Hydrogen-like ions (any atomic nucleus with one electron) have orbitals with typical size scale $a = 4\pi\epsilon_0\hbar^2/\mu Ze^2$ and reduced mass $\mu = Mm_e/(M + m_e)$ (where M is the mass of the charge $+Ze$ nucleus, and m_e is the mass of the charge $-e$ electron). This system has energy levels $E_n = -\hbar^2/(2\mu a^2 n^2)$. In Hydrogen, $E_1 = -13.6$ eV, $a = a_H$ and $\mu_H \approx m_e$.

(a) Calculate the typical size scale in units of a_H for Hydrogen-like iron ($Z = 26$, mass of the nucleus is $55.8m_p$). Calculate the energies (in eV) of E_1 and the $n = 1 \to 2$, $n = 1 \to 3$ and $n = 1 \to 4$ transitions (1st three Lyman series) for this ion. [3 marks]

(b) Calculate the typical size scale in units of a_H and energy E_1 (in units of the E_1 value for hydrogen) for a bound state made from a proton and muon (muon charge is $-e$, mass is $200m_e$). [2 marks]

(c) Any hydrogen-like ion has a ground state wavefunction of the form $\psi_{100} = (\pi a^3)^{-1/2}e^{-r/a}$. This state has $l = 0$, and so the kinetic energy operator simplifies to

$$T = \frac{p^2}{2\mu} = -\frac{\hbar^2}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$$

calculate $\langle T\rangle$ (express your answer in terms of μ and a). [4 marks]

(d) Compare $\langle T\rangle$ in (c) above with $\langle V\rangle$ for this state, where $V(r) = -Ze^2/(4\pi\epsilon_0 r) = -\hbar^2/(a\mu r)$. [1 mark]

Useful Integrals

$$\int_0^\infty x^p e^{-qx}dx = \frac{p!}{q^{p+1}}$$