# Mathematical Methods II
# Weekly problem set 5

(1) Solve the following differential equation

$$y'' - y = x,$$

using the Wronskian method.

**Solution**

$$y'' - y = x$$

Auxiliary equation from subbing $y = Ae^{\lambda x}$

$$\lambda^2 - 1 = 0 \to \lambda = \pm 1$$

Thus

$$y_c = k_1 e^x + k_2 e^{-x}$$

For particular solution, try Wronskian

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = (e^x)(-e^{-x}) - (e^x)(e^{-x}) = -2$$

$$k_1' = \frac{-x}{-2} e^{-x} \to k_1 = \int \frac{xe^{-x}}{2} dx = -\frac{x+1}{2e^x} + c_1$$

$$k_2' = \frac{x}{-2} e^{x} \to k_2 = -\int \frac{xe^{x}}{2} dx = -\frac{(x-1)e^x}{2} + c_2$$

So general solution is

$$\begin{aligned} y_c &= \left(-\frac{x+1}{2e^x} + c_1\right) e^x + \left(-\frac{(x-1)e^x}{2} + c_2\right) e^{-x} \\ &= c_1 e^x + c_2 e^{-x} - (x+1)/2 - (x-1)/2 \\ &= c_1 e^x + c_2 e^{-x} - x \end{aligned}$$

(2) Consider the following equation

$$(1 - x^2)y'' - 2xy' + 30y = 0. \tag{1}$$

(a) Identify the type of this equation, stating its general form.

**Solution** This is a Legendre equation, with general form

$$(1 - x^2)y'' - 2xy' + vy = 0$$

where $v = \ell(\ell + 1)$, and $\ell$ is a constant.

(b) Find the expression for the Legendre polynomial solution of this equation by applying the Rodrigues formula.

*Hint:* the Rodrigues formula for Legendre polynomials is given by

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$

We know that $v = 30 = \ell(\ell + 1) = 5(6) \Rightarrow \ell = 5$. So

$$\begin{aligned} P_5(x) &= \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell \\ &= \frac{1}{(2^5)(5!)} \frac{d^5}{dx^5} (x^2 - 1)^5 \\ &= \frac{1}{32 \times 120} \frac{d^5}{dx^5} (x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1) \\ &= \frac{480}{3840} (63x^5 - 70x^3 + 15x) \\ &= \frac{1}{8} (63x^5 - 70x^3 + 15x) \end{aligned}$$