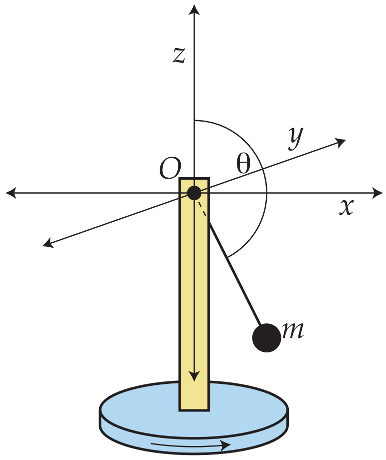
CM2 Solutions: Lagrangian of a Spinning Plane Pendulum



1. **(2 marks total)** Where r is the distance from the origin, θ is the angle of elevation, and ϕ is the azimuthal angle, these spherical coordinates are defined through $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$ [1 mark for writing exactly these equations down]. Hence

$$\dot{x} = \dot{r}\cos\phi\sin\theta - r\dot{\phi}\sin\phi\sin\theta + r\dot{\theta}\cos\phi\cos\theta, \tag{1}$$
$$\dot{y} = \dot{r}\sin\phi\sin\theta + r\dot{\phi}\cos\phi\sin\theta + r\dot{\theta}\sin\phi\cos\theta, \tag{2}$$
$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \tag{3}$$

[1 mark for determining exactly these equations].

2. **(3 marks total)** Squaring $\dot{x}, \dot{y}, \dot{z}$

$$\begin{aligned}\dot{x}^2 =& \dot{r}^2\cos^2\phi\sin^2\theta + r^2\dot{\phi}^2\sin^2\phi\sin^2\theta + r^2\dot{\theta}^2\cos^2\phi\cos^2\theta \\ &- 2r\dot{r}\dot{\phi}\cos\phi\sin\phi\sin^2\theta + 2r\dot{r}\dot{\theta}\cos^2\phi\sin\theta\cos\theta - 2r^2\dot{\phi}\dot{\theta}\sin\phi\cos\phi\sin\theta\cos\theta,\end{aligned} \tag{4}$$
$$\begin{aligned}\dot{y}^2 =& \dot{r}^2\sin^2\phi\sin^2\theta + r^2\dot{\phi}^2\cos^2\phi\sin^2\theta + r^2\dot{\theta}^2\sin^2\phi\cos^2\theta \\ &+ 2r\dot{r}\dot{\phi}\sin\phi\cos\phi\sin^2\theta + 2r\dot{r}\dot{\theta}\sin^2\phi\sin\theta\cos\theta + 2r^2\dot{\phi}\dot{\theta}\cos\phi\sin\phi\sin\theta\cos\theta,\end{aligned} \tag{5}$$
$$\dot{z}^2 = \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta \tag{6}$$

[1 mark for getting this far, i.e., before cancellations and simplifications]. Hence, making abundant use of the identity $\cos^2\alpha + \sin^2\alpha = 1$, and cancellations,

$$T \equiv \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) \tag{7}$$

[1 mark to arrive at this result]. **It is not sufficient to simply state the above result — a credible amount of working, such as in Eqs. (4), (5), (6) must be given.** More straightforwardly, $V = mgr\cos\theta$ [1mark].

3. **(2 marks total)** The constraint equations are $r = R$ and $\phi = \omega t$ [1 mark], from which it immediately follows $\dot{r} = 0$, $\dot{\phi} = \omega$. Hence,

$$T = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta), \quad V = mgR\cos\theta, \quad L = T - V = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta) - mgR\cos\theta \tag{8}$$

[1 mark for the correct Lagrangian (or correctly combining the previously determined T, V, and the constraint equations, if there are errors in any of these)].

4. **(3 marks total)** Now

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \Rightarrow mR^2\ddot{\theta} = mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta \Rightarrow \ddot{\theta} = \omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta \tag{9}$$

[1 mark for the correct equation of motion, whether or not it is divided through by mR^2]. Constant θ implies $\dot{\theta} = 0, \ddot{\theta} = 0$, hence, such an orbit requires $\omega^2\sin\theta\cos\theta + (g/R)\sin\theta \Rightarrow \cos\theta = -g/R\omega^2$ [1 mark]. This means $\cos\theta$ must be negative, and so θ must be between $\pi/2$ and π [1 mark].