Lecture 4:
Stellar power source –
Hydrostatic equilibrium and conditions in the stellar core

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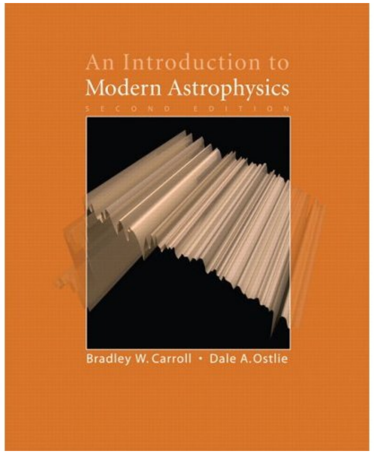
Chapter 10 of Carroll and Ostlie



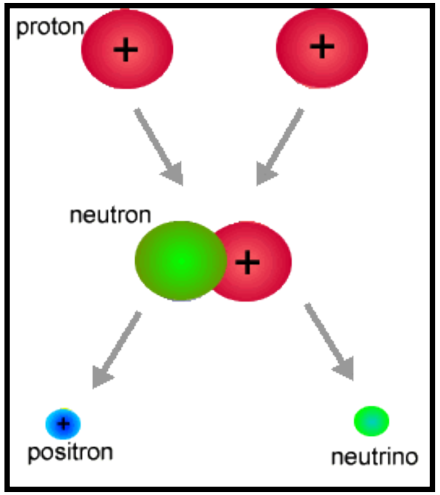
The power source of stars

Overview:

(1) Hydrostatic equilibrium (lecture 4)
conditions in the core: pressure, temperature, gas

(2) Virial theorem (lecture 5)

(3) Gravitational collapse (lecture 5)
Kelvin-Helmholtz time



(4) Nuclear fusion and other potential
power sources (lectures 5 and 6)

(5) Conditions required for nuclear fusion
(lecture 6)

(6) Nuclear reaction processes (lecture 7)
PP chain, other reaction chains, and neutrinos

Aims of lecture

Key concept: hydrostatic equilibrium

Aims:

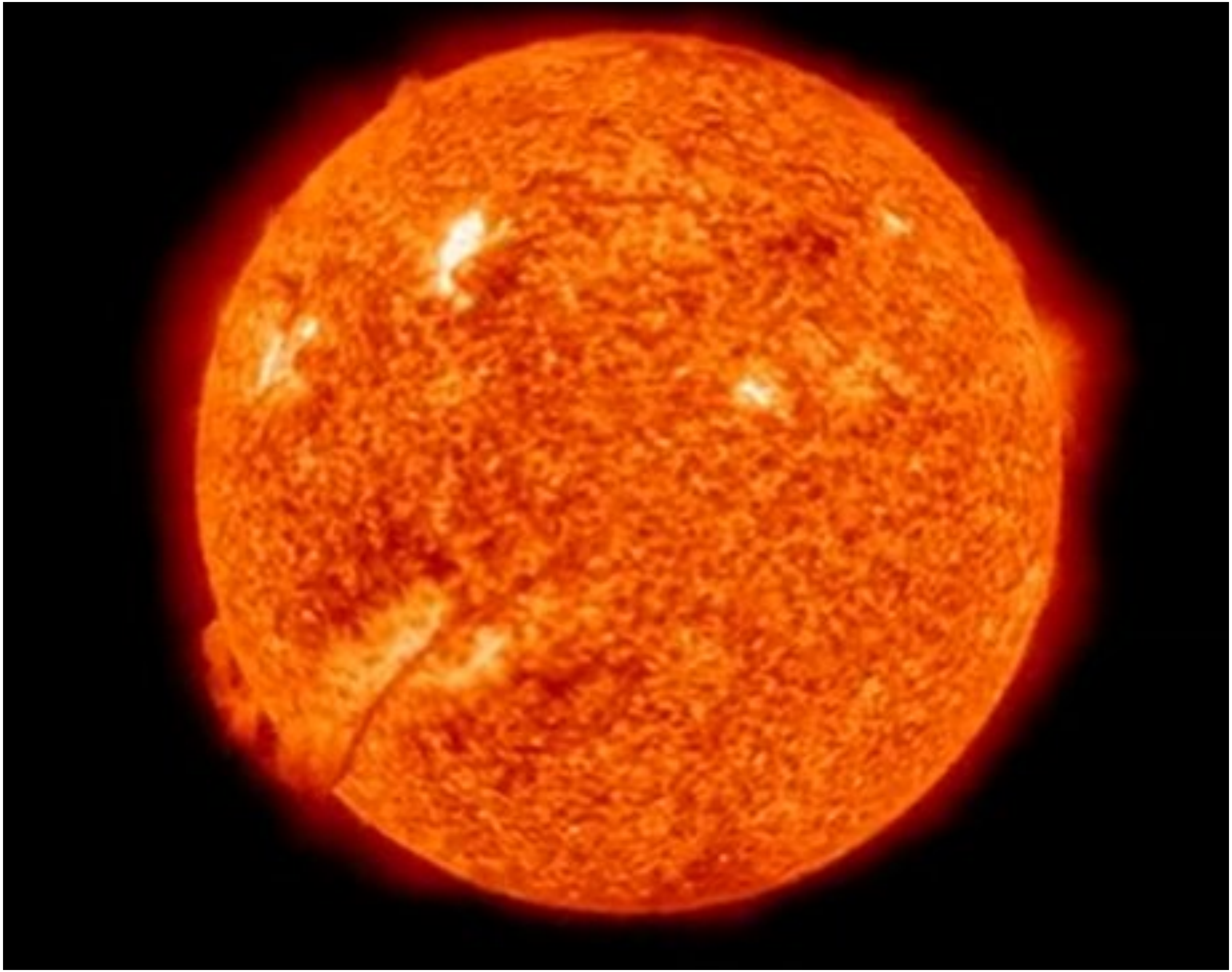
- Understand hydrostatic equilibrium and why it applies to stars
- Understand the conditions in the cores of stars, the definition of mean molecular mass, and the forms of pressure in stars
- Know and be able to use:

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

Hydrostatic equilibrium

$$P = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4$$

Internal pressure

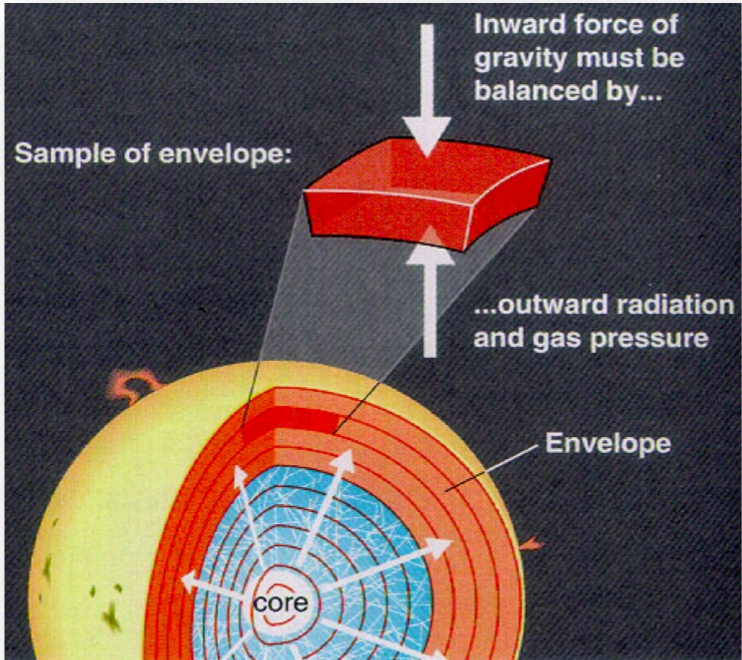


Hydrostatic equilibrium and application to stars

Stars do not (typically) noticeably contract or expand, they are luminous, and they live for a long time (~10^6-10^{12} years); therefore they require:

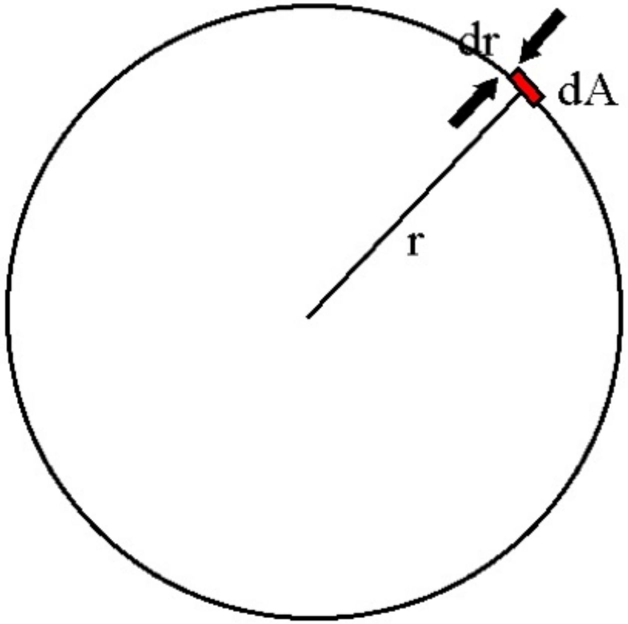
- **Balance between gravity and pressure to provide stability**
- **Continuous luminous energy source to provide long-lasting emission**
  - **the energy from the Sun in 1 sec is equivalent to that the total energy consumption of the world over ~500,000 years (at current energy rates)!**

Hydrostatic equilibrium: balance between gravity and pressure



Equation of hydrostatic equilibrium

Derivation of hydrostatic equilibrium equation:



$P = P(r) =$ pressure

$M_r =$ mass in sphere of radius r

$\rho = \rho(r) =$ density

Pressure force on element dA = [P(r+dr)-P(r)]dA

Gravitational force = ρ dr dA g where, for a spherical body

$$g = \frac{GM_r}{r^2}$$

Providing that dr is small, we may write the pressure force = dPdA

For hydrostatic equilibrium, set the pressure force equal to the gravitational force:

$$dPdA = -\rho drdAg$$

and substitute for g:

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

Equation 6

Hydrostatic equilibrium must occur at all radii – what would happen if not?

Estimate the pressure at the stellar core

Using the hydrostatic equation we can estimate the stellar-core pressure (assuming constant density)

We know that:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

Equation 7

Integrating with constant density gives:

$$M_r = \frac{4}{3}\pi r^3 \rho$$

Substituting M_r in the equation of hydrostatic equilibrium, then:

$$\frac{dP}{dr} = -G\frac{4\pi}{3}\frac{\rho^2 r^3}{r^2}$$

$$= -\frac{4}{3}\pi G \rho^2 r$$

Estimate the pressure at the stellar core

Integrate up once more from the center to R, and let the pressure at the core be P_c. Then:

$$P = P_c - \frac{2\pi}{3} G\rho^2 R^2$$

Now, when r=R, P=0, so:

$$P_c = \frac{2\pi}{3} G\rho^2 R^2$$

Substituting for ρ:

$$= \frac{2\pi}{3} GR^2 \left(\frac{3M}{4\pi R^3} \right)^2$$

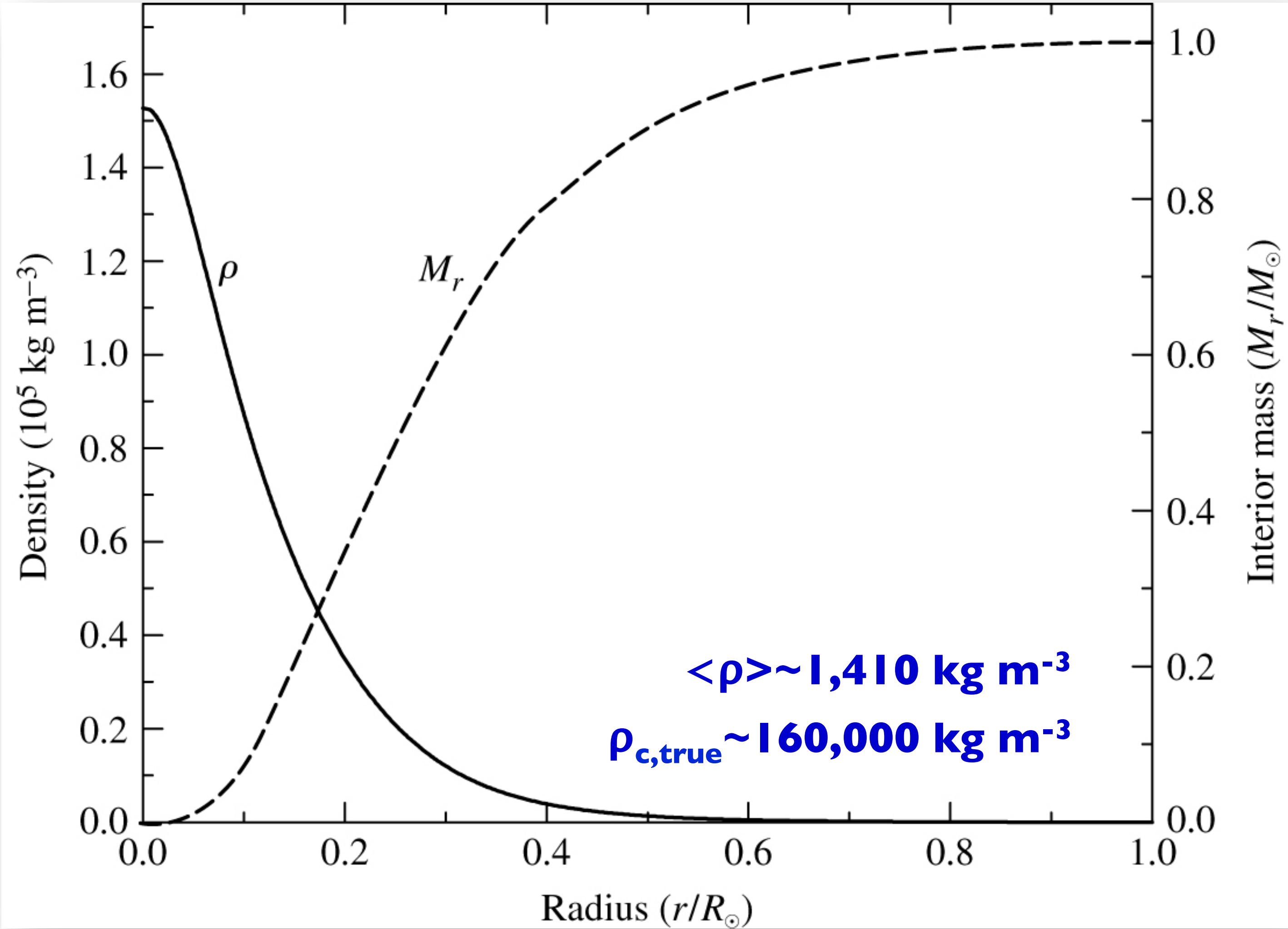
$$P_c = \frac{3}{8\pi} \frac{GM^2}{R^4}$$

Equation 8

Investigate the pressure at the centre of the Sun

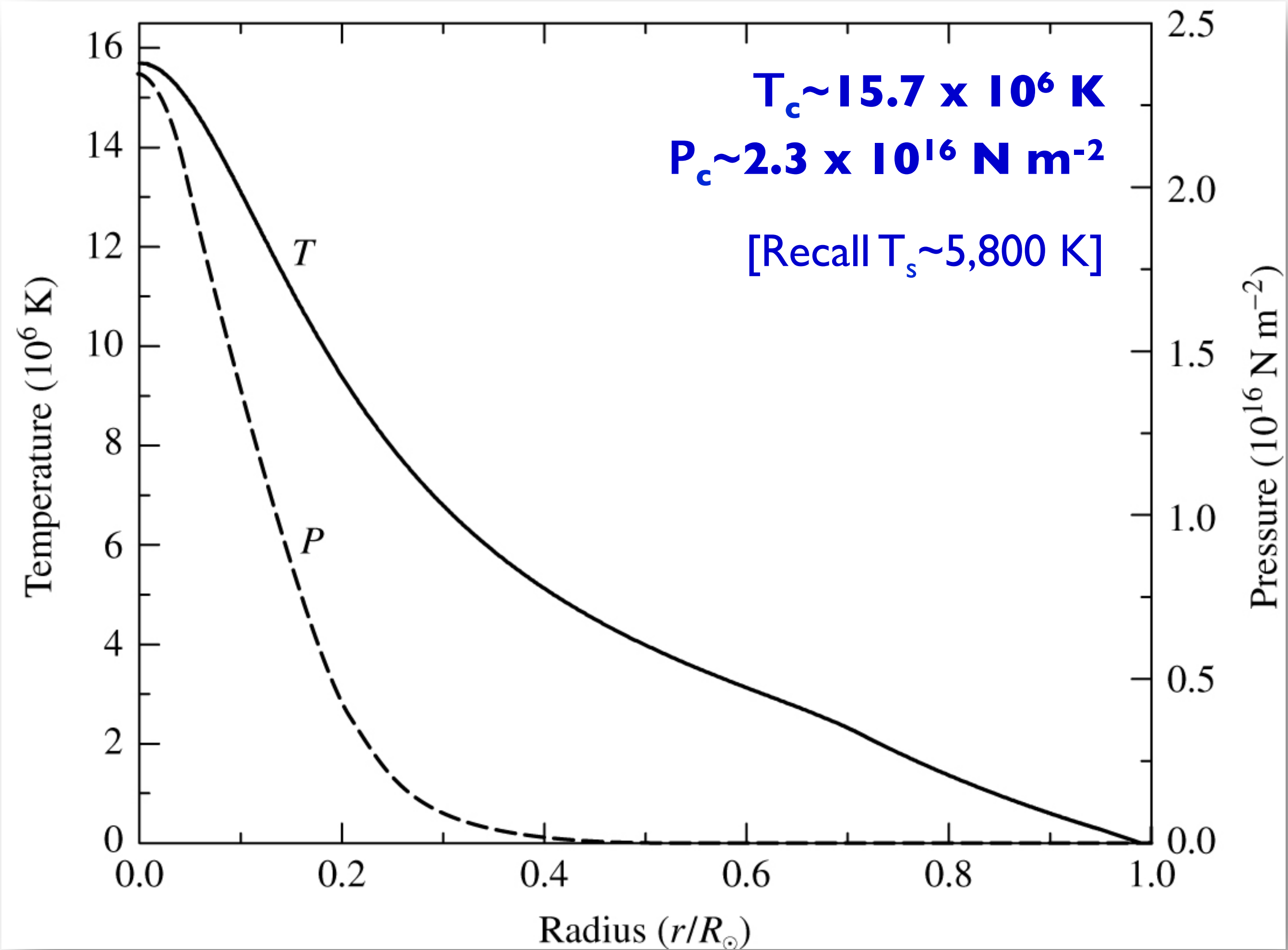
What is a major limitation of our approach?

Predictions for the density profile of the Sun



These are from a computer simulation: the density of gas towards the centre of the Sun is much higher than the gas density towards the surface

...and the pressure (and temperature) profiles



Pressure and temperature profiles of the Sun, calculated from a computer simulation under the assumption of hydrostatic equilibrium, where ρ changes as function of r - notice how much higher the core temperature (T_c) is than the surface temperature (T_s)

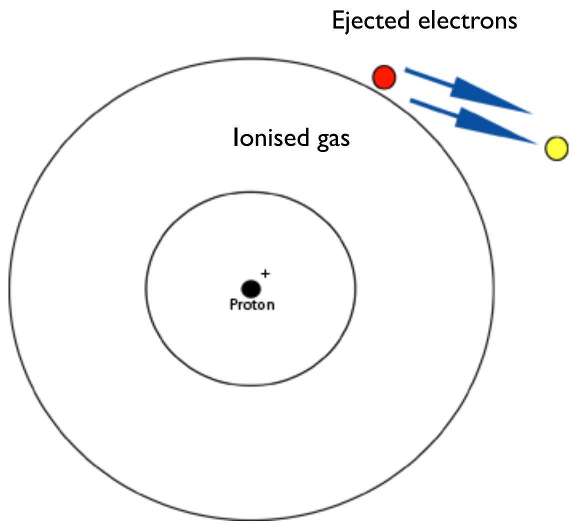
Gaseous state of stellar material

We are left with a problem – the central pressures of stars are very high, which leads one to think that the material must be solid. Yet, the measured temperatures of stars are also high **(as we just saw)** - far too high for stars to be solid.

The important point about stellar material is that it is in the **plasma** state. This may be defined as follows: -

> A plasma is a gas consisting of ions, electrons, and neutral particles; where the behavior of the gas is dominated by the electromagnetic interaction between the charged particles.

The temperatures inside stars are so high that all but the most tightly bound electrons are separated from their atoms (Saha equation). This makes possible a very much greater compression of the stellar material without deviation from a perfect gas law, because the nuclear dimension is 10^{-15} m compared with a typical atomic dimension of 10^{-10} m.



Another important point to bear in mind is that, in stellar interiors, radiation is in thermal equilibrium with matter. Just as the gas particles exert a pressure, so do the photons.

Defining the gaseous state: mean molecular mass

The mean molecular mass is the average mass of a free particle in the gas in units of m_H:

$$\mu = \frac{\bar{m}}{m_H}$$

For heavier elements replace m_p with the mass of the nucleus (e.g, the mass of 2 neutrons + 2 protons for Helium etc)

For just ionised Hydrogen:

$$\bar{m} = \frac{m_p + m_e}{2} \approx \frac{m_H}{2} \quad \text{so} \quad \mu = 0.5$$

More generally we would define μ based on the fraction of Hydrogen (*X*), Helium (*Y*), and metals (*Z*; particles more massive than Helium):

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

So for a typical composition of *X*=0.70, *Y*=0.28, and *Z*=0.02:

$$\frac{1}{\mu} \approx 1.62 \quad \text{and therefore} \quad \mu = 0.62$$

Gaseous state of stellar material: form of pressure

In stellar interiors, radiation is in thermal equilibrium with matter. Just as the gas particles exert a pressure, so do the photons.

So we have:

$$P_{gas} = nkT$$

Gas pressure (ideal gas law): note the form of this equation – lower-case n not capital N

$$P_{rad} = \frac{1}{3}aT^4$$

Radiation pressure: note the T^4 dependence – this is the photon flux

where:

n = number of particles per m^{-3}

a = radiation density constant = 7.57×10^{-16} J m^{-3} K^{-4}

Although radiation pressure is of vital importance in some stars, generally speaking it is not of great significance, and $P_{rad}/P_{gas} \sim$ few x 10^{-4} (as we demonstrate below). However, the true form of pressure inside a star is:

$$P = nkT + \frac{1}{3}aT^4$$

Equation 9

Gaseous state of stellar material: form of pressure

We can alternatively write this equation as:

$$P = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4$$

where:

μ = mean molecular weight (for a fully ionized plasma of hydrogen =0.5)

Calculate the pressure at the core of the Sun assuming the ideal gas law

Verify our assumption that radiation pressure is negligible is valid

In what types of stars will radiation pressure dominate?