Mathematical Methods in Physics

Weekly Problems 7. Solution

7.1

To show that $\mathbf{L}$ is conserved, we show that its time derivative vanishes, i.e.

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) = \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt} + \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} \\ &= \mathbf{r}\times f(r)\mathbf{r} = f(r)\,(\mathbf{r}\times\mathbf{r}) = 0. \qquad \boxed{\text{1 marks}}\end{aligned}$$

7.2

a) $\mathbf{r}(t) = (1-2t)\mathbf{i} + t\mathbf{j} + t\mathbf{k}$ $\boxed{\text{1 mark}}$

b) Hyperbole: $xz = 1, \quad y = 0$ $\boxed{\text{1 mark}}$

7.3

a) $f_2 = c_x\,x + c_y\,y + c_z\,z$, hence $\nabla\,(f_2) = \mathbf{c}$, $\boxed{\text{1 mark}}$

b) Using the chain rule

$$\nabla\,(f_4) = f_4'(r)\nabla r = \left(-\frac{(\alpha r+1)}{r^2}\,e^{-\alpha\,r}\right)\frac{\mathbf{r}}{r} = -\frac{(\alpha r+1)}{r^3}\,e^{-\alpha\,r}\,\mathbf{r}. \quad \boxed{\text{2 marks}}$$

7.4

Let us use $F = -\nabla\phi$. Then

$$\frac{\partial\phi}{\partial x} = GMm\,\frac{x}{R^3} \longrightarrow \phi = \frac{GMm}{R^3}\,\frac{x^2}{2} + g(y,z).$$

In addition

$$\frac{\partial\phi}{\partial y} = GMm\,\frac{y}{R^3} \longrightarrow \frac{\partial g}{\partial y} = GMm\,\frac{y}{R^3} \longrightarrow g = \frac{GMm}{R^3}\,\frac{y^2}{2} + h(z).$$

Finally

$$\frac{\partial\psi}{\partial z} = -GMm\,\frac{y}{R^3} \longrightarrow \frac{\partial h}{\partial z} = -GMm\,\frac{y}{R^3} \longrightarrow h = -\frac{GMm}{R^3}\,\frac{z^2}{2} + \text{constant}.$$

Hence the potential is $\phi = \frac{GMm}{R^3}\,\frac{(x^2+y^2-z^2)}{2} + \text{constant}.$ $\boxed{\text{4 marks}}$