Quantum Theory - Worksheet 3

Problem 1

The unit element of information in Quantum Computing is the qubit (quantum bit). Any single qubit can be represented by a complex 2-component column vector of unit norm, and conversely, any complex 2-component column vector of unit norm can represent a qubit. (In terms of what can be made in a lab, a single qubit can be the spin state of a spin-1/2 particle, or something completely different as long as the quantum states of that something can be mapped to 2-component column vectors.) Recall that any 2-component column vectors can always be written as a linear combination of the two orthonormal vectors χ_+ and χ_-, where

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A quantum computation involves transforming the state of a system of one or several qubits into another state of the same system. The relevant operations include, in particular, the Hadamard transformation, which transforms χ_+ into $(\chi_+ + \chi_-)/\sqrt{2}$ and χ_- into $(\chi_+ - \chi_-)/\sqrt{2}$, and the Y transformation, which transforms χ_+ into $i\chi_-$ and χ_- into $-i\chi_+$. In the language of operators, these transformations are operators H and Y such that $H\chi_+ = (\chi_+ + \chi_-)/\sqrt{2}$, $H\chi_- = (\chi_+ - \chi_-)/\sqrt{2}$, $Y\chi_+ = i\chi_-$ and $Y\chi_- = -i\chi_+$.

(a) Represent each of the operators H and Y by a 2×2 matrix in the basis $\{\chi_+, \chi_-\}$.

(b) Suppose that you transform a qubit in the state $\alpha\chi_+ + \beta\chi_-$ by a Hadamard transformation. What do you get?

(c) Apply a second Hadamard transformation to the result obtained in (b). Show that you are back to $\alpha\chi_+ + \beta\chi_-$.

(d) Part (c) shows that $H(H\chi) = \chi$ for any qubit χ. Hence the product HH must be the identity operator. Accordingly, multiplying a matrix representing H by itself must give the unit matrix. Is this what you obtain with the matrix found in (a)?

Problem 2

You may remember to have encountered the ladder operators a_+ and a_- when you looked at the energy levels of a linear harmonic oscillator in the Term 1 course. These two operators were defined as follows:

$$a_+ = (2\hbar m\omega)^{-1/2} \left(m\omega x - \hbar \frac{\mathrm{d}}{\mathrm{d}x} \right),$$

$$a_- = (2\hbar m\omega)^{-1/2} \left(m\omega x + \hbar \frac{\mathrm{d}}{\mathrm{d}x} \right),$$

where m is the mass of the oscillator and ω is its angular frequency. Show that these two operators are the adjoint of each other — i.e., that $a_+ = a_-^\dagger$ and $a_- = a_+^\dagger$. To this effect, simply show that

$$\int_{-\infty}^{\infty} \phi^*(x) a_+ \psi(x)\,\mathrm{d}x = \left(\int_{-\infty}^{\infty} \psi^*(x) a_- \phi(x)\,\mathrm{d}x \right)^*$$

for any function $\psi(x)$ and $\phi(x)$ going to 0 for $x \to \pm\infty$ and such that these integrals exist and are finite. [Hints: Think about integrating the term in $\mathrm{d}/\mathrm{d}x$ by parts. Since $(A^\dagger)^\dagger = A$ for any operator A, the fact that $a_+ = a_-^\dagger$ implies that $a_+^\dagger = a_-$.]

Problem 3

Show that if $\psi(x)$ and $\phi(x)$ are two square-integrable functions on $(-\infty, \infty)$ and that the derivatives of $\psi(x)$ and $\phi(x)$ exist for all x and are also square-integrable on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} \phi^*(x) \left(-i\hbar \frac{\mathrm{d}\psi}{\mathrm{d}x} \right) \mathrm{d}x = \left[\int_{-\infty}^{\infty} \psi^*(x) \left(-i\hbar \frac{\mathrm{d}\phi}{\mathrm{d}x} \right) \mathrm{d}x \right]^*.$$

[Note: this result shows that the operator $-i\hbar\mathrm{d}/\mathrm{d}x$ (the momentum operator) is Hermitian in $L^2(-\infty, \infty)$.]

Problem 4

What is the adjoint of the identity operator, I? Is the adjoint of the inverse of an operator the inverse of the adjoint of this operator? (Explain your reasoning!)

Problem 5

Suppose that an operator $\hat{A}$ and the vector $|\psi\rangle$ is represented, respectively, by a matrix A and a column vector c. Suppose, also, that $|\psi\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue λ. Accordingly, c is an eigenvector of the matrix A with the same eigenvalue: $\mathsf{Ac} = \lambda\mathsf{c}$. Using the method oulined below, show that

$$\mathsf{c}^\dagger \mathsf{A}^\dagger = \lambda^* \mathsf{c}^\dagger.$$

In this equation, $\mathsf{c}^\dagger$ is the row vector whose elements are the complex conjugate of the elements of the column vector c, and $\mathsf{A}^\dagger$ is the conjugate transpose of the matrix A.

[Hint: One way of showing this result is to start from the equation $\sum_j A_{ij} c_j = \lambda\, c_i$, which says the same as the equation $\mathsf{Ac} = \lambda\mathsf{c}$. Taking the complex conjugate of this equation gives $\sum_j c_j^* A_{ij}^* = \lambda^*\, c_i^*$. Note at this point that A_{ij}^* is the ji element of $\mathsf{A}^\dagger$...]