$$\sigma = \frac{2\pi}{v} |M_{fi}|^2 \rho(E') V$$

Matrix element

$$M_{fi} = \langle \psi_f | \mathcal{H}_{int} | \psi_i \rangle = \int d^3\vec{x} \; \psi_f^*(\vec{x}) \, \mathcal{H}_{int} \, \psi_i(\vec{x})$$

Initial and final states are plane-waves

$$\psi_i = \frac{1}{\sqrt{V}} e^{i\vec{p}\cdot\vec{x}} \qquad \psi_f = \frac{1}{\sqrt{V}} e^{i\vec{p}'\cdot\vec{x}}$$

The Interaction Hamiltonian is the electromagnetic potential

$$\mathcal{H}_{int} = ze\,\phi(\vec{x}) = \frac{e^2 z Z}{|\vec{x}-\vec{x}_0|}$$

$$M_{fi} = \frac{1}{V} \int d^3\vec{x} \; e^{-i\vec{p}'\vec{x}} \frac{e^2 z Z}{|\vec{x}-\vec{x}_0|} e^{i\vec{p}\cdot\vec{x}}$$

$$\vec{q} = \vec{p} - \vec{p}'$$

$$= \frac{e^2 z Z}{V} \int d^3\vec{x} \; e^{i\vec{q}\cdot\vec{x}} \frac{1}{|\vec{x}-\vec{x}_0|}$$

Problem: This integral does not converge, because the e/m potential does not vanish quickly enough.

Solution: Use modified potential

$$\phi(\vec{x}) \rightarrow \frac{Ze}{|\vec{x}-\vec{x}_0|} e^{-M|\vec{x}-\vec{x}_0|}$$

$M > 0$, but later $M \rightarrow 0$.

$$\mathcal{M}_{fi} = \frac{e^2 z Z}{V} \int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{x}} \frac{1}{\underbrace{|\vec{x}-\vec{x}_0|}_{\vec{x}'}} e^{-M|\vec{x}-\vec{x}_0|} \qquad \frac{dx}{dx'} = 1$$

$$= \frac{e^2 z Z}{V} \int d^3\vec{x}'\, e^{i\vec{q}\cdot\vec{x}_0} e^{i\vec{q}\cdot\vec{x}' - M|\vec{x}'|} \frac{1}{|\vec{x}'|}$$

$$= \frac{e^2 z Z}{V} e^{i\vec{q}\cdot\vec{x}_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^\infty \cancel{r^2}\, dr \int_{-1}^{1} d\cos\theta \frac{1}{\cancel{r}} e^{i|\vec{q}|r\cos\theta - Mr}$$

$$= \frac{e^2 z Z}{V} e^{i\vec{q}\cdot\vec{x}_0}\, 2\pi \int_0^\infty \cancel{r}\, dr \frac{1}{i|\vec{q}|\cancel{r}} e^{i|\vec{q}|r\cos\theta - Mr}\Big|_{\cos\theta=-1}^{\cos\theta=+1}$$

$$= \frac{2\pi e^2 z Z}{i|\vec{q}|V} e^{i\vec{q}\cdot\vec{x}_0} \int_0^\infty dr \left[e^{(i|\vec{q}|-M)r} - e^{(-i|\vec{q}|-M)r} \right]$$

$$= \frac{2\pi e^2 z Z}{i|\vec{q}|V} e^{i\vec{q}\cdot\vec{x}_0} \left[\frac{e^{(i|\vec{q}|-M)r}}{i|\vec{q}|-M} - \frac{e^{(-i|\vec{q}|-M)r}}{-i|\vec{q}|-M} \right]_0^\infty$$

$$= \frac{2\pi e^2 z Z}{i|\vec{q}|V} e^{i\vec{q}\cdot\vec{x}_0} \left[0 - \frac{1}{i|\vec{q}|-M} - 0 + \frac{1}{-i|\vec{q}|-M} \right]$$

$$= \frac{2\pi e^2 z Z}{V i|\vec{q}|} e^{i\vec{q}\cdot\vec{x}_0} \frac{i|\vec{q}| + \cancel{M} + i|\vec{q}| - \cancel{M}}{|\vec{q}|^2 + M^2}$$

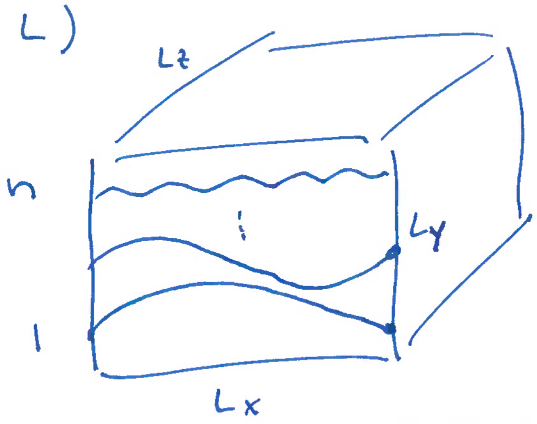
$$= \frac{4\pi e^2 z Z}{V} \frac{1}{|\vec{q}|^2 + M^2} e^{i\vec{q}\cdot\vec{x}_0}$$

$$\lim_{M\to 0} \mathcal{M}_{fi} = \frac{4\pi e^2 z Z}{V} e^{i\vec{q}\cdot\vec{x}_0} \frac{1}{|\vec{q}|^2}$$

$\rho(E')$: density of final states

Assume our experiment takes place in a finite volume V. Let V be a cube of length L, and impose periodic boundary conditions

$\psi(x_i) = \psi(x_i + L)$



$L = n \cdot \lambda$

$\lambda = \frac{2\pi\hbar}{p'}$

1-dim

$$n_x = \frac{L_x}{\lambda} \Rightarrow n_x = \frac{L_x p_i'}{2\pi\hbar}$$

3-dim (here choose $\hbar = 1$)

$$n = \frac{V\vec{p}'}{(2\pi)^3} \Rightarrow dn = \frac{V}{(2\pi)^3} d^3\vec{p}'$$

in spherical coordinates $d^3\vec{p}' = d\Omega\, |\vec{p}'|^2 d|\vec{p}'|$

$$dn(|\vec{p}'|) = \frac{V}{(2\pi)^3} d\Omega\, |\vec{p}'|^2 d|\vec{p}'|$$

$$d\rho(E') = \frac{dn}{dE'} = \frac{V}{(2\pi)^3}\, d\Omega\, |\vec{p}'|^2\, \frac{d|\vec{p}'|}{dE'}$$

$|\vec{p}'| \approx E'$ in the high-energy limit

$$= \frac{V}{(2\pi)^3}\, d\Omega\, |\vec{p}'|^2$$

Putting everything together

$$d\sigma = \frac{2\pi}{v_i}\, |\mathcal{M}_{fi}|^2\, V\, d\rho(E')$$

$$= \frac{2\pi}{v_i} \left| \frac{4\pi e^2 z Z}{\cancel{V}}\, \cancel{e^{i\vec{q}\cdot\vec{x}_0}}\, \frac{1}{|\vec{q}|^2} \right|^2 \cdot \cancel{V} \cdot \frac{\cancel{V}}{(2\pi)^3}\, d\Omega\, |\vec{p}'|^2$$

high-energy limit

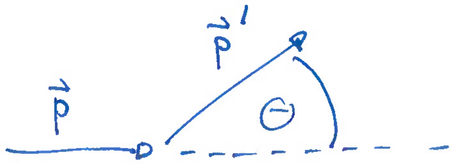
$v_i = c = 1$

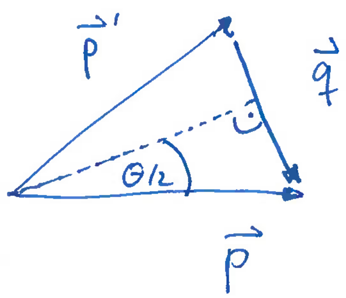
$|\vec{p}'| = E'$

$$= \frac{4 e^4 z^2 Z^2}{|\vec{q}|^4}\, d\Omega\, E'^2$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{4 e^4 z^2 Z^2 E'^2}{|\vec{q}|^4}$$

use





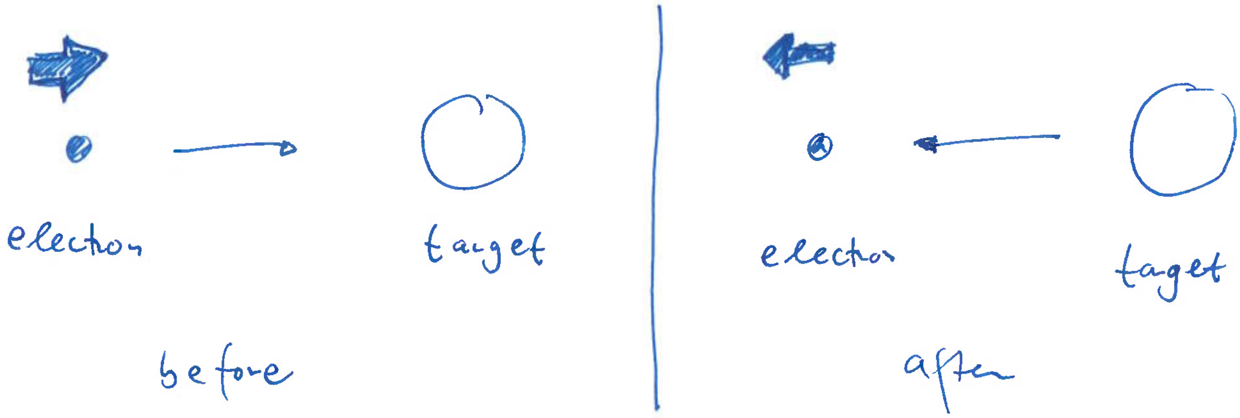
$$|\vec{q}| = 2|\vec{p}| \sin\frac{\Theta}{2}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{e^4 z^2 Z^2 E'^2}{4\,|\vec{p}|^4 \sin^4 \theta/2} = \frac{e^4 z^2 Z^2}{4 E^2 \sin^4 \theta/2}}$$

Elastic scattering $E = E'$

3 Mott cross section

The Rutherford neglects spin. For relativistic projectiles spin becomes relevant.



Spin in the direction of travel is conserved

helicity : $$h \equiv \frac{\vec{S}\cdot\vec{p}}{|\vec{S}|\,|\vec{p}|} = \pm 1$$

for spin $1/2$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott, no recoil}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \left(1 - \beta^2 \sin^2 \theta/2\right)$$

$\beta = v/c$

$$\overset{v\to c}{=} \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \cdot \cos^2 \frac{\theta}{2}$$

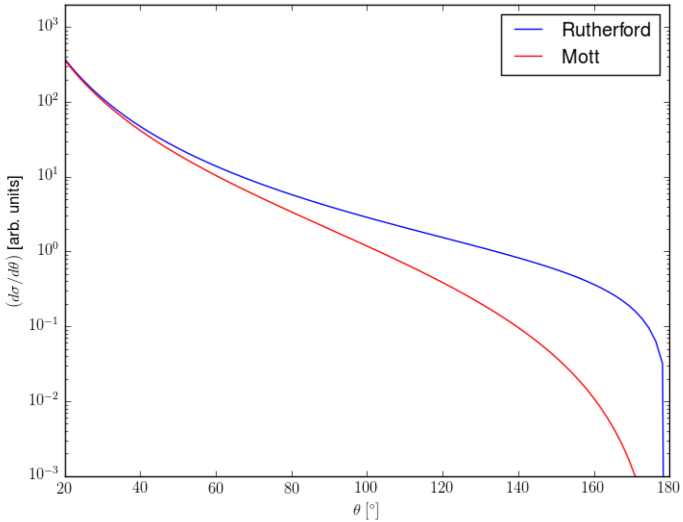


Figure 11: Rutherford and Mott differential cross sections as a function of the scattering angle. The Mott cross section is suppressed at large angles compared with the Rutherford cross section.