University of Durham

EXAMINATION PAPER

May/June 2016 Examination code: PHYS3631-WE01

FOUNDATIONS OF PHYSICS 3B

SECTION A. Statistical Physics
SECTION B. Condensed Matter Physics part 1
SECTION C. Condensed Matter Physics part 2

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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SECTION A. STATISTICAL PHYSICS

Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) Consider a system of N free particles, with fixed total energy U, occupying a volume V. Each particle is in a single-particle state i with (nondegenerate) energy ϵ_i. The distribution $\{n_1, n_2, \ldots\}$ gives the number of particles n_i in the single-particle state i, with $\sum_i n_i = N$.

   Give the number of microstates $\Omega(\{n_i\})$ of the distribution $\{n_i\}$, stating whether all distributions compatible with (N, U, V) have the same number of microstates. Give the statistical entropy of the distribution $\{n_i\}$ according to Boltzmann, in the limit of large numbers, $N, n_i \gg 1$. [4 marks]

   [Hint: The Stirling formula for $k \gg 1$ is: $\ln k! = k \ln k - k$.]

   (b) For the system in part(a), explain briefly how the number of microstates $\Omega(\{n_i\})$ is modified when the N particles are indistinguishable. How is the statistical entropy modified accordingly? [4 marks]

   (c) A classical system Σ is divided into two independent subsystems A and B. Show that the partition function of Σ is the product of the partition functions of the subsystems: $Z_\Sigma = Z_A \times Z_B$. What is the generalisation of this result for the partition function of a system Σ composed of N independent distinguishable particles. [4 marks]

   [Hint: When A is in state i with energy E_i^A and B is in state j with energy E_j^B, then Σ is in state (i, j) with energy $E_i^A + E_j^B$.]

   (d) Calculate the partition function and the free energy of a system of N spin-one particles localised on a lattice, in thermal equilibrium at temperature T. The energy levels of each particle are $-\epsilon, 0, \epsilon$. [4 marks]

   (e) A particle can be in one of six degenerate states, $i = 1, 2, \ldots, 6$. The probability that the particle is in state i is p_i and the probabilities are normalised: $\sum_{i=1}^{6} p_i = 1$. The Gibbs statistical entropy of the particle for the probability distribution $\{p_1, p_2, \ldots, p_6\}$ is $S(\{p_i\}) = -k_\mathrm{B} \sum_{i=1}^{6} p_i \ln p_i$. Show that the set of probabilities that maximise the entropy, under the constraint of normalisation, satisfy $p_i = 1/6$. Use no other constraint. Comment on the principle of equal *a priori* probabilities for a microcanonical ensemble. [4 marks]

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2. (a) Show that the density of states in wavevector (k) space for a free particle of mass m, in three dimensions, enclosed in a box of volume V is

$$g(k)\,\delta k \propto k^2\,\delta k.$$

Obtain the constant of proportionality. [4 marks]

(b) Consider the single-particle partition function for a gas of N free classical particles of mass m, in a volume V at a temperature T with $Z_1 = C\,V$. The constant C is independent of V. Without deriving C, give its units and state which quantities of the system it must depend upon. Without deriving C, specify the ideal gas law for the system. [2 marks]

[Hint: $P = -[\partial F/\partial V]_T$, with F the Helmholtz free energy.]

(c) Show that the most probable speed v_0 and the root-mean-square speed $v_{\rm rms}$ for a gas of free particles of mass m in a box of volume V at temperature T, are:

$$v_0 = \sqrt{\frac{2k_{\rm B}T}{m}}\,, \quad v_{\rm rms} = \sqrt{\frac{3k_{\rm B}T}{m}}$$

[4 marks]

[Hint: $\int_0^\infty dx\,\exp(-ax^2) = (1/2)\sqrt{\pi/a}$, $\int_0^\infty dx\,x\,\exp(-ax^2) = 1/(2a)$. Differentiate w.r.t. a to obtain $\int_0^\infty dx\,x^n\,\exp(-ax^2)$, for $n = 2, 3, \ldots$.]

(d) What is the most probable speed and root-mean-square speed for neutrons in thermal equilibrium at 77 K? [2 marks]

(e) Calculate the average reciprocal speed $\langle 1/v\rangle$ for a gas of free particles of mass m in a box of volume V at a temperature T. The De Broglie wavelength of a particle with momentum p is h/p. What is the average De Broglie wavelength $\langle h/p\rangle$ of the gas particles? Comment on a related quantity that was discussed in lectures (you do not need to give its exact definition). [4 marks]

(f) Using the average De Broglie wavelength $\langle h/p\rangle$ above, give a criterion for the quantum (or degenerate) limit and for the classical (or dilute gas) limit for a system of N free particles of mass m in a box of volume V at temperature T. [4 marks]

[Hint: Consider the average separation of the gas particles $\sim (V/N)^{1/3}$.]

3. (a) A system of N particles is in thermal equilibrium with a heat bath at a temperature T. The single-particle energy levels ϵ_i are non-degenerate and the single-particle partition function is $Z_1 = \sum_i \exp(-\beta\,\epsilon_i)$, where $\beta = +1/(k_B T)$. The average energy per particle is $\langle\epsilon\rangle = \sum_i p_i\,\epsilon_i$, where p_i is the probability that the i-th single-particle state is occupied.

Show that $\langle\epsilon\rangle$ is given by

$$\langle\epsilon\rangle = -\frac{\partial \ln Z_1}{\partial\beta}$$

What is the internal energy U of the system? [2 marks]

(b) Show that the second derivative of $\ln Z_1$ is equal to the variance in the single-particle energy, $\sigma_\epsilon^2 = \langle\epsilon^2\rangle - \langle\epsilon\rangle^2$:

$$\frac{\partial^2 \ln Z_1}{\partial\beta^2} = \sigma_\epsilon^2$$

where $\langle\epsilon^2\rangle = \sum_i p_i\,\epsilon_i^2$. [4 marks]

(c) Show that the heat capacity under constant volume is given by:

$$C_V = \frac{N\,\sigma_\epsilon^2}{k_B\,T^2}$$

[2 marks]

$$[\,\text{Hint: First show:}\ \frac{\partial}{\partial T} = -\frac{1}{k_B T^2}\frac{\partial}{\partial\beta}\,]$$

(d) Calculate the average energy per particle $\langle\epsilon\rangle$ and the standard deviation σ_ϵ, for a system of N particles, with single-particle energy levels, $\epsilon_1 = 0$, $\epsilon_2 = \Delta$ with $\Delta > 0$. Comment briefly on the limit of $\langle\epsilon\rangle$, σ_ϵ for high and for low temperatures. [4 marks]

(e) For the system of particles in part(d), show that the entropy S is given by

$$S = N k_B \left[\frac{\beta\Delta\,\exp(-\beta\Delta)}{1+\exp(-\beta\Delta)} + \ln[1+\exp(-\beta\Delta)]\right]$$

[4 marks]

[Hint: Use the Gibbs expression for the statistical entropy.]

(f) The energy gap Δ between the two energy levels of the previous system is (i) reduced adiabatically to half its initial value, (ii) increased adiabatically to twice its initial value. Find the change in temperature in the two cases. Give a physical example of this effect. [4 marks]

SECTION B. CONDENSED MATTER PHYSICS part 1
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) Write an expression for the group velocity of Bloch electrons. By considering the current carried by a collection of electrons in a band show that a completely filled band makes no contribution to the current carried by a crystal. [4 marks]

   (b) In the presence of an applied magnetic field B the periodicity $\Delta(1/B) = 2\pi e/(\hbar S)$ of the de Haas-van Alphen oscillations measures the extremal cross-sectional area S in k-space of the Fermi surface normal to the direction of $\mathbf{B}$. Calculate the period expected for a single crystal of rubidium metal within the free electron model given that the Fermi wavevector is 0.70×10^{10} m^{-1}. [4 marks]

   (c) Calculate the energy difference between the spin states of an isolated spin-only electron in the presence of a magnetic field of $B = 10.0$ Tesla. Take the Bohr magneton to be $\mu_{\rm B} = 9.27 \times 10^{-24}$ J T^{-1}. [4 marks]

   (d) Sketch the form of the temperature dependence of the magnetic susceptibility, χ, of an antiferromagnetic single crystal assuming that χ is measured in a small magnetic field. Identify on your sketch the Neèl temperature, $T_{\rm N}$, and briefly explain the origin of the features of χ at a temperature of $T = 0$ K. [4 marks]

   (e) What is indirect exchange? Explain how the RKKY exchange interaction can give rise to either ferromagnetism or antiferromagnetism in the rare earth ($4f$) elements and alloys. [4 marks]

5. (a) All materials show some degree of diamagnetism, a small temperature-independent negative magnetic susceptibility. Diamagnetism does not require the atoms to have a non-zero magnetic moment in the absence of an applied magnetic field. Explain in simple terms the origin of diamagnetism. [4 marks]

(b) Using either Langévin's classical approach, or a quantum mechanical derivation, show that the diamagnetic susceptibility, χ_d, of a solid composed of N ions per unit volume (each with Z electrons) is given by

$$\chi_d = -\frac{\mu_0 N Z e^2}{6 m_{\mathrm{e}}} \langle r^2 \rangle,$$

where $\langle r^2 \rangle$ is the mean square distance of the electrons from the nucleus. [6 marks]

(c) Lead ($Z = 82$, atomic mass = 207.2 u) is a diamagnetic material with a density of 11.35×10^3 kg m^{-3}. Calculate the magnetic susceptibility of this element given that the root mean square atomic radius is 0.175 nm. [6 marks]

(d) A magnetic field strength of 5000 A m^{-1} is applied to a sample of lead of mass 10 g. Calculate the magnetic moment induced in the sample by this applied field. [4 marks]

6. (a) Calculate the quantum numbers of the atomic spin, S, the orbital angular momentum, L, and the total angular momentum, J, of the ground state of the paramagnetic Dy^{3+} ion ($4f^9$) stating any assumptions that you make. [6 marks]

(b) Calculate the magnitudes of S, L and J of the Dy^{3+} ion in its ground state (in units of $\hbar$). The effective g-factor, known as the Landé g-factor, is given by

$$g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}.$$

What are the magnitudes of the three corresponding magnetic moments (in units of the Bohr magneton μ_{B})? [4 marks]

(c) State the number of energy levels that contribute to the magnetic moment of a Dy^{3+} ion in the presence of a magnetic field. Calculate the smallest energy separation between two of these levels if the applied magnetic field strength is $H = 2.0$ x 10^6 A m^{-1}. What frequency of electromagnetic radiation could be used to excite a transition between two of these levels? Take the Bohr magneton to be $\mu_{\mathrm{B}} = 9.27 \times 10^{-24}$ J T^{-1}. [4 marks]

(d) Assuming the ions are magnetically isolated from one another in the solid, comment on the applicability of Curie's law to the paramagnetic susceptibility at 1 K and 300 K. [4 marks]

[Hint: Use the condition adopted to obtain Curie's law from the Brillouin function form of the magnetisation].

(e) Calculate the maximum measureable magnetic moment of a solid consisting of 1 mole of Dy^{3+} ions. [2 marks]

SECTION C. CONDENSED MATTER PHYSICS part 2
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) A semiconductor material has electron energy profiles for the valence (VB) and conduction bands (CB) given by

$$\epsilon_{\mathrm{VB}} = -10^{-37} k^2 \,\mathrm{J\,m^2}$$

$$\epsilon_{\mathrm{CB}} = 2 \times 10^{-37} k^2 \,\mathrm{J\,m^2} + 2.0 \times 10^{-19}\,\mathrm{J}$$

where k is the magnitude of the wavevector. An electron is excited from the state $\underline{k} = 10^9\,\mathrm{m}^{-1}\,\hat{\underline{i}}$ from the valence band to the conduction band by a photon. What is the photon frequency? Comment on whether the effective mass of the electron in the CB is larger or smaller than the hole in the VB. [4 marks]

(b) An infinitely long hollow cylinder is placed in an external magnetic field $\underline{B_e}$. For simplicity, assume the magnetic field is uniform and points in the z-direction, and the long axis of the cylinder coincides with the z-axis. Under this external field, the hollow cylinder is cooled down such that it becomes superconducting. The inner and outer radii of the cylinder are at $r = a$ and $r = b$ respectively. Assume $a, b, b - a \gg \lambda$ (λ is the London penetration depth). What are the values of the magnetic field for $r < a$ and $a < r < b$? Is there a surface current at $r = a$ or at $r = b$? [4 marks]

(c) The Gibbs free energy per unit volume of a material in its superconducting state at applied magnetic field of magnitude B_e and temperature T is given by

$$G_s(B_e, T) = G_0(T) - \frac{\alpha(T)^2}{2\beta} + \frac{B_e^2}{2\mu_0},$$

where $\alpha(T)$ and β take their usual meaning in Ginzburg-Landau theory and $G_0(T)$ is the corresponding Gibbs free energy of the material in its normal state at temperature T. Experiments on Gallium yield values for the temperature dependence of the critical field $B_c(T)$ which can be fitted to $B_c(T) = B_0\left[1 - (T/T_c)^2\right]$, where T_c is the critical transition temperature at zero applied magnetic field. Show that the latent heat, L, per unit volume at the field-induced normal-to-superconducting transition is given by

$$L = -\frac{2B_0^2 T^2}{\mu_0 T_c^2}\left[1 - \left(\frac{T}{T_c}\right)^2\right].$$

[4 marks]

[Hint: these relations can be used without any derivation, $L = T\Delta S$ and $S = -(\partial G/\partial T)|_{B_e}$.]

(d) Sketch how the real part of the polarizability of an ionic solid depends on frequency. The ionic solid has no permanent dipoles. For each contribution, label whether it is of resonance or relaxation type. [4 marks]

(e) A ferroelectric material can be described by the following Landau free energy

$$G(P,T) = g_0 + \frac{1}{2}g_2P^2 + \frac{1}{4}g_4P^4,$$

where P is the polarization, $g_2 = \gamma(T - T_0)$, T is the temperature of the system, and T_0 is the so-called Curie temperature. γ, g_0 and g_4 are positive constants in the model. Determine whether the phase transition from the dielectric to the ferroelectric state is first or second order. [4 marks]

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8. (a) A semiconductor has conduction and valence band effective densities of states N_C and N_V respectively. Write down expressions for the electron concentration in the conduction band and the hole concentration in the valence band in terms of the effective densities of states N_C and N_V, the energy band gap E_g, the chemical potential μ, and temperature T. [2 marks]

(b) A sample of silicon is purified until it only contains donor impurities with concentration $N_D = 1.00 \times 10^{18}$ donors/m^3. The energy band gap is $E_g = 1.10$ eV, and the conduction and valence band effective densities of states are $N_C = 4.00 \times 10^{25}$ m^{-3} and $N_V = 1.00 \times 10^{25}$ m^{-3} respectively. For simplicity, assume N_C and N_V to be constant.

Derive how the dominant charge carriers and the chemical potential depend on temperature at:
(i) low temperature [3 marks]
(ii) room temperature [3 marks]
(iii) high temperature [3 marks] .

(c) Correspondingly, sketch the chemical potential, electron and hole concentrations as a function of temperature. [6 marks]

(d) Compute the lowest temperature above which the material shows an intrinsic behaviour. [3 marks]

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9. A p-n junction is constructed by combining a p-type and an n-type germanium semiconductor. Assuming electrons and holes recombine in the depletion layer, this leaves net charges of

$$\rho(x) = \begin{cases} -eN_\mathrm{A} & -w_\mathrm{p} < x < 0, \\ +eN_\mathrm{D} & 0 < x < w_\mathrm{n}, \\ 0 & \text{elsewhere,} \end{cases}$$

where N_A is the density of the acceptor impurities in the p-type region and N_D is the density of donor impurities in the n-type region. Here w_p and w_n are the widths of the depletion layer in the p- and n-type regions respectively.

(a) Starting from Gauss' law $\underline{\nabla} \cdot \underline{E} = \rho/(\epsilon_\mathrm{r}\epsilon_0)$ in one dimension, show that the electric field across the p-n junction is given by

$$E(x) = \begin{cases} -eN_\mathrm{A}(x + w_\mathrm{p})/(\epsilon_\mathrm{r}\epsilon_0) & -w_\mathrm{p} < x < 0, \\ +eN_\mathrm{D}(x - w_\mathrm{n})/(\epsilon_\mathrm{r}\epsilon_0) & 0 < x < w_\mathrm{n}, \\ 0 & \text{elsewhere.} \end{cases}$$

[4 marks]

(b) Show that the electric potential follows the relation

$$\phi(x) = \begin{cases} eN_\mathrm{A}(x + w_\mathrm{p})^2/(2\epsilon_\mathrm{r}\epsilon_0) & -w_\mathrm{p} < x < 0, \\ \phi_i - eN_\mathrm{D}(x - w_\mathrm{n})^2/(2\epsilon_\mathrm{r}\epsilon_0) & 0 < x < w_\mathrm{n}. \end{cases}$$

ϵ_r and ϵ_0 are the relative and vacuum permittivity respectively. $\phi_i = \phi(w_\mathrm{n}) - \phi(-w_\mathrm{p})$ is the built-in voltage of the p-n junction. For simplicity, take $\phi(-w_\mathrm{p})$ to be equal to zero. The neutrality condition also requires that $N_\mathrm{A}w_\mathrm{p} = N_\mathrm{D}w_\mathrm{n}$. [4 marks]

(c) Show that the widths of the depletion layer can be expressed as [4 marks]

$$w_\mathrm{n} = \left(\frac{2\epsilon_\mathrm{r}\epsilon_0 N_\mathrm{A}\phi_i}{eN_\mathrm{D}(N_\mathrm{A} + N_\mathrm{D})}\right)^{1/2}, \quad w_\mathrm{p} = \left(\frac{2\epsilon_\mathrm{r}\epsilon_0 N_\mathrm{D}\phi_i}{eN_\mathrm{A}(N_\mathrm{A} + N_\mathrm{D})}\right)^{1/2}.$$

[4 marks]

(d) The p-n junction is used as a variable capacitor and an external bias voltage is applied to the junction. Derive the expressions for the bias voltage such that the capacitance is (i) half and (ii) twice the value of the capacitance at zero bias potential. [4 marks]

(e) For the values of the bias voltage obtained in part (d), what are the corresponding widths of the depletion layer. [4 marks]

The built-in voltage for the p-n junction is 0.30 V at 300 K. The donor and acceptor impurities are $N_\mathrm{A} = N_\mathrm{D} = 1.00 \times 10^{20}$ donors m^{-3}. The relative permittivity for germanium is $\epsilon_\mathrm{r} = 16.0$.

