University of Durham

EXAMINATION PAPER

May/June 2017 Examination code: PHYS3661-WE01

THEORETICAL PHYSICS 3

SECTION A. Relativistic Electrodynamics
SECTION B. Quantum Theory 3

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50\ \times\ 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09\ \times\ 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99\ \times\ 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84\ \times\ 10^{26}$ W

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SECTION A. RELATIVISTIC ELECTRODYNAMICS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Consider two 4-vectors a^μ, b^μ. Define what it means for a^μ and for b^μ to be time-like. Show that if $a^\mu b_\mu = 0$, then both a^μ and b^μ cannot be time-like. [4 marks]

   (b) Show that $a^\mu v_\mu = 0$ where a^μ is the four-acceleration and v_μ is the four-velocity of a point particle. [4 marks]

   (c) Determine the speed of a particle (relative to c) if its kinetic energy is equal to its rest mass energy. [4 marks]

   (d) Show that the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval. [4 marks]

   (e) An electron with velocity $\underline{v}$ collides with an anti-electron with velocity $-\underline{v}$ producing a muon and its antiparticle. Given that the muon mass is roughly 200 times the electron mass, what is the minimal magnitude of the velocity of the incoming electron? [4 marks]

   (f) Write down the gauge transformation of the 4-potential A^μ in contravariant form and use the transformation to show that the field strength tensor, $F^{\mu\nu}$, is gauge invariant. [4 marks]

   (g) The Lienard-Wiechert potential of a point charge q with 4-velocity u^μ is

$$A^\mu = \frac{q}{4\pi\epsilon_0}\frac{u^\mu}{u^\nu R_\nu},$$

   where R_ν is the 4-distance between the observer and the point charge. The right-hand side of the expression must be evaluated at the retarded time $t_{\rm ret}$. Evaluate this expression in the instantaneous rest frame of the point charge and show that you obtain the expected result. [4 marks]

2. Consider a point charge q moving in an inertial frame S with constant velocity v along the x-axis.

   (a) Write down the electric $\underline{E}'$ and magnetic $\underline{B}'$ fields at a position $\underline{r}'$ from the charge in the rest frame S' of the point charge. [4 marks]

   (b) The transformations of the electric $\underline{E}$ and magnetic $\underline{B}$ fields as measured in two inertial frames S and S' in the standard configuration (i.e. S' moves with velocity v along the x-axis and at $t = t' = 0$ the two frames coincide) are given by

   $$E'_x = E_x; \quad E'_y = \gamma(E_y - vB_z); \quad E'_z = \gamma(E_z + vB_y);$$

   $$B'_x = B_x; \quad B'_y = \gamma(B_y + \frac{v}{c^2}E_z); \quad B'_z = \gamma(B_z - \frac{v}{c^2}E_y);$$

   Use these transformation properties to compute the $\underline{E}$ and $\underline{B}$ fields of the point charge in S at the time $t = 0$ for the point P with Cartesian coordinates $(0, b, 0)$. [12 marks]

   (c) What value is measured in the inertial frame S for $\underline{E} \cdot \underline{B}$ at this point P at $t = 0$? [4 marks]

3. a) Give the definition of the electric, $\underline{E}$, and magnetic, $\underline{B}$, fields in terms of the scalar, Φ, and vector, $\underline{A}$, potentials. [4 marks]

b) State the definition of the field-strength tensor in terms of the 4-potential $A^\mu = (\Phi, c\underline{A})$. [2 marks]

c) Show that the field-strength tensor can be written in terms of the electric and magnetic fields as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

[6 marks]

Let S' be the rest frame of a medium with a charge-density of ρ_0 and a current $\underline{J}'$. In this frame, the electric field, $\underline{E}'$ is related to the current $\underline{J}'$ by Ohm's law $\underline{J}' = \sigma \underline{E}'$, where σ is the conductivity.

The frame S' moves with velocity $\underline{v}$ with respect to the frame S. In S the 4-current is

$$j^\mu = a v^\mu + \frac{\sigma}{c} F^{\mu\nu} v_\nu,$$

where a is a constant and $v^\mu = \gamma(c, \underline{v})$ is the 4-velocity of the medium with $\gamma = 1/\sqrt{1 - \frac{|\underline{v}|^2}{c^2}}$.

d) Compute a by calculating $j^\mu v_\mu$, using that in the frame S', $j'^\mu = (\rho_0 c, \sigma \underline{E}')$. [2 marks]

e) Calculate the 3-current $\underline{J}$ in S in terms of $\underline{v}$ and the electric, $\underline{E}$, and magnetic fields, $\underline{B}$ (as measured in S). Interpret your result. [6 marks]

4. The relativistic generalisation of Larmor's formula for the power radiated from an accelerated, charged, point-like particle can be written as

$$\mathcal{P} = \frac{d\mathcal{W}}{dt} = \frac{\gamma^2 q^2}{6\pi\epsilon_0 m^2 c^3}\left[\left|\frac{d\underline{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right],$$

where q and m are the charge and mass of the particle, $p = |\underline{p}|$ is the magnitude of the relativistic three-momentum $\underline{p} = \gamma m \underline{v}$ of the particle, $\beta = v/c$ where v is speed of the particle, and γ is the standard relativistic factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Consider the scattering of a particle of charge q off a stationary charge q' at the origin and with a fixed Coulomb potential (evaluated in the rest frame of q')

$$\underline{E}(\underline{r}) = \frac{q'}{4\pi\epsilon_0}\frac{\underline{r}}{r^3}.$$

During the scattering, the incident particle will emit electromagnetic radiation due to the acceleration it experiences.

Assume that the impact parameter b of the scattering is large enough that the potential energy is small compared to the relativistic kinetic energy, such that the trajectory is nearly a straight line with constant velocity $\underline{v} \approx \underline{\beta} c$, and the acceleration experienced is a perturbation from this.

(a) Sketch the scattering with the fixed charge at the origin, and a segment of the straight line movement of the scattering charge. The impact parameter b must be indicated on the sketch, along with the axes of the coordinate system. [2 marks]

b) Use the Lorentz Force to write $\frac{dp}{dt}$ in terms of the $\underline{E}$ and q. [4 marks]

c) Consider a small infinitesimal period of time Δt. Write $(\Delta\underline{p})^2$ in terms of Δt, and by keeping just the first term in Δt show that $\frac{\Delta p^2}{\Delta t} = 2\underline{p}\cdot\underline{F}$, where $\underline{F}$ is the force applied to the particle. Use this to deduce that $\frac{dp}{dt} = \hat{\underline{p}}\cdot\underline{F}$. [4 marks]

d) Using the approximations above, show that the total energy radiated $\mathcal{W}$ during the collision is given by

$$\mathcal{W} = \frac{\gamma^2 q^4 q'^2}{192\pi^2\epsilon_0^3 m^2 c^4 b^3 \beta}\left(1 - \frac{\beta^2}{4}\right).$$

[10 marks]

$$\left[\text{Hint: } \int_{-\infty}^{\infty}\frac{1}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3} \quad ; \quad \int_{-\infty}^{\infty}\frac{1}{(x^2+b^2)^2}\frac{x^2}{x^2+b^2}\,dx = \frac{\pi}{8b^3}\right]$$

SECTION B. QUANTUM THEORY 3

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) In a given scattering problem the $l = 4$ phase shift δ_4 can be expressed by $\cot \delta_4 = 2(E_0 - E)/\Gamma$. Given that

$$\sigma_\lambda = \frac{4\pi}{k^2}(2\lambda + 1) \sin^2 \delta_\lambda,$$

compute the $l = 4$ partial cross-section σ_4 in terms of E_0, E and Γ. Sketch this as a function of E, and give a physical interpretation of the parameters E_0 and Γ. [4 marks]

(b) What do the symbols ρ and $\underline{j}$ represent in the continuity equation for probability, $\partial_t\rho + \underline{\nabla} \cdot \underline{j} = 0$? Given an expression $\rho = \Psi^\dagger\Psi$ for a spinor Ψ which is a solution to the free Dirac equation $i\partial_t\Psi = H_D\Psi$ where $H_D = -i\underline{\alpha} \cdot \underline{\nabla} + m\beta$, derive an expression for $\underline{j}$. [4 marks]

(c) Given the expression for the scattering amplitude in terms of phase shifts,

$$f(k, 0, 0) = \sum_{\lambda=0}^{\infty} \frac{2\lambda + 1}{k} e^{i\delta_\lambda} \sin \delta_\lambda,$$

derive the *optical theorem* connecting $f(k, 0, 0)$ to the total scattering cross-section $\sigma = \sum_{\lambda=0}^{\infty} \sigma_\lambda$. Give a physical explanation of this relation. [4 marks]

(d) Describe what is meant by the *micro-canonical*, *canonical* and *grand canonical* ensembles. In each case, explain what is assumed constant and what is allowed to vary, and how these ensembles relate to each other. [4 marks]

(e) The Klein-Gordon equation can be written $(\partial_\mu\partial^\mu + m^2)\,\Phi(\underline{x}, t) = 0$. Find a relation connecting the energies and momenta of plane wave solutions for this equation, and show that the energy can be positive or negative. Explain the problems this would cause if Φ can emit photons. [4 marks]

(f) The Lippmann-Schwinger equation for scattering can be expressed as

$$\Psi_{\underline{k}}(\underline{r}') = e^{i\underline{k}\cdot\underline{r}'} + \int G_0(k, \underline{r}' - \underline{r}'')U(\underline{r}'')\Psi_{\underline{k}}(\underline{r}'')d\underline{r}''.$$

Explain how this can be used to derive the *Born series*. Define the *first Born approximation* to the scattering amplitude and derive an expression for the scattering amplitude in this approximation in terms of a Fourier transform. You can use assume that G_0 can be written as

$$G_0(k, \underline{r}' - \underline{r}'') \approx -\frac{e^{ikr'}}{4\pi r'} e^{-i\underline{k}'\cdot\underline{r}''}.$$

[4 marks]

(g) Give an expression for the density matrix in terms of a set of states $|n\rangle$ and their statistical weights w_n. Now consider a quantum system where 10% of particles are in state $|1\rangle$, 30% are in $|2\rangle$ and the rest are in $|3\rangle$. Write down the density matrix describing this system, and find the expectation value of an operator O, given that the states satisfy $O|n\rangle = n^2|n\rangle$. [4 marks]

(h) For $\underline{\pi} = -i\underline{\nabla} - e\underline{A}$ where $\underline{A}$ is the electromagnetic vector potential, show that $(\underline{\sigma} \cdot \underline{\pi})^2 = \underline{\pi}^2 - e\underline{\sigma} \cdot \underline{B}$ where $\underline{\sigma}$ denotes the Pauli matrices and $\underline{B} = \underline{\nabla} \times \underline{A}$ is the magnetic field. You can assume the identity $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$ without proof. [4 marks]

6. A particle undergoes scattering from a strongly localised potential shell described by

$$V(\underline{r}) = -C\delta(r - r_0),$$

where C and r_0 are positive real constants and $r = |\underline{r}|$. Let $\hbar = c = 1$ throughout this question. The first spherical Bessel and spherical Neumann functions are given by

$$j_0(x) = \frac{\sin x}{x} \qquad \text{and} \qquad n_0(x) = -\frac{\cos x}{x}.$$

(a) Show that the time independent Schrödinger equation for an initial particle of momentum $\underline{k}$ scattering in this potential can be written as

$$\left[\nabla^2 + \underline{k}^2 - U(r)\right] \Psi_{\underline{k}}(\underline{r}) = 0,$$

giving an explicit expression for $U(r)$. [3 marks]

(b) The Laplacian in spherical coordinates can be written

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{\underline{L}^2(\theta, \phi)}{r^2}.$$

Separate your equation from part (a) into radial and angular equations. What is the most general solution for $\Psi_{\underline{k}}(\underline{r})$ in a region where the potential is zero? [4 marks]

(c) What boundary condition must be satisfied at the origin? Use this to show that the s-wave wavefunction in the region $r < r_0$ can be written as $\Psi_{r<r_0}(\underline{r}) = A_1 \sin(kr)/r$. [3 marks]

(d) What boundary conditions must be satisfied in the limit $r \to \infty$? Use this to show that the s-wave wavefunction in the region $r > r_0$ can be written as $\Psi_{r>r_0}(\underline{r}) = A_2 \sin(kr + \delta_0)/r$. [3 marks]

(e) At $r = r_0$, the potential $V(\underline{r})$ alters the normal boundary conditions. Instead of continuity of the first derivative, we require that the radial function's first derivative satisfies,

$$\left.\frac{d(r\Psi_{r>r_0})}{dr}\right|_{r=r_0} - \left.\frac{d(r\Psi_{r<r_0})}{dr}\right|_{r=r_0} = -2mCr_0\Psi(r_0).$$

Use the boundary conditions at $r = r_0$ to derive an expression for $\tan(kr_0 + \delta_0)$. [4 marks]

(f) Compute the scattering cross-section in the low-k limit under two assumptions i) $C \to 0$ and ii) $C \to \infty$. Describe physically the behaviour in these limits. [3 marks]

7. A Dirac spinor is a four component object Ψ which satisfies the differential equation
$$\left(-i\sum_{i=1}^{3}\alpha_i\nabla_i+\beta m\right)\Psi(\vec{x},t)=i\frac{\partial\Psi}{\partial t},$$
where α_1, α_2, α_3 and β are four-dimensional matrices. Let $\hbar=c=1$ throughout this question.

   (a) From the assumption that any solution of the Dirac equation is also a solution of the Klein-Gordon equation $(\partial_\mu\partial^\mu+m^2)\Psi=0$, find a set of algebraic constraints which the matrices α_i and β must satisfy. [3 marks]

   (b) Show how the Dirac equation given above can be written in covariant form by introducing the four gamma matrices, $\gamma^0=\beta$ and $\gamma^i=\beta\alpha^i$ for $i\in\{1,2,3\}$. Show that the gamma matrices satisfy
$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}.$$
   where μ and ν are spacetime indices taking the values $0,1,2,3$ and $g^{\mu\nu}$ is the Minkowski metric. [4 marks]

   (c) Under a Lorentz transformation, an electromagnetic vector potential A_μ and a Dirac spinor Ψ transform as
$$\Psi'=S\Psi \qquad \text{and} \qquad A'_\mu=\Lambda_\mu^{\ \nu}A_\nu,$$
   such that $\Lambda_\mu^{\ \nu}$ denotes the normal Lorentz transformation matrix obeying $\Lambda_\alpha^{\ \beta}g_{\beta\gamma}\Lambda_\delta^{\ \gamma}=g_{\alpha\delta}$. Explain what is meant by the term *covariant*. Show that for the Dirac equation to be covariant under Lorentz transformations, the gamma matrices must satisfy
$$S^{-1}\gamma^\mu S=\gamma^\nu(\Lambda^{-1})_\nu^{\ \mu}.$$
   [4 marks]

   (d) A specific infinitesimal Lorentz transformation can be written as
$$S=I+\omega_{\mu\nu}\sigma^{\mu\nu},$$
   where I is the 4-dimensional identity matrix, $\sigma^{\mu\nu}=[\gamma^\mu,\gamma^\nu]$ and $\omega_{\mu\nu}$ are real parameters. Show that $\gamma^0S^\dagger\gamma^0=S^{-1}$ to first order in $\omega_{\mu\nu}$. You may use the identity $\gamma^0(\gamma^\mu)^\dagger\gamma^0=\gamma^\mu$ without proof. [5 marks]

   (e) You can now assume that $\gamma^0S^\dagger\gamma^0=S^{-1}$ holds for all Lorentz transformations S. Show that for two four vectors A_μ and B_ν, the spinor product
$$\left(\overline{\Psi}\sigma^{\mu\nu}\Psi\right)A_\mu B_\nu,$$
   is invariant under Lorentz transformations, where $\overline{\Psi}=\Psi^\dagger\gamma^0$. [4 marks]

8. (a) What properties must a matrix satisfy to be a valid density matrix? Do the following three matrices represent valid density matrices? For each matrix, explain your reasoning.

$$\rho_{\mathrm{A}} = \begin{pmatrix} 0 & -i\frac{\sqrt{3}}{4} & 0 \\ i\frac{\sqrt{3}}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \rho_{\mathrm{B}} = \begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{2}} & \frac{1}{5} \\ \frac{1}{\sqrt{2}} & \frac{1}{4} & -i2 \\ \frac{1}{5} & i2 & \frac{1}{5} \end{pmatrix}, \rho_{\mathrm{C}} = \begin{pmatrix} \frac{7}{12} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{5}{12} \end{pmatrix}.$$

[5 marks]

(b) For two valid density matrices ρ_a and ρ_b, show that their linear combination

$$\rho = \lambda\rho_a + (1-\lambda)\rho_b,$$

is only a valid density matrix for a range of the real parameter λ which you should specify. [5 marks]

(c) Define the terms *pure* and *mixed* states. [2 marks]

(d) The previously introduced density matrices can be written as

$$\rho_a = |\Psi_a\rangle\langle\Psi_a| \qquad \text{and} \qquad \rho_b = |\Psi_b\rangle\langle\Psi_b|,$$

where $|\Psi_a\rangle$ and $|\Psi_b\rangle$ are normalized but not necessarily orthogonal. Show that the density matrix ρ satisfies

$$\mathrm{Tr}\left(\rho^2\right) = 1 + 2\lambda(1-\lambda)\left(|\langle\Psi_a|\Psi_b\rangle|^2 - 1\right).$$

[4 marks]

(e) Using the above relation or otherwise, deduce the conditions for which ρ is a pure state. Give a physical interpretation of each of the situations for which ρ represents a pure state. [4 marks]