University of Durham

EXAMINATION PAPER

May/June 2016 Examination code: PHYS2581-WE01

FOUNDATIONS OF PHYSICS 2A

SECTION A. Quantum Mechanics 2
SECTION B. Electromagnetism

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A. QUANTUM MECHANICS 2

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) An electron has wavefunction Ae^{kx} for $x \leq 0$ and Ae^{-kx} for $x > 0$ where k is real and positive. Sketch the wavefunction. Find A. What is the probability of finding the electron between $0 < x < 1/k$? Give your answer to 3 significant figures. [4 marks]

   (b) An electron in the ground state of a one-dimensional infinite square well potential, where $V = 0$ for $0 < x < L$, has wavefunction $\psi(x, t = 0) = \sqrt{2/L}\sin(\pi x/L)$. Write down the appropriate form of the momentum operator, p, and give an expression for $\langle p \rangle$. Evaluate $\langle p \rangle$ by the symmetry properties of the integrand. [4 marks]

   (c) Consider the one-dimensional Hamiltonian $H = T + V(x)$, where $T = p^2/2m$ is the kinetic energy operator. Show that $[H, x] = -i\hbar p/m$, given the fundamental commutator $[x, p] = i\hbar$. [4 marks]

   (d) The generalised Uncertainty Principle between any two operators A and B can be written as

   $$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle [A, B] \rangle\right)^2.$$

   Write down the Uncertainty Principle for energy and position using $[H, x] = -i\hbar p/m$. What is the uncertainty in energy, σ_H, if the system is in one of the energy eigenfunctions, and σ_x is finite? What does this imply for $\langle p \rangle$, and explain the physical significance of this. [4 marks]

   (e) An electron constrained to move on the surface of a sphere has wavefunction given by the spherical harmonic Y_{11}. The normalised eigenfunctions of L_x are $\psi_1 = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}Y_{11} + Y_{10} + \frac{1}{\sqrt{2}}Y_{1-1})$ for eigenvalue $\hbar$, $\psi_0 = \frac{1}{\sqrt{2}}(Y_{11} - Y_{1-1})$ for eigenvalue 0, and $\psi_{-1} = \frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}}Y_{11} + Y_{10} - \frac{1}{\sqrt{2}}Y_{1-1})$ for eigenvalue $-\hbar$. What is the probability that a measurement of L_x results in the value $\hbar$? [4 marks]

   (f) The energy eigenfunctions of a hydrogen atom, ψ_{nlm}, have energies $E_n = E_1/n^2$. An electron is in the superposition state $\psi = \frac{1}{\sqrt{6}}(\psi_{100} + 2\psi_{210} + \psi_{321})$. Write down the possible results of a measurement of energy, and the associated probability of each. What is $\langle E \rangle$? Give your answer in terms of E_1.

   Following a measurement of L_z which returns the value $\hbar$, what are the possible results of a measurement of energy and their probabilities. [4 marks]

   (g) A hydrogen atom is in the state

   $$\psi_{311}(r, \theta, \phi) = R_{31}(r)Y_{11}(\theta, \phi) = -R_{31}(r)\sqrt{\frac{3}{(8\pi)}}\sin\theta e^{i\phi},$$

   where all the symbols have their usual meanings. Write down the probability of finding the electron within a volume dV in spherical polar coordinates of position r, θ, ϕ. Hence calculate the probability distribution, $D(\mu)$, where $\mu = \cos\theta$ (so $d\mu = -\sin\theta d\theta$). At what value(s) of μ does this probability distribution peak? [4 marks]

(h) A particle in a one-dimensional infinite square well potential, where $V = 0$ for $0 < x < a$, has eigenfunctions $\psi_n^0(x) = \sqrt{2/a}\,\sin(n\pi x/a)$. The system is subject to a perturbation $H' = \alpha\delta(x - a/2)$, giving a first order change in energy for each state $E_n^1 = \langle\psi_n^0|H'\psi_n^0\rangle$. Write down the general form for E_n^1, and calculate this separately for odd and even n. Give a physical reason for your answer for even n. [4 marks]

2. An electron in free space has normalized wavefunction given by a Gaussian wavepacket

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+iT)}}{(1+iT)^{1/2}}, \quad \text{where } T = 2\hbar at/m.$$

(a) Calculate the expectation values $\langle x\rangle$ and $\langle x^2\rangle$. The uncertainty in position is given by $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$. Show that $\sigma_x = \sqrt{(1+T^2)/(4a)}$. [8 marks]

$$\left[\text{Hint}: \quad \int_{-\infty}^{+\infty} x e^{-qx^2} dx = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} x^2 e^{-qx^2} dx = \frac{1}{2}\left(\frac{\pi}{q^3}\right)^{1/2}. \right]$$

(b) Evaluate $p\Psi$, where p is the momentum operator, giving your answer in terms of Ψ. Hence calculate $\langle p\rangle$. Since p is Hermitian, $\langle p^2\rangle = \int (p\Psi)^*(p\Psi)dx$. Use this to calculate $\langle p^2\rangle$ in terms of $\langle x^2\rangle$, and hence show that the uncertainty in momentum, $\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$, is equal to $\sqrt{a}\hbar$. [7 marks]

(c) Sketch the behaviour of the product $\sigma_x\sigma_p$ as a function of T. At what time, t_{min}, is the system closest to the Heisenberg Uncertainty Principle limit? What is the time, t_2, for $\sigma_x\sigma_p$ to double from its minimum value? Evaluate t_2 to 2 significant figures for an initial size of $a^{-1/2} = 10^{-10}$ m. [5 marks]

3. Electron spin has total square angular momentum operator S^2 and z-component S_z, with common eigenfunctions X_+ (spin-up) and X_- (spin-down). Nuclear spin is similarly represented by operators I^2 and I_z, with common eigenfunctions $Z_\pm$. S^2 and S_z have eigenvalues $s(s+1)\hbar^2 = (3/4)\,\hbar^2$ for $s = 1/2$ and $m_s\hbar$, respectively, with $m_s = 1/2$ (spin up) or $m_s = -1/2$ (spin down). Similarly, I^2 and I_z have eigenvalues $(3/4)\,\hbar^2$ and $m_I\hbar$ with $m_I = \pm 1/2$. Ladder operators, S_- and S_+, switch spin states, and $S_-X_+ = \hbar X_-$, $S_+X_- = \hbar X_+$, $I_-Z_+ = \hbar Z_-$, $I_+Z_- = \hbar Z_+$.

   All S and I operators act only on $X_\pm$ and $Z_\pm$, respectively, and S^2, I^2, S_z, I_z all commute.

   The coupling of the electron and proton spin gives a hyperfine perturbation of the ground state with $H'_{hf} = A\,\underline{S}\cdot\underline{I}$, where A is a constant.

   (a) Given that $\underline{S}\cdot\underline{I} = \frac{1}{2}(S_+I_- + S_-I_+) + S_zI_z$, evaluate the operation of $\underline{S}\cdot\underline{I}$ on each of the wavefunctions $\psi_1 = X_+Z_+$, $\psi_2 = X_+Z_-$, $\psi_3 = X_-Z_+$, $\psi_4 = X_-Z_-$. Express your answers in terms of ψ_i. Which of the ψ_i are eigenfunctions of $\underline{S}\cdot\underline{I}$? [7 marks]

   (b) The four eigenfunctions, ψ_i, are degenerate without considering $\underline{S}\cdot\underline{I}$ coupling. Hence any linear combination, $\chi = \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3 + \delta\psi_4$, also has the same energy. The hyperfine perturbation gives a first-order correction to the ground state energy, E^1_{hf}, which is given by the solution of the matrix equation,

$$\frac{A\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = E^1_{hf}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

   Solve for E^1_{hf} and write down the corresponding wavefunctions which follow the perturbation. [9 marks]

   (c) Total angular momentum is given by $\underline{F} = \underline{I} + \underline{S}$. Calculate $F^2 = \underline{F}\cdot\underline{F}$ and use this to write down the eigenvalues of $\underline{S}\cdot\underline{I}$, given that the eigenvalues of F^2 are $f(f+1)\hbar^2$. What are the are the possible values of f, and what are the possible values of $\underline{S}\cdot\underline{I}$? How does spin-spin coupling change the degeneracy of an electron in the ground state? [4 marks]

SECTION B. ELECTROMAGNETISM

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Scientists in Durham have discovered a mineral, which they think may be highly insulating and almost non-magnetic. Describe the measurements that you would recommend they do, to test their expectations about the mineral's properties. [4 marks]

   (b) Explain what the 'skin effect' is, in the context of highly conducting materials. [4 marks]

   (c) Provide a sketch of the magnetic field lines around a magnetic monopole. Suggest the form of the mathematical equation that describes the magnetic field lines you have sketched. [4 marks]

   (d) Explain what 'displacement current density' is and briefly discuss why it is important in the context of Ampère's law. [4 marks]

   (e) Explain why Fresnel's equations are useful. [4 marks]

   (f) Describe the difference between polar and non-polar dielectrics. [4 marks]

   (g) One of Maxwell's equations describes the electric field that is produced by a changing magnetic field. Explain how this equation can be tested experimentally. [4 marks]

5. (a) Explain what a dispersion relation is and why such relations are useful. [5 marks]

(b) The general dispersion relation for an electromagnetic wave propagating in a non-magnetic, conducting medium is given by:

$$k^2 = \mu_0 \varepsilon \omega^2 + i\omega\mu_0\sigma_N,$$

where k is the wavevector, ω is the angular frequency, σ_N is the electrical conductivity and $\varepsilon = \varepsilon_r \varepsilon_0$ where ε_r is the relative permittivity. Consider some astronauts who have encountered a gas cloud 20 km thick. They have reported that when the light from a laser, with a frequency of 5×10^{14} Hz, was directed into the dense gas cloud, the light had a wavelength of 0.2 μm. By assuming that the gas cloud is an insulating dielectric, calculate an approximate value for the relative permittivity of the gas in the cloud. [5 marks]

(c) After the light passed through the gas cloud, it had reduced in amplitude by only one part in 10^{10}. Hence the gas cloud can be considered a poor conductor. Calculate an approximate value for the conductivity of the gas. [6 marks]

(d) Some scientists suggested that the energy of each of the photons associated with the light may be different when the light is in the gas compared with when it is in vacuum. Explain whether or not this could be possible. [4 marks]

6. (a) Use Maxwell's equations to derive the boundary conditions for the $\underline{E}$-field parallel and orthogonal to the interface between two electrically insulating, dielectric materials. [4 marks]

(b) For an electromagnetic wave at normal incidence to the planar interface between two insulating dielectric media, the power reflection coefficient, R, is given by:

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 ,$$

where n_1 and n_2 are the refractive indices of the media. At what values of n_1/n_2 does the power transmitted across the interface equal that reflected from it? [5 marks]

(c) Sketch the form of R against n_1/n_2 showing where R= 1/2. [5 marks]

(d) A thick sheet of insulating dielectric material with refractive index of 2.00 is in vacuum with an electromagnetic wave directed at it at normal incidence. By including the effect of at least one internal reflection, calculate, to an accuracy of better than 1 %, what percentage of the incident energy is reflected. [6 marks]

