University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 043661/01

LEVEL 3 PHYSICS: THEORETICAL PHYSICS 3

SECTION A. RELATIVISTIC ELECTRODYNAMICS
SECTION B. QUANTUM THEORY 3

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. RELATIVISTIC ELECTRODYNAMICS
Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Give the definition and physical interpretation of lightlike, timelike and spacelike separations. [4 marks]

   (b) State the definition of a covariant and contravariant 4-vector and the relation between them. Show that the sum of two covariant 4-vectors is also a covariant 4-vector. [4 marks]

   (c) Show that $a^\mu v_\mu = 0$ where a^μ is the four-acceleration and v^μ is the four-velocity. [4 marks]

   (d) A muon with velocity v collides with an antimuon with velocity $-v$ producing a tau lepton and its antiparticle. Given that the tau lepton mass is 17 times the muon mass, what is the minimal magnitude of the velocity of the incoming muon? [4 marks]

   (e) An observer in a particular inertial frame notes that the angle between the $\underline{E}$ and $\underline{B}$ fields is larger than $\pi/2$ radians. Show that, in this case, the angle between the electric and the magnetic field is larger than $\pi/2$ radians in *any* inertial frame. [4 marks]

   (f) Write down the gauge transformation of the 4-potential A^μ in covariant form. Show that the field strength tensor $F^{\mu\nu}$ is gauge invariant. [4 marks]

   (g) Express the 0-component of the Maxwell equation $\partial_\mu F^{\mu\nu} = j^\nu/(c\epsilon_0)$ in terms of the electric and magnetic fields. [4 marks]

   [Hint: See question 2 for the definition of $F^{\mu\nu}$ in terms of the fields.]

   (h) The 4-potential for a parallel-plate capacitor at rest and oriented normal to the y axis is $A^\mu = (Ey, 0, 0, 0)$ where E is the electric field strength between the plates and y the distance from the negatively charged plate. Show that, in a frame moving relative to the capacitor with velocity v in the x-direction, there is a magnetic field in the z-direction and calculate its magnitude. [4 marks]

2. Consider a point charge q of rest mass m in an electromagnetic field. The field strength tensor is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

The 4-force f^μ acting on the point charge is defined by

$$f^\mu \equiv \frac{dp^\mu}{d\tau},$$

where τ is the proper time and p^μ the 4-momentum of the point charge.

(a) Show that f^μ is a 4-vector, and demonstrate how it is related to the usual force $\underline{F} = d\underline{p}/dt$. [4 marks]

(b) The 4-force acting on the point charge due to the electromagnetic field is given by

$$f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu,$$

where u_ν is the 4-velocity of the point charge. Show that the spatial components of this equation correspond to the Lorentz force law. Also, interpret the equation obtained from the 0-component of the above equation. [8 marks]

(c) The point charge moves under the influence of a uniform magnetic field $\underline{B}$. Starting with the assumption that the velocity $\underline{v}$ of the point charge is perpendicular to $\underline{B}$, show that the point charge moves with constant speed in a circle. Compute the radius of this circle in terms of the magnitude of the magnetic field, the mass, m, and speed, v, of the point charge. [6 marks]

What happens if the initial velocity is not perpendicular to $\underline{B}$? [2 marks]

3. A photon with frequency ν moves along the z-axis before it scatters off an electron, with mass m, which is initially at rest. Compute the frequency of the scattered photon, ν', as a function of the scattering angle of the photon, i.e. the angle the outgoing photon makes with the z-axis, and show that the photon always loses energy in the collision. [6 marks]

   Now consider the process where the electron also moves, with velocity $\underline{v} = (0, 0, -v)$ head-on towards the incoming photon which is travelling in the opposite direction. Compute the frequency ν' of the scattered photon as a function of the scattering angle. [8 marks]

   Find the minimal velocity of the electron such that the process results in a gain of energy for the photon. [6 marks]

4. The electromagnetic fields generated by a point charge q in arbitrary motion are given by

$$\underline{E}(\underline{r},t) = \frac{q}{4\pi\epsilon_0}\frac{R}{(\underline{R}\cdot\underline{u})^3}[(c^2-v^2)\underline{u} + \underline{R}\times(\underline{u}\times\underline{a})],$$

$$\underline{B}(\underline{r},t) = \frac{1}{c}\hat{\underline{R}}\times\underline{E}(\underline{r},t),$$

where $\underline{R}$ is the vector between the point charge and the observer, $\underline{v}$ is the velocity of the point charge, $\underline{u} = c\hat{\underline{R}} - \underline{v}$, and $\underline{a}$ is the acceleration of the point charge. $\underline{R}, \underline{u}, \underline{v}$, and $\underline{a}$ are all evaluated at the retarded time.

Consider a point charge q in the frame where, at the time t, it is instantaneously at rest but undergoing an acceleration $\underline{a}$.

(a) Identify the electric radiation field from the equations above, and show that it is given by

$$\underline{E}_{\text{rad}}(\underline{r},t) = \frac{\mu_0\, q}{4\pi R}\left[\left(\hat{\underline{R}}\cdot\underline{a}\right)\hat{\underline{R}} - \underline{a}\right].$$

[4 marks]

(b) Show that the Poynting vector for the radiation-fields is given by

$$\underline{S}_{\text{rad}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin\theta}{R}\right)^2\hat{\underline{R}},$$

where θ is the angle between $\underline{R}$ and $\underline{a}$. [4 marks]

(c) Calculate the total power radiated to infinity by the point charge at the time t, by suitably integrating the Poynting vector, and check that your answer is in agreement with the general result by Liénard for a point charge

$$P = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \left|\frac{\underline{v}\times\underline{a}}{c}\right|^2\right).$$

[6 marks]

Consider now a new system, consisting of *two* point charges instantaneously at rest and located at the origin of the coordinate system. The first point charge $+q$ undergoes an acceleration $\underline{a}$ as in the example above. The second point charge $-q$ undergoes an acceleration $-\underline{a}$.

(d) Find the total power radiated to infinity by this new system at the instant when the acceleration starts. [6 marks]

[Hint: $\underline{A}\times(\underline{B}\times\underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B})$
$\underline{S} = (\underline{E}\times\underline{B})/\mu_0$]

SECTION B. QUANTUM THEORY 3

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) State the Klein-Gordon and time-dependent Schrödinger equations for a free particle. In what situations are these two equations applicable? What is the possible spin value of a particle satisfying the Klein-Gordon equation? Give an example of a particle whose behaviour is described by the Klein-Gordon equation. [4 marks]

   (b) State the two Weyl equations for a free particle and describe the meaning of each term. [4 marks]

   (c) State the Dirac equation for a free particle. Why is it Lorentz invariant? What kind of functions are solutions of this equation? [4 marks]

   (d) Which algebra do the γ-matrices satisfy? How many γ-matrices are there? Which of these matrices are hermitian and which are antihermitian? [4 marks]

   (e) Show that $Tr(\gamma_\mu \gamma_\nu) = Tr(g_{\mu\nu}) = 4\delta_{\mu\nu}$, where $g_{\mu\nu}$ is the spacetime metric. [4 marks]

   (f) Solve

$$\frac{d^2}{dx^2}\psi + k^2\psi = 0,$$

   with k a constant and ψ a function. What type of solution is this? [4 marks]

   (g) Compute the differential cross section in the first Born approximation for the elastic scattering from a Coulomb potential

$$V(r) = \frac{Z_1 Z_2 e^2}{r}$$

   remembering that

$$\frac{d\sigma}{d\Omega} = \frac{A}{q^2}\left| \int_0^\infty r'\, V(r')\, \sin(qr')\, dr' \right|^2 .$$

   What does q represent?

   [4 marks]

6. (a) Write the time-dependent Schrödinger equation for a free particle and formulate the Hamiltonian using α and β matrices so as to describe relativistic particles. Give their dimension. Specify all the elements for β. [6 marks]

   (b) Derive the relations between these matrices. What type of relations are they? [6 marks]

   (c) Derive the Dirac equation in a Lorentz invariant form. [4 marks]

   (d) What is the spin of the particles whose behaviour is described by the Dirac equation? Give an example of such a particle. What is the Dirac equation for a charged particle? [4 marks]

7. (a) Derive the Klein-Gordon equation for a free (spin-0) particle from basic principles. [6 marks]

(b) Give the non-relativistic limit of the Hamiltonian and show that in this limit the Klein-Gordon equation reduces to the Schrödinger equation. [4 marks]

$$\left[\text{Hint: use } \psi = e^{-i\frac{E}{\hbar}t}\psi' \text{ in a non-relativistic limit.}\right]$$

(c) Derive the Klein-Gordon equation for a particle of charge e in the presence of an electromagnetic field described by a vector potential

$$A_\mu = (A_0, A_i).$$

[6 marks]

(d) Give an example of a particle that would satisfy such a Klein-Gordon equation in the presence of an electromagnetic field. Explain your choice. [4 marks]

8. We want to study the type of solutions that the time-dependent Schrödinger equation admits for a particle of mass m when the potential $(V(r))$ vanishes at large distance ($\lim_{r\to\infty} V(r) = 0$).

   In spherical coordinates (r,θ,ϕ) the Laplacian reads:

$$\Delta = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\right)$$

   where L^2 is the angular momentum operator.

   (a) Show that the Schrödinger equation is

$$\left(-\frac{\hbar^2}{2mr^2}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{l(l+1)\psi}{2mr^2} + V(r)\psi\right) = i\hbar\frac{d\psi}{dt}.$$

   [2 marks]

   (b) Describe physically what happens when $r \to \infty$. [2 marks]

   (c) Solve this equation in the limit $r \to \infty$ given that

$$x^2X''(x) + 2xX'(x) + (x^2 - \alpha(\alpha+1))X(x) = 0$$

   has as its solutions $X(x)$ two Bessel functions denoted $j_\alpha(x)$ and $y_\alpha(x)$ respectively. [8 marks]

   (d) Rewrite your solution using the approximations

$$j_\alpha(x) \sim \frac{\sin(x - \alpha\pi/2)}{x}$$

   and

$$y_\alpha(x) \sim -\frac{\cos(x - \alpha\pi/2)}{x}$$

   when $x \to \infty$.

   Find the phase shift. [4 marks]

   (e) Give a physical interpretation of the notion of cross section. Give the relation between cross section and phase shift. [4 marks]