Cosmology Part I: Workshop IV

1. A model Universe comprises ordinary matter (present-day density parameter Ω_M) and relativistic particles (present-day density parameter Ω_{rad}). The latter comprises the $T_0 = 2.73$ K CMB photons and a population of massless particles whose energy density is 0.68 times that of the CMB. Show that the expansion factor at the epoch when the energy densities of these two contributions are equal is:

$$a_{eq} = 7.4 \times 10^{-5} \Omega_M^{-1} h_{75}^{-2},$$

where h_{75} is the dimensionless Hubble constant H_0 in units of 75 km s^{-1} Mpc^{-1}.

Hint: The total energy density in the CMB is given by is $4\sigma T_0^4/c$, where $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

2. Show that at $a << a_{eq}$ the Friedmann equation may be written:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_{eq}^2 \left(\frac{a}{a_{eq}}\right)^{-4},$$

and evaluate H_{eq} if $H_0 = 75$ km s^{-1} Mpc^{-1} and $\Omega_M = 0.3$. Evaluate the CMB temperature and its instantaneous rate of change (in K yr^{-1}) at $a = 0.1 a_{eq}$. How old was the Universe at this point?

3. (a) Consider a hypothetical cosmological dark energy fluid with the equation of state $P_{DE} = -\frac{2}{3}\rho_{DE}c^2$. Use the fluid equation

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

to show that for this material $\rho_{DE} \propto a^{-1}$.

(b) If the current density parameters in this fluid and normal matter are Ω_{DE} and Ω_M, respectively, show that the Hubble parameter, H, at arbitrary expansion factor a is given by:

$$\frac{H^2}{H_0^2} = a^{-3}\left[\Omega_{DE}(a^2 - a) + \Omega_M(1 - a) + a\right].$$

(c) If $\Omega_{DE} = 0$ and $\Omega_M = 3$, by what additional factor will the Universe expand before the expansion comes to a halt?

(d) If $\Omega_M = 3$ but $\Omega_{DE} > 0$, calculate the range of values of Ω_{DE} for which: (i) the Universe will nevertheless still re-collapse at some future time; (ii) the Universe started with a Big Bang but will expand forever; (iii) the Big Bang did not occur and the Universe will expand forever.