Foundations 3A - QM, Worksheet 5

Problem 1

One can produce "circularly polarized" electromagnetic waves, in which the electric field vector rotates about the direction of propagation of the wave. Within the dipole approximation, such a wave can be represented by the electric field vector

$$\boldsymbol{\mathcal{E}}(t) = \frac{\mathcal{E}_0}{2}\left[\hat{\boldsymbol{\epsilon}} \exp(-i\omega t) + \hat{\boldsymbol{\epsilon}}^* \exp(i\omega t)\right],$$

where $\hat{\boldsymbol{\epsilon}}$ is a complex unit vector. If the wave propagates in the positive z-direction, then $\hat{\boldsymbol{\epsilon}} = (\hat{\mathbf{x}} - i\hat{\mathbf{y}})/\sqrt{2}$, $\hat{\boldsymbol{\epsilon}}^* = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$ for "right-circular" polarization and $\hat{\boldsymbol{\epsilon}} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$, $\hat{\boldsymbol{\epsilon}}^* = (\hat{\mathbf{x}} - i\hat{\mathbf{y}})/\sqrt{2}$ for "left-circular" polarization (these two cases correspond to opposite senses of rotation of the electric field vector). Here $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors in, respectively, the x- and the y-directions.

- What is the TD Hamiltonian term $H'(\mathbf{r}, \mathbf{t})$ that gives rise to $\boldsymbol{\mathcal{E}}(t)$? (We want $-\nabla H'(\mathbf{r}, t) = \text{force} = q\boldsymbol{\mathcal{E}}(t)$.)

Suppose that an atom of hydrogen, initially in a bound state with magnetic quantum number m_a, makes a transition to a bound state with magnetic quantum number m_b under the effect of a right-circularly polarized field.

- What should the difference $m_b - m_a$ be for the corresponding transition probability to be non-zero?

Hint: The wave functions of these two states, $\psi_a(\mathbf{r})$ and $\psi_b(\mathbf{r})$ are products of a spherical harmonic and a function of the radial variable r only: $\psi_a(\mathbf{r}) = R_{n_a}(r)Y_{l_a m_a}(\theta, \phi)$ or just $|n_a, l_a, m_a\rangle$ and $\psi_b(\mathbf{r}) = R_{n_b}(r)Y_{l_b m_b}(\theta, \phi)$ or just $|n_b, l_b, m_b\rangle$.

You may use the commutation relations : $\left[L_z, x + iy\right] = \hbar\,(x + iy)$, $\quad \left[L_z, x - iy\right] = -\hbar\,(x - iy)$

Problem 2 (See Griffiths Example 1.12)

We start with the two-level example of the lectures (the two states are ψ_a, ψ_b).
A particle of mass m is initially in state a with wave function $\psi_a(\mathbf{r})$. The particle interacts with a potential $H'(t)$ that does not change with time, after it is switched on at $t = 0$.

$$H'(t) = \begin{cases} 0 & t < 0 \\ \mathcal{V}(\mathbf{r}) & t \geq 0 \end{cases} \tag{1}$$

At time $t > 0$ the particle is in a linear combination of the two states a, b with wavefunction:

$$\psi(\mathbf{r}, t) = c_a(t)\,\psi_a(\mathbf{r}) \exp[-iE_a\,t/\hbar] + c_b(t)\,\psi_b(\mathbf{r}) \exp[-iE_b\,t/\hbar]$$

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Remember that for a sinusoidal perturbation, $\mathcal{V}(\mathbf{r})\cos(\omega t)$, (Lectures 10-11), in first order TD PT, the amplitudes $c_a(t)$, $c_b(t)$ are

$$c_a^{(1)}(t) = 1 \quad c_b^{(1)}(t) = -\frac{\mathcal{V}_{ba}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0 - \omega}\right] \simeq -\frac{i\mathcal{V}_{ba}}{\hbar}\, e^{i(\omega_0-\omega)t/2}\,\frac{\sin[(\omega_0 - \omega)t/2]}{\omega_0 - \omega}$$

with $\hbar\,\omega_0 = E_b - E_a$. The probability for transition $a \to b$ is $P_{a\to b} = |c_b^{(1)}(t)|^2$:

$$P_{a\to b}(\omega, t) = \frac{|\mathcal{V}_{ba}|^2}{\hbar^2}\,\frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

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• [a] What are the amplitudes c_a, c_b and the probability for transition $P_{a\to b}(t)$ when the perturbation is time-independent (1) and acts for time $0 \le t' \le t$?

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We now consider that the initial state of the particle has wave vector $\mathbf{k}'$ and is represented by the plane wave

$$\psi_i(\mathbf{r}) = \frac{1}{\sqrt{V}}\, e^{i\mathbf{k}'\cdot\mathbf{r}}$$

[V is the total volume. We follow Griffiths and use box normalisation for plane waves, which is not rigorous.]

The probability current, $\mathbf{j}(\mathbf{r})$, for a particle gives the probability the particle will cross an area, per unit area per unit time. When the particle is described by the wf $\psi(\mathbf{r})$, the probability current is given by:

$$\mathbf{j}(\mathbf{r}) = -\frac{i\hbar}{2m}\big(\psi^*(\mathbf{r})\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla\psi^*(\mathbf{r})\big)$$

• [b] What is the probability current $\mathbf{j}_i(\mathbf{r})$ for the incident particle described by ψ_i?

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Because of the interaction with $\mathcal{V}(\mathbf{r})$, the state of the particle makes a transition to a bunch of plane wave states

$$\psi_f(\mathbf{r}) = \frac{1}{\sqrt{V}}\, e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{2}$$

The density of states in energy $\rho(E)$ of final states ψ_f with energy $E = \hbar^2 k^2/(2m)$ travelling in a solid angle $d\Omega$ is:

$$\rho(E) = V\,\frac{\sqrt{2m^3 E}}{8\pi^3\hbar^3}\, d\Omega \tag{3}$$

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Remember, from Fermi's Golden Rule, in Lecture 13-14 (for a sinusoidal perturbation), the rate at which particles are scattered in continuum states (2) into a solid angle $d\Omega$ is:

$$R_{i\to d\Omega} = \frac{\pi|\mathcal{V}_{if}|^2}{2\hbar}\rho(E_f)$$

• [c] What is the rate of scattering $R_{i\to d\Omega}$ into a solid angle $d\Omega$, when the perturbation (1) is independent of time, after it is switched on at $t = 0$?

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Extra Problem: Born Approximation

• [d] Find the differential scattering cross-section $d\sigma/d\Omega$, which gives the rate at which particles are scattered into a solid angle $d\Omega$, per solid angle $d\Omega$ and divided by the magnitude of the incoming probability current:

$$\frac{d\sigma}{d\Omega} = \frac{R_{i\to d\Omega}}{J_i\, d\Omega}$$