Mathematical Methods in Physics

Workshop 1

1.1

Show that:

a) $\epsilon_{ij\,k}\,\epsilon_{ij\,m} = 2\delta_{km}$.

b) $\epsilon_{ij\,k}\,\epsilon_{ij\,k} = 6$.

1.2

Making use of the identity

$$\epsilon_{ij\,k}\,\epsilon_{ilm} = \delta_{j\,l}\delta_{km} - \delta_{j\,m}\delta_{kl},$$

and the fact that the i-component of a vector product can be written as follows

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ij\,k}\, a_j\, b_k,$$

show that the vector identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

holds.

1.3

Make use of the formula

$$\frac{d}{dt}\,(\mathbf{a} + t\mathbf{b})^2 = 2\mathbf{b} \cdot (\mathbf{a} + t\mathbf{b}), \qquad (\mathbf{a} + t\mathbf{b})^2 = (\mathbf{a} + t\mathbf{b}) \cdot (\mathbf{a} + t\mathbf{b}),$$

in order to prove that the shortest vector $\mathbf{v}$ between a point on the line described by the equations $\mathbf{r} = \mathbf{a}_1 + \lambda_1 \mathbf{d}_1$ and another point on the line described by $\mathbf{r}' = \mathbf{a}_2 + \lambda_2 \mathbf{d}_2$ is such that $\mathbf{v}$ is perpendicular to both $\mathbf{d}_1$ and $\mathbf{d}_2$.

1.4

Determine whether V is a vector space. If V fails to be a vector space, state an axiom which fails to hold.

a) The set of the following functions

$$V = \{f(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)\, c_1, c_2 \in \mathbb{R}\}$$

with the usual definition of addition of functions and multiplication by a scalar.

b) All vectors (x, y) such that $x + y = 0$, i.e.

$$V = \{(x, y) |\, y = -x, x, y \in \mathbb{R}\}$$

with the usual definition of addition of vectors and multiplication by a scalar.

c) The set of the following $(n \times n)$ matrices A such that $A^2 = I$, i.e.

$$V = \{A_{n\times n} |\, A^2 = I\}$$

where I is the identity matrix, with the usual matrix operations.

1.5

For a set of linearly independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the reciprocal set of vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ is defined as follows

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \qquad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \qquad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}.$$

a) Using the properties of the triple scalar product convince yourself that:

$$\mathbf{a}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{b} = \mathbf{c}' \cdot \mathbf{c} = 1$$

$$\mathbf{a}' \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{c} = \mathbf{b}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{c} = \mathbf{c}' \cdot \mathbf{a} = \mathbf{c}' \cdot \mathbf{b} = 0.$$

b) What are the reciprocal vectors of the set $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$?

c) Show that the volume of the parallelepiped spanned by the three reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$, i.e. $\mathbf{V}' = \mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}')$, is the inverse of the volume of that spanned by the original vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, i.e. $\mathbf{V} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

[*Hint: Use the formula:* $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$.]