L2 Foundation of Physics 2B Optics 2019-20

O.10 Summary

February 13, 2020

Learning outcomes:

1. To understand the link between Fraunhofer diffraction and Fourier transforms [Optics $f2f$ p 78 and Appendix B (in brief!!]
2. To understand properties of Fraunhofer diffraction patterns such as translational invariance. [Optics $f2f$ Ex. 5.3]
3. To apply the Fraunhofer diffraction formula to the case of a double slit. [Optics $f2f$ Ex. 5.3 and 5.4 on p. 80-81].

Key equations: For a single slit, the Fraunhofer diffraction integral gives

$$\mathcal{I}^{(z)} = \frac{\mathcal{I}_0}{\lambda z}\left|\int_{-a/2}^{a/2} \mathrm{e}^{-\mathrm{i}kxx'/z}\mathrm{d}x'\right|^2 = \frac{\mathcal{I}_0 a^2}{\lambda z}\mathrm{sinc}^2\left(\frac{\pi a x}{\lambda z}\right) \; . \tag{1}$$

The first zeros of the sinc-squared function are at $x = \pm(\lambda/a)z$. The **angular width** is defined at the half the angular spread between the first zeros, i.e., $\Delta\theta = \lambda/a$.

The Rayleigh distance, d_R, (discussed in Workshop 6) is defined as when the angular spread, $\Delta\theta z$ is equal to the initial size, a, which gives $z = d_\mathrm{R} = a^2/\lambda$, see [Optics $f2f$ Fig. 5.15 on p. 80].

(a) The **far-field** is defined as $z \gg d_\mathrm{R}$, i.e., propagation distances more than an order of magnitude larger than the Rayleigh distance. The Fraunhofer diffraction formula is only a good approximation in the far-field.

(b) The **near-field** is defined as $z < d_\mathrm{R}$. The Fraunhofer approximation is no longer applicable.

In the Fraunhofer approximation, translating the slit in the input plane does not change the diffraction pattern! We can prove this as follows (see [Optics $f2f$ Ex. 5.3 on p. 80])

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0\mathrm{e}^{\mathrm{i}kz}}{\sqrt{\mathrm{i}\lambda z}}\int_{d/2-a/2}^{d/2+a/2} \mathrm{e}^{-\mathrm{i}kxx'/z}\mathrm{d}x' = \mathrm{e}^{-\mathrm{i}kxd/2z}\frac{\mathcal{E}_0\mathrm{e}^{\mathrm{i}kz}}{\sqrt{\mathrm{i}\lambda z}}a\mathrm{sinc}\left(\frac{\pi a x}{\lambda z}\right) \; , \tag{2}$$

and the intensity given by the modulus squared is unchanged. This is approximate for far-field diffraction but exact in focal plane of a lens, see [Optics $f2f$ Fig. 5.16 on p. 81] For a double slit, see [Optics $f2f$ Ex. 5.4 on p. 81], we obtain two phasor terms giving a cosine:

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0\mathrm{e}^{\mathrm{i}kz}}{\sqrt{\mathrm{i}\lambda z}}\int_{-\infty}^{\infty} \mathrm{f}(x')\mathrm{e}^{-\mathrm{i}kxx'/z}\mathrm{d}x' = \frac{\mathcal{E}_0\mathrm{e}^{\mathrm{i}kz}}{\sqrt{\mathrm{i}\lambda z}}2\cos\left(\frac{\pi d x}{\lambda z}\right)a\mathrm{sinc}\left(\frac{\pi a x}{\lambda z}\right) \; . \tag{3}$$

The intensity distribution is a cosine-squared interference pattern with a sinc-squared envelope. The sinc-squared envelope suppressed the $m = d/a$ order of the interference pattern.

Outlook: In the next lecture, we shall look at diffraction in two transverse directions and laser beams [Optics $f2f$ Sec. 5.9 on p. 83-84].