University of Durham

EXAMINATION PAPER

May/June 2016 Examination code: PHYS3651-WE01

PLANETS AND COSMOLOGY 3

SECTION A. Cosmology
SECTION B. Planetary Systems

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}
Astronomical Unit:	AU = 1.50×10^{11} m
Parsec:	pc = 3.09×10^{16} m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A. COSMOLOGY

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) A galaxy at a distance of 15 Mpc follows the Hubble expansion for which $H_0 = 70$ km s^{-1} Mpc^{-1}. Calculate the galaxy's redshift and the observed wavelength of the Hα spectral line emitted at 656.3 nm. [4 marks]

   (b) The outskirts of a disk galaxy rotate at a constant 200 km s^{-1} relative to the centre. Calculate the radius in Mpc where the enclosed mass density in the galaxy's halo is 10 times the cosmic mean. Assume spherical symmetry, $H_0 = 70$ km s^{-1} Mpc^{-1} and matter density parameter $\Omega_M = 0.3$. [4 marks]

   (c) Calculate the rate of change of the temperature of the 2.73 K cosmic microwave background in K Gyr^{-1} at redshift $z = 3$ if $H_0 = 70$ km s^{-1} Mpc^{-1} where the current matter and cosmological constant density parameters are $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$, respectively. You may use the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

   where a is the expansion factor, ρ is mass-energy density and k is a constant. [4 marks]

   (d) In part (c), what is the physical meaning of k? [2 marks]

   A universe contains only matter, radiation and a cosmological constant, with present-day fractional density parameters $\Omega_M + \Omega_R = 0.25$ and $\Omega_\Lambda = 0.65$ respectively. Assuming $H_0 = 70$ km s^{-1}Mpc^{-1}, evaluate k and express it in units of Mpc^{-2}. [2 marks]

   (e) Two objects A and B of the same physical size but at redshifts $z_A = 1$ and $z_B = 9$ respectively, have the same angular size to an observer at $z = 0$. Find the ratio of the luminosity distances between these two objects and the observer. [4 marks]

   (f) A universe contains three species of active neutrinos all with zero mass, which decoupled from other matter species when the photon temperature T_γ was given by $k_{\rm B}T_\gamma = 1$ MeV. The annihilation of electrons and positrons happened when $k_{\rm B}T_\gamma = 0.5$ MeV, and the annihilation of other particles all happened at $k_{\rm B}T_\gamma \gg 2$ MeV. At the present day, the temperatures of photons and neutrinos (T_ν) satisfy $T_\nu = (4/11)^{1/3}T_\gamma$.

   Assuming instantaneous neutrino decoupling and electron-positron annihilation, find the ratio of T_ν/T_γ at the following three values of T_γ: (i) $k_{\rm B}T_\gamma = 0.3$ MeV, (ii) $k_{\rm B}T_\gamma = 0.8$ MeV, (iii) $k_{\rm B}T_\gamma = 1.2$ MeV. [4 marks]

(g) A universe contains matter, radiation and a cosmological constant with fractional density parameters $\Omega_M = 0.29$, $\Omega_R = 1 \times 10^{-4}$ and $\Omega_\Lambda = 0.70$ today, respectively. Let $\Omega(a)$ be the total density parameter when the expansion factor is a ($a = 1$ today), which includes contributions from all three components. Evaluate the value of $|\Omega(a) - 1|$ at $a = 10^{-50}$ (note that matter and cosmological constant can be approximately neglected at this early time). [2 marks]

Based on your numerical result, describe the flatness problem of the standard Big Bang model. [2 marks]

(h) At the end of Big Bang nucleosynthesis (BBN), all free neutrons ended up in Helium-4 (^{4}He) to a good approximation. Neglecting all other elements than H and ^{4}He synthesised during BBN, find the Helium-4 mass fraction and express your answer in terms of N_n/N_p, the number ratio of free neutrons and protons before the nuclear reactions leading to Helium-4 were triggered. [2 marks]

The next most abundant element produced during BBN is Deuterium (D). A tiny fraction $f \ll 1$ of neutrons were used to form Deuterons, so that the H and ^{4}He abundances are approximately unaffected by including D in the calculation. Assuming that the remaining neutrons again ended up in ^{4}He, express the number ratio of D and H, $N_{\mathrm{D}}/N_{\mathrm{H}}$, after BBN in terms of f and N_n/N_p. [2 marks]

2. (a) A galaxy cluster of radius r_0 contains galaxies with a uniform space density, n_0. The galaxies are each of mass m and the gravitational potential energy of the cluster is given by

$$V = -\frac{3}{5}\frac{GM^2}{r_0},$$

where M is its total mass, which may be assumed to be the sum of the constituent galaxy masses. If $n_0 = 500\ \mathrm{Mpc}^{-3}$, $m = 10^{12}\ \mathrm{M}_\odot$ and $r_0 = 200$ kpc, estimate the line-of-sight velocity dispersion in km s^{-1} of the galaxies in the cluster, stating any assumptions made. [5 marks]

(b) In another cluster, the combined density profile of dark matter and galaxies is $\rho(r) = \rho_0(r_0/r)^2$, where $\rho_0 = mn_0$, and m, n_0 and r_0 have the same numerical values as in (a). The intracluster medium comprises diffuse hot gas of density profile $\rho_g(r) \propto r^{-1.5}$. If the total mass (gas, galaxies and dark matter) within a radius of 2 Mpc is $5.6 \times 10^{14}\ \mathrm{M}_\odot$, show that the enclosed mass within radius r is of the form

$$M(< r) = A(r/r_0)^{1.5} + B(r/r_0),$$

and evaluate the constants A and B in solar masses. [6 marks]

(c) In hydrostatic equilibrium the intracluster gas follows the equation

$$\frac{dP}{dr} = -\frac{GM(< r)\rho_g}{r^2},$$

where P is gas pressure and $M(< r)$ is total mass within radius r. For the cluster of part (b), verify that this equation is satisfied by a temperature profile of the form $T(r) = T_0[1 + \alpha(r/r_0)^{0.5}]$ and evaluate T_0 in K and the quantity α if the gas is pure ionized hydrogen. [9 marks]

3. (a) Explain what is meant by *comoving distance*. In an expanding Universe, light emitted by a source at arbitrary time t_1 is received by an observer at time t_2. Show that the comoving distance between the source and observer is given by:

$$X_{CM} = c \int_{a_1}^{a_2} \frac{da}{a\dot{a}} \ ,$$

where $a_1 = a(t_1)$ and $a_2 = a(t_2)$, and a is expansion factor with $a(t_0) = a_0 = 1$ at present. [6 marks]

(b) A model universe comprises dark energy with equation of state $p = -\frac{4}{3}\rho c^2$, where p and ρ are fluid pressure and density, respectively. Use the fluid equation,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\Big(\rho + \frac{p}{c^2}\Big) = 0,$$

to derive $\rho(a)$ if $\rho(a_0) = \rho_0$ at present. Hence show that $\dot{a} = H_0 a^{3/2}$ if the density parameter in this fluid $\Omega_{\rm DE} = 1$ and H_0 is the Hubble constant. [5 marks]

(c) In this universe, a supernova is observed to explode in a galaxy at redshift $z = 1$, as seen from Earth at present. Using the above results, calculate the co-moving radius of the sphere, in units of (c/H_0), from the supernova to the most distant observers who could ever see it. [4 marks]

(d) If this sphere is populated with galaxies, calculate their maximum and minimum redshifts, as observed now from Earth. [5 marks]

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4. A spatially flat universe contains baryons, dark matter, photons and a cosmological constant. There are two components of dark matter: cold dark matter (CDM) and hot dark matter (HDM), the latter of which is made up of three species of active neutrinos with equal nonzero mass $m_\nu = 0.1\ \mathrm{eV}/c^2$ and present-day temperature $T_{\nu,0} = 1.945$ K. The neutrino temperature evolves as $T_\nu \propto a^{-1}$. Neutrinos decoupled when $k_\mathrm{B} T_\nu \approx 1$ MeV, where k_B is the Boltzmann constant.

   (a) Using the definition of $\Omega_\nu \equiv \rho_\nu / \rho_{crit}$, where ρ_ν is the neutrino mass density and $\rho_{crit} \equiv 3H^2/8\pi G$ is the critical density, show that at the present day

$$\Omega_\mathrm{HDM} = \Omega_\nu \approx \frac{m_\nu c^2}{30\ \mathrm{eV}} \left[\frac{H_0}{100\ \mathrm{km\ s^{-1} Mpc^{-1}}} \right]^{-2},$$

   where H_0 is the current value of the Hubble expansion rate H. You can use $1\,\mathrm{J} = 6.242 \times 10^{18}$ eV in your calculation. Note that for relativistic fermion species the density is given by

$$\rho = \frac{1}{2} \times g_* \times \frac{7}{8} \times \frac{4\sigma T^4}{c^3},$$

   where g_* is the number of bosonic degrees of freedom and T is the temperature. The factor 7/8 only exists for fermions, and not for bosons (such as photons). [6 marks]

   (b) Assuming that photons have a temperature $T_\gamma = 2.725$ K today, and $H_0 = 70\ \mathrm{km\ s^{-1} Mpc^{-1}}$, show that the present-day photon fractional density parameter $\Omega_\gamma \approx 5.0 \times 10^{-5}$. [4 marks]

   (c) If the present-day fractional densities of baryons and the cosmological constant are respectively $\Omega_\mathrm{baryons} = 0.05$ and $\Omega_\Lambda = 0.7$, use the results of parts (a) and (b) to find the fractional densities of CDM and HDM, Ω_CDM and Ω_HDM, today. [4 marks]

   (d) Find the value of the expansion factor a when these neutrinos experienced the transition from relativistic to non-relativistic, and show that they were highly relativistic at the time of matter-radiation equality, $a = a_\mathrm{eq}$. Estimate the value of a_eq. [6 marks]

SECTION B. PLANETARY SYSTEMS

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) Sketch the locations of the main constituents of the Solar System, showing how the dominant objects change at different radii from the Sun. Where would you expect to find the majority of the dwarf planets? [4 marks]

(b) A spacecraft is in a wide circular orbit around Neptune, with a radius of 5.00×10^5 km. Calculate the time taken for it to transfer to a second circular orbit just above the Neptunian atmosphere, with radius 3.00×10^4 km, via a Hohmann transfer orbit. The mass of Neptune is 1.02×10^{26} kg. [4 marks]

(c) Sketch the positions of the Lagrange points relative to a planet-moon system. For the Pluto-Charon system, indicate which – if any – of these points are stable, given the masses $M_{\rm Pluto} = 1.31 \times 10^{22}$ kg and $M_{\rm Charon} = 1.52 \times 10^{21}$ kg. [4 marks]

(d) The effective temperature at the top of an atmosphere, $T_{\rm e}$, is given by

$$T_{\rm e} = \left(\frac{(1-A)L_{\odot}}{16\pi r^2 \epsilon \sigma} \right)^{\frac{1}{4}},$$

for a planet with albedo A and emissivity ϵ a distance r from the Sun. Given that Venus' orbit has a semi-major axis of $a = 7.23 \times 10^{-1}$ AU and eccentricity of $e = 6.77 \times 10^{-3}$, and Venus has an albedo of $A = 0.75$, calculate the range of temperature expected at the top of the Venusian atmosphere for an emissivity $\epsilon = 1$. Why is the mean surface temperature of 737 K is so much higher than the equilibrium temperature range? [4 marks]

(e) The effective gravity experienced at Earth's equator, $g_{\rm eq} = 9.78$ m s^{-2}, is smaller than that at its poles, $g_{\rm pole} = 9.83$ m s^{-2}. Assuming that the Earth is spherical in shape, use these values to calculate the length of an Earth day. Comment on the origin of any discrepancies between your result and the length of a sidereal day ($\approx$ 23 hours 56 minutes). The radius of the Earth is 6.37×10^6 m. [4 marks]

(f) Starting from the *vis viva* equation, show that the observed velocity amplitude K for the reflex motion of a star co-orbiting with a planet in a circular orbit is given by

$$K = \left(\frac{2\pi G}{P} \right)^{\frac{1}{3}} \frac{M_{\rm p} \sin i}{(M_* + M_{\rm p})^{\frac{2}{3}}},$$

where M_* is the mass of the star, $M_{\rm p}$ is the mass of the planet, and the system has an orbital period P and an inclination i to our line-of-sight. [4 marks]

(g) The discovery of 'hot jupiters', gas giants in close proximity to their parent stars, was an unexpected early result of extra-solar planet studies. Explain why this is the case, in the context of the solar nebula model for the formation of our own Solar System. [4 marks]

6. (a) Sketch the path taken by a body, relative to a second body, in the cases that they are in (i) an elliptical orbit; (ii) a parabolic orbit; and (iii) a hyperbolic orbit. In each case show the position of the second body, and state the possible range of values of the eccentricity e. [4 marks]

(b) A new comet in the 'sungrazing' class is observed to pass a mere 1.45×10^9 m above the Sun's surface at perihelion. Given that the comet has an orbital period of 253 years, show that the eccentricity of its orbit is $e = 0.99964$ (to 5 significant figures). Where in the Solar System did this comet originate? The radius of the Sun $R_\odot = 6.96 \times 10^8$ m. [4 marks]

(c) (i) Calculate the time the comet takes to move from a mean anomaly $M = -10°$ to its perihelion passage. [2 marks]

(ii) Calculate the time the comet takes to move from a true anomaly $\theta = -10°$ to its perihelion passage. Contrast this time with the value calculated for the mean anomaly change in part (i) and explain the origin of any differences, with reference to Kepler's laws. [6 marks]

[The true anomaly θ is related to the eccentric anomaly E by

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{(1+e)}{(1-e)}}\tan\left(\frac{E}{2}\right),$$

and the mean anomaly M relates to the eccentric anomaly via Kepler's equation, $M = E - e\sin E$.]

(d) The comet does not survive its perihelion passage, starting to break up at a distance of 2.10×10^9 m above the Sun's surface. What does this tell us about the composition of the comet? [4 marks]

[The Roche radius $r_{\rm R}$ around an object of radius R and density $\rho_{\rm M}$, is

$$r_{\rm R} = 2.44\left(\frac{\rho_{\rm M}}{\rho_{\rm m}}\right)^{\frac{1}{3}} R,$$

for an object of density $\rho_{\rm m}$.]

7. (a) Summarise the observable characteristics of planets that can be used to constrain planetary interiors, and briefly discuss how they are used to do that. [5 marks]

(b) The moment of inertia factor $\alpha_{\rm p}$ for a planet can be calculated from the planet's radial density profile $\rho(r)$ as

$$\alpha_{\rm p} = \frac{\frac{8}{3}\pi \displaystyle\int_0^{R_{\rm p}} \rho(r) r^4 dr}{4\pi R_{\rm p}^2 \displaystyle\int_0^{R_{\rm p}} \rho(r) r^2 dr},$$

where r is the radial distance from the centre of the planet, and $R_{\rm p}$ is the radius at which the pressure drops to zero.

The density profile of Jupiter's hydrogen/helium envelope can be approximated by

$$\rho(r) = \rho_{\rm c} \left(\frac{\sin(C_{\rm K} r)}{C_{\rm K} r} \right),$$

where $\rho_{\rm c}$ is the central pressure, and the constant $C_{\rm K} = \pi/R_{\rm p}$. Show that the moment of inertia factor for Jupiter $\alpha_{\rm p} = 0.261$. [11 marks]

(c) The actual value of the moment of inertia factor for Jupiter is $\alpha_{\rm p} = 0.254$. Discuss the implications of the difference between this value and that you calculated above in terms of the interior composition of Jupiter. [4 marks]

8. (a) (i) Explain briefly why discoveries of planets using the astrometric method have not been made from large ground-based observatories. Suggest methods by which such detections might be made in the future. [3 marks]

(ii) It can be shown that observational limits on the minimum detectable astrometric shift lead to a limit on the detectable mass of a planet that is inversely proportional to the semi-major axis of that planet's orbit. Similarly, the minimum detectable radial velocity shift for a star leads to a limit on the mass of a detectable planet that is proportional to the square root of its semi-major axis. Explain, with the aid of a graph, how this implies that the two methods of planet detection are complementary. [3 marks]

(b) The Kepler 454 system hosts a planet (Kepler 454-b) that has been detected both by the radial velocity and transit methods. The following observational characteristics have been found.

Kepler 454-b		
Lower limit on planet's mass	$M_{\rm p} \sin i$	4.08×10^{25} kg
Transit depth	$\Delta L/L$	4.10×10^{-4}
Transit time	τ	4670 s
Orbital period	P	10.6 days
Kepler 454		
Mass of star	M_*	$1.03 M_\odot$
Radius of star	R_*	$1.07 R_\odot$

i is the inclination of the planetary system to our line-of-sight. Calculate the density of the planet, and so infer its planetary class. [10 marks]

[The radius of the Sun $R_\odot = 6.96 \times 10^8$ m. In transits, the transit time τ is related to the radius of the planet $R_{\rm p}$, its orbital semi-major axis a and the latitude of the transit across the parent star δ as

$$\tau = \frac{P(R_{\rm p} + R_*)\cos\delta}{\pi a},$$

and the transit latitude relates to the inclination as
$a \cos i = (R_{\rm p} + R_*) \sin\delta$.]

(c) Calculate the variation in the timing of the start of the Kepler 454-b transit, in seconds, if it has a moon with a mass of 2.34×10^{23} kg in a circular orbit with a semi-major axis of 4.52×10^5 km, coplanar with the orbit of the planet around Kepler 454. [4 marks]