CM5: Orbiting electrons

In a classical model of a multi-electron atom, electrons of mass m are assumed to move in a modified electrostatic potential, with a potential energy given by

$$V(r) = -\frac{k}{r}e^{-r/a},$$

where k and a are positive constants and r is the distance from the centre of the atom.

1. Using spherical polar coordinates (r, θ, ϕ), write down an expression for the kinetic energy, T. Show that, for motion in a plane the Lagrangian can be written as

   $$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{k}{r}e^{-r/a},$$

   explaining your reasoning.

2. Noting that ϕ is an ignorable coordinate, determine the associated constant of motion, J, and show that the total energy may be written as $E = (m/2)\dot{r}^2 + V_{\text{eff}}(r)$, where

   $$V_{\text{eff}}(r) = \frac{J^2}{2mr^2} - \frac{k}{r}e^{-r/a}.$$

3. Sketch $V_{\text{eff}}(r)$ for the cases where a is either very large or very small.

4. Determine the condition on r/a for a stable circular orbit to exist.