University of Durham
EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2019	PHYS3651-WE01

Title:
Planets and Cosmology 3

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Attempt **all** questions. The short-answer questions at the start of each section carry 50% of the total marks for the paper. The remaining 50% of the marks are carried by the longer questions, which are equally weighted.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Cosmology
Section B: Planetary Systems

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A: COSMOLOGY

1. (a) In an expanding Universe, explain how the physical separation, $r(t)$, between any two points following the expansion is related to their comoving separation, r_0, and the scale or expansion factor, $a(t)$. Show that this relation is consistent with a local Hubble expansion law. [4 marks]

   (b) Starting from the Friedmann equation for a matter-only Universe,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

   where ρ is the matter density and k is a constant, derive an expression for the density parameter, $\Omega(a)$, in terms of the scale factor a and the present-day density parameter $\Omega_0 = \Omega(a = 1)$. For cases with $\Omega_0 > 1$, show that $\Omega(a) \to \infty$ at some future epoch and explain physically why this occurs. [4 marks]

   (c) An edge-on rotating disk galaxy of radius 8 kpc emits Hα line emission with observed wavelengths in the range 799.3 to 800.9 nm. Estimate the redshift of the galaxy and its mass in solar masses, given that the rest wavelength of Hα is 656.3 nm. [4 marks]

   (d) A cosmological fluid has a mass-energy density that scales with the expansion factor as $\rho \propto a^\alpha$, where α is a constant. Use thermodynamic arguments for a thermally-isolated expanding Universe to derive the equation of state relating the fluid pressure, P, to ρ. Deduce the range of values of α for which a Universe containing this fluid could undergo accelerated expansion. [4 marks]

   (e) In a flat Universe containing non-relativistic matter and a positive cosmological constant Λ, the acceleration equation can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{3P_m}{c^2}\right) + \frac{1}{3}\Lambda c^2,$$

   where ρ_m and P_m are respectively the density and pressure of matter. Using the definition of the deceleration parameter $q \equiv -\left(\ddot{a}/a\right)/\left(\dot{a}/a\right)^2$, derive an expression of $q(a)$ in terms of $\Omega_m(a)$, the density parameter for matter at a. [4 marks]

   (f) The Friedmann-Robertson-Walker metric can be written as

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\mathrm{d}r^2 + S_k^2(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right],$$

   where

$$S_k(r) = \begin{cases} \sin\left(\sqrt{k}r\right)/\sqrt{k}, & \text{if } k > 0 \\ r, & \text{if } k = 0. \\ \sinh\left(\sqrt{-k}r\right)/\sqrt{-k}, & \text{if } k < 0 \end{cases}$$

   Consider an object at redshift z with a comoving distance $r = r(z)$. Would this object appear larger or smaller in an open Universe than in the case the Universe is flat? You can assume that z and $r(z)$ are the same in both Universes regardless of k. You can also assume that $\sqrt{|k|}r \ll 1$, and do a Taylor expansion of $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$. [4 marks]

(g) When the temperature of the Universe was given by $k_{\mathrm{B}}T = 10$ MeV, the relativistic species of particles were photons, three species of massless neutrinos, electrons and their anti-particles. If a cosmologist incorrectly assumes that neutrinos have two internal degrees of freedom (handedness), what is the value of g_* (the number of bosonic degrees of freedom) that they calculate at this temperature? You can use the expression

$$g_* = \sum_{\text{boson}} g_{\text{boson}} + \frac{7}{8} \sum_{\text{fermion}} g_{\text{fermion}},$$

where g_{boson} and g_{fermion} are respectively the numbers of relativistic degrees of freedom in bosonic and fermionic particle species. [4 marks]

(h) When primordial nucleosynthesis started, there were N_n free neutrons and N_p free protons in the Universe. Assume that a fraction $f \ll 1$ of these free neutrons ended up in Deuterium (^{2}H) by the end of primordial nucleosynthesis, while the remaining ended up in Helium-4 (^{4}He). Show that fN_n protons ended up forming Deuterium.

Also show that the mass ratio between ^{2}H and ^{4}He at the end of primordial nucleosynthesis is approximately equal to f. You can assume that the proton mass m_p and the neutron mass m_n satisfy $m_n = m_p$ in this calculation. [4 marks]

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2. (a) Consider a Universe containing matter and radiation with present-day density parameters $\Omega_{M,0}$ and $\Omega_{\gamma,0}$, respectively, and Hubble constant H_0. Explain what is meant by the eras of radiation-dominated and matter-dominated cosmic expansion, and derive an expression for the scale factor of matter-radiation equality, a_{eq}. [4 marks]

(b) In the early Universe, prior to the production of the cosmic microwave background (CMB) at $a = a_{dec} \simeq 10^{-3}$, the Universe contained a coupled baryon-photon plasma in which density perturbations propagated at a speed close to $c/\sqrt{3}$.

(i) Show that the maximum comoving distance travelled by the perturbations prior to CMB production is given by:

$$r_{s,CM}(a_{dec}) \simeq \frac{c}{H_0\sqrt{3\Omega_{M,0}}} \int_0^{a_{dec}} \frac{da}{\sqrt{a + a_{eq}}}.$$

In numerical terms, you may assume that $\Omega_{M,0} \gg a_{dec}$ and that $\Omega_{M,0} \gg \Omega_{\gamma,0}$. [10 marks]

(ii) Using the definition of angular diameter distance, derive a general expression for the angular size on the sky, θ_s, of CMB features with a comoving radius of $r_{s,CM}(a_{dec})$. Give your answer in terms of $r_{s,CM}(a_{dec})$ and the angular diameter distance, $d_A(z_{dec})$, to the CMB epoch at redshift z_{dec}. Evaluate θ_s in degrees for the case of a flat Universe in which $\Omega_{M,0}$ is close to unity, $a_{eq} \ll a_{dec}$ and the comoving distance to z_{dec} is approximately $2c/H_0$. [4 marks]

(c) $r_{s,CM}\,(a_{dec})$ is usually taken as the maximal distance within which matter could have had interactions to achieve thermal equilibrium by a_{dec}. Briefly describe the horizon problem in this context. [5 marks]

(d) The horizon problem can be resolved by assuming a period of fast expansion at the beginning of the Universe. The expansion is often assumed to be exponential, driven by a large cosmological constant. However, a cosmologist thinks that the expansion rate during inflation takes the following power-law form

$$a(t) \propto \left(\frac{t}{t_i}\right)^n,$$

where $t_i = 10^{-43}$ s is the time when inflation started and $n > 1$ is a positive integer. Let $t_f = 10^{-34}$ s be the time when inflation ended. The size of the horizon, $l_H(t_f)$, (i.e. the physical distance travelled by light during the inflationary epoch) in a typical inflation model is of order 10^{16} m. Find a minimum value of the integer n to achieve this value of $l_H(t_f)$ in this cosmologist's model. [7 marks]

[Hint: The comoving distance travelled by a signal of constant speed v between cosmic times t_1 and t_2 is given by:

$$X_{CM} = v \int_{t_1}^{t_2} \frac{dt}{a(t)}$$
]

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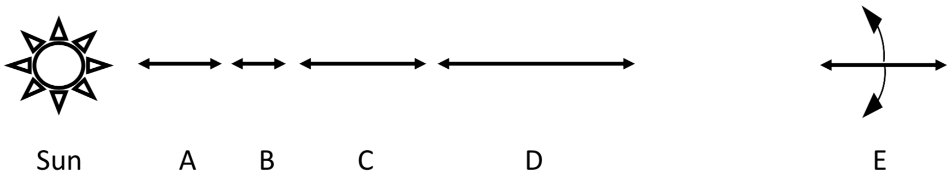
SECTION B: PLANETARY SYSTEMS

3\. (a) The following schematic outlines the main regions of the Solar System (not to scale!). Regions A and C are inhabited by the terrestrial and gas giant planets respectively. Identify regions B, D and E and note which region contains most of the known dwarf planets. What single criterion distinguishes dwarf planets from planets? [4 marks]



(b) Sketch the paths of three elliptical orbits that all have the same instantaneous distance between one focus and the position of a body, $\underline{r}$, the same magnitude of velocity $|\underline{v}|$ at $\underline{r}$, but values of the angle ϕ between the radial vector $\underline{r}$ and the velocity vector $\underline{v}$ of (i) $\phi = \pi/2$, (ii) $\phi = \pi/4$ and (iii) $\phi = -\pi/8$. [4 marks]

(c) Kepler's equation is $M = E - e \sin E$, where M is the mean anomaly, E is the eccentric anomaly and e is the orbital eccentricity. Calculate the eccentric anomaly of Uranus to the nearest hundredth of a degree four Earth years after its perihelion passage. [4 marks]

[1 Uranian year = 84.0 Earth years, and the eccentricity of Uranus' orbit is $e = 0.046$.]

(d) The equation for the effective temperature of a planetary atmosphere, $T_{\rm e}$, can be written as

$$T_{\rm e} = \left(\frac{(1 - A) L_\odot}{16 \pi r^2 \epsilon \sigma} \right)^{\frac{1}{4}},$$

where A is the planet's albedo, r its distance from the Sun and ϵ the planet's infrared emissivity.

Using this equation, the Earth can be shown to have an effective temperature at the top of its atmosphere of 254 K, with a greenhouse effect of +34 K keeping the Earth's surface above freezing at the current epoch. However, at certain points in the past it is thought the whole of the Earth's surface was covered by ice (the "Snowball Earth" hypothesis). By reference to the different variables in the effective temperature equation, deduce why once in this Snowball state it becomes difficult for the Earth to quickly escape it. Suggest a mechanism by which the Earth has escaped this state in the past. [4 marks]

(e) Describe how rotation affects the shape of planetary bodies. Suggest an observational method by which this effect could be measured. [4 marks]

(f) In radial velocity experiments, the observed maximum line-of-sight velocity K from a circular orbit is related to the orbital period T, planetary mass $M_{\rm p}$, stellar mass M_* and orbital inclination i as

$$K = \left(\frac{2\pi G}{T}\right)^{\frac{1}{3}} \frac{M_{\rm p}\sin i}{(M_* + M_{\rm p})^{\frac{2}{3}}}.$$

A planet with mass $10^{-3}M_\odot$ orbits a Sun-like star on a circular orbit with a 30 day period. Calculate the range of values of orbital inclination i over which this planet would be detected in a radial velocity experiment that is sensitive to line-of-sight velocities ≥ 10 m s^{-1}. [4 marks]

(g) Rossiter-McLaughlin effect measurements of extra-solar planets show that some giant planets have an orbit that is not coplanar with the equator of their parent star. Explain why this result is not predicted by solar nebula theory, and suggest a mechanism whereby planets end up on these orbits. [4 marks]

4. In late 2017 astronomers identified the first interstellar object to be seen passing through our Solar System, and named it 'Oumuamua. Observations showed it to be no more than a kilometre in length, with a very elongated shape.

   (a) Explain briefly why small planetary bodies such as 'Oumuamua are not spherical in shape. [3 marks]

   (b) Given that the velocity of 'Oumuamua is 26.3 km s^{-1} at large distances from the Sun, use the vis viva equation to show that the semi-major axis of its hyperbolic orbit is $a = -1.28$ AU. Hence, calculate the perihelion distance of 'Oumuamua given the equation for a hyperbolic orbit

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta},$$

   where r is the distance from the focus of the orbit, $e = 1.20$ is the eccentricity of the orbit and θ is the true anomaly. [6 marks]

   (c) It is found that 'Oumuamua originates from a nearby stellar system. Transit observations show this system to contain at least two planets. Data for the star and the largest planet are given below.

Star	Mass $M_* = 0.760 M_\odot$, radius $R_* = 4.87 \times 10^8$ m
Planet B	Mass $M_\mathrm{p} = 7.60 \times 10^{27}$ kg, radius $R_\mathrm{p} = 6.51 \times 10^7$ m, orbit: semi-major axis $a = 1.89$ AU, eccentricity $e = 0.06$

   (i) Sketch the observed luminosity of the star as a function of time as Planet B transits the star, evaluating the depth of the transit. [3 marks]

   (ii) Describe the factors that could affect the length of time for which Planet B is observed to transit the star. Hence, calculate the range of possible lengths of transit time for Planet B, as a multiple of the time taken for an equatorial transit for a similar-sized object in a circular orbit with the same semi-major axis. [6 marks]

   (iii) Planet A lies on a circular orbit with semi-major axis $a = 0.835$ AU. Show that the shortest Hohmann transfer orbit between the planets would take 314 Earth days to travel. [4 marks]

   (iv) The velocity change from a rocket Δv can be calculated as $\Delta v = v_\mathrm{ex} \ln(m_0/m)$, where v_ex is the exhaust velocity, and m_0 and m are the total spacecraft mass before and after the rocket has been fired. Calculate the fraction of the initial mass of a spacecraft in the form of rocket fuel that is required to manoeuvre a spacecraft from the orbit of Planet B to that of Planet A on the Hohmann transfer orbit from (iii). You may ignore the gravity of the planets, and assume an exhaust velocity $v_\mathrm{ex} = 3.00$ km s^{-1} for the rocket. [8 marks]