# University of Durham

## EXAMINATION PAPER

May/June 2014 Examination code: 042581/01

**LEVEL 2 PHYSICS: FOUNDATIONS OF PHYSICS 2A**

**SECTION A**. Quantum Mechanics 2
**SECTION B**. Electromagnetism

**Time allowed: 3 hours**

**Examination material provided: None**

**Calculators:** The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

**ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

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**Information**

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|---|---|
| Elementary charge: | $e = 1.60 \times 10^{-19}$ C |
| Speed of light: | $c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$ |
| Boltzmann constant: | $k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$ |
| Electron mass: | $m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg |
| Gravitational constant: | $G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$ |
| Proton mass: | $m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg |
| Planck constant: | $h = 6.63 \times 10^{-34}$ J s |
| Permittivity of free space: | $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$ |
| Magnetic constant: | $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$ |
| Molar gas constant: | $R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$ |
| Avogadro's constant: | $N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$ |
| Gravitational acceleration at Earth's surface: | $g = 9.81\ \mathrm{m\,s^{-2}}$ |
| Stefan-Boltzmann constant: | $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$ |
| Astronomical Unit: | AU $= 1.50 \times 10^{11}$ m |
| Parsec: | pc $= 3.09 \times 10^{16}$ m |
| Solar Mass: | $M_{\odot} = 1.99 \times 10^{30}$ kg |
| Solar Luminosity: | $L_{\odot} = 3.84 \times 10^{26}$ W |

**SECTION A**. QUANTUM MECHANICS 2

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Write down the form of the one dimensional momentum operator, $p$. Use this to derive the form of the kinetic energy operator, $T = p.p/(2m)$. Write down the form of the time dependent energy operator, $E$. Hence derive the one dimensional, time dependent Schroedinger equation from the energy equation $T\psi(x,t) + V\psi(x,t) = E\psi(x,t)$. [4 marks]

   (b) The wavefunction for hydrogen in the ground state is $\psi_{100} = (\pi a^3)^{-1/2} e^{-r/a}$, where $a = 4\pi\epsilon_0\hbar^2/(me^2)$ is the Bohr radius, with energy $E_1 = -\hbar^2/(2ma^2)$. The potential $V(r) = -e^2/(4\pi\epsilon_0 r)$.

   Show that the maximum classically allowed radius, $r_c = 2a$.

   What is the probability that the electron can be found with $r \geq r_c$? Evaluate your answer to 3 significant figures. [4 marks]

$$\left[ \quad \int_c^d x^2 e^{-2x/a} dx = -\frac{a}{4} |(a^2 + 2ax + 2x^2) e^{-2x/a}|_c^d . \quad \right]$$

   (c) For the hydrogen ground state wavefunction in (b) above, calculate $\langle r \rangle$ and $\langle V \rangle$. Write down $\langle E \rangle$, and compare this with $\langle V \rangle$. [4 marks]

$$\left[ \quad \int_0^\infty x^p e^{-qx} dx = \frac{p!}{q^{p+1}} . \quad \right]$$

   (d) Hydrogen is in the state $\psi_{2,1,-1} = R_{21} Y_{1,-1} = R_{21}\sqrt{3/(8\pi)} \sin\theta e^{-i\phi}$, where all the symbols have their usual meanings. Write down the probability of finding the electron within a volume $dV$ of $r, \theta, \phi$. Hence calculate the probability distribution as a function of $\mu = \cos\theta$ (so $|d\mu| = \sin\theta d\theta$). Where is this maximal? [4 marks]

   (e) An operator $A$ has two normalised eigenstates $\psi_1$ and $\psi_2$ with eigenvalues $a_1$ and $a_2$. Operator $B$ has two eigenstates $\phi_1$ and $\phi_2$ with eigenvalues $b_1$ and $b_2$. The eigenstates are related by

$$\psi_1 = \frac{1}{5}(3\phi_1 + 4\phi_2) \quad \psi_2 = \frac{1}{5}(4\phi_1 - 3\phi_2).$$

   A measurement of $A$ in this system results in the value $a_1$. What is the state of the system immediately after the measurement? If $B$ is now measured, what are the possible results and their probabilities?

   What is the probability of getting $a_1$ if $A$ is remeasured immediately after the measurement of $B$ gives the value $b_1$? [4 marks]

   (f) The three dimensional anisotropic harmonic potential has energies $E_{n_x,n_y,n_z} = (n_x + 1/2)\hbar\omega_x + (n_y + 1/2)\hbar\omega_y + (n_z + 1/2)\hbar\omega_z$.

   Write down the energy and degeneracy of the ground state and first excited state if $\omega_y = \omega_x$ and $\omega_z = 2\omega_x$.

   Give a qualitative description of the values of $\omega_x, \omega_y$ and $\omega_z$ which maximize the degeneracy of the first excited state. What values minimize the degeneracy? [4 marks]

  

(g) The infinite square well potential with $V = 0$ for $0 < x, y, z < a$ has unperturbed ground state wavefunction

$$\psi_{1,1,1} = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{a}\right) \quad \text{with } E_{1,1,1} = \frac{3\pi^2\hbar^2}{2ma^2}.$$

The system is subject to a perturbation

$$H' = V_0 \delta(x - a/4)\delta(y - a/2)\delta(z - 3a/4).$$

For this perturbation, calculate the first order change to the ground state energy, $E_1^1 =< \psi_{1,1,1}|H'\psi_{1,1,1} >$, and hence estimate the new ground state energy. [4 marks]

(h) Three states $\psi_1, \psi_2$ and $\psi_3$ are degenerate, so any linear combination $\psi = \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3$ also gives the same energy. A small perturbation gives a first order correction to the energy $E^1$ given by the solution of the matrix equation

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E^1 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Solve for $E^1$, and write down the corresponding wavefunctions which follow the perturbation. [4 marks]

2. A particle of mass $\mu$ is in the ground state of the one dimensional infinite square well potential where $V = 0$ for $0 < x < a$, with wavefunction $\psi = \sqrt{2/a}\sin(\pi x/a)$ and energy $E = \pi^2\hbar^2/(2\mu a^2)$. Suddenly, the well expands to twice its size, so the right wall moves from $a$ to $2a$, leaving the wavefunction momentarily undisturbed. The double width system has energy eigenfunctions $\psi_n = \sqrt{1/a}\sin[n\pi x/(2a)]$ with associated energy $E_n = n^2\pi^2\hbar^2/(8\mu a^2)$.

   (a) General energy eigenfunctions, $\phi_n$, form a complete basis, so any arbitrary wavefunction can be written as $\psi = \sum_n c_n\phi_n$. Use the orthonormal property of energy eigenfunctions, where $\int \phi_n^*\phi_m dx = \delta_{nm}$, to show that $c_n = \int \phi_n^*\psi dx$. [2 marks]

   (b) Calculate the $c_n$ for the initial wavefunction $\psi$ in the double width square well potential described above, given the standard integral,

$$\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{n\pi x}{2a}\right)dx = \frac{4a\sin(n\pi/2)}{(4\pi - n^2\pi)}.$$

   Evaluate your result for both $n = 1$ and $n = 3$, and write down the general case for odd $n$ without any trigonometric functions. Evaluate your result for even $n$, treating the pathological case of $n = 2$ separately by considering $n = 2 + \epsilon$ and then taking the limit as $\epsilon \to 0$ to show that $c_2 = 1/\sqrt{2}$. [10 marks]

   (c) What is the most probable result for a measurement of the energy, and what is its probability? [2 marks]

   (d) Write down, but do not evaluate, the expectation value for the energy of the electron in the double width potential as an infinite sum. The expectation value of this energy can also be written as $\langle H\rangle = \int \psi^* H\psi dx$ where $H$ is the Hamiltonian. Evaluate the integral by inspection to determine $\langle H\rangle$, and comment on the physical significance of your result by comparing it to the energy of the electron in the single width potential. [6 marks]

3. The unperturbed wavefunctions of hydrogen including spin can be written as $\psi^0_{n,l,m,m_s} = R_{nl} Y_{lm} X_\pm$ where $X_+$ (spin up) and $X_-$ (spin down) are the eigenfunctions of the spin operators $S^2$ and $S_z$ with eigenvalues $s(s+1)\hbar^2 = 3/4\hbar^2$ for $s = 1/2$, and $m_s\hbar$ with $m_s = \pm\frac{1}{2}$. Similarly, $Y_{lm}$ are the eigenfunctions of $L^2$ and $L_z$ with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$, and $\psi^0_{n,l,m,m_s}$ are the eigenfunctions of $H^0$ with eigenvalues $E_n$.

   Spin-orbit coupling in hydrogen gives rise to a perturbation $H' \propto \underline{L}.\underline{S}$ where $\underline{L}$ and $\underline{S}$ are the orbital and electron spin angular momentum operators, respectively. $H^0$, $L^2, L_z, S^2$ and $S_z$ all commute with each other.

   (a) The total angular momentum operator is $\underline{J} = \underline{L} + \underline{S}$. Evaluate $J^2 = (\underline{L} + \underline{S}).(\underline{L} + \underline{S})$ to show that $\underline{L}.\underline{S} = (1/2)\,(J^2 - L^2 - S^2)$ [2 marks]

   (b) $\underline{L}.\underline{S} = L_x S_x + L_y S_y + L_z S_z$. Show that this can be rewritten using the ladder operators $L_\pm = L_x \pm iL_y$ and $S_\pm = S_x \pm iS_y$ as

$$\underline{L}.\underline{S} = \frac{1}{2}(L_+ S_- + L_- S_+) + L_z S_z.$$

   [3 marks]

   (c) Ladder operators raise and lower their associated angular momentum quantum number by unity, so $S_- X_+ = aX_-$ and $S_+ X_- = aX_+$ where $a = \hbar/\sqrt{2}$, and $L_\pm Y_{lm} = A_{lm} Y_{l,m\pm1}$ where $A_{lm} = \hbar\sqrt{l(l+1) - m(m\pm1)}$. All angular momenta have a maximum value beyond which the upwards ladder operator gives zero, and a minimum below which the downwards ladder operator gives zero.

   Use this together with the definition of $\underline{L}.\underline{S}$ in terms of ladder operators given in (b) to show that

$$\underline{L}.\underline{S}\ \psi^0_{2,1,-1,1/2} = \frac{\hbar^2}{2}(\psi^0_{2,1,0,-1/2} - \psi^0_{2,1,-1,1/2}).$$

   Are the unperturbed energy eigenfunctions of hydrogen also eigenfunctions of $\underline{L}.\underline{S}$? [9 marks]

   (d) Use $\underline{L}.\underline{S} = L_x S_x + L_y S_y + L_z S_z$, together with the standard commutation relations for any general angular momenta $[J_x, J_y] = i\hbar J_z$, $[J_y, J_z] = i\hbar J_x$, $[J_z, J_x] = i\hbar J_y$, to show that $[\underline{L}.\underline{S}, L_z] \neq 0$ and $[\underline{L}.\underline{S}, S_z] \neq 0$ but that $[\underline{L}.\underline{S}, J_z] = [\underline{L}.\underline{S}, L_z + S_z] = 0$.

   Which set of quantum numbers allow the effect of the perturbation to be calculated using non-degenerate perturbation theory? [6 marks]

## SECTION B. ELECTROMAGNETISM

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Using standard complex notation, the wavevector, $\underline{k_0}$, of a wave is given by $\underline{k_0} = (5+3i)\hat{\underline{k}}$ m$^{-1}$, where $\hat{\underline{k}}$ is a unit vector that points in the direction of propagation. What is the wavelength of the wave ? [4 marks]

(b) Show that one of the consequences of Maxwell's equations is that charge cannot be created or destroyed. [4 marks]

(c) The general dispersion relation for an electromagnetic wave propagating in a non-magnetic, conducting medium is given by

$$k^2 = \mu_0 \epsilon \omega^2 + i\omega\mu_0\sigma_N,$$

where $k$ is the modulus of the wavevector, $\omega$ is the angular frequency, $\sigma_N$ is the electrical conductivity and $\epsilon = \epsilon_r\epsilon_0$ where $\epsilon_r$ is the relative permittivity. Can a material which has a relative permittivity of 2 and an electrical conductivity of $5 \times 10^8\ \Omega^{-1}\ \mathrm{m}^{-1}$ be considered a good conductor at a frequency of $10^9$ Hz? [4 marks]

(d) Provide a brief outline of Ampère's model, which describes how a magnetic material generates a magnetic field. [4 marks]

(e) A brilliant scientist in Durham claims to have discovered a new insulating strongly magnetic material. Briefly describe which measurements you would do to characterize the important properties of the material. [4 marks]

(f) Briefly describe how a radio transmitter works. [4 marks]

(g) Use a diagram to explain the relationships between the electric field, the magnetic field and the direction of propagation of a plane electromagnetic wave propagating in free space. [4 marks]

5. (a) Use Maxwell's equations to derive the boundary conditions for the $\underline{B}$-field parallel and orthogonal to the interface between two magnetic materials. [4 marks]

(b) For an electromagnetic wave at normal incidence to the planar interface between two insulating magnetic media, the power reflection coefficient, $R$, is given by:

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 ,$$

where $n_1$ and $n_2$ are the refractive indices of the media. At what values of $n_2/n_1$ does the power transmitted across the interface equal one quarter of that reflected from it? [5 marks]

Sketch the form of the transmission coefficent, $T$, against $n_2/n_1$, where $T = (1 - R)$. [5 marks]

(c) An electromagnetic plane wave is fired at a thin sheet of magnetic material in vacuum at normal incidence. By including the effect of internal reflections, calculate an approximate value for the refractive index of the magnetic material if about 4 percent of the incident energy is reflected. [6 marks]

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6. (a) Briefly describe two different examples of a plasma and in each case explain how the plasma arises. [4 marks]

(b) In a plasma inside a tokamak where only the motion of the electrons is important, the equation of motion for the electrons in an electric field, $\underline{E}$, where $t$ is the time, is given by:

$$m_{\mathrm{e}}\frac{d\underline{v}}{dt} = e\underline{E} - \frac{m_{\mathrm{e}}\underline{v}}{\tau},$$

where $\underline{v}$ is the electron velocity and $\tau$ is the scattering time for the electrons. Show that the electrical conductivity, $\sigma_{\mathrm{e}}$, at angular frequency $\omega$ can be written in the form:

$$\sigma_{\mathrm{e}} = \frac{n_{\mathrm{e}}e^2}{m_{\mathrm{e}}(\tau^{-1} - i\omega)},$$

where $n_{\mathrm{e}}$ is the number of electrons per unit volume. [3 marks]

Given that the electron charge carrier concentration is $10^{28}$ m$^{-3}$ and the conductivity at low frequencies is $10^6$ $\Omega^{-1}$ m$^{-1}$, calculate a value for $\tau$. [3 marks]

(c) It is found that the conductivity of the plasma is lower by a factor of 2 for the electromagnetic waves used to heat the plasma than the low frequency conductivity. Calculate a value for the frequency of the electromagnetic waves. [5 marks]

Calculate the approximate phase difference between the electric field and the magnetic field for the electromagnetic wave inside the plasma. [5 marks]