Mathematical Methods II
Weekly problem set 1

Solve problems 14.1, 14.2(a), 14.3(a) and 14.3(b) in Ripley, Hobson and Bence, Mathematical Methods for Physics and Engineering. For your convenience these problems are written below:

(14.1) A radioactive isotope decays in such a way that the number of atoms present at a given time, N(t), obeys the equation

$$\frac{dn}{dt} = -\lambda N.$$

If there are initially N_0 atoms present, find $N(t)$ at a later time.

Solution

$$\frac{dn}{dt} = -\lambda N$$

Separate the variables

$$\frac{dn}{N} = -\lambda dt$$

Integrate

$$\ln N = -\lambda t + c$$

Take exponentials

$$N = e^{-\lambda t} e^{c}$$

$$N = Ae^{-\lambda t}$$

We are given that $N(t = 0) = N_0$, so $N(0) = A = N_0$, therefore

$$N(t) = N_o e^{-\lambda t}$$

(14.2) Solve the following equation by the separation of variables method

(a) $y^{'} - xy^3 = 0$

Solution Separate variables

$$\frac{dy}{y^3} = xdx$$

Integrate and rearrange

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c$$

$$\frac{2y^2}{1} = \frac{2}{-x^2 - 2c}$$

$$y = \pm \frac{1}{\sqrt{-x^2 - 2c}}$$

(14.3) Show that the following equations either are exact or can be made exact, and solve them:

(a) $y(2x^2y^2+1)y^{'}+x(y^4+1)=0$,

Solution Rewrite in usual form by $\times dx$

$$x(y^4+1)dx+y(2x^2y^2+1)dy=0$$

Here we define $x(y^4+1)=A$, $y(2x^2y^2+1)=B$.

Check if exact,

$\partial_y A=4xy^3$ and $\partial_x B=4xy^3 \rightarrow$ exact!

Find U,

$\partial_x U=A$ and $\partial_y U=B$ so

$$U=\int Adx+F(y)=\frac{x^2}{2}(y^4+1)+F(y)$$

To fix $F(y)$

$$B=\partial_y U=2x^2y^3+F^{'}(y)=y(2x^2y^2+1)$$

Thus

$$F^{'}(y)=y$$

$$F(y)=\frac{y^2}{2}+c$$

and

$$U(x,y)=\frac{x^2}{2}(y^4+1)+\frac{y^2}{2}+c=\text{constant}$$

which is constant since $du=0$ so y defined by

$$x^2y^4+x^2+y^2=K$$

This is sufficient. but can close the solution with some extra work, giving

$$y^2=-\frac{1\pm\sqrt{1+8Kx^2-4x^4)}}{2x^2}=y_{\pm}$$

Four solutions are therefore $y=\pm\sqrt{y_+}$ and $y=\pm\sqrt{y_-}$

(b) $2xy^{'}+3x+y=0$,

Solution Rewrite in usual form by $\times dx$

$$(3x+y)dx+2xdy=0$$

Here we define $(3x+y)=A$, $2x=B$.

Check if exact,

$\partial_y A=1$ and $\partial_x B=2 \rightarrow$ not exact!

Look for simple integration factor,

$$\frac{1}{B}(\partial_y A - \partial_x B) = \frac{1}{2x}(1-2) = -\frac{1}{2x} = f(x)$$

So integration factor given by

$$\frac{dn}{n} = f(x)$$

$$\ln n = \int f(x)$$

$$n = e^{\int f(x)dx} = e^{-\int \frac{1}{2x}dx} = e^{-\frac{1}{2}\ln x} = \frac{1}{e^{\ln x^{1/2}}} = \frac{1}{\sqrt{x}}$$

$$2\sqrt{x}y^{'} + 3\sqrt{x} + \frac{y}{\sqrt{x}} = 0 \rightarrow \text{exact!}$$

Alternatively, this is a linear equation

$$y^{'} + \frac{1}{2x}y = -\frac{3}{2}$$

$$n = e^{\int \frac{1}{2x}dx} = \sqrt{x}$$

$$\sqrt{x}y^{'} + \frac{y}{2\sqrt{x}} = -\frac{3}{2}\sqrt{x}$$

or

$$2\sqrt{x}y^{'} + 3\sqrt{x} + \frac{y}{\sqrt{x}} = 0$$

Same equation as above, using a different integration factor.

Consider,

$$2\sqrt{x}y^{'} + 3\sqrt{x} + \frac{y}{\sqrt{x}} = 0$$

$$(2\sqrt{x})dy + \left(3\sqrt{x} + \frac{y}{\sqrt{x}}\right)dx = 0$$

Here,

$$A = 3\sqrt{x} + \frac{y}{\sqrt{x}} \rightarrow \partial_y A = \frac{1}{\sqrt{x}}$$

$$B = 2\sqrt{x} \rightarrow \partial_x B = \frac{1}{\sqrt{x}} = \partial_y A$$

Find U,

$\partial_x U = A$ and $\partial_y U = B$. B is simpler than A, start from

$$\partial_y U = 2\sqrt{x} \rightarrow U = 2\sqrt{x}y + F(x)$$

Now

$$A = 3\sqrt{x} + \frac{y}{\sqrt{x}} = \partial_x U = \frac{y}{\sqrt{x}} + F^{'}(x)$$

$$F^{'}(x) = 3\sqrt{x}$$

$$F = 2x^{3/2} + c$$

$$U = 2\sqrt{x}y + 2x^{3/2} + c = \text{constant}$$

$$\sqrt{x}y = K - x^{3/2}$$

$$y = \frac{K}{\sqrt{x}} - x$$