University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 042591/01 or 043671/01

LEVEL 2 PHYSICS: FOUNDATIONS OF PHYSICS 2B
LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3C

SECTION A. THERMODYNAMICS
SECTION B. CONDENSED MATTER PHYSICS
SECTION C. MODERN OPTICS

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_B = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_e = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_p = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_A = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. THERMODYNAMICS

Question 1 is compulsory. Question 2 is optional.

1. (a) A 50.0 kg block of iron, having heat capacity of 22.5 kJ K^{-1}, is removed from a blast furnace with a temperature of 1470 K and allowed to cool in the surroundings, at a temperature of 300 K. What is the irreversibility of this process, assuming that the temperature of the surroundings doesn't change? [4 marks]

   (b) What information does a distribution function provide about the energy levels of a system? Write down the distribution function which describes fermions, $f_{FD}(\varepsilon)$. Sketch this distribution function at two temperatures, $T > 0$ K and $T = 0$ K. [4 marks]

   (c) Write down the Nernst statement of the third law of thermodynamics. By defining the heat capacity, C_L, at constant length, L, of an elastic rod, use this statement of the third law to determine the behaviour of this heat capacity in the vicinity of absolute zero. By extending this result to the behaviour of both C_p and C_V, what can be said about the validity of applying the ideal gas law to describe the behaviour of gases at very low temperatures? [4 marks]

   [Hint: You may find it useful to consider Mayer's equation, $C_p = C_V + R$.]

2. (a) An infinitesimal amount of work done, δW, is termed an *inexact differential*, whilst an equivalent infinitesimal change in the entropy of a system, dS, is termed an *exact differential.* What is meant by the terms exact and inexact differential in this context? Which form of differential is used to describe a system property in thermodynamics? [3 marks]

   (b) The entropy of a substance can be written as a function of pressure and volume, $S = S(p, V)$. By writing the total differential of this function, derive the third TdS equation

$$TdS = C_p \left(\frac{\partial T}{\partial V}\right)_p dV + C_V \left(\frac{\partial T}{\partial p}\right)_V dp.$$

   [6 marks]

   [Hint: You may wish to consider the reciprocal and reciprocity theorems and all the standard Maxwell relations.]

$$\left[\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V ; \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p ; \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V ;\right]$$

$$\left[\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .\right]$$

   (c) By defining the isobaric expansivity, α_p, and isothermal compressibility, κ_T, show that the result from (b) can be written as

$$TdS = \frac{C_p}{V\alpha_p} dV + \frac{\kappa_T C_V}{\alpha_p} dp.$$

   [4 marks]

   (d) By applying the above equation to an adiabatic process on an ideal gas, what standard result is obtained? [4 marks]

   (e) What is the entropy change as one mole of a diatomic, ideal gas, for which $C_p = 29.1$ J K^{-1}, quadruples in volume at constant pressure? [3 marks]

SECTION B. CONDENSED MATTER PHYSICS
Question 3 is compulsory. Question 4 is optional.

3. (a) For a simple cubic lattice, sketch the planes with the Miller indices (111) and (122). Include the x, y and z axes in your diagram. If the lattice constant a is 0.4 nm, determine the spacing for each of these two families of planes. [4 marks]

(b) A metal comprises a single atom basis on each lattice point of a simple cubic lattice with a lattice constant of $a = 0.2$ nm. An X-ray scattering measurement is performed using X-rays of wavelength 0.1542 nm. Calculate the Bragg angles for scattering of X-rays from the (100) and (200) planes. Would you expect to see X-ray peaks from all possible planes in this crystal? [4 marks]

(c) Describe the motion of atoms in a unit cell for transverse and longitudinal waves each having acoustic and optical modes of vibration. Illustrate your answer with a sketch of the phonon dispersion relation for the first Brillouin zone. [4 marks]

(d) Silver has a free electron density of $n = 5.9 \times 10^{28}$ m^{-3} and an electrical conductivity of $\sigma = 6.2 \times 10^7$ S m^{-1}. From the Drude model determine the mean time between collisions, τ, for electrons in this metal. Consider the motion of an individual electron under an applied electric field $\underline{E}$. Describe the behaviour of the velocity of this electron with time. [4 marks]

(e) Explain the terms *Fermi energy* and *Fermi surface* in the context of the Sommerfeld free-electron model. What determines the value of the *Fermi energy* in a bulk 3D metal? [4 marks]

(f) The ionisation energy of atomic hydrogen is given by

$$-\frac{e^4 m_{\mathrm{e}}}{2(4\pi\epsilon_0\hbar)^2}.$$

From this determine the value of the shallow donor binding energy in a semiconductor which has an electron effective mass of 0.06 m$_e$ and a relative permittivity of $\epsilon_r = 14$. Draw an energy level diagram illustrating the position of this energy level with respect to the energy band gap. [4 marks]

4. Describe the origin of the effective mass in a crystal. Explain why the effective mass can be both positive and negative. [4 marks]

   Use an $E(k)$ diagram for a nearly-free electron to illustrate the behaviour of the effective mass from the Brillouin zone centre to the boundary of the first Brillouin zone. [4 marks]

   A crystalline solid is described by an energy-wavevector relation given by:

   $$E(k) = A(k^2 - wk^4),$$

   where A and w are constants. Determine the value of the constant w in terms of the crystal lattice constant, a. [8 marks]

   Find values for the effective masses at the Brillouin zone centre $k = 0$ and the first Brillouin zone boundary $k = \pi/a$ in terms of the constant A. Show that the effective mass at the Brillouin zone boundary is -0.5 times the effective mass at the centre of the Brillouin zone. [4 marks]

SECTION C. MODERN OPTICS

Question 5 is compulsory. Questions 6 and 7 are optional.

5. (a) The Fourier transforms of $g(x)$ and $h(x)$ are

$$G(k_x) = \int_{-\infty}^{\infty} g(x) \exp(-\mathrm{i}k_x x)\, dx \text{ and } H(k_x) = \int_{-\infty}^{\infty} h(x) \exp(-\mathrm{i}k_x x)\, dx,$$

respectively. Give expressions for the Fourier transforms of: (i) $g(x) + h(x)$, (ii) $g(x/2)$, (iii) $g(x-d)$ and (iv) $g(x) \otimes h(x)$. [4 marks]

(b) The Fourier transform of the two dimensional function $\mathrm{Gauss}(\rho/w) = \exp(-\rho^2/w^2)$ is $\pi w^2 \mathrm{Gauss}(k_\rho w/2)$ where $\rho^2 = x^2 + y^2$ and $k_\rho^2 = k_x^2 + k_y^2$. Find the two dimensional Fourier transform of the function $f(x, y) = \exp(\mathrm{i}k\rho^2/(2z))$. [4 marks]

(c) What diffracts faster, light downstream of a single slit of width a or a Gaussian beam with beam waist a? Estimate the spot size (i.e., the beam radius) of a laser beam on the moon produced by a red laser pointer with wavelength 670 nm and a beam waist $w_0 = 1$ mm on earth. The distance to the moon is 384 000 km. [4 marks]

(d) The propagation of a monochromatic light field over a distance z is given by the Fresnel integral

$$\mathcal{E}^{(z)} = \frac{\exp(\mathrm{i}k\rho^2/(2z))}{\mathrm{i}\lambda z} \mathcal{F}\left[\mathcal{E}^{(0)} \exp\left(\mathrm{i}k\rho'^2/(2z)\right)\right],$$

where $\mathcal{E}^{(0)}$ is the input field, $\lambda = 2\pi/k$ is the wavelength, ρ' and ρ are the radial coordinates in the input and output planes, and $\mathcal{F}$ denotes a two-dimensional Fourier transform. How is the input field modified if a thin lens with focal length f is placed in the input plane? Re-write the Fresnel integral with the lens present and comment on any special case. [4 marks]

(e) In a $4f$ spatial filter set up, an aperture with the shape of the letter p in the input plane is illuminated normally by uniform monochromatic light. Sketch the intensity distribution in the Fourier plane. What filter is required to change a p into a d? [4 marks]

(f) A mask is punctured with 6 holes in a 2 by 3 array (similar to the 6 on a dice with the 3 in the vertical direction). Sketch the far–field intensity distribution along the horizontal and vertical axes if the mask is illuminated normally by uniform monochromatic light. [4 marks]

6. Sketch the $4f$ set-up used for optical spatial filtering. [6 marks]

The field in the Fourier plane has the functional form $\mathrm{Gauss}(\rho/w)\otimes\mathrm{comb}_5(x/d)$, where

$$\mathrm{Gauss}(x) = \mathrm{e}^{-x^2}\ ,\ \mathrm{comb}_N\left(\frac{x}{d}\right) = \sum_{n=-(N-1)/2}^{(N-1)/2} \delta\left(x - nd\right)\ ,$$

w and d are both distances and $w \ll d$. Sketch the intensity distribution along the x and y axes. [4 marks]

Sketch the intensity distribution in the output plane along the x and y axes. [3 marks]

A mask is placed in Fourier plane that blocks the central Gaussian 'spot'. Draw phasor diagrams corresponding to the field in the output plane at different positions along the x-axis. Show the phasor diagram corresponding to (i) a principal maximum, (ii) the first zero, (iii) another zero, and (iv) a position where the intensity is 1/4 of the maximum. [5 marks]

Sketch the modified intensity distribution in the output plane along the x axis. [2 marks]

7. The equation for a Gaussian beam with wavelength $\lambda = 2\pi/k$ and beam waist w_0 is

$$\mathcal{E}^{(z)}(x, y, z) \;=\; \mathcal{E}_0 \frac{w_0}{w} \exp(\mathrm{i}kz) \exp(-\mathrm{i}\alpha) \exp\left(\mathrm{i}k\rho^2/(2R)\right) \exp\left(-\rho^2/w^2\right) \ ,$$

where $\rho^2 = x^2 + y^2$, the beam radius $w = w_0(1 + z^2/z_{\mathrm{R}}^2)^{1/2}$, the radius of wavefront curvature $R = z + z_{\mathrm{R}}^2/z$ with $z_{\mathrm{R}} = \pi w_0^2/\lambda$, and α is the Gouy phase.

Sketch the beam radius as a function of z and indicate the wavefront curvature at $z = 0$, $z = z_{\mathrm{R}}$ and $z > z_{\mathrm{R}}$. [5 marks]

Sketch on the same axes the intensity as a function of position along the x axis at (i) $z = 0$ and (ii) $z = z_R$. Indicate w_0 on the plot. [4 marks]

Sketch on the same axes the x-component of the photon momentum probability distribution at $z = 0$ and $z = z_{\mathrm{R}}$. Relate your answer to a fundamental law. [4 marks]

A laser cavity is formed from one plane mirror and one mirror with radius of curvature R_{m}. Find an expression for the beam waist at the position of optimal stability. [5 marks]

Assuming that the beam waist remains constant draw a sketch to indicate what happens if the plane mirror is moved either (i) farther away from, or (ii) closer to the curved mirror. [2 marks]