Mathematical Methods in Physics

Examination May/June 2018

Question 1

(a) (Unseen)

(i) $a_{1i}a_{2i} = a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = -5.$ [1 mark]

(ii) $a_{i1}a_{2i} = -6.$ [1 mark]

(iii) $a_{1i}a_{2j}\delta_{ij} = a_{1i}a_{2i} = -5.$ [1 mark]

(iv) $\delta_{ij}\,\epsilon_{ij\,k} = \epsilon_{iik} = 0.$ [1 mark]

(b) (Unseen)

(i) No. For instance, it is not closed with respect to multiplication by a scalar. Consider $\alpha \neq 1$. Then $(\alpha A)^2 = \alpha^2 A^2 = \alpha^2 I \neq I.$ [2 marks]

(ii) No. For instance, it is not closed with respect to addition. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad |A| = 0, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad |B| = 0.$$

Then $A + B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, with $|A + B| = -1 \neq 0.$

[2 marks]

(c) (Unseen)

(i) The determinant is $a - 7$. Therefore it is zero if $a = 7$. [2 marks]

(ii) It is zero because the second column is twice the first column. [2 marks]

(d) (Unseen)

(i) $\delta(2x) = \delta(x)/2$. Therefore

$$I_1 = \int_{-\infty}^{\infty} \frac{\delta(x)}{2}(e^{2(x-1)} + e^{-2(x-1)})\,dx = \cosh 2.$$

[2 marks]

(ii) $g = x^2 - 3x - 4 = 0$ if $x = \{4, -1\}$. In addition $g' = 2x - 3$. Therefore

$$I_2 = \int_{-2}^{2} x^4 \left(\frac{\delta(x-4)}{|5|} + \frac{\delta(x+1)}{|-5|} \right) dx = \int_{-2}^{2} x^4 \, \frac{\delta(x+1)}{5} \, dx = \frac{1}{5}.$$

[2 marks]

(e) (Unseen)

(i) $\bar{f}(s) = \int_0^\infty e^{3t} e^{-st} dt = 1/(s-3)$ with $s > 3$. [2 marks]

(ii) $\bar{f}(s) = \int_0^\infty (2t+1) e^{-st} dt = (2/s + 1) \int_0^\infty e^{-st} dt = 2/s^2 + 1/s$ with $s > 0$. [2 marks]

(f) (Unseen)

$$\begin{aligned} \underline{a}(\underline{r}(u)) &= (1+u)^2 \, \hat{\underline{i}} + 4u(1-3u) \, \hat{\underline{j}} + 4u \, \hat{\underline{k}}, \qquad \frac{d\underline{r}}{du} = \hat{\underline{i}} + 4 \, \hat{\underline{j}} - 3 \, \hat{\underline{k}}. \\ \underline{a} \cdot \frac{d\underline{r}}{du} &= (1+u)^2 + 16u(1-3u) - 12u. \end{aligned}$$

[2 marks]

$$I = \int_0^1 \underline{a} \cdot (d\underline{r}/du) \, du = \int_0^1 ((1+u)^2 + 4u - 16 \cdot 3u^2) \, du = -35/3.$$

[2 marks]

(g) (Unseen)

(i) $\nabla \times \underline{a}_1 = (4yz - 4yz) \, \hat{\underline{i}} + (2x - 2x) \, \hat{\underline{j}} = 0$. It is conservative. [2 marks]

(ii) $\nabla \times \underline{a}_2 = 2(\nabla \times \underline{r}) = 0$. It is conservative. [2 marks]

(h) (Unseen)

$$\begin{aligned} \frac{\partial \underline{r}}{\partial \phi} &= -z^2 \sin\phi \, \hat{\underline{i}} + z^2 \cos\phi \, \hat{\underline{j}}, \\ \frac{\partial \underline{r}}{\partial z} &= 2z \cos\phi \, \hat{\underline{i}} + 2z \sin\phi \, \hat{\underline{j}} + \hat{\underline{k}}. \end{aligned}$$

[1 mark]

(Bookwork)

$$dS = |d\underline{S}| = \left| \left(\frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial z} \right) \right| \, dz \, d\phi.$$ [1 mark]

(Unseen)

$$
\begin{aligned}
d\underline{S} &= (z^2\,\cos\phi\,\hat{\underline{i}} + z^2\,\sin\phi\,\hat{\underline{j}} - 2z^3\,\hat{\underline{k}})\,dzd\phi \\
dS &= z\sqrt{(z^2+4z^4)}\,dzd\phi.
\end{aligned}
$$

[2 marks]

Mathematical Methods in Physics

Examination May/June 2018

Question 2

(a) (Unseen)

Eigenvalues:

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (2-\lambda)(\lambda^2 - 5\lambda + 4) = 0.$$

Hence $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 4.$ [4 marks]

Eigenvectors.

$\lambda_1 = 1:$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Hence $\underline{x_1}^T = (x, -x, -x)$.

$\lambda_2 = 2:$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Hence $\underline{x_2}^T = (0, y, -y)$.

$\lambda_3 = 4:$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Hence $\underline{x_3}^T = (2z, z, z)$.

Possible normalised eigenvectors: $\underline{x_1}^T = (1, -1, -1)/\sqrt{3}$, $\underline{x_1}^T = (0, 1, -1)/\sqrt{2}$, $\underline{x_1}^T = (2, 1, 1)/\sqrt{6}$.

[4 marks]

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ S = \begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix},\ S^{-1} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix},$$

with $S^{-1} = S^T$. [3 marks]

Since $D = S^{-1}AS$, then $A = SDS^{-1}$ and $A^{-1} = SD^{-1}S^{-1}$, i.e. $A^{-1} = SD^{-1}S^T$. [3 marks]

(b) (Unseen)

On the left hand side:

$(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{d}) = ((\underline{a} \times \underline{b}) \cdot \underline{d})\underline{c} - ((\underline{a} \times \underline{b}) \cdot \underline{c})\underline{d} = [\underline{d}, \underline{a}, \underline{b}]\underline{c} - [\underline{c}, \underline{a}, \underline{b}]\underline{d}$. [2 marks]

On the right hand side:

$-(\underline{c} \times \underline{d}) \times (\underline{a} \times \underline{b}) = -((\underline{c} \times \underline{d}) \cdot \underline{b})\underline{a} + ((\underline{c} \times \underline{d}) \cdot \underline{a})\underline{b} = -[\underline{b}, \underline{c}, \underline{d}]\underline{a} + [\underline{a}, \underline{c}, \underline{d}]\underline{b}$. [2 marks]

Together:

$\underline{c}\,[\underline{d}, \underline{a}, \underline{b}] - \underline{d}\,[\underline{c}, \underline{a}, \underline{b}] + \underline{a}\,[\underline{b}, \underline{c}, \underline{d}] - \underline{b}\,[\underline{a}, \underline{c}, \underline{d}] = 0,$

which becomes

$\underline{c}\,[\underline{d}, \underline{a}, \underline{b}] - \underline{d}\,[\underline{a}, \underline{b}, \underline{c}] + \underline{a}\,[\underline{b}, \underline{c}, \underline{d}] - \underline{b}\,[\underline{c}, \underline{d}, \underline{a}] = 0$

by using the cyclic permutation property of the scalar triplet product.

[2 marks]

Mathematical Methods in Physics

Examination May/June 2018

Question 3

(a) (Unseen)

On the left hand side.

$\nabla \times \underline{a} = -2xz\,\hat{\underline{j}} - 2y\,\hat{\underline{k}}$. [2 marks]

$d\underline{S} = \hat{\underline{k}}\,dxdy$ with $z = 1$. Hence

$$\int_S (\nabla \times \underline{a}) \cdot d\underline{S} = \int_1^3 dx \int_1^2 dy\,(-2y) = -6.$$

[3 marks]

On the right hand side.

$$\begin{aligned} \underline{r}_{AB} &= \hat{\underline{i}}dx\ (y = 1, z = 1); \qquad \underline{r}_{BC} = \hat{\underline{j}}dy\ (x = 3, z = 1); \\ \underline{r}_{CD} &= -\hat{\underline{i}}dx\ (y = 2, z = 1); \qquad \underline{r}_{DA} = -\hat{\underline{j}}dx\ (x = 1, z = 1). \end{aligned}$$

[4 marks]

Hence

$$\begin{aligned} \int_C \underline{a} \cdot d\underline{r} &= \int_1^3 dx\,(1 + x) + \int_1^2 dy\,y - \int_1^3 dx\,(4 + x) - \int_1^2 dx\,y \\ &= \left[x + \frac{x^2}{2}\right]_1^3 + \left[\frac{y^2}{2}\right]_1^2 - \left[4x + \frac{x^2}{2}\right]_1^3 - \left[\frac{y^2}{2}\right]_1^2 = -6. \end{aligned}$$

[4 marks]

(b) (Unseen)

$s^2 - s - 12 = (s - 4)(s + 3) = 0$. [1 marks]

Hence

$$\bar{f}(s) = \frac{5s + 1}{s^2 - s - 12} = \frac{5s + 1}{(s - 4)(s + 3)} = \frac{A}{(s - 4)} + \frac{B}{(s + 3)},$$

with $A = 3$ and $B = 2$. [4 marks]

Then

$$\mathcal{L}^{-1}\left[\frac{3}{(s-4)} + \frac{2}{(s+3)}\right](t) = \mathcal{L}^{-1}\left[\frac{3}{(s-4)}\right](t) + \mathcal{L}^{-1}\left[\frac{2}{(s+3)}\right](t) = 3\,e^{4t} + 2\,e^{-3t}.$$

[2 marks]

MATHEMATICAL METHODS, PART B: SOLUTIONS

QUESTION 4

a) THE DIFFERENTIAL EQUATION IS SEPARABLE:

$$\frac{dy}{dx} = x(4-y) \longrightarrow \frac{dy}{4-y} = x\,dx .$$

INTEGRATING BOTH SIDES, ONE GETS

$$-\ln(4-y) = \frac{x^2}{2} + \tilde{c}, \quad \text{WHICH IMPLIES}$$

$$4-y = c\,e^{-\frac{x^2}{2}}, \quad \text{OR} \quad y = 4 - c\,e^{-x^2/2}.$$

ASKING $y(0)=5$ LEADS TO $4-c=5$, SO

$$\boxed{y = 4 + e^{-x^2/2}}$$

UNSEEN, 3 MARKS.

SUBSTITUTING BACK WE GET $y' = -x\,e^{-x^2/2}$ AND

$$-x\,e^{-x^2/2} \stackrel{?}{=} x\left(4-4-e^{-x^2/2}\right) \quad ✓$$

UNSEEN, 1 MARK

b) I START FROM THE HOMOGENEOUS PART. SUBSTITUTING $y = A e^{\lambda x}$ ONE GETS THE AUXILIARY EQUATION FOR λ: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2$ REPEATED ROOT. THEN THE GENERAL SOLUTION TO THE HOMOGENEOUS PROBLEM IS

$y_H = c_1 e^{2x} + c_2 x e^{2x}$ [WHICH INDEED SOLVES $y'' - 4y' + 4y = 0$]

UNSEEN, 2 MARKS

FOR THE PARTICULAR FUNCTION, ~~[illegible] e^{2x} [illegible] $x e^{2x}$ [illegible] $x e^{2x}$~~. TRY $f = a + bx$. SUBSTITUTING:

$-4b + 4(a+bx) = 4x-4 \longrightarrow b=1$ AND HENCE $a=0$. THE GENERAL SOLUTION OF THE INHOMOGENEOUS PROBLEM IS THEN

$$\boxed{y = x + c_1 e^{2x} + c_2 x e^{2x}}$$

UNSEEN, 2 MARKS

c) $\mathcal{L}[f(x)](s) = \bar{f}(s)$, THEN $\mathcal{L}[f'(x)](s) = -f(0) + s\,\bar{f}(s)$

BOOKWORK, 2 MARKS

$$\mathcal{L}[\cos x] = \mathcal{L}[\sin'(x)] = -\sin(0) + s\,\mathcal{L}[\sin(x)] = \boxed{\frac{s}{1+s^2}}$$

BOOKWORK, 2 MARKS

D) USING THE HINT; ONE CAN FIND THE PARTICULAR SOLUTION USING

$$f_P = -f_1 \int \frac{f_2(x)\,h(x)}{W(x)}\,dx + f_2 \int \frac{f_1(x)\,h(x)}{W(x)}\,dx$$

IN OUR CASE: $W = \begin{vmatrix} e^{-x} & e^{-4x} \\ -e^{-x} & -4e^{-4x} \end{vmatrix} = -3e^{-5x}$, SO $\frac{h(x)}{W(x)} = -1$

AND $f_P(x) = +e^{-x}\int dx\, e^{-4x} - e^{-4x}\int dx\, e^{-x} = -\frac{e^{-5x}}{4} + e^{-5x}$ AND

THE GENERAL SOLUTION IS

$$\boxed{f(x) = \frac{3}{4}e^{-5x} + C_1 e^{-x} + C_2 e^{-4x}}$$

UNSEEN, 4 MARKS

E) THE EULER EQUATION READS: $a_2 x^2 \frac{d^2 f}{dx^2} + a_1 x \frac{df}{dx} + a_0 f = g(x)$

NOW: $x = e^t$; $\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx} = \frac{df}{dt}\cdot\frac{1}{x}$ AND $\frac{d^2 f}{dx^2} = \frac{d}{dx}\left[\frac{df}{dt}\cdot\frac{1}{x}\right] =$

$= \frac{1}{x^2}\frac{d^2 f}{dt^2} - \frac{1}{x^2}\frac{df}{dt}$. SUBSTITUTING, THE EULER EQUATION BECOMES

$a_2\left[\frac{d^2 f}{dt^2} - \frac{df}{dt}\right] + a_1\frac{df}{dt} + a_0 f = g(e^t)$, LINEAR WITH CONSTANT

COEFFICIENTS. BOOKWORK, 4 MARKS

F) I LOOK FOR $p = p(x,y)$ SUCH THAT $u(x,y) = f(p)$. SUBSTITUTING:

$\frac{du}{dp}\frac{\partial p}{\partial x} + 2x\frac{du}{dp}\frac{\partial p}{\partial y} = 0 \longrightarrow \frac{\partial p}{\partial x} + 2x\frac{\partial p}{\partial y} = 0$. ALSO:

$dp = 0 = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$. COMBINING THE TWO: $-2x\,dx + dy = 0$, OR

2

$Y = X^2 + C$ AND THEN $u(X,Y) = f(Y - X^2)$. TO SPECIFY THE BOUNDARY VALUE : $\boxed{u(x,y) = g(Y - X^2) + 5 \text{ , WITH } g(0) = 0}$ UNSEEN, 4 MARKS

6) $-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$ BOOKWORK, 1 MARK

SEPARATION OF VARIABLES: $\Psi(x,t) = \psi(x)\varphi(t)$ AND

$$i\hbar \left(\frac{\partial \varphi}{\partial t}\right) \psi(x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} \varphi(t) + V(x)\psi(x)\varphi(t) \text{ , SO}$$

$$\begin{cases} i\hbar \frac{1}{\varphi} \frac{\partial \varphi}{\partial t} = \text{CONSTANT, } E. \\ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\,\psi(x) \end{cases}$$

SOLUTION FOR THE TIME-DEPENDENCE: $\frac{1}{\varphi}\frac{\partial \varphi}{\partial t} = \frac{-iE}{\hbar} \longrightarrow$

$\boxed{\varphi(t) = C\, e^{-iE/\hbar \cdot t}}$ BOOKWORK, 3 MARKS

MATHEMATICAL METHODS, PART B: SOLUTIONS

QUESTION 5

a) BY WRITING

$$f''(x) + \frac{2}{x} f'(x) + \left(1 - \frac{\alpha(1+\alpha)}{x^2}\right) f(x) = 0$$ AND COMPARING

WITH THE GENERAL CASE $f''(x) + p(x) f'(x) + q(x) f(x) = 0$:

p, q NOT REGULAR IN $x \to 0 \Rightarrow$ SINGULAR POINT. BUT

$x\,p(x)$ AND $x^2 q(x)$ REGULAR $\longrightarrow$ $x=0$ IS A REGULAR SINGULAR POINT

BOOKWORK, 2 MARKS

b) IN GENERAL, p AND q ARE NOT-ANALYTIC AT $x=0$, SO WE DO NOT EXPECT A TAYLOR EXPANDABLE SOLUTION. WE NEED A MORE GENERAL FORM FOR THE SOLUTION, SO WE ASK $f(x) = x^\sigma \sum_{i=0}^{\infty} x^i a_i$. BOOKWORK, 2 MARKS

c) TO FIND THE INDICIAL EQUATION, I PLUG THE FROBENIUS SERIES IN THE ORIGINAL EQUATION AND MULTIPLY BY $x^{\sigma-2}$:

$$x^{\sigma-2}\left[\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1)\, x^{n+\sigma-2} a_n + 2(n+\sigma)\, x^{n+\sigma-2} a_n + \right.$$

$$\left. + \left(x^2 - \alpha(1+\alpha)\right) x^{n+\sigma-2} a_n \right] = \sum_{n=0}^{\infty} a_n (n+\sigma)(n+\sigma-1)\, x^n + 2(n+\sigma)\, x^n a_n +$$

$+ (x^2 - \alpha(1+\alpha))\, x^n a_n = 0$. I NOW SET $x=0$, AND ARE LEFT WITH THE INDICIAL EQUATION

$\sigma(\sigma-1) + 2\sigma - \alpha(1+\alpha) = 0$, WHICH HAS SOLUTION $\sigma = \alpha$, $\sigma = -1-\alpha$ AND FOR $\alpha = 0$ WE GET $\sigma = 0, -1$

UNSEEN, 4 MARKS

d) SINCE $\sigma = -1$ IS A SOLUTION, I LOOK FOR SOLUTIONS OF THE FORM $f(x) = g(x)/x$.

SUBSTITUTING: $f' = \frac{g'}{x} - \frac{g}{x^2}$, $f'' = \frac{g''}{x} - \frac{g'}{x^2} - \frac{g'}{x^2} + 2\frac{g}{x^3}$

SUBSTITUTING, WE GET

$$\frac{g''}{x} - \frac{2g'}{x^2} + \frac{2g}{x^3} + \frac{2g'}{x^2} - \frac{2g}{x^3} + \frac{g}{x} = 0, \quad \text{OR} \quad \underline{g'' + g = 0}$$

UNSEEN, 3 MARKS

THE SOLUTION OF THIS EQUATION IS WELL-KNOWN: $e^{\pm i x}$ OR $\cos x$, $\sin x$.

THE GENERIC SOLUTION FOR f IS THEN

$$f(x) = C_1 \frac{\sin x}{x} + C_2 \left(- \frac{\cos x}{x} \right)$$

WHERE THE MINUS SIGN AT THIS POINT IS JUST BY CONVENTION.

THE WRONSKIAN OF THE TWO SOLUTIONS IS $\neq 0$, SO THEY ARE INDEPENDENT

UNSEEN, 3 MARKS

c) USING $\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r} \right) = 2r \frac{\partial \psi}{\partial r} + r^2 \frac{\partial^2 \psi}{\partial r^2}$, IT IS IMMEDIATE TO SEE

THAT THE EQUATION WE SOLVED IS THE RADIAL SCHRÖDINGER EQUATION, UP TO A RESCALING, FOR $l = 0$. INDEED, IT IS STRAIGHTFORWARD TO OBTAIN

$$r^2 \psi''(r) + 2r \psi'(r) + \frac{2(E-V) m r^2}{\hbar^2} \psi(r) = 0$$ FOR THE SCHRÖDINGER

EQUATION WITH $l = 0$. DEFINING ALSO HERE $\psi(r) = \frac{\varphi(r)}{r}$, ONE GETS AS BEFORE

$$\varphi''(r) + \frac{2(E-V)m}{\hbar^2} \varphi(r) = 0.$$ SETTING $V = 0$, ONE GETS

$$\varphi''(r) + \frac{2Em}{\hbar^2} \varphi = 0,$$ WHOSE SOLUTIONS ARE $\varphi = \sin\left(\sqrt{\frac{2Em}{\hbar^2}}\, r \right) C_1 +$

$+ \cos\left(\sqrt{\frac{2Em}{\hbar^2}} \right) C_2$, AND THEN $\psi(r) = C_1 \frac{\sin kr}{r} + C_2 \left[- \frac{\cos kr}{r} \right]$,

WHERE I DEFINED $k = \sqrt{\frac{2Em}{\hbar^2}}$ UNSEEN, 4 MARKS

ASKING THE SOLUTION TO BE REGULAR IN $z=0$ FORCES US TO SET

$C_2 = 0$. ASKING $\Psi(L) = 0$ THEN IMPLIES $KL = n\pi$, AND THEN

$$E = \frac{\hbar^2 n^2 \pi^2}{2mL^2},$$ QUANTIZED. UNSEEN, TWO MARKS

MATHEMATICAL METHODS, PART B: SOLUTIONS

QUESTION 6

a) $y'' + \frac{1}{2x} y' - \frac{1}{4x} y \equiv y'' + p(x) y' + q(x) y = 0.$

p AND q ARE NON ANALYTIC IN $x=0$, WHICH IS THEN A SINGULAR POINT.
HOWEVER, $x\,p(x)$ AND $x^2 q(x)$ ARE ANALYTIC, SO $x=0$ IS A REGULAR SINGULAR POINT. BOOKWORK, 2 MARKS

b) THE APPROPRIATE SERIES NEAR A REGULAR SINGULAR POINT IS

$$y = x^{\sigma} \sum_{n=0}^{\infty} a_n x^n$$ BOOKWORK, 1 MARK

USING $y' = \sum_{n=0}^{\infty} (\sigma+n)\, a_n x^{n-1+\sigma}$, $y'' = \sum_{n=0}^{\infty} (\sigma+n)(\sigma+n-1)\, a_n x^{n-2+\sigma}$,

WE GET FOR THE DIFFERENTIAL EQUATION

$$\sum_{n=0}^{\infty} a_n \left[(\sigma+n)(\sigma+n-1)\, x^{n+\sigma-2} + \frac{n+\sigma}{2} x^{n+\sigma-2} - \frac{x^{n+\sigma-1}}{4} \right] = 0$$

UNSEEN, 1 MARK

c) I MULTIPLY THE SERIES BY $x^{2-\sigma}$, TO GET

$$\sum_{n=0}^{\infty} a_n x^n \left[(\sigma+n)\left(\sigma+n-\tfrac{1}{2}\right) - \frac{x}{4} \right]$$

SETTING $x=0$, ONLY THE $n=0$ TERM SURVIVES BOOKWORK, 2 POINTS

SINCE $a_0 \neq 0$ BY CONSTRUCTION, WE HAVE

$$a_0\, \sigma\left(\sigma - \tfrac{1}{2}\right) = 0 \longrightarrow \sigma = 0 \text{ AND } \sigma = \tfrac{1}{2}$$

THEY ARE DISTINCT, AND DON'T DIFFER BY AN INTEGER SO THEY WOULD LEAD TO INDEPENDENT SOLUTIONS UNSEEN, 4 MARKS

d) CONSIDER $\sum_{n=0}^{\infty} a_n x^n \left[(\sigma+n)\left(\sigma+n-\frac{1}{2}\right) - \frac{\cancel{x}}{4} \right] = 0.$

EQUATING TERMS WITH THE SAME POWER OF x, WE GET THE RECURSION

$$(\sigma+n)(\sigma+n-1/2)\, a_n - \frac{a_{n-1}}{4} = 0.$$

FOR $\sigma=0$, THIS BECOMES $n\left(n-\frac{1}{2}\right) a_n - \frac{a_{n-1}}{4} = 0$ UNSEEN, 2 MARKS

FOR $\sigma=\frac{1}{2}$: $n\left(n+\frac{1}{2}\right) a_n - \frac{a_{n-1}}{4} = 0$ UNSEEN, 2 MARKS

e) USING $n=1$: $(\sigma+1)(\sigma+1-1/2)\, a_1 = +\frac{a_0}{4}$

$n=2$ $(\sigma+2)(\sigma+2-1/2)\, a_2 = +\frac{a_1}{4}$

$n=3$ $(\sigma+3)(\sigma+3-1/2)\, a_3 = +\frac{a_2}{4}$

IN THE $\sigma=0$ CASE, WE GET: $a_1 = \frac{a_0}{2}$

$$a_2 = \frac{a_1}{12} = \frac{a_0}{24}$$

$$a_3 = \frac{a_2}{30} = \frac{a_0}{720}$$

IN THE $\sigma=\frac{1}{2}$ CASE, WE GET: $a_1 = \frac{a_0}{6}$

$$a_2 = \frac{a_1}{20} = \frac{a_0}{120}$$

$$a_3 = \frac{a_2}{42} = \frac{a_0}{5040}$$

UNSEEN, 4 MARKS

FINALLY, WE CAN WRITE THE SOLUTION AS

$$y = c_1\left[1 + \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{720} + O(x^4)\right] +$$

$$+ c_2\sqrt{x}\left[1 + \frac{x}{6} + \frac{x^2}{120} + \frac{x^3}{5040} + O(x^4)\right]$$

UNSEEN, 2 MARKS

(INCIDENTALLY, IT IS POSSIBLE TO RESUM THESE SERIES. THE FIRST TERM SUMS TO $\cosh\sqrt{x}$, THE SECOND TO $\sinh\sqrt{x}$)