Lecture 5:
Stellar power source –
Virial theorem and gravitational collapse

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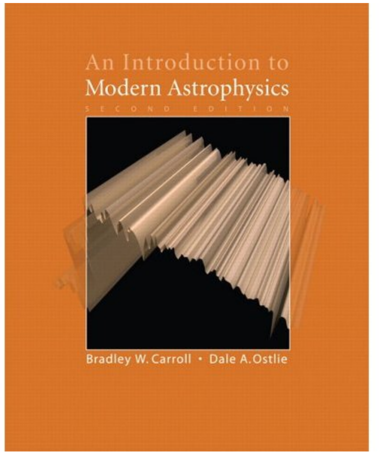
Chapter 10 of Carroll and Ostlie

(general approach for virial theorem from Dina Prialnik "Theory of stellar structure and evolution")



Aims of lecture

Key concept: energy from gravitational collapse

Aims:

- Understand the virial theorem as it applies to stars
- Be able to calculate the energy available from gravitational collapse and to be able to calculate the Kelvin-Helmholtz time
- Know and be able to use:

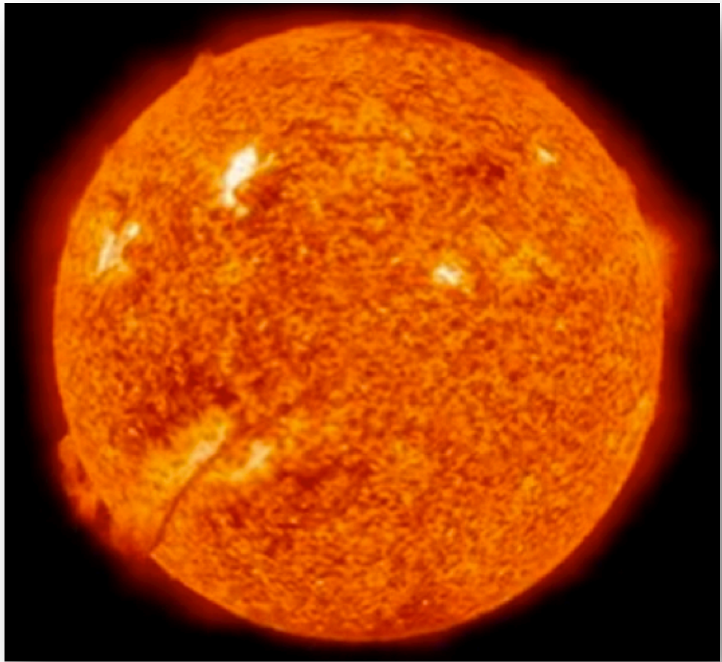
$$K = -\frac{1}{2}U$$

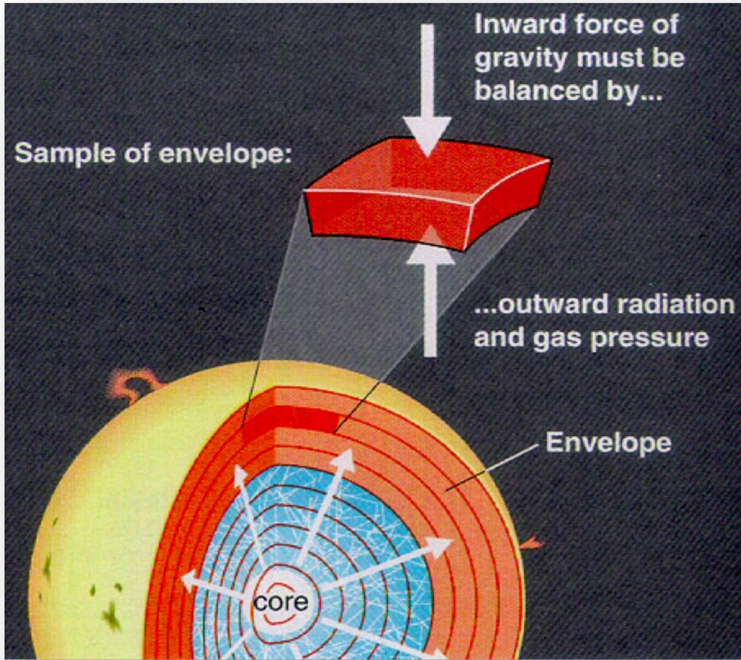
Virial theorem

$$E = \frac{3}{10}\frac{GM^2}{R}$$

Radiated energy from gravitational collapse

Hydrostatic Equilibrium





$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

In this lecture we will understand how the gravitational potential energy is connected to the kinetic energy in stars

The Virial Theorem

The virial theorem describes the overall connection between the gravitational potential energy and the kinetic energy of a system

We start with the hydrostatic equilibrium equation and multiply it by volume:

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$ and since $$V = \frac{4}{3}\pi r^3$$ then

$$V\frac{dP}{dr} = -\frac{GM\rho}{r^2}\frac{4\pi r^3}{3}$$ which taking account of Equation 7 (*dm/dr*) becomes

$$V\frac{dP}{dr} = -\frac{1}{3}\frac{GM}{r}\frac{dm}{dr}$$

$$\int_0^{P(R)} V\,dP = -\frac{1}{3}\int_0^M \frac{GM}{r}\,dm$$

(the integral on the RHS will give the gravitational potential of the star, which is sometimes referred to as Ω but we will define as U)

The Virial Theorem

Integrate the LHS by parts:

$$\int_0^{P(R)} V\,dP = [PV]_0^R - \int_0^{V(R)} P\,dV$$

Since $P=0$ at $r=R$ and $V=0$ at $r=0$ the first term on the RHS vanishes and so

$$-3\int_0^{V(R)} P\,dV = U$$ which since $$dV = \frac{dm}{\rho}$$ becomes

$$-3\int_0^{M} \frac{P}{\rho} dm = U$$

This is the generalised form of the virial theorem: we will now define the internal energy of the star

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Assume the ideal gas law:

$$P = nkT = \frac{\rho kT}{\mu m_H}$$

and since the kinetic energy/particle is: $\frac{3}{2}kT$

(i.e., Maxwell-Boltzmann distribution)

Therefore $$E_{KE} = \frac{3}{2}\frac{kT}{\mu m_H} = \frac{3}{2}\frac{P}{\rho}$$ is the kinetic energy/kg

The Virial Theorem

$$-3\int_0^M \frac{P}{\rho}dm = U$$ with $$\frac{P}{\rho} = E_{KE}\frac{2}{3}$$ gives:

$$\int_o^M E_{KE}\,dm = -\frac{1}{2}U$$

(the integral on the LHS will give the internal energy of the star, which we will define as K)

Therefore: $$K = -\frac{1}{2}U$$ **Equation 10**

The virial theorem implies that as the star collapses under gravity only half of the potential energy is radiated away

The virial theorem doesn't just apply to stars – it applies more generally to systems in equilibrium and has many applications

Energy available from gravitational collapse

How long can a star shine through gravitational collapse? This was first explored as a viable mechanism for the Sun by Kelvin and Helmholtz

We will also use this approach for protostars and supernovae

The gravitational potential of a point mass is:

$$dU_{g,i} = -\frac{GM_r dm_i}{r}$$

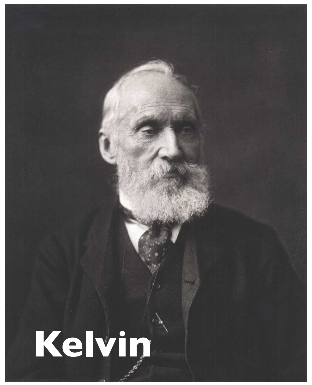
If we assume a shell of thickness dr and mass dm then

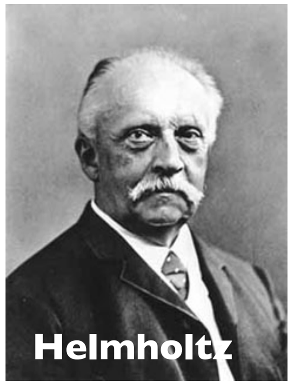
$$dm = 4\pi r^2 \rho dr$$

and therefore:

$$dU_g = \frac{-GM_r 4\pi r^2 \rho}{r} dr$$

Postulated as mechanism for powering Sun





Energy available from gravitational collapse

Then integrate over all shells assuming a constant density for M_r:

$$U_g = -4\pi G \int_0^R M_r \rho r dr$$

where

$$M_r = \frac{4}{3}\pi r^3 \bar{\rho}$$

(assuming a constant density, again!)

Which gives:

$$U_g = -\frac{16\pi^2}{15} G \bar{\rho}^2 R^5$$

Converting from density back to mass gives:

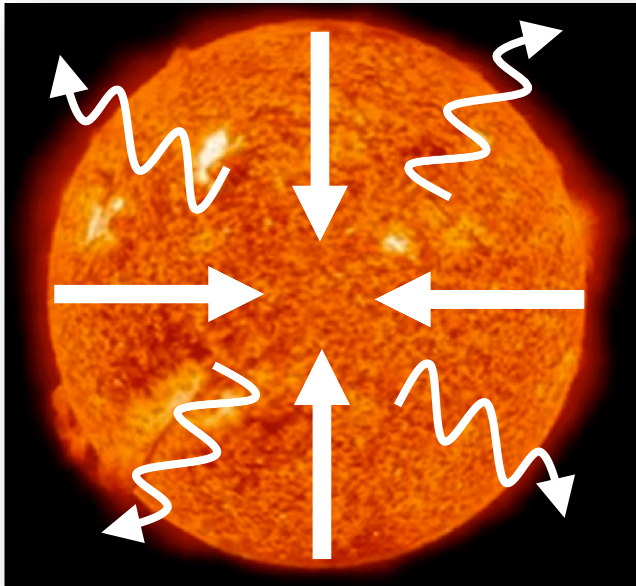
$$U_g \sim -\frac{9}{15}\frac{GM^2}{R}$$

This is the total energy from the gravitational collapse. However, how much will be emitted as thermal energy?

Energy available from gravitational collapse

For a system in equilibrium, such as a star, the virial theorem applies, which states that the kinetic energy (K) is half the time-averaged potential energy (U):

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle$$



Therefore, thermal energy is released as the star collapses (i.e., as potential energy decreases) but only half of the potential energy is radiated – the rest effectively goes into heating the star

Applying the virial theorem therefore gives us the radiated energy:

$$E \sim \frac{3GM^2}{10}\left[\frac{1}{R} - \frac{1}{R_{initial}}\right]$$ which is

Equation 11

$$E \sim \frac{3}{10}\frac{GM^2}{R}$$ when $R << R_{initial}$

How long can the Sun emit at its current luminosity via gravitational collapse?

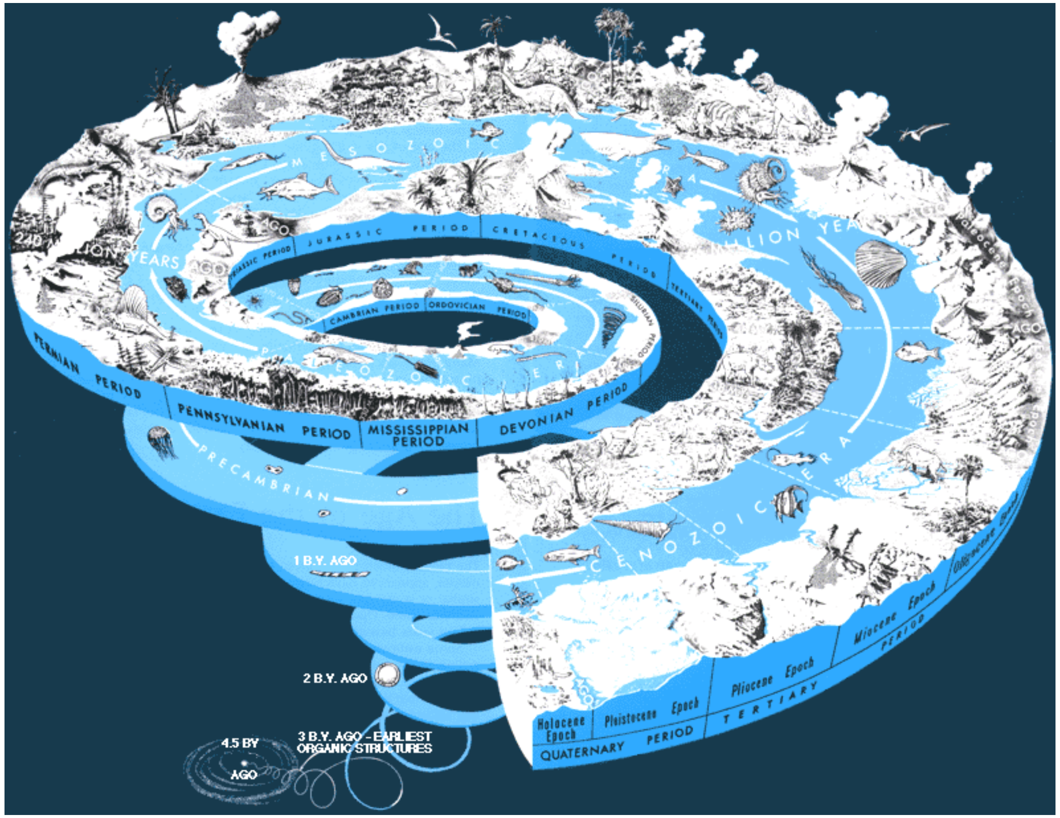
Radiative timescale: gravitational
(Kelvin Helmholtz)

The energy radiated from the gravitational collapse of the Sun is:

$$E \sim \frac{3}{10}\frac{GM_{Sun}^2}{R_{Sun}} \sim 1.1\times 10^{41} J$$

Therefore the lifetime of the Sun via gravitational collapse is:

$$t_{KH} \sim \frac{E}{L_{Sun}} \sim 10^7 \, years$$



Radiative timescale: chemical and mineral(!)

Chemical reactions are based on interactions between electrons in atoms and therefore the amount of energy available is modest (of order 1-10 eV; e.g., Hydrogen atom energy levels). Therefore the total energy release for chemical reactions for an element with mass m_{el} is:

$$E = M/m_{el}\,\Delta E_{el}$$

(assume a chemical reaction involving Hydrogen where ΔE_{el}=10 eV)

$$E = n_{el}\,\Delta E_{el} \sim 2\times10^{39}\ \text{J}$$

$$t \sim E/L \sim 160{,}000\ \text{years}$$

The timescale is even shorter if the Sun was powered by burning coal (molten rocks).

Assuming 3.5×10^{7} J/kg is released from burning coal, the lifetime would be just ~6000 years!

Radiative timescale: nuclear fusion

The energies involved through nuclear processes (e.g., fusion) are typically MeV (i.e., ~10^6 times larger than the energies associated with electron orbits)

$$E = \Delta m\, c^2$$

e.g., proton mass=$1.672621\text{x}10^{-27}$ kg; total E=938.272 MeV
electron mass=$9.109383\text{x}10^{-31}$ kg; total E=510.999 keV

where Δm is the mass converted to energy in the nuclear reaction

On the basis that 0.7% of the mass of Hydrogen is converted to energy when forming a Helium nucleus, the amount of energy available from the sun by converting 10% of its mass into Helium is:

$$E = (0.1\text{x}0.007) \text{ x } M c^2 = 1.3\text{x}10^{44} \text{ J}$$

This gives a nuclear timescale of $t \sim E/L$ or $\sim 10^{10}$ years

This looks more plausible as an energy source!