Mathematical Methods in Physics

Workshop 3

3.1

A matrix is said to be idempotent if $A = A^2$.

a) Explain why all eigenvalues of A must be 0 or 1.

[*Hint: Start by considering the expression* $A\mathbf{x} = \lambda\mathbf{x}$.]

b) Show that the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{pmatrix}$$

is idempotent and find its eigenvalues and eigenvectors. If possible, choose a set of linearly independent eigenvectors.

c) Confirm that $\mathrm{Tr}A = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1\lambda_2\lambda_3$, where λ_i are the eigenvalues.

3.2

The same similarity transformation diagonalise each of two matrices A and B. Show that the matrices A and B must commute i.e. $AB = BA$.

3.3

Diagonalise the matrix A

$$A = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix},$$

and calculate its exponential i.e. the matrix $B = e^A$.

[*Hint: You can find a matrix* S *that is unitary.*]

3.4

Consider a matrix

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

and a column vector $\mathbf{b}$ whose components are b_1, b_2, b_3.

a) If $A\mathbf{b} = \mathbf{c}$, what is the relationship between the vectors $\mathbf{b}$ and $\mathbf{c}$?

b) Find the eigenvalues of A and show that $\mathbf{a}$, whose components are a_1, a_2, a_3, is one of its eigenvectors, that is $A\mathbf{a} = \lambda\mathbf{a}$, where λ is one of the eigenvalues. Explain why this must be so.

3.5

An elastic membrane in the xy-plane with boundary circle $x^2 + y^2 = 1$ is stretched so that a point $P = (x, y)$ goes over into the point $P' = (x', y')$ given by $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix},$$

and $\mathbf{x}$, $\mathbf{x}'$ are the position vectors of P and P' respectively.

a) Find the *principal directions*, i.e the directions of $\mathbf{x}$ for which $\mathbf{x}'$ is the same or exactly opposite.

b) What is the shape of the boundary circle of the membrane after the deformation?

[*Hint: This is an eigenvalue problem.*]