University of Durham

EXAMINATION PAPER

May/June 2012 Examination code: 042611/01

LEVEL 2 PHYSICS: MATHEMATICAL METHODS IN PHYSICS

SECTION A. MATHEMATICAL METHODS PART 1
SECTION B. MATHEMATICAL METHODS PART 2

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. MATHEMATICAL METHODS PART 1

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Compute the determinant of the following matrix. [4 marks]

$$M = \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -1 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 4 \end{pmatrix} .$$

(b) Give the equation of the plane in which the curve

$$\underline{u}(t) = \left(t^2 - 1, 2t - 4, 1 - 3t^2\right)$$

is embedded. [4 marks]

(c) Given the scalar field

$$\phi(x, y, z) = (y^3 - xz - 3)$$

and the vector field $\underline{a}$

$$\underline{a}(x, y, z) = (3xz, z^2 - y, x\sqrt{z}) ,$$

compute the quantities $\underline{\nabla}\phi$ and $\underline{\nabla} \times \underline{a}$. [4 marks]

(d) Define the cylindrical coordinates ρ, ϕ, h in terms of the cartesian coordinates x, y, z. What is the infinitesimal volume element in cylindrical coordinates? When considering the surface defined by the radius $\rho = R$, where R is a constant, what is the infinitesimal surface element in cylindrical coordinates? When considering the surface defined by the height $h = H$, where H is a constant, what is the infinitesimal surface element in cylindrical coordinates? [4 marks]

(e) State the divergence theorem, draw a sketch to illustrate it for the case of a sphere of radius a and explain all symbols you are using. [4 marks]

(f) Define the Fourier series of a periodic even function with period L. What are the Fourier coefficients a_p and b_p of the function

$$f(t) = 3 + \sin(t)(1 - \cos(t))$$

with period 2π? Remember the relation $2\cos(\theta)\sin(\theta) = \sin(2\theta)$. [4 marks]

(g) Compute the Fourier transform of the following function.

$$g(x) = \begin{cases} x & \text{if } 0 < x < 3, \\ 0 & \text{otherwise.} \end{cases}$$

[4 marks]

(h) Give the definition of the Dirac delta-function and compute the following integrals.

$$I_1 = \int_{-\infty}^{\infty} x^2 \delta(x - 2)\, dx \qquad \text{and} \qquad I_2 = \int_{-\infty}^{\infty} h(x)\delta(x^2 - a)\, dx ,$$

where a is a positive constant. [4 marks]

2. (a) Give the definition of the Fourier transform and its inverse for a one-dimensional function $f(x)$ and a three-dimensional function $f(x, y, z)$. [4 marks]

(b) Compute the Fourier transform of the function f given by

$$f(x) = \begin{cases} \exp(-ax) & \text{if } 0 \le x < b \\ 0 & \text{otherwise,} \end{cases}$$

where a and b are positive constants. [2 marks]

(c) Show the following properties of the Fourier transform:

$$\mathcal{F}\left[f(t+a)\right](\omega) = e^{i\omega a}\mathcal{F}\left[f(t)\right](\omega) \ ,$$

$$\mathcal{F}\left[f(at)\right](\omega) = \frac{1}{a}\mathcal{F}\left[f(t)\right]\left(\frac{\omega}{a}\right) \ ,$$

by using an appropriate parameter transformation in the definition of the Fourier transform. [4 marks]

(d) Use the properties of (c) and your result for (b) to compute the Fourier transform of the following function,

$$g(t) = \begin{cases} \exp(3(-t+1)) & \text{if } t > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Verify your result through direct computation. [4 marks]

(e) Compute the following three-dimensional integral

$$\int d^3\underline{r}\ \delta(\underline{r} - \underline{r_0})\frac{1}{\underline{p}\cdot\underline{r}}$$

over the entire three-dimensional space for $\underline{r_0} = (2, -1, 3)$ and $\underline{p} = (5, 1, -2)$. [2 marks]

(f) How can you rewrite the following Dirac delta-function

$$\delta(h(x))$$

if you know the zeros x_i of the function h? [1 mark]

(g) Show that the integral of the function

$$k(x) = \exp(-|x|)\delta\left(\sin(x)\right)$$

over the entire real axis is given by the following result.

$$\int_{-\infty}^{\infty} k(x)\, dx = \frac{e^{\pi}+1}{e^{\pi}-1} \ .$$

[3 marks]

$$\left[\text{Hint: Remember the summation formula } \sum_{i=n}^{\infty} y^i = \frac{y^n}{1-y} \ .\right]$$

3. Consider the surface S given by the parametric equations

$$\underline{r}(\phi, u) = (\sqrt{1-u}\cos\phi, \sqrt{1-u}\sin\phi, u) \ , \qquad 0 \leq u \leq 1 \ , \quad 0 \leq \phi < 2\pi \ .$$

(a) Sketch the surface S. [2 marks]

(b) Compute the surface element $d\underline{S} = \frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial u} d\phi\, du$. [3 marks]

(c) Compute the area of the surface S. [3 marks]

(d) State Stokes' theorem, draw a sketch to illustrate it for the case of the surface of a half-sphere and explain all symbols you are using. [4 marks]

(e) Compute the integral

$$I = \int_S (\underline{\nabla} \times \underline{a}) \cdot d\underline{S}$$

explicitly for the field $\underline{a}(x, y, z) = (2x - yz, xz + 2y, z^2)$. [3 marks]

(f) Verify your result for (e) using Stokes' theorem. [3 marks]

(g) Compute the volume between the surface and the $z = 0$ plane using cylindrical coordinates. [2 marks]

SECTION B. MATHEMATICAL METHODS PART 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Consider the ordinary differential equation

$$\frac{dy}{dx} - \cos x = -y \ \cos x.$$

- Find the solution to the homogeneous equation. [2 marks]
- Find the solution to the inhomogeneous equation by using the varying constant method. [2 marks]

(b) Solve the ordinary differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2\ e^{2x}$$

by using the varying constant method. [4 marks]

(c) Laplace transform the ordinary differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2\ e^{2x}.$$

Use the initial conditions $y(0) = y'(0) = 0$ (where the prime denotes the derivative with respect to x) and indicate explicitly the Laplace transform of the derivatives of y. [2 marks]

Solve the equation thus obtained and find $y(x)$. [2 marks]

$$\left[\text{Hint: You may want to use the Laplace transform } L[e^{ax}x^n] = \frac{n!}{(p-a)^{n+1}}\right]$$

(d) Solve the ordinary differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

using the appropriate method for this type of equation. [2 marks]

Taylor expand (up to $O(x^3)$) your result around $x = 0$. [1 mark]

Using a power series decomposition, write the relation between a_0, a_1 and a_2. Assuming that $a_0 = 0$, find a_2 in terms of a_1 and check that this leads to the same result as in the Taylor expansion that you obtained. [1 mark]

(e) The Rodrigues formula for Legendre polynomials can be written as

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$$

Compute $P_0(x)$, $P_1(x)$, $P_2(x)$. [2 marks]

Use the recurrence relation

$$(n+1)P_{n+1}(x) - (2n+1)\ x\ P_n(x) + n\ P_{n-1}(x) = 0$$

to check your result for $P_2(x)$. [1 mark]

Find $P_3(x)$ using the recurrence relation. [1 mark]

(f) Consider the ordinary differential equation

$$\frac{du}{dt} = u(t) + Ku(t)^2,$$

where K is a constant.
What type of equation is this? [1 mark]
Solve it by using the change of variable

$$u(t) = \frac{1}{X(t)}.$$

[3 marks]

(g) Spherical harmonics are given by

$$Y_l^m \;=\; A\; P_l^m(\cos\theta)\; e^{i\,m\,\phi},$$

where A is a constant and

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

(where $x = \cos\theta$) with

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$$

Compute the real part of Y_1^1. [1 mark]
Explain where on a sphere of unit radius the real part of Y_1^1 would be positive or negative. [3 marks]

5. Consider a sinusoidal wave of frequency ω in a spherical cavity of radius R. Its propagation obeys the equation

$$\frac{1}{r}\frac{d^2}{dr^2}(rf) + \frac{\omega^2}{c^2}f(r) = 0,$$

where $f(r)$ is a function which depends on the radial coordinate r.

(a) Solve this equation and show that

$$f(r) = \frac{A}{r}\cos(kr) + \frac{B}{r}\sin(kr),$$

where A, B are constants and $k = \omega/c$. [5 marks]

(b) The pressure felt by the wave in this cavity at a given time t, is given by

$$p(r,t) = \mathrm{Re}[f(r)\ e^{iwt}],$$

find the expression for p. Make sure to keep only the physical terms. [3 marks]

(c) The variation of velocity of the wave obeys the equation

$$\rho_0\frac{\partial \underline{v}}{\partial t} = -\underline{\nabla} p$$

where ρ_0 is a constant. Solve this equation in order to find the radial component $v = v_r$. [4 marks]

(d) Show that the velocity thus obtained is finite for $r \to 0$. [2 marks]

(e) Find the condition that the radius R times frequency must satisfy in order for v to equal zero at the border of the cavity. [2 marks]

(f) Represent this condition graphically. [2 marks]

(g) Hence, from this graph, determine the condition that the frequency must satisfy. Use in particular the intersection between $\tan(x)$ and x. [2 marks]

6. We want to study the following equation:

$$\Delta u = 0,$$

where Δ is given by

$$\Delta = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{L^2}{r^2}\right)$$

with

$$L^2 = \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

and u is a function of r, θ and ϕ.

(a) What is the name of the operator Δ? Which system of coordinates is this? [2 marks]

(b) Use the method of separation of variables to rewrite u in terms of three functions R, Θ and Φ. Specify the variable (r,θ or ϕ) that each of these functions depend on. [1 mark]

(c) Write the two equations that the functions R and Y which we define as

$$u = RY$$

and

$$Y = \Theta\,\Phi$$

must satisfy. [2 marks]

(d) Solve the equation for R. [4 marks]

(e) Determine the equation satisfied by Θ if

$$\Phi = M\cos(n\phi) + N\sin(n\phi)?$$

[5 marks]

[Hint: You may want to use the relation $d\cos\theta = \sin\theta d\theta$.]

(f) Name the equation thus obtained for Θ. Rewrite it for the case where $n = 0$, and describe the dependence of Φ on ϕ for this case. [3 marks]

(g) What is the name of the function Θ? Justify your answer. [3 marks]