Solutions

1.

a) The energy release can be written as

$$\begin{aligned}E_\alpha &= M(A,Z) - M(A-4,Z-2) - M(4,2)\\ &= ZM(^1H) + (A-Z)M(n) - B(A,Z) - (Z-2)M(^1H)\\ &\quad - (A-Z-2)M(n) + B(A-4,Z-2) - 2M(^1H) - 2M(n) + B(4,2)\\ &= -B(A,Z) + B(A-4,Z-2) + B(4,2)\\ &= -4a_V - a_s((A-4)^{2/3} - A^{2/3}) - a_c\left(\frac{(Z-2)^2}{(A-4)^{1/3}} - \frac{Z^2}{A^{1/3}}\right)\\ &\quad - a_a\left(\frac{(A-2Z)^2}{4(A-4)} - \frac{(A-2Z)^2}{4A}\right) + B(4,2)\end{aligned}$$

b) For $Z = A/2$, the asymmetry term vanishes and the expression simplifies to

$$\begin{aligned}E_\alpha &\approx -4a_V - a_s((A-4)^{2/3} - A^{2/3}) - \frac{a_c}{4}\left((A-4)^{5/3} - A^{5/3}\right) + B(4,2)\\ &= -4a_V + a_s\frac{8}{3}A^{-1/3} + \frac{5a_c}{3}A^{2/3} + B(4,2)\,.\end{aligned}$$

c) Using the numerical values given,

$$\begin{aligned}E_\alpha &= -4(15.84\,\mathrm{MeV}) + (18.33\,\mathrm{MeV})\frac{8}{3}A^{-1/3} + \frac{5(0.71\,\mathrm{MeV})}{12}A^{2/3} + 28.3\,\mathrm{MeV}\\ &= -35.06 + 1.18A^{2/3} + 48.88A^{-1/3}\,\mathrm{MeV}\,,\end{aligned}$$

which becomes larger than 0 for $A = 92.8$.

d) We find

$$\begin{aligned}E_P &= M(A,Z) - M(A-1,Z-1) - M(1,1)\\ &= ZM(^1H) + (A-Z)M(n) - B(A,Z) - (Z-1)M(^1H)\\ &\quad - (A-Z-1)M(n) + B(A-1,Z-1) - M(^1H) - M(n)\\ &= -B(A,Z) + B(A-1,Z-1)\\ &= -a_V - a_s((A-1)^{2/3} - A^{2/3}) - a_c\left(\frac{(Z-1)^2}{(A-1)^{1/3}} - \frac{Z^2}{A^{1/3}}\right)\\ &\quad - a_a\left(\frac{(A-2Z+1)^2}{4(A-1)} - \frac{(A-2Z)^2}{4A}\right)\,.\end{aligned}$$

In the limit $Z \to A/2$,

$$E_P = -a_V - a_s((A-1)^{2/3} - A^{2/3}) - \frac{a_c}{4}\left(\frac{(A-2)^2}{(A-1)^{1/3}} - A^{5/3}\right) - \frac{a_a}{4(A-1)}$$
$$\approx -a_V + a_s\frac{2}{3}A^{-1/3} + a_c\frac{11}{12}A^{2/3} - \frac{a_a}{4A}\,.$$

The asymmetry term does not vanish, because taking away a single proton changes the neutron-proton balance. Numerically,

$$E_P = -15.84\,\text{MeV} + 18.33\,\text{MeV}\frac{2}{3}A^{-1/3} + 0.71\,\text{MeV}\frac{11}{12}A^{2/3} - \frac{92.80\,\text{MeV}}{4A}\,,$$
$$= -15.84 + 0.65A^{2/3} + 12.22A^{-1/3} - 23.2A^{-1}\,\text{MeV}$$

which for $A = 92.8$ gives $E_P = -0.05$ MeV and since the leading term proportional to $A^{2/3}$ has a larger coefficient in E_α compared to E_P, $E_\alpha > E_P$ for all nuclei with $A > 93$.

2. The ${}^{236}_{92}\text{U}$ nucleus has

$$Z_0 = 92\,,\quad A_0 = 236\,,\quad R_0 = 1.2\cdot 10^{-13}A_0^{1/3}\,.$$

When it splits up in two equal parts, each daughter nucleus has

$$Z = \frac{1}{2}Z_0\,,\quad A = \frac{1}{2}A_0\,,\quad R = 1.2\cdot 10^{-13}A^{1/3}\,.$$

Using the formula for the electrostatic energy of a sphere with uniformly distributed charge is given by $E_{\text{stat}} = \frac{3}{5}\frac{Q^2}{4\pi R}$, the electrostatic energy release in the fission process is

$$\Delta E = \frac{3}{5}\left[\frac{(Z_0 e)^2}{4\pi R_0} - 2\frac{(Ze)^2}{4\pi R}\right]$$
$$= \frac{3}{5}\frac{Z_0^2 e^2}{4\pi R_0}\left[1 - \frac{1}{2^{2/3}}\right] = 0.222\frac{Z_0^2\alpha}{R_0}$$
$$= 0.222\frac{92^2}{137\,\times 236^{1/3}\times 1.2\cdot 10^{-13}\text{cm}}(\hbar c)$$
$$= \frac{1.85\cdot 10^{13}}{\text{cm}}(200\,\text{MeV fm})$$
$$= 369.9\,\text{MeV}$$

This reduction is the source of the energy released in uranium fission. However, to calculate the actual energy release, some other factors should also be considered such as the increase of surface energy on fission. The real value is $\Delta E = 210$ MeV.