University of Durham
EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2018	PHYS2581-WE01

Title:
Foundations of Physics 2A

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper.
- Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** the answers that are not to be marked.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Quantum Mechanics 2
Section B: Electromagnetism

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A: QUANTUM MECHANICS 2

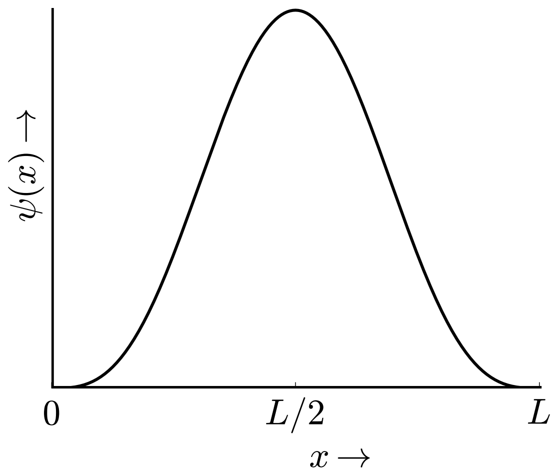
Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) A particle is described by the wavefunction

$$\psi(x) = \begin{cases} A(b^2 - x^2) & |x| < b \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of the constant, A, such that the wavefunction is correctly normalized and find the probability of finding the particle within $b/2$ of the origin. [4 marks]

(b) A particle confined to the interval $0 < x < L$ by an infinite square well has the wavefunction $\psi(x)$ shown in the figure.



This wavefunction can be expressed by the expansion

$$\psi(x) = \sum_{n=1}^{\infty} c_n \, \phi_n(x),$$

where $\phi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$ are the usual eigenfunctions of the infinite square well. Which of the coefficents c_1, c_2, c_3 and c_4 are zero? Which has the largest magnitude? Which are positive and which are negative? Give reasons for your answers. [4 marks]

(c) A particle has wavefunction

$$\psi(x) = c\,\phi_1(x) + 2c\,\phi_2(x),$$

where ϕ_n are the normalized eigenstates with corresponding energies E_n. Find the normalization constant c. [2 marks]

What is the probability of measuring E_1? [1 mark]

What happens if after measuring E_1, you repeat the measurement on the same system? [1 mark]

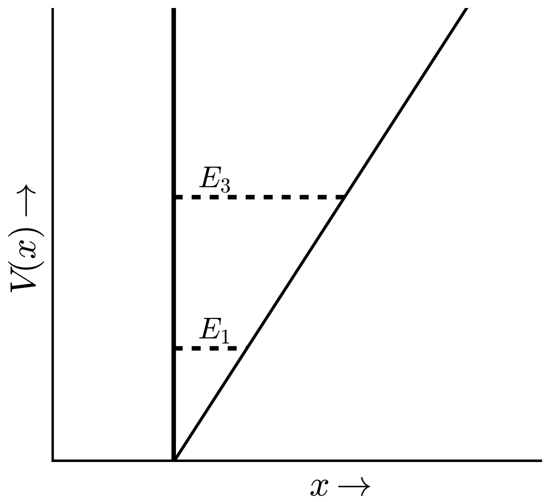
(d) The virial theorem can be written as

$$2\langle T\rangle = \langle\, \underline{r}.\underline{\nabla} V(\underline{r})\,\rangle,$$

where T is the kinetic energy and $V(\underline{r})$ the potential energy. Show that for the Coulomb potential $2\langle T\rangle + \langle V\rangle = 0$. [2 marks]

Hydrogen in its ground state has a binding energy of $E_0 = -13.6$ eV. Find the kinetic energy of the electron. [2 marks]

(e) Sketch the wavefunctions of the ground ($n = 1$, E_1) and 2nd excited state ($n = 3$, E_3) of the potential shown in the figure.



[4 marks]

(f) What property of the energy levels of the harmonic oscillator follows from the commutator $[H, a_+] = \hbar\omega a_+$, where a_+ is the raising operator, i.e. $\phi_{n+1} \propto a_+\phi_n$ and H is the Hamiltonian operator. Explain your answer. [4 marks]

(g) The radial wavefunction of an excited state of hydrogen is

$$R(r) = \frac{1}{\sqrt{24}}\, a^{-3/2} \left(\frac{r}{a}\right) \exp\left(-\frac{1}{2}\frac{r}{a}\right),$$

where a is a constant. In terms of a, find the radial distance r at which the electron is most likely to be found. [4 marks]

(h) The eigenfunctions of an infinite square well in the interval $0 < x < L$ are

$$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin(n\pi x/L).$$

The system is perturbed by an additional potential $V'(x) = \epsilon\delta(x - L/2)$, where ϵ is small and $\delta(x)$ is the Dirac delta function. Calculate the first order corrections to all the energy levels. [4 marks]

2. A system described by a Hamiltonian, H^0, has non-degenerate eigenfunctions ψ_n^0 with energies E_n^0. A small perturbation, H', to the potential, V, gives a first-order correction, $E_n^1 = \int \psi_n^{0*} H' \psi_n^0 d\mathcal{V}$ to the energy, where $d\mathcal{V}$ denotes the volume element for the integral.

   The relativistic correction to the kinetic energy of a hydrogen atom gives a first-order perturbation

$$H' = -\left[E_n^0 - V(r)\right]^2 / (2mc^2),$$

   where $V(r)$ is the potential energy of the interaction between the electron and proton, m is the mass of the electron and c is the velocity of light. This perturbation commutes with all orbital and spin angular momentum operators, so corrections to the energy levels can be calculated from non-degenerate perturbation theory.

   (a) Show that non-degenerate perturbation theory gives

$$\frac{E_n^1}{E_n^0} = \frac{\alpha^2}{4n^2} - \frac{\hbar^2}{am^2c^2}\langle r^{-1}\rangle + \frac{\hbar^4 n^2}{\alpha^2 a^2 m^4 c^4}\langle r^{-2}\rangle$$

   for a hydrogen atom state with principal quantum number n, where

$$E_n^0 = -\alpha^2 mc^2/(2n^2),$$

   $\alpha = 1/137$ is the fine structure constant, and $a = (4\pi\epsilon_0\hbar^2)/(me^2)$ is the Bohr radius. [8 marks]

   (b) Calculate E_1^1/E_1^0 for relativistic energy correction to the ground state of hydrogen, where $\psi_1^0 = (\pi a^3)^{-1/2} e^{-r/a}$. Numerically evaluate your answer given that $a = 5.29 \times 10^{-11}$ m. [12 marks]

$$\left[\text{Hint}: \int_0^\infty x^p e^{-qx} dx = \frac{p!}{q^{p+1}}\right]$$

3. A particle trapped in the region $0 < x < L$ by a one-dimensional infinite square well potential has a wave function at time $t = 0$ given by

$$\Psi(x, t = 0) = [\psi_1(x) + \psi_2(x)]/\sqrt{2},$$

where $\psi_n = \sqrt{2/L}\sin(n\pi x/L)$ are the normalized energy eigenfunctions corresponding to energies $E_n = n^2\hbar\omega$, n is a positive integer and $\omega = \pi^2\hbar/(2mL^2)$.

(a) Write down the time-dependent wavefunction, $\Psi(x, t)$, and show that the probability density, $|\Psi(x, t)|^2$, can be written as

$$|\Psi(x,t)|^2 = \frac{1}{L}\left[\sin^2(\pi x/L) + \sin^2(2\pi x/L) + 2\sin(\pi x/L)\sin(2\pi x/L)\cos(3\omega t)\right].$$

[7 marks]

(b) Calculate the expectation value of the position, $\langle x\rangle$, as a function of time. What is the frequency and amplitude of the oscillation? [8 marks]

(c) Calculate the expectation value of the energy, $\langle E\rangle$, as a function of time by explicitly evaluating the integral $\int \Psi^*(x,t)H\Psi(x,t)dx$, where H is the Hamiltonian. Explain physically why this is not time dependent. [5 marks]

$$\left[\begin{array}{c} \text{Hint: } \int_0^L x\sin^2(n\pi x/L)dx = L^2/4, \\ \int_0^L x\sin(\pi x/L)\sin(2\pi x/L)dx = -8L^2/(9\pi^2). \end{array}\right]$$

SECTION B: ELECTROMAGNETISM

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Describe what a plasma is and give two examples. [4 marks]

   (b) Explain the difference between polar and non-polar dielectrics. [4 marks]

   (c) Provide an order of magnitude estimate of the size of the electric field, at the mid-point of the straight edge of a semi-circular sheet of charge. Provide your answer in terms of the charge per unit area of the sheet, ρ_A, and the length of the straight edge, $2r_0$. [4 marks]

   (d) Show that the general vector identity

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{C}) = -\nabla^2 \underline{C} + \underline{\nabla}(\underline{\nabla}.\underline{C})$$

   is true for the specific vector field $\underline{C} = x^3 y \widehat{\underline{k}}$. [4 marks]

   (e) One of the consequences of Maxwell's equations is that charge cannot be created or destroyed. This conservation law can be written mathematically as follows:

$$\underline{\nabla}.\underline{J} = -(\frac{\partial \rho}{\partial t})$$

   where the symbols have their usual meanings. By integrating this equation or otherwise, provide an interpretation of both the left-hand side and the right-hand side and hence explain why it is a law of conservation. [4 marks]

   (f) Using standard complex notation, the wavevector, $\underline{k_0}$, for a wave is given by $\underline{k_0} = (3+2i)\hat{\underline{k}}$ m^{-1}, where $\hat{\underline{k}}$ is a unit vector that points in the direction of propagation. What is the wavelength of the wave ? [4 marks]

   (g) A cylinder of dielectric material has an electric field of 1 V m^{-1} applied along its axis by a pair of capacitor plates. This produces a net field inside the material, along its axis, of 0.2 V m^{-1}. Calculate the relative permittivity of the material. [4 marks]

5. (a) The standard expression for the electric field in the spherical polar coordinate system, at a large distance r and at an angle θ from the axis of a full-wave transmitting aerial carrying a peak current I_0 is

$$\underline{E} = -\frac{i\mu_0 c I_0}{2\pi r}\frac{\sin(\pi\cos\theta)}{\sin\theta}e^{i(kr-\omega t)}\hat{\theta}$$

where ω is the angular frequency. Sketch the variation of the amplitude of the electric field with θ on a polar diagram. [4 marks]

(b) Add to your sketch an indication of the relative directions of the electric field and the magnetic field. [4 marks]

(c) At an angle of 60 degrees above the equatorial plane, at a distance of 20 km, the amplitude of the electric field is found to be 5 V m^{-1}. Calculate the value of the peak current in the transmitting aerial. [4 marks]

(d) The wave passes through a thin sheet of a reversible weakly paramagnetic material that is also insulating. It is several wavelengths long and wide and orientated such that the electric field of the wave is in the plane of the sheet. You may assume that the atomic spacing in the magnetic material is neglibly small. Provide a snapshot sketch (i.e. a sketch at an instant in time) of the relative directions of the electric and magnetic fields of the wave as well as the currents flowing in the interior of the magnetic material. Explain the underlying physics for your sketch. [8 marks]

6. The wave equation for an electromagnetic wave propagating in vacuum is given by:

$$\nabla^2 \underline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}.$$

(a) Calculate the phase velocity of an electromagnetic pulse obeying this wave equation. [4 marks]

(b) It is known that one possible solution for the wave equation within the interior of a infinitely long hollow square metallic tube (known as a waveguide) of width a is:

$$\underline{E} = \underline{E}_0 \sin\left(\frac{6\pi x}{a}\right) \sin\left(\frac{6\pi z}{a}\right) \exp[i(ky - \omega t)],$$

where x, y and z are the conventional Cartesian coordinates, ω is the angular frequency, k is the wavevector, $\underline{E}_0$ gives the magnitude and polarisation of the wave, the bottom corner of the waveguide is located at $(x,z) = (0,0)$, and the axis of the tube is parallel to the y-axis. Given that $a = 0.1$ m, find the minimum frequency at which this particular wave can propagate along the waveguide. [6 marks]

(c) Write down another solution to the wave equation which describes a different wave that can propagate through the waveguide with a lower minimum frequency and calculate the value of the minimum frequency. [4 marks]

(d) A scientist fills the waveguide with a dieletric. For the wave you have proposed in (c), discuss how the minimum frequency and the wavelength would change. [6 marks]

