University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 042611/01

LEVEL 2 PHYSICS: MATHEMATICAL METHODS IN PHYSICS

SECTION A. MATHEMATICAL METHODS PART 1
SECTION B. MATHEMATICAL METHODS PART 2

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. MATHEMATICAL METHODS PART 1

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Compute the eigenvectors and eigenvalues of the following matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} . \qquad \text{[4 marks]}$$

(b) What is the equation of the plane tangent to the surface

$$x^2 - y - z^3 = 12$$

at the point $(5, 5, 2)$? [4 marks]

(c) Given the following coordinate system with coordinates (t, ϕ, u),

$$\underline{r} = \Big(\exp(t)\cos\phi, \exp(t)\sin\phi, ut^2\Big) ,$$

calculate the infinitesimal volume element. Calculate the infinitesimal surface element for surfaces of constant u. [4 marks]

(d) Give the definition of the complex form of the Fourier series of a periodic function of period L and give the formula to obtain the Fourier coefficients. Show the orthogonality identity

$$\int_0^L \exp\left(\frac{2\pi i r x}{L}\right) \exp\left(-\frac{2\pi i p x}{L}\right) dx = 0$$

for integers $r \neq p$. [4 marks]

(e) Given the scalar field

$$\phi(x, y, z) = x^2 + y^3 - z^4$$

and the vector field $\underline{a}$

$$\underline{a}(x, y, z) = (xz, y^2 + x, xyz) ,$$

compute the quantities $\underline{\nabla} \cdot \underline{a}$ and $(\underline{a} \cdot \underline{\nabla})\phi$. [4 marks]

(f) State the divergence theorem and explain all the symbols you use. [4 marks]

(g) Compute the Fourier transform of the function

$$f(x) = \begin{cases} \sin(bx) & \text{for } |x| < a \\ 0 & \text{otherwise,} \end{cases}$$

where a and b are positive constants. [4 marks]

(h) Compute the following integrals containing the Dirac δ function.

$$I_1 = \int_{-\infty}^{\infty} \delta(x-3) f(x+2) dx , \qquad I_2 = \int_{-2}^{2} \left(\delta(4x)\cos\omega x + \delta(4(x-\pi))\cos(\omega x)\right) dx ,$$

$$I_3 = \int_{-4}^{4} \delta(\sin(x)) g(x) dx , \qquad I_4 = \int_{-\infty}^{\infty} \delta(x^2 - 1) e^{ix} dx . \qquad \text{[4 marks]}$$

2. Consider the surface S given by the following parametric equations

$$\underline{r}(\phi, z) = \begin{pmatrix} \sqrt{z}\cos\phi \\ \sqrt{z}\sin\phi \\ z \end{pmatrix} \qquad 0 \le z \le 2\,, \quad 0 \le \phi < 2\pi\,.$$

(a) Compute the infinitesimal surface element for this surface. [3 marks]

(b) Using your result for (a) compute the area of the surface. [3 marks]

(c) Using cylindrical coordinates, compute the volume enclosed between this surface and the $z = 2$ plane. [2 marks]

(d) We now take the surface S to be a mirror and consider a beam with direction $\underline{b} = (0, 0, -1)$ hitting the mirror at a position $P = (\sqrt{z_0}, 0, z_0)$. What is the direction of the reflected beam? [3 marks]

(e) Show that all such beams incoming with a direction parallel to the z axis are reflected in such a way that they intersect the z axis at the same special point (called focal point) independently of z_0 and give the location of this point.[2 marks]

(f) State Stokes' theorem, explaining all of the symbols used.[3 marks]

(g) Verify Stokes' theorem for the surface S and the vector field

$$a(x, y, z) = (y, -x, z)$$

by computing both sides of the equation explicitly. [4 marks]

3. (a) Write the definition of the Fourier transform and the inverse Fourier transform for a two-dimensional function $f(x, y)$. [4 marks]

(b) Show that the Fourier transform of the following function

$$g_a(x) = \begin{cases} 1 & -a < x < a \\ 0 & \text{otherwise,} \end{cases}$$

is given by

$$\widetilde{g_a}(\omega) = \frac{2\sin(a\omega)}{\omega\sqrt{2\pi}} . \qquad [2 \text{ marks}]$$

(c) Show that the Fourier transform of $g(t) \equiv f(t)\cos(bt)$ is related to the Fourier transform $\tilde{f}(\omega)$ of $f(t)$ through

$$\tilde{g}(\omega) = \frac{1}{2}\left(\tilde{f}(\omega + b) + \tilde{f}(\omega - b)\right),$$

and use it and the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$ to show that the Fourier transform of the function

$$i(x) = \begin{cases} \cos bx & -a < x < a \\ 0 & \text{otherwise,} \end{cases}$$

is given by

$$\tilde{i}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{-2b\sin(ab)\cos(a\omega) + 2\omega\cos(ab)\sin(a\omega)}{\omega^2 - b^2} . \qquad [6 \text{ marks}]$$

(d) Use the results of (b) and (c) to show that the Fourier transform of the function

$$h(x) = \begin{cases} 1 + \cos x & -\pi < x < \pi \\ 0 & \text{otherwise,} \end{cases}$$

is given by

$$\tilde{h}(\omega) = \frac{2}{\sqrt{2\pi}} \frac{\sin(\pi\omega)}{\omega(1-\omega^2)} . \qquad [2 \text{ marks}]$$

(e) Given the definition of the convolution $f * g$ of two functions f and g,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(t - x)\, dt ,$$

show that the Fourier transform of the convolution is given by

$$\mathcal{F}[f * g](\omega) = \sqrt{2\pi}\tilde{f}(\omega)\tilde{g}(\omega) . \qquad [3 \text{ marks}]$$

(f) The function $h(x)$ in (d) is obtained through the convolution of a function j with the function g_a above (part (b)) with $a = \pi/2$

$$h(x) = \int_{-\infty}^{\infty} g_{\pi/2}(t)j(t - x)\, dt.$$

What is the function $j(x)$? [3 marks]

$$[\sin(\alpha) = 2\sin(\alpha/2)\cos(\alpha/2).]$$

SECTION B. MATHEMATICAL METHODS PART 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Solve the ordinary differential equation

$$\frac{dy}{dx} + \frac{y}{x} + 2\,x = 0,$$

where y is a function of x. Indicate all the steps. [3 marks]

Insert the solution you have obtained into the differential equation and verify that it is correct. [1 mark]

(b) Consider the ordinary differential equation

$$2\,\frac{d^2y}{dx^2} + 12\,\frac{dy}{dx} + 4\,y = 0.$$

What is the generic form of the solution for this equation? [1 mark]

Solve this equation (give all the steps) and show that the solution can be written as

$$y = k_1\,\exp\left((-3+\sqrt{7})x\right) + k_2\,\exp\left(-(3+\sqrt{7})x\right).$$

[3 marks]

(c) Use the Wronskian method to show that the ordinary differential equation

$$2\,\frac{d^2y}{dx^2} + 12\,\frac{dy}{dx} + 4\,y = 2$$

has the solution

$$y = k_1\,\exp\left((-3+\sqrt{7})x\right) + k_2\,\exp\left(-(3+\sqrt{7})x\right) + \frac{1}{2}.$$

[4 marks]

[Hint: Remember that if $y = k_1\,y_1 + k_2\,y_2$, then $k_1' = -\left(h(x)/W(x)\right)\,y_2$ and $k_2' = \left(h(x)/W(x)\right)\,y_1$ where $h(x)$ is the inhomogeneous term written so that the coefficient in front of d^2y/dx^2 is unity and $W(x)$ is the Wronskian.]

(d) Consider the ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2} + 16\,x\,\frac{dy}{dx} + 14\,y = x.$$

What is the generic name for this type of equation? Specify the form of the solution of this equation. [1 mark]

Solve this equation explicitly. [3 marks]

(e) Find the *power series* solution to the equation given in (d):

$$x^2\,\frac{d^2y}{dx^2} + 16\,x\,\frac{dy}{dx} + 14\,y = x.$$

Show that you can recover part of the solution of part (d). [3 marks]

Explain why you cannot recover the full solution. [1 mark]

(f) Solve the Partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial f}{\partial y} = 0,$$

if f is a function of the x, y Cartesian coordinates. Show that it oscillates along the x-axis and diverges with y. [4 marks]

(g) Sketch the representation of the real part of the spherical harmonic Y_2^0 on a sphere of radius unity. [4 marks]

$\left[\text{Hint: recall } Y_l^m = A\ P_l^m(\cos\theta)\exp(\mathrm{i}m\phi)\,, \text{ where } A \text{ is a constant and } P_2^0(\cos\theta) = \frac{1}{2}\left(3\cos\theta^2 - 1\right).\right]$

5. Consider the equation of anti-diffusion for the function $u(x,t)$

$$\frac{\partial u}{\partial t} = -\alpha \frac{\partial^2 u}{\partial x^2},$$

where α is a positive constant. The variable x is defined in the range $x \in [0, L]$ (L being finite). The boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$.

(a) Rewrite this equation for a function u defined in cartesian coordinates. Which operator describes the spatial dependence of the equation? Why is this equation describing a process of anti-diffusion? [3 marks]

(b) Solve

$$\frac{\partial u}{\partial t} = -\alpha \frac{\partial^2 u}{\partial x^2},$$

and find the possible time and spatial dependence of the solution $u(x,t)$ before applying the boundary conditions. [8 marks]

(c) Apply the boundary conditions and determine the expression for $u(x,t)$. [2 marks]

(d) Describe the behaviour of the solution $u(x,t)$ when $t = 0$. [2 marks]

(e) Describe the behaviour of the time dependent solution $u(x,t)$. Is your solution physical or not? Explain why. [2 marks]

(f) Would your conclusions for part (e) be different if we had considered the equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

instead?

Explain why. [3 marks]

6. Consider Hermite's equation

$$\frac{d^2y}{dx^2} - 2\,x\,\frac{dy}{dx} + 2\,k\,y = 0,$$

where $y(x)$ is a function, x is a variable and k is a constant.

(a) Explain why it is reasonable to assume that the solution is of the form $y = \sum_n a_n x^n$ instead of the form $y = x^r$ or $y = \exp(rx)$. [2 marks]

(b) Find the recurrence relation between the coefficients. [6 marks]

(c) Inspecting the form of the recurrence relation, what kind of solutions do you expect? [4 marks]

(d) Give the solutions for $k = 3$, focusing on the odd coefficients. [2 marks]

(e) Using the expression

$$H_n(z) = (-1)^n \exp\left(z^2\right) \frac{d^n}{dz^n} \exp\left(-z^2\right),$$

compute H_0, H_1, H_2, H_3. What are the $H_n(z)$ functions? [6 marks]