# ELECTROMAGNETISM - Workshop 6th Set (Qns) Materials and Waves

**Professor D P Hampshire – 2nd Year Physics Lecture Course**

The material for this workshop is split into just two parts. Part I gives some worked examples. Part II gives some additional unseen questions.



## 1 Worked examples

If you are asked to write down equations or definitions in an exam, make sure that every symbol you use is properly defined using vector or scalar notation and good scientific English.

### 1.1 Questions

**1.** Give a pictorial description of the Drude model.
**2.** Write down Ohm's law in intensive variables.
**3.** Give a pictorial representation and a mathematical definition of an elementary electric dipole moment.
**4.** Give definitions for: polarization and relative dielectric constant.
**5.** Give a pictorial representation and a mathematical definition of an elementary magnetic moment.
**6.** Give definitions for: magnetization; magnetic field strength; magnetic susceptibility and permeability
**7.** Write down the London equations for a superconductor.
**8.** Show that the general dispersion relation for an infinite, linear, isotropic homogeneous material which is conducting with electrical conductivity $\sigma_n$, magnetic with permeability $\mu_r$, and a dielectric with permittivity $\varepsilon_r$ is given by,

$$k^2 = \mu_o\mu_r\varepsilon_o\varepsilon_r\omega^2 + i\omega\mu_o\mu_r\sigma_n$$

Hence show that in the limit that a material is highly conducting, the wavelength $\lambda$, of an electromagnetic wave in the material is independent of the permittivity of the material.
**9.** Write down the London brother's two equations that describe superconductors in the Meissner state. Using Maxwell's equations, show that typically an applied magnetic field penetrates a distance equal to the London penetration depth, $\lambda_L$ , into the material where $\lambda_L = \left(\frac{m_e}{\mu_0 N_s e^2}\right)^{\frac{1}{2}}$.

**10.** Given that for a plasma, the phase velocity, $v_{phase}$ , and the group velocity, $v_{group}$ are given
by:

$v_{phase} = \frac{c}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}$ (for $\omega > \omega_p$) and $v_{group} = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$ (for $\omega > \omega_p$), sketch them as a function of angular frequency.

**11.** Given that for a dielectric such as glass or water, the real and imaginary parts of the relative dielectric constant are given by:

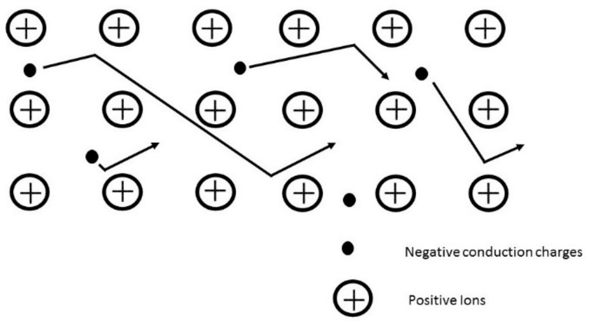
$$\varepsilon_{real} = 1 + \frac{Nq^2}{m\varepsilon_0}\frac{(\omega_0^2-\omega^2)}{\left((\omega_0^2-\omega^2)^2+\left(\frac{\omega}{\tau}\right)^2\right)}$$

$$\varepsilon_{imaginary} = \frac{Nq^2}{m\varepsilon_0\tau}\frac{\omega}{\left((\omega_0^2-\omega^2)^2+\left(\frac{\omega}{\tau}\right)^2\right)}$$

sketch them as a function of angular frequency.

## 1.2 Answers

**1. Drude model**




The Drude model - The electric field accelerates the charges which then collide with the scattering sites. Note we have shown negative charge being accelerated but by convention we assume the current density is the flow of positive charges.

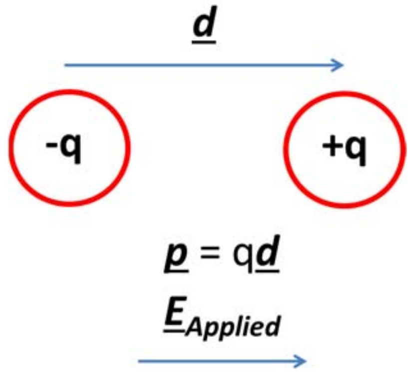
**2.** Ohm's law:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{J}}\rho_n \text{ or } \underline{\boldsymbol{J}} = \sigma_n\underline{\boldsymbol{E}}, \qquad 1\text{-}1$$

| Dielectrics | Magnetic materials |
|---|---|
| $\underline{\boldsymbol{p}} = q\underline{\boldsymbol{d}}$ | $\underline{\boldsymbol{m}} = IA\hat{\boldsymbol{n}}$ |
| $\underline{\boldsymbol{P}} = N\underline{\boldsymbol{p}}$ | $\underline{\boldsymbol{M}} = N\underline{\boldsymbol{m}}$ |
| $\underline{\boldsymbol{P}} = \varepsilon_o(\varepsilon_r - 1)\underline{\boldsymbol{E}}$ | $\underline{\boldsymbol{B}} = \mu_o\mu_r\underline{\boldsymbol{H}}$ |
| $\underline{\boldsymbol{P}} = \varepsilon_o\chi_e\underline{\boldsymbol{E}}$ | $\underline{\boldsymbol{M}} = \chi\underline{\boldsymbol{H}}$ |
| $\underline{\boldsymbol{D}} = \varepsilon_o\underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$ | $\underline{\mathbf{B}} = \mu_o[\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}]$ |

Table 1 : Useful definitions for dielectrics and magnetic materials.

**3.** An electric dipole moment. $\underline{\boldsymbol{p}}$,




Definition of the elementary electric dipole moment $\underline{\boldsymbol{p}}$:

$$\underline{\boldsymbol{p}} = \mathrm{q}\underline{\boldsymbol{d}} \qquad 1\text{-}2$$

**4.** Definition of the polarisation, $\underline{\boldsymbol{P}}$ ($\mathrm{Cm}^{-2}$):

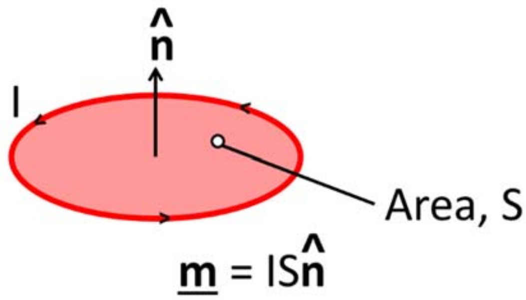
$$\underline{\boldsymbol{P}} = \mathrm{N}\underline{\boldsymbol{p}} \qquad 1\text{-}3$$

where N is the number of dipoles per unit volume($\mathrm{m}^{-3}$).

Definition of the relative dielectric constant ($\varepsilon_r$):

$$\underline{\boldsymbol{P}} = \varepsilon_o(\varepsilon_r - 1)\underline{\boldsymbol{E}} \qquad 1\text{-}4$$

**5.** A magnetic dipole moment.




Definition of the magnetic dipole moment ($\underline{\boldsymbol{m}}$):

$$\underline{\boldsymbol{m}} = IA\hat{\boldsymbol{n}} \qquad 1\text{-}5$$

where $I$ is the current flowing around a loop of area $A$.

**6.** Definition of the magnetization (***M***):

$$\underline{\boldsymbol{M}} = N\underline{\boldsymbol{m}} \qquad 1\text{-}6$$

where $N$ is the number of magnetic dipoles per unit volume.

Definition of the magnetic field strength $\underline{\boldsymbol{H}}$:

$$\underline{\mathbf{B}} = \mu_o[\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}] \qquad 1\text{-}7$$

Definition of the magnetic susceptibility ($\chi$):

$$\underline{\boldsymbol{M}} = \chi\underline{\boldsymbol{H}} \qquad 1\text{-}8$$

Definition of relative permeability $\mu_r$:

$$\underline{\boldsymbol{B}} = \mu_o \mu_r \underline{\boldsymbol{H}} \qquad 1\text{-}9$$

**7.**

$$\underline{\boldsymbol{E}} = \mu_0 \lambda_L^2 \frac{\partial \underline{\boldsymbol{J}}}{\partial t} \text{ – Zero resistance} \qquad 1\text{-}10$$

and

$$\underline{\boldsymbol{B}} = -\mu_0 \lambda_L^2 \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{J}} \text{ – The Meissner effect} \qquad 1\text{-}11$$

**8.** Maxwell's fourth equation is:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \qquad 1\text{-}12$$

The current density produced by conducting, magnetic and dielectric materials is given by:

$$\underline{\boldsymbol{J}} = \sigma_n \underline{\boldsymbol{E}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial t} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \qquad 1\text{-}13$$

Do not get confused: $\sigma_n$ is the electrical conductivity from Ohm's law.

This gives:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \left( \sigma_n \underline{\boldsymbol{E}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial t} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \right) + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \qquad 1\text{-}14$$

which can be rearranged as

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{B}} - \mu_0 \underline{\boldsymbol{M}}) = \mu_0 \sigma_n \underline{\boldsymbol{E}} + \mu_0 \frac{\partial}{\partial t} (\underline{\boldsymbol{P}} + \varepsilon_0 \underline{\boldsymbol{E}}) \qquad 1\text{-}15$$

Using the materials properties definitions for $\varepsilon_r$ , $\underline{\boldsymbol{H}}$, $\mu_r$:

$$\underline{\boldsymbol{P}} = \varepsilon_o (\varepsilon_r - 1) \underline{\boldsymbol{E}} \qquad 1\text{-}16$$

$$\underline{\mathbf{B}} = \mu_o [\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}] = \mu_o \mu_r \underline{\boldsymbol{H}} \qquad 1\text{-}17$$

Which gives:

$$\underline{\boldsymbol{\nabla}} \times \frac{\underline{\mathbf{B}}}{\mu_r} = \mu_0 \sigma_n \underline{\boldsymbol{E}} + \mu_0 \frac{\partial}{\partial t} (\varepsilon_r \varepsilon_0 \underline{\boldsymbol{E}}) \qquad 1\text{-}18$$

We substitute this into the curl of Maxwell's 3rd equation to find:

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) = -\frac{\partial}{\partial t} \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = -\frac{\partial}{\partial t} \left[ \mu_o \mu_r \left( \sigma_n \underline{\boldsymbol{E}} + \varepsilon_o \varepsilon_r \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \right) \right] \qquad 1\text{-}19$$

Given that for a wave travelling through a conducting medium, $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\boldsymbol{\nabla} \cdot \boldsymbol{D}}{\varepsilon_r \varepsilon_0} = \frac{\rho_{free}}{\varepsilon_r \varepsilon_0} = 0$ and using the vector identity $\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) = \underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}) - \nabla^2 \underline{\boldsymbol{E}}$ gives,

$$\Rightarrow \nabla^2 \underline{\boldsymbol{E}} - \mu_o \mu_r \sigma_n \frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \mu_o \mu_r \varepsilon_o \varepsilon_r \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} = 0 \qquad \text{1-20}$$

We can substitute a plane wave solution into the wave equation.

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \text{expi}(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t) \text{ - plane wave equation} \qquad \text{1-21}$$

Hence, we find the dispersion relation (which is by definition the relationship between ω and k).

$$k^2 = \mu_o \mu_r \varepsilon_o \varepsilon_r \omega^2 + i\omega \mu_o \mu_r \sigma_n \qquad \text{1-22}$$

General dispersion relation for an infinite, linear, isotropic, homogenous medium

For highly conducting media where $\sigma_N$ is very large

So $k^2 = +i\mu_o\mu_r\sigma_N\omega$, and $k = \left(\frac{\mu_o\mu_r\sigma_N\omega}{2}\right)^{1/2} + i\left(\frac{\mu_o\mu_r\sigma_N\omega}{2}\right)^{1/2}$. Substituting back into the travelling wave gives:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \text{expi}\left(\frac{(\mu_o\mu_r\sigma_N\omega)^{\frac{1}{2}}}{\sqrt{2}} x - \omega t\right) \cdot \exp - \frac{(\mu_o\mu_r\sigma_N\omega)^{1/2}}{\sqrt{2}} x$$

[Note if $k = k_{real} + ik_{imaginary}$ then $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \text{expi}((k_{real} + ik_{imaginary}) \cdot x - \omega t) = \underline{\boldsymbol{E}}_o \text{expi}(k_{real} \cdot x - \omega t)\exp(-k_{imaginary} \cdot x)$]

So the wavelength λ of an electromagnetic wave in a highly conducting material is $\lambda = \frac{2\pi}{k_{real}} = 2\pi\left(\frac{2}{\mu_o\mu_r\sigma_N\omega}\right)^{1/2}$ which is independent of the permittivity of the material.

**9.** The London equations are:

$$\underline{\boldsymbol{E}} = \mu_0 \lambda_L^2 \frac{\partial \boldsymbol{J}}{\partial t} \text{ – Zero resistance}$$

$$\underline{\boldsymbol{B}} = -\mu_0 \lambda_L^2 \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{J}} \text{ – The Meissner effect}$$

Substituting Maxwell IV for $\boldsymbol{J}$ into the second London equation gives

$$\underline{\boldsymbol{B}} = -\lambda_L^2 \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}$$

We find, using the vector relation: $\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}) = \underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}}) - \nabla^2 \underline{\boldsymbol{B}}$ **,** that

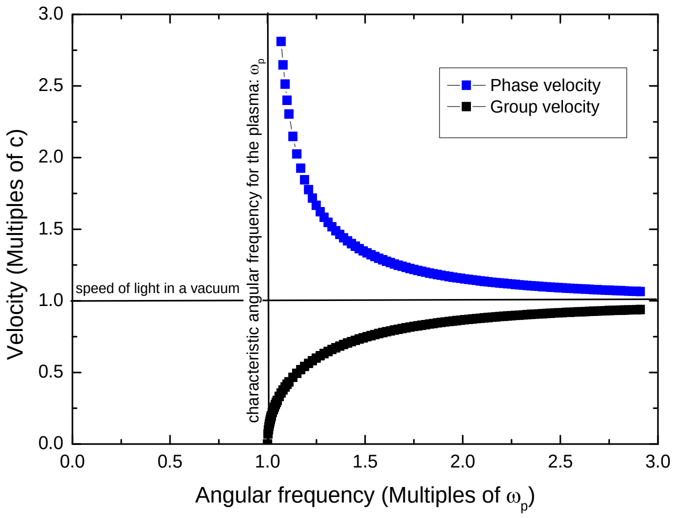
$$\nabla^2 \underline{\boldsymbol{B}} = \frac{1}{\lambda_L^2} \underline{\boldsymbol{B}}$$

For a semi-infinite slab, this equation has solutions of the form:

$$\underline{\boldsymbol{B}}(x) = \underline{\mathbf{B}}_0 \exp(-x/\lambda_L) \quad \text{for} \quad x > 0,$$

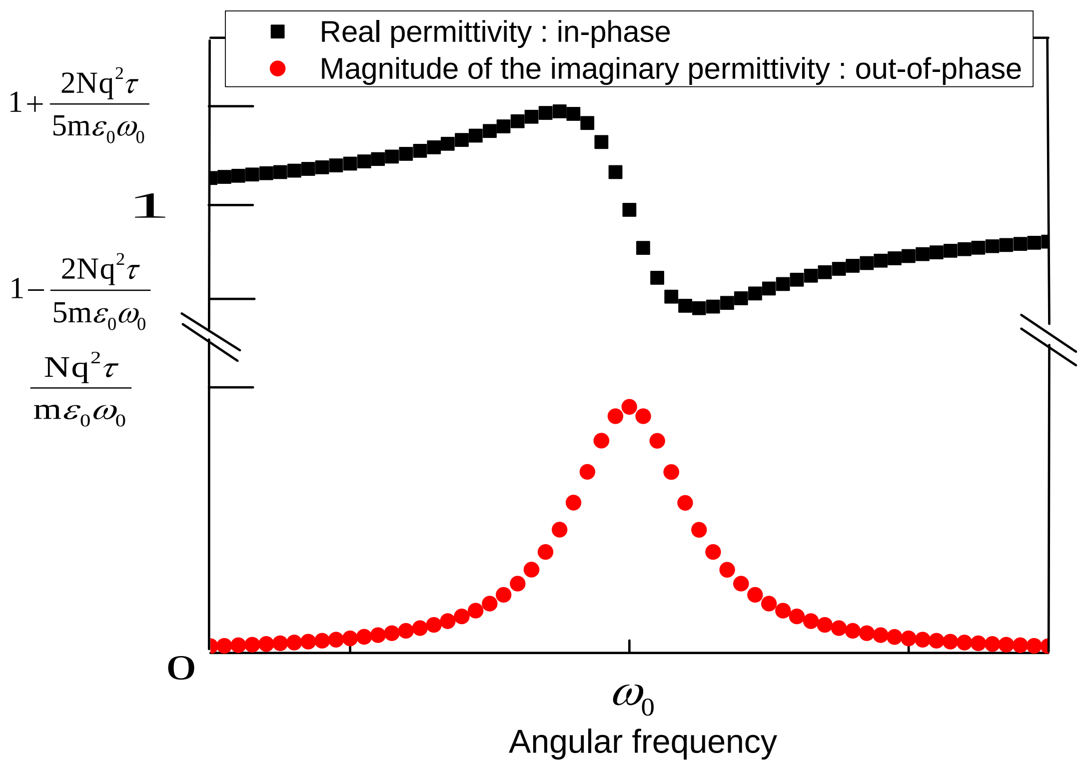
and describes the Meissner effect.

**10.**




The velocity of an electromagnetic wave in a plasma as a function of angular frequency. Electromagnetic waves propagating through the plasma are attenuated at angular frequencies below the angular plasma frequency ($\omega_p$).

**11.**




The variation of relative permittivity with angular frequency near a resonance.

## 2 Unseen problems

Dielectrics are an important class of materials - including such materials as glass which of course are critical in glasses, telescopes…. and water, which gives us rainbows. It can be argued that the use of glass was the pivotal activity in the development of the modern technological world since it enabled both widespread reading and an understanding of our solar system.

**1.** Point charges $q_1 = -4.5$ nC and $q_2 = +4.5$ nC are separated by 3.1 mm, forming an electric dipole, find the electric dipole moment.

**2.** In the problem above, if the dipole is placed in a uniform electric field whose direction makes an angle of $36.9^o$ with the line connecting the charges. What is the magnitude of the electric field if the torque exerted on the dipole has magnitude $7.2 \times 10^{-9}$Nm.

**3.** The polarisation of a dielectric is given by $\underline{\boldsymbol{P}} = \mathrm{x^2y\hat{i} - xy\hat{j} + 4z\hat{k}}$. What is the value of the volume charge density at the point (1, 2, -3) from the polarized material?

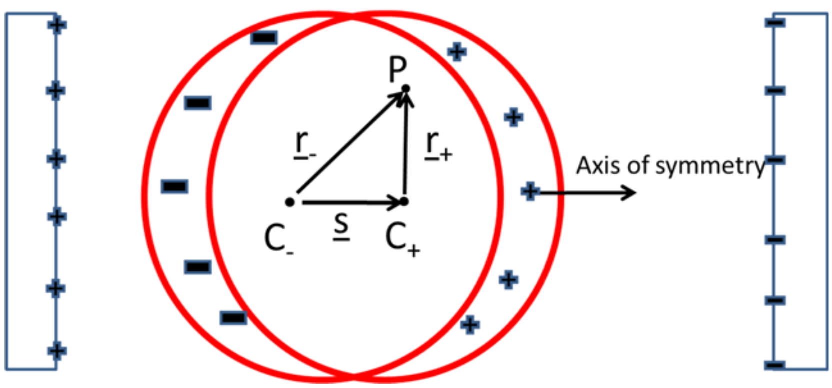
**4.** The polarisation of a dielectric is given by $\underline{\boldsymbol{P}} = \frac{\mathrm{k}}{\mathrm{r^2}}\hat{\mathbf{r}}$, where $k$ is a constant. Find the volume charge density induced by the polarisation.
Note:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

**5.** A long straight wire, carrying uniform line charge $\lambda$ , is surrounded by rubber insulation out to a radius $a$. Find the electric field inside the insulator if the relative permittivity is $\varepsilon_r$.

**6.** The definition of the electrical displacement vector $\underline{\boldsymbol{D}} = \varepsilon_0\underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$ . The definition of the dielectric constant ($\varepsilon_r$) is given by $\underline{\boldsymbol{P}} = \varepsilon_o(\varepsilon_r - 1)\underline{\boldsymbol{E}}_{applied}$. Inside a dielectric, the surface charge density is $\underline{\boldsymbol{P}} \cdot \hat{\boldsymbol{n}}$ , the volume charge density is $-\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{P}}$ and the current density is $\frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}}$.
Show that the speed of light in an insulating dielectric medium of relative dielectric constant $\varepsilon_r$ is given by $c/\sqrt{\varepsilon_r}$.

**7.**




Consider a spherical insulating dielectric, radius $R$, which is placed in a uniform electric field, $\mathrm{E_{external}}$. The positive and negative charges in the sphere are then displaced by the effect of the external electric field. The model for the uniformly polarized sphere is two overlapped spheres of charge – a positive sphere and negative sphere as shown in the figure.

a) Assume that each sphere has the same charge $q$, but opposite sign i.e. the total charge is zero as expected, and the displacement from the centre of the negative sphere to the centre of the positive sphere is $\underline{\boldsymbol{s}}$. Show (using superposition and vector addition or otherwise) that the field in the overlapped region is constant and given by, $\underline{\boldsymbol{E}} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\underline{\mathbf{s}}$.
b) Using the definition the electric dipole and polarisation, find the expression for the polarisation, $\underline{\boldsymbol{P}}$, in term of $\underline{\boldsymbol{s}}$, $q$ and $R$.
c) Using the expression in part b), re-write the electric field in part a) in term of $\underline{\boldsymbol{P}}$.
d) Show that the net electric field is given by $\underline{\boldsymbol{E}}_{\text{net}} = \underline{\boldsymbol{E}}_{\text{external}} - \frac{\boldsymbol{P}}{3\varepsilon_0}$.

**8.** An LIH material has a relative permittivity of 85.0 (similar to that of water), a relative permeability of 1.2 and a conductivity of $3 \times 10^{-4}\Omega^{-1}\text{m}^{-1}$.
a) Should the above material be considered a good or poor conductor at a frequency of 100 MHz?
b) Using the above frequency, estimate the value of the refractive index and phase velocity in this material.
c) Assuming that the material may be considered a good conductor at a frequency of 100 Hz, what is the value of the skin depth and by what percentage would B be reduced in travelling through 1 km of this material?

**9.** Propagation of EM through conductors and insulators The wave equation for electromagnetic wave propagating through ILIH (infinite, linear, isotropic, homogeneous) media is given by,

$$\nabla^2\underline{\boldsymbol{E}} - \mu\sigma_{\text{N}}\frac{\partial\underline{\boldsymbol{E}}}{\partial t} - \mu\varepsilon\frac{\partial^2\underline{\boldsymbol{E}}}{\partial t^2} = 0.$$

Where $\mu = \mu_0\mu_r$ and $\varepsilon = \varepsilon_0\varepsilon_r$. By assuming the wave solution is in the form of $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \exp i(kx - \omega t)$, show that,

$$k^2 = \omega^2\mu\varepsilon + i\mu\sigma_N\omega.$$

Using $k^2 = (k_{real} + ik_{imaginary})^2$ and equating real and imaginary parts (or otherwise)

show that, $v_{phase} = \frac{\omega}{k_{real}} = \frac{1}{\sqrt{\mu\varepsilon}}\left[\frac{2}{1+\left(1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{\frac{1}{2}}}\right]^{\frac{1}{2}}$.

Demonstrate that the expected result for the phase velocities in an insulator is obtained in the appropriate limiting case.

**10.** A small dipole centred at the origin of the coordinates has Cartesian vector components (0, p) in the $x$-$z$ plane where $p = qa$, $q$ is the charge and $a$ is the separation. By differentiating the expression for the potential at the point $(x,z)$, find the $x$ and $z$ components for the electric field when $(x^2 + z^2) >> a^2$.

**11.** A non-magnetic, non-conducting sugar solution is characterised by the equation $\underline{\boldsymbol{P}} = \beta\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}$ where $\beta$ is a real constant. Show that the wave-equation of the sugar is given by: $\nabla^2\underline{\boldsymbol{E}} - \beta\mu_0\underline{\boldsymbol{\nabla}} \times \frac{\partial^2\underline{\boldsymbol{E}}}{\partial t^2} - \mu_0\varepsilon_0\frac{\partial^2\underline{\boldsymbol{E}}}{\partial t^2} = 0$. The sugar is optically active. Show that for right-handed circularly polarized light when $\beta$ is small, the refractive index is given by $n = 1 + \beta\mu_0\omega c/2$ and calculate the refractive index for left-handed circularly polarised light.