CM6 Solutions: An application of Hamiltonian mechanics

1. **(2 marks total)** The velocity of the bob is $(\dot{x}_0 + l\dot{\theta}\cos\theta, l\dot{\theta}\sin\theta)$. Hence the kinetic energy is $T = m(\dot{x}_0^2 + 2\dot{x}_0 l\dot{\theta}\cos\theta + l^2\dot{\theta}^2)/2$. Defining the zero of potential at the top of the pendulum, the potential energy is $V = -mgl\cos\theta$. Hence, $L = T - V$ as required.

2. **(3 marks total)** $p_\theta = \partial L/\partial\dot{\theta} = m(\dot{x}_0 l\cos\theta + l^2\dot{\theta})$.
   For small θ and $\dot{x}_0$, $p_\theta \approx m(l\dot{x}_0 + l^2\dot{\theta})$. This uses $\cos\theta \sim 1$, because including the next term ($-\theta^2/2$) would lead to a term that is cubic in small quantities, and for oscillations we only need up to quadratic order. Rearrange for $\dot{\theta}$, use the Legendre transformation, and ignore terms higher than second order in $\theta, \dot{\theta}, \dot{x}_0$ to give

   $$H = \frac{p_\theta^2}{2ml^2} - \frac{p_\theta \dot{x}_0}{l} - mgl\left(1 - \frac{\theta^2}{2}\right).$$

   This is not equal to the total energy, because $p_\theta\dot{\theta} \neq 2T$.

3. **(2 marks total)** The Lagrangian is $L \approx m\dot{x}^2/2 + mgl[1 - (x - x_0)^2/(2l^2)]$.
   $p_x = m\dot{x}$.
   Hence the required Hamiltonian follows, which is the total energy of the system.

4. **(3 marks total)**

   $$\dot{x} = \frac{p_x}{m}, \quad \dot{p}_x = -\frac{mg}{l}(x - x_0).$$

   Differentiating the first of these equations w.r.t. time and using the second leads to $\ddot{x} = -g(x - x_0)/l$ or $\ddot{x} + \omega^2 x = \omega^2 x_0(t)$, where $\omega^2 = g/l$.

   This represents a driven, undamped oscillator. Green's functions represent the response of a system to an impulsive force. Thus, as the equation is linear, the general solution can be found by splitting the driving force into an infinite sequence of impulsive forces and integrating over the responses to each of these impulsive forces.

   As the oscillator is undamped, if $\omega_0 = \omega$, the natural frequency of the oscillator, then the amplitude of oscillation will grow without limit, i.e. resonance. For other driving frequencies, the motion will be an erratic combination of motions at the two frequencies, ω_0 and ω, with no net energy input provided by the driving force averaged over a long period.