University of Durham

EXAMINATION PAPER

May/June 2014 Examination code: 043651/01

LEVEL 3 PHYSICS: PLANETS AND COSMOLOGY 3

SECTION A. Cosmology
SECTION B. Planetary Systems

Time allowed: 3 hours

Examination material provided: None

Calculators: The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. COSMOLOGY

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) For a Hubble parameter of $H_0 = 68$ km s^{-1} Mpc^{-1}, by how much will the physical separation of two galaxies currently 10 Mpc apart change in 100 yr due to the Hubble flow? Give your answer in parsec. [2 marks]

   If one of these galaxies emits an emission line at a rest wavelength of 500.7 nm, at what wavelength will the line be observed by the other galaxy? [2 marks]

$$[1 \text{ yr} = 3.16 \times 10^7 \text{ s}]$$

   (b) The Friedmann equation for a Universe containing non-relativistic matter of density ρ and cosmological constant Λ is

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3},$$

   where a is the expansion factor, H is the Hubble parameter and k is a constant.

   For arbitrary time, derive an expression for the density parameter Ω_Λ in terms of Λ and H. [2 marks]

   If, at the present time, the density parameter in matter is $\Omega_{\rm M,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$, evaluate the redshift at which $\Omega_{\rm M} = \Omega_\Lambda$. [2 marks]

   (c) If the energy density in the 2.73 K cosmic microwave background (CMB) radiation is currently 1.5×10^{-4} times that in ordinary non-relativistic matter, calculate the temperature of the CMB and the redshift at which the combined energy density in CMB and relativistic particles equals that in non-relativistic matter. [4 marks]

   [Hint: the present-day energy density in relativistic particles is 0.68 times that of the CMB]

   (d) Evaluate the comoving distance to a galaxy at $z = 1$ in a critical density universe that is dominated by relativistic particles. For such a universe, the Friedmann equation is

$$H^2 = H_0^2 a^{-4}$$

   where H is the Hubble parameter and a is the expansion factor. Assume that H_0, the value of the Hubble parameter at the present day, is $75\,\mathrm{kms^{-1}Mpc^{-1}}$. [4 marks]

   (e) Assume that the Universe is flat and dominated by non-relativistic matter such that the comoving distance, r, is related to redshift, z, by

$$r = \frac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right]$$

   where c is the speed of light and $H_0 = 75\,\mathrm{kms^{-1}Mpc^{-1}}$ is the present-day value of the Hubble parameter. Calculate the physical diameter of a galaxy observed at $z = 0.7$ to have an angular diameter of 0.3 arcseconds. [4 marks]

(f) Define the deceleration parameter q_0. [2 marks]
Using the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

(where a is the expansion factor, ρ is the density and P is the pressure), show that, in a universe containing radiation and non-relativistic matter (but no cosmological vacuum energy),

$$q_0 = \Omega_\gamma + \Omega_m/2$$

where Ω_γ and Ω_m are the density parameters of radiation and non-relativistic matter. [2 marks]

(g) At the end of the quark era, the temperature of the Universe is $T \sim 10^{12}K$, and the energy density of the relativistic plasma is

$$\epsilon = \frac{g_*}{2}\frac{4\sigma T^4}{c}.$$

Assuming that the effective number of Bosonic degrees of freedom, g_*, is 100, estimate the age of the Universe at this time. State the assumptions you have made. [4 marks]

(h) Briefly explain what is meant by the Horizon length, l_H, and show that during Inflation (when the universe is dominated by a large vacuum energy density) the physical size of the Horizon increases with time, t, as

$$l_H = \frac{c}{H_{\text{infl}}}\left(\exp\left(H_{\text{infl}}t\right) - 1\right)$$

where H_{infl} is the value of the Hubble parameter during inflation. [4 marks]
[Hint: during inflation, the Friedmann equation can be written

$$\left(\frac{\dot{a}}{a}\right)^2 = H_{\text{infl}}^2 = \frac{\Lambda}{3}$$

where Λ is a constant.]

2. (a) Starting from the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

where ρ is mass-energy density, a is the expansion factor and k is a constant, define what is meant by the critical density and density parameter. How is the geometry of the Universe linked to its fate via the constant k for a matter-only Universe? [5 marks]

(b) If the density ρ includes contributions for non-relativistic matter and a cosmological constant with present-day (i.e. when $a = a_0 = 1$) density parameters $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, respectively, show that the Hubble parameter, H, at arbitrary expansion factor a is given by

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\Lambda,0}\left(1 - a^{-2}\right) + \Omega_{M,0}\left(a^{-3} - a^{-2}\right) + a^{-2},$$

where H_0 is the present-day value of the Hubble parameter. [5 marks]

(c) For Universes with $\Omega_{\Lambda,0} > 0$, use the result of (b) to discuss whether the Universe will expand to infinite size for the three separate cases of $\Omega_{M,0} < 1$, $\Omega_{M,0} = 1$ and $\Omega_{M,0} > 1$. [Your discussion should be qualitative; you are not required to solve the cubic equation.] [4 marks]

(d) At arbitrary expansion factor, show that $\Omega_{tot} = \Omega_\Lambda + \Omega_M$ satisfies the following equation

$$\Omega_{tot} = \frac{\Omega_{M,0} + \Omega_{\Lambda,0}a^3}{\Omega_{\Lambda,0}\left(a^3 - a\right) + a\left(1 - \Omega_{M,0}\right) + \Omega_{M,0}}.$$

For Universes with $0 < \Omega_M < 1$, evaluate Ω_{tot} in the limits $t \to 0$ and $t \to \infty$, for the two cases $\Omega_{\Lambda,0} = 0$ and $\Omega_{\Lambda,0} > 0$. [In the latter case, you may assume that $\Omega_{\Lambda,0}$ is sufficiently small that a big bang occurred]. With reference to the behaviour of Ω_{tot} as $t \to 0$, briefly discuss how a period of cosmic inflation can overcome the spatial flatness problem. [6 marks]

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3. (a) Explain the roles of Cepheid variable stars and the Tully-Fisher and Type Ia supernova distance indicators in the Hubble Space Telescope Key Project to measure the Hubble constant, H_0. Discuss the remaining sources of uncertainty in this determination of H_0. [6 marks]

(b) The baryonic Tully-Fisher relation is an observed correlation between the total baryonic mass, M_b, (i.e. stellar mass plus cold gas mass) of a galaxy and the maximum amplitude, V_f, of its rotation curve. The relation takes the form

$$M_b = \alpha V_f^4,$$

where α is a constant. The rotation curves of such galaxies are flat at values of $\pm V_f$ beyond the luminous galaxy and out into their dark matter haloes. If the outer radius, $r = r_{vir}$, of a dark matter halo is defined as the point at which the mean enclosed total matter density (assuming spherical symmetry) is 100 times the Universe's critical mass density, show that

$$r_{vir} = \frac{V_f}{\sqrt{50}H_0}.$$

[6 marks]

(c) If $\alpha = 47$ M$_\odot$ km^{-4} s^4 and $H_0 = 70$ km s^{-1} Mpc^{-1}, show that the fraction of the total mass within r_{vir} which is in the form of baryons (i.e. the baryon mass fraction) is given by

$$f_b \simeq 10^{-4} V_f,$$

when V_f is given in units of km s^{-1}. [4 marks]

(d) Assuming that galaxies initially form with the universal baryon mass fraction of $f_b = \Omega_b/\Omega_M = 0.17$, for the case of the Milky Way galaxy ($V_f =$ 225 km s^{-1}) calculate the fraction of its initial baryonic mass which has been lost since it formed 13 Gyr ago. If, at all times, the rate at which the Milky Way loses baryonic mass is proportional to its baryonic mass, how long will it take to lose a further 50 per cent of its baryonic mass? [4 marks]

4. (a) The light elements (Deuterium, Helium, Lithium and Beryllium) are thought to be produced in the Big Bang. Briefly explain why the measured abundance of these elements is strong evidence in favour of the Big Bang model. [4 marks]

(b) Briefly explain what is meant by "Freeze Out" and why it plays an important role in determining the abundance of Helium. [4 marks]

The weak interaction mediates the conversion of neutrons into protons and vice-versa. Assuming that the interaction timescale is

$$t_w = \hbar K_{\mathrm{F}}^{-2} (k_{\mathrm{B}} T)^{-5}$$

(where $K_{\mathrm{F}} = 3.5 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is a constant), estimate the temperature at which freeze-out occurs, and show that the ratio of the number of neutrons to protons is approximately 0.16 at this time. [5 marks]

[Hints: At the freeze-out temperature, the effective number of Bosonic degrees of freedom, $g_* = 10.75$ and the Universe is radiation dominated. The expansion timescale of the Universe is $a/\dot{a} = 1/H$. The mass difference between the neutron and proton is $1.3\,\mathrm{MeV}/c^2$.]

(c) After freeze-out, neutrons decay into protons until the number density of high energy photons is sufficiently low that neutrons and protons can combine to form nuclei such as Deuterium and Helium. A key reaction in this process is

$$n + p \Rightarrow D + \gamma$$

which produces a photon with energy 2.2 MeV.

Show that expansion factor of the universe when nuclei form, a_{nuc}, depends on the present-day temperature of the cosmic microwave background, $T_{\mathrm{CMB},0}$, and the Baryon density parameter, $\Omega_{b,0}$. [5 marks]

Estimate a_{nuc} for a universe in which $H_0 = 75\,\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$, $T_{\mathrm{CMB},0} = 10\,\mathrm{K}$ and $\Omega_{b,0} = 1$. [2 marks]

[Hints: Assume that the number density of photons is conserved, ie., you may ignore the production of photons by processes such as e^+–e^- annihilation. Assume that the neutron–proton ratio is close to 0.1.]

SECTION B. PLANETARY SYSTEMS

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) Briefly describe the locations and distributions of objects in the Asteroid Belt, the Kuiper belt and the Oort Cloud, and note the type of objects found in each of these regions. [4 marks]

(b) Comet Ison was trailed as one of the major astronomical events of the past few years; but ultimately it disintegrated on perihelion passage and fizzled out. Given that at perihelion Ison was a distance 0.01244 AU from the centre of the Sun, calculate a limit on its density given the Roche limit $r_{\rm R}$ for an object of density ρ_2 orbiting an object of density ρ_1 and radius R_1 is

$$r_{\rm R} = 2.44 \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{3}} R_1.$$

What does this suggest about the composition of comet Ison? [4 marks]

(c) Describe, with the aid of a diagram, how a gravitational slingshot can be used to accelerate a spacecraft. [4 marks]

(d) The mass-radius relation for a planet of mass $M_{\rm p}$ and radius $R_{\rm p}$, composed of atoms with atomic number Z and atomic mass number A is given by

$$2\beta \left(\frac{M_{\rm p} Z^5}{A^5}\right)^{\frac{1}{3}} = R_{\rm p} \left(\frac{\alpha Z^2}{A^{\frac{4}{3}}} + \gamma M_{\rm p}^{\frac{2}{3}}\right),$$

where α, β & γ are all constants. Derive a simple proportionality relationship between $M_{\rm p}$ and $R_{\rm p}$ in the limit of both small and large planetary masses, and hence sketch the mass-radius relationship for a wide range of mass. [4 marks]

(e) The second gravitational moment of a planet, J_2, is related to its ellipticity f and Ω, the ratio between its equatorial centrifugal force and gravitational force, as

$$J_2 = \frac{2}{3}\left[f - \frac{\Omega}{2}\right].$$

Given the following data calculate the polar radius $R_{\rm pole}$ for the innermost Galilean moon of Jupiter, Io, to the nearest kilometre.

Mass (kg)	8.933×10^{22}
Equatorial radius (km)	1821
J_2	1.860×10^{-3}
Rotation period (days)	1.769

[4 marks]

(f) The observed velocity semi-amplitude, K, of a star's reflex motion due to a planet is given by

$$K = \left(\frac{2\pi G}{T}\right)^{\frac{1}{3}} \frac{M_{\rm P} \sin i}{(M_* + M_{\rm p})^{\frac{2}{3}}} \frac{1}{(1-e^2)^{\frac{1}{2}}},$$

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for an orbital period T, stellar mass M_*, planetary mass $M_{\rm p}$, orbital inclination i and orbital eccentricity e. A large gas giant planet ($M_{\rm p} = 6.23 \times 10^{27}$ kg) orbits a solar-mass star in an eccentric orbit ($e = 0.6$), with the distance of the planet from the star at the orbital pericentre being 0.067 AU. Given that the orbital inclination is $i = 60°$, calculate the maximum wavelength shift $\Delta\lambda$ that would be observed for the stellar Na I absorption line at 589.2 nm if the system were subjected to a radial velocity study. [4 marks]

(g) Briefly describe the main stages in the formation of terrestrial planets from a protoplanetary disc. [4 marks]

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6. (a) Sketch the positions of the Lagrange points relative to the Sun-Jupiter system, and note which – if any – of these are stable points. Provide an example of an object that orbits in close proximity to a Lagrange point in the Solar System. [5 marks]

[Jupiter has a mass of 1.90×10^{27} kg.]

(b) A new telecommunications satellite is on a Hohmann transfer orbit between a circular low Earth orbit, 400 km above the Earth's surface, and a geostationary orbit with a radius of 4.22×10^4 km. However, a problem occurs when it is at a true anomaly $\theta = 132°$, when the ground station systems sending commands to the mission fail. Given that a command to fire the satellite's rockets to circularise the orbit is required at apogee (when the satellite is furthest from the Earth), how long do mission control have to repair their ground stations? Will it be the end of the mission if they do not do it in time? [11 marks]

[The mass of the Earth is 5.97×10^{24} kg, and its radius is 6370 km. The true anomaly θ, eccentric anomaly E and mean anomaly M are related as

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \qquad \text{and} \qquad M = E - e\sin E,$$

where e is the eccentricity of the orbit.]

(c) Fortunately, the ground station is fixed before apogee is reached, and the command is sent at the correct time. Given the mass of the satellite (without fuel) is 1500 kg, calculate the mass of fuel that must be burnt at apogee to complete the Hohmann transfer if its rockets have an exhaust velocity of 3 km s^{-1}. [4 marks]

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7. (a) Briefly explain how the carbon-silicate cycle stabilises temperatures on Earth, and why a similar cycle does not occur on Venus. [4 marks]

(b) Starting from the condition of hydrostatic equilibrium, show that the pressure p varies with altitude h as

$$p = p_0 \exp(-h/H),$$

for an isothermal atmosphere with surface pressure p_0 composed of an ideal gas, defining the pressure scale height of the atmosphere H in the process. Given that Venus has a pressure scale height H of 16 km and surface temperature $T = 737$ K, show that the dominant component of this atmosphere is carbon dioxide (CO_2). [7 marks]

[Carbon dioxide has mean atomic mass of 44 atomic mass units.]

(c) Estimate a lower limit to the pressure at the centre of Venus, expressing it in units of the surface atmospheric pressure $p_s = 9.2 \times 10^6$ Pa. [5 marks]

(d) Describe two different sets of observations that could demonstrate that the density of Venus increases towards its centre. [4 marks]

[Venus has a mass of 4.87×10^{24} kg and radius of 6050 km.]

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8. (a) The direct imaging of extra-solar planets is extremely challenging, not least due to the $> 10^9$ difference in optical flux between stars and their planets. Suggest several methods by which the direct imaging of planets can be made a more practical proposition. [4 marks].

(b) The duration of a microlensing event (in days) is given by

$$t_{\rm E} = 69.9 \left(\frac{M_{\rm L}}{M_\odot}\right)^{\frac{1}{2}} \left(\frac{D_{\rm S}}{8\ {\rm kpc}}\right)^{\frac{1}{2}} [(1-d)d]^{\frac{1}{2}} v_{200}^{-1},$$

where $M_{\rm L}$ is the mass of the body creating the lens, $D_{\rm S}$ is the distance of the background star being lensed, $d = D_{\rm L}/D_{\rm S}$ where $D_{\rm L}$ is the distance to the lensing body, and v_{200} is the transverse velocity of the lensing body across the background star in units of 200 km s^{-1}.

An O star at a distance of 4.5 kpc is seen to brighten in a fashion consistent with being gravitationally lensed, for a total of 50.9 days. Subsequent observations identify a foreground G4 star of mass 0.98 $M_\odot$ at a distance of 200 pc as the candidate lens. Continued monitoring of this object shows a further brightness enhancement one year later, lasting for 2.1 days. If this is the result of lensing by a planet in a wide orbit around the G4 star, what is the mass of this planet, in units of Jupiter masses? [6 marks]

[The mass of Jupiter is $M_{\rm J} = 1.90 \times 10^{27}$ kg.]

(c) A detailed reanalysis of the lensing light curve for the G4 star shows two further planets were detected, with masses 1.01 $M_{\rm J}$ and 3.95×10^{-3} $M_{\rm J}$ respectively. By an extraordinary stroke of luck, the G4 star is within the *Kepler* field-of-view; and, furthermore, transit signals are detected for both planets. The larger planet has a period of 5.86 days, and causes a 2.96 per cent drop in the light from the star whilst in transit; the smaller object has a period of 354 days and causes a diminution of 1.05×10^{-2} per cent in the starlight in transit. Calculate the density ρ and the orbital semi-major axis a of both planets. [6 marks]

From this data suggest the class of each planet. [2 marks]

[The G4 star has a radius $R_* = 6.68 \times 10^8$ m.]

(d) The larger of the two close-in planets is close enough to the star to provide both strong transit and radial velocity signals. Suggest further observational tests that would improve our characterisation of this planet and its orbit. [2 marks]