Mathematical Methods II
PDF 1

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Key Points

- Recall various standard notations.
- Recall fundamental definitions related to differential equations.
- Solving 1^{st} order ODEs by seperation of variables.
- Detemining if an equation is exact.
- Solving 1^{st} order ODEs using integrating factors.

Notation

- Recall the differences in 'change' notation:
  - Δ: Relatively large change
  - δ: Relatively small change
  - d: Infinitesimal change (i.e. in the limit $x \to 0$), total derivative
  - ∂: Infinitesimal change (i.e. in the limit $x \to 0$), partial derivative
- Recall the different forms of derivative notation:
  - Total derivatives: The following notations are frequently encountered.
    - Leibniz's notation: d^2y/dx^2
    - Lagrange's notation: $f''(x)$
    - Euler's notation: D^2f
    - Newton's notation: $\ddot{y}$
  - Partial derivatives: The following notations are equivalent

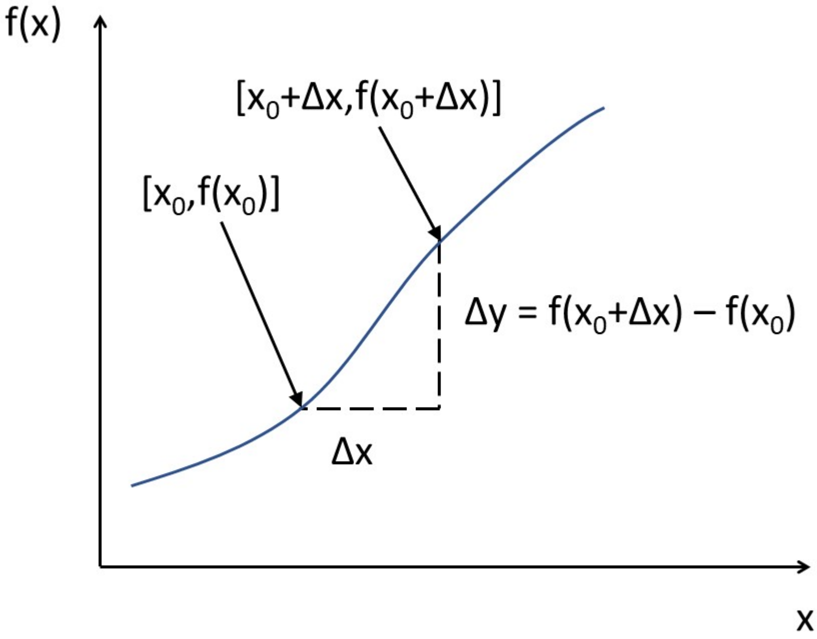
$$\frac{\partial f}{\partial x} = f_x$$

$$\frac{\partial f}{\partial x \partial y} = f_{xy} = \partial_{xy} f$$

Definitions

- **Differential equation**: an equation involving derivatives of a function or functions.
- **Derivative**: Shows a rate of change of a variable(s) w.r.t another variable(s).

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



- **Solution**: A solution to a differential equation is an equation that contains no derivatives, typically given in a form such as $y = f(x)$. A given DE may have more than one solution.
- **Classification by type**
  - **ODE**: Ordinary Differential Equation. Contains one or more dependent variables differentiated w.r.t one independent variable. e.g.

    $$\frac{d^2u}{dx^2} + \frac{du}{dx} + u = e^2$$

    where u is the *dependent* variable and x is the *independent* variable.
  - **PDE**: Partial Differential Equation. Contains one or more dependent variables differentiated w.r.t two or more independent variables. e.g.

    $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = e^{xy}$$

    where u is the *dependent* variable and x and y are the *independent* variables.
  - Recall the relationship between partial and total derivatives. For $u(x, y)$ we have

    $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

- **Classification by order/degree**
  - **Order of a DE**: Determined by the highest derivative
  - **Degree of a DE**: Determined by the exponent of the highest derivative (all exponents should be made integers first).
    e.g. the equation
    $$\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{dy}{dx}\right)^2 + y = 0$$
    is of 3^{rd} order and 4^{th} degree.
- **Classification as linear/non-linear**
  - **Linear DEs**: The dependent variables and their derivatives are of 1^{st} degree and each coefficient depends only on the independent variable.
    i.e. Linear equations are of the form
    $$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$
    e.g.
    $$\frac{d^2y}{dx^2} + y = \sin x$$
  - **Non-linear DEs**: The dependent variables and their derivatives are *not* of 1^{st} degree e.g.
    $$\frac{d^2y}{dx^2} + y^2 = \sin x$$

Solving 1^{st} order ODEs

Not all equations can be solved. In some cases it is possible to prove that an equation is unsolvable. Additionally, there exists no single method for solving all ODEs; thus we require a variety of techniques so we can apply the most appropriate "tool" for the job.

Generally, ODEs become trickier to solve as they increase in order. To begin with we will examine how to solve 1^{st} order ODEs.

- **Separable ODE**: If an equation of two variables and their derivatives can be arranged such that one variable appears only on the LHS of the equation and the other appears only on the RHS it allows us to integrate them independently. Suppose
  $$f(x, y) = \frac{dy}{dx} = u(x)v(y)$$
  This equation is separable, since we can write it as follows
  $$\frac{dy}{v(y)} = u(x)dx$$
  This can be solved by integrating both sides
  $$\int \frac{1}{v(y)}dy = \int u(x)dx$$

e.g. PDF1.1 Solve the following equation, given that $y(0) = 3$

$$\frac{dy}{dx} = \frac{y\cos x}{2}.$$

Separate the functions and integrate

$$\int \frac{2}{y} dy = \int \cos x dx$$

$$2\ln y = \sin x + c_1$$

$$\ln y = \frac{\sin x}{2} + c_2$$

$$y = c_3 e^{\sin x/2}$$

Use the initial condition to find c_3

$$c_3 = \frac{y}{e^{\sin x/2}} = \frac{3}{1} = 3$$

Hence the solution is

$$y = 3e^{\sin x/2}$$

- **Exact ODE**: Exact and inexact ODEs are common in physics, and are of particular relevance in thermodynamics.

  Exact ODEs describe state/point functions - functions describing quantities in a system that do not depend on the path taken, such as internal energy, enthalpy and entropy. These values quantitatively describe the equilibrium state of a thermodynamic system, regardless of how the system arrived in that state.

  Inexact ODEs describe path/process functions - functions describing quantities in a system that do depend on the path taken, such as mechanical work and heat. These values quantitatively describe the transition between equilibrium states of a thermodynamic system.

  . The following form is useful in determining whether an ODE is exact.

$$A(x,y)dx + B(x,y)dy = 0.$$

  Where A and B are arbitrary functions of x and y. An ODE is exact if the following condition is met

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.$$

  Otherwise it is inexact. If the ODE is exact a function $u(x,y)$ exists such that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = Adx + Bdy = 0$$

  We can solve $\partial u/\partial x = A$ and $\partial u/\partial y = B$ to find $u(x,y)$. Since $du = 0$, $u(x,y) = c$, a constant.

e.g. PDF1.2 Determine whether the following equation is exact, and solve it

$$2xy\frac{dy}{dx} + y^2 + x^2 = 0.$$

Start by $\times dx$ to give the standard form

$$(y^2 + x^2)dx + (2xy)dy = 0$$

So $A = y^2 + x^2$ and $B = 2xy$. Check if the equation is exact.

$$\frac{\partial A}{\partial y} = 2y$$

$$\frac{\partial B}{\partial x} = 2y$$

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

Exact! Now, since $A = \partial u/\partial x$ and $B = \partial u/\partial y$, integrate either A w.r.t x or B w.r.t y to find $u(x, y)$. Here the latter appears easier.

$$u(x, y) = \int 2xydy = xy^2 + F(x)$$

To find $F(x)$ we examine the unused function for A

$$A = \frac{\partial u}{\partial x} = y^2 + F^{'}(x) = y^2 + x^2$$

Thus

$$F^{'}(x) = x^2$$

$$F(x) = \int x^2dx = \frac{x^3}{3} + c_1$$

So, substituting back into u

$$u(x, y) = xy^2 + \frac{x^3}{3} + c_1 = c_2$$

Since $du = 0$, $u = c_2$, a constant. Hence the solution, where $c_3 = c_2 - c_1$, is

$$xy^2 + \frac{x^3}{3} = c_3$$

$$y = \sqrt{\frac{c_3 - \frac{x^3}{3}}{x}}$$

- **Inexact ODEs and integrating factors**: If an equation is not exact it can be made exact by multiplying it with an integrating factor, μ. Using the same form as earlier

$$A(x, y)dx + B(x, y)dy = 0,$$

an equation is inexact if

$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x},$$

and can be made exact by multiplying by μ,

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x},$$

giving us the corrected form of

$$\mu A(x,y)dx + \mu B(x,y)dy = 0.$$

If $\mu = \mu(x,y)$ then there is no general method (other than possibly inspection) to find μ. But if $\mu = \mu(x)$ or $\mu = \mu(y)$ then we can test for it. Let's assume $\mu = \mu(x)$, then the last equation becomes,

$$\mu\frac{\partial A}{\partial y} = \mu\frac{\partial B}{\partial x} + B\frac{d\mu}{dx}.$$

We can rearrange this to give

$$B\frac{d\mu}{dx} = \mu\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right),$$

$$\frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)dx = f(x)dx,$$

where $f(x)$ is a function of x alone, as required by our initial condition. Integrating this equation gives us our integrating factor

$$\ln\mu = \int f(x)dx,$$

$$\mu = \exp\left(\int f(x)dx\right),$$

where

$$f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right).$$

Similarly for $\mu(y)$ where $g(y)$ is a function of y alone

$$\mu = \exp\left(\int g(y)dy\right),$$

where

$$g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right).$$

$f(x)$ and $g(y)$ can be used to test if the integrating factor is a factor of x or y alone.

e.g. PDF1.3 Show that the following equation is not exact. Determine an appropriate integrating factor and hence, solve the equation.

$$(xy + y^2 + y)dx + (x + 2y)dy = 0$$

First, show that the equation is not exact. Here $A = xy + y^2 + y$ and $B = x + 2y$.

$$\frac{\partial A}{\partial y} = x + 2y + 1$$

$$\frac{\partial B}{\partial x} = 1$$

$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

Not exact! Let's assume there is an integrating factor containing only x and rewrite our equation to include the factor

$$\mu(x)(xy + y^2 + y)dx + \mu(x)(x + 2y)dy = 0$$

Let $C = \mu(x)(xy+y^2+y)$ and $D = \mu(x)(x+2y)$. Assuming our integrating factor works then it should be true that

$$\frac{\partial C}{\partial y} = \frac{\partial D}{\partial x}$$

So differentiate C w.r.t y and D w.r.t. x.

$$C_y = \mu(x)(x + 2y + 1)$$

$$D_x = \mu^{'}(x)(x + 2y) + \mu(x)$$

We can equate these equations and solve them by using the separable equation method.

$$\mu(x)(x + 2y + 1) = \mu^{'}(x)(x + 2y) + \mu(x)$$

$$\mu(x)(x + 2y) = \mu^{'}(x)(x + 2y)$$

$$\mu(x) = \mu^{'}(x) \Rightarrow \mu(x) = e^x$$

So our equation, modified with out integrating factor, is now

$$e^x(xy + y^2 + y)dx + e^x(x + 2y)dy = 0$$

Test if it is exact

$$\frac{\partial C}{\partial y} = xe^x + 2ye^x + e^x$$

$$\frac{\partial D}{\partial x} = e^x(x + 2y) + e^x$$

$$\frac{\partial C}{\partial y} = \frac{\partial D}{\partial x}$$

Exact! The solution can now be found as previously shown, by finding u(x,y), giving

$$(xy + y^2)e^x = k$$

where k is a constant.