L2 Foundation of Physics 2B Optics 2018-19

O.WP.2 Spatial frequencies and paraxial plane waves

January 28, 2020

The harmonic wave solution of Maxwell's wave equation is

$$\underline{E}(\boldsymbol{r},t) \quad = \quad \underline{E}_0(\boldsymbol{r},t)\cos(\underline{k}\cdot\underline{r}-\omega t)\ . \tag{1}$$

In optics, it is sometimes convenient to use a paraxial plane wave solution of the form:

$$E = E_0 \mathrm{e}^{\mathrm{i}2\pi(ux+vy)}\mathrm{e}^{\mathrm{i}2\pi z/\lambda}\mathrm{e}^{-\mathrm{i}\pi(u^2+v^2)\lambda z}\ . \tag{2}$$

1. What are u and v, and what are their units? [2 marks]

2. List four steps needed to re-write the harmonic wave solution in the form of a paraxial plane wave. [4 marks]

3. Calculate u and v for a plane wave with wavelength $\lambda = 500$ nm, propagating at an angle $\theta = 30.0^\circ$ to the z axis in the xz plane. [4 marks]

Solution:

1. $u = k_x/2\pi$ and $v = k_y/2\pi$ are the spatial frequencies of the plane wave in the x and y directions, respectively. [1]

   The unit of spatial frequency is inverse length, e.g., m^{-1}. [1]

2. Any four of the following 5 are correct:
   - Complex representation: $\underline{E}(\underline{r},t) = \underline{E}_0\mathrm{e}^{\mathrm{i}(\underline{k}\cdot\underline{r}-\omega t)}$. [1]
   - Neglect time dependence: $\underline{E}(\underline{r},t) = \underline{E}_0\mathrm{e}^{\mathrm{i}\underline{k}\cdot\underline{r}}$. [1]
   - Scalar approximation: replace $\underline{E}_0$ by E_0. [1]
   - E_0 is a constant independent of position and time. Not necessarily true of the harmonic wave. In this case we can write the full x, y and z dependence as $E = E_0\mathrm{e}^{\mathrm{i}k_x x + k_y y + k_z z}$.
   - Small-angle approximation, $k_z = k - (k_x^2 + k_y^2)/2k = 2\pi/\lambda - \pi(u^2+v^2)\lambda$. [1]

3. In the yz the propagation angle [1] is zero so $v = 0$. [1] For the xz plane

$$u = \frac{1}{\lambda}\sin\theta\ [1] = \frac{0.500}{5.00\times 10^{-7}} = 1.00\times 10^{6}\ \mathrm{m}^{-1}\ .\ [1]$$

   One mark for formula and one for correct answer, remember to pay attention to significant figures and units.