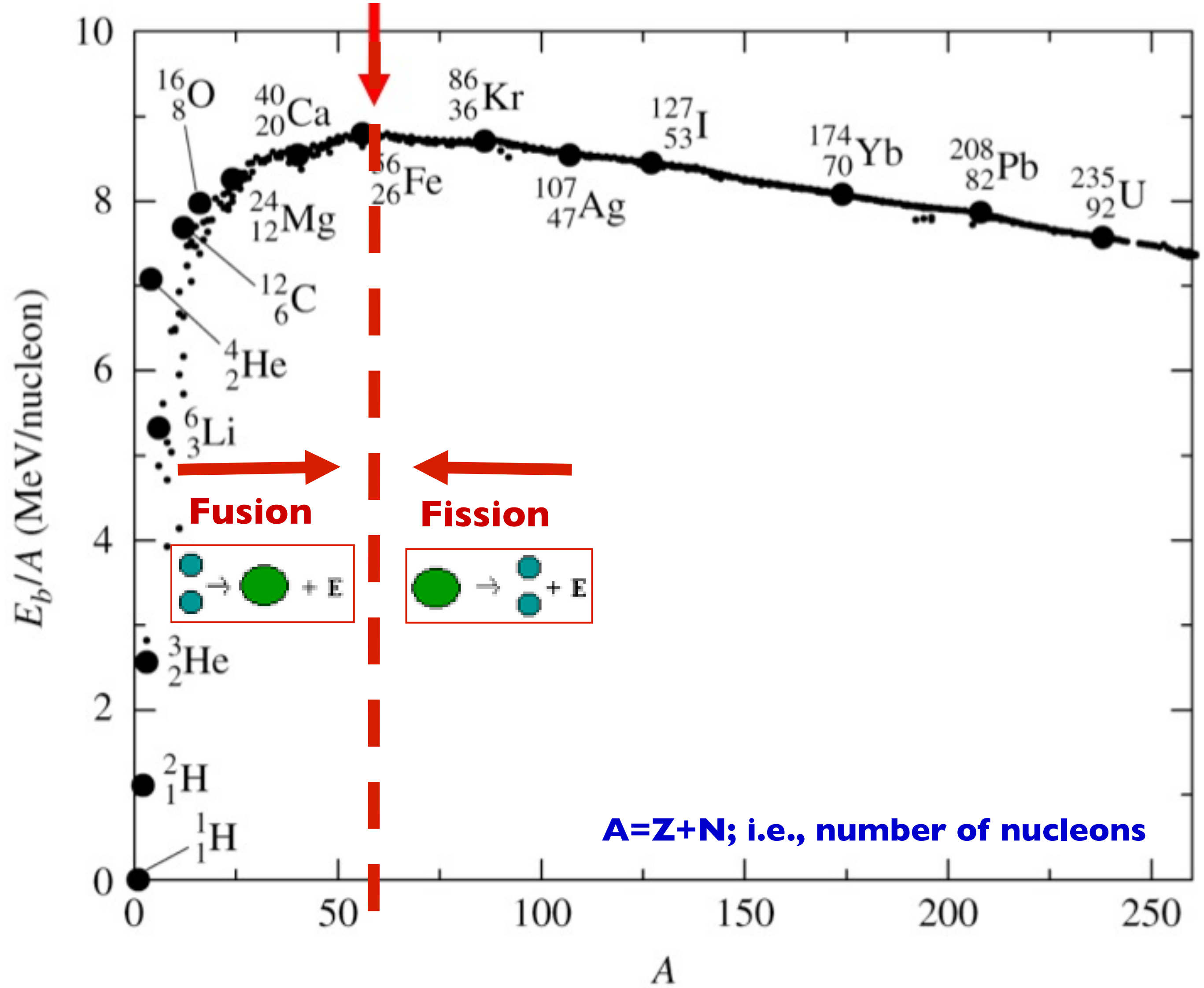
The difference in mass between the nucleus and the constituent nucleons (the binding energy) is the basis behind nuclear fusion

For example, the energy released from the fusion of four protons to form Helium 4 is:

$$E_b(4,0) = \left[4m_p - m_{He-4}\right]c^2 = 26.731 \quad \text{MeV}$$

This 26.731 MeV release is the mass difference and the origin of the 0.7% efficiency

Binding energy/nucleon for different elements



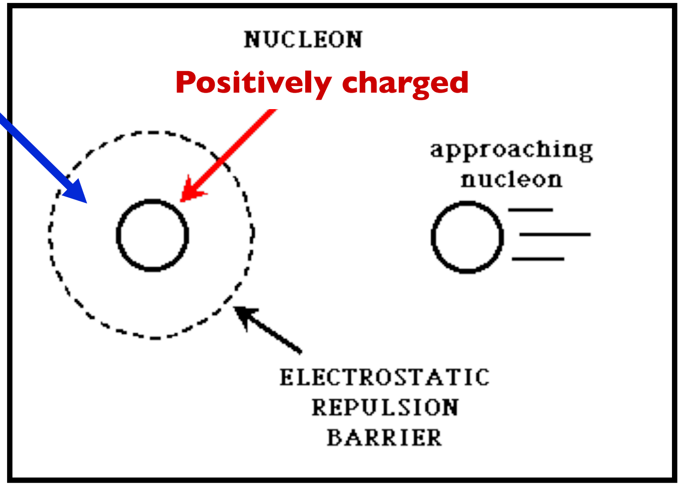
Fusion - when the fusing of elements releases energy (binding energy); fission - when heavier elements break into lighter elements (basis behind our power stations)

Conditions required for nuclear fusion

The probability of a nuclear reaction occurring is a product of two factors:

1. The probability of two particles approaching close enough for the nuclear force to become important.
2. The probability that a nuclear reaction will then occur.

Nuclear (strong) force dominates



Factor 1 depends on the masses and charges of the particles, the number of particles present and the temperature. The nuclei must overcome their Coulomb barriers and get close enough to have a chance of interacting, which as we will see requires quantum mechanical tunneling.

Factor 2 depends on the detailed properties of the nuclei involved. We wont derive the formula for nuclear reaction rates as it is complicated and long-winded. But we will explore the key properties involved in determining the nuclear reaction rate.

Coulomb barrier and classical temperature

Relate particle kinetic energy to the thermal energy and the Coulomb barrier energy

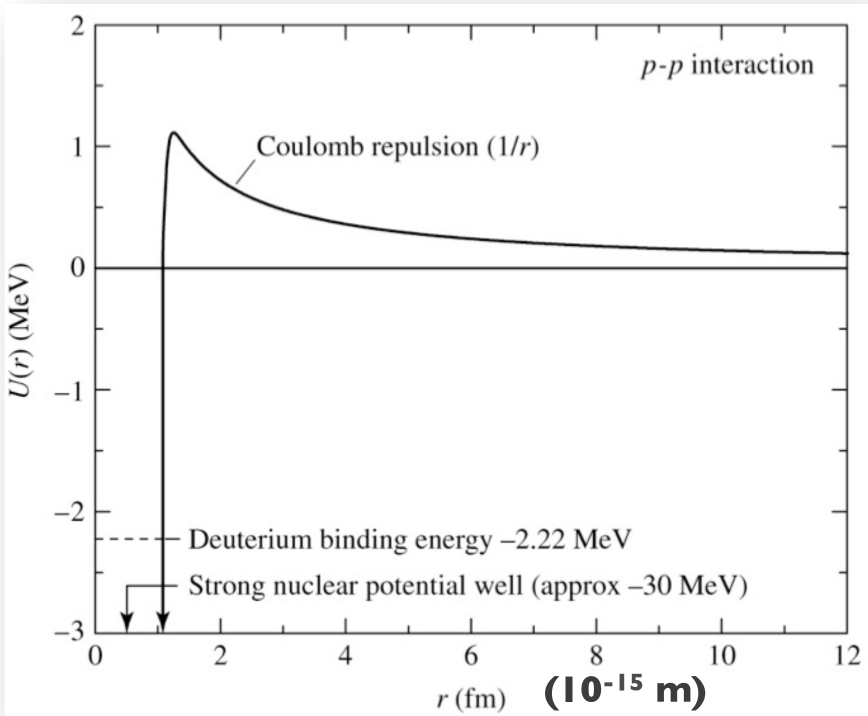
$$\frac{1}{2}\mu_m \bar{v}^2 = \frac{3}{2}kT_{classical} = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e_c^2}{r}$$

Z_1 and Z_2 are the number of protons for each interacting particle 1 and 2
μ_m is the reduced mass **(~0.5*m_H for Hydrogen=8.3675x10^{-28} kg)**
e_c is the elementary electrical charge (1.6022x10^{-19} C)
ε_0 is the permittivity of free space (8.8542x10^{-12} F m^{-1})
r is the distance of separation
k is the Boltzmann constant

Rearranging gives

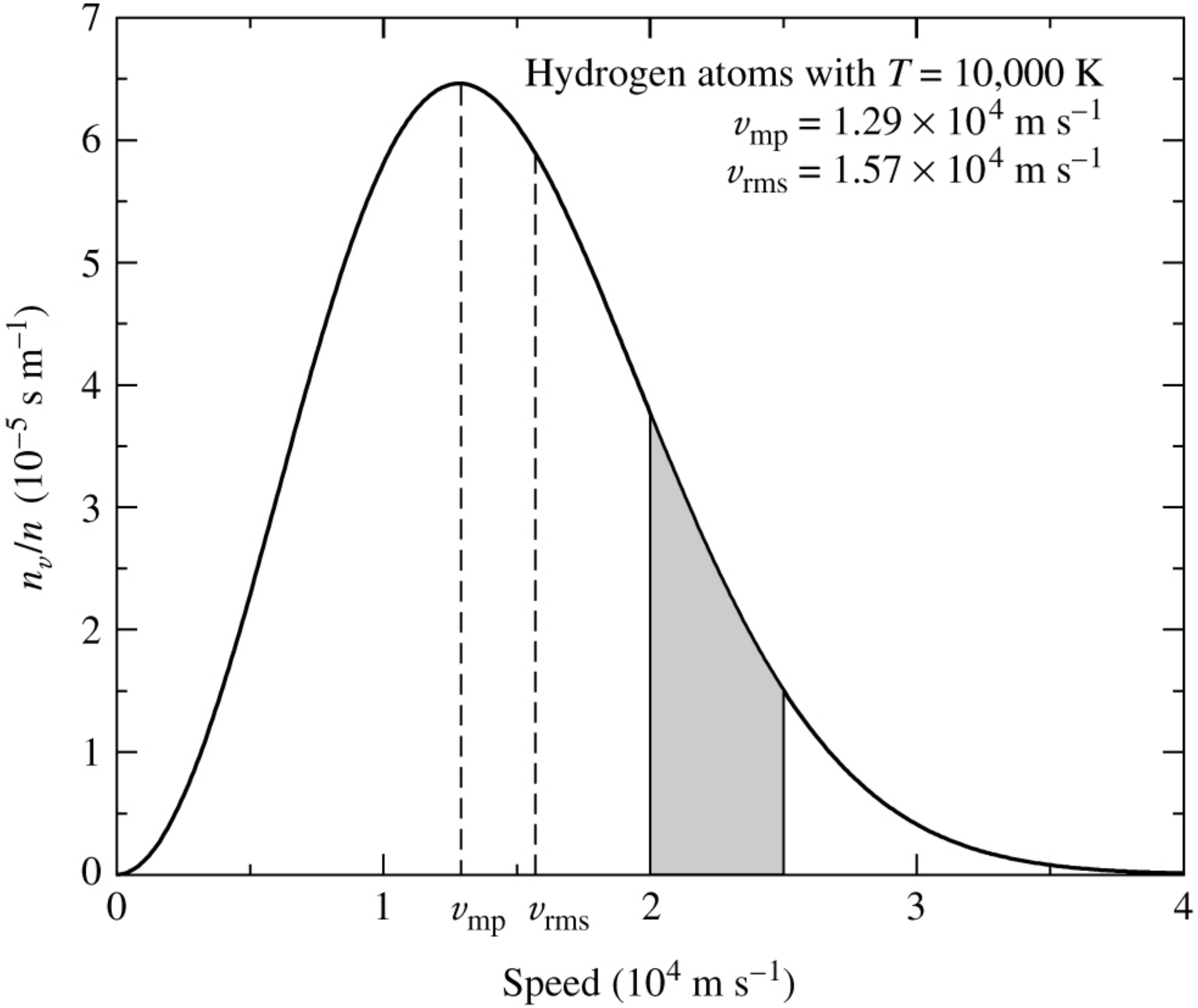
$$T_{classical} = \frac{Z_1 Z_2 e_c^2}{6\pi\varepsilon_0 kr}$$ **Equation 12**

Which for r=1 fm (10^{-15} m; typical radius of a nucleus) gives $T_{classical}$~10^{10} K **(for Hydrogen-Hydrogen)**



Based on this, does nuclear fusion appear plausible in the sun?

Gas velocities: Maxwell-Boltzmann distribution



Most-probable velocity

$$v_{mp} = \sqrt{\frac{2kT}{m}}$$

$$\frac{1}{2}mv^2 = kT$$

Mean velocity

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

For fusion to occur "classically" requires particles with velocities far from the peak of the velocity distribution – the velocity would need to be ~30x higher (energy ~1000x higher) than the most probable velocity – very improbable: ~e^{-1000}!

Coulomb barrier and quantum mechanics

All is not lost - quantum mechanics and the Heisenberg uncertainty principle:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} \qquad \text{where } \hbar = \frac{h}{2\pi}$$

Louis de Broglie (in 1927) postulated that the wave-particle duality extends to particles (as well as photons), which has since been experimentally proven. Therefore particles can be considered as waves (characterised as the de Broglie wavelength):

$$\lambda = \frac{h}{p}$$

Even we have a de Broglie wavelength but it is extremely small (~10^{-36} m)! However, for a particle the de Broglie wavelength is similar to the size of an atom (~10^{-10} m), which is very convenient given the height of the Coulomb barrier in atomic nuclei.

Lets assume that the uncertainty in the particle position is the de Broglie wavelength (λ); i.e., we'll assume that the particles are behaving as waves

Coulomb barrier and quantum mechanics

We can rewrite the kinetic energy equation in terms of momentum (p) and λ:

$$\frac{1}{2}\mu_m v^2 = \frac{p^2}{2\mu_m} = \frac{h^2}{\lambda^2}\frac{1}{2\mu_m}$$

and define the Coulomb barrier energy equation in terms of λ:

$$\lambda = \frac{4\pi\varepsilon_0 h^2}{Z_1 Z_2 e_c^2 2\mu_m}$$

Replacing r with λ in $T_{classical}$:

$$T_{quantum} = \frac{Z_1^2 Z_2^2 e_c^4 \mu_m}{12\pi^2\varepsilon_0^2 k h^2}$$ which gives $T_{quantum}$~10^7 K **Equation 13**

While the classical temperature was too high, the quantum temperature is consistent with that estimated at the solar core (lecture 4): ~10^7 K – this is approximately the minimum temperature required for nuclear fusion

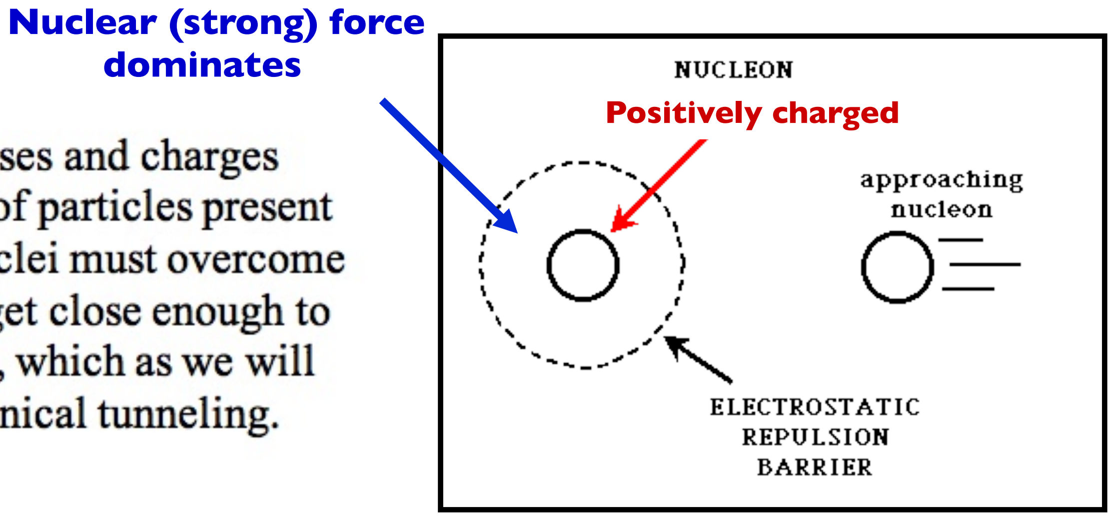
This process is referred to as <u>quantum mechanical tunneling</u>

Conditions required for nuclear fusion

The probability of a nuclear reaction occurring is a product of two factors:

1. The probability of two particles approaching close enough for the nuclear force to become important.
2. The probability that a nuclear reaction will then occur.

Factor 1 depends on the masses and charges of the particles, the number of particles present and the temperature. The nuclei must overcome their Coulomb barriers and get close enough to have a chance of interacting, which as we will see requires quantum mechanical tunneling.



Factor 2 depends on the detailed properties of the nuclei involved. We wont derive the formula for nuclear reaction rates as it is complicated and long-winded. But we will explore the key properties involved in determining the nuclear reaction rate.

Nuclear reaction probability: factors

Consideration of quantum-mechanical tunneling demonstrates that some nuclear fusion can occur at T~10^7 K (recall we assumed the mean velocities of the particles). However, there is still only a finite probability that a reaction will occur:

The tunnelling probability is based on the Coulomb barrier, the probability of an interaction, and the particle kinetic energy:

$$\sigma(E) \propto e^{-bE^{-1/2}} \equiv \frac{number\left(reactions / nucleus / time\right)}{number\left(particles / area / time\right)}$$ where

$$b \equiv \frac{\pi \mu_m^{1/2} Z_1 Z_2 e_c^2}{2^{1/2} \varepsilon_0 h}$$

Tunnelling probability (interaction probability distribution)

while the particle velocity probability is from the Maxwell-Boltzmann distribution:

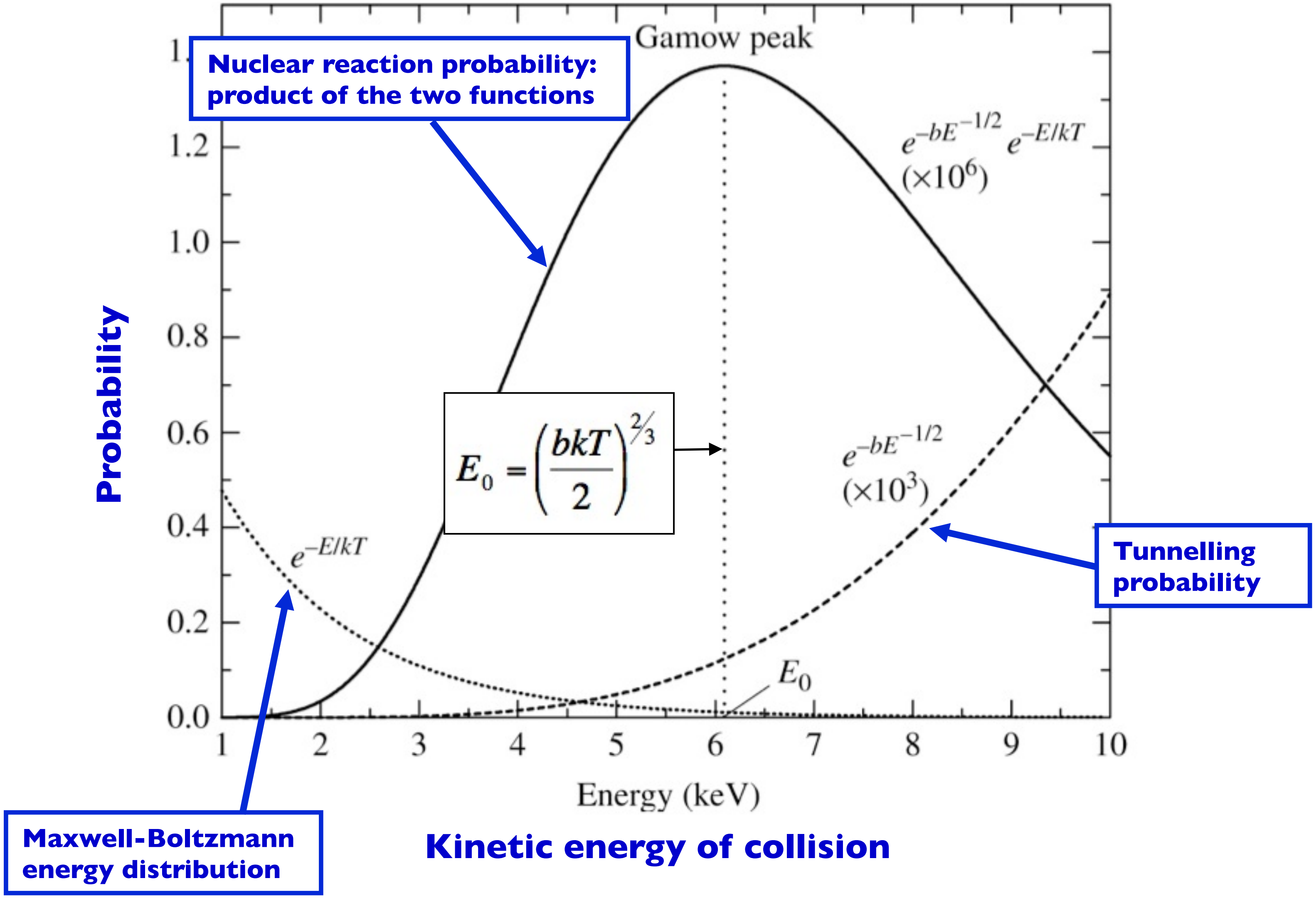
$$P(E) \propto e^{-\frac{mv^2}{2kT}} \propto e^{-\frac{E}{kT}}$$

Maxwell Boltzmann energy distribution (particle velocity probability distribution)

The nuclear fusion probability is the product of these two functions (see next slide)

Note you don't need to remember these equations

Nuclear reaction probability: Gamow peak



Reaction rate:

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/kT} dE$$

Nuclear reaction rates

We can characterise the rate of nuclear reactions in the form of a power law centred at a particular temperature. For a two-particle reaction (particles i and x), the rate is

$$r_{ix} \cong r_0 X_i X_x \rho^{\alpha'} T^{\beta}$$

r_0 is a constant, X_i and X_x are the mass fractions of the particles
α' and β are determined from the power-law expansion of the reaction-rate equations (see section 10.3 of CO book); $\alpha' = 2$ for a two-body collision

---

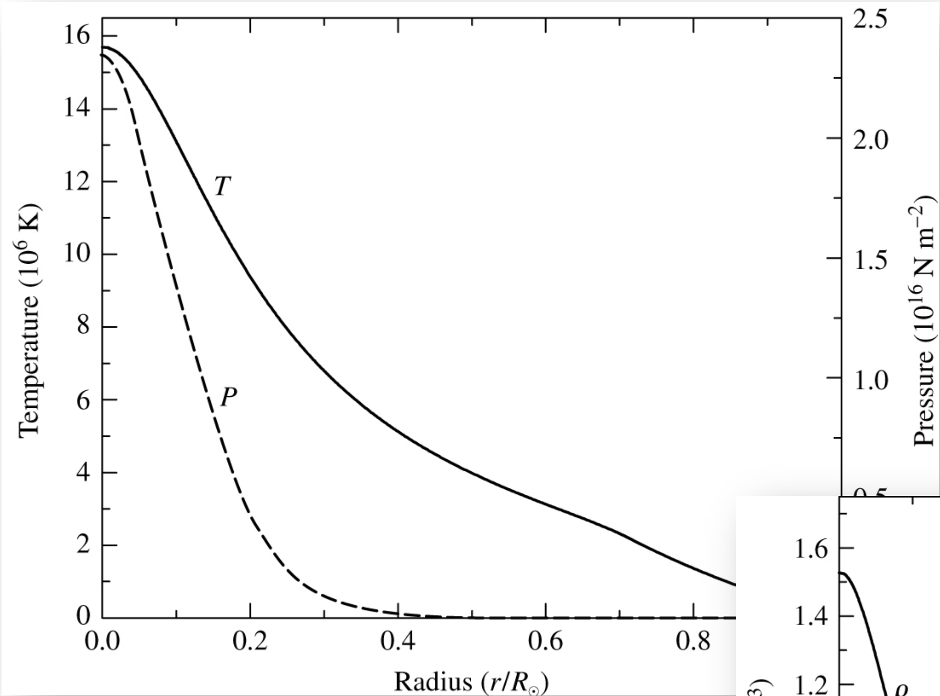
By combining this equation with the energy released per reaction we can determine the amount of energy liberated per kilogram of material per second (W kg^{-1}) as

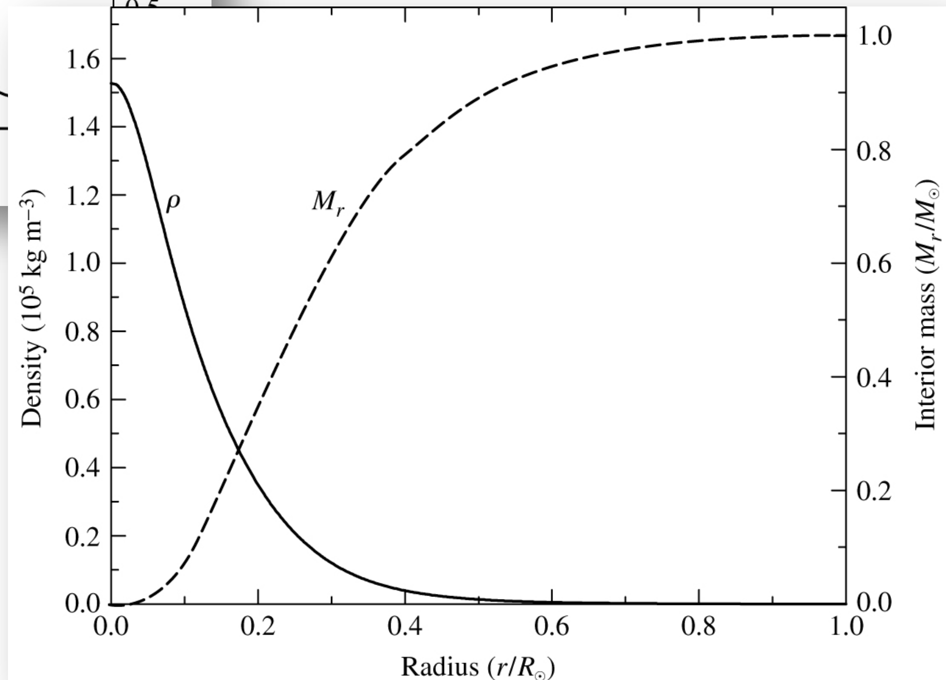
$$\varepsilon_{ix} = \left(\frac{\varepsilon_0}{\rho}\right) r_{ix}$$ where ε_0 is the amount of energy released/reaction and

$$\varepsilon_{ix} = \varepsilon_0' X_i X_x \rho^{\alpha} T^{\beta}$$ where $\alpha = \alpha' - 1$ **Equation 14**

What are the main factors for significant nuclear fusion?

Temperature, density, pressure gradients in Sun





Where will most of the nuclear fusion be occurring?