ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 18+19+20

14 Electromagnetic fields and waves crossing interfaces

Pauli: "God made the bulk, but the devil has the surfaces"

14.1 Boundary Conditions across the interfacial plane between two dielectrics

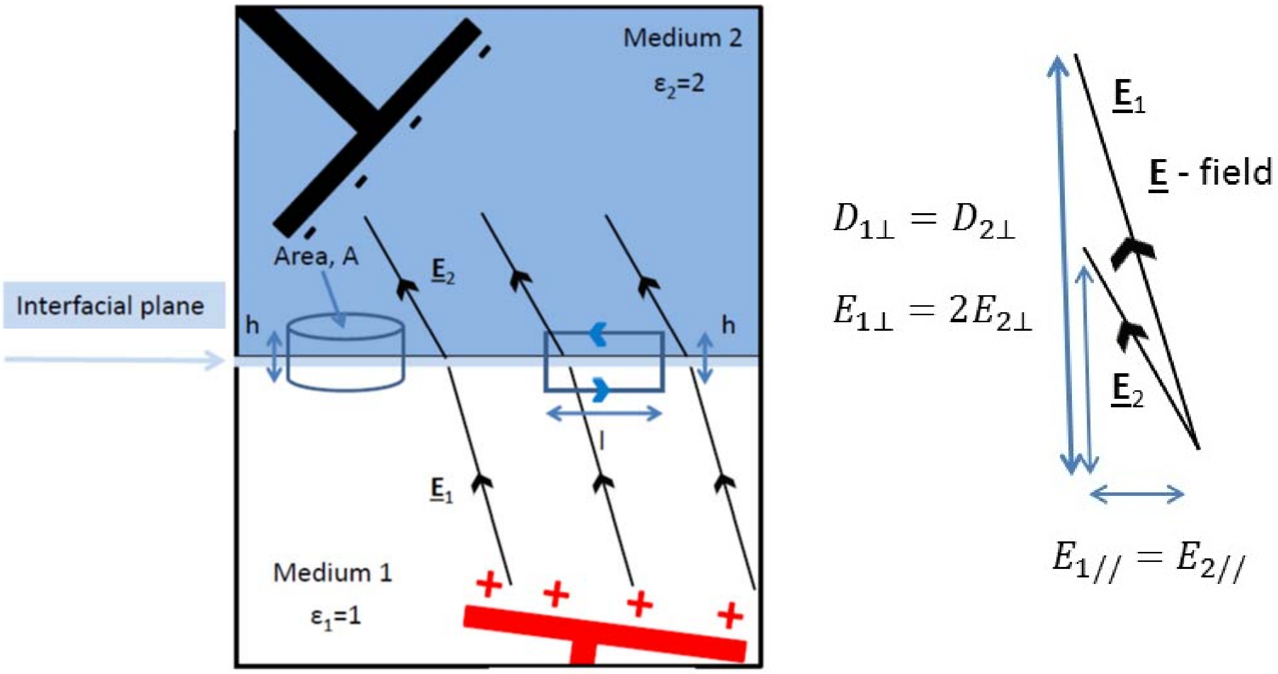

Figure 1 : (LHS) Two charged capacitor plates immersed in two dielectrics. The direction and magnitude of a dc. electric field is different from one side of the dielectric interface to the other. The arrows give the direction of the dc. E-fields on either side of the interfacial plane (Nb. This is not a ray diagram). (RHS). Maxwell's equations require that the displacement field orthogonal to the interfacial plane is continuous (i.e. $D_{1_\perp} = D_{2\perp}$ or equivalently $\varepsilon_1 E_{1_\perp} = \varepsilon_2 E_{2\perp}$) and that the electric field parallel to the interfacial plane is continuous (i.e. $E_{1//} = E_{2//}$).

A) For an insulating material Maxwell I

$$\underline{\nabla} \cdot \underline{D} = \rho_{\text{free}} = 0 \qquad 14\text{-}1$$

$$\oint \underline{\boldsymbol{D}}_1 \cdot d\underline{\boldsymbol{S}} = -\oint \underline{\boldsymbol{D}}_2 \cdot d\underline{\boldsymbol{S}} \Rightarrow D_{1_\perp} = D_{2\perp} \qquad 14\text{-}2$$

or equivalently

$$\varepsilon_1 E_{1_\perp} = \varepsilon_2 E_{2\perp}. \quad (\underline{\boldsymbol{D}} = \varepsilon_o \varepsilon_r \underline{\boldsymbol{E}}) \tag{14-3}$$

B) Using Maxwell III (consider path B)

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \tag{14-4}$$

$$\oint \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{l}} = -\frac{\partial}{\partial t} \int \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{S}} \tag{14-5}$$

$$E_{1//} l - E_{2//} l = -\frac{\partial \phi}{\partial t} = 0 \tag{14-6}$$

The loop area tends to zero

$$\Rightarrow E_{1//} = E_{2//} \tag{14-7}$$

$D_\perp$(or equivalently $\varepsilon_r E_\perp$) and $E_{//}$ are continuous across the interfacial plane between two dielectrics.

14.2 Boundary Conditions across the interfacial plane between two magnetic materials

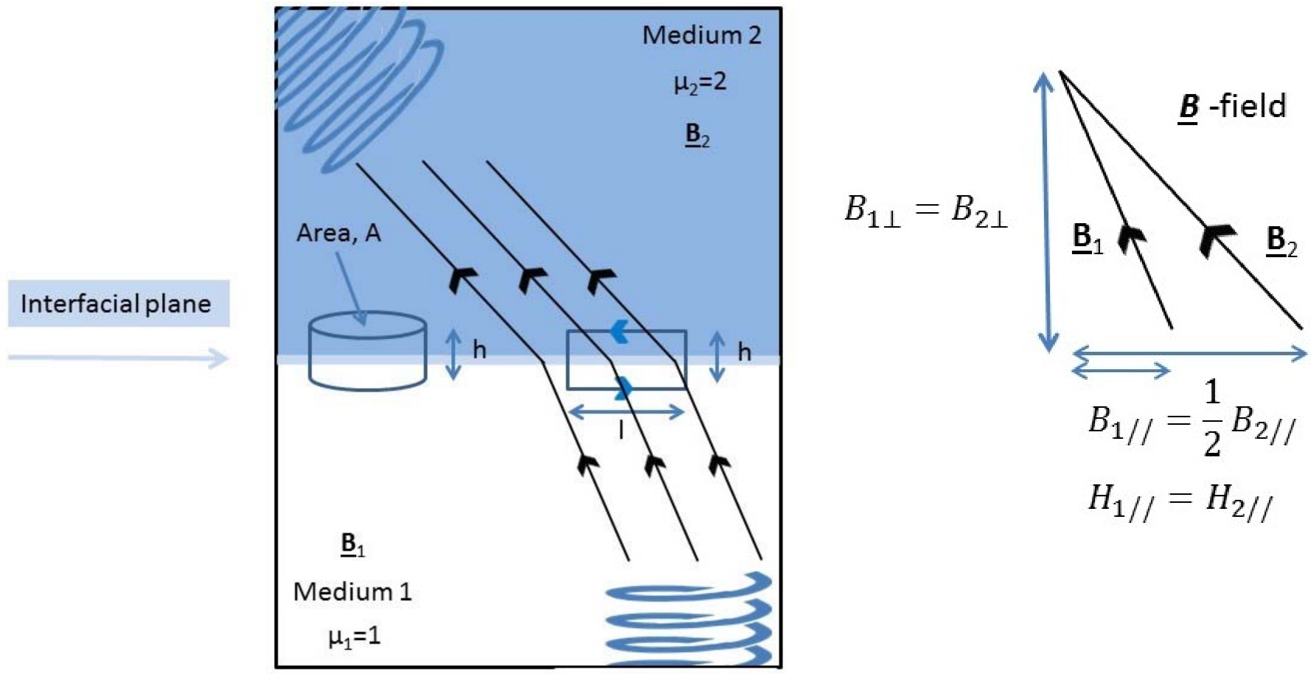

Figure 2 Two magnets immersed in two magnetic materials. The direction and magnitude of a dc. magnetic field from opne side of the magnetic interface to the other. The arrows give the direction of the dc. B-field on either side of the interfacial plane (Nb. This is not a ray diagram). (RHS). Maxwell's equations require that the magnetic field orthogonal to the interfacial plane is continuous (i.e. $B_{1_\perp} = B_{2\perp}$) and that the field strength parallel to the interfacial plane is continuous (i.e. $\mathbf{H_{1//}} = \mathbf{H_{2//}}$ or equivalently $B_{1//}/\mu_1 = B_{2//}/\mu_2$)).

Maxwell II

$$\int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = 0 \Rightarrow \mathrm{B}_{1\perp} = \mathrm{B}_{2\perp} \qquad \text{14-8}$$

Maxwell IV

$$\underline{\nabla} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}}_{\text{total}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad \text{14-9}$$

$$\nabla \times \underline{\boldsymbol{H}} = \underline{\boldsymbol{J}}_{\mathbf{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial \mathrm{t}} \qquad \text{14-10}$$

$$\int (\underline{\nabla} \times \underline{\boldsymbol{H}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \oint \underline{\boldsymbol{H}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \underbrace{\int \underline{\boldsymbol{J}}_{\text{free}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \int \frac{\partial \nabla . \underline{\boldsymbol{D}}}{\partial \mathrm{t}} \cdot \mathrm{dV}}_{\substack{-\text{no free current density} \\ -\text{no free charge density}}} \qquad \text{14-11}$$

$$\Rightarrow \mathrm{H}_{1//}\mathrm{l} = \mathrm{H}_{2//}\mathrm{l} \quad \Rightarrow \mathrm{H}_{1//} = \mathrm{H}_{2//} \qquad \text{14-12}$$

$\therefore$ $\mathrm{B}_{\perp}$, $\mathrm{H}_{//}$ are continuous across the boundary.

14.3 The Laws of geometrical optics

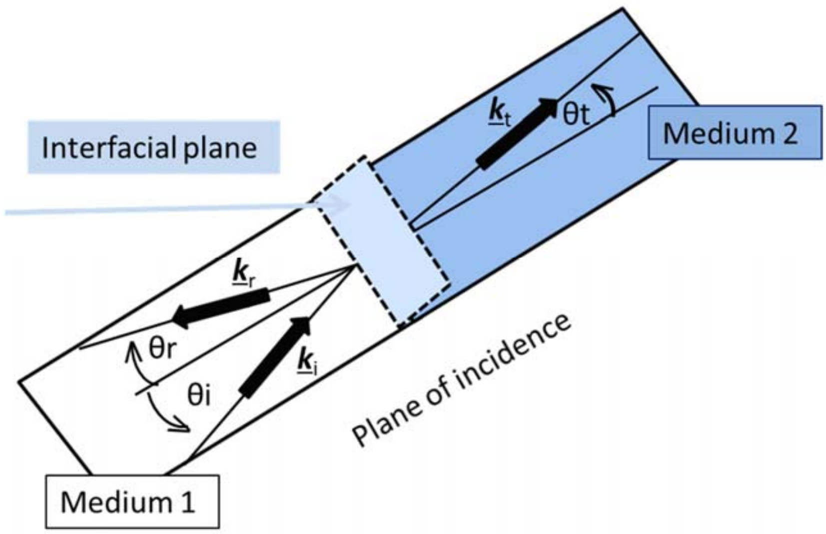

Figure 3 : An electromagnetic wave incident on an interface – it is partially reflected and partially transmitted.

$$\underline{\tilde{\boldsymbol{E}}}_{\mathrm{i}}(\underline{\boldsymbol{r}},t) = \underline{\mathbf{E}}_{0\mathrm{i}}\mathrm{expi}(\underline{\mathbf{k}}_{\mathrm{i}} \cdot \underline{\boldsymbol{r}} - \omega_{\mathrm{i}}t) \qquad \text{14-13}$$

$$\underline{\tilde{\boldsymbol{E}}}_{\mathrm{r}}(\underline{\boldsymbol{r}},t) = \underline{\mathbf{E}}_{0\mathrm{r}}\mathrm{expi}(\underline{\mathbf{k}}_{\mathrm{r}} \cdot \underline{\boldsymbol{r}} - \omega_{\mathrm{r}}t) \qquad \text{14-14}$$

$$\underline{\tilde{\boldsymbol{E}}}_{\mathrm{t}}(\underline{\boldsymbol{r}},t) = \underline{\mathbf{E}}_{0\mathrm{t}}\mathrm{expi}(\underline{\mathbf{k}}_{\mathrm{t}} \cdot \underline{\boldsymbol{r}} - \omega_{\mathrm{t}}t) \qquad \text{14-15}$$

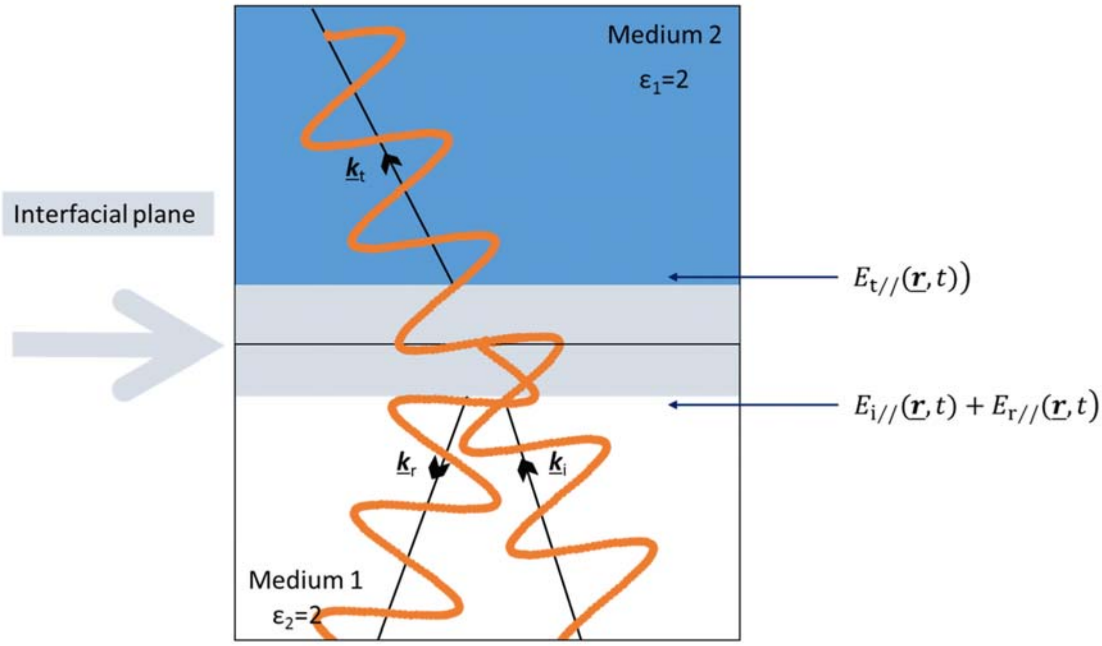

Figure 4 : An electromagnetic wave incident on an interface. After the incident and reflected waves have left the interfacial plane, they superimpose in medium 1. The transmitted wave is the only wave in medium 2.

Across the interfacial plane, the general condition for the continuity of $E_{//}$ (i.e E parallel to the interface) requires:

$$E_{\mathrm{i}//}(\underline{\boldsymbol{r}},t) + E_{\mathrm{r}//}(\underline{\boldsymbol{r}},t) = E_{\mathrm{t}//}(\underline{\boldsymbol{r}},t) \qquad \text{14-16}$$

a) $\underline{\boldsymbol{r}} = 0$

$$\mathrm{E}_{0\mathrm{i}//}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{i}}\mathrm{t}} + \mathrm{E}_{0\mathrm{r}//}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{r}}\mathrm{t}} = \mathrm{E}_{0\mathrm{t}//}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{t}}\mathrm{t}} \qquad \text{14-17}$$

where

$\mathrm{E}_{0\mathrm{i}//}$, $\mathrm{E}_{0\mathrm{r}//}$, $\mathrm{E}_{0\mathrm{t}//}$ are all real constants 14-18

$$E_{0i//}\cos\omega_r t + E_{0r//}\cos\omega_i t = E_{0t//}\cos\omega_t t \qquad 14\text{-}19$$

$$\Rightarrow \omega_i = \omega_r = \omega_t \qquad 14\text{-}20$$

b) $t = 0$

$$E_{0i//}e^{i\underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}}} + E_{0r//}e^{i\underline{\mathbf{k}}_r\cdot\underline{\boldsymbol{r}}} = E_{0t//}e^{i\underline{\mathbf{k}}_t\cdot\underline{\boldsymbol{r}}} \qquad 14\text{-}21$$

$$\Rightarrow \underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_r\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_t\cdot\underline{\boldsymbol{r}} \qquad 14\text{-}22$$

Consider reflection (i.e. $\underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_r\cdot\underline{\boldsymbol{r}}$):

$$(\underline{\mathbf{k}}_i - \underline{\mathbf{k}}_r)\cdot r = 0 \qquad 14\text{-}23$$

$$\Rightarrow (\underline{\mathbf{k}}_i - \underline{\mathbf{k}}_r) \text{ is normal to the interface} \qquad 14\text{-}24$$

$$k_{//i} = k_{//r} \qquad 14\text{-}25$$

$$|\underline{\mathbf{k}}_i| = |\underline{\mathbf{k}}_r| \qquad 14\text{-}26$$

$$\Rightarrow \theta_i = \theta_r \ - \ \text{1st law of geometrical optics: Angle of incidence} = \text{Angle of reflection.} \qquad 14\text{-}27$$

Consider transmission (i.e. $\underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_t\cdot\underline{\boldsymbol{r}}$):

$$(\underline{\mathbf{k}}_i - \underline{\mathbf{k}}_t)\cdot\underline{\boldsymbol{r}} = 0 \qquad 14\text{-}28$$

$$\Rightarrow k_{//i} = k_{//t} \qquad 14\text{-}29$$

$$k_i\sin\theta_i = k_t\sin\theta_t \qquad 14\text{-}30$$

Using the definition of the refractive index, $n_i = \frac{ck_i}{\omega}$ and $n_t = \frac{ck_t}{\omega}$.

$$\Rightarrow \frac{n_i}{n_t} = \frac{k_i}{k_t} = \frac{\sin\theta_t}{\sin\theta_i} - \text{2nd law of geometrical optics} \ - \ \text{Snell's law.} \qquad 14\text{-}31$$

Note that Snell's law is valid for all polarisations.

From Snells' law, one can if $n_i > n_t$, given $\sin\theta_t \le 1$

$$\sin\theta_i = \frac{n_t}{n_i} - \text{ is the condition for the critical angle.} \qquad 14\text{-}32$$

Further analysis shows one has an attenuated (evanescent) waves.

Since we can also write down $\underline{\mathbf{k}}_r \cdot \underline{\boldsymbol{r}} = \underline{\mathbf{k}}_t \cdot \underline{\boldsymbol{r}}$

$$(\underline{\mathbf{k}}_r - \underline{\mathbf{k}}_t) \cdot \underline{\boldsymbol{r}} = 0 \qquad \text{14-33}$$

which gives

$$k_r \sin\theta_r = k_t \sin\theta_t \qquad \text{14-34}$$

The constraints on the components of the incident, reflected and transmitted wavevectors parallel to the interfacial plane leads and on the magnitude of the wavevectors in both media leads to:

- 3rd law of geometrical optics. The incident, reflected and transmitted wave are all in the plane of incidence:

14.4 Fresnel's equations - E-field normal to the plane of incidence.

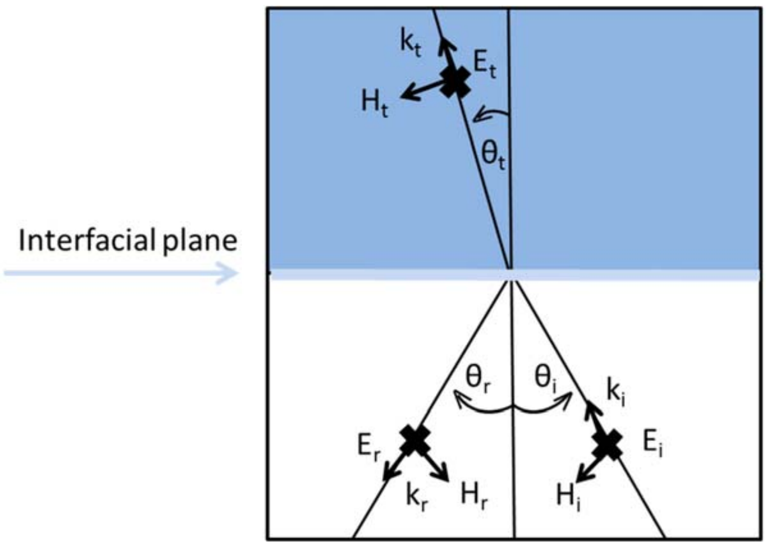

Figure 5 :An electromagnetic wave incident on an interfacial plane where $\underline{\boldsymbol{E}}$ is normal to the plane of incidence.

The continuity of $\underline{\boldsymbol{E}}$ parallel to interfacial plane gives (with $\underline{\boldsymbol{r}} = 0$, $t = 0$)

$$E_{0i} + E_{0r} = E_{0t} \ (1) \qquad \text{14-35}$$

The continuity of $\underline{\boldsymbol{H}}$ parallel to the interfacial plane gives:

$$H_{0i}\cos\theta_i - H_{0r}\cos\theta_r = H_{0t}\cos\theta_t \ (2) \qquad \text{14-36}$$

Using $\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \Rightarrow \underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = \omega \underline{\boldsymbol{B}}$:

$$B = \frac{E}{v_{phase}} \qquad 14\text{-}37$$

The refractive index, *n*, by definition is:

$$n = \frac{c}{v_{phase}} = \sqrt{\varepsilon_r \mu_r} \qquad 14\text{-}38$$

Using $\underline{\boldsymbol{B}} = \mu_r\mu_0\underline{\boldsymbol{H}} = \mu\underline{\boldsymbol{H}}$:

$$H = \frac{En}{c\mu}. \quad (3) \qquad 14\text{-}39$$

$$(n_i E_{0i} - n_r E_{0r})\cos\theta_i = n_t E_{0t}\cos\theta_t \quad (4) \qquad 14\text{-}40$$

$$\frac{E_{0r}}{E_{0i}} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} \qquad 14\text{-}41$$

$$\frac{E_{0t}}{E_{0i}} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t} \qquad 14\text{-}42$$

Fresnel’s first 2 equations –***E*** normal to plane of incidence.

14.5 Fresnel’s equations - E-field parallel to the plane of incidence

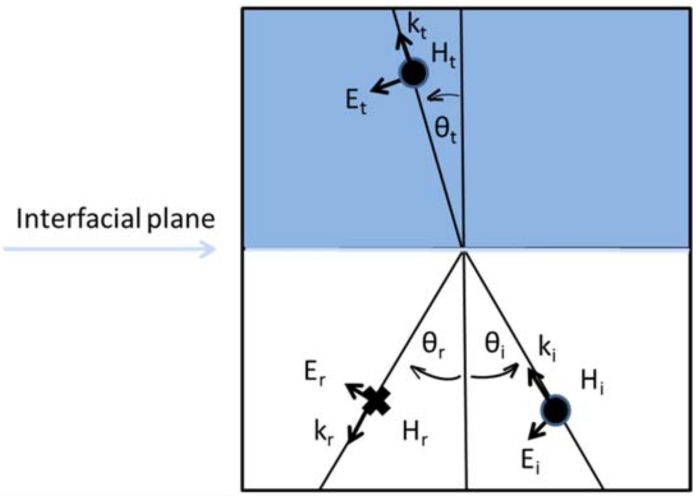

Figure 6 : An electromagnetic wave incident on an interfacial plane where $\underline{\boldsymbol{E}}$ is parallel to the plane of incidence

The continuity equations for $H_{//}$ and $E_{//}$ are now,

$$(E_{0i} + E_{0r})\cos\theta_i = E_{0t}\cos\theta_t \qquad 14\text{-}43$$

and

$$H_{0i} - H_{0r} = H_{0t} \qquad 14\text{-}44$$

Substituting in for H_0:

$$\Rightarrow n_i(E_{0i} - E_{0r}) = n_t E_{0t} \qquad 14\text{-}45$$

Again eliminating E_{OT}, E_{OR} in turn gives,

$$\frac{E_{or}}{E_{oi}} = \frac{n_i \cos\theta_t - n_t \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} \qquad 14\text{-}46$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} \qquad 14\text{-}47$$

Fresnel's equations for $E_{//}$ to the plane of incidence

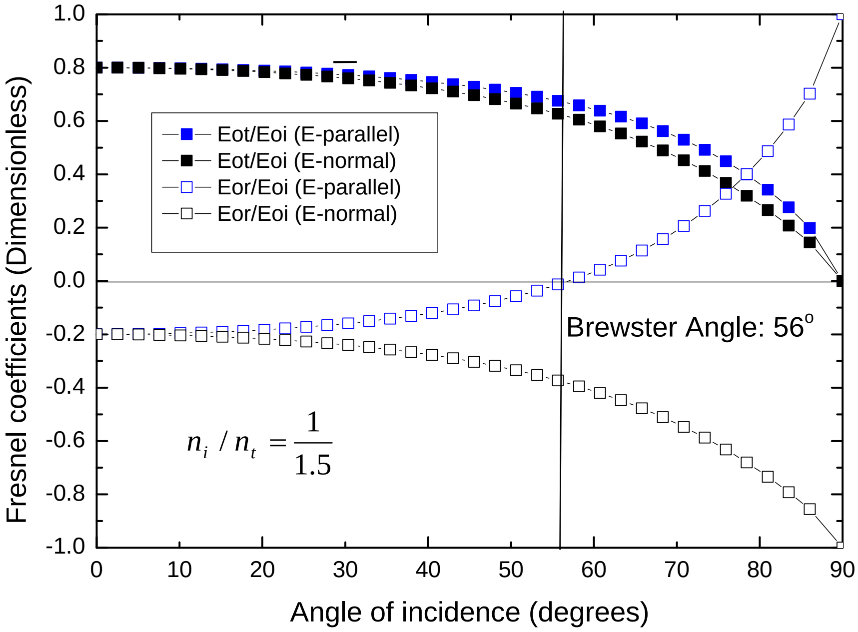

Figure 7 : The Fresnel coefficients as a function of the angle of incidence for the ***E*** - field parallel and normal to the plane of incidence. The Brewster angle is 56 degrees for an air/glass boundary

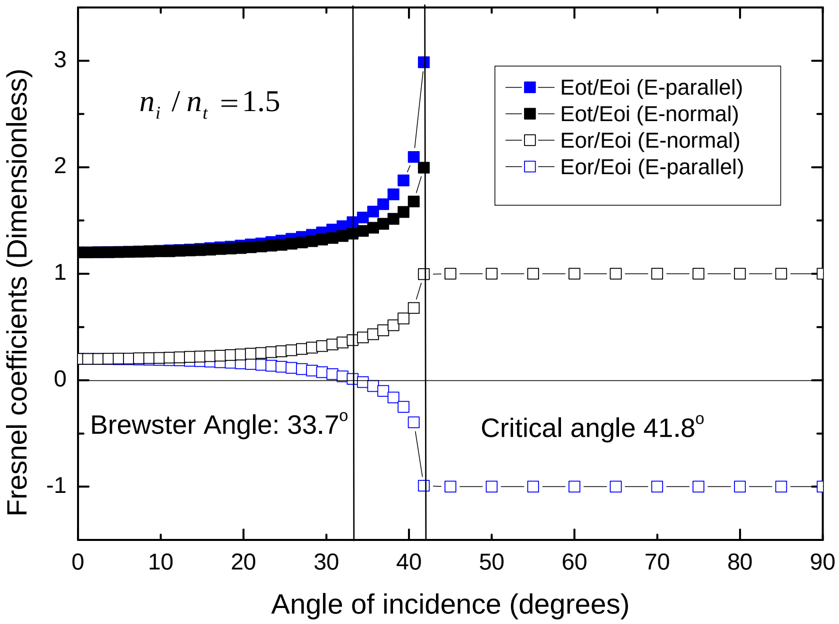

Figure 8 : The Fresnel coefficients as a function of the angle of incidence for the ***E*** - field parallel and normal to the plane of incidence. The Brewster angle is 33.7 degrees for an glass/air boundary. Above the critical angle of 41.8 degrees, there is no transmission.

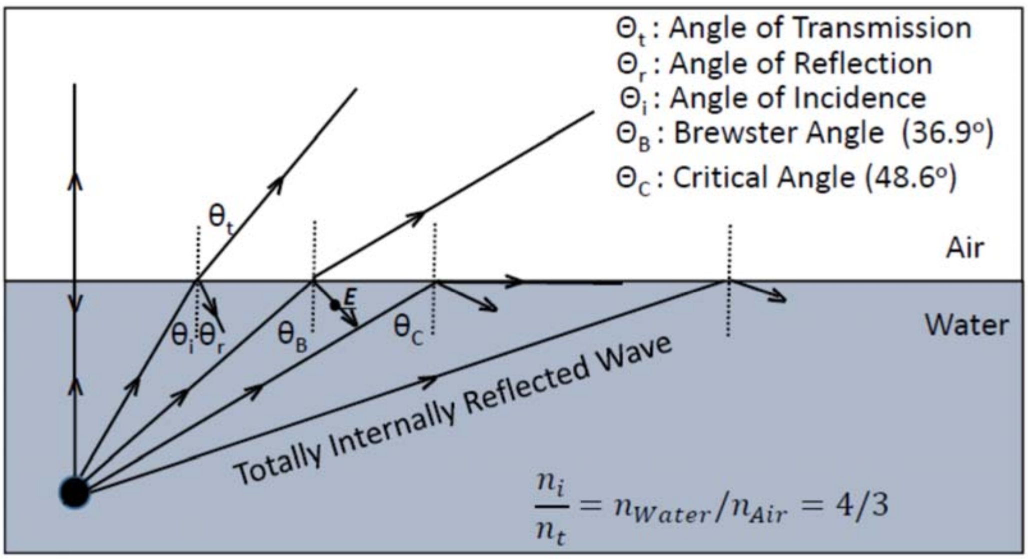

Figure 9 : The reflection and transmission of light into the air that has originated in the water. The Brewster angle is 36.9 degrees for an glass/air boundary so the light that is reflected is polarized normal to the angle of incidence. Above the critical angle of 48.6 degrees, there is no transmission.