Lecture 12. Tests of Cosmological Models - II

In the previous lecture we introduced the use of energy fluxes we receive from distant sources to test cosmological models. In this lecture, we shall focus the use of angular sizes of objects for this purpose. This is possible because the expansion of space also alters the relations between the angular and physical sizes of objects, and the shapes of past and future lightcones.

12.1 A static Euclidean universe

In a static Euclidean space, for small angles we have the well-known result that the transverse physical length of an object at distance d that subtends an angle Θ (measured in radians) is (see Figure 1)

$$l = d\Theta. \tag{1}$$

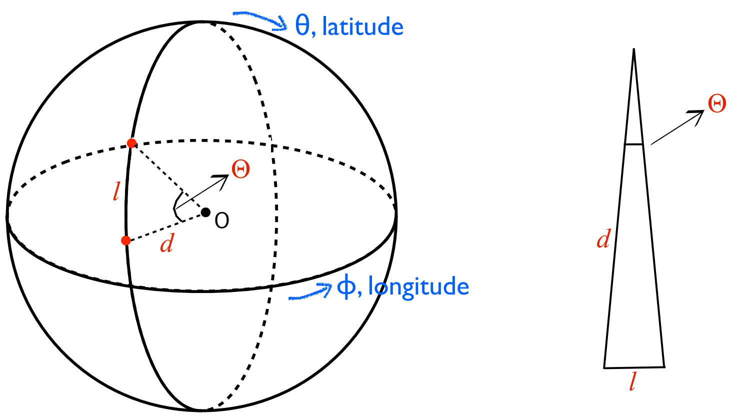

Figure 1: *Left panel*: An object whose two ends lie on the same longitude (so that $d\phi = 0$) of a spherical surface of radius d, and which subtends an angle of Θ ($d\theta = \Theta$) to the observer at the centre O. *Right panel*: small angle approximation used to derive Eq. (1).

12.2 An expanding Friedmann-Robertson-Walker (FRW) universe

In the real Universe, both the expansion of space and the spatial curvature $k \neq 0$ can modify the relation between the physical size, angular size and distance of a distant object of finite extension (for a point-like object we cannot talk about angular size).

To quantify this relation, let us introduce the **angular diameter distance**, $d_{\rm A}$, defined by the inversion of Eq. (1),

$$d_{\rm A}(z) \equiv \frac{l}{\Theta}, \tag{2}$$

where l is the transverse *physical* size of an object at redshift z and Θ is the angle it subtends. From Eqs. (1, 2) we can see that for a static Euclidean universe $d_{\rm A} = d$.

Next we shall find an expression of $d_{\rm A}(z)$ like we did for $d_{\rm L}(z)$ in the previous lecture. Again, we use the FRW metric, Eq. (10.2):

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r)(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{3}$$

Let us first consider the **physical** separation of two points both at the same redshift separated on the sky by an angle Θ (cf. left panel of Figure 1). Since both points are at the same redshift and observed by us at the same time, this implies that $dt = 0$ and $dr = 0$ between these two *events*[1]. We have also chosen the orientation of our angular coordinate system so that $d\phi = 0$ and $d\theta = \Theta$ (cf. Figure 1). Thus their **physical** space separation, as read from the metric, is

$$l = |ds| = a(t)S_k(r)\Theta. \tag{4}$$

Hence from Eq. (2)

$$d_{\rm A}(z) = a(t)S_k(r) = \frac{S_k(r)}{(1+z)}, \tag{5}$$

with r the comoving distance between the observer and source (Lecture 4; note the different symbol). $S_k(r)$ is defined in Eq. (10.3), and from the Friedmann equation k is given

$$kc^2 = H_0^2(\Omega_{m0} + \Omega_{\Lambda 0} - 1) \tag{6}$$

If the universe is flat, $k = 0$ ($\Omega_{m0} + \Omega_{\Lambda 0} = 1$), then $S_k(r) = r$ and so $d_{\rm A}(z) = r/(1+z)$. In other words, in a flat universe, the angular diameter distance is equal to the physical distance of an object at redshift z, i.e., the object's redshift.

Note also that the angular diameter and luminosity distances are related by the following **distance duality relation**, which is independent of k:

$$d_{\rm L}(z) = (1+z)^2 d_{\rm A}(z). \tag{7}$$

12.3 Test of cosmological models using angular diameter distances

Eq. (5) is the key equation of this lecture. As the comoving distance r between an object at redshift z and an observer at $z = 0$ is given by (see Lecture 7.1.2 for a specific example)

$$r(z) = c\int_0^z \frac{dz'}{H(z')}, \tag{8}$$

[1] Remember that an event has both time and space coordinates. Here, we talk about the *event* of the emission, by these objects, of the photons that reach the observer later.

with $H(z)$ given by the Friedmann equation, for any cosmological model with specific matter content, curvature k and value of cosmological constant, one can use Eq. (5) to predict $d_A(z)$. Comparing this *theoretical* prediction with the *observational* measurements of $d_A(z)$ based on Eq. (2) can serve as a test of the correctness of the assumed cosmological model.

However, in Eq. (2) one only measures the angular size of the object, and without *a priori* knowledge about l (its physical size) one cannot determine $d_A(z)$.

Astronomical objects whose physical (or comoving) size is known are called **standard rulers**. Not all objects are standard rulers (most are not). However, standard rulers do exist and have been observed. Here we give one example, with another example given in Lecture 18.

At $z > 1100$, the temperature was high enough to keep hydrogen atoms ionised, so that the Universe contained protons, free electrons and photons (in addition to other matter). Photons interacted with free electrons through Thompson scattering, and electrons interacted with protons by Coulomb interaction, so that these three particle species formed a strongly coupled plasma. The sound wave speed (the speed at which ripples caused by perturbations propagate) in this plasma was approximately $c_s = c/\sqrt{3}$, where c is the speed of light.

This situation lasted until $z \sim 1100$, when the temperature was low enough so that electrons combined with protons to form neutral hydrogen atoms. This **recombination** stopped the interactions between photons and electrons (**photon decoupling**), after which the Universe became transparent for photons, which can travel towards the observer, becoming the cosmic microwave background (CMB) detected by the latter. From Lecture 7 we have learned that the CMB tells us the temperature fluctuation on the **last scattering surface**, which is the furthest distance we can see in the Universe.

The photon-electron-baryon plasma ceased to exist at $z_{\rm rec} \sim 1100$, so sound waves in the early Universe had a finite time window to propagate – between the Big Bang and $z_{\rm rec}$. With sound speed c_s, the maximal distance sound waves could have travelled by $z_{\rm rec}$ can be shown to be roughly ~ 145 Mpc in comoving size. This is a **characteristic length scale** that, with a careful statistical analysis, can be extracted from the CMB map, and used as standard ruler.

Figure 2 shows the CMB power spectrum from the Planck satellite, where the lower horizontal axis is the angular scale in degrees. The first CMB peak corresponds to the angular size ($\sim 0.6°$ or 0.01 radian) of the mentioned characteristic length of ~ 145 comoving Mpc.

In-class example: Assume that the Universe is flat, which is matter dominated between $z_{\rm rec} = 1089$ and today ($z = 0$). There is no cosmological constant Λ. Find the angular diameter distance to the last-scattering surface which is at $z_{\rm rec}$. Use $H_0 = 70$ km/s/Mpc.

Solution: The comoving distance is given by Eq. (8), so that

$$r\,(z_{\rm rec}) = c\int_0^{z_{\rm rec}} \frac{{\rm d}z}{H(z)} = c\int_0^{z_{\rm rec}} \frac{{\rm d}z}{H_0(1+z)^{3/2}} = \frac{2c}{H_0}\left[(1+z)^{-1/2}\right]^0_{z_{\rm rec}} \approx 8306.27\ \text{Mpc}. \tag{9}$$

where in the last step we have used $z_{\rm rec} = 1089$ and $H_0 = 70$ km/s/Mpc. Therefore the angular diameter distance is

$$d_A(z_{\rm rec}) = \frac{r\,(z_{\rm rec})}{1+z_{\rm rec}} \approx 7.62\ \text{Mpc},$$

where in the first step we have used that for a flat universe $k = 0$ and $S_k(r) = r$. □

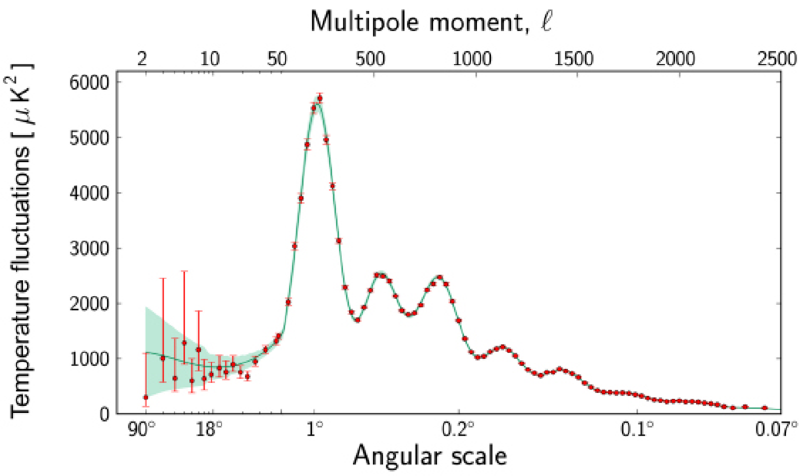

Figure 2: CMB power spectrum as measured by the Planck satellite.

The result is $d_A = 7.62$ Mpc, and so the characteristic length 145 comoving Mpc (or physical size $l = 145/(1 + z_{rec}) \approx 0.133$ Mpc) at z_{rec} has an angular size of

$$\Theta = \frac{l}{d_A(z_{rec})} \approx 0.017 \text{ radian} \approx 1.0°.$$

This is incompatible with the observed angular size of the first CMB peak in Figure 2, 0.6°, implying that the assumed cosmological model is incorrect. Indeed, for a flat universe with $\Omega_{m0} = 0.3$, $\Omega_{\Lambda 0} = 0.7$ and $H_0 = 70$ km/s/Mpc, one has[2] $r(z_{rec}) \approx 14546.73$ Mpc and $d_A(z_{rec}) \approx 13.35$ Mpc. The angular size of the characteristic length is then

$$\Theta = \frac{l}{d_A(z_{rec})} \approx 0.01 \text{ radian} \approx 0.57°,$$

which is about right.

In the example above we have assumed $k = 0$, but what happens if $k \neq 0$? Taking a closed universe ($k > 0$) as an example, one has

$$d_A(z) = \frac{S_k[r(z)]}{1+z} = \frac{1}{1+z}\frac{\sin\sqrt{k}r(z)}{\sqrt{k}} < \frac{r(z)}{1+z}. \tag{10}$$

This indicates that the angular diameter distance to the same redshift is smaller in a closed universe than in a flat universe, so that an object of the same *physical* size would appear 'larger' in the closed universe than in the flat one. The opposite applies to an open universe. Actually, Planck's 2018 result gives $\Omega_k = 0.001 \pm 0.002$, suggesting that the universe is very close to flat. One can see from Figure 2 that the position of the first CMB peak predicted by the best-fit model with $\Omega_k = 0$ agrees very well with data, while having $k > 0$ ($k < 0$) will shift the peak to the left (right), causing tension with data.

[2]This can only be done by performing the integral in Eq. (8) numerically.

Key Takeaway Points of Lecture 12

• The observational foundation to exploit angular diameter distance is

$$d_{\rm A}(z) \equiv \frac{l}{\Theta},$$

through which $d_{\rm A}(z)$ can be determined using the physical and angular size of an object at z.

• The theoretical foundation to exploit angular diameter distance is

$$d_{\rm A}(z) = \frac{S_k(r)}{(1+z)} = \frac{S_k[r(z)]}{(1+z)},$$

through which $d_{\rm A}(z)$ can be predicted by assuming a cosmological model (which enables the calculation of $r(z)$).

• Standard rulers, which are cosmological objects whose physical size is known, are needed.

• Acoustic oscillation patterns on the last scattering surface are an excellent standard ruler, which has been used to constrain cosmological models[3].

[3] A cosmological model is a model of the Universe, with its various properties or parameters specified, e.g., k, Ω_{m0}, H_0, the nature of dark energy, the theory of gravity, etc.