Mathematical Methods II
Weekly problem set 3

(1) Use the method of Laplace transforms to solve

$$y'' + 5y' + 6y = 0,$$

subject to the boundary condition $y(0) = 1$, $y'(0) = -4$.

Solution

$$\mathcal{L}[y''] = s^2\bar{y} - sy(0) - y'(0) = \bar{s}\bar{y}^2 - s + 4$$

$$\mathcal{L}[y'] = s\bar{y}(s) - y(0) = \bar{s}\bar{y} - 1$$

Sub into ODE

$$(s^2 + 5s + 6)\bar{y} - s + 4 - 5 = 0$$

$$(s + 2)(s + 3)\bar{y} = s + 1$$

Partial fractions

$$\bar{y}(s) = \frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = \lim_{s\to -2}[\bar{y}(s+2)] = -1$$

$$B = \lim_{s\to -3}[\bar{y}(s+3)] = 2$$

So

$$\bar{y} = -\frac{1}{s+2} + \frac{2}{s+3}$$

Reversing the transform

$$y(x) = -e^{-2x} + 2e^{-3x}$$

(2) Consider the following differential equation

$$y'' + 3y' + 2y = 10\cos(x).$$

(a) Compute the complementary function,

$$y_c(x) = c_1y_1(x) + c_2y_2(x).$$

Solution Auxiliary equation

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{9-4}}{22} = -1, -2$$

$$y_c = c_1e^{-x} + c_2e^{-2x}$$

(b) Use the Wronskian method to solve the inhomogeneous problem. *Hint*: if $y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$, then $c_1' = -\frac{h(x)}{W(x)}y_2$ and $c_2' = \frac{h(x)}{W(x)}y_1$ where W is the Wronskian and h is the inhomogeneous term.

Solution Calculate the Wronskian

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

Since $10\cos(x)$, using the hints

$$k_1' = -\frac{f(x)}{W} y_2 = -(10\cos x)(-e^{3x})(e^{-2x}) = 10\cos x e^{x}$$

$$k_2' = \frac{f(x)}{W} y_1 = (10\cos x)(-e^{3x})(e^{-x}) = -10\cos x e^{2x}$$

Integrate by parts

$$k_1 = 10\int \cos x e^{x} dx = 5e^{x}(\sin x + \cos x) + c_3$$

$$k_2 = -10\int \cos x e^{2x} dx = -2e^{2x}\left(\sin x + 2\cos x\right) + c_4$$

So

$$\begin{aligned} y_p = k_1 y_1 + k_2 y_2 &= \left(5e^{x}\left(\sin x + \cos x\right) + c_3\right) e^{-x} + \left(-2e^{2x}\left(\sin x + 2\cos x\right) + c_4\right) e^{-2x} \\ &= c_3 e^{-x} + c_4 e^{-2x} + 3\sin x + \cos x \end{aligned}$$

Finally, the general solution

$$y = y_c + y_p = c_5 e^{-x} + c_6 e^{-2x} + 3\sin x + \cos x$$