University of Durham

EXAMINATION PAPER

May/June 2015 Examination code: PHYS3661WE01

THEORETICAL PHYSICS 3

SECTION A. Relativistic Electrodynamics
SECTION B. Quantum Theory 3

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A. RELATIVISTIC ELECTRODYNAMICS
Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Consider two 4-vectors a^μ, b^μ with $a^\mu b_\mu = 0$. Show that they cannot both be time-like. [4 marks]

   (b) Show that $a^\mu v_\mu = 0$ where a^μ is the four-acceleration and v_μ is the four-velocity of a point particle. [4 marks]

   (c) What is the speed of a particle (relative to c) if its kinetic energy is equal to twice its rest mass energy? [4 marks]

   (d) The rapidity of a particle with energy, E, and momentum in the z direction, p_z, in an inertial frame S is defined to be

$$y = \frac{1}{2}\ln\left(\frac{E + cp_z}{E - cp_z}\right).$$

   Show that in an inertial frame S' moving with velocity $\underline{v}$ in the z-direction with respect to S the rapidity

$$y' = \frac{1}{2}\ln\left(\frac{E' + cp'_z}{E' - cp'_z}\right) = y + f(\beta),$$

   where E' and p'_z are the energy and momentum in the z-direction in S', respectively. $f(\beta)$ is a function of $\beta = \frac{|\underline{v}|}{c}$ only which should be determined. [3 marks]

   If an observer in S measures the difference between the rapidities of two particles Δy, what value does an observer in S' measure? [1 mark]

   (e) A muon with velocity $\underline{v}$ collides with an antimuon with velocity $-\underline{v}$ producing a tau lepton and its antiparticle. Given that the tau lepton mass is 17 times the muon mass, what is the minimal magnitude of the velocity of the incoming muon? [4 marks]

   (f) Write down the gauge transformation of the 4-potential A^μ in contravariant form and use the transformation to show that the field strength tensor, $F^{\mu\nu}$, is gauge invariant. [4 marks]

   (g) Three identical point charges of charge q are arranged at the corners of an equilateral triangle. Within the inertial frame that moves with velocity v along one of the sides of the triangle, what value is measured for $\underline{E} \cdot \underline{B}$ in the centre of the triangle? [4 marks]

   (h) The 4-potential for a parallel-plate capacitor at rest and oriented normal to the y axis is $A^\mu = (Ey, 0, 0, 0)$ where E is the electric field strength between the plates and y the distance from the negatively charged plate. Show that, in a frame moving relative to the capacitor with velocity v in the x-direction, there is a magnetic field in the z-direction and calculate its magnitude. [4 marks]

2. Consider a point charge q moving in an inertial frame S with constant velocity v along the x-axis.

   (a) Write down the electric $\underline{E}'$ and magnetic $\underline{B}'$ fields at a position $\underline{r}'$ from the charge in the rest frame S' of the point charge. [4 marks]

   (b) The transformations of the electric $\underline{E}$ and magnetic $\underline{B}$ fields as measured in two inertial frames S and S' in the standard configuration (i.e. S' moves with velocity v along the x-axis and at $t = t' = 0$ the two frames coincide) are given by

$$E'_x = E_x; \quad E'_y = \gamma(E_y - vB_z); \quad E'_z = \gamma(E_z + vB_y);$$

$$B'_x = B_x; \quad B'_y = \gamma(B_y + \frac{v}{c^2}E_z); \quad B'_z = \gamma(B_z - \frac{v}{c^2}E_y);$$

   Use these transformation properties to compute the $\underline{E}$ and $\underline{B}$ fields of the point charge in S at the time $t = 0$ and at the point P with Cartesian coordinates $(0, b, 0)$. [12 marks]

   (c) What value is measured in the inertial frame S for $\underline{E}^2 - c^2\underline{B}^2$ at this point P at $t = 0$? [4 marks]

3. The wave 4-vector for a light wave with frequency f and wavevector $\underline{k}$ is

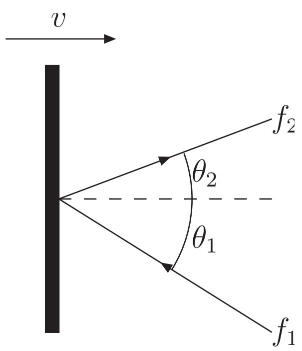
$$k^\mu = \left(\frac{2\pi f}{c}, \underline{k}\right).$$

a) Show that the plane wave $\Phi = \exp(ik \cdot x)$, where x^μ is the position 4-vector, is a solution of the wave equation $\partial^\mu \partial_\mu \Phi = 0$. [4 marks].

b) A light source in the frame S emits light with a frequency f at an angle θ with respect to the x-axis. Using the Lorentz transformation of the wave 4-vector, or otherwise, calculate the frequency measured by an observer in the frame S' moving with velocity $\underline{v}$ in the x direction with respect to S. [6 marks]

c) What is the frequency for $\theta = 0°$ and $\theta = 90°$. [2 marks]

A mirror with its normal in the x direction is at rest in S'. An observer in S sees a ray of light with frequency f_1 strike the mirror at an angle of incidence θ_1. The light is then reflected with frequency f_2 at an angle of reflection θ_2 as shown below.



d) By boosting the wave 4-vectors of the light into S' and using the normal relations for reflection in that frame, or otherwise, show that f_2/f_1 can be written in all of the following forms

$$\frac{f_2}{f_1} = \frac{\sin\theta_1}{\sin\theta_2} = \frac{c\cos\theta_1 + v}{c\cos\theta_2 - v} = \frac{c + v\cos\theta_1}{c - v\cos\theta_2}.$$

[8 marks]

4. A particle of charge q and rest mass m travels with a relativistic velocity $\underline{v}_0$ as it enters a medium, where it is slowed down by a force proportional to the velocity $\underline{v}$ of the particle at any given time, $\underline{F} = -\alpha \underline{v}$, $\alpha > 0$. In the following you should ignore the effects of radiation back reaction.

   a) Show that for motion along one direction the following holds

$$\frac{\mathrm{d}(\gamma m v)}{\mathrm{d}t} = m\gamma^3 \frac{\mathrm{d}v}{\mathrm{d}t} = m\gamma^3 a.$$

   Use this to show that after the particle has entered the medium the acceleration $\underline{a} = \frac{\mathrm{d}\underline{v}}{\mathrm{d}t}$ is related to the velocity of the particle. [3 marks]

The electromagnetic fields generated by a point charge q in vacuum and in arbitrary motion are given by

$$\underline{E}(\underline{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{R}{(\underline{R} \cdot \underline{u})^3} [(c^2 - v^2)\underline{u} + \underline{R} \times (\underline{u} \times \underline{a})],$$

$$\underline{B}(\underline{r}, t) = \frac{1}{c} \hat{\underline{R}} \times \underline{E}(\underline{r}, t),$$

where $\underline{R}$ is the vector between the point charge and the observer, $\underline{v}$ is the velocity of the point charge, $\underline{u} = c\hat{\underline{R}} - \underline{v}$, and $\underline{a}$ is the acceleration of the point charge. $\underline{R}, \underline{u}, \underline{v}$, and $\underline{a}$ are all evaluated at the retarded time. $\hat{\underline{R}}$ is a unit vector in the direction of $\underline{R}$: $\hat{\underline{R}} = \underline{R}/|\underline{R}|$. In the following you can assume that within the medium the radiation propagates as in vacuum.

   (b) Identify the electric radiation field from the equations above, and show that it is given by

$$\underline{E}_{\mathrm{rad}}(\underline{r}, t) = \frac{c}{4\pi\epsilon_0} \frac{q}{R} \frac{1}{(\hat{\underline{R}} \cdot \underline{u})^3} \left[\left(\hat{\underline{R}} \cdot \underline{a} \right) \hat{\underline{R}} - \underline{a} \right].$$

   [3 marks]

   (c) Show that the Poynting vector for the radiation fields is given by

$$\underline{S}_{\mathrm{rad}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{1}{R^2} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^6} \hat{\underline{R}},$$

   where θ is the angle between $\underline{R}$ and $\underline{v}$, and $\beta = v/c$. [4 marks]

(d) Calculate the total power radiated to infinity by the point charge at the retarded time t_{ret}, by evaluating the surface integral

$$P = \oint_S \left(\frac{\underline{R} \cdot \underline{u}}{Rc}\right) \underline{S}_{\text{rad}} \cdot \mathrm{d}\underline{a}$$

for the closed sphere of radius r, and with the Poynting vector evaluated at time t. Here, $\underline{u} = c\hat{\underline{R}} - \underline{v}$, where $\hat{\underline{R}}$ and $\underline{v}$ are evaluated at the retarded time. Check that your answer is in agreement with the general result by Liénard for the power radiated by a point charge

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(\underline{a}^2 - \left|\frac{\underline{v} \times \underline{a}}{c}\right|^2\right).$$

[5 marks]

$$\left[\text{Hint: With } \gamma = 1/\sqrt{1-\beta^2} \text{ we have } \int_{-1}^{1} \frac{1-x^2}{(1-\beta x)^5}\,\mathrm{d}x = \frac{4}{3}\frac{1}{(1-\beta^2)^3} = \frac{4}{3}\gamma^6.\right]$$

(e) Use the results from (a) and (d) to calculate the total energy emitted as electromagnetic radiation by the particle as it slows down from the initial velocity v_0 to complete rest. [5 marks]

SECTION B. QUANTUM THEORY 3

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) The spherical Bessel functions $j_l(\rho)$ are defined by

$$j_l(\rho) = (-1)^l \rho^l \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^l \frac{\sin\rho}{\rho} \, .$$

Give the explicit expression for $j_2(\rho)$. Expand $j_2(\rho)$ for small values of ρ and write down the leading term. [4 marks]

(b) State the quantum mechanical equation that describes a relativistic, spin-less, free particle moving in three dimensions. What is the name of this equation? [4 marks]

(c) The Green's function $G(\underline{k}, \underline{x})$ is defined by

$$\left[\underline{\nabla}^2 + \underline{k}^2\right] G(\underline{k}, \underline{x}) = \delta^{(3)}(\underline{x}) \, .$$

Derive the result for the Fourier transform $\tilde{G}(\underline{k}, \underline{q})$ of the Green's function $G(\underline{k}, \underline{x})$. [4 marks]

(d) State the quantum mechanical equation that describes a non-relativistic, spin-less, particle moving in three dimensions within a potential given by $U(\underline{r})$. What is the name of this equation? Write down a general ansatz for the solution in the case of a spherical potential, by separating time-dependence and radial and angular coordinates. What functions describe the angular dependence? [4 marks]

(e) Write down the definition of the density operator in terms of state vectors $|\beta\rangle$. Express the ensemble average of an operator A in terms of the density matrix. Write down the equation that governs the time evolution of the density operator. [4 marks]

(f) Write down the Bose-Einstein distribution function, the Fermi-Dirac distribution function and the Maxwell-Boltzmann distribution function. Sketch the energy dependence of these three distribution functions. [4 marks]

(g) The Dirac equation can be expressed in terms of α- and β-matrices. Express the α- and β-matrices in terms of the γ-matrices. Write down the algebra of the α- and β-matrices. How many independent γ-matrices are there? Which of the γ-matrices are hermitian and which are antihermitian? [4 marks]

6. A massive particle with spin $\underline{s}$ and mass m has a magnetic moment $\underline{\mu}$, where

$$\underline{\mu} = 2\mu_{\rm B}\underline{s}\ ,$$

with $\mu_{\rm B} = e\hbar/(2mc)$. In a magnetic field $\underline{B}$ such a particle has energy $\epsilon = -\underline{\mu}\cdot\underline{B}$. Consider a system of N identical particles with spin $s = 1$ that only interact via their spin with an external magnetic field $\underline{B}$. Assume no other interactions.

   (a) What are the possible energies of each of these particles, if the magnetic field is in the z-direction? [2 marks]

   (b) Derive the total energy of the N-particle system. [2 marks]

   (c) Derive the partition function of this system. [4 marks]

   (d) What is the probability that the spin of one of the particles is parallel or anti-parallel to the magnetic field? [4 marks]

   (e) Calculate the average magnetic moment of one of the particles, showing explicitly the dependence on the magnetic field. [4 marks]

   (f) Give an expression for the energy of the whole system using the average magnetic moment of one of the particles. [4 marks]

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7. (a) Write down the Dirac equation for an electron coupled to the electromagnetic field via minimal coupling.

[2 marks]

(b) Decompose the four-component Dirac-spinor Ψ into two two-component spinors $\tilde{\phi}$ and $\tilde{\chi}$ as

$$\Psi = \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} .$$

Insert this ansatz in the Dirac equation of part (a) and express the result in terms of the canonical momentum $\underline{\pi} = \underline{p} - e/c\underline{A}$.

[3 marks]

(c) To investigate the non-relativistic limit, split off the dominant energy dependence via

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} \phi \\ \chi \end{pmatrix} .$$

Insert this ansatz into the result of part (b) to yield an equation for both ϕ and χ.

[4 marks]

(d) What is the relation between χ and ϕ, if you assume

$$\begin{aligned} i\hbar\partial_t\chi &\ll 2mc^2\chi \, , \\ e\Phi\chi &\ll 2mc^2\chi \, ? \end{aligned}$$

[2 marks]

(e) Insert the result for χ obtained in part (d) to derive an equation for ϕ.

[3 marks]

(f) Use the identity

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \, ,$$

to simplify the equation for ϕ from part (e) and give the Pauli equation in terms of the vector potential $\underline{A}$ and the magnetic field $\underline{B}$.

[6 marks]

8. A particle of mass m and momentum $\underline{p} = \hbar \underline{k}$ is scattered by the following potential:

$$V(\underline{x}) = \frac{V_0 l_0}{x e^{\frac{x}{l_0}}} ,$$

where $l_0 > 0$ and V_0 are real constants and $\underline{x}$ is the position vector.

(a) Show that the scattering amplitude in the Born approximation can be written as

$$f^B(k,\theta) \quad = \quad -\frac{2mV_0 l_0}{\hbar^2} \frac{1}{\Delta^2 + \frac{1}{l_0^2}} , \qquad (1)$$

where the momentum transfer is denoted by $\Delta = |\underline{k} - \underline{k}'|$ and where θ is the scattering angle. [10 marks]

(b) Use the result from part (a) to calculate the differential cross section in the Born approximation in terms of the scattering angle θ. [2 marks]

(c) Calculate the total cross section in the Born approximation. Perform all integrations explicitly. [8 marks]

