Mathematical Methods in Physics

Examination 2016

Question 1

(a) (Unseen)

(i) $a_{1i}b_i = a_{11}b_1 + a_{12}b_2 + a_{13}b_3 = 2.$ [1 mark]

(ii) $a_{ji}a_{i1}b_j = (a_{ji}b_j)\, a_{i1} = (a_{1i}b_1 + a_{2i}b_2 + a_{3i}b_3)\, a_{i1} = 11 + 8 + 30 = 49.$ [3 marks]

(b) (Unseen)

(i) Yes. Zero element: $(0, 0, 0)$. Inverse element: $(-x_1, -x_2, 0)$. Closed with respect to both operations. [2 marks]

(ii) No. Zero element: $(0, 0, 0)$ not in V. Inverse element: $(-x_1, -x_2, -1)$ not in V. [2 marks]

(c) (Unseen)

It is neither symmetric nor antisymmetric. [1 mark]

The matrix is orthogonal since $AA^T = I$. [1 mark]

Since it is orthogonal the inverse matrix exists and $A^{-1} = A^T$. Therefore, it is not singular. [2 marks]

Note that students could have calculated the determinant in order to find out whether the matrix is singular or not. The determinant is $+1$.

(d) (Bookwork/Unseen)

(i) $\bar{f}(s) = \int\limits_0^\infty H(t-4)\, e^{-st}dt = \int\limits_4^\infty e^{-st}dt = e^{-4s}/s.$ [2 marks]

(ii) $\bar{f}(s) = \int\limits_0^\infty t^2\delta(t-2)\, e^{-st}dt = 4\, e^{-2s}.$ [2 marks]

(e) (Unseen)

The function is odd. Hence $a_n = 0$ for all n. [1 mark]

$$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} x\sin(nx)\, dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\frac{\cos(nx)}{n}\, dx = -\frac{2\,\pi(-1)^n}{n}.$$

[2 marks]

Therefore

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

[1 mark]

(f) (Unseen)

$$\begin{aligned} \underline{a}(\underline{r}(u)) &= (u+u^3)\,\hat{\underline{i}} + u^3\,\hat{\underline{j}} + (u-u^2)\,\hat{\underline{k}}, \qquad \frac{d\underline{r}}{du} = \hat{\underline{i}} + 2u\,\hat{\underline{j}} + 3u^2\,\hat{\underline{k}}. \\ \underline{a}\cdot\frac{d\underline{r}}{du} &= u + 4u^3 - u^4. \end{aligned}$$

[2 marks]

$$I = \int_0^2 \underline{a}\cdot(d\underline{r}/du)\,du = \int_0^2 (u+4u^3-u^4)\,du = [u^2/2 + u^4 - u^5/5]_0^2 = 58/5.$$

[2 marks]

(g) (Unseen)

(i) $\nabla f = e^{xyz}(yz\,\hat{\underline{i}} + xz\,\hat{\underline{j}} + xy\,\hat{\underline{k}})$. [2 marks]

(ii) $\nabla f = f'(r)\nabla r = -2\underline{r}/(2+r^2)^2$. [2 marks]

(h) (Unseen)

$$\begin{aligned} \frac{\partial\underline{r}}{\partial u} &= v\cos\phi\,\hat{\underline{i}} + v\sin\phi\,\hat{\underline{j}} + u\,\hat{\underline{k}}, \\ \frac{\partial\underline{r}}{\partial v} &= u\cos\phi\,\hat{\underline{i}} + u\sin\phi\,\hat{\underline{j}} - v\,\hat{\underline{k}}, \\ \frac{\partial\underline{r}}{\partial\phi} &= -uv\sin\phi\,\hat{\underline{i}} + uv\cos\phi\,\hat{\underline{j}} - v \end{aligned}$$

[1 mark]

(Bookwork)

$dV = \left|\frac{\partial\underline{r}}{\partial u}\cdot\left(\frac{\partial\underline{r}}{\partial v}\times\frac{\partial\underline{r}}{\partial\phi}\right)\right| dudvd\phi.$ [1 mark]

Note that expressions for dV obtained by any permutation of the vectors is also fine.

(Unseen)

$$\begin{aligned} \left(\frac{\partial\underline{r}}{\partial v}\times\frac{\partial\underline{r}}{\partial\phi}\right) dvd\phi &= (uv^2\cos\phi\,\hat{\underline{i}} + uv^2\sin\phi\,\hat{\underline{j}} + u^2v\,\hat{\underline{k}})\,dvd\phi \\ dV &= uv(u^2+v^2)\,dudvd\phi. \end{aligned}$$

[2 marks]

Mathematical Methods in Physics

Examination 2016

Question 2

(a) (Unseen)

Since $(AB - BA) = 2iC$ then $(AB - BA)(AB - BA) = -4C^2$. Expanding this expression and using the properties of the matrices A and B we have $-4 = -4C^2$ i.e. $C^2 = 1$

[4 marks]

Take $(AB - BA) = 2iC$. Multiply on the left by B. You get: $-2A = 2iBC$. Then multiply the original expression on the right by B. You get: $2A = 2iCB$. Subtract the two expressions obtained. Then $BC - CB = 2iA$.

[4 marks]

(b) (Unseen)

Eigenvalues: $\lambda_1 = \lambda_2 = 1$. Eigenvectors: $\underline{x}^T = (2y, y)$. [3 marks]

The matrix A is not diagonalisable because it does not exist a basis of eigenvectors. In other words, there are not two eigenvectors which are linearly independent.

[3 marks]

(c) (Bookwork)

$\hat{h}(\omega) = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$. [1 mark]

(Unseen)

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\delta(t+2) + \delta(t-2))\, e^{-i\omega t}\, dt = \sqrt{\frac{2}{\pi}}\, \cos 2\omega.$$

[2 marks]

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} 2\, e^{-i\omega t}\, dt = 2\sqrt{\frac{2}{\pi}} \frac{\sin 2\omega}{\omega}.$$

[2 marks]

Hence $\hat{h}(\omega) = 2\sqrt{2/\pi}\, \sin(4\omega)/\omega$. [1 mark]

Mathematical Methods in Physics

Examination 2016

Question 3

(a) (Bookwork)

Gauss' theorem: $\int_V (\nabla \cdot \underline{a})\, dV = \int_S \underline{a} \cdot d\underline{S}$. [2 marks]

(Unseen)

On the left hand side.

$\nabla \cdot \underline{a} = 2(1+z)$. [1 mark]

Then

$$\int_V (\nabla \cdot \underline{a})\, dV = \int_0^{2\pi} d\phi \int_0^2 \rho\, d\rho \int_2^4 2(1+z)\, dz = 64\pi.$$

[3 marks]

On the right hand side.

On the cylinder surface S_1 :

$\underline{r}_1 = 2\cos\phi\, \hat{\underline{i}} + 2\sin\phi\, \hat{\underline{j}} + z\, \hat{\underline{k}}$. [1 mark]

$$\begin{aligned} d\underline{S}_1 &= \left(\frac{\partial \underline{r}_1}{\partial \phi} \times \frac{\partial \underline{r}_1}{\partial z}\right) d\phi\, dz = (2\cos\phi\, \hat{\underline{i}} + 2\sin\phi\, \hat{\underline{j}})\, d\phi\, dz, \\ \underline{a}(\underline{r}_1) &= 2\cos\phi\, \hat{\underline{i}} + 2\sin\phi\, \hat{\underline{j}} + z^2\, \hat{\underline{k}}, \qquad \underline{a} \cdot d\underline{S}_1 = 4\, d\phi\, dz. \end{aligned}$$

$I_1 = \int_o^{2\pi} d\phi \int_2^4 4\, dz = 16\pi$. [4 marks]

On the surface S_2 of the circle at $z = 2$:

$$\underline{r}_2 = \rho\cos\phi\, \hat{\underline{i}} + \rho\sin\phi\, \hat{\underline{j}} + 2\, \hat{\underline{k}}, \qquad d\underline{S}_2 = \left(\frac{\partial \underline{r}_2}{\partial \phi} \times \frac{\partial \underline{r}_2}{\partial \rho}\right) d\rho\, d\phi = -\rho\, \hat{\underline{k}}.$$

$I_2 = \int_0^{2\pi} d\phi \int_0^2 d\rho (\underline{a}(\underline{r}_2) \cdot d\underline{S}_2) = -\int_0^{2\pi} d\phi \int_0^2 (4\rho) d\rho = -16\pi$. [3 marks]

Similarly, on the surface S_3 of the circle at $z = 4$:

$$d\underline{S}_3 = \rho\,\underline{\hat{k}}, \qquad \underline{a}(\underline{r}_3)\cdot d\underline{S}_3 = 16\rho.$$

$I_3 = \int\limits_0^{2\pi} d\phi \int\limits_0^{2} (16\rho) d\rho = 64\pi$. It follows that $I_1 + I_2 + I_3 = 64\pi$. [2 marks]

(b) (Unseen)

Because of the Gauss'theorem

$$\frac{1}{3}\int\limits_S \underline{r}\cdot d\underline{S} = \frac{1}{3}\int\limits_V (\nabla\cdot\underline{r})\,dV, \quad \text{with} \quad \partial V = S.$$

[2 marks]

It follows that

$$\frac{1}{3}\int\limits_{\partial V} \underline{r}\cdot d\underline{S} = \frac{1}{3}\int\limits_V 3\,dV = \int\limits_V dV.$$

[2 marks]

(a) We are looking for a generic solution y such that the combination of the 3 terms x^2y'', $x\ y'$ and y eventually cancel out for any value of x.

The terms x^2y'' and $x\ y'$ respectively lead to $x^2\ y$ and $x\ y$ (which implies 3 different powers of x in the equation). Hence $y = e^{rx}$ cannot be a general solution to this equation. [2 marks, seen]

The right solution is $y = x^r$ since $x^2 \frac{d^2}{dx^2} y] \propto y$, thus leading to $r = \pm 1$ and therefore

$$y = Ax + \frac{B}{x}.$$

[2 marks, seen]

(b) The number of individuals of species X is a function of t and T. Hence the equation which governs the evolution of the number of individuals given the time and temperature dependence is a partial equation and must be in its simplest form of the kind

$$\frac{\partial n}{\partial t} = sign_1 \frac{\partial^2 n}{\partial T^2}$$

[2 marks, seen]

with $n \equiv n(t, T)$. Using the separation of variables $n = \hat{t}(t)\ \hat{T}(T)$, we have:

$$\frac{1}{\hat{t}(t)} \frac{\partial \hat{t}(t)}{\partial t} = sign_1 \frac{1}{\hat{T}(T)} \frac{\partial^2 \hat{T}(T)}{\partial T^2} = sign_2\ m^2$$

so

$$\hat{t}(t) = A\ e^{sign_2\ m^2\ t}$$

and

$$\hat{T}(T) = A\ e^{\sqrt{\frac{sign_2}{sign_1}}\ m\ T} + B\ e^{-\sqrt{\frac{sign_2}{sign_1}}\ m\ T}$$

which leads us to deduce that $sign_2/sign_1$ must be negative (that is $\hat{T}(t) = C\sin(mT) + D\cos(mT)$). Indeed for $D = 0$, one can satisfy the initial condition $n(\hat{T} = 0) = 0$. So either $sign_2 = -1$ and $sign_1 = +1$ or $sign_2 = +1$ and $sign_1 = -1$. The first case causes the extinction of the species with time. The second would imply an increase with time while, for some values of the temperature, it would go extinct. This second case is not really physical.

[2 marks, unseen]

(c) To use the Wronskian method we first need to solve the homogeneous equation:

$$3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

which leads to

$$y = \ k_1\ e^x \ + \ k_2\ e^{x/3}$$

and the Wronskian:

$$W(x) = -\frac{2}{3}\ e^x\ e^{\frac{x}{3}} \quad [1 \text{ mark, unseen}]$$

Hence the constants are

$$k_1' = -\frac{e^x}{3\ (-2/3)e^x\ e^{\frac{x}{3}}} y_2 = \frac{1}{2\ e^{\frac{x}{3}}} e^{x/3} = \frac{1}{2}$$

and

$$k_2' = \frac{1}{-2\ e^{\frac{x}{3}}} e^x = -\frac{1}{2} e^{2x/3}$$

That is

$$k_1 = \frac{x}{2} + k_3 \quad [1 \text{ mark, unseen}]$$

and

$$k_2 = -\frac{3}{4} e^{2x/3} + k_4 \quad [1 \text{ mark, unseen}]$$

leading to the solution

$$y = (\frac{x}{2} + k_3)\, e^x + (-\frac{3}{4} e^{2x/3} + k_4)\, e^{x/3} \quad [1 \text{ mark, unseen}]$$

(d) To describe the variations of temperature on the surface of the Earth, we can use spherical harmonics. Indeed the radius is fixed and the Earth being a sphere the variations of temperature only depend on the 2 spherical angles θ and ϕ. Hence we can write

$$T = \sum_l \sum_m T_{ml}\ P_l^m(\cos\theta)\ e^{i\,m\,\phi}$$

[2 marks, seen]

To determine the size of the regions where the temperature varies, one can use correlation functions

$$c(n_1, n_2) = \langle T(n_1)\ T(n_2)\rangle.$$

[1 mark, seen]

The CMB is another physical quantity well described by spherical harmonics as the maps of the light coming from the Big Bang is located at equal distance from us in any direction. [1 mark, seen]

(e) The equation

$$(1-x^2)y'' - 2xy' + 2y = 0$$

is a Legendre equation and the last term that corresponds to $-vy = -l(l+1)y$ indicates that $l = 1$. Hence the solution is a Legendre polynomial P_1. [1 mark, seen]

We thus have $P_1(x) = a_1\ x$. [1 mark, seen]

To determine a_1 we can use the normalisation condition, that is $a_1 x_{|x=1} = 1$ so $a_1 = 1$. [1 mark, seen]

When $x = \cos\theta$, we have $P_1(\cos\theta) = \cos\theta$, which is either positive from $\theta \in [0, \pi/2]$ and $[3\pi/2, 2\pi]$ or negative for the complementary θ values. So this gives a dipole. [1 mark, seen]

(f) The time dependent Schrödinger equation for a free particle reads

$$H\psi = E\psi$$

that is $\frac{p^2}{2m}\psi = E\psi$. Using the substitution $p \to i\hbar\partial_x$ and $E \to i\hbar\partial_t$ we have

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = i\hbar\partial_t\psi.$$

[2 marks, seen]

The massless KG equation reads

$$\frac{\partial^2}{\partial t^2}\psi = \frac{\partial^2}{\partial x^2}\psi$$

with $\psi = \psi(t,x) = \hat{t}\ \hat{X}$ so we have

$$\frac{1}{\hat{t}}\frac{\partial^2}{\partial t^2}\hat{t} = \frac{1}{\hat{X}}\frac{\partial^2}{\partial x^2}\ \hat{X} = sign\ m^2$$

leading to

$$\hat{t} = A\ e^{\sqrt{sign}\ m\ t} + B\ e^{-\sqrt{sign}\ m\ t}$$

and

$$\hat{X} = A\ e^{\sqrt{sign}\ m\ x} + B\ e^{-\sqrt{sign}\ m\ x}$$

If the sign is negative then we have a plane wave solution, like for Schrödinger.

[2 marks, unseen]

(g) To obtain a logarithmic solution of the equation

$$A\,x^2\,y'' + 8\;x\;y' + y = 0$$

we need the discriminant of

$$A\,r(r-1) + 8\;r + 1 = 0\;\;\Rightarrow A\,r^2 + (8-A)\;r + 1 = 0$$

to be zero (since this is an Euler equation and the solution has the form $y \propto x^r$). [1 mark, seen]

Hence

$$\Delta = (8-A)^2 - 4A = 0 \Rightarrow A^2 - 20A + 64 = 0$$

that is $A = 4$ or $A = 16$. [1 mark, unseen]

Since the discriminant is null the solution can be written as

$$y = x^{-1/2}\;(A\ln x + B).$$

[2 marks, seen]

(a) The differential equation

$$x^2y'' + x(p+1)\ y' + 2y = 0$$

is an Euler equation and can be solved by postulating that

$$y = x^r$$

is the solution.

This leads to the characteristics equation

$$r^2 + r\ p + 2 = 0$$

[1 mark, seen] which has two solutions

$$r_\pm = (-p \pm \sqrt{p^2 - 8})/2$$

[2 marks, unseen]

and lead to

$$y = Ax^{r_+} + Bx^{r^-}$$

[1 mark, seen]

(b) Writing a possible solution as

$$y = \sum a_n x^{n+\rho}$$

we have

$$y' = \sum_n a_n(n+\rho)\ x^{n+\rho-1}$$

and

$$y'' = \sum_n\ a_n\ (n+\rho)\ (n+\rho-1)\ x^{n+\rho-2}$$

[2 marks, seen]

and so the equation becomes

$$x^2 \sum_n\ a_n\ (n{+}\rho)\ (n{+}\rho{-}1)\ x^{n+\rho-2} + x\ (p{+}1)\ \sum_n a_n(n{+}\rho)x^{n+\rho-1}{+}2\ \sum a_n x^{n+\rho} = 0$$

that is

$$\sum_n a_n\,(n+\rho)\,(n+\rho-1)\,x^{n+\rho} + (p+1)\sum_n a_n(n+\rho)x^{n+\rho} + 2\sum a_n x^{n+\rho} = 0$$

For $n = 0$, the lowest power of x is x^ρ [1 mark, seen] so we have:

$$a_0\; x^\rho\,[\rho\;(\rho-1) + \;(p+1)\,\rho + 2] = 0$$

[1 mark, unseen]
leading to

$$a_0\; x^\rho\,\left[\rho^2 + \;p\;\rho\;\; + 2\right] = 0$$

[1 mark, unseen]
which is the same as the previous characteristic equation. Hence

$$\rho_\pm = (-p \pm \sqrt{p^2 - 8})/2$$

[1 mark, unseen]

(c) We now have

$$y = \sum a_n x^{n+\rho}$$

with

$$\rho_\pm = (-p \pm \sqrt{p^2 - 8})/2.$$

Let us choose the smallest value for ρ. This gives

$$y = \sum a_n\; x^{n-\,p/2\,-\,\sqrt{p^2-8}/2}$$

So for our solution to be the same as in (a) we need to require either

$$n = 0$$

or

$$n + \rho_- = \rho_+$$

that is

$$n = \sqrt{p^2 - 8},$$

assuming values of p such that n is an integer. [2 marks, unseen]

(d) To solve

$$x^2 y'' + 4\,x\;y' + 2y = e^x$$

we can use the result obtained above for the homogeneous equation, with $p = 3$. This leads to $\rho = -1, -2$ and $r = -1, -2$. The solution is therefore

$$y = \frac{A}{x} + \frac{B}{x^2}$$

[1 mark, seen]

To find the solution of the inhomogeneous equation we can now use the Wronskian.

$$W(x) = \left[\begin{array}{cc} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{array}\right] = \frac{1}{x^4}$$

[2 marks, seen]

We then had

$$k_1' = -e^x x$$

leading to

$$k_1 = -xe^x + e^x + k_3$$

and

$$k_2' = e^x$$

leading to

$$k_2 = e^x + k_4$$

[2 marks, unseen]

So finally the solution is

$$y = \frac{k_3}{x} + \frac{k_4}{x^2} + \frac{e^x}{x^2}$$

[1 mark, unseen]

(e) Another method to solve

$$x^2 y'' + x(p+1)\;y' + 2y = 0$$

is to use a change of variables and use $x = e^t$. This has the advantage to transform the above Euler equation into an equation with constant coefficients that it is easy to solve. [2 marks, seen]

(a) By setting $h = r/R$ and $x = \cos\theta$ we immediately find

$$V = \left(1 - 2hx + h^2\right)^{-1/2}$$

[2 marks, unseen]

(b) We can take the derivative of

$$V = \left(1 - 2hx + h^2\right)^{-1/2}$$

remembering that V is a function of two variables h, x and so
$dV = \frac{\partial V}{\partial h} dh + \frac{\partial V}{\partial x} dx$.
[1 mark, seen]
This leads to

$$\frac{\partial V}{\partial h} = \left(1 - 2hx + h^2\right)^{-3/2} (x - h)$$

and

$$\frac{\partial V}{\partial x} = h \left(1 - 2hx + h^2\right)^{-3/2}$$

[2 marks, unseen]
Noticing that

$$\frac{\partial V}{\partial h} = \left(1 - 2hx + h^2\right)^{-3/2} (x - h) = \left(1 - 2hx + h^2\right)^{-1} \; V \; (x - h)$$

we then have

$$\left(1 - 2hx + h^2\right) \; \frac{\partial V}{\partial h} = \left(1 - 2hx + h^2\right)^{-3/2} (x - h) = V \; (x - h).$$

[1 mark, unseen]
The potential thus written is actually the generating function of Legendre polynomials.
[2 marks, seen]

(c) We can now use the decomposition

$$V = \sum_l h^l X_l(x)$$

where $X_l(x)$ are the coefficients. [1 mark, seen]

(d) Plugging in the decomposition into the equation for $\frac{\partial V}{\partial h}$, we obtain:

$$\left(1 - 2hx + h^2\right) \sum_{l=1} l\ h^{l-1} X_l(x) = (x - h) \sum_l h^l X_l(x)$$

[1 mark, seen]

Reindexing, we obtain:

$$\left(1 - 2hx + h^2\right) \sum_{l=0} (l+1)\ h^l X_{l+1}(x) = (x - h) \sum_l h^l X_l(x)$$

[2 marks, unseen]

Hence we obtain for the highest power h^{l+2}:

$$(l+3)X_{l+3} - 2x(l+2)X_{l+2} - xX_{l+2} + (l+1)X_{l+1} + X_{l+1} = 0$$

[2 marks, seen]

(e) The equation

$$(l+3)X_{l+3} - 2x(l+2)X_{l+2} - xX_{l+2} + (l+1)X_{l+1} + X_{l+1} = 0$$

can be rewritten as

$$(l+2)X_{l+2} - 2x(l+1)X_{l+1} - xX_{l+1} + lX_l + X_l = 0$$

leads for $l = 0$ to

$$2X_2 - 3xX_1 + X_0 = 0$$

[1 mark, unseen] and therefore :

$$X_2 = \frac{1}{2}(3x^2 - 1)$$

[1 mark, seen]

While for $l = 1$:

$$3X_3 - 5xX_2 + 2X_1 = 0$$

[1 mark, unseen]

leading to

$$X_3 = \frac{1}{2}(5x^3 - 3x)$$

[1 mark, unseen]

Note that one gets the same results by starting with

$$\left(1 - 2hx + h^2\right) \sum_{l=1} l\ h^{l-1} X_l(x) = (x - h) \sum_l h^l X_l(x)$$

and retaining the coefficients in h and h^2 so I will accept this solution too.

(f) The X_l are Legendre polynomials and V is their generating function. [2 marks, unseen]