Lecture 11. Tests of Cosmological Models - I

As the Universe is expanding, and space may be curved, the relationships between physical sizes and angular sizes, and the energy flux received from distant sources, are modified. These effects can be exploited to constrain cosmological parameters, e.g., k, Ω_{m0}, $\Omega_{\Lambda 0}$. This lecture concerns the use of energy fluxes received from distant sources to test cosmological models.

11.1 A static Euclidean universe

Static Euclidean geometry is the geometry that we are used to, in which angles of triangles add up to 180°, the Pythagorean theorem holds, and parallel lines remain parallel forever.

If an object at distance d emits radiation isotropically with a constant luminosity, L, in unit of Js^{-1}, where J is joule and s is second, then the flux we receive from it is (Lectures 2 & 5)

$$f = \frac{L}{4\pi d^2}\ \mathrm{Js}^{-1}\,\mathrm{m}^{-2}. \tag{1}$$

This can be seen most directly by realising that, if we sum up the energy across the full spherical surface of radius d centred at the source, it must be equal to the total luminosity.

11.2 An expanding Friedmann-Robertson-Walker (FRW) universe

In an expanding universe, objects moving with the Hubble Flow (i.e., with fixed comoving positions) have divergent trajectories. If the spatial curvature $k \neq 0$, the usual relation between the area and radius of a spherical surface is also different.

These distortions alter the flux-distance relation, Eq. (1). To define these modified relations let us introduce the **luminosity distance**, d_{L}, by inverting Eq. (1):

$$d_{\mathrm{L}}(z) \equiv \sqrt{\frac{L}{4\pi f}}, \tag{2}$$

where f is the flux received from an isotropically emitting object at redshift z of luminosity L. We notice from Eqs. (1, 2) that, for a static Euclidean universe, $d_{\mathrm{L}} = d$.

To determine the expression of d_{L}, we use the FRW metric, Eq. (10.2)[1]

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + S_k(r)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{3}$$

[1] Recall that the metric specifies the rules to define distances in a given space.

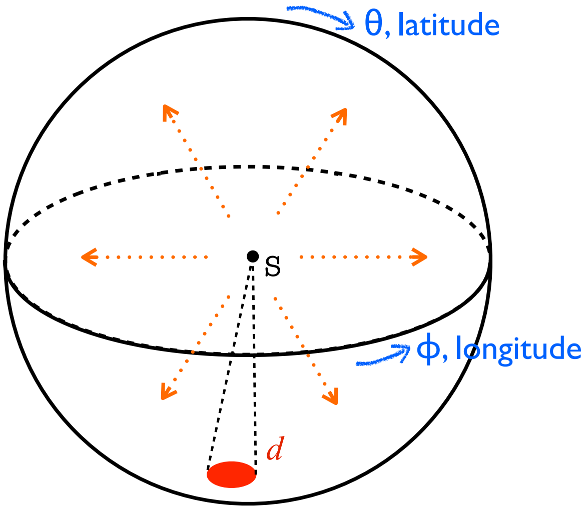

Figure 1: An illustration of the flux-luminosity relation. The source s at the centre emits isotropically at a constant luminosity L Js^{-1}. The red circular region on the sphere with radius d is the area of a detector that receives a flux of f Js^{-1}m^{-2}. Energy conservation implies $L = 4\pi d^2 f$.

The terms in the *brackets* describe the distance element $\mathrm{d}l^2$ between two points with comoving (since the expansion factor $a(t)$ is not included) coordinates (r, θ, ϕ) and $(r + \mathrm{d}r, \theta + \mathrm{d}\theta, \phi + \mathrm{d}\phi)$. Even when $k \neq 0$ so that $S_k(r) \neq r$, the total solid angle remains 4π as in Euclidean geometry, since $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ with $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$.

To relate luminosity and flux we need to calculate the physical area of the spherical surface illustrated in Fig. 1 at the present day ($t = t_0$ and $a(t) = 1$). From the above discussion, we have

$$A = 4\pi S_k(r)^2. \tag{4}$$

The photons emitted by the source are spread uniformly across this area. However, to calculate the corresponding energy flux and relate this to the intrinsic luminosity of the source, L, we must consider two more effects.

First, a photon emitted with a wavelength λ_e by a source redshift z will be **redshifted** and its observed wavelength will be $\lambda_o = \lambda_e(1 + z)$. Since photon energy $\epsilon = h\nu = hc/\lambda$, this reduces the observed energy flux by a factor of $(1 + z)$.

Second, if, in the rest frame of the source, N photons are emitted a second, **time dilation** (a general relativistic effect that says time passes at different rates in places of different curvature) means that in the observer's frame the emission rate is reduced by a factor of $(1 + z)$, namely $\Delta t_o = \Delta t_e(1 + z)$ (Lecture 4).

As a result, the overall *energy* passing through the entire sphere *per second* is reduced by *two* factors of $(1+z)$ – one for energy and one for time interval – and the observed flux is given by

$$f = \frac{L}{4\pi S_k(r)^2(1+z)^2}, \tag{5}$$

which implies

$$d_\mathrm{L}(z) = S_k(r)(1+z) = S_k[r(z)](1+z). \tag{6}$$

11.3 Test of cosmological models using luminosity distances

Eq. (6) is the key equation of this lecture. Because the comoving distance r between an object at redshift z and an observer at $z=0$ is given by (see Lecture 4, note again the different symbol)

$$r(z) = c\int_0^z \frac{\mathrm{d}z'}{H(z')}, \tag{7}$$

with $H(z)$ given by the Friedmann equation, for any cosmological model with specific matter content, curvature k and value of cosmological constant, one can use Eq. (6) to predict $d_\mathrm{L}(z)$. Comparing this *theoretical* prediction with the *observational* measurements of $d_\mathrm{L}(z)$ based on Eq. (2) serves as a test of the correctness of the assumed cosmological model.

However, in Eq. (2) one only measures the flux (i.e., the energy received by a detector per unit time per unit detector area) and reshift. Without *a priori* knowledge about L (the intrinsic luminosity of the source), one cannot determine $d_\mathrm{L}(z)$.

Astronomical objects whose luminosity L is known and is independent of redshift are called as **standard candles** (Lecture 4). They allow the luminosity distance d_L of such objects to be measured and used to test cosmological models. One candidate for standard candle is type Ia supernova – the explosion caused by a white dwarf accreting materials from its neighbourhood so that its mass exceeds the **Chandrasekhar mass** of $1.44M_\odot$ and electron degeneracy pressure can longer balance the gravity caused by its mass – the star then collapses, triggering nuclear reactions in its core and releasing energy that blows it up. The calibrated light curves of type Ia supernovae are found to be very similar, making these objects standard candles.

Figure 2 shows the comparison between the observed $d_\mathrm{L}(z)$ from samples of type Ia supernovae (symbols with error bars) and the predicted $d_\mathrm{L}(z)$ for various cosmological models with different values of $\Omega_{m0}, \Omega_{\Lambda 0}$ and k. The vertical axis of the top panel is the apparent magnitude (how bright a supernova appears to be), which is related to d_L as $m = 5\log_{10} d_\mathrm{L} + b$ with b being a constant. A larger m means the object is fainter and has a larger luminosity distance from us. The bottom panel shows the residual of m with respect to the prediction by the best-fit model with $\Omega_{m0} = 0.28$, $\Omega_{\Lambda 0} = 0.72$ and $k=0$; the observed data scatter evenly around 0.

In general, calculating $d_\mathrm{L}(z)$ requires evaluation of the integral in Eq. (7), which cannot be done analytically. Here we make the small-z approximation to derive an approximate analytic expression (by neglecting z^2 and higher order terms), which allows us to gain some insight:

$$H(z) \approx H_0 + \left.\frac{\mathrm{d}H}{\mathrm{d}z}\right|_{z=0} z \approx H_0 + \left[\frac{\mathrm{d}H}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}z}\right]_{z=0} z \approx H_0\left[1 + \left(1+q_0\right)z\right], \tag{8}$$

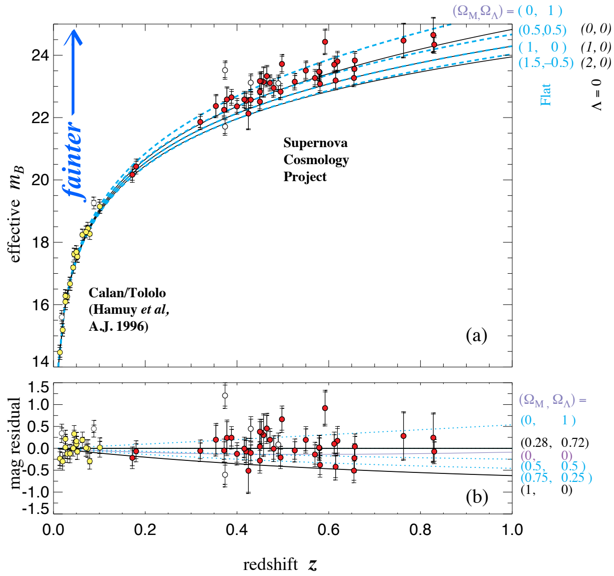

Figure 2: *Upper panel*: The apparent magnitudes m of type Ia supernovae at various redshifts z from observations (symbols with error bars) and predictions by various cosmological model (lines). *Lower panel*: residuals with respect to the best-fit model, with $\Omega_{m0} = 0.28$, $\Omega_{\Lambda 0} = 0.72$, $k = 0$.

where we have used $1+z = 1/a$ and so $\mathrm{d}z/\mathrm{d}t = -\dot{a}/a^2 = -(1+z)H$ and $\mathrm{d}H/\mathrm{d}t = \ddot{a}/a - (\dot{a}/a)^2 \equiv -(\dot{a}/a)^2(1+q)$. Here

$$q \equiv -\frac{\ddot{a}/a}{(\dot{a}/a)^2} = -\frac{a\ddot{a}}{\dot{a}^2}, \tag{9}$$

is the dimensionless **deceleration parameter**. Substituting Eq. (8) into Eq. (7), one obtains

$$r(z) \approx \frac{c}{H_0}\left[z - \frac{1}{2}(1+q_0)z^2\right] \tag{10}$$

so that for a flat universe ($k = 0$ and $S_k(r) = r$) containing only matter and Λ:

$$d_\mathrm{L}(z) = (1+z)r(z) \approx \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2\right]. \tag{11}$$

Using the **acceleration equation** $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{1}{3}\Lambda c^2$, the deceleration parameter can be expressed as

$$q = -\frac{\ddot{a}/a}{(\dot{a}/a)^2} = -\frac{1}{(\dot{a}/a)^2}\left[-\frac{4\pi G}{3}\rho_m + \frac{1}{3}\Lambda c^2\right] = \frac{1}{2}\Omega_m - \Omega_\Lambda, \tag{12}$$

where in the last step we have used the definition of **density parameters** (Lecture 3). Therefore, having a cosmological constant, $\Omega_\Lambda > 0$, makes q smaller, $d_\mathrm{L}(z)$ larger and the supernova appear less brighter (m larger) than would be predicted by a model with $\Omega_m = 1$. This dimming of distant type Ia supernovae offered the first firm evidence of $\Omega_\Lambda > 0$ or the existence of the still mysterious dark energy.

Key Takeaway Points of Lecture 11

• The observational foundation to exploit luminosity distance is

$$d_{\rm L}(z) \equiv \sqrt{\frac{L}{4\pi f}},$$

through which $d_{\rm L}(z)$ can be determined using the luminosity and flux received from an object at z.

• The theoretical foundation to exploit luminosity distance is

$$d_{\rm L}(z) = (1+z)S_k(r) = (1+z)S_k[r(z)],$$

through which $d_{\rm L}(z)$ can be predicted by assuming a cosmological model (which enables the calculation of $r(z)$).

• Standard candles, which are cosmological objects whose luminosity is known, are needed.

• Type Ia supernovae are great standard candles, which have been used to constrain cosmological models[2].

[2] A cosmological model is a model of the Universe, with its various properties or parameters specified, e.g., k, Ω_{m0}, H_0, the nature of dark energy, the theory of gravity, etc.