University of Durham

EXAMINATION PAPER

May/June 2011 Examination code: 043522/01

LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3 PAPER 1

SECTION A. CLASSICAL MECHANICS
SECTION B. STATISTICAL PHYSICS
SECTION C. MODERN OPTICS

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Nuclear magneton:	$\mu_\mathrm{N} = 5.05 \times 10^{-27}\ \mathrm{J\,T^{-1}}$
Molar Gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$

SECTION A. CLASSICAL MECHANICS
Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) How do the Lagrangian and Hamiltonian formulations differ in terms of the variables they use to describe a mechanical system? For a system with N degrees of freedom, how many differential equations and of what type describe the dynamics in the two different formulations? [4 marks]

   (b) What does a Green's function represent? Explain, briefly, how they can be used to find the motion of a driven linear oscillator. [4 marks]

   (c) Describe, briefly, Noether's theorem and give an example of its use. [4 marks]

   (d) What type of force is the Euler force and how do such forces arise? Why is the Euler force usually neglected when considering motion on Earth? [4 marks]

   (e) Draw views from three orthogonal directions of an oblate symmetric top. State in which directions the principal axes lie, and whether or not this choice is unique for your chosen oblate symmetric top. [4 marks]

2. A monkey of mass M climbs at a known irregular rate, $\dot{\phi}(t)$, up a light inextensible rope. $\phi(t)$, the distance the monkey has climbed up the rope, is a specified function of time and therefore not a dynamical variable. The rope goes over a light, frictionless pulley with a mass m at the other end. The monkey and mass are at heights $z_1(t)$ and $z_2(t)$ respectively above the ground, both of which are positive. The monkey is initially resting at the end of the rope, being held at height h.

(a) (i) Write down the kinetic and potential energies of this system, and hence determine its Lagrangian in terms of z_1, z_2, $\dot{z}_1$ and $\dot{z}_2$. [4 marks]

(ii) Write down a rheonomic constraint relating the two generalised coordinates z_1 and z_2. It may help to consider the heights of the end points of the inextensible rope. State why the constraint is not scleronomic. [4 marks]

(iii) Show that the Lagrangian for the system can be written as

$$L = (1/2)[M\dot{z}_1^2 + m(\dot{\phi} - \dot{z}_1)^2] - g[Mz_1 + m(\phi - z_1)].$$

[2 marks]

(b) (i) Using the Euler-Lagrange equation for a generalised coordinate q:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

show that

$$\ddot{z}_1 = \frac{m\ddot{\phi} - (M - m)g}{M + m}.$$

[2 marks]

(ii) Making clear what initial conditions you have used, integrate the expression for $\ddot{z}_1$ to find $z_1(t)$. [6 marks]

(c) In the case $M = 2m$ and $\ddot{\phi}(t) = 5g/3$, how long will it take the monkey to reach height $2h$? [2 marks]

3. A particle of mass m moves in a central force field with potential energy given by $V(r)$. In spherical polar coordinates (r, θ, ϕ) the Lagrangian of the particle can be written as

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\ \dot{\phi}^2) - V(r).$$

(a) (i) Using the fact that the azimuthal angle ϕ is an ignorable coordinate, determine the associated constant of the motion J. What does this constant represent? [3 marks]

(ii) Express the total energy of the particle as a function of the single variable r in a way that incorporates the effective potential energy

$$V_{\text{eff}} = V(r) + \frac{J^2}{2mr^2}$$

and explain why no θ dependence is required. [3 marks]

(b) (i) Assuming that $V(r) = Ar^{n+1}/(n+1)$, for each of the cases $n+1 < -2$, $-2 < n+1 < 0$ and $n+1 > 0$, sketch the radial variation of the effective potential. In each case, mark the radius r_0 where the potential is stationary with respect to radius and a circular orbit exists, and state whether or not the orbit is stable or unstable. [7 marks]

(ii) Calculate an expression for the radius r_0. [2 marks]

(iii) By performing a Taylor series expansion of the effective potential about the point $r = r_0$, show that, for small perturbations away from a stable circular orbit, the radius of the particle performs simple harmonic motion with an angular frequency

$$\omega = \sqrt{n+3}\frac{J}{mr_0^2}.$$

[5 marks]

SECTION B. STATISTICAL PHYSICS
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) Some defective dice have been manufactured that contain one blank side and the numbers 1-5 on the other five sides. What is the most likely total when rolling three of these dice and the corresponding probability? [4 marks]

   (b) Write down expressions for the Fermi-Dirac and Bose-Einstein distributions. Show that in the high temperature limit the statistical weights of each distribution become equivalent to the classical Boltzmann distribution. [4 marks]

   (c) The volume occupied by a ^{3}He atom in liquid helium is 46×10^{-30} m^3. Use this information to give numerical estimates for the Fermi energy and the Fermi temperature for ^{3}He atoms in liquid ^{3}He. [4 marks]

   (d) Is a negative temperature 'hotter' (i.e., corresponding to a higher energy) or 'colder' than positive temperatures? Suggest how a negative temperature might be produced in an experiment. [4 marks]

   (e) State the definition of the chemical potential μ of a system containing a large number, N, of weakly interacting particles. What limiting values does the chemical potential have at low temperatures for

      (i) fermions and

      (ii) bosons? [4 marks]

5. (a) A paramagnetic solid consists of a dilute collection of N spin $\frac{1}{2}$ ions. In a magnetic field B, each spin may be either parallel to the field and have an energy $-\mu_{\mathrm{B}}B$, or anti-parallel with energy $+\mu_{\mathrm{B}}B$. The entropy of the system is given by

$$S = Nk_{\mathrm{B}} \ln Z_1 + \frac{U}{T}$$

where

$$U = Nk_{\mathrm{B}}T^2 \frac{1}{Z_1} \frac{\partial Z_1}{\partial T},$$

at temperature, T.

Derive expressions for the single particle partition function Z_1 and thus the total energy U. [6 marks]

(b) Show that the entropy of the system is zero at low temperatures and $Nk_{\mathrm{B}} \ln 2$ at high temperature. [4 marks]

(c) Sketch a graph of the variation of entropy against temperature at two different applied magnetic fields. Use this to explain how a dilute paramagnetic system can be cooled. [6 marks]

(d) A sample is at a temperature of 2.0 K and in an external magnetic field of 2.0 Tesla. Calculate the final temperature, if the magnetic field is adiabatically reduced to 1.0×10^{-3} Tesla. [4 marks]

[Hint: the ratio of the two spin populations is $\frac{n_2}{n_1} = \exp\left(\frac{-2\mu_{\mathrm{B}}B}{k_{\mathrm{B}}T}\right)$].

6. (a) Show, using equipartition theorem or otherwise, that the heat capacity of a classical monoatomic gas consisting of N atoms is

$$C_V = \frac{3}{2} N k_{\mathrm{B}}. \qquad \text{[4 marks]}$$

(b) The Bose Einstein condensate (BEC) transition in a Bose gas of N particles of mass m in a volume V occurs at a temperature $T = T_C$ when

$$\frac{N}{V}\left(\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}\right)^{\frac{3}{2}} = \frac{4}{\sqrt{\pi}}\int_0^\infty \frac{x^2 \mathrm{dx}}{e^{x^2}-1} = 2.612.$$

The energy of a system below T_C obtained by summing over the energy levels ϵ of the occupied states is

$$U = V\frac{2\pi(2m)^{3/2}}{h^3}\int_0^\infty \frac{\epsilon^{3/2}\mathrm{d}\epsilon}{e^{\epsilon/k_{\mathrm{B}}T}-1}.$$

Show that the temperature dependence of the heat capacity of a Bose gas at low temperatures is

$$C_V = \frac{\mathrm{dU}}{\mathrm{dT}} = 1.93 N k_{\mathrm{B}}\left(\frac{T}{T_C}\right)^{3/2} \qquad \text{[8 marks]}.$$

$$\text{[Hint: } \int_0^\infty \frac{x^{3/2}\mathrm{d}x}{e^x - 1} = 1.78\text{]}$$

(c) Calculate the Bose temperature of ^{4}He given that the experimental mass density of liquid helium at low temperatures is $(mN/V) = 0.146 \times 10^3$ kg m^{-3}. [4 marks]

(d) Discuss the experimental heat capacity of liquid ^{4}He and the extent to which the low temperature phase can be considered to be a BEC. [4 marks]

SECTION C. MODERN OPTICS

Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) The ‘forward’ and ‘inverse’ Fourier transforms are defined as

$$\mathcal{F}\left[f(x)\right] = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \ , \qquad (1)$$

$$\mathcal{F}^{-1}\left[f(k)\right] = \int_{-\infty}^{\infty} f(k) e^{ikx} \frac{dk}{2\pi} \ , \qquad (2)$$

respectively. Evaluate the forward transform of $f(x) = 1$ and the inverse of $f(k) = 1$, and explain, briefly, the physical meaning of the result. [4 marks]

(b) A slit of width a (with edges at $x = \pm a/2$) is illuminated by monochromatic light propagating along the z axis with wavelength λ and uniform intensity. A thin lens of focal length f is placed immediately after the slit. Sketch the electric field as a function of x in a plane at distances $z = f$ and $z > a^2/\lambda$ downstream of the slit. Use a common axis for both curves. [4 marks]

(c) Give an expression for the directions of darkness for (i) a square aperture with side dimension a, and (ii) a circular aperture with diameter a. Which is easier to resolve, two squares illuminated by red light with wavelength $\lambda = 610$ nm or two circles illuminated by green light with wavelength $\lambda = 500$ nm? [4 marks]

(d) Sketch the far-field intensity distributions produced by diffraction of a uniform monochromatic light wave by a circular aperture of diameter a and an opaque disk of diameter a. Explain, briefly, the main differences. [4 marks]

(e) A Gaussian beam propagating along the z axis has a beam waist w_0 at $z = 0$. Write an expression for the electric field at the waist $E(x, y, 0)$, and the angular spectrum, $|E(k_x, k_y; z)|^2$, at $z = 0$ and $z \gg w_0$. [4 marks]

8. Describe, briefly, the form of the Fresnel integral and the condition that distinguishes between Fresnel and Fraunhofer diffraction. [4 marks]

   Give an equation which expresses the translation property of the Fourier transform. [2 marks]

   Four slits each of width a and separated in the x direction by a distance d in the $z = 0$ plane are illuminated by a monochromatic light with uniform amplitude E_0 propagating along z. For $a \ll d$ write an approximate expression for the electric field in the plane of the slits, i.e., $E(x, 0)$. [2 marks]

   Write an expression for the resulting Fraunhofer diffraction pattern. [2 marks]

   Sketch the spatial dependence of the intensity distribution corresponding to the Fraunhofer diffraction limit. [2 marks]

   Sketch four phasor diagrams corresponding to (i) the principal maxima, (ii) the neighbouring zeros, (iii) a subsidiary maximum and (iv) the intermediate zero midway between the principal maxima. [8 marks]

9. If $f(x)$ and $g(x)$ have Fourier transforms of $F(k)$ and $G(k)$, respectively, write an expression for the Fourier transform of the convolution of $f(x)$ and $g(x)$. [2 marks]

   Write a general expression for the Fourier transform of the two dimensional rect–function with dimensions a and b in the x and y directions, respectively. [2 marks]

   Write an expression describing the letter H using the rect–function and a convolution. Assume that the separation of the vertical bars is equal to their height. [3 marks]

   Using the above, write an expression for the Fourier transform of the letter H. [3 marks]

   Sketch the intensity distribution corresponding to the far field diffraction pattern produced by an aperture with a shape matching the letter H. [2 marks]

   Describe, briefly, the main difference between the Fourier transform of the letter L and the letter T. [2 marks]

   Design and sketch an optical system to convert a letter L into a letter T. What optical components in addition to lenses could be used? [6 marks]