Appendix B: The Fourier transform and the Dirac delta function

In principle, this appendix contains nothing that you have not already seen previously. Please refer to your maths courses, to the term 1 Quantum Mechanics course, to a maths or Quantum Mechanics textbook and/or to reliable online material if you are unfamiliar with any of the results stated below.

The Fourier transform

Let $\psi(x)$ be a function integrable on $(-\infty, \infty)$. The Fourier transform of $\psi(x)$ is the function $\phi(k)$ defined by the following equation:

$$\phi(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(-ikx)\psi(x)\,\mathrm{d}x. \tag{B.1}$$

If $\phi(k)$ is the Fourier transform of $\psi(x)$ then $\psi(x)$ is the inverse Fourier transform of $\phi(k)$ and

$$\psi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(ikx)\phi(k)\,\mathrm{d}k. \tag{B.2}$$

The Dirac delta function

The Dirac delta function $\delta(x - x')$ is a mathematical object such that

$$\int_{-\infty}^{\infty} \delta(x - x') f(x')\,\mathrm{d}x' = f(x) \tag{B.3}$$

for any integrable function $f(x)$ of a real variable x. (The Dirac delta function is not defined for complex arguments.) There is no difference between $\delta(x - x')$ and $\delta(x' - x)$, so that Eq. (B.3) can also be written

$$\int_{-\infty}^{\infty} \delta(x' - x) f(x')\,\mathrm{d}x' = f(x).$$

It is customary to refer to $\delta(x - x')$ as a function. However, the mathematical symbol $\delta(x - x')$ does *not* represent a function. There is no function $\delta(x - x')$, in the usual sense of the word function, such that Eq. (B.3) could hold for any x and any $f(x)$. The delta “function” belongs to a different class of mathematical objects called distributions (or generalized functions).

The delta function $\delta(x - x')$ can be represented by various mathematical expressions. In particular,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[ik(x - x')]\,\mathrm{d}k = \delta(x - x'). \tag{B.4}$$

☞ Eq. (B.4) can be justified by the following argument: In view of Eq. (B.1), Eq. (B.2) can also be written as

$$\psi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(ikx) \left[\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(-ikx')\psi(x')\,\mathrm{d}x' \right] \mathrm{d}k.$$

Permuting the two integrals appearing in this equation yields

$$\begin{aligned} \psi(x) &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) \exp(-ikx')\,\mathrm{d}k \right] \psi(x')\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[ik(x-x')]\,\mathrm{d}k \right] \psi(x')\,\mathrm{d}x'. \end{aligned} \tag{B.5}$$

Eq. (B.4) follows.

The mathematical properties of the delta function is the main topic of one of the workshops. Please refer to the corresponding worksheet for further details.