(a) $\theta = 1.22 \times \lambda \,/\, \mathrm{D}$ (where $\mathrm{D} = 4\,\mathrm{m}$) [1 mark seen]
$\theta = 1.22 \times 550\,\mathrm{nm} \,/\, 4\,(\times\, 206265 \times 1000)$
$\theta = 34.6$ milli-arcseconds [1 mark partly seen]

$\theta = \lambda \,/\, \mathrm{D}$ (where D is now $\mathrm{D} = 25\,\mathrm{m}$) [1 mark seen]
$\theta = 550\,\mathrm{nm} \,/\, 25\,(\times\, 206265 \times 1000)$
$\theta = 4.50$ milli-arcsec [1 mark partly seen]

(b) The effective focal length, f is

$f = f/\# \times D$

$f = 17 \times 40$

$\underline{f = 680\,\mathrm{m}}$ [1 mark partly seen]

The field of view, θ, relates to the size of the detector, s, and the focal length f via

$d\theta = ds \,/\, f$ [1 mark bookwork]

$d\theta = 0.1 \,/\, 680$

$d\theta = 0.15$ milli-rads $(\times\, 206265'')$

$\underline{d\theta = 30.3\,\text{arcseconds}}$ [1 mark partly seen]

$p = 30.3'' \,/\, 4096$ pixels

$\underline{p = 0.007\,\text{arcsec}\,/\,\text{pixel}}$ [1 mark partly seen]

(c) If the gain is 10, then the number of photons detected is

$\mathrm{N}_\gamma = 100 \times 10$ [1 mark partly seen]

$\mathrm{N}_\gamma = 1000$

and the noise is

$\sqrt{N_\gamma} = \sqrt{1000}$

Hence

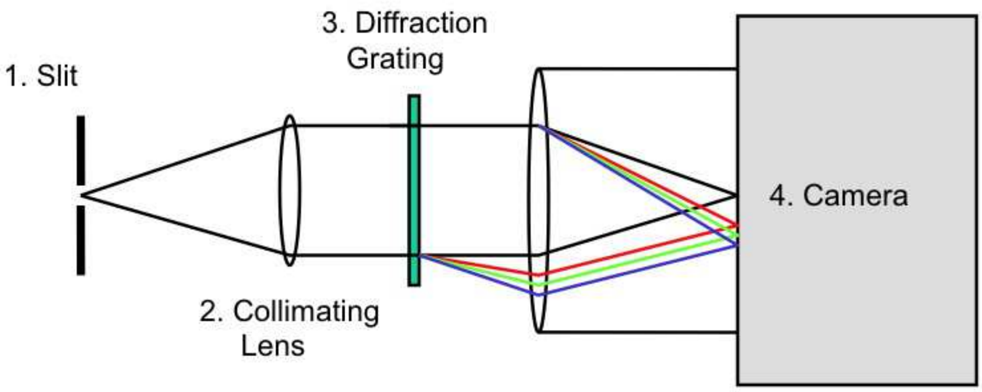
$\mathrm{S} \,/\, \mathrm{N} = 1000 \,/\, \sqrt{1000}$

$\underline{\mathrm{S} \,/\, \mathrm{N} = 31.6}$ [1 mark partly seen]

For a 12-bit detector with gain, g, the number of photoelectrons required to saturate is $2^{12} \times g = 4096\,g$.

For a 16-bit detector with gain g, the number of photoelectrons required to saturate is $2^{16} \times g = 65{,}536\, g$.

Therefore the relative exposure time required for a star of fixed count rate is $65{,}536\, g \,/\, 4096\, g = 16$. [2 marks unseen]



(a) Slit – required to block part of image so that spectra from different parts of image do not overlap. [1 mark bookwork]

Collimator – required so that incident light on diffraction grating is parallel [1 mark bookwork]

Diffraction Grating – required to disperse the light in to its constituent wavelength. [1 mark bookwork]

Camera – required to focus and record the dispersed spectrum. [1 mark bookwork]

(b) For a practical spectrograph, the resolution is given by $\mathrm{R} = \frac{n\,\rho\,\lambda\,W}{\chi\,D_T}$.

The resolution is therefore set by:

the ruling density of the lines on the grating, ρ

the grating size, W

the wavelength of observation, λ

the seeing (or slit width), χ

the size of the telescope, D_{T}

[4 marks for any combination (bookwork)]

(c) The condition for constructive interference is:

$\mathrm{d}\sin\theta = \mathrm{n}\lambda$ [1 mark seen]

Differentiating θ with respect to λ gives

$\mathrm{d} \times \cos\theta \, d\theta = \mathrm{n} \, d\lambda$

hence

$\underline{d\theta \,/\, d\lambda = \mathrm{n} \,/\, \mathrm{d} \cos\theta}$ [1 mark seen]

(d) using

$d\theta \,/\, d\lambda = (d\theta \,/\, dx) \times (dx \,/\, d\lambda)$ [1 mark seen]

and

$d\theta \,/\, dx = 1 \,/\, f$ where f is the focal length

gives:

$\underline{d\lambda \,/\, dx = \mathrm{d} \cos\theta \,/\, (\mathrm{n}f)}$ [1 mark seen]

(e) $n = 2$

$\theta = 0°$

$d\lambda \,/\, dx = 33.0\text{Å}\,\mathrm{mm}^{-1}$

$d\lambda \,/\, dx = 33.0 \times 10^{-10} \,/\, 10^{-3}$

$d\lambda \,/\, dx = 3.30 \times 10^{-6}$ [1 mark seen]

$f = 500\,\mathrm{mm}$

$\mathrm{d} = nf \, (d\lambda \,/\, dx) \,/\, \cos\theta$

$\mathrm{d} = 2 \times (500 \times 10^{-3}) \times 3.30 \times 10^{-6} \,/\, \cos(0°)$

$\mathrm{d} = 3.30 \times 10^{-6}\,\mathrm{m}$ [1 mark unseen]

$\rho = 1 \,/\, d$

$\rho = 1 \,/\, 3.30 \times 10^{-6}$

$\rho = 303030\,\mathrm{lines} \,/\, \mathrm{m}$

$\underline{\rho = 303\,\mathrm{lines} \,/\, \mathrm{mm}}$ [1 mark unseen]

(f) In the theoretical limit:

$R = nN$ where n is the order and N is the number of lines [1 mark seen]

For:

$n = 2$

$R \; = 10\,000$

then

$N = R\,/\,n$

$N = 5000$ lines

Grating size $= 5000\,\text{lines}/\,303\,\text{lines}\,\text{mm}^{-1}$

$\underline{\text{Grating size} = 16.5\,\text{mm}}$ [1 mark partly seen]

(g) $R = \lambda\,/\,\Delta\lambda$ [1 mark seen]

and

$\Delta\lambda\,/\,\lambda = v\,/\,c$ [1 mark seen]

so

$v = c\,/\,R$

$v = 3 \times 10^5\,/\,10\,000\,\text{km}\,\text{s}^{-1}$

$\underline{v = 30\,\text{km}\,\text{s}^{-1}}$ [1 mark unseen]

L2, Stars and Galaxies 2016 exam

David Alexander

June, DMA, Q3

7 short questions:

a) Calculate the radius of a star with a luminosity that is 10^3 times that of the Sun and an effective temperature of 6,000 K. Briefly comment on whether you would expect this star to lie on the main sequence. [4 marks]

Solution

$L = 4\pi R^2 \sigma T_e^4$ therefore

$R = \sqrt{\dfrac{L}{4\pi\sigma T_e^4}}$ so

R=2x10^{10} m for 1,000 times the luminosity of the Sun and T=6,000 K

This star would not lie on the main sequence, it would lie above the main sequence. The student doesn't need to remember the details of the main sequence – in this case it should be clear given the temperature is the same as that of the Sun, which lies on the main sequence, but the luminosity is much higher.

[Seen: essentially 1 mark for each stage; i.e., 3 marks for the basic rearrangement of the equation and radius calculation and 1 mark for bringing up that the star doesn't lie on the main sequence]

b) Calculate the temperature at which radiation pressure exceeds gas pressure in the core of a star. In your calculation assume a particle density of $n=10^{32}$ m^{-3}.

[Hint: the radiation constant is given as $a = \frac{4\sigma}{c} = 7.57 \times 10^{-16}\, Jm^{-3}K^{-4}$]

Solution

$$P = nkT \qquad \textit{and} \qquad P = \frac{1}{3}aT^4 \qquad \textit{where } a = \frac{4\sigma}{c} = 7.57\times10^{-16}\, Jm^{-3}K^{-4}$$

so

$$nkT = \frac{1}{3}aT^4 \quad \textit{therefore} \quad T^3 = \frac{3nk}{a} \quad \textit{and} \quad T = \left(\frac{3nk}{a}\right)^{1/3}$$

$$T = \left(\frac{3\times10^{32}\times1.38\times10^{-23}}{7.57\times10^{-16}}\right)^{1/3} = 2\times10^{8}\, K$$

[3 marks for method, 1 mark for correct answer; Unseen]

c) What is opacity? A strong continuum break (at 365nm) is seen in the optical spectrum of a star. Briefly describe the opacity process that causes this continuum break and explain why there is a deficit of photons at wavelengths shorter than the break wavelength. [4 marks]

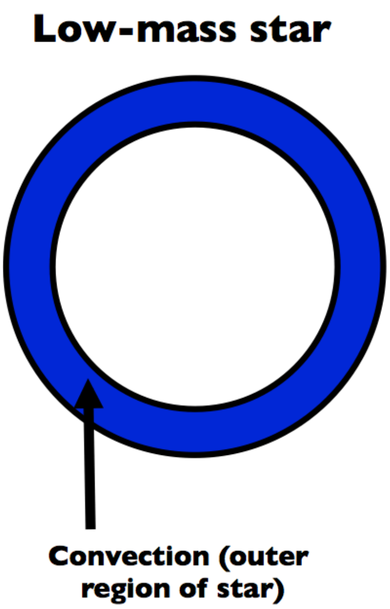
Solution

Opacity is the resistance to the flow of radiation from material. It is responsible for the transfer of radiation. [1 mark; Seen]

The opacity process is bound free, which is also called photo ionisation. This occurs when a photon interacts with an atom and has sufficient energy to eject an electron from the atom. The break wavelength occurs at the minimum energy required to eject the electron (i.e., the ionisation energy). Since any photon with wavelengths shorter than this break have sufficient energy to eject the electron from the atom a deficit of photons are seen short ward of the continuum break. [3 marks; Seen]

d) The Schwarzschild condition for convection is satisfied in different regions for different masses of stars. Draw a cross section through a star with a mass equivalent to that of the Sun and highlight where convection is believed to occur in this star. Why does convection occur in this region? [4 marks]

Solution



[2 marks; Seen]

Convection occurs when there is a steep temperature gradient. In low mass stars this is found in the surface layers where the opacity increases with increasing temperature, and so radiation can be trapped in the lower layers, resulting in convection. [2 marks; Seen]

e) Calculate the horizontal-branch lifetime of the Sun. Assume that the efficiency of the triple-alpha fusion process is 0.07%, that 10% of the mass of the Sun is used during the horizontal-branch lifetime, and that the Sun is 5 times more luminous than on the main sequence. Give your answer in years. [4 marks]

Solution

$$t = \frac{E}{L}$$

The luminosity is 5 times the solar luminosity and so just need to determine the energy E:

$$E = X\xi Mc^2 = 0.1 \times 0.0007 \times 1.99 \times 10^{30} \times \left(3.00 \times 10^8\right)^2 = 1.25 \times 10^{43}\, J$$

$$t = \frac{1.25 \times 10^{43}}{5 \times 3.84 \times 10^{26}} = 6.51 \times 10^{15}\, s = 2.06 \times 10^8\, yrs$$

[2 marks for method, 2 marks for correct numerical answer; Unseen]

f) Describe the origin of the pressure that prevents a white dwarf and a neutron star from collapsing. How does the physical origin of this pressure differ between a white dwarf and a neutron star? [4 marks]

Solution

White dwarfs and neutron stars are prevented from collapse due to"degeneracy pressure". For the white dwarf it is electron degeneracy pressure and for the neutron star it is neutron degeneracy pressure. The degeneracy pressure is the pressure exerted by either electrons or neutrons due to their tight confinement, which gives rise to high momentum (i.e., as expected given the Heisenberg Uncertainty principle and the Pauli Exclusion principle). [4 marks; Seen]

g) By equating the gravitational acceleration with the centripetal acceleration, show that the minimum rotation period of a spherical spinning pulsar of uniform density can be written as $P_{\min} = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}$, where ρ is the density. [4 marks]

Solution

The maximum angular velocity ω of a spinning star is determined by the centripetal acceleration on a mass at the equator. Assume the star is spherical, with radius R. The gravitational and centripetal acceleration are balanced when:

$$\omega_{\max}^2 R = \frac{GM}{R^2}$$

If we assume uniform density, then:

$$\omega_{\max}^2 R = \frac{G}{R^2}\rho\frac{4}{3}\pi R^3$$

and remembering ω=2πf and that f=1/P, then

$$\frac{4\pi^2 R}{P^2} = \frac{4}{3}G\rho\pi R$$

Rearranging:

$$P_{\min} = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}$$

[Seen: 2 marks for correctly giving the equations and 2 marks for the correct derivation]

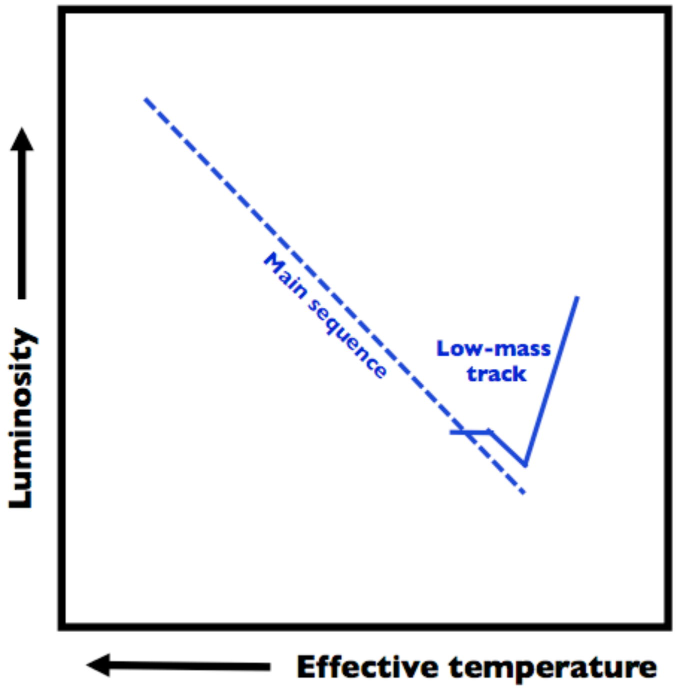
L2, Stars and Galaxies 2016 exam

David Alexander

June, DMA, Q4

a) Hayashi tracks trace the paths that protostars take on the Hertzsprung-Russell diagram before joining the main sequence. Draw a Hertzsprung-Russell diagram and highlight the main sequence and the key features of the Hayashi track for a low-mass (~1 $M_{\odot}$) star. [4 marks]

Solution



[1 mark for giving the correct axes and 1 mark for drawing on the correct main sequence. 2 marks for drawing the low-mass track, indicating the key features; Seen]

b) Derive an expression for the energy (E) available from the gravitational collapse of a spherically symmetrical protostar of uniform density to show that $E \sim -\frac{3}{10}\frac{GM^2}{R}$, where M is the mass and R is the post-collapse (final) radius. Assume that the protostar is in virial equilibrium but state any other assumptions that you make in your derivation. [8 marks]

[Hint: the gravitational potential of a point mass is $dU = -\frac{GMdm}{r}$, where $dm = 4\pi r^2 \rho dr$ is a shell at radius r of density ρ and thickness dr]

Solution

Taking $dU_g = -GM4\pi r\rho dr$ *integrate over all shells assuming constant density for M*

$$U_g = -4\pi G \int_0^R M\rho r dr$$

where $M = \frac{4}{3}\pi r^3 \rho$ *which gives* *[Seen: 3 marks]*

$$U_g = -\frac{16\pi^2 G\rho^2}{3}\int_0^R r^4 dr \quad \textit{and} \quad U_g = -\frac{16\pi^2}{15}G\rho^2 R^5$$ *[Seen: 2 marks]*

Converting back from density to mass where

$$\rho^2 = \frac{M^2}{16/9\,\pi^2 R^6} \quad \textit{which gives} \quad U_g = -\frac{9}{15}\frac{GM^2}{R}$$

Now applying the virial theorem (where only half of the potential energy is liberated in the gravitational collapse) finally gives

$$E \sim -\frac{3}{10}\frac{GM^2}{R}$$ *[Seen: 3 marks]*

c) Calculate the Kelvin-Helmholtz time for a 2 $M_{\odot}$ protostar with the following properties: an average luminosity of 2 $L_{\odot}$, an initial radius of $R_i=10^{10}$ m, and a final radius of $R_f=10^9$ m. Give your answer in years. [3 marks]

Solution

The Kelvin-Helmholtz time is $t_{KH} = \frac{E}{L}$, *where E is the energy released through gravitational collapse. The energy released through gravitational collapse is* $E = -\frac{3}{10}\frac{GM^2}{R}$ *(i.e.,* $\frac{1}{2}U$ *). To determine the total amount of energy emitted it is necessary to calculate the amount of energy at the initial radius and also the final radius.*

Therefore the energy released from the collapse is:

$$\Delta E = E_f - E_i = -\frac{3GM^2}{10}\left[\frac{1}{R_f} - \frac{1}{R_i}\right]$$

$$\Delta E = -\frac{3 \times 6.67 \times 10^{-11} \times \left(1.99 \times 10^{30}\right)^2}{10}\left[\frac{1}{10^9} - \frac{1}{10^{10}}\right] = 2.85 \times 10^{41} J$$

Note, if don't take account of the initial radius then the energy release is slightly higher at $3.17 \times 10^{41} J$.

$$t_{KH} = \frac{E}{L} = \frac{2.85 \times 10^{41}}{2 \times 3.84 \times 10^{26}} = 3.7 \times 10^{14} s = 12 Myrs$$

[3 marks but only receive 1 mark if do not take account of the initial radius; Seen]

d) The protostar core has the following properties: fully ionized hydrogen with a pressure of P=3x10^{16} N m^{-2} and a density of ρ=10^{5} kg m^{-3}. Calculate the temperature of the gas at the core of the protostar. Briefly comment on whether this temperature is high enough for nuclear fusion via the proton-proton chain. [5 marks]

Solution

The temperature is calculated using the ideal gas law:

$$P = nkT = \frac{\rho kT}{\mu m_H} \qquad \textit{so} \qquad T = P\frac{\mu m_H}{\rho k} = \frac{3\times10^{16}\times0.5\times1.67\times10^{-27}}{100000\times1.38\times10^{-23}} = 2\times10^{7}\,K$$

The temperature at the core is comparable to that at the centre of the Sun and hence nuclear fusion via the proton-proton chain can occur (note the proton-proton chain requires T>4x10^{6} K).

[2 marks for the method, 2 marks for correct quantitative answer (which requires assuming the correct mean-molecular mass) and 1 mark for getting the correct qualitative result; Unseen]

L2, Stars and Galaxies 2016 exam

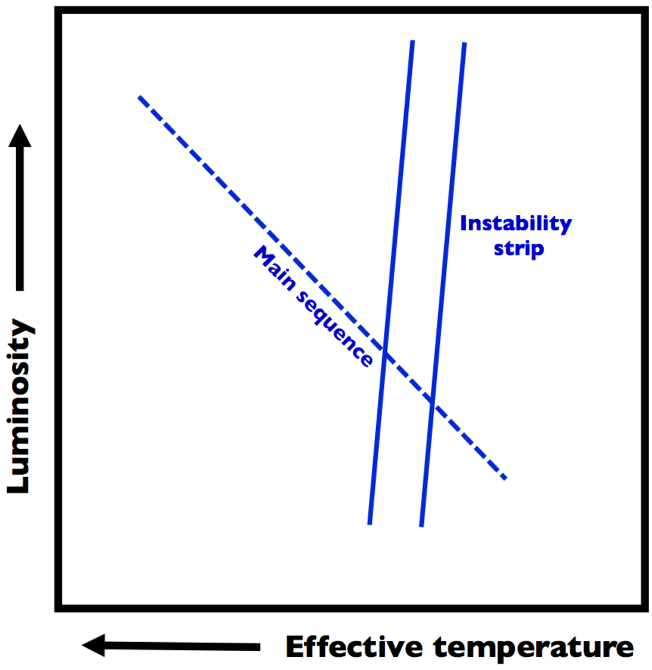
David Alexander

June, DMA, Q5

Long question:

a) The instability strip is the region on the Hertzsprung-Russell diagram, where the majority of pulsating stars are found to lie. Draw a Hertzsprung-Russell diagram and highlight the main sequence and the instability strip. [4 marks]

Solution:



[1 mark for giving the correct axes, 1 mark for drawing on the correct main sequence, and 2 marks for correctly highlighting the instability strip, which is broadly constant in temperature; Seen]

b) Assuming a uniform density and using the sound-speed equation for an adiabatic gas, show that the period of a radially pulsating star is given by

$$\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}},$$

where Π is the pulsation period, γ is the ratio of specific heats and ρ is the density. State any other assumptions that you make. [8 marks]

[Hint: $$\int_0^R \frac{dr}{\sqrt{(R^2 - r^2)}} = \left[\sin^{-1}\left(\frac{r}{R}\right)\right]_0^R.$$]

Solution:

The adiabatic sound speed is given by:

$$v_s = \sqrt{\frac{\gamma P}{\rho}}$$

where P=pressure and γ=ratio of specific heats.

[2 marks; Seen]

Now, we know how to find the pressure: we can use the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

Let's assume that the star has constant density (not a very good assumption!). Then:

$$\frac{dP}{dr} = -\frac{G\left(\frac{4}{3}\pi r^3\rho\right)\rho}{r^2}$$

$$\frac{dP}{dr} = -\frac{4}{3}\pi G\rho^2 r$$

Now integrate this using the boundary condition that P=0 at the surface of the star:

$$P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$$

[3 marks; Seen]

This gives us a pressure as a function of r. The pulsation period will be given roughly by the distance divided by the speed, or:

$$\Pi \approx 2\int_0^R \frac{dr}{v_s}$$

$$\Pi \approx 2\int_0^R \frac{dr}{\sqrt{\tfrac{2}{3}\gamma\pi G\rho\left(R^2 - r^2\right)}}$$

$$\Pi \approx 2 \times \sqrt{\frac{3}{2\gamma\pi G\rho}} \times \int_0^R \frac{dr}{\sqrt{\left(R^2 - r^2\right)}}$$

$$\Pi \approx 2 \times \sqrt{\frac{3}{2\gamma\pi G\rho}} \times \left[\sin^{-1}\left(\frac{r}{R}\right)\right]_0^R$$

So:

$$\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

[3 marks, Seen]

c) What region and what physical process drives radial pulsations in stars? Why doesn't the Sun show strong radial pulsations? [4 marks]

Solution:

Radial pulsations originate in the partial ionisation zone. The partial ionisation of the hydrogen and helium gas in this region allows the stellar material to be strongly compressed – the energy produced through compression goes into ionising the gas rather than significantly raising the gas temperature and pressure. [2 marks; Seen]

The Sun doesn't pulsate radially because convection in the outer regions dampens the radial pulsations. [2 marks; Seen]

d) Assuming that pulsating stars radiate as black bodies, show that the luminosity of pulsating stars of the same effective temperature and mass is proportional to $\Pi^{4/3}$. [4 marks]

Solution:

We are only looking to show proportionality. The key to showing this is that

$$\rho = \frac{3M}{4\pi R^3} \qquad \text{and} \qquad L = 4\pi R^2 \sigma T^4 \qquad \text{[2 marks]}$$

Therefore since $\Pi = \sqrt{\dfrac{3\pi}{2\gamma G\rho}}$ *then* $\Pi \propto R^{3/2}$ *[1 mark]*

We also know that $R \propto L^{1/2}$ *therefore* $\Pi \propto L^{3/4}$ *and* $L \propto \Pi^{4/3}$
[1 mark]

If the student derives the full equation (i.e., not just the proportionality) then it is:

Replacing density for radius in $\Pi = \sqrt{\dfrac{3\pi}{2\gamma G\rho}}$ *gives*

$$\Pi = \sqrt{\frac{3\pi}{2\gamma G}} \times \sqrt{\frac{4\pi R^3}{3M}} = \sqrt{\frac{4\pi^2 R^3}{2\gamma GM}} \qquad (\text{i.e., } \Pi \propto R^{3/2})$$

Replacing radius for luminosity in $\Pi = \sqrt{\dfrac{4\pi^2 R^3}{2\gamma GM}}$ *gives*

$$\Pi = \sqrt{\frac{4\pi^2}{2\gamma GM}} \sqrt{\left(\frac{L}{4\pi\sigma T^4}\right)^{3/2}} = \sqrt{\frac{4\pi^2}{2\gamma GM}} \left(\frac{L}{4\pi\sigma T^4}\right)^{3/4} \qquad (\text{i.e., } \Pi \propto L^{3/4} \text{ and } L \propto \Pi^{4/3})$$

[4 marks: Unseen]

Solution to 2015/16 Stars and Galaxies exam Q6

(a) Spiral galaxies have star formation, ellipticals do not. The massive young stars that form are blue, and contribute to the blue light in spirals. [1 mark, bookwork]
students need to mention star formation and link it to young blue stars.
The stars in spiral galaxies typically have lower metallicity than ellipticals. This makes the stars bluer. [1 mark, bookwork]
Spiral galaxies contain dust, which reddens stars [1 mark, bookwork]
When the galaxy is seen edge on, the path-length through the dust increases, making the galaxy appear redder [1 mark, unseen]

(b) The three components are disk, bulge and stellar halo. 1 mark per correct answer; [3 marks, bookwork]
The Milky Way's disk is the most massive. [1 mark, bookwork]

(c) A standard candle is an intrinsic property of an object that can be inferred observationally, and used to determine the distance to the object. [2 marks, bookwork]
Restricting this to 'objects of known luminosity' so that distance can be determined from flux, also gets 2 marks.

The rotation speed of a galaxy can be measured, and the TF used to yield the luminosity L. Measuring the flux yields the distance. [2 marks, bookwork]
Students need to mention that flux needs to be measured, otherwise lose two marks.

(d) The total mass of the cluster can be measured in several ways:

1. from the velocity dispersion of the galaxies, assuming the virial theorem
2. from the X-ray profile of hot gas, and assuming this is in hydrostatic equilibrium
3. using gravitational lensing

[1 mark per correct answers to a maximum of 3 marks, bookwork]
The inferred value is higher than that in hot gas and in stars, suggesting some invisible dark matter dominates the total mass [1 mark, bookwork]

(e) Gravitational lensing is the deflection of light due an intervening mass distribution bending space-time. [2 marks, bookwork]
Not mentioning bending of space-time loses a mark.

Write these distances in terms of x as $D_{\rm OL} = xD_{\rm OS}$, $D_{\rm LS} = (1-x)\,D_{\rm OS}$, then the signal is signal $=$ const $x\,(1-x)D_{\rm OS}$, where const and $D_{\rm OS}$ are constants. This expression is maximum when $x = 1/2$ since

$$\frac{\mathrm{d}}{\mathrm{d}x}x(1-x) = 1 - 2x$$

is zero for $x = 1/2$ and the second derivative is positive (so it is a maximum). [2 marks, unseen]

Level 2 Paper 4 Question 7

(a) There are several correct ways to demonstrate this, that were discussed in the lectures. The easiest way is to equate the acceleration along a circular orbit of radius r, V_c^2/r, to the gravitational force, $GM(<r)/r^2$, where $M(<r)$ is the mass enclosed by the orbit. This yields

$$V_c^2\, r = GM(<r)\,.$$

[2 marks, bookwork]
Taking the derivative with respect to r yields $V_c^2 = G(dM/dr) = G\,4\pi\rho\, r^2$, since V_c is constant and $dM/dr = 4\pi\rho\, r^2$. [2 marks, bookwork]

(b) The observed surface brightness of the Milky Way drops very rapidly with radius (exponentially), and hence the mass distribution in stars is very different from the required $1/r^2$. [2 marks, bookwork]
There is no evidence that observed gas could make-up the deficit.

(c) Using the result from (a) yields for the number density $n = V_c^2/(4\pi\,\mathrm{G}\, r_\odot^2\, m_\mathrm{H})$ [2 marks, unseen] $= 0.5\times 10^6$ m^{-3} [1 mark, unseen]

(d) The mass enclosed by the density distribution from (a) in a sphere of radius r_h is $M_\mathrm{h} = V_c^2\, r_\mathrm{h}/\mathrm{G}$ [2 marks, unseen] $= 0.80\times 10^{12}$ $\mathrm{M}_\odot$ as given in the question.
Solving for r_h yields $r_h = \mathrm{G}\, M_\mathrm{h}/V_c^2 = 71$ kpc. [2 marks, unseen]
Incorrect rounding, or not expressing the answer in kpc, loses 1 mark each time.

(e) The specific energy for a particle moving with the escape speed, v_esc, is zero, $0 = E/m = (1/2)v_\mathrm{esc}^2 + \Phi$, with Φ the gravitational potential. [1 mark, unseen]
The gradient of the potential is the gravitational acceleration, $\mathrm{d}\Phi/\mathrm{d}r = GM(<r)/r^2 = V_c^2/r$, using the result from (a) for the gravitational acceleration [1 mark, unseen]
Integrating the gradient of the potential to the edge of the halo yields,

$$\int_{r_\odot}^{r_\mathrm{h}} \frac{\mathrm{d}\Phi}{\mathrm{d}r}\, dr = \Phi(r_\mathrm{h}) - \Phi(r_\odot) = V_c^2 \log(r_\mathrm{h}/r_\odot)\,,$$

[1 mark, bookwork], hence $\Phi(r_\odot) = \Phi(r_\mathrm{h}) - V_c^2\log(r_\mathrm{h}/r_\odot) = -V_c^2(1+\log(r_\mathrm{h}/r_\odot))$.
[1 mark, bookwork]
Substituting the value from (d) yields $v_\mathrm{esc} = (2\Phi(r_\odot))^{1/2} = 560$ km s^{-1}. [1 mark, unseen]

(f) The rms velocity of hydrogen atoms in a gas of temperature T is $v^2 = 2k_\mathrm{B}T/m_\mathrm{H}$ [1 mark, unseen] $= 406$ km s^{-1}.
Therefore only the atoms in the tail of the Gaussian distribution have $v > v_\mathrm{esc}$ and can escape. [1 mark, unseen]
Answers such as ‘most gas cannot escape’ get full marks if it is made sufficiently clear what that answer is based on (i.e. that the gas particles have a distribution of velocities). Note that adding the speed of the SN in quadrature does not change the answer substantially.