University of Durham

EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2019	PHYS2611-WE01

Title:
Mathematical Methods in Physics

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Attempt **all** questions. The short-answer questions at the start of each section carry 50% of the total marks for the paper. The remaining 50% of the marks are carried by the longer questions, which are equally weighted.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Mathematical Methods part 1
Section B: Mathematical Methods part 2

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A: MATHEMATICAL METHODS PART 1

1. (a) Determine whether the given sets, together with the specified operations of addition and scalar multiplication, are vector spaces. If they are not, for each of them state an axiom that fails to hold.

   (i) The set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in R^3 such that $3x - 4y + z = 1$, with the usual vector addition and scalar multiplication.

   (ii) The set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in R^3 such that $3x - 4y + z \geq 0$, with the usual vector addition and scalar multiplication.

   [4 marks]

   (b) State and give a reason whether the given sets of vectors are linearly dependent or independent.

   (i) $\{(0, -1, 0), (1, 0, 1), (2, 1, 0)\}$.
   (ii) $\{(-1, -2, -3), (-6, 2, 5), (1/2, 1, 3/2)\}$.
   (iii) $\{(1, 0, 0), (0, 0, -1), (0, 1, 0), (-1, -1, -1)\}$.
   (iv) $\{(1, 3, 5), (2, 0, 2), (6, 7, 8)\}$.

   [4 marks]

   (c) Given that the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

   has two eigenvectors $\underline{v}$ of the form $\underline{v}^T = (1, y, 1)$, use the eigenvalue equation $A\underline{v} = \lambda\underline{v}$ to find the corresponding eigenvalues λ. Then deduce the third eigenvalue. [4 marks]

   (d) Evaluate the line integral

$$I = \int_{\mathcal{C}} \left(\frac{x}{y}\, dx + \frac{y}{x}\, dy \right)$$

   over the curve $\mathcal{C}$ whose parametrisation is $\underline{r}(\theta) = 2\cos\theta\, \hat{\underline{i}} + 3\sin\theta\, \hat{\underline{j}}$, $0 \leq \theta \leq 2\pi$. [4 marks]

   (e) Evaluate the surface integral

$$I = \int_{\mathcal{S}} \underline{a} \cdot d\underline{S}, \quad \underline{a} = xy\, \hat{\underline{i}} + z\, \hat{\underline{k}},$$

   over the surface $\mathcal{S}$ whose parametrisation is $\underline{r}(u, v) = u\, \hat{\underline{i}} + v\, \hat{\underline{j}} + u^2\, \hat{\underline{k}}$, $-1 \leq u \leq 1$, $0 \leq v \leq 2$. [4 marks]

   (f) State the divergence theorem and explain all symbols you use. [4 marks]

(g) Given that $\mathcal{L}[e^{ct}](s) = 1/(s-c)$, $s > c$, verify that

$$\mathcal{L}\left[\frac{\cos at - \cos bt}{(b^2 - a^2)}\right] = \frac{s}{(s^2+a^2)(s^2+b^2)},$$

where a, b are constants such that $a^2 \neq b^2$. [4 marks]

(h) The Fourier transform for a function $f(t)$ is defined as follows

$$\mathcal{F}[f(t)](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-it\omega} dt.$$

Calculate the Fourier transform for the function

$$f(t) = (H(t) - H(t-\pi))\, e^{i\alpha t},$$

where α is a constant and $H(t)$ is the Heaviside step function defined as follows

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

[4 marks]

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2. (a) The work done by a force $\underline{F}$ along a path $\mathcal{C}$ is given by the following integral

$$I = \int_{\mathcal{C}} \underline{F} \cdot d\underline{r},$$

where $\underline{r}$ represents the parametrisation of the path $\mathcal{C}$. Determine the work done along the paths provided for the two forces described below. For each of them, verify whether the force is conservative. If it is, determine its potential i.e. the scalar function ϕ for which $\nabla\phi = \underline{F}$.

(i) $\underline{F}_1 = y\,\hat{\underline{i}} + \hat{\underline{j}} + x\,\hat{\underline{k}}$. The path is a straight line from $(0,0,0)$ to $(1,1,1)$. [7 marks]

(ii) $\underline{F}_2 = (xy^2 + z)\,\hat{\underline{i}} + (x^2y + 1)\hat{\underline{j}} + x\,\hat{\underline{k}}$. The path is the curve curve $x = t$, $y = 1/t$, $z = 2$ from (1,1,2) to $(2, 1/2, 2)$. [7 marks]

(b) Using the fact that

$$(\underline{a} \times \underline{b})_i = \epsilon_{ijk} a_j b_k, \qquad \epsilon_{ij\,k}\, \epsilon_{jlm} = \delta_{i\,m}\delta_{kl} - \delta_{i\,l}\delta_{km},$$

prove the following vector identity

$$(\underline{u} \times \underline{v}) \times \underline{w} = (\underline{u} \cdot \underline{w})\underline{v} - (\underline{v} \cdot \underline{w})\underline{u}.$$

Notice that it suffices to prove the identity for a single component

$$((\underline{u} \times \underline{v}) \times \underline{w})_i.$$

[5 marks]

(c) If $f(t) = t^a$, and $g(t) = t^b$, with a, b integers greater than or equal to zero,

(i) show that the convolution integral

$$h(t) = (f * g)(t) = \int_0^t u^a\,(t-u)^b\,du$$

is equal to

$$t^{a+b+1} \int_0^1 x^a\,(1-x)^b\,dx,$$

[4 marks]

(ii) then, given that $\mathcal{L}[t^n](s) = n!/s^{n+1}$, and using the convolution theorem for the Laplace transform applied to $h(t)$, show that

$$\int_0^1 x^a\,(1-x)^b\,dx = \frac{a!\,b!}{(a+b+1)!}.$$

[7 marks]

SECTION B: MATHEMATICAL METHODS PART 2

3. (a) Find the solution $y(x)$ of the ordinary differential equation

$$\frac{2y}{x}\frac{dy}{dx} = 4y^2 + 3xy^2,$$

subject to the boundary condition $y(1) = 2e^{3/2}$. [4 marks]

(b) Find a solution of the second-order inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3x^2 + x + 2,$$

by finding the roots of the auxiliary equation and using the method of trial functions. [4 marks]

(c) Use the Wronskian method to solve the ordinary differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 20x^3e^{4x}.$$

[4 marks]

$$\left[\begin{array}{c} \text{Hint: Remember that if } y = k_1y_1 + k_2y_2 \text{ then,} \\ k_1' = \frac{-h(x)}{W(x)}y_2 \text{ and } k_2' = \frac{h(x)}{W(x)}y_1 \\ \text{where } h(x) \text{ is the inhomogeneous term and } W(x) \text{ is the Wronskian.} \end{array}\right]$$

(d) State the name given to the following type of equation and write down its general form [2 marks]

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 56y = 0.$$

Write down a polynomial solution to this equation, with undetermined coefficients. [2 marks]

(e) The definition of a Laplace transform is

$$\bar{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt.$$

Show that the Laplace tranform of $f(t) = e^{at}$ is

$$\bar{f}(s) = \frac{1}{s-a}.$$

[4 marks]

(f) The 1D wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Use the method of separation of variables to find a solution to this equation, with a negative separation constant. [4 marks]

(g) State the name given to the following type of equation [1 mark]

$$2x^2y'' + 2xy' - 8y = 0.$$

Solve the equation to find $y(x)$. [3 marks]

4. (a) Write down the total derivitive ds/dx in terms of partial derivatives, where $s = s(x, y, z)$. [2 marks]

(b) Consider the partial differential equation

$$2x\frac{\partial u}{\partial x} - 8x^4\frac{\partial u}{\partial y} = 0,$$

where $u = u(x, y)$. Find the most general solution, such that $u = 5$ on the curve $y = -x^4$. [4 marks]

(c) Consider a second order PDE of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0,$$

where $u = u(x, y)$ and A, B and C are constants.
Assume there is a solution to this PDE of the form $u(x, y) = f(p)$, where p is a linear function of x and y.

(i) Show that $f(p)$ can be eliminated from the LHS of the PDE, resulting in the quadratic equation

$$Aa^2 + Bab + Cb^2 = 0,$$

where $a = \partial p/\partial x$ and $b = \partial p/\partial y$. [5 marks]

(ii) By solving the quadratic equation, show that the general solution to the PDE is given by

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y),$$

where f and g are arbitrary functions and λ_1 and λ_2 are solutions to the quadratic equation. [3 marks]

(d) The Laplace equation has the form of the PDE given in part (c). Find the general solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 24x + 12y^2$$

by first solving the equation assuming the RHS is equal to zero, and then finding a particular integral for the RHS shown. [9 marks]

(e) Changes in the three-dimensional spatial and temporal distribution of a chemical with concentration $u(\underline{r}, t)$ can be described by the diffusion equation

$$k\nabla^2 u = \frac{\partial u}{\partial t},$$

where ∇^2 is the Laplace operator and k is the diffusivity, a constant.

(i) State the order of the spatial and temporal derivatives of the diffusion equation. [2 marks]

(ii) Write down the diffusion equation for a one-dimensional problem, where $u = u(x, t)$. [2 marks]

(iii) What are the dimensions of the constant k? [1 mark]

(iv) Explain why the method used in part (c) will not provide a solution to the diffusion equation. [2 marks]