Mathematical Methods in Physics

Examination 2015

Question 1

(a) (Unseen)

(i) Linearly dependent. [1 mark]

(ii) Linearly dependent. [1 mark]

(iii) Linearly dependent. [1 mark]

(iv) Linearly independent. [1 mark]

In order to established whether they are sets of linearly independent or dependent vectors, student can use determinant - if applicable -, inspection - for instance in (i) the third vector is the sum of the first two vectors hence they are not linearly independent - or use the definition of what a set of linearly independent vectors is.

(b) (Unseen)

(i) $A = A^\dagger, \qquad U^{-1} = U^\dagger$, then

$$(U^{-1}AU)^\dagger = (U^\dagger AU)^\dagger = U^\dagger A^\dagger U = U^{-1}AU.$$

[2 marks]

(ii) $U^{-1} = U^\dagger$ with $U = A + iB$, $A = A^\dagger, \quad B = B^\dagger$ and $AB = BA$ then

$$I = UU^\dagger = (A + iB)(A - iB) = A^2 + B^2.$$

[2 marks]

(c) (Unseen)

$$\left(\frac{ds}{dt}\right)^2 = \frac{dr}{dt} \cdot \frac{dr}{dt} = \left(1 + \frac{9}{4}t\right).$$

[2 marks]

Then

$$L = \int_0^4 \sqrt{\left(1 + \frac{9t}{4}\right)}\, dt = \frac{8}{27}\left[\left(1 + \frac{9}{4}t\right)^{3/2}\right]_0^4 = \frac{8}{27}\left(10^{3/2} - 1\right).$$

[2 marks]

(d) (Unseen)

(i) $\nabla \times \underline{F} = 0$. The field is conservative. [1 mark]

Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2xy^3z^4 \longrightarrow f = x^2y^3z^4 + g(y,z), \\
\frac{\partial f}{\partial y} &= 3x^2y^2z^4 \longrightarrow g(y,z) = g(z), \\
\frac{\partial f}{\partial z} &= 4x^2y^3z^3 \longrightarrow g(z) = c.
\end{aligned}$$

Then $f = x^2y^3z^4 + c$ where c is a constant. [2 marks]

(ii) $\nabla \times \underline{F} = (xy\,\underline{\hat{i}} - yz\underline{\hat{j}} - 2\,\underline{\hat{k}}) \neq 0$. The field is not conservative. [1 mark]

(e) (Unseen)

A suitable parametrisation for the path is:

$$\underline{r}(t) = 2\cos t\,\underline{\hat{i}} + 2\sin t\,\underline{\hat{j}}, \qquad 0 \leq t \leq \pi.$$

[2 marks]

Then

$$\begin{aligned}
I &= \int_0^\pi \underline{F} \cdot \underline{r}'\,dt = 6\int_0^\pi (-2^4\cos^4 t \sin t + 2^6 \sin^6 t \cos t)\,dt \\
&= 6\left(-2^4\left[-\frac{\cos^5 t}{5}\right]_0^\pi + 2^6\left[\frac{\sin^7 t}{7}\right]_0^\pi\right) = -\frac{192}{5}.
\end{aligned}$$

[2 marks]

(f) (Unseen)

$\nabla \times \underline{F} = -x^2\,\underline{\hat{k}} = -(a\sin\theta\cos\phi)^2\,\underline{\hat{k}}$. [1 mark]

The surface element is:

$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\right) = a\sin\theta\,\underline{r}\,d\theta d\phi.$$

[2 mark]

Then

$$I = -a^4\int_0^{2\pi} \cos^2\phi\,d\phi \int_0^{\pi/2} \sin^3\theta\cos\theta\,d\theta = -a^4\pi\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} = -\frac{a^4\pi}{4}.$$

[1 mark]

(g) (Unseen)

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{2i}\frac{1}{\sqrt{2\pi}}\int_0^\infty \left(e^{t(-\beta+i\alpha-iw)} - e^{t(-\beta-i\alpha-iw)}\right)\,dt \\
&= \frac{1}{2i}\frac{1}{\sqrt{2\pi}}\left[\frac{e^{t(-\beta+i\alpha-iw)}}{-\beta+i\alpha-iw} + \frac{e^{t(-\beta-i\alpha-iw)}}{\beta+i\alpha+iw}\right]_0^\infty = \frac{1}{\sqrt{2\pi}}\,\frac{\alpha}{\alpha^2+(\beta+iw)^2}.
\end{aligned}$$

[4 marks]

(h) (Unseen)

(i)

$$I_1 = 2\pi\left(\int_{-\infty}^{\infty}\frac{\delta(x-\pi)}{|2\pi|}\,e^{2x} + \int_{-\infty}^{\infty}\frac{\delta(x+\pi)}{|-2\pi|}\,e^{2x}\right) + 2 = e^{2\pi} + e^{-2\pi} + 2 = (2\cosh\pi)^2.$$

[3 marks]

(ii)

$$I_1 = \frac{1}{\underline{a}\cdot\underline{b}-2} = \frac{1}{3}.$$

[1 marks]

Mathematical Methods in Physics

Examination 2015

Question 2

(a) (Unseen)

(i) The eigenvalues are: $\lambda_1 = 1$, $\lambda_2 = 1 + (\alpha\beta)^{1/2}$ and $\lambda_3 = 1 - (\alpha\beta)^{1/2}$. The forms of the corresponding eigenvectors are: $v_1 = (0,0,z)^T$, $v_2 = (x, x(\beta/\alpha)^{1/2}, 0)^T$ and $v_3 = (x, -x(\beta/\alpha)^{1/2}, 0)^T$. Hence a possible choice for the eigenvectors is: $v_1 = (0,0,1)^T$, $v_2 = (1, (\beta/\alpha)^{1/2}, 0)^T$ and $v_3 = (1, -(\beta/\alpha)^{1/2}, 0)^T$. [4 marks]

(ii) The eigenvalues are real if the product $\alpha\beta$ is real and positive. The scalar product between pairs of different eigenvectors must be equal to zero in order for the eigenvectors to be orthogonal. This leads to the constraint $|\alpha| = |\beta|$. [3 marks]

(iii) The matrix A is Hermitian if $A = A^\dagger$. This leads to the constraint $\alpha^* = \beta$, which implies the constraints in (ii) since $|\alpha| = |\beta|$ and $\alpha\beta = \alpha\alpha^* = |\alpha|^2 > 0$. [3 marks]

(b) (i) (Bookwork)

The divergence theorem states that

$$\int\limits_V (\nabla \cdot \underline{F}) dV = \int\limits_S \underline{F} \cdot d\underline{S}.$$

The integral on the left is the integral of the divergence of the vector field $\underline{F}$ over the volume enclosed by the surface S. The integral of the right is the integral of the vector field $\underline{F}$ over the surface S. The symbol ∇ is a vector differential operator and $d\underline{S}$ is a differential vector perpendicular to the surface S. [4 marks]

(ii) (Unseen)

For the integral on the left $\nabla \cdot \underline{F} = 3x^2 + 2yz + 2z$. Hence the volume integral is:

$$\int_{-2}^{2} dy \int_0^3 dz \int_{-1}^{1} 3x^2\, dx + \int_{-1}^{1} dx \left(\int_0^3 dz \int_{-2}^{2} 2yz\, dy \right) + \int_{-1}^{1} dx \int_{-2}^{2} dy \int_0^3 2z\, dz$$

$$= \; 12\left[x^3\right]_{-1}^{1} + 2\left(\int_0^3 dz \left[zy^2\right]_{-2}^{2} \right) + 8\left[z^2\right]_0^3 = 24 + 72 = 96.$$

[3 marks]

For the integral on the right there are six different $d\underline{S}$, they are:

$$d\underline{S} = \pm\hat{\underline{i}}\, dydz, \quad d\underline{S} = \pm\hat{\underline{j}}\, dxdz, \quad d\underline{S} = \pm\hat{\underline{k}}\, dxdy.$$

Hence the integral becomes

$$\int_{-2}^{2} dy \int_{0}^{3} dz\, x^3 \,|_{x=1} - \int_{-2}^{2} dy \int_{0}^{3} dz\, x^3 \,|_{x=-1} + \int_{-1}^{1} dx \int_{0}^{3} dz\, y^2 z \,|_{y=2}$$
$$- \int_{-1}^{1} dx \int_{0}^{3} dz\, y^2 z \,|_{y=-2} + \int_{-1}^{1} dx \int_{-2}^{2} dy\, z^2 \,|_{z=3} - \int_{-1}^{1} dx \int_{-2}^{2} dy\, z^2 \,|_{z=0} = 96.$$

[3 marks]

Mathematical Methods in Physics

Examination 2015

Question 3

(a) (Unseen)

(i) The even extension between 0 and $-\pi$ corresponds to the function $f(x) = -\sin x$. The period is 2π. The Fourier coefficients are:

$$a_0 = \frac{1}{\pi} 2 \int_0^\pi \sin x \, dx = \frac{4}{\pi}.$$

[1 mark]

$$\begin{aligned} a_n &= \frac{1}{\pi} 2 \int_0^\pi \sin x \cos(nx) \, dx = \frac{1}{\pi} \int_0^\pi (\sin(1+n)x + \sin(1-n)x) \, dx \\ &= -\frac{1}{\pi} \frac{((-1)^{1+n} - 1)}{1+n} - \frac{1}{\pi} \frac{((-1)^{1-n} - 1)}{1-n} = -\frac{4}{\pi(n^2-1)} \qquad \text{for } n \text{ even.} \end{aligned}$$

[4 marks]

Hence

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1}.$$

[1 mark]

(ii) For $x = 0$

$$0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1},$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}.$$

For $x = \pi/2$

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

The sum of this expression and the previous one gives

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1} = \frac{1}{2} - \frac{\pi}{8}.$$

[4 marks]

(b) (Unseen)

(i)

$$h(u) = \int_0^t ue^{-3(t-u)}\,du = -\frac{1}{9} + \frac{t}{3} + \frac{e^{-3t}}{9}.$$

[3 marks]

(ii) Using the hint

$$\bar{h}(s) = -\frac{1}{9s} + \frac{1}{3s^2} + \frac{1}{9(s+3)} = \frac{1}{s^2(s+3)}.$$

On the other hand

$$\bar{f}(s) = \frac{1}{s^2}, \qquad \bar{g}(s) = \frac{1}{(s+3)},$$

hence $\bar{h}(s) = \bar{f}(s)\bar{g}(s)$. [3 marks]

(iii) The roots of the polynomial at the denominator are $0, 2, -3$. Therefore

$$\bar{m}(s) = \frac{A_1}{s} + \frac{A_2}{(s-2)} + \frac{A_3}{(s+3)},$$

with $A_1 = -1/6$, $A_2 = 3/10$ and $A_3 = -2/15$. Then

$$\mathcal{L}[\bar{m}(s)](t) = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}.$$

[4 marks]

(a)

$$x^2\ \frac{d^2y}{dx^2} +\ x\ \frac{dy}{dx} -\ 9\ y =\ 5\ x^2$$

The solution ($y \propto x^r$) to the homogeneous equation

$$x^2\ \frac{d^2y}{dx^2} +\ x\ \frac{dy}{dx} -\ 9\ y = 0$$

is

$$y = Ax^3 + Bx^{-3}$$

[2 marks, bookwork]

Given the RHS, the smart ansatz thus needs to be of the form $y\ =\ D\ x^2$ leading to $D = -1$. [2 marks, unseen]

(b)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6\ x\ e^x$$

Using $y \propto e^{p\ x}$ to solve this equation, we find $p = 1$ (double root), thus leading to

$$y = k_1\ x\ e^x + k_2\ e^x$$

[1 mark, bookwork]

The Wronskian is

$$W(x) = -e^{2x}$$

and therefore

$$k_1' = 6\ x$$

and

$$k_2' = -6\ x^2.$$

[2 marks, unseen]

Integrating, we obtain $k_1 = 3\ x^2 + A$ and $k_2 = -\ 2\ x^3 + B$ and therefore

$$y = (3\ x^2\ +\ A)\ x\ e^x + (-\ 2\ x^3\ +\ B)\ e^x$$

[1 mark, unseen]

(c)

$$(1-2x)\frac{dy}{dx} + 2y = 0$$

immediately leads to

$$y = k\ \exp\left(-\int^x \frac{dx'}{1-2x'}\right)$$

with k a constant so we obtain

$$y = k\ (1 - 2x).$$

[2 marks, unseen]

The equation

$$(1-2x)\frac{dy}{dx} + 2y + y^3 = 0$$

is a Bernoulli equation (non linear) and we can use $v = y^{3-1}$ to solve it. [2 marks, bookwork]

(d)

$$\frac{d^4x}{dy^4} - 16x = 0$$

$x \propto e^{r\ y}$ leads to $r = \pm 2$ and $r = \pm 2\,i$ and therefore

$$x = \alpha e^{2y} + \beta e^{-2y} + \gamma e^{2iy} + \delta e^{-2iy}$$

[2 marks, unseen]

A possible RHS is of the exponential form, i.e.

$$\frac{d^4x}{dy^4} - 16x = Ce^{By}$$

in which case a solution is $y = D\,e^{By}$ with $D = C/(B^4 - 16)$ but we could have also

$$\frac{d^4x}{dy^4} - 16x = f_x(y)$$

with $f_x(y) = -16\ A\ y^n$ and $n < 4$ (with $x = A\,y^n$ a solution of this equation). [2 marks, unseen]

(e) The Cartesian surface and lack of dissipation imply

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = C\frac{\partial^2\psi}{\partial t^2}$$

with C a coefficient. [2 marks, bookwork] Therefore the solution is of the form

$$y = A\cos(k(x+y) \pm wt) + B\sin(k(x+y) \pm wt)$$

[2 marks, bookwork]

(f) The generic expression of a Legendre polynomial of order n, where n is an odd number is

$$P_{l=n} = \sum_{i=odd}^{n} \alpha_i\ x^i$$

Therefore the polynomial

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{2}(5x^3-3x)$$

which contains only odd powers of x and ends at x^3 is as expected.

[4 marks,bookwork]

(g) The convention for spherical harmonics is Y_{lm} and the first ($l=0$), second ($l=1$), third ($l=2$) harmonics are called monopole (or s-wave), dipole (or p-wave), quadrupole (or d-wave). The quadrupole is a pair dipole and can therefore be represented as 2 orthogonal '8' shapes. [4 marks,bookwork]

(a) For the oscillations to be sustainable with time, must contain a second derivatives in t. Hence

$$k^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

with k a constant. [2 marks,bookwork]

(b)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial t}$$

contains a first order term $\frac{\partial u}{\partial t}$ which implies that there is dissipation with time. [2 marks,bookwork]

(c) We can solve the above equation using the separation of variables:

$$u = X(x)\, T(t)$$

[2 marks,bookwork]

(d) Using

$$u = X(x)\, T(t)$$

we find:

$$T''\, X = c^2\, T\, X'' - k\, T'\, X$$

which gives

$$\frac{T''}{T} + k \frac{T'}{T} = c^2 \frac{X''}{X}$$

once we divide by u. [1 mark,bookwork]

From the above equation, we conclude that

$$\frac{T''}{T} + k \frac{T'}{T} = sign \lambda^2 = c^2 \frac{X''}{X}$$

[1 mark,unseen] meaning that

$$c^2 \frac{X''}{X} = \lambda^2$$

and therefore

$$X = A \ \sin(\frac{\lambda}{c} x) + B \ \cos(\frac{\lambda}{c} x) \text{ if sign} = -1$$

and

$$X = A \ \sinh(\frac{\lambda}{c} x) + B \ \cosh(\frac{\lambda}{c} x) \text{ if sign} = +1$$

[2 marks,unseen]

The boundary condition $u(x=0,t) = u(x=L,t) = 0$ means that $B = 0$ and the solution must be

$$X = A \ \sin(\frac{\lambda}{c} x)$$

with $\frac{\lambda}{c} = \frac{2\,\pi}{L}$ which also selects the negative sign.

[2 marks,bookwork]

(e) Using

$$\frac{T''}{T} + k\,\frac{T'}{T} = -\lambda^2 = c^2\,\frac{X''}{X}$$

we obtain the following equation for the time dependence:

$$T'' + k\,T' + \lambda^2 T = 0$$

[1 mark,unseen]

which admits a solution in $T \propto e^{r\,t}$ [1 mark,bookwork] with

$$r_{\pm} = \frac{-k \pm \sqrt{k^2 - 4\lambda^2}}{2}$$

[2 marks,unseen]

and therefore $T \propto A\ e^{r_+\,t} + B\ e^{r_-\,t}$

(f) The vibrations are thus described by the function:

$$u \propto \ \sin(\frac{\lambda}{c} x) \left(A\ e^{r_+\,t} + B\ e^{r_-\,t}\right)$$

with

$$r_{\pm} = \frac{-k \pm \sqrt{k^2 - 4\lambda^2}}{2}$$

When $k = 2\,\lambda$, $r_{\pm} < 0$ and the vibration is exponentially damped with time.

When $k > 2\,\lambda$, the term in $e^{-(\frac{k+\sqrt{k^2-4\lambda^2}}{2})\,t}$ decreases with time while the other exponential vanishes so the solution is damped.

When $k < 2\,\lambda$, the exponentials contain an imaginary part but the oscillations are damped.

At last when $k = 0$, there is no damping, just oscillations.

[4 marks,unseen]

(a) To solve

$$x^2y'' + (2p+1)\,x\,y' + (\alpha^2 x^{2r} + \beta^2)y = 0$$

we need to use the Frobenius method where

$$y = x^\rho \sum_n a_n\, x^n$$

Inserting y and its derivatives in the equation we obtain for $n = 0$ (and lowest order x^ρ) [2 marks, bookwork]:

$$a_0[\rho(\rho-1) \;+\; (2p+1)\,\rho + \beta^2] = 0$$

which gives

$$\rho^2 + 2\,p\,\rho + \beta^2$$

[1 mark, unseen] and therefore

$$\rho_\pm = -p \pm q$$

with $q = \sqrt{p^2 - \beta^2}$ [2 marks, unseen]

Hence one solution of this equation has the form

$$y = x^{-p}[x^{-q} \sum_n a_n\, x^n].$$

[1 mark, unseen]

(b) Writing the above solution as $y = x^{-p}\; v$ and inserting the derivatives

$$y' = -\,p\,x^{-p-1}\,v \;+\; x^{-p}\,v'$$

[1 mark,bookwork]

$$y'' = p\,(p+1)\,x^{-p-2}\,v \;-2\,p\,x^{-p-1}\,v' \;+\; x^{-p}\,v''$$

[2 marks, bookwork]

we find

$$v''\; x^{-p+2} \;+\; v'x^{-p+1} \;+\; v\left(-p^2 + \beta^2 + \alpha^2\,x^{2r}\right)\; x^{-p}$$

[1 mark, unseen]

that is equivalent to

$$x^2\,v'' + x\,v' + v\;(\alpha^2\,x^{2r} - \;(p^2 - \beta^2)) = 0$$

(c) When $r = 1$ the equation reads

$$x^2 y'' + (2p+1)\, x\, y' + (\alpha^2 x^2 + \beta^2) y = 0$$

which has for solution

$$y = x^{-p} v$$

with v solution to the Bessel equation [2 marks, bookwork]

$$x^2\, v'' + x\, v' + v\, \left(\alpha^2\, x^2 - \ (p^2 - \beta^2)\right) = 0$$

that is

$$v = A\, J_q(\alpha x) \ + \ B\, Y_q(\alpha x)$$

[2 marks,bookwork]

implying that

$$y = x^{-p}\, (A\, J_q(\alpha\, x) + \ B\, Y_q(\alpha\, x))$$

(d) Multiplying by t^2 the equation for the background in presence of axions, we obtain:

$$t^2 \ddot{\phi} + \ 3\, h\, t\, \dot{\phi} + m^2\, t^2\, \phi = 0$$

[2 marks, bookwork] which can be written as

$$x^2 y'' + (2p+1)\, x\, y' + (\alpha^2 x^{2r} + \beta^2) y = 0$$

providing that $3\, h = (2p+1)$, $\alpha = m$, $r = 1$ and $\beta = 0$. [2 marks,unseen]

Therefore the solution is

$$\phi = t^{-p}\, (A\, J_p(m\, t) + \ B\, Y_p(m\, t))$$

with $p = \frac{3\, h - 1}{2}$. [2 marks,unseen]