FD distribution function

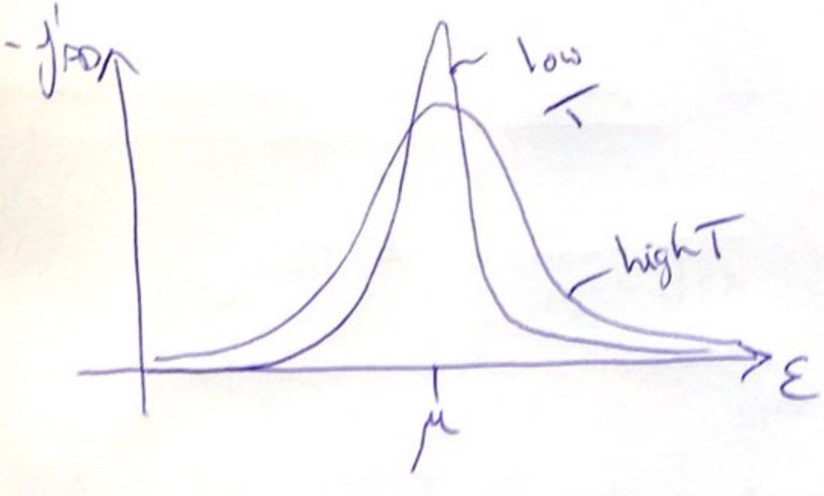
$$f_{FD}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \Rightarrow f'_{FD}(\varepsilon) = \frac{-\beta e^{\beta(\varepsilon-\mu)}}{\left(e^{\beta(\varepsilon-\mu)}+1\right)^2}$$

Note that $f'_{FD}(\varepsilon)$ is an even function centred at the Fermi level, μ.

and also $-\int_{-\infty}^{\infty} f'_{FD}(\varepsilon)\, d\varepsilon = +1$, so that $-f'_{FD}(\varepsilon)$ appears to be a probability distribution. It has mean μ and

$$\sigma^2 = \int_{-\infty}^{\infty} f'_{FD}(\varepsilon)(\varepsilon-\mu)^2\, d\varepsilon = \frac{\pi^2}{3\beta^2} \Rightarrow \sigma = \frac{\pi}{\sqrt{3}} k_B T \approx 1.81\, k_B T$$

So the standard deviation is directly proportional to temperature.



Combining this with previous integrals (last lecture) we have for any function $F(\varepsilon)$ then

$$\int_{-\infty}^{\infty} F'(\varepsilon)\, f_{FD}(\varepsilon)\, d\varepsilon = F(\mu) + \frac{\pi^2}{6\beta^2} F''(\mu) + \cdots$$

for low T, i.e. $k_B T \ll \mu$.

Example. Calculate the internal energy of a (3D) Fermi gas at temperature T up to order T^2.

We have $g(\varepsilon) = V \frac{4\pi}{h^3} (2M)^{3/2} \varepsilon^{1/2}$

We know that if we have N particles then

$$N = \int_0^\infty g(\varepsilon)\, f_{FD}(\varepsilon)\, d\varepsilon$$

[Note - the integral becomes $\displaystyle\int_0^\infty \frac{\varepsilon^{1/2}}{\left(e^{\beta(\varepsilon-\mu)}+1\right)}\, d\varepsilon$ - this is difficult!]

Comparing to above, let $F'(\varepsilon) = \varepsilon^{1/2}$ (so $F(\varepsilon) = \frac{2}{3}\varepsilon^{3/2}$, $F''(\varepsilon) = \frac{1}{2}\varepsilon^{-1/2}$)

then $$N \propto \int_0^\infty d\varepsilon\, \varepsilon^{1/2} f_{FD}(\varepsilon) \simeq \frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12\beta^2\mu^{1/2}}$$

For internal energy: $U = \displaystyle\int_0^\infty \varepsilon\, g(\varepsilon)\, f_{FD}(\varepsilon)\, d\varepsilon$, again difficult.

Let $F'(\epsilon) = \epsilon^{3/2} \Rightarrow F''(\epsilon) = \frac{3}{2}\epsilon^{1/2}$ and $F(\epsilon) = \frac{2}{5}\epsilon^{5/2}$

giving $$U \propto F(\mu) + \frac{\pi^2}{6\beta^2}F''(\mu) + \dots$$

$$= \frac{2}{5}\mu^{5/2} + \frac{5\pi^2}{12\beta^2}\mu^{1/2}$$

<u>Example</u> Ultra-high temperature Fermions.

By this we mean so high that we are in an extreme relativistic limit such that $k_B T \gg mc^2$ and so that $k_B T \gg \mu$.

Evaluate the internal energy of ultra-relativistic Fermions.

$$i = \sum_{\substack{i \\ \{\text{degeneracy}\}}} 1 . \varepsilon_i \, g_i \, f_i$$

$$= \int_0^\infty d\varepsilon \; \varepsilon \; g(\varepsilon) \; f_{FD}(\varepsilon).$$

$$= \int_0^\infty dk \; \varepsilon(k) \; g(k) \; f_{FD}(k)$$

$$\sqrt{p^2c^2 + m_0^2c^4} = \sqrt{\hbar^2c^2k^2 + m_0^2c^4}$$

$$\simeq \hbar c k \text{ (very relativistic).}$$

$$\frac{k^2 dk}{(\;/a)^3} \; f_{FD}(k) \; dk.$$

$$\frac{U}{V} = \frac{\hbar c}{2\pi^2}\int_0^\infty \frac{k^3}{e^{(\hbar c k-\mu)/k_BT}+1}$$

Note we can ignore μ term because $\mu \ll k_B T$

Note that $\int_0^\infty \frac{x^3}{e^x+1}\,dx = \frac{7\pi^4}{15}$ hence we get

$$\frac{U}{V} = \frac{\pi^2}{30}\frac{(k_BT)^4}{(\hbar c)^3}\cdot\frac{7}{8}.$$

<u>Aside.</u> If we asked the same question for Bosons, we'd do the same but use $f_{BE}(\varepsilon)$. It would look very similar but ~~the~~ we'd end up

with $\int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{8\pi^4}{15}$ giving $\frac{U}{V} = \frac{\pi^2}{30}\frac{(k_BT)^4}{(\hbar c)^3}\cdot 1$

Where can we find such a highly relativistic regime?

The early universe - $1\,ms$, $T \sim 10^{13}\,K$ $> m_0c^2$ for protons, neutrons.

$1\,s$, $T \sim 10^{10}\,K$ $> m_0 c^2$ for electrons.

Here $\mathcal{U}/V = \frac{\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3} \times$

1 (Bosons)
(2 photons
+ 2 × 8 gluons
$g\bar{g}$)

$7/8$ (Fermions.
2×2 $q\bar{q}$, 2 spins
+ 3×3 3 colours, 3 flavours
+ $2 \times 2 \times 6$ 2 spins, particle/antiparticle of each of
$e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$.