# Relativistic Electrodynamics, Workshop 1

## Exercise 1

Let $q^\mu = (q^0, q^1, q^2, q^3)$ and $r^\mu = (r^0, r^1, r^2, r^3)$ be two contravariant 4-vectors, $g^{\mu\nu}$ the metric and $\Lambda^\mu_\nu$ a boost along the $z$-axis with velocity $\beta = v/c$.

Compute:

| | | | |
|---|---|---|---|
| (a) $q^\mu r_\mu$ | (b) $r_\mu\, q^\mu$ | (c) $q_\mu\, r^\mu$ | (d) $q^\mu r_\nu$ |
| (e) $g^\mu{}_\mu\, g^\nu{}_\nu$ | (f) $g^\mu{}_\nu\, g^\nu{}_\mu$ | (g) $g^{\mu\nu} r_\mu \Lambda^\rho{}_\nu$ | (h) $g^{\mu\rho} q^\sigma r_\mu\, g_{\nu\rho}\, g^\nu{}_\sigma$ |

Which of the above quantities are invariant under an arbitrary Lorentz transformation?

## Exercise 2

We define the (contravariant) partial derivative as

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

(a) Show that $\partial^\mu$ is a contravariant 4-vector.

(b) Express all four components of $\partial^\mu$ in terms of derivatives with respect to $t$, $x$, $y$ and $z$. What is $\partial_\mu$?

(c) Compute $\partial^\mu x_\mu$, $\partial^\mu x_\nu$ and $\partial^\mu x^\nu x_\nu$.

# Solutions

## Exercise 1

There are two important points to learn in this exercise. First, how to raise (lower) indices (= labels), second, how to translate expressions written in component notation to expressions written in terms of $4 \times 4$ matrices.

Let's start with raising (lowering) indices. Consider an expression written in component notation (i.e. with labels $\mu, \nu \ldots$) and assume it contains a lower (i.e. covariant) label $\nu$. How is this expression related to the same expression where the label is an upper (i.e. contravariant) label? We know that for 4-vectors this relation is

$$x^\mu = g^{\mu\nu} x_\nu$$

with $g$ being the metric. Thus, multiplication with $g^{\mu\nu}$ transforms the covariant index $\nu$ to a contravariant label $\mu$. It is also often said that the index $\nu$ is *contracted.* Given the explicit form of the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we see that $x^0 = x_0$ and $x^i = -x_i$ for $i \in \{1, 2, 3\}$. The point is that this relation is the same for any expression. Thus, we have

$$T^{\mu_1\mu_2}{}_{\mu_3\mu_4\ldots} = g^{\mu_1\nu_1} T_{\nu_1}{}^{\mu_2}{}_{\mu_3\mu_4\ldots} = g^{\mu_1\nu_1} g^{\mu_2\nu_2} T_{\nu_1\nu_2\mu_3\mu_4\ldots}$$

where $T$ is any tensor (for the moment think of a tensor simply as any expression with an arbitrary number of labels). Again, comparing the first and the last term in the above equation, we would say that the indices $\nu_1$ and $\nu_2$ have been contracted.

Lowering indices is done in a similar way. The only difference is that we multiply by $g_{\mu\nu}$ rather than $g^{\mu\nu}$ to transform a contravariant index $\nu$ to a covariant index $\mu$.

The second crucial point to note is the following: if we have a tensor $T^{\mu\nu}$ that is represented by a $4 \times 4$ matrix $\mathbf{T}$ and a tensor $U_{\nu\rho}$ represented by the $4 \times 4$ matrix $\mathbf{U}$ then the tensor $T^{\mu\nu} U_{\nu\rho}$ is represented by the $4 \times 4$ matrix $\mathbf{T} \cdot \mathbf{U}$. In other words, $T^{\mu\nu} U_{\nu\rho} = (T \cdot U)^\mu{}_\rho$. Note this is true only if the repeated indices (in this case $\nu$) are adjacent. Thus, we have $T^{\mu\nu} U_{\nu\rho} = U_{\nu\rho} T^{\mu\nu} = (T \cdot U)^\mu{}_\rho \neq (U \cdot T)^\mu{}_\rho$. Thus if an expression is written in index notation, the order of the terms does not matter at all. After all, all these terms are simply numbers. However, going from the index notation to "matrix" notation (where the order definitely does matter!) we have to be careful. We can only convert a sum (in index notation) to a "dot"-multiplication (in matrix notation) if the corresponding repeated labels are adjacent. If you have problems in understanding this, it might be worthwhile to write down the sums explicitly to convince yourself that only $T^{\mu\nu}U_{\nu\rho}$ but not $T^{\nu\mu}U_{\nu\rho}$ corresponds to matrix multiplication (rows $\times$ columns).

The above holds not only for tensors with two labels (matrix multiplication), but also for a matrix acting on a vector. Thus we have $\Lambda^\mu{}_\nu x^\nu = x^\nu \Lambda^\mu{}_\nu$ and this can be computed either by performing the sum (over $\nu$ explicitly) or by acting with the matrix $\Lambda$ on the 4-vector $x$ (in fact, this is exactly the same as performing the sums explicitly).

We are now ready to actually do the exercises.

**part (a)** First note, there is a sum over $\mu$, since this index is repeated. Also, there is no free (i.e. non-repeated) index in the expression. Performing the sum we get

$$\begin{aligned} q^\mu r_\mu &= \left(\sum_{\nu=0}^{3} q^\mu r_\mu =\right) q^0 r_0 + q^1 r_1 + q^2 r_2 + q^3 r_3 \\ &= q^0 r^0 - q^1 r^1 - q^2 r^2 - q^3 r^3 \end{aligned}$$

The minus signs in the last step come from converting the covariant (spatial) indices of $r$ to contravariant (spatial) indices.

**part (b)** As noted above, the order of terms does not matter if we use index notation, thus

$$q_\mu r^\mu = q^\mu r_\mu$$

and we get the same as in (a)

**part (c)** Again, we have to do a sum

$$\begin{aligned} q_\mu r^\mu &= \left(\sum_{\nu=0}^{3} q_\mu r^\mu =\right) q_0 r^0 + q_1 r^1 + q_2 r^2 + q_3 r^3 \\ &= q^0 r^0 - q^1 r^1 - q^2 r^2 - q^3 r^3 \end{aligned}$$

This time, the minus signs in the last step come from converting the covariant (spatial) indices of $q$ (not $r$ as in part (a)) to contravariant (spatial) indices. However, in the end it does not matter where the minus sign comes from, we get still the same answer as in (a). This is generally true! For a repeated index there is always *one upper and one lower index*, but it does not matter which of the two is the upper and which one is the lower index. Or, in other words, the contraction of an index always involves one upper (contravariant) and one lower (covariant) index. However, there are *never* two equal lower or two equal upper indices in an equation.

**part (d)** This looks similar enough to (a) but actually is completely different. This time we have no repeated index, but two free indices. So there is no sum to be performed, but we have to give $4 \times 4 = 16$ quantities, since $\mu$ as well as $\nu$ run independently from 0 to 3.

Let's start with the fist of these 16 terms, $\mu = 0,\ \nu = 0$. We get $q^0 r_0 = q^0 r^0$. For the next three quantities, $\mu = 0,\ \nu = i$ with $i \in \{1, 2, 3\}$ we get $q^0 r_i = -q^0 r^i$. Three more quantities $\mu = i,\ \nu = 0$ are given by $q^i r_0 = q^i r^0$. finally, the remaining nine terms, $\mu = i,\ \nu = j$ with $i, j \in \{1, 2, 3\}$ we get $q^i r_j = -q^i r^j$.

**part (e)** Let's first concentrate on ${g^\mu}_\mu$. Again we have to keep in mind the sum over $\mu$. To start with we look at ${g^\mu}_{\mu'}$ and worry about the sum (i.e. setting $\mu' \to \mu$ later. We note

$${g^\mu}_{\mu'} = g^{\mu\sigma} g_{\sigma\mu'}$$

Since the repeated index, $\sigma$ is adjacent, this corresponds to matrix multiplication of the $4 \times 4$ matrices

$$g^{\mu\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad g_{\sigma\mu'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus

$${g^\mu}_{\mu'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or, written in a more compact way ${g^\mu}_{\mu'} = {\delta^\mu}_{\mu'}$, where $\delta$ is the Kronecker symbol. Now we set $\mu' \to \mu$ which means we have to sum over $\mu = 0 \ldots 3$. We have

$${g^\mu}_\mu = {g^0}_0 + {g^1}_1 + {g^2}_2 + {g^3}_3 = 1 + 1 + 1 + 1 = 4$$

Of course, we also have ${g^\nu}_\nu = 4$ and finally get ${g^\mu}_\mu {g^\nu}_\nu = 4 \cdot 4 = 16$

**part (f)** We immediately get ${g^\mu}_\nu {g^\nu}_\mu = {g^\mu}_\mu = 4$

**part (g)** To start with we note $g^{\mu\nu} r_\mu {\Lambda^\rho}_\nu = g^{\nu\mu} r_\mu {\Lambda^\rho}_\nu = r^\nu {\Lambda^\rho}_\nu = {\Lambda^\rho}_\nu r^\nu$ where in the first step we used $g^{\mu\nu} = g^{\nu\mu}$. It is generally not true that indices can simply be swapped. However the metric is a rather special tensor and as can be seen from its explicit form, allows for interchanging the indices. This property is called '*symmetric*", i.e. the statement "the metric is symmetric" means precisely $g^{\mu\nu} = g^{\nu\mu}$.

Our expression now is ${\Lambda^\rho}_\nu r^\nu$ and since the repeated (= contracted) indices are adjacent can be written as a $4 \times 4$ matrix acting on the 4-vector $r^\nu$

$$\Lambda^\rho{}_\nu \, r^\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} r^0 \\ r^1 \\ r^2 \\ r^3 \end{pmatrix} = \begin{pmatrix} \gamma r^0 - \gamma\beta r^3 \\ r^1 \\ r^2 \\ -\gamma\beta r^0 + \gamma r^3 \end{pmatrix}$$

Thus as expected (since there is one free index) the result consists of four terms which are given on the rhs of the above equation.

**part (h)** This looks horribly complicated to start with. However, we first note that all indices are contracted (= repeated). Thus, the result is simply a number. Using the symmetry of the metric $g_{\nu\rho} = g_{\rho\nu}$ we get

$$\begin{aligned} &g^{\mu\rho} \, q^\sigma \, r_\mu \, g_{\nu\rho} \, g^\nu{}_\sigma = g^{\mu\rho} \, q^\sigma \, r_\mu \, g_{\rho\nu} \, g^\nu{}_\sigma = g^{\mu\rho} \, q^\sigma \, r_\mu \, g_{\rho\sigma} \\ &g^{\mu\rho} \, g_{\rho\sigma} \, q^\sigma \, r_\mu = g^{\mu\rho} \, q_\rho r_\mu = q^\mu r_\mu \end{aligned}$$

and we get the same result as in part (a).

All of the above quantities that do not have a free index are invariant under arbitrary Lorentz transformations, i.e. (a), (b), (c), (e), (f) and (h). The expression in (d) has two free indices (i.e. involves $4^2 = 16$ quantities) and is said to be a tensor of rank 2. The expression in (g) has one free index (i.e. involves $4^1 = 4$ quantities) and is said to be a tensor of rank 1, or simply a vector.

This pattern is generally true as long as all quantities involved are tensors, i.e. we can build higher or lower rank tensors by multiplying tensors and contracting indices.

## Exercise 2

### (a)

During the lectures we showed that the Lorentz transformation for covariant vectors

$$q_\mu \to q'_\mu = \Lambda_\mu{}^\nu \, q_\mu, \qquad \text{with } \Lambda_\mu{}^\nu \equiv \frac{\partial x'_\mu}{\partial x_\nu} \tag{1}$$

could also be written as

$$\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}. \tag{2}$$

We can therefore write the rules for transforming the covariant coordinates in terms of derivatives of the contra-variant coordinates, but the rôle of the primed and unprimed coordinates are swapped.

We therefore find

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \;\to\; \partial'^\mu = \frac{\partial}{\partial x'_\mu} = \frac{\partial}{\partial\left(\Lambda_\mu{}^\nu x_\nu\right)} = \frac{1}{\Lambda_\mu{}^\nu}\frac{\partial}{\partial x_\nu} = \frac{1}{\partial x^\nu/\partial x'^\mu}\frac{\partial}{\partial x_\nu} \tag{3}$$

$$= \partial x'^\mu/\partial x^\nu \frac{\partial}{\partial x_\nu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu} = \Lambda^\mu{}_\nu \partial^\nu, \tag{4}$$

where we used the general result that

$$\frac{1}{\partial y/\partial x} = \frac{\partial x}{\partial y}. \tag{5}$$

We have therefore shown that the partial derivative with respect to the covariant coordinate $\partial^\mu = \frac{\partial}{\partial x_\mu}$ transforms as a contravariant 4-vector.

## (b)

We first express the covariant 4-vector in terms of $t, x, y$ and $z$ as follows: $x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$. Then we get for the four components $\mu \in \{0, 1, 2, 3, \}$ of

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

the following results.

$$\begin{aligned}
\partial^0 &\equiv \frac{\partial}{\partial x_0} = \frac{\partial}{\partial ct} = \frac{1}{c}\frac{\partial}{\partial t} \\
\partial^1 &\equiv \frac{\partial}{\partial x_1} = \frac{\partial}{\partial(-x)} = -\frac{\partial}{\partial x} \\
\partial^2 &\equiv \frac{\partial}{\partial x_2} = \frac{\partial}{\partial(-y)} = -\frac{\partial}{\partial y} \\
\partial^3 &\equiv \frac{\partial}{\partial x_3} = \frac{\partial}{\partial(-z)} = -\frac{\partial}{\partial z}
\end{aligned}$$

These four equations can be summarized as

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

We obtain the corresponding covariant 4-vector as

$$\partial_\mu = g_{\mu\nu}\,\partial^\nu = \sum_{\nu=0}^{3} g_{\mu\nu}\,\partial^\nu$$

where the metric $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

Thus, the time component (i.e. the 0-component) of the 4-vector does not change, whereas the spatial components get a minus sign

$$\partial_\mu = \left( \frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla} \right)$$

This corresponds to

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

### (c)

We next compute $\partial^\mu x_\mu$. First note that there is a sum over the label $\mu$ (since it is repeated). Thus, there is no free (= non-repeated) label and the expression will be a scalar. Performing the calculation we get

$$\begin{aligned} \partial^\mu x_\mu &= \partial^0 x_0 + \partial^1 x_1 + \partial^2 x_2 + \partial^3 x_3 \\ &= \frac{1}{c}\frac{\partial}{\partial t}(ct) + \left(-\frac{\partial}{\partial x}\right)(-x) + \left(-\frac{\partial}{\partial y}\right)(-y) + \left(-\frac{\partial}{\partial z}\right)(-z) = 4 \end{aligned}$$

For the calculation of $\partial^\mu x_\nu$ we observe that there are two free indices (there is no repeated index, thus there is no sum). Since both, $\mu$ and $\nu$ can be either 0, 1, 2 or 3, we have to compute $4 \times 4 = 16$ quantities. In other words, $\partial^\mu x_\nu$ is a tensor of rank two with one contravariant ($\mu$) and one covariant ($\nu$) index. We also note that if $\mu \neq \nu$ we get 0, since e.g.

$$\frac{\partial}{\partial x} y = 0$$

On the other hand, for $\mu = \nu$ we get 1. This can all be summarized as follows

$$\partial^\mu x_\nu = g^\mu{}_\nu \tag{7}$$

where in accordance with eq.(6)

$$g^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

From eq.(7) we also immediately conclude

$$\partial_\mu x^\nu = g_\mu{}^\nu; \quad \partial^\mu x^\nu = g^{\mu\nu}; \quad \partial_\mu x_\nu = g_{\mu\nu}$$

It is generally true that if an equation between tensor holds, the position of the indices (i.e. whether upper or lower) does not matter, as long as they are *the same* on both sides of the equation. Note that eq.(6) and eq.(8) are *not* equations between tensors. These equations are a shorthand to write $g^{00} = 1,\ g^{01} = 0 \ldots$. Equations between tensors have *always matching* labels, i.e. if there is a free (non-repeated) upper label $\mu$ on the left hand side of the equation, there must be a free (non-repeated) upper label $\mu$ on the right hand side of the equation.

Finally we compute $\partial^\mu x^\nu x_\nu$. This is a 4-vector (tensor of rank one) since there is one free index. We can either use

$$x^\nu x_\nu = (ct)^2 - x^2 - y^2 - z^2$$

(the superscripts 2 denote powers and are not labels!) and perform the four differentiations for $\mu = 0, 1, 2, 3$ or we use the previous result and the product rule:

$$\partial^\mu x^\nu x_\nu = (\partial^\mu x^\nu)\, x_\nu + x^\nu \partial^\mu x_\nu = g^{\mu\nu} x_\nu + x^\nu g^\mu{}_\nu = 2x^\mu$$

Note once more that at each stage the labels fit perfectly, term by term!!!