University of Durham

EXAMINATION PAPER

May/June 2011 Examination code: 042521/01

LEVEL 2 PHYSICS: MATHEMATICAL METHODS IN PHYSICS

SECTION A. MATHEMATICAL METHODS PART 1
SECTION B. MATHEMATICAL METHODS PART 2

Time allowed : 2 hours and 30 minutes

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 60% of the total marks for the paper. Answer **one** optional question from **each** of the two sections. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Nuclear magneton:	$\mu_{\mathrm{N}} = 5.05 \times 10^{-27}\ \mathrm{J\,T^{-1}}$
Molar Gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$

SECTION A. MATHEMATICAL METHODS PART 1

Answer question 1 and **either** question 2 **or** question 3.

1. (a) Find the determinant, eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} a & b \\ b-a & a \end{pmatrix} .$$

[4 marks]

(b) Compute the volume of the parallelipiped whose sides are given by the vectors $\underline{a} = (1, 4, 2)$, $\underline{b} = (-2, 1, 3)$ and $\underline{c} = (-1, -1, 2)$. [4 marks]

(c) At which angle do the two curves

$$\underline{f}_1(t) = (t, t^2, t^3) \qquad \text{and} \qquad \underline{f}_2(u) = (1, 1-u, \sqrt{1+u^3})$$

cross each other? [4 marks]

(d) Given the scalar field

$$\phi(x, y, z) = x^2 y^3 z^4$$

and the vector field $\underline{a}$

$$\underline{a}(x, y, z) = (-2z, y^2 - x^2, yz) ,$$

compute the quantities $\underline{\nabla} \cdot \underline{a}$ and $(\underline{a} \cdot \underline{\nabla})\phi$. [4 marks]

(e) Explain the geometrical interpretation of the gradient of a scalar field and find a normal vector $\underline{n}$ to the surface defined by the equation

$$x^2 + 2y^2 + 4z^4 = 21$$

at the point $P = (3, 2, 1)$. [4 marks]

(f) Define the Fourier transform and its inverse. [4 marks]

(g) Using the representation of the cosine function as exponentials of complex arguments

$$\cos t = \frac{\exp(it) + \exp(-it)}{2} ,$$

compute the Fourier transform of the function

$$f(t) = \cos(t) \exp(|t|) .$$

[4 marks]

(h) Compute the following integrals involving the Dirac δ-function:

$$I_1 = \int_{-5}^{5} h(x)\delta(x-4)\mathrm{d}x ,$$

$$I_2 = \int_{-3}^{3} x^3 \delta(x+2)\mathrm{d}x ,$$

$$I_3 = \int_{-2}^{2} \sin(x)\delta(x+4)\mathrm{d}x ,$$

$$I_4 = \int_{-\infty}^{\infty} g(x)\delta(x^2-4)\mathrm{d}x .$$

[4 marks]

2. (a) The Fourier series for a periodic function f with period L is given by

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} a_r \cos\left(\frac{2r\pi x}{L}\right) + \sum_{r=1}^{\infty} b_r \sin\left(\frac{2r\pi x}{L}\right).$$

Give formulae for the computation of the coefficients a_0, a_r and b_r. [4 marks]

(b) Compute the Fourier series of the function $g(x)$ given by

$$g(x) = \sin(x/2) \quad \text{for } -\pi \le x < \pi,$$

and repeating itself with period 2π. [3 marks]

[Hints: Is $g(x)$ even or odd? Does this impact on the coefficients of the Fourier series?
Use (without proof) $\int_{-\pi}^{\pi} \sin\left(\frac{x}{n}\right)\sin(rx)dx = \frac{2(-1)^r n^2 r \sin\left(\frac{\pi}{n}\right)}{1-n^2r^2}$.]

(c) Check the integral

$$\int_{-\pi}^{\pi} \cos\left(\frac{x}{n}\right)\cos(rx)dx = \frac{2(-1)^r n \sin\left(\frac{\pi}{n}\right)}{1-n^2r^2}.$$

using integration by parts and the integral in the hint for (b). [2 marks]

(d) Using (c), compute the Fourier series of

$$h(x) = -\cos(x/2) \quad \text{for } -\pi \le x < \pi\ ,$$

repeating itself with period 2π. [3 marks]

[Hint: Is $h(x)$ even or odd? Does this impact on the coefficients of the Fourier series?]

(e) Verify your result for (d) by integrating your result for (b) term by term. [4 marks]

(f) Choosing an appropriate value for x, use the Fourier series of $h(x)$ to compute the sums

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{1-4r^2} \qquad \text{and} \qquad \sum_{r=1}^{\infty} \frac{(-1)^r}{1-4(2r)^2}.$$

[4 marks]

3. Consider the surface W given by the parametric equations

$$\underline{r}(u, \theta) = (\sqrt{1+u^2}\cos\theta, \sqrt{1+u^2}\sin\theta, u) \ ,$$

for $0 \leq \theta < 2\pi$ and $-1 \leq u \leq 1$ and the two discs D_1 and D_2 of radius $\sqrt{2}$ parallel to the x-y plane with centre at positions $(0, 0, 1)$ for D_1 and $(0, 0, -1)$ for D_2.

(a) Sketch the surface W. [2 marks]

(b) Compute the surface element $d\underline{S} = \left[\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial \theta}\right] du d\theta$. [3 marks]

(c) Using the result of (b) compute the area of the surface W. [4 marks]

$$\left[\text{Hint: } \frac{d}{dx}\left(x\sqrt{1+ax^2} + \frac{1}{\sqrt{a}}\text{arcsinh}(\sqrt{a}x)\right) = 2\sqrt{1+ax^2}\right]$$

(d) State the divergence theorem and explain all the symbols you use. [4 marks]

(e) Compute the surface integral

$$\int_{D_1} \underline{a} \cdot \underline{dS} \quad \text{and} \quad \int_{D_2} \underline{a} \cdot \underline{dS}$$

of the field $\underline{a}(x, y, z) = (-2x, y - x, 1 + z)$ for the discs D_1 and D_2. [4 marks]

$$\left[\begin{array}{l} \text{Hint: Show first that } \int_{D_i} \underline{a} \cdot \underline{dS} = \int_{D_i} a_z dS \\ \\ \text{where } a_z \text{ is the } z \text{ component of } a \text{ and } i = 1, 2. \end{array}\right]$$

(f) Consider the surface T obtained by adding to W the two discs D_1 and D_2. Use the divergence theorem to compute the surface integral

$$\int_{W} \underline{a} \cdot \underline{dS}$$

for the same vector field $\underline{a}$ over W. [3 marks]

SECTION B. MATHEMATICAL METHODS PART 2
Answer question 4 and **either** question 5 **or** question 6.

4. (a) Identify the type of the differential equation,

$$\frac{dx}{dt} + 3x = 2t,$$

and solve it. [4 marks]

(b) Identify the type of the differential equation,

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 6t,$$

and solve it. [4 marks]

(c) Solve the differential equation,

$$x\frac{dy}{dx} = (x+y).$$

[4 marks]

(d) Consider the differential equation,

$$4xy'' + 2y' - y = 0,$$

where the primes denote differentiation with respect to x.
Use the series expansion $y = \sum_n a_n x^n$ to determine the coefficients a_1 and a_2 as a function of a_0. Once these coefficients have been determined, write the first three terms of the series expansion. [4 marks]

(e) Write down the heat equation and give the form of the solutions. [4 marks]

(f) Using the definition of the Laplace transform, determine the Laplace transform of $\cos(wx)$,

$$F(p) = L[\cos(wx)].$$

[3 marks]
This Laplace transform is often used in electronics. Why? [1 mark]

(g) Using the change of variables $y(x) = x^\lambda f(x)$, transform the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \lambda^2)y = 0.$$

[2 marks]
Deduce the solution when $\lambda = -1/2$ and the initial condition is $f'(0) = 0$. [2 marks]

5. The radial component of an electric field in an optical fibre, $f(r, \phi)$, obeys the wave equation (written in cylindrical coordinates r, ϕ)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r,\phi)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f(r,\phi)}{\partial^2 \phi} + \kappa^2\ f(r,\phi) = 0,$$

where κ is a constant.

(a) Use the method of separation of variables $f(r, \phi) = R \times \Phi$ (where you need to specify the variables of R and Φ respectively) to transform this equation. [2 marks]

(b) Write the equation obtained in (a) in the form

$$\Phi'' + X(r)\ \Phi = 0,$$

where $X(r)$ is a function of r and show that $X(r)$ can be written as $X(r) = R^{-1}\,(r^2 R'' + rR' + R\kappa^2 r^2)$. [2 marks]

(c) Solve $\Phi'' + X(r)\ \Phi = 0$. Find an expression for Φ. [2 marks]

(d) Using (c), write the equation that $R(r)$ satisfies. Name this equation and give an expression for R. [4 marks]

(e) Give an expression for $f(r, \phi)$. [2 marks]

(f) Solve $X(r) = \alpha^2$ by using a power series expansion for $R = \sum_n a_n r^{n+s}$. What is the recurrence relation between the coefficient of the series? [4 marks]

(g) Identify the two expressions for R that you obtained in (d) and (f). What information does this give? [2 marks]

What is the "first" zero of the R function? [2 marks]

6. Consider the differential equation,

$$y'(x) + \int_0^x y(t)dt = \sin(\omega x),$$

that satisfies the initial condition, $y(0) = 1$, where ω is a constant and the prime denotes differentiation with respect to x.

(a) Laplace transform this equation, denoting the Laplace transform of $y(x)$ as $Y(p)$. [8 marks]

(b) Using (a), show that

$$Y(p) = \frac{pw}{(p^2+\omega^2)(p^2+1)} + \frac{p}{p^2+1}.$$

[2 marks]

(c) Find the Laplace transform of $\cos(at)e^{st}$. [4 marks]

(d) Find $L^{-1}[Y(p)]$ where $Y(p)$ is the result in (b). [4 marks]

(e) Take the derivative of $y'(x) + \int_0^x y(t)dt = \sin(\omega x)$ and assume $\omega = 0$. Solve this differential equation with the boundary conditions: $y(0) = 1$ and $y'(0) = 0$. [2 marks]