Mathematical Methods in Physics

Weekly Problems 3. Solution

3.1

(i) A is Hermitian, (ii) A is not anti-Hermitian, (iii) $|A| = 0$, hence A is singular, (iv) since A is singular it does not have an inverse. Hence its adjoint cannot be its inverse. The matrix A is not unitary. $\boxed{\text{2 mark}}$

3.2

The eigenvalues are $\lambda_1 = 3 + i$ and $\lambda_2 = 3 - i$.

The eigenvectors have the form $\mathbf{x}_1 = \begin{pmatrix} (-1+i)x \\ x \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} (-1-i)x \\ x \end{pmatrix}$.

A possible choice is $\mathbf{x}_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$. $\boxed{\text{2 marks}}$

Then $D = \begin{pmatrix} 3+i & 0 \\ 0 & 3-i \end{pmatrix}$, $S = \begin{pmatrix} -1+i & -1-i \\ 1 & 1 \end{pmatrix}$, $S^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1-i \\ i & 1+i \end{pmatrix}$. $\boxed{\text{2 marks}}$

In order to find the matrix S^{-1} you can use the Gauss-Jordan method or remember the following simple formula, which holds for a (2×2) matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

3.3

a) The matrix B is symmetric. The eigenvalues are $\lambda_{1/2} = 0$ (twice degenerate) and $\lambda_3 = -2$.

The eigenvectors have the form $\mathbf{x}_{1/2}^T = (x, x, z)$ with x and z arbitrary and $\mathbf{x}_3^T = (x, -x, 0)$. A set of orthonormal vector is $\mathbf{x}_1^T = (1,1,0)/\sqrt{2}$, $\mathbf{x}_2^T = (0,0,1)$, $\mathbf{x}_3^T = (1,-1,0)/\sqrt{2}$. $\boxed{\text{2 marks}}$

b) The matrix S that diagonalizes B is

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

The matrix S is unitary (actually orthogonal since it is real). Therefore

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \qquad \boxed{\text{2 marks}}$$