University of Durham

EXAMINATION PAPER

May/June 2015 Examination code: PHYS2611WE01

MATHEMATICAL METHODS IN PHYSICS

SECTION A. Mathematical Methods part 1
SECTION B. Mathematical Methods part 2

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\,\mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\,\mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\,\mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

  Continued Next Page

SECTION A. MATHEMATICAL METHODS PART 1

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) State whether the given vectors are linearly independent or dependent:

   (i) $(3,-1,4),(6,7,5),(9,6,9)$.

   (ii) $(4,3,9),(0,0,0),(1,3,2)$.

   (iii) $(3,0,2,4,5),(7,2,6,1,0),(1,2,2,-7,-10)$.

   (iv) $(9,0,9),(0,6,6),(3,3,0)$.

   [4 marks]

   (b) (i) Prove that $U^{-1}AU$ is Hermitian if A is Hermitian and U is a unitary matrix.

   (ii) It can be proved that if a unitary matrix U can be written as $A+iB$, where A and B are Hermitian, then A and B commute, that is $AB = BA$. Making use of this information, prove that the matrices A and B satisfy the relation $A^2+B^2 = I$, where I is the identity matrix.

   [4 marks]

   (c) Find the length of the following curve written in parametric form

$$\underline{r}(t) = t\,\hat{\underline{i}} + t^{3/2}\,\hat{\underline{j}},$$

   from the point $(0,0,0)$ to the point $(4,8,0)$. [4 marks]

   (d) Determine which of the following vector fields, $\underline{F}$, is conservative. If the field is conservative find its scalar potential f such that $\underline{F} = \nabla f$.

   (i) $\underline{F} = 2xy^3z^4\,\hat{\underline{i}} + 3x^2y^2z^4\,\hat{\underline{j}} + 4x^2y^3z^3\,\hat{\underline{k}}$.

   (ii) $\underline{F} = y\,\hat{\underline{i}} - x\,\hat{\underline{j}} + xyz\,\hat{\underline{k}}$.

   [4 marks]

   (e) Compute the line integral

$$I = \int_C \underline{F}\cdot d\underline{r},$$

   where F is the vector field $\underline{F} = 3\,x^4\,\hat{\underline{i}} + 3\,y^6\,\hat{\underline{j}}$ and C is the path of integration. This is anticlockwise from $(2,0)$ to $(-2,0)$ along the curve $x^2+y^2 = 4$.

   (f) Calculate the following surface integral

$$I = \int_S (\nabla\times\underline{F})\cdot d\underline{S},$$

   where $\underline{F} = (x^2y)\,\hat{\underline{i}}$ and the surface S is the hemisphere $x^2+y^2+z^2 = a^2$, $z \geq 0$ given parametrically by

$$\underline{r} = a\sin\theta\cos\phi\,\hat{\underline{i}} + a\sin\theta\sin\phi\,\hat{\underline{j}} + a\cos\theta\,\hat{\underline{k}}, \qquad 0\leq\phi\leq 2\pi, \quad 0\leq\theta\leq\pi/2.$$

   (g) Given the definition of the Fourier transform

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-iwt}dt,$$

compute the Fourier transform of the function

$$f(t) = \begin{cases} e^{-\beta t} \sin(\alpha\, t) & t > 0 \\ 0 & t < 0 \end{cases},$$

where $\beta(> 0)$ and α are constants. [4 marks]

(h) Compute the following integrals containing the Dirac δ function

(i)

$$I_1 = 2\pi \int_{-\infty}^{\infty} \delta(x^2 - \pi^2)\, e^{2x}\, dx + 4 \int_{-\infty}^{\infty} \delta(2x) \cos(2x)\, dx.$$

(ii)

$$I_2 = \int \delta(\underline{r} - \underline{a}) \left(\frac{1}{\underline{r} \cdot \underline{b} - 2} \right) d^3\underline{r},$$

defined over the entire three-dimensional space for $\underline{a} = (2, -1, 0)$ and $\underline{b} = (3, 1, 2)$.

[4 marks]

2. (a) Consider the following matrix

$$A = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where α and β are complex numbers different from zero.

(i) Find its eigenvalues and eigenvectors. [4 marks]

(ii) Find the constraints on α and β such that the eigenvalues are real and the eigenvectors are orthogonal. [3 marks]

(iii) Find the constraints on α and β such that the matrix A is Hermitian and verify that they coincide with the constraints found in part (ii). [3 marks]

(b) (i) State the divergence theorem and explain all symbols you use. [4 marks]

(ii) Verify the divergence theorem for the vector field

$$\underline{F} = x^3\,\hat{\underline{i}} + y^2 z\,\hat{\underline{j}} + z^2\hat{\underline{k}}$$

and the surface of the box $-1 \leq x \leq 1$, $-2 \leq y \leq 2$, $0 \leq z \leq 3$, by computing explicitly both sides of the equation representing the divergence theorem. [6 marks]

3. (a) Consider the function

$$f(x) = \sin x, \quad 0 \le x \le \pi.$$

(i) Show that the Fourier series of its even extension on the interval $-\pi \le x \le \pi$ is

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1}.$$

Note that the Fourier coefficients are:

$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi r x}{L}\right) dx, \qquad b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi r x}{L}\right) dx,$$

where L is the period and x_0 is arbitrary. [6 marks]

[Hint: $\sin\alpha\cos\beta = (\sin(\alpha+\beta) + \sin(\alpha-\beta))/2.$]

(ii) By setting $x = 0$ and $x = \pi/2$ find the value of the following sums

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}, \qquad S_2 = \sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}.$$

[4 marks]

[Hint: Note that in order to find the sum S_2 you need to combine the values of the Fourier series at $x = 0$ and $x = \pi/2$.]

(b) The Laplace transform for a function $f(t)$ is defined as follows

$$\mathcal{L}[f](s) \equiv \bar{f}(s) = \int_0^\infty f(t)\, e^{-ts} dt.$$

Consider two functions

$$f(t) = t, \qquad g(t) = e^{-3t}.$$

(i) Calculate the integral

$$h(u) = \int_0^t f(u) g(t-u) du.$$

[3 marks]

(ii) Given that

$$\mathcal{L}[1](s) = 1/s, \quad \mathcal{L}[t](s) = 1/s^2, \quad \mathcal{L}[e^{at}](s) = 1/(s-a),$$

compute the Laplace transform of the function $h(u)$ above. Verify your result using the convolution theorem

$$\bar{h}(s) = \bar{f}(s)\bar{g}(s).$$

[3 marks]

(iii) By using partial fraction decomposition and the Laplace transforms in part (ii) find the inverse Laplace transform of

$$\bar{m}(s) = \frac{s+1}{s^3 + s^2 - 6s}.$$

[4 marks]

SECTION B. MATHEMATICAL METHODS PART 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Solve the ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2} + \;x\;\frac{dy}{dx} - \;9\;y = \;5\;x^2$$

using a smart ansatz. [4 marks]

(b) Use the ***Wronskian*** method to solve the ordinary differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6\;x\;e^x.$$

[4 marks]

[Hint: Remember that if $y = k_1\,y_1 + k_2\,y_2$, then

$$k_1' = -\frac{h(x)}{W(x)}y_2 \quad \text{and} \quad k_2' = \frac{h(x)}{W(x)}y_1$$

with $W(x)$ the Wronskian and $h(x)$ the inhomogeneous term.]

(c) Solve the ordinary differential equation

$$(1-2x)\frac{dy}{dx} + 2y = 0.$$

Identify the following equation,

$$(1-2x)\frac{dy}{dx} + 2y + y^3 = 0.$$

How would you solve it? [4 marks]

(d) Solve the equation

$$\frac{d^4y}{dx^4} - 16y = 0.$$

[2 marks]

What would be another possible right hand side for this equation so that there is still a solution? [2 marks]

(e) Consider a wave propagating on a Cartesian surface with neither dissipation nor amplification. Give the equation describing its spatial and temporal evolution. [4 marks]

(f) Compute the Legendre polynomial $P_3(x)$ using the expression:

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

and check that it is consistent with the generic expression of a Legendre polynomial of order n, where n is an odd number. [4 marks]

(g) Recall the conventional symbol for spherical harmonics and give the names of the first, second, third harmonics. What is a quadrupole? Sketch an example. [4 marks]

5. Consider a vibrating string.

   (a) What would be the equation of motion describing the string's oscillations in a plane, with only one polarisation state, if they were undamped with time? [2 marks]

   (b) In this context, what is the simplest interpretation of the following equation (u is a function of time and space and c, k are two constants):

   $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial t}.$$

   [2 marks]

   (c) What is the best method to solve this equation? [2 marks]

   (d) Show that if u vanishes at $x = 0$ and $x = L$ the spatial dependence of the function u is given by $u \propto \sin(A\ x)$ where A is a constant that you need to determine. [6 marks]

   (e) Find the time dependence of u [4 marks]

   (f) Assume that the equation

   $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial t}$$

   is a correct description of the vibrating string. Discuss the form of the time evolution of the vibration as a function of k. [4 marks]

Continued Next Page

6. We want to understand how structure formation is affected by the presence of (almost) massless particles in the Universe, called axions and predicted by string theories. For this purpose we need to solve the following differential equation

$$\ddot{\phi} + \frac{3\,h}{t}\,\dot{\phi} + m^2\phi = 0,$$

which describes how the axion energy density (ϕ) evolves with time t. In this equation m is the axion mass and h a constant. To solve this problem, we will start with a similar (but more generic) equation:

$$x^2y'' + (2p+1)\,x\,y' + (\alpha^2x^{2r} + \beta^2)y = 0$$

where p, α, β, r are constants.

(a) Show that one solution to this equation has the form

$$y = x^{-p}[x^{-q}\sum_n a_n\,x^n].$$

Give the expression for q. [6 marks]

(b) Writing the above solution as $y = x^{-p}\ v$ show that the equation

$$x^2y'' + (2p+1)\,x\,y' + (\alpha^2x^{2r} + \beta^2)y = 0$$

is equivalent to the equation

$$x^2\,v'' + x\,v' + v\ (\alpha^2\,x^{2r} -\ (p^2 - \beta^2)) = 0.$$

[4 marks]

(c) Hence show that the solution of

$$x^2y'' + (2p+1)\,x\,y' + (\alpha^2x^{2r} + \beta^2)y = 0,$$

when $r = 1$ can be written as

$$y = x^{-p}\,(A\,J_q(\alpha\,x) +\ B\,Y_q(\alpha\,x))\,,$$

with A, B two constants and J_q and Y_q are two functions you need to identify. [4 marks]

(d) Hence deduce the solution of the equation

$$\ddot{\phi} + \frac{3\,h}{t}\,\dot{\phi} + m^2\phi = 0.$$

[6 marks]

