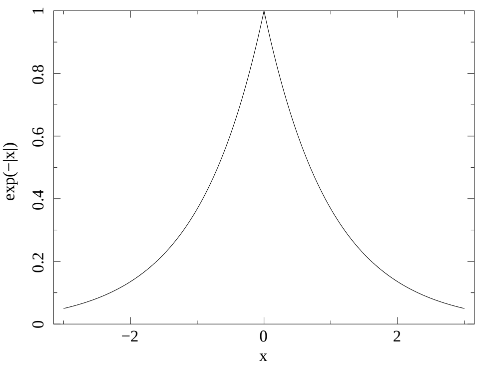
(a) [U:1 mark]



$\int_{-\infty}^{+\infty} \psi^* \psi dx = 1$ so $2A^2 \int_{-\infty}^{0} e^{2kx} dx = 2A^2/(2k)$ so $A = \sqrt{k}$ [U:1 mark]

prob $= A^2 \int_0^{1/k} e^{-2kx} dx = k/(-2k)[e^{-2kx}]_0^{1/k}$ [1 mark]

$= -1/2[e^{-2} - 1] = 0.432$ [U:1 mark]

(b) $p = -i\hbar d/dx$ [S:1 mark]

$\langle p \rangle = \int_0^L 2/L \sin \pi x/L(-i\hbar)d/dx(\sin \pi x/L)dx$ [S:1 mark]

$= -i\hbar(2/L) \int_0^L \sin \pi x/L \cos \pi x/L dx$ [S:1 mark]

$= 0$ as cos is odd about L/2 sin is even [S:1 mark]

(c) $[H, x] = [p^2/2m + V, x] = [p^2/2m, x] + [V, x]$ [S:1 mark]

$V(x)$ is a function only of x so $[V, x] = 0$ [S:1 mark]

$= [p^2/2m, x] = \frac{1}{2m}[p^2, x] = \frac{1}{2m}(p[p, x] + [p, x]p)$ [S:1 mark]

$= \frac{1}{2m}(-i\hbar p - i\hbar p) = -i\hbar p/m$ [S:1 mark]

(d)

$$\sigma_H^2 \sigma_x^2 \geq \left(\frac{1}{2i}\langle [H, x]\rangle\right)^2$$

$$= \left(\frac{1}{2i}\langle -i\hbar p/m\rangle\right)^2 = \left(\frac{-\hbar}{2m}\langle p\rangle\right)^2$$

$= (\hbar^2/4m^2)\langle p\rangle^2$ [U:2 marks] if in energy eigenstate, the energy is deterministic and $\sigma_H = 0$. This means that $\langle p \rangle = 0$ if σ_x is finite [U:1 mark]

physically, the energy eigenfunctions are standing waves [U:1 mark]

(e) $\psi_1 - \psi_{-1} = \frac{1}{\sqrt{2}}(\frac{2}{\sqrt{2}}Y_{11} + \frac{2}{\sqrt{2}}Y_{1-1})$ so $Y_{11} + Y_{1-1} = \psi_1 - \psi_{-1}$ [U:1 mark]

$\psi_0 = \frac{1}{\sqrt{2}}(Y_{11} - Y_{1-1})$ so $\sqrt{2}\psi_0 = Y_{11} - Y_{1-1}$

$\psi_1 - \psi_{-1} + \sqrt{2}\psi_0 = 2Y_{11}$

hence $Y_{11} = \frac{1}{2}(\psi_1 - \psi_{-1} + \sqrt{2}\psi_0)$ [U:2 marks]

prob to get L_x=$\hbar$ is 1/4 [U:1 mark]

(f) prob 1/6 for energy E_1, 4/6=2/3 for E_2 1/6 for energy E_3 [U:1 mark]

$\langle E \rangle = 1/6E_1 + 2/3E_1/4 + 1/6E_1/9 = E_1/6(2 + 1/9) = 19/54E_1$ [U:1 mark]

L_z has eigenvalues $m\hbar$ so $\hbar$ means $m = 1$ so then it is in state ψ_{321} so we get E_3 with probability 1 [U:2 marks]

(g) Prob is $\psi^*_{311}\psi_{311}dV = \psi^*_{311}\psi_{311}r^2 \sin\theta dr d\theta d\phi$ [S:1 mark]

want prob distribution as function of μ so integrate over ϕ and r $D(\mu)d\mu = \int_0^\infty R^*_{31}R_{31}r^2dr \int_0^{2\pi} e^{-i\phi}e^{i\phi}d\phi \;\; 3/(8\pi)\sin^2\theta d\mu$ [U:1 mark]

$= 2\pi \times 3/(8\pi)(1-\mu^2)d\mu = 3/4(1-\mu^2)d\mu$ as R_{nl} are normalised in their space [U:1 mark]

max when $dD(\mu)/d\mu = 0$ so $2\mu = 0$ so $\mu = 0$ (so $\cos\theta = 0$ and $\theta = \pi/2$) [U:1 mark]

(h) $\int_0^a \sqrt{\frac{2}{a}}\sin(n\pi x/a)\alpha\delta(x - a/2)\sqrt{\frac{2}{a}}\sin(n\pi x/a)dx$

$= \frac{2}{a}\alpha\sin^2(n\pi/2)$ [S:1 mark]

odd n then $\sin^2(n\pi/2) = 1$ so $E^1_n = \frac{2}{a}\alpha$ [S:1 mark]

even n and $\sin^2(n\pi/2) = 0$ so $E^1_n = 0$ [S:1 mark]

the perturbation only affects the wavefunction at $x = a/2$ and all even n have zero probability of being found at $x = a/2$ for there is no correction term for these. [S:1 mark]

(a) $\langle x\rangle = \int \Psi(x,t)^* x \Psi(x,t) dx$ [S:1 mark]

$$= \left(\frac{2a}{\pi}\right)^{1/2} \int \frac{e^{-ax^2/(1-iT)}}{(1-iT)^{1/2}} x \frac{e^{-ax^2/(1+iT)}}{(1+iT)^{1/2}} dx$$

$\propto \int e^{-qx^2} x dx = 0$ or just write it down from symmetry! [U:1 mark]

$\langle x^2\rangle = \int \Psi(x,t)^* x^2 \Psi(x,t) dx$ [S:1 mark]

$$= \left(\frac{2a}{\pi}\right)^{1/2} \int \frac{e^{-ax^2/(1-iT)}}{(1-iT)^{1/2}} x^2 \frac{e^{-ax^2/(1+iT)}}{(1+iT)^{1/2}} dx$$ [U : 1 mark]

$$= \left(\frac{2a}{\pi}\right)^{1/2} \int \frac{e^{-2ax^2/(1+T^2)}}{(1+T^2)^{1/2}} x dx = 0$$ [U : 1 mark]

$$= \left(\frac{2a}{\pi}\right)^{1/2} \left(\frac{1}{1+T^2}\right)^{1/2} \frac{1}{2} \left(\frac{\pi(1+T^2)^3}{(2a)^3}\right)^{1/2}$$ [U : 1 mark]

$= (1/2)(1+T^2)/2a = (1+T^2)/(4a)$ [U:1 mark]

hence $\sigma_x = \sqrt{(1+T^2)/(4a) - 0} = \sqrt{(1+T^2)/(4a)}$ as required[U:1 mark]

(b) $p\Psi = -i\hbar d/dx A e^{-ax^2/(1+iT)}$

$= -i\hbar A[-2ax/(1+iT)]e^{-ax^2/(1+iT)}$ [U:1 mark]

$= i\hbar \frac{2ax}{1+iT} \Psi$ [U:1 mark]

$\langle p\rangle = \int \Psi^* p \Psi dx = \int \Psi^* \frac{2ax}{1+iT} \Psi dx$ [U:1 mark]

this is $\propto \int e^{-qx^2} x dx = 0$ [U:1 mark]

$\langle p^2\rangle = \int (p\Psi)^*(p\Psi) dx$

$= \int -i\hbar \frac{2ax}{1-iT} \Psi^* i\hbar \frac{2ax}{1+iT} \Psi dx$ [U:1 mark]

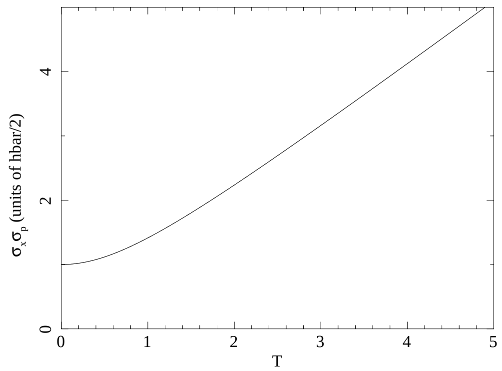
$= \hbar^2 \frac{4a^2}{1+T^2} \int \Psi^* x^2 \Psi dx$

$= \hbar^2 \frac{4a^2}{1+T^2} \langle x^2\rangle = \hbar^2 a$ [U:1 mark]

$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle} = \sqrt{a}\hbar$ [U:1 mark]

(c) $\sigma_x \sigma_p = \sqrt{a}\hbar(1+T^2)^{1/2}/(2\sqrt{a})$

$= (\hbar/2)(1+T^2)^{1/2}$ [U:1 mark]



sketch of $\sigma_x\sigma_p$ [S:1 mark]

minimised at $T = 0$ so $t = 0$, and is at the heisenburg uncertainty limit. [U:1 mark]

$\sigma_x\sigma_p = \hbar$ at t_2 so $\hbar = \hbar/2(1 + T_2^2)^{1/2}$

so timescale is $(1+T_2^2) = 4$ so $T_2 = \sqrt{3}$ so $t_2 = \sqrt{3}m/(2\hbar a)$. [U:1 mark]

$a^{-1/2} = 10^{-10}$ so $a = 10^{20}$

$$\frac{\sqrt{3} \times 9.1 \times 10^{-31}}{2 \times (6.63 \times 10^{-34}/2\pi) \times 10^{20}} = 7.5 \times 10^{-17}\ s$$

[U:1 mark]

(a) $\psi_1 = X_+Z_+$ so $I.S\psi_1 = \frac{1}{2}(I_+S_- + I_-S_+)X_+Z_+ + I_ZS_zX_+Z_+$
$= \frac{1}{2}(I_+Z_+)(S_-X_+) + \frac{1}{2}(I_-Z_+)(S_+X_+) + (I_ZZ_+)(S_zX_+)$
$S_+X_+ = 0$ and $I_+Z_+ = 0$ [U:1 mark]
so $= (\hbar/2Z_+)(\hbar/2X_+) = \hbar^2/4X_+Z_+ = \hbar^2/4\psi_1$ so this is an eigenfunction [U:1 mark]
$\psi_2 = X_+Z_-$ so $I.S\psi_2 = \frac{1}{2}(I_+S_- + I_-S_+)X_+Z_+I_ZS_zX_+Z_-$
$= 1/2(I_+Z_-)(S_-X_+) + 0 + (I_ZZ_-)(S_zX_+)$
$= 1/2(\hbar Z_+)(\hbar X_-) + (-\hbar/2)(\hbar/2)X_+Z_-$ [U:1 mark]
$= \hbar^2/4(-X_+Z_- + 2X_-Z_+) = \hbar^2/4(-\psi_2 + 2\psi_3)$ so not an eigenfunction [U:1 mark]
$\psi_3 = X_-Z_+$ so $I.S\psi_3 = \frac{1}{2}(I_+S_- + I_-S_+)X_-Z_+ + I_ZS_zX_-Z_+$
$= 0 + 1/2(I_-Z_+)(S_+X_-) + (I_ZZ_+)(S_zX_-)$
$= 1/2(\hbar Z_-)(\hbar X_+) + (1/2\hbar)(-\hbar/2)X_-Z_+$
$= 1/2\hbar^2 Z_-X_+ - \hbar^2/4X_-Z_+$ [U:1 mark]
$= \hbar^2/4(2X_+Z_- - X_-Z_+) = \hbar^2/4(2\psi_2 - \psi_3)$ not an eigenfunction [U:1 mark]
$\psi_4 = X_-Z_-$ so $I.S\psi_1 = 0 + 0 + I_ZS_zX_-Z_-$
$= (-\hbar/2)Z_-(-\hbar/2)X_- = \hbar^2/4X_-Z_- = \hbar^2/4\psi_4$ which is an eigenfunction. [U:1 mark]

(b) let $w = 4E^1/A\hbar^2$

$$\begin{pmatrix} 1-w & 0 & 0 & 0 \\ 0 & -1-w & 2 & 0 \\ 0 & 2 & -1-w & 0 \\ 0 & 0 & 0 & 1-w \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 0 \quad [\mathrm{U:1\ mark}]$$

non trivial solutions where det=0 [U:1 mark]

$$(1-w)\begin{vmatrix} -1-w & 2 & 0 \\ 2 & -1-w & 0 \\ 0 & 0 & 1-w \end{vmatrix} = 0$$

$= (1-w)[(-1-w)(-1-w)(1-w) - 2(2(1-w))] = (1-w)^2[(-1-w)^2 - 4]$ [U:1 mark] solns are $w = 1$ (twice) and $-1 - w = \pm 2$ so $w = -1 \pm 2$ so $w = 1$ and -3 so $E^1_{hf} = A\hbar^2/4$ and $-3A\hbar^2/4$ [U:1 mark]

$w = 1$ (3 solutions) is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 0$$ [U : 1 mark]

so need 3 eigenvectors (one for each solution). the constrained one is $-2\beta + 2\gamma = 0$ so $\chi_1 = \frac{1}{\sqrt{2}}(\psi_2 + \psi_3)$ [U:1 mark]

unconstrained $\chi_2 = \psi_1$ and $\chi_3 = \psi_4$ [U:1 mark]

$w = -3$ (1soln) is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 0$$ [U : 1 mark]

so $\alpha = 0$ and $\delta = 0$ and only $2\beta + 2\gamma = 0$ so $\gamma = -\beta$ and $\chi_4 = \frac{1}{\sqrt{2}}(\psi_2 - \psi_3)$ [U:1 mark]

(c) $F^2 = (\underline{I} + \underline{S}).(\underline{I} + \underline{S})$

$= I^2 + S^2 + 2\underline{IS}$

so $\underline{IS} = \frac{1}{2}(F^2 - I^2 - S^2)$ [U:1 mark]

eigenvalues $\frac{\hbar^2}{2}(f(f+1) - \frac{3}{4} - \frac{3}{4}) = \frac{\hbar^2}{2}(f(f+1) - \frac{3}{2})$ [U:1 mark]

f can go from $|s - s| = 0$ to $s + s = 1$ so $\underline{S}.\underline{I}$ has 2 values, $f = 0$ is $-\frac{3}{4}\hbar^2$ and $f = 1$ is $\frac{1}{4}\hbar^2$ [U:1 mark]

without it was degeneracy 2 as 2 spins gave the same energy. now got 2 separate energies, for the 2 electron states, so degeneracy is completely lifted. [S:1 mark]

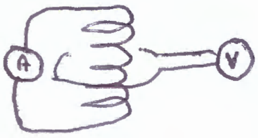
Electromagnetism

Prof. Hampshire June 16, Qn1.

a). $\underline{E} = \rho_N \underline{J}$ Apply a current through the mineral and measure the electric field generated across it. High insulating $\Rightarrow \rho_N \to \infty$.
$\underline{M} = \chi \underline{H}$ Apply a magnetic field to the sample and measure the additional field the material produces in response. Non magnetic $\Rightarrow \chi = 0$.

4 Marks Seen /Unseen

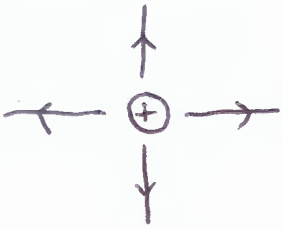
g). $\underline{\nabla} \times \underline{E} = -\partial \underline{B}/\partial t \Rightarrow V = -\partial \phi/\partial t$.



Measure the ac. voltage across a loop that surrounds a coil producing an ac. B-field.

4 Marks Seen.

c).



Magnetic monopole.

$$\underline{B} = \frac{q_m \hat{r}}{r^2}$$

q_m : magnetic charge
r : distance to point of observation.

4 Marks Seen/ Unseen.

d) $\underline{J}_D = \varepsilon_0 \partial \underline{E}/\partial t$. The displacement current density does not involve the movement of charge. However it shows that a changing electric field has a magnetic field associated with it and is required to generalise Ampère's law.

4 Marks Seen.

e). Fresnel's equation describe the reflection and transmission of electromagnetic waves across an interface. They are useful for describing the propagation of EM from one medium to another.

4 Marks Seen

f). In polar dielectrics, there are permanent dipole moments that rotate in response to an applied $\underline{E}$-field. In non-polar dielectrics, there is the displacement of charge to produce dipoles.

4 Marks Seen

b) The skin effect is associated with currents or EM waves ~~beta~~ being constrained to the surface layer or 'skin depth' of a conducting material. For EM waves, ohmic currents flow that dissipate the energy of the wave.

4 Marks Seen.

Electromagnetism

Prof. Hampshire June 2016 Qn 2.

a) By definition, dispersion relations give ω (angular frequency) versus k (wavevector). They are useful because they relate energy to momentum, or phase velocity to frequency, or group velocity to frequency. — 5 Marks Seen/Unseen

b) $$k^2 = \mu_0 \varepsilon_0 \varepsilon_r \omega^2 + \cancel{i\omega\mu_0\sigma_N}$$ insulating ($\sigma_N = 0$). — 5 Marks Unseen.

$$\sqrt{\varepsilon_r} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \cdot \frac{k}{\omega} = 3\times10^8 \cdot \frac{2\pi}{2\pi \cdot 5\times10^4 \cdot 0.2\times10^{-6}} = 3$$
$$\Rightarrow \varepsilon_r = 9.$$

c). In the poor conductor limit:
$$k^2 = \mu_0\varepsilon_0\varepsilon_r\omega^2\left(1 + i\sigma_N/\varepsilon_0\varepsilon_r\omega\right)$$ 2 Marks
$$k \approx \sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}\,\omega\left(1 + \frac{i\sigma_N}{2\varepsilon_0\varepsilon_r\omega}\right) = k_{REAL} + k_{IMAG}$$

Given $$\underline{E} = \underline{E}_0 \exp i(k_{REAL}x - \omega t)\exp -(k_{IMAG}x)$$

$$\Rightarrow k_{IMAG} \cdot x = 10^{-10} = \frac{\sqrt{\mu_0\varepsilon_0}\,\sigma_N x}{2\varepsilon_0\sqrt{\varepsilon_r}} = \frac{\sigma_N c\, x\, \mu_0}{2\sqrt{\varepsilon_r}}$$ 4 Marks.

$$\Rightarrow \sigma_N = \frac{10^{-10}\, 2.3}{2\times10^4 \cdot 3\times10^8 \cdot 4\pi\times10^{-7}} = 8\times10^{-17}\,\Omega^{-1}m^{-1}$$

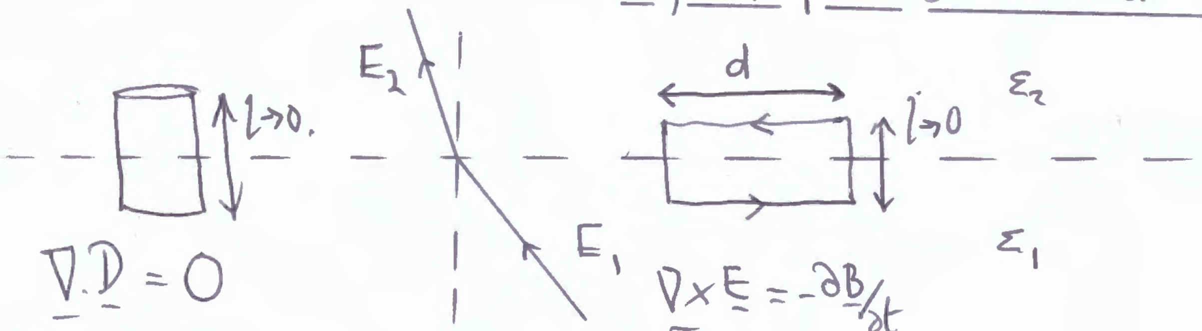
— 6 Marks Unseen

d) The energy of the light is determined by the energy of the photons. The frequency of the photons is unaffected by the gas hence the energy is unaffected. $\Rightarrow$ Not possible. — 4 Marks Unseen.

Electromagnetism

Prof. Hampshire June 2016 Qn 3.

a)



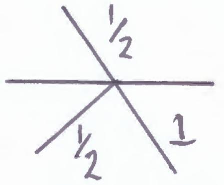
$\nabla \cdot D = 0$ $\qquad$ $\nabla \times E = -\partial B/\partial t$

$D_{1\perp} = D_{2\perp}$ $\qquad$ $E_{1\parallel}\, l = E_{2\parallel}\, l$

$\Rightarrow \varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$ $\qquad$ $\Rightarrow E_{1\parallel} = E_{2\parallel}$.

4 Marks Seen/Unseen.

b)



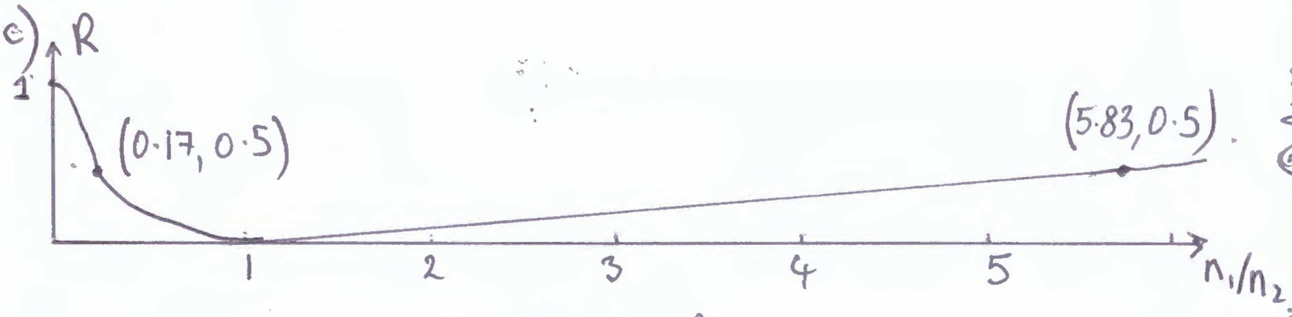
$\alpha = n_1/n_2$, $R = \left(\dfrac{\alpha - 1}{\alpha + 1}\right)^2 = \dfrac{1}{2} \Rightarrow \sqrt{2}(\alpha - 1) = \pm(\alpha + 1)$ $\quad$ 1

$(\sqrt{2} - 1)\alpha = \sqrt{2} + 1 \Rightarrow \alpha = (\sqrt{2}+1)/(\sqrt{2}-1) = 5.83.$ $\quad$ 2

$(\sqrt{2} + 1)\alpha = \sqrt{2} - 1 \Rightarrow \alpha = (\sqrt{2}-1)/(\sqrt{2}+1) = 0.17.$ $\quad$ 2

5 Marks Unseen

c)

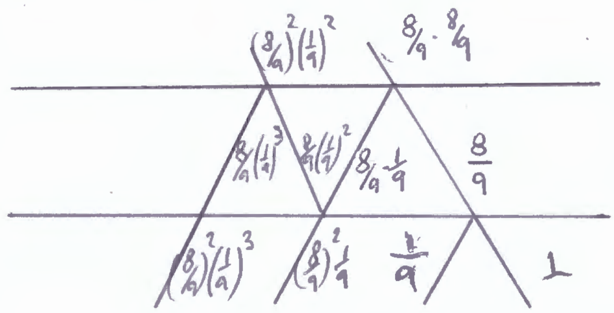


2 Axes

3 Plot.

5 Marks Unseen

d)



3

$n = 2$, $R = \left(\frac{1}{3}\right)^2 = 1/9$.

Fraction reflected $= \frac{1}{9} + \left(\frac{8}{9}\right)^2\left(\frac{1}{9}\right) + \left(\frac{8}{9}\right)^2\left(\frac{1}{9}\right)^3$

$= 0.111 + 0.087 + 0.001 \approx 0.199$ $\quad$ 3

20% reflected.

6 Marks Unseen