Mathematical Methods II
Weekly problem set 2

(1) A simple harmonic oscillator experiences an oscillating driving force $f(t) = ma\cos(\omega t)$. Its equation of motion is then

$$\frac{d^2x}{dt^2} + \omega_0^2 x = a\cos(\omega t). \tag{1}$$

At $t = 0$ the initial displacement and velocity are zero. The goal of this exercise is to find the function $x(t)$ that satisfies the equation of motion and the boundary condition, using two different techniques.

(a) *Trial function method.* First, solve the complementary equation. Then, find a particular solution using a trial function motivated by the structure of the inhomogeneous term. Finally, fix the integration constants using the given boundary conditions.

Solution

$$\ddot{x} + \omega_0^2 x = a\cos(\omega t)$$

Note: $\dot{x} \equiv dx/dt$ Find the auxiliary equation by substituting $x = Ae^z$ and setting RHS to zero (i.e. solve as homogeneous equation)

$$\frac{d^2}{dt^2}Ae^z + \omega_0^2 Ae^z = 0$$

$$Ae^z(\lambda^2 + \omega_0^2) = 0$$

$Ae^z = x = 0$ is trivial case, so assume $Ae^z \neq 0$, hence

$$\lambda^2 + \omega_0^2 = 0$$

Find the roots

$$\lambda = \pm i\omega_0$$

Complex roots, thus solution is of the form $\alpha \pm i\beta$ where $\alpha = 0$ and $\beta = \omega_0$

$$x_c(t) = c_1 e^{i\omega_0} + c_2 e^{-i\omega_0} = d_1 \cos\omega_0 t + d_2 \sin\omega_0 t = B\sin(\omega_0 t + \phi)$$

Find the particular integral by trying $x_p = C\cos\omega t$, since same form as inhomogeneous RHS.

$$\dot{x_p} = -C\omega\sin\omega t$$

$$\ddot{x_p} = -C\omega^2\cos\omega t$$

Sub this into original equation

$$\ddot{x_p} + \omega_0^2 x_p = -C\omega^2 \cos\omega t + \omega_0^2 C \cos\omega t = a\cos(\omega t)$$

$$C = \frac{a}{\omega_0^2 - \omega^2}$$

The Wronskian (see later lectures) can be used to show that this is an independent solution w.r.t x_c, so general solution is

$$x(t) = x_c + x_p = d_1 \cos\omega_0 t + d_2 \sin\omega_0 t + \frac{a}{\omega_0^2 - \omega^2}\cos\omega t$$

To fix the integration constants

$$x(0) = 0 \to d_2 + \frac{a}{\omega_0^2 - \omega^2} = 0 \to d_2 = -\frac{a}{\omega_0^2 - \omega^2}$$

$$\dot{x}(0) = 0 \to \omega_0 d_1 = 0 \to d_1 = 0$$

So given the boundary conditions, the general solution is

$$x(t) = \frac{a}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t)$$

(b) *Laplace transform method.* Solve Eq. (1) using the Laplace transform method. You may find the following Laplace transforms useful

$$\mathcal{L}[\sin(\alpha x)](s) = \frac{\alpha}{\alpha^2 + s^2}, \qquad \mathcal{L}[\cos(\alpha x)](s) = \frac{s}{\alpha^2 + s^2}. \tag{2}$$

Solution

$$\ddot{x} + \omega_0^2 x = a\cos(\omega t)$$

First, note the required subsitutions

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right](s) = s^2\bar{x}(s) - sx(0) - \dot{x}(0)$$

$$\mathcal{L}[x](s) = \bar{x}(s)$$

Also, since a is constant transforming the product is simple

$$\mathcal{L}[a\cos(\omega t)](s) = \frac{as}{\omega^2 + s^2}$$

Our boundary conditions give us $x(0) = 0$ and $\dot{x}(0) = 0$, so

$$s^2\bar{x} + \omega_0^2\bar{x} = (\omega_0^2 + s^2)\bar{x} = \frac{as}{\omega^2 + s^2}$$

$$\bar{x} = as\frac{1}{(\omega^2 + s^2)(\omega_0^2 + s^2)}$$

Solve using partial fractions

$$\frac{1}{(\omega^2 + s^2)(\omega_0^2 + s^2)} = \frac{A}{\omega^2 + s^2} + \frac{B}{\omega_0^2 + s^2}$$

(Note: we could start with the more general

$$\frac{1}{(\omega^2+s^2)(\omega_0^2+s^2)} = \frac{Cs+A}{\omega^2+s^2} + \frac{Ds+B}{\omega_0^2+s^2}$$

$$1 = (Cs+A)(\omega^2+s^2) + (Ds+B)(\omega_0^2+s^2)$$

but notice how there are no s terms on the LHS? So $C = D = 0$, leaving us with the partial fraction solution above)

$$\frac{\omega^2+s^2}{(\omega^2+s^2)(\omega_0^2+s^2)} = \frac{1}{(\omega_0^2+s^2)} = A + \frac{B(\omega^2+s^2)}{\omega_0^2+s^2}$$

In the limit where $s^2 \to -\omega^2$ we find that

$$A = \frac{1}{\omega_0^2-\omega^2}$$

Similarly, in the limit where $s^2 \to -\omega_0^2$ we find that

$$B = \frac{1}{\omega^2-\omega_0^2}$$

Hence

$$\frac{1}{(\omega^2+s^2)(\omega_0^2+s^2)} = \frac{1}{\omega_0^2-\omega^2}\left[\frac{1}{\omega^2+s^2} - \frac{1}{\omega_0^2+s^2}\right]$$

So

$$\bar{x}(s) = \frac{a}{\omega_0^2-\omega^2}\left[\frac{s}{\omega^2+s^2} - \frac{s}{\omega_0^2+s^2}\right]$$

which transforms using $\mathcal{L}[\cos(at)]$ to

$$x(t) = \frac{a}{\omega_0^2-\omega^2}[\cos\omega t - \cos\omega_0 t]$$

i.e. the same result.

(c) Study the behavior of $x(t)$ for $\omega \sim \omega_0$. What is the physical interpretation? *Hint:* you may find the following relation useful

$$\cos(ax) - \cos(bx) = 2\sin\left(\frac{a+b}{2}x\right)\sin\left(\frac{b-a}{2}x\right).$$

Solution

$$x(t) = \frac{a}{(\omega_0+\omega)(\omega_0-\omega)}[\cos\omega t - \cos\omega_0 t]$$

Using the hint in the question

$$x(t) = \frac{2a}{(\omega_0+\omega)}\sin\left[\frac{\omega_0+\omega}{2}t\right]\frac{\sin[(\omega_0-\omega)t/2]}{\omega_0-\omega}$$

Since for $x \to 0$ we have $\sin(x)/x \to 1$, for $\omega \to \omega_0$ we have

$$x(t) = \frac{a}{\omega_0}\sin\omega_0 t.\frac{t}{2} = \frac{at\sin\omega_0 t}{2\omega_0}$$

Thus for $\omega \to \omega_0$ the system becomes resonant and the amplitude grows with time.

(2) The hyperbolic sine is defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{3}$$

(a) Use

$$\mathcal{L}[e^{\alpha x}](s) = \frac{1}{s - \alpha} \tag{4}$$

to find the Laplace transform of $\sinh(x)$.

Solution

$$\mathcal{L}[\sinh x] = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = \frac{1}{2}.\frac{s+1-(s-1)}{(s+1)(s-1)} = \frac{1}{s^2-1}$$

(b) Using the derivative rule for the Laplace transform, compute

$$\bar{f}(s) \equiv \mathcal{L}\left[\frac{d\sinh(x)}{dx}\right]. \tag{5}$$

Solution

$$\mathcal{L}[f^{'}](s) = s\bar{f} - f(0)$$

Since $\sinh(0) = 0$

$$\mathcal{L}[d\sinh(x)/dx](s) = \frac{s}{s^2-1}$$

(c) Check that the inverse Laplace transform of $\bar{f}(s)$ Eq. (5) is the hyperbolic cosine

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \tag{6}$$

Solution

$$\frac{s}{s^2-1} = \bar{f}(s)$$

$$\bar{f}(s) = \frac{s}{(s+1)(s-1)} = \frac{1}{2}\left[\frac{1}{s+1} + \frac{1}{s-1}\right]$$

Using the formula for $\mathcal{L}[e^{ax}]$ this implies

$$f(x) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = \frac{e^{-x}+e^x}{2} = \cosh(x)$$