ELECTROMAGNETISM - Workshop 3rd Set
Complex Numbers
for Solving Differential Equations

Professor D P Hampshire – 2nd Year Physics Lecture course

1 Complex Numbers and the Argand Diagram

Complex numbers provide an essential mathematical tool for solving differential equations. They are useful for finding wave-like solutions that are ubiquitous when solving Maxwell's equations.

1.1 The complex plane or Argand diagram

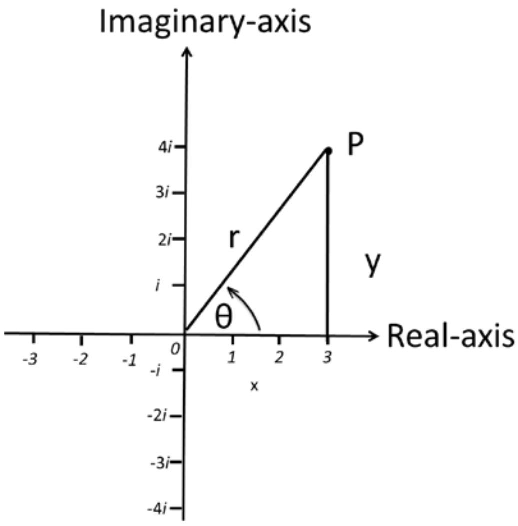


Figure 1 : A complex number z can be described on an Argand diagram using either Cartesian coordinates (x, y) or polar coordinates (r, θ).

The complex number z is

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)\,, \qquad 1\text{-}1$$

where r is the modulus of z,

$$r = |z| = \sqrt{x^2 + y^2}\;, \qquad 1\text{-}2$$

where θ is called the argument of z,

$$\theta = \arg(z) = \arctan\left(\frac{y}{x}\right)\;, \qquad 1\text{-}3$$

where i is the imaginary number given by $i = \sqrt{-1}$.

Euler found the incredible equation,

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta\;, \qquad 1\text{-}4$$

from which it follows

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = \mathrm{r}\mathrm{e}^{\mathrm{i}\theta}. \qquad 1\text{-}5$$

<u>Proof of Euler's equation:</u>

Consider the function

$$f(\theta) = \mathrm{e}^{-\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta)\;, \qquad 1\text{-}6$$

where θ is real. Differentiating and using the product rule gives

$$f'(\theta) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta) + \mathrm{e}^{-\mathrm{i}\theta}(-\sin\theta + \mathrm{i}\cos\theta) = 0. \qquad 1\text{-}7$$

Hence, f is constant for all θ. Since $f(0) = 1$, it follows that $f(\theta) = 1$ for all θ and therefore

$$\mathrm{e}^{-\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta) = 1 \qquad 1\text{-}8$$

for all θ. QED.

2 Propagating waves – mathematical formulism.

2.1 Sign conventions – waves

We will use the sign conventions from quantum mechanics. The energy operator (E_{op}) is given by

$$E_{op} = i\hbar\frac{\partial}{\partial t}. \qquad 2\text{-}1$$

Hence to ensure that waves have positive energy and travel along the positive x-direction (as time increases), we must describe a complex planar wave propagating along the x–direction using

$$\psi(x,t) = \psi_o \exp i(kx - \omega t)\;. \qquad 2\text{-}2$$

This ensures the energy (E_ψ) is positive, given by

$$E_\psi = +\hbar\omega. \qquad 2\text{-}3$$

We also note that as the time increases, if we sit on the wave (i.e. at constant phase), x increases (i.e. the wave propagates along the x-direction).

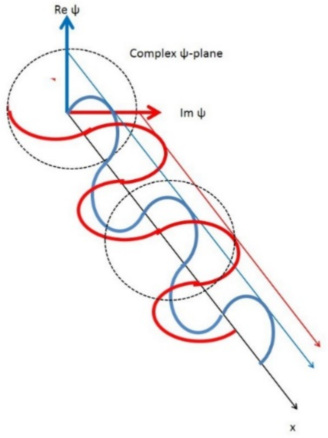


Figure 2 : The disturbance produced by an infinite complex wave in the complex plane as a function of distance at a fixed time, say t=0.

Many older optic books use: $\exp i(\omega t - kz)$. Unfortunately, if we substitute this expression into Equation 2-1 we end up with negative energy which is not physically meaningful. This choice also leads to a different sign in the general dispersion relation (which we shall consider later). We use the convention that leads to $k^2 = \varepsilon\mu\omega^2 \boxed{+} i\omega\mu\sigma_N$, whereas the boxed sign is negative in older optics books.

2.2 Sign conventions – oscillating voltages

When we are considering voltages that are for example driving electrical circuits (or forces driving simple harmonic oscillators), the time dependence is often taken to be of the form $V(t) = V_0\cos(\omega t)$ and we don't need a spatial dependence in the functional form. This naturally leads to a starting point where $V(t) = V_o e^{i\omega t}$. Don't be misled by the different choices made for the sign in the argument of $V(t)$ and $\underline{\boldsymbol{E}}(x,t)$. The different sign choice often changes the signs in some equations en route to solving a specific differential equation. However the physical results are (i.e. the Physics is) of course unchanged.

2.3 The phase difference between waves

If we have two waves with the same frequency and wavevector given by

$$W = W_o \exp i(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t) \qquad 2\text{-}4$$

and

$$X = X_o \exp i(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t + \delta). \qquad 2\text{-}5$$

We can plot (either) the real (or the imaginary) parts of W and X as a function of time at say x=0, (cf Figure 14) to find that in both cases if δ is positive, X lags W. If δ is negative, X leads W.

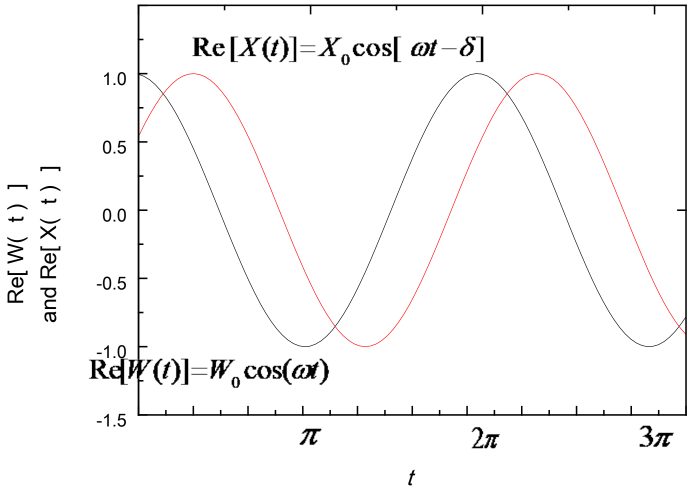


Figure 3 : The magnitude of the real part of X and W at a fixed point (say x = 0) as a function of time. In this graph, X lags W.

Hence to determine the phase difference between two complex waves we can write down the ratio:

$$\frac{X}{W} = \frac{X_0}{W_0}\exp i\delta. \qquad \text{2-6}$$

If δ is positive, X lags W. If δ is negative, X leads W.

3 Worked examples

3.1 Questions

1. Find the square root of i.

2. Draw an Argand diagram and show the complex number 3 + 4i . Sketch on the diagram the two roots of this number.

3. Using complex numbers (or otherwise), find the net motion, x_N, from the superposition of two oscillatory motions, given

$$x_N = x_1 + x_2$$

where $x_1 = A\cos\omega t$ and $x_2 = A\cos\left(\omega t + \frac{\pi}{3}\right)$.

Some students (recommended route) notice that they can simplify the algebra by rewriting the equation using complex notation using

$$\tilde{x}_N = \tilde{x}_1 + \tilde{x}_2$$

Where $\tilde{x}_1 = Ae^{i\omega t}$ and $\tilde{x}_2 = Ae^{i\left(\omega t+\frac{\pi}{3}\right)}$. We have added the tilde to make clear that the variable $\tilde{x}$ is complex. Having solved for $\tilde{x}_N$, the approach is to then use x_N=Re $(\tilde{x}_N)$.

Other students (less recommended route), like to write down a second equation of the form,

$$y_N = y_1 + y_2$$

where $y_1 = A\sin\omega t$ and $y_2 = A\sin\left(\omega t + \frac{\pi}{3}\right)$. Adding x_N to iy_N gives, as before.

$$\tilde{z}_N = x_N + iy_N = \tilde{z}_1 + \tilde{z}_2$$

where $\tilde{z}_1 = Ae^{i\omega t}$ and $\tilde{z}_2 = Ae^{i\left(\omega t + \frac{\pi}{3}\right)}$. Again the approach is to find $\tilde{z}_N$ and then use $x_N =$ Re $(\tilde{z}_N)$. Note that the solution for x_N only contains amplitude, frequency and phase information.

4. The equation of motion for the driven harmonic oscillator is given by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + {\omega_o}^2 x = F(t)\,,$$

where $F(t)$ is the force on the oscillator equal to $F_0\cos(\omega t)$ and γ and ω are constants.

Show that the magnitude of the oscillation is $\frac{F_o}{\sqrt{(\omega_o^2-\omega^2)^2+\omega^2\gamma^2}}$ and derive the phase difference (δ) between the driving force and the oscillation.

Our approach is to write the differential equation down as a complex differential equation where

$$\frac{\mathrm{d}^2 \tilde{x}}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} + {\omega_o}^2 \tilde{x} = \tilde{F}(t)\,,$$

and then derive

$$(-\omega^2 + i\omega\gamma + \omega_o^2)\tilde{x} = \tilde{F}$$

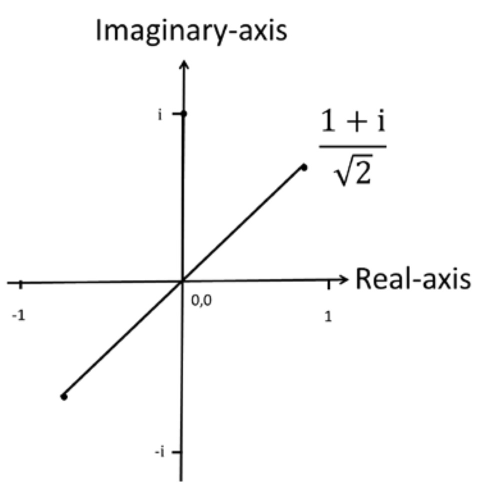
Then use $\tilde{F}(t) = F_0 \mathrm{e}^{i\omega t}$ so that

$$F(\mathrm{t}) = \mathrm{Re}\{\tilde{F}(t)\} = F_0\cos(\omega t)$$

and

$$x(t) = \mathrm{Re}\,[\,\tilde{x}\,].$$

3.2 Answers



1. There are three steps to find the square root of any complex number.

1) Re-write the complex number in polar form using the Euler equation
2) Find the general solution that repeats every 2π where $re^{i\theta} = re^{i(\theta+2n\pi)}$.
3) The square root of the complex number is found by identifying the square root of the magnitude and half the angle i.e. $\sqrt{r}\mathrm{e}^{i(\frac{\theta}{2}+n\pi)}$.

Substituting either n = 0 or 1, will give us the two square roots of the complex number. In most cases, all the Physics is contained in the n = 0 solution. The other solution may for example represent a wave travelling in the opposite direction.

First re-write i in Euler's form, $i = \mathrm{e}^{i\frac{\pi}{2}}$.
Since the complex number repeats its value every $2n\pi$, we have

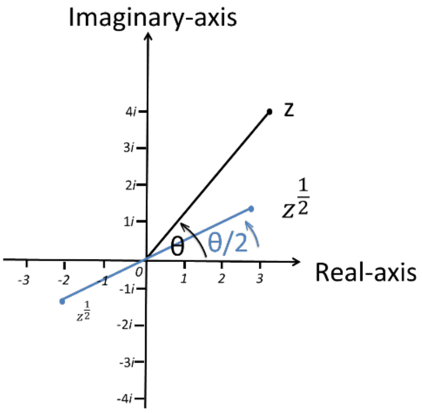
$$i = e^{i\frac{\pi}{2}} = e^{i(\frac{\pi}{2}+2n\pi)}.$$

We can identify the square root by taking the root of the magnitude and half the angle of the complex number so that:

$$i^{\frac{1}{2}} = e^{i(\frac{\pi}{4}+n\pi)},$$

For n = 0, $z_0 = \mathrm{e}^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$. For n = 1, $\mathrm{z}_1 = \mathrm{e}^{i\frac{5\pi}{4}} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1+i}{\sqrt{2}}$.
Note that the solution for n = 2 is the same as n = 0 etc.

2.



Note that the two roots both have a magnitude that is the square root of the magnitude of 3 + 4i . One root has half the angle of 3 + 4i and the other root has half the angle + π radians.

3.

Given $\tilde{z}_N = \tilde{z}_1 + \tilde{z}_2$ and $\tilde{z}_1 = Ae^{i\omega t}$ and $\tilde{z}_2 = Ae^{i\left(\omega t+\frac{\pi}{3}\right)}$.

$$\tilde{z}_N = Ae^{i\omega t} + Ae^{i(\omega t+\frac{\pi}{3})}$$

Of course $\tilde{z}_N$ is not the resultant motion – how can it be ? It is complex.

$$= Ae^{i\omega t}(1 + e^{i\frac{\pi}{3}})$$
$$= Ae^{i\omega t}\left(1 + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = Ae^{i\omega t}(\frac{3}{2} + i\frac{\sqrt{3}}{2})$$

$$= A\sqrt{3}e^{i\omega t}e^{i\frac{\pi}{6}} = \sqrt{3}Ae^{i(\omega t+\frac{\pi}{6})}$$

Then taking the real part of the final answer:

$$x_N = Re(\sqrt{3}Ae^{i(\omega t+\frac{\pi}{6})})$$
$$= \sqrt{3}A\cos(\omega t + \frac{\pi}{6})$$

4.

The strategy here is to appreciate that we can rewrite the equation of motion as a complex equation where:

$$\frac{\mathrm{d}^2\tilde{\mathrm{x}}}{\mathrm{dt}^2} + \gamma\frac{\mathrm{d}\tilde{\mathrm{x}}}{\mathrm{dt}} + \omega_o{}^2\tilde{\mathrm{x}} = \tilde{\mathrm{F}}(\mathrm{t}),$$

and that the complex differential equation has all the information required to find how the solutions for $\tilde{x}$ and $\tilde{F}(t)$ are related (in magnitude and phase). We have used the tilde (i.e. $\sim$) to denote a complex variable.

We assume that in steady state both $\tilde{x}$ and $\tilde{\mathrm{F}}$ are oscillating of the form:

$$\tilde{x}, \tilde{\mathrm{F}} \propto \exp - \mathrm{i}\omega\mathrm{t}$$

Hence we have

$$(-\omega^2 + i\omega\gamma + \omega_o^2)\tilde{x} = \tilde{F}$$

We can now take the Force as the independent variable where it is the real part pf the complex force so

$$F(t) = \mathrm{Re}\{\tilde{F}(t)\} = F_0\cos(\omega t)$$

and the displacement is

$$x = Re\{\tilde{x}\} = Re\left\{\frac{\tilde{F}}{(-\omega^2 + i\omega\gamma + \omega_o^2)}\right\} = \mathrm{Re}\{\frac{(\omega_o^2 - \omega^2 - i\omega\gamma)\tilde{F}}{((\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2)}\}$$

Taking the real part gives:

$$x = \frac{(\omega_o^2 - \omega^2)F_0\cos(\omega t)}{((\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2)} + \frac{(\omega\gamma)F_0\sin(\omega t)}{((\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2)}$$

So comparing the expressions for the force F and the displacement x, the magnitude of the oscillation is $\frac{F_o}{\sqrt{(\omega_o^2-\omega^2)^2+\omega^2\gamma^2}}$ and the phase difference between the driving force and the oscillation is $\theta = \tan^{-1}\frac{\omega\gamma}{\omega_o^2-\omega^2}$ where if $\frac{\omega\gamma}{\omega_o^2-\omega^2}$ is positive, the displacement lags the force.

One can always add to the steady state motion, $x_s(t)$, the solution associated with the initial conditions. This is found by solving the differential equation without a driving force and is of the form : $x_s(t) = Ae^{-\gamma t/2}(\frac{F_0}{((-\omega^2)^2+\omega^2)^{1/2}}\cos\left[\omega_0\left(1-\frac{\gamma^2}{4\omega_o^2}\right)^{1/2}t\right]$ when $4\omega_o^2 > \gamma^2$.

4 Unseen problems

4.1 Complex numbers

1. Plot these complex numbers on the Argand diagram: $3+4i, 2-i, 3-7i, -1-3i$ and find the modulus of these complex numbers.

2. Find the square root of $7+24i$.

4.2 Solving Differential Equations

3. The wave equation for an electromagnetic wave travelling through a highly conducting infinite linear isotropic medium is given by

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_o \sigma_n \frac{\partial \underline{\boldsymbol{E}}}{\partial t},$$

where μ_o is a fundamental constant and σ_n is electrical conductivity. Assume the electric field component of the wave is described mathematically by a travelling wave solution of the form

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \exp i(kx - \omega t)$$

When an electromagnetic wave penetrates a highly conducting media, the $\underline{E}$-field is exponentially attenuated with a characteristic decay length δ. Show that $\delta = \sqrt{2}/(\mu_o \sigma_N \omega)^{1/2}$. Look up (on the internet) values for the relevant materials constants for copper and hence calculate how thin a copper sheet would need to be, for you to see through it.

4. From Maxwell's equations, one can derive a wave equation for a conducting dielectric of the form

$$\nabla^2 \underline{\boldsymbol{E}} - \mu_o \varepsilon_r \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} - \mu_o \sigma_n \frac{\partial \underline{\boldsymbol{E}}}{\partial t} = 0$$

where ε_r is the relative dielectric constant and σ_N is the electrical conductivity. Hence by substituting a travelling wave solution into the wave equation, derive a dispersion relation (by definition a relation between ω and k) of the form

$$k^2 = \mu_o \varepsilon_o \varepsilon_r \omega^2 + i \mu_o \sigma_n \omega$$

where k is the wavevector and ω is the angular frequency.

A scientist attaches a transmitter to one side of a large elephant in order to track its movements. Calculate the change in the received signal when there is a clear line of sight between the transmitter and the receiver compared to when the elephants body is in the way. The relative permittivity of the elephant is ~ 90, the conductivity of the elephant tissue is ~ $10/4\pi\ \Omega^{-1}\text{m}^{-1}$, the frequency of transmission is 1 GHz and the elephant is about 1 m wide. You may ignore reflections at interfaces including those from the elephant's skin.

The biologist needs to get slightly better reception whether the elephant's body is between the transmitter and the receiver or not. Suggest how this could be achieved?

5. Find an approximate value (a) $\sqrt{[2i + 10^{-6}]}$ and (b) $\sqrt{[10^{-12} + 6i]}$ accurate to say 1 %.

6. For a series LCR circuit, we can apply Kirchoff's law to find the differential equation that relates the alternating current in the circuit to the driving alternating voltage where:

$$V = L\frac{dI}{dt} + IR + \frac{Q}{C}$$

where these constants have their usual meanings and $V = V_0 \cos(\omega t)$. Using $I = dQ/dt$, show that the peak current (I_0) in the circuit is given by:

$$I_0 = \frac{V_0}{\{R^2 + (\omega L - 1/\omega C)^2\}^{1/2}}$$

where V_0 is the magnitude of the voltage.

7. The currents I_1 and I_2 in the primary and secondary circuits (respectively) of a transformer with a resistance (R) and an inductance (L) in the primary circuit and an inductance (L) and a capacitance (C) in the secondary circuit with a mutual inductance (M) between the two circuits are described by :

$$V = L\frac{dI_1}{dt} + I_1 R + M\frac{dI_2}{dt}$$

$$0 = L\frac{dI_2}{dt} + M\frac{dI_1}{dt} + \frac{Q_2}{C}$$

Where $V = V_0 \cos(\omega t)$. By assuming the currents in the primary and secondary circuits are sinusoidal, show that the peak current in the primary circuit is given by

$$|I_1|_{Peak} = \frac{V_0}{\left\{R^2 + \left(\omega L + \frac{\omega^3 M^2}{\frac{1}{C} - \omega^2 L}\right)^2\right\}^{1/2}}.$$

Also, find an expression for the peak current in the secondary circuit.