Mathematical Methods in Physics

Weekly Problems 8. Solution

8.1

a) The straight line between A and B could be represented as follows (mind you, it is not unique)

$$\mathbf{r}(t) = (1,2,3)+t((4,5,9)-(1,2,3)) \longrightarrow \mathbf{r}(t) = (1+3t)\,\mathbf{i}+(2+3t)\,\mathbf{j}+(3+6t)\,\mathbf{k} \qquad 0 \le t \le 1.$$

$\boxed{\text{1 mark}}$

Then

$$\mathbf{a}(\mathbf{r}(t)) = (4+9t)\,\mathbf{i} + (5+9t)\,\mathbf{j} - \,\mathbf{k}, \qquad \mathbf{r}'(t) = 3\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}.$$

The line integral is given by

$$I = \int_C \mathbf{a}\cdot \mathrm{d}\mathbf{r} = \int_0^1 (\mathbf{a}\cdot\mathbf{r}')\,dt = \int_0^1 (54t+21)\,dt = 48. \qquad \boxed{\text{1 mark}}$$

b) Set $t = x$ then $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ with $0 \le x \le 2$. $\boxed{\text{1 mark}}$

In addition $\mathbf{a}(\mathbf{r}(t)) = (t+t^3)\,\mathbf{i} + (t^3+t^2)\,\mathbf{j} + (t-t^2)\,\mathbf{k}$ and $\mathbf{r}' = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$.

The line integral is

$$I = \int_0^2 \big((t+t^3) + 2t(t^3+t^2) + 3t^2(t-t^2)\big)\,dt = \frac{98}{5}. \qquad \boxed{\text{2 marks}}$$

8.2

The vector area element is given by

$$d\mathbf{S} = \left(\frac{d\mathbf{r}}{d\phi}\times\frac{d\mathbf{r}}{dz}\right)d\phi dz = \begin{pmatrix} -\sqrt{z}\sin\phi \\ \sqrt{z}\cos\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\phi/(2\sqrt{z}) \\ \sin\phi/(2\sqrt{z}) \\ 1 \end{pmatrix} d\phi dz = \begin{pmatrix} \sqrt{z}\cos\phi \\ \sqrt{z}\sin\phi \\ -1/2 \end{pmatrix} d\phi dz,$$

hence the scalar area element is $dS = (z+1/4)^{1/2} d\phi\, dz$. $\boxed{\text{1 mark}}$

a)

$$I_1 = \int_S dS = 2\pi \int_0^2 dz \sqrt{z + \frac{1}{4}}.$$

Set $t = (z + 1/4)^{1/2}$. Then $dt = dz/2t$ and the integral becomes

$$I_2 = 2\pi \int_{1/2}^{3/2} 2\,t^2 dt = \frac{13\pi}{3}. \qquad \boxed{\text{2 marks}}$$

b)

$$\nabla \times \mathbf{a} = -2\,\mathbf{k},$$

and the integral is

$$I_2 = \int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \int_0^2 dz \int_0^{2\pi} d\phi = 4\pi. \qquad \boxed{\text{2 marks}}$$