Workshop 5: Central Forces

A particle of mass m moves in a central force field with potential energy given by $V(r)$. In spherical polar coordinates, the Lagrangian of the particle can be written as

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) - V(r).$$

(a) Using the fact that the azimuthal angle, ϕ, is an ignorable coordinate, determine the associated constant of the motion, J. What does this constant represent? Express the total energy of the particle as a function of the single variable, r, in a way that incorporates the effective potential energy

$$V_{\text{eff}} = V(r) + \frac{J^2}{2mr^2}$$

and explain why no θ dependence is required.

(b) Assuming that $V(r) = Ar^{n+1}/(n+1)$ where $A > 0$, for each of the cases $n+1 < -2$, $-2 < n+1 < 0$ and $n+1 > 0$, sketch the radial variation of the effective potential. In each case, mark the radius, r_0, where the potential is stationary with respect to radius and a circular orbit exists, and state whether or not the orbit is stable or unstable. Calculate an expression for the radius r_0.

(c) By performing a Taylor series expansion of the effective potential about the point $r = r_0$, show that, for small perturbations away from a stable circular orbit, the radius of the particle performs simple harmonic motion with an angular frequency

$$\omega = \sqrt{n+3}\frac{J}{mr_0^2}.$$

(d) By comparing this angular frequency with that for a circular orbit at radius r_0, determine which values of n give rise to closed orbits. Describe the orbits for the cases $n = -2$ and $n = 1$.