(a) An ignorable coordinate is one that does not appear explicitly in the Lagrangian while its time derivative does. **[2 marks,Knowledge]**
An ignorable coordinate will have an associated canonically conjugate momentum that is a constant of the motion. **[2 marks,Knowledge]**
(b) The Lagrangian can be written as a quadratic in both the generalised coordinate and velocity, ie $L = \dot{q}^2 + Aq^2$, where A is a constant. (Other constant factors are fine too.) **[2 marks,Knowledge]**
If $A > 0$ then the potential energy decreases when moving from equilibrium, representing unstable equilibrium. Conversely, $A < 0$ leads to oscillatory motion around a position of stable equilibrium. **[2 marks,Comprehension]**

(c) A Green's function describes the response of a system to the application of an impulsive force, i.e. the evolution of a dynamical system subsequent to the application of an impulsive force. **[1 mark,Knowledge]**
The Green's function for an underdamped linear oscillator will be of the form

$$G(t - t') \propto e^{-\alpha(t-t')} \sin[\omega(t - t')],$$

where t' represents the time at which the oscillator was struck. (Anything showing appreciation of the exponential decay, the oscillations and the continuity of G during the application of the impulsive force is fine.) **[3 marks,Comprehension]**

(d) A central force is one that acts along the direction between the point where the force originates and the location of the particle feeling the force. It has a magnitude that depends only on the distance between these two positions. **[2 marks,Knowledge]**
The translational invariance of the Lagrangian for the system implies that the centre-of-mass coordinate is ignorable and the linear momentum of the system is conserved. Hence the centre of mass moves like a free particle. **[2 marks,Comprehension]**

(e) Hamilton's equations imply that $\dot{q} = 2ap$ and $\dot{p} = -3bq^2$. Combining these leads to $\ddot{q} = -6abq^2$. **[3 marks,Application]**
The particle will be accelerated away at an ever-increasing rate in the direction of negative q. **[1 mark,Comprehension]**

(f) $q = -2pQ$ and $P = -p^2$. The first of these implies $Q = -q/(2p)$. **[2 marks,Application]**
The Poisson bracket $\{Q, P\} = [-1/(2p)][-2p] - 0 = 1$, hence F produces a canonical transformation. **[2 marks,Application]**

(g) The angular velocity is defined such that $\underline{v} = \underline{\omega} \times \underline{r}$. We are told that

$$\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ \omega_z \\ -\omega_y \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}.$$

Hence

$$\underline{\omega} = \begin{pmatrix} \omega_x \\ -1 \\ 7 \end{pmatrix}.$$

The y and z components are defined by the information provided, but the x component is not. **[4 marks,Application]**

(h) The Euler force is an inertial force that appears due to considering dynamics in a non-inertial, rotating frame. $\dot{\underline{\omega}}$ represents the time derivative of the angular velocity of the rotating frame and $\underline{r}$ is the coordinate of the mass relative to the rotation axis. **[3 marks,Knowledge]**
As the Earth is slowing down, $\dot{\underline{\omega}}$ points out through the south pole of the Earth and the (tiny) Euler force acts to the east in Durham. **[1 mark,Comprehension]**

(a) [4 marks total] **(Application)**
The kinetic energy of the system is $T = (l^2/2)(M\dot{\theta}^2 + m\dot{\phi}^2)$. **[1 mark]**
The potential energy can be approximated as $V = [gl(M\theta^2 + m\phi^2) + kl^2(\phi - \theta)^2]/2$, using $\cos\theta \approx 1 - \theta^2/2$, etc. **[2 marks]**
$L = T - V$ yields the required result. **[1 mark]**

(b) [8 marks total] **(Application)**
The definitions of $\hat{\tau}$ and $\hat{\upsilon}$ lead to

$$\hat{\tau} = \frac{l^2}{2}\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}$$

[3 marks]

and

$$\hat{\upsilon} = \frac{l^2}{2}\begin{pmatrix} M\omega^2 + k & -k \\ -k & m\omega^2 + k \end{pmatrix}.$$

[3 marks]

Differentiating the trial solution twice leads to $\underline{\ddot{q}} = -\lambda^2 \underline{q}$. **[1 mark]**
Substituting into the matrix formulation of the Euler-Lagrange equation leads to the required equation. **[1 mark]**

(c) [7 marks total] **(Application)**
Non-trivial solutions to the generalised eigenvalue problem require

$$\begin{vmatrix} M(\omega^2 - \lambda^2) + k & -k \\ -k & m(\omega^2 - \lambda^2) + k \end{vmatrix} = 0,$$

from which

$$\begin{aligned} (M(\omega^2 - \lambda^2) + k)(m(\omega^2 - \lambda^2) + k) - k^2 &= 0 \\ Mm(\omega^2 - \lambda^2)^2 + k(M + m)(\omega^2 - \lambda^2) &= 0 \\ (\omega^2 - \lambda^2)[Mm(\omega^2 - \lambda^2) + k(M + m)] &= 0. \end{aligned}$$

Hence, the normal mode frequencies are
$\lambda_1^2 = \omega^2$ and $\lambda_2^2 = \omega^2 + k(M + m)/(Mm)$. **[3 marks]**
Plugging these normal mode frequencies back into the matrix equation in part (b) gives, for $\lambda_1^2 = \omega^2$,

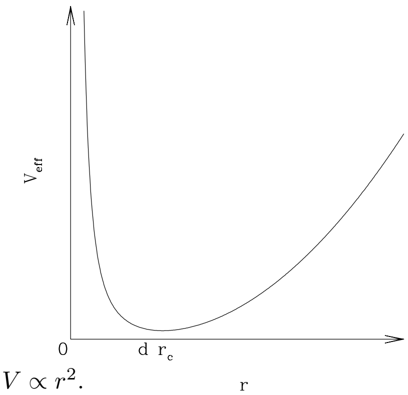
$$b_{1,\theta} - b_{1,\phi} = 0, \quad \text{i.e.} \quad \underline{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and, for $\lambda_2^2 = \omega^2 + k(M + m)/(Mm)$,

$$-\frac{kM}{m} b_{2,\theta} - k b_{2,\phi} = 0, \quad \text{i.e.} \quad \underline{b}_2 = \begin{pmatrix} 1 \\ -M/m \end{pmatrix},$$

[4 marks]

(d) (i) [6 marks total] **(Application)**
At small r, $V \propto r^{-2}$. **[1 mark]**



At large r, $V \propto r^2$. **[1 mark]**
Minimum $V_{\rm eff} > 0$ and at $r > d$. **[2 marks]**

Circular orbit occurs at $\mathrm{dV_{eff}/dr} = 0$. **[1 mark]**
Evaluating this gives

$$-\frac{J^2}{\mu r_{\mathrm{c}}^3} + k(r_{\mathrm{c}} - d) = 0.$$

[1 mark]

(ii) [5 marks total] **(Application)**
The distance of m from the centre of mass is $rM/(M+m)$. Hence, the moment of inertia is

$$I = r^2 \frac{(mM^2 + Mm^2)}{(M+m)^2} = \mu r^2.$$

[2 marks]

The angular velocity satisfies $\omega = J/I = J/\mu r^2$. **[1 mark]**
In the rotating reference frame with angular velocity ω, the particles are stationary and have the centrifugal force balancing that from the spring. Hence, for m, $m\omega^2[r_{\mathrm{c}}M/(M+m)] = J^2/(\mu r_{\mathrm{c}}^3) = k(r_{\mathrm{c}} - d)$ (cf previous part).
[2 marks]

Academic year 2018/19 — Theoretical Physics 2 — May/June 2019 paper — Question 3

(a) *[Knowledge.]* A Hilbert space is a complete vector space equipped with an inner product. [3 marks; a mention of completeness is not required] For example, the functions integrable on $-\infty < x < \infty$ with the inner product defined by the equation

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} f^*(x) g(x)\,\mathrm{d}x$$

form a Hilbert space. [1 mark for any correct example, or 2 marks if the first part of this question would not have been awarded any mark]

(b) *[Knowledge.]* An operator $\hat{A}$ is said to be Hermitian when

$$\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}|\phi\rangle^*$$

for any vectors $|\phi\rangle$ and $|\psi\rangle$ this operator acts on. [3 marks] For example, the spin operator represented by the matrix

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a Hermitian operator. [1 mark for any correct example, or 2 marks if the first part of this question would not have been awarded any mark]

(c) (i) *[Knowledge.]* These functions are said to be "normalized to a delta function in momentum space" because

$$\int_{-\infty}^{\infty} \phi_p^*(x)\phi_{p'}(x)\,\mathrm{d}x = \delta(p' - p).$$

[1 mark]

(ii) *[Knowledge.]* The word "generalized" refers to the fact that these functions do not belong to the Hilbert space of square-integrable functions on $(-\infty, \infty)$. [1 mark]

(iii) *[Knowledge.]*

$$P_x \equiv -i\hbar\,\frac{\partial}{\partial x}.$$

[2 marks]

(d) (i) *[Knowledge.]* An operator $\hat{U}$ is unitary if $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \hat{I}$, where $\hat{I}$ is the unit operator. [2 marks]

(ii) *[Application.]* For this matrix to be a unitary operator, it is necessary that

$$\begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ x^* & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & x \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 1/2+1/2 & x/\sqrt{2}-i/2 \\ x^*/\sqrt{2}+i/2 & x^*x+1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which means that x can only be equal to $i/\sqrt{2}$. [2 marks] (Alternative solution, also worth full marks: x must be such that the determinant of the matrix is 1, i.e., such that $1/2 - ix/\sqrt{2} = 1$, which implies that $x = i/\sqrt{2}$.)

(e) (i) *[Knowledge.]* $\hat{a}|\phi_n\rangle = \sqrt{n}|\phi_{n-1}\rangle$ for $n > 0$, while $\hat{a}|\phi_0\rangle = 0$ (the zero vector) for $n = 0$. [2 marks]

(ii) *[Knowledge.]* $\hat{a}^\dagger$ is the adjoint of $\hat{a}$. [1 mark]
[Knowledge.] In general, the adjoint of an operator $\hat{A}$ is the operator $\hat{A}^\dagger$ such that

$$\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle^*$$

for any vectors $|\phi\rangle$ and $|\psi\rangle$ these operators act on. [1 mark]

(f) *[Knowledge.]* The quantum number j_1 refers to the operator $\underline{J}_1^2$, where $\underline{J}_1$ is the angular momentum operator for system 1. The corresponding eigenvalue is $\hbar^2 j_1(j_1+1)$. The quantum number m_1 refers to the z-component of $\underline{J}_1$. The corresponding eigenvalue is $\hbar m_1$. The quantum number J refers to the operator $\underline{J}^2$, where $\underline{J}$ is the sum of the angular momentum operator for system 1 and the angular momentum operator for system 2. The corresponding eigenvalue is $\hbar^2 J(J+1)$. The quantum number M refers to the z-component of $\underline{J}$. The corresponding eigenvalue is $\hbar M$. [4 marks]

(g) *[Application.]* (i) This Clebsch-Gordan coefficient is necessarily zero since $M \neq m_1 + m_2$ when $m_1 = 1$, $m_2 = 3/2$ and $M = 3/2$. (ii) $m_1 = 1, m_2 = 3/2$ is the only combination of values of m_1 and m_2 for which $M = m_1 + m_2$ when $M = 5/2$. Therefore the Clebsch-Gordan coefficient $\langle 1, 3/2, m_1, m_2 \,|\, 5/2, 5/2\rangle$ is non-zero only for $m_1 = 1$ and $m_2 = 3/2$, which means that this coefficient is necessarily equal to 1. [1 mark for noting that M must be equal to $m_1 + m_2$, 3 marks for the rest of the reasoning]

Academic year 2018/19 — Theoretical Physics 2 — May/June 2019 paper — Question 4

[This question is entirely application.]

(a) Step by step,

$$\begin{aligned}\langle\phi_\alpha|\phi_\alpha\rangle &= [\langle 1|1\rangle + \langle 1|2\rangle + i\langle 1|3\rangle + \langle 2|1\rangle + \langle 2|2\rangle + i\langle 2|3\rangle - i\langle 3|1\rangle - i\langle 3|2\rangle + \langle 3|3\rangle]/3 \\ &= [1+0+0+0+1+0+0+0+1]/3 = 1. \\ \langle\phi_\alpha|\phi_\beta\rangle &= [\langle 1|1\rangle - \langle 1|2\rangle + \langle 2|1\rangle - \langle 2|2\rangle - i\langle 3|1\rangle + i\langle 3|2\rangle]/\sqrt{6} \\ &= [1-0+0-1+0+0]/\sqrt{6} = 0.\end{aligned}$$

[2 marks for using the orthonormality of the basis vectors, 2 marks for getting the $-i$ right in $\langle\phi_\alpha|$, and 2 marks for completing the calculations.]

(b) The question says that $|\phi_\alpha\rangle$, $|\phi_\beta\rangle$ and $|\phi_\gamma\rangle$ must be orthonormal. Hence $\langle\phi_\alpha|\phi_\gamma\rangle = 0$, which means, proceeding as in (a), that $[1 + 1 - ix]/\sqrt{18} = 0$. Hence $x = -2i$. [3 marks] This value of x also ensures that $|\phi_\gamma\rangle$ is normalized. Proceeding as above, or by inspection, one also sees that $|\phi_\gamma\rangle$ is orthogonal to $|\phi_\beta\rangle$. [1 mark]

(c) (i) This probability is $|\langle\phi_\beta|1\rangle|^2$. [3 marks] That is, $|[\langle 1|1\rangle - \langle 2|1\rangle]/2^{1/2}|^2 = |1-0|^2/2 = 1/2$. [1 mark]

(ii) This probabilty is $|\langle\phi_\alpha|1\rangle|^2 + |\langle\phi_\gamma|1\rangle|^2 = 1/3 + 1/6 = 1/2$. [2 marks]

(d) (i) The question says that the system is the state $|\phi_\beta\rangle$ at $t = 0$. This state is represented by the column vector

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis. Hence

$$\begin{aligned}\langle\phi_\beta|\hat{A}|\phi_\beta\rangle &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{pmatrix} = 2.\end{aligned}$$

The uncertainty is zero since $|\phi_\beta\rangle$ is an eigenstate of $\hat{A}$. [4 marks]

(ii) The matrix given shows that the Hamiltonian operator reduces to E_0 times the unit operator in the space spanned by these three basis vectors. Hence $\hat{A}$ commutes with $\hat{H}$, which implies that in the Heisenberg representation $\hat{A}(t \neq 0) = \hat{A}(t = 0)$. Hence the expectation value and the uncertainty obtained in (i) are constant in time. [2 marks]

(e) The ket $|\,3\rangle$ can be written as a linear combination of the kets $|\phi_\alpha\rangle$ and $|\phi_\gamma\rangle$. Hence $|\,3\rangle$ is also an eigenstate of $\hat{B}$ with eigenvalue μ, which implies that a measurement of B in that state would give μ with zero uncertainty. [4 marks]

(f) The matrix representing this projector in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis is

$$\begin{pmatrix} \langle 1|\phi_\alpha\rangle\langle\phi_\alpha|1\rangle & \langle 1|\phi_\alpha\rangle\langle\phi_\alpha|2\rangle & \langle 1|\phi_\alpha\rangle\langle\phi_\alpha|3\rangle \\ \langle 2|\phi_\alpha\rangle\langle\phi_\alpha|1\rangle & \langle 2|\phi_\alpha\rangle\langle\phi_\alpha|2\rangle & \langle 2|\phi_\alpha\rangle\langle\phi_\alpha|3\rangle \\ \langle 3|\phi_\alpha\rangle\langle\phi_\alpha|1\rangle & \langle 3|\phi_\alpha\rangle\langle\phi_\alpha|2\rangle & \langle 3|\phi_\alpha\rangle\langle\phi_\alpha|3\rangle \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & -i/3 \\ 1/3 & 1/3 & -i/3 \\ i/3 & i/3 & 1/3 \end{pmatrix}.$$

Owing to the orthonormality of $|\psi_\alpha\rangle$, $|\psi_\beta\rangle$ and $|\psi_\gamma\rangle$, the only element of the matrix representing this projector in the $\{|\phi_\alpha\rangle, |\phi_\beta\rangle, |\phi_\gamma\rangle\}$ basis is $\langle\phi_\alpha|\phi_\alpha\rangle\langle\phi_\alpha|\phi_\alpha\rangle$, which is 1. This matrix thus reads

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

[2 marks for the principle of the calculation, 2 marks for getting the final results correctly]