

7 Tensor products of Hilbert spaces

7.1 Bipartite and multipartite quantum systems

Example 1

Let us start with a brief study of the simple harmonic oscillator in Quantum Mechanics (a single particle in a quadratic potential well). You are familiar with the Hamiltonian operator for a linear harmonic oscillator of mass m and angular frequency ω ,

$$H_{1D} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2. \quad (7.1)$$

Similarly, for a 3D isotropic oscillator,

$$H_{3D} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2). \quad (7.2)$$

Alternatively, in spherical polar coordinates,

$$H_{3D} = -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m\omega^2 r^2. \quad (7.3)$$

(Isotropic means “the same in every direction”: the potential energy depends only on r , the distance of the particle to the point of equilibrium, not on the polar angles θ and ϕ describing its angular position.) For a 2D isotropic oscillator,

$$H_{2D} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} m\omega^2 (x^2 + y^2). \quad (7.4)$$

We observe that H_{2D} can also be written as the sum of two 1D Hamiltonians, one in x and one in y :

$$H_{2D} = H_{1Dx} + H_{1Dy}, \quad (7.5)$$

where H_{1Dx} and H_{1Dy} are given by the following equations:

$$H_{1Dx} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2, \quad H_{1Dy} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m\omega^2 y^2. \quad (7.6)$$

(We write H_{1Dx} and H_{1Dy} in terms of partial derivatives, contrary to what we did in Eq. (7.1), because we are now dealing with several independent variables. It is customary to write derivatives as total derivatives rather than partial derivatives when there is only one independent variable.)

Since the eigenenergies of a 1D harmonic oscillator are always of the form $\hbar\omega(n + 1/2)$, where n is any non-negative integer, the eigenenergies of H_{1Dx}

and of H_{1Dy} can be written as $\hbar\omega(n+1/2)$ and $\hbar\omega(n'+1/2)$, respectively, with $n, n' = 0, 1, 2, \dots$. Let us denote by $\psi_n(x)$ a normalized eigenfunction of H_{1Dx} with eigenenergy $E_n = \hbar\omega(n+1/2)$ and by $\psi_{n'}(y)$ a normalized eigenfunction of H_{1Dy} with eigenenergy $E_{n'} = \hbar\omega(n'+1/2)$ ($n, n' = 0, 1, 2, \dots$):

$$H_{1Dx}\psi_n(x) = E_n\psi_n(x), \quad H_{1Dy}\psi_{n'}(y) = E_{n'}\psi_{n'}(y). \quad (7.7)$$

The sets $\{\psi_n(x)\}$ and $\{\psi_{n'}(y)\}$ are both orthonormal since we assume that the $\psi_n(x)$'s and $\psi_{n'}(y)$'s are normalized and different values of n or n' correspond to different eigenenergies:

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x) dx = \delta_{nm}, \quad \int_{-\infty}^{\infty} \psi_{n'}^*(y)\psi_{m'}(y) dy = \delta_{n'm'}. \quad (7.8)$$

Since $\psi_n(x)$ is an eigenfunction of H_{1Dx} with eigenenergy E_n and $\psi_{n'}(y)$ is an eigenfunction of H_{1Dy} with eigenenergy $E_{n'}$, the product $\psi_n(x)\psi_{n'}(y)$ is an eigenfunction of H_{2D} with eigenenergy $E_n + E_{n'}$:

$$\begin{aligned} H_{2D}\psi_n(x)\psi_{n'}(y) &= [H_{1Dx}\psi_n(x)]\psi_{n'}(y) + \psi_n(x)[H_{1Dy}\psi_{n'}(y)] \\ &= E_n\psi_n(x)\psi_{n'}(y) + E_{n'}\psi_n(x)\psi_{n'}(y) \\ &= (E_n + E_{n'})\psi_n(x)\psi_{n'}(y). \end{aligned} \quad (7.9)$$

In fact, one can show that any eigenfunction of the Hamiltonian H_{2D} is either a product of the form $\psi_n(x)\psi_{n'}(y)$ or a linear combination of such products.

Given Eq. (7.8), it is easy to see that the products $\psi_n(x)\psi_{n'}(y)$ form an orthonormal set:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\psi_n(x)\psi_{n'}(y)]^* [\psi_m(x)\psi_{m'}(y)] dx dy \\ = \int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x) dx \int_{-\infty}^{\infty} \psi_{n'}^*(y)\psi_{m'}(y) dy = \delta_{nm}\delta_{n'm'}. \end{aligned} \quad (7.10)$$

It is also possible to show that given any square-integrable function $f(x, y)$, there always exists a set of constants $c_{nn'}$ such that

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} c_{nn'} \psi_n(x)\psi_{n'}(y). \quad (7.11)$$

The products $\psi_n(x)\psi_{n'}(y)$ thus form an orthonormal basis spanning the space of the functions square-integrable on the xy -plane. The coefficients $c_{nn'}$ do not depend on x or y but may depend on other variables, e.g., time. In particular, any time-dependent wave function $\Psi(x, y, t)$ can be written as an expansion of form

$$\Psi(x, y, t) = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} c_{nn'}(t) \psi_n(x)\psi_{n'}(y). \quad (7.12)$$

Note what we are doing here: we combine two 1D system into a single 2D system, and write the wave functions of the latter in terms of wave functions of the former.

Now, rather than a single particle confined to the xy -plane, consider two particles confined to the x -axis — i.e., a particle of mass m_A and coordinate x_A and a particle of mass m_B and coordinate x_B . We denote the Hamiltonian of the first particle by H_A and the Hamiltonian of the second particle by H_B , and take

$$H_A = -\frac{\hbar^2}{2m_A} \frac{\partial^2}{\partial x_A^2} + \frac{1}{2} m_A \omega_A^2 x_A^2, \quad (7.13)$$

$$H_B = -\frac{\hbar^2}{2m_B} \frac{\partial^2}{\partial x_B^2} + \frac{1}{2} m_B \omega_B^2 x_B^2. \quad (7.14)$$

It might well be possible to treat these two harmonic oscillators as if they were completely on their own, and, doing this, describe the quantum state of the first one by a certain wave function $\Psi_A(x_A, t)$ and the quantum state of the other by a certain wave function $\Psi_B(x_B, t)$. However, it would be necessary to treat them as forming a single 2-particle system, rather than two 1-particle systems, if they were interacting with each other. Typically, an interaction between the two oscillators would depend on the position of particle B relative to particle A and would be represented by a potential energy term $V(x_A - x_B)$ in the Hamiltonian of the joint system, H_{AB} :

$$H_{AB} = H_A + H_B + V(x_A - x_B). \quad (7.15)$$

Treating the two oscillators as a single system implies that the quantum state of this system is described by a wave function $\Psi(x_A, x_B, t)$ rather than by separate wave functions $\Psi_A(x_A, t)$ and $\Psi_B(x_B, t)$.

At this point, we note that H_{AB} reduces to the sum $H_A + H_B$ in the absence of this interaction. H_{AB} is then mathematically equivalent to the Hamiltonian H_{2D} of Eq. (7.5), apart from a trivial change of notation and the unimportant difference that the mass and angular frequency of oscillators A and B may not be the same. Proceeding as above, we can introduce a complete set of normalized eigenfunctions of H_A and a complete set of normalized eigenfunctions of H_B , respectively $\{\psi_{An}(x_A), n = 0, 1, 2, \dots\}$ and $\{\psi_{Bn'}(x_B), n' = 0, 1, 2, \dots\}$, such that

$$H_A \psi_{An}(x_A) = \hbar \omega_A (n_A + 1/2) \psi_{An}(x_A), \quad (7.16)$$

$$H_B \psi_{Bn'}(x_B) = \hbar \omega_B (n'_B + 1/2) \psi_{Bn'}(x_B), \quad (7.17)$$

We then write the 2-particle wave function $\Psi(x_A, x_B, t)$ as an expansion in products of these 1-particle eigenfunctions:

$$\Psi(x_A, y_B, t) = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} c_{nn'}(t) \psi_{An}(x_A) \psi_{Bn'}(x_B). \quad (7.18)$$

Considering the two harmonic oscillators as a single system may be necessary even if they are not interacting. For example, take the wave function $\psi_{A0}(x_A)\psi_{B1}(x_B)$, which describes a state where the first oscillator is in its ground state and the second in its lowest excited state, and the wave function $\psi_{A1}(x_A)\psi_{B0}(x_B)$, which describes a state where the first oscillator is in its lowest excited state and the second in its ground state. (We do not indicate a dependence on time, here, to keep the notation as simple as possible. This dependence is not important for our discussion. Assume, e.g., that we are considering these wave functions at the instant $t = 0$.) These two wave functions describe possible states of the system formed two non-interacting oscillators. Hence, by the Principle of Superposition, any linear combination of these wave functions [(e.g., $\psi_{A0}(x_A)\psi_{B1}(x_B) + \psi_{A1}(x_A)\psi_{B0}(x_B)$)] also describes a possible state of this system. Such linear combinations link the state of the first oscillator to the state of the second oscillator; therefore they do not describe quantum states in which the state of one oscillator can be treated independently from the state of the other.

Terminology

In relation to this first example:

- These two oscillators, when considered as a single system, are said to form a bipartite quantum system (i.e., a quantum system composed of two distinct parts which can be considered jointly or in isolation, depending on the circumstances). Systems composed of more than two distinct parts are called multipartite systems.
- A quantum state of the joint system described by a wave function which can be factorized in a product of the form $\Psi_A(x_A, t)\Psi_B(x_B, t)$ is called a separable (or product) state (e.g., the state described by the wave function $\psi_{A0}(x_A)\psi_{B1}(x_B)$ is separable).
- Quantum states that are not separable are called entangled states (e.g., the state described by the wave function $\psi_{A0}(x_A)\psi_{B1}(x_B) + \psi_{A1}(x_A)\psi_{B0}(x_B)$ is entangled).

Entangled states vs. separable states

Suppose that the two oscillators of our example do not interact with each other, and that measurements are made on them. E.g., suppose that Alice checks whether oscillator A is in the ground state or in the lowest excited state and that Bob does the same on oscillator B . Clearly, the result will depend on which states A and B are in at the time of these measurements.

First, let us assume that their joint system is in a separable state of wave function $\phi_{AB}(x_A, x_B)$, with

$$\phi_{AB}(x_A, x_B) = \left[\sqrt{\frac{1}{3}} \psi_{A0}(x_A) + \sqrt{\frac{2}{3}} \psi_{A1}(x_A) \right] \left[\sqrt{\frac{2}{5}} \psi_{B0}(x_B) + \sqrt{\frac{3}{5}} \psi_{B1}(x_B) \right] \quad (7.19)$$

$$= \left[\sqrt{\frac{2}{15}} \psi_{A0}(x_A) \psi_{B0}(x_B) + \sqrt{\frac{3}{15}} \psi_{A0}(x_A) \psi_{B1}(x_B) + \right. \\ \left. \left[\sqrt{\frac{4}{15}} \psi_{A1}(x_A) \psi_{B0}(x_B) + \sqrt{\frac{6}{15}} \psi_{A1}(x_A) \psi_{B1}(x_B) \right] \right]. \quad (7.20)$$

(As above, we do not indicate a dependence on time as it is not essential.) We have already seen that the products $\psi_{Ai}(x_A) \psi_{Bj}(x_B)$ are normalized (in fact, they are orthonormal), and it is easy to verify that $\phi_{AB}(x_A, x_B)$ is also normalized. The probability $\text{Pr}(0, 0; \phi_{AB})$ that both Alice and Bob find their oscillator to be in the ground state is given by the Born rule as

$$\text{Pr}(0, 0; \phi_{AB}) = \left| \int_{-\infty}^{\infty} [\psi_{A0}(x_A) \psi_{B0}(x_B)]^* \phi_{AB}(x_A, x_B) dx_A dx_B \right|^2. \quad (7.21)$$

The integral is readily calculated using the orthonormality of the products $\psi_{Ai}(x_A) \psi_{Bj}(x_B)$, with the result that $\text{Pr}(0, 0; \phi_{AB}) = 2/15$. Similarly, there is a probability of 4/15 that Alice finds her oscillator to be in the excited state and Bob finds his to be in the ground state, of 3/15 that Alice finds hers to be in the ground state and Bob finds his to be in the excited state, and of 6/15 that both Alice and Bob find their oscillator to be in the excited state:

$$\begin{aligned} \text{Pr}(0, 0; \phi_{AB}) &= 2/15, & \text{Pr}(1, 0; \phi_{AB}) &= 4/15, \\ \text{Pr}(0, 1; \phi_{AB}) &= 3/15, & \text{Pr}(1, 1; \phi_{AB}) &= 6/15. \end{aligned} \quad (7.22)$$

It is worth noting that there is no correlation between the results found by Alice and those found by Bob: whether Bob finds his oscillator to be in the ground state or in the excited state, the probability that Alice finds hers to be in the ground state is half the probability that she finds it to be in the excited state, and similarly, whatever Alice finds for her oscillator, the probability that Bob finds his to be in the ground state is 2/3 the probability that he finds it to be in the excited state.

We would arrive to the same conclusion for any separable state: in such states, the results of any measurement on A are completely independent from the results of any measurement on B .

Instead, let us now assume that these two oscillators are in an entangled state of normalized wave function $\psi_{AB}(x_A, x_B)$, with

$$\psi_{AB}(x_A, x_B) = \frac{1}{\sqrt{2}} [\psi_{A0}(x_A)\psi_{B1}(x_B) + \psi_{A1}(x_A)\psi_{B0}(x_B)]. \quad (7.23)$$

The probability that both Alice and Bob find their oscillator to be in the ground state is now zero, since $\psi_{A0}(x_A)\psi_{B0}(x_B)$ is orthogonal to both $\psi_{A0}(x_A)\psi_{B1}(x_B)$ and $\psi_{A1}(x_A)\psi_{B0}(x_B)$. Repeating the calculation for the other product functions gives

$$\begin{aligned} \Pr(0, 0; \psi_{AB}) &= 0, & \Pr(1, 0; \psi_{AB}) &= 1/2, \\ \Pr(0, 1; \psi_{AB}) &= 1/2, & \Pr(1, 1; \psi_{AB}) &= 0. \end{aligned} \quad (7.24)$$

Instead of an absence of correlation, we now observe a perfect correlation between the results found by Alice and those found by Bob (or rather, a perfect anticorrelation): either Alice finds “ground” and Bob “excited” or Alice finds “excited” and Bob “ground”. There is a zero probability that they both find “ground” or both find “excited”, hence these two outcomes are impossible and will not occur (assuming, of course, that the measurements are not erroneous).

Imagine that Alice and Bob make their measurement on a large number of pairs of oscillators, all prepared in the state $\psi_{AB}(x_A, x_B)$. They write down the result they obtain for each of the oscillators they measure. They will each find random results, on average half of them “ground” and half of them “excited” (in the same way that you obtain a random distribution of “heads” and “tails” when you toss a coin repeatedly). Should they compare their lists, however, they would not find a single instance where the two oscillators of a same pair were both in the ground state or both in the excited state.

☞ At first sight, the perfect anticorrelation between the results found by Alice and by Bob in the measurements mentioned above is nothing different than what you observe in everyday life. E.g., if Alice and Bob were tossing a coin between them, there would be a perfect anticorrelation between Alice’s wins and Bob’s wins: Bob would lose each time Alice would win and Bob would win each time Alice would lose. This anticorrelation is the same as that found in the measurements on the harmonic oscillators. However, there is a profound difference between the quantum correlations observed in measurements on entangled states and the classical correlations observed in everyday life. This difference does not manifest in the measurements discussed above but may be (and has been) revealed by well chosen experiments.

Saying more about this difference would take us far beyond the scope of the course. However, its conceptual importance can be appreciated from the following. When you are tossing a coin, you take it for granted that one of the two sides faces upwards before you check which one does, and if you find this side to be head then you take it for granted that it was head before you checked. By analogy, you may think that if Alice finds her oscillator to be in the ground state rather than in the excited state, and Bob finds the opposite for his oscillator, this must have been because these two oscillators *were* in these states before they checked them. The alternatives seem bizarre: if they were not in these states, then perhaps the measurement itself would force them to be in these states through some interaction between the two oscillators; however, such an interaction would need to propagate faster than the speed of light since Alice and Bob may be millions of kilometers away. Or perhaps it is not possible to say that the oscillator A is truly separated from oscillator B , however far apart they might be? Every day intuition suggests that instead, if A is found to be in the ground state and B in the excited state, it is because A was in the ground state and B in the excited state immediately before the measurement. However, an in depth discussion of correlations between results of measurements on entangled states shows that this view is untenable: when in the state $\psi_{AB}(x_A, x_B)$, neither of the two oscillators can be assigned a definite state (ground or excited) prior to the measurement. But if this is the case for a quantum object, why wouldn't it also be the case for a coin? Could it be that which of the two sides faces up becomes determined only when someone looks at it? Or instead, could it be that all what the wave functions $\psi_{AB}(x_A, x_B)$, $\psi_{A0}(x_A)$, $\psi_{B1}(x_B)$, etc., refer to is what we can say about the system, irrespective of what may actually be the case?

These issues of interpretation of Quantum Mechanics are deep, difficult and unsettled.

Example 2

The two oscillators of Example 1 are distinct particles and are spatially separated. However, what we have seen above in regards to the entanglement of the states of separated particles also applies to the entanglement of different degrees of freedom of a same particle — e.g., entanglement of its position and its spin, as we will now see.

Recall the Stern-Gerlach experiment: a beam of silver atoms divides into two branches when passing through an inhomogeneous magnetic field \mathbf{B} . The beam split into two because each atom has a magnetic dipole moment $\boldsymbol{\mu}$ and experiences a force $\nabla \boldsymbol{\mu} \cdot \mathbf{B}$ when passing through the field, and because the component of $\boldsymbol{\mu}$ in the direction of \mathbf{B} has only two possible values. As these

two values are spin-dependent, the interaction with the magnetic field couples the spin state of each atom to its spatial wave function. This coupling makes it necessary to consider spin and position together rather than separately (in the same way as an interaction between two oscillators makes it necessary to consider them together rather than separately, i.e., as a single system rather than as two separate systems).

For simplicity, we represent each atom by a mass point of coordinate \mathbf{R} , ignoring its internal structure. We describe its position by a time-dependent wave function $\Psi(\mathbf{R}, t)$ and its spin state by a column vector χ . Let us assume that

$$\chi = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (7.25)$$

(We have seen previously that the two column vectors appearing in the right-hand side describe the states of spin up and of spin down.) Before the atom has entered the magnetic field, $\Psi(\mathbf{R}, t)$ and χ are uncoupled. For simplicity, we take the joint spatial and spin state of the atom to be described by the product $\Psi(\mathbf{R}, t)\chi$, thus by

$$a\Psi(\mathbf{R}, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\Psi(\mathbf{R}, t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The interaction with the magnetic field transforms this state into one of the form

$$a\Psi_\alpha(\mathbf{R}, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\Psi_\beta(\mathbf{R}, t) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where the wave functions multiplying the two spin states now describe distinct distributions of position. Clearly, this transformed space + spin wave function does not describe a separable state since it cannot be written as the product of a wave function depending on \mathbf{R} with a spin state: the interaction with the magnetic field entangles the atom's spatial and spin degrees of freedom.

Suppose that the probability density $|\Psi_\alpha(\mathbf{R}, t)|^2$ is practically zero everywhere except in a certain region \mathcal{A} , and that $|\Psi_\beta(\mathbf{R}, t)|^2$ is practically zero everywhere except in a certain region \mathcal{B} . If these two regions do not overlap, atoms found in the region \mathcal{A} are necessarily in a state of spin up and those found in the region \mathcal{B} are necessarily in a state of spin down. There would be no correlation between spin and position if the joint spatial and spin state of the atom was separable: instead, finding the position of the atom would reveal nothing about its spin. For the above entangled state, measuring the position of an atom is in effect measuring whether the atom is in a state of spin up or a state of spin down.

Example 3

Our last example is the system formed by the two electrons of a helium atom. You will see in the level 3 Quantum Mechanics course that in the ground state of helium, the two electrons are in a joint spin state described by the following combination of column vectors:

$$\psi_{12}^- = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right]. \quad (7.26)$$

Here the subscript attached to each column vector indicates whether this column vector represents a spin state of the first electron or one of the second electron. (The superscript $-$ is traditional for that state and is a reminder of the minus sign in the right-hand side. We neglect spin-orbit coupling here.) Thus ψ_{12}^- is a linear combination of a state in which electron 1 is spin up and electron 2 is spin down with a state in which electron 1 is spin down and electron 2 is spin up. It is not possible to write ψ_{12}^- as a single product of a spin state of electron 1 with a spin state of electron 2; therefore ψ_{12}^- describes an entangled state.

It is important to realize that the products of column vectors appearing in Eq. (7.26) are not dot products or inner products of some kind. They represent *pairs* of column vectors, in which one of these vectors pertain to one part of the system (electron 1) and the other to another part (electron 2). Note the analogy with Eq. (7.23), in which the two products $\psi_{A0}(x_A)\psi_{B1}(x_B)$ and $\psi_{A1}(x_A)\psi_{B0}(x_B)$ also represent states of individual parts of the joint system.

To illustrate this formalism, let us imagine a thought experiment in which you would prepare a pair of electrons in the state ψ_{12}^- and measure whether electron 1 is or is not in a state of spin up and electron 2 is or is not in the spin state represented by the normalized column vector

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}_2.$$

From the Born rule, the probability of finding electron 1 in the state of spin up and electron 2 in that particular spin state is $|(\phi_{12}, \psi_{12}^-)|^2$, the square of the modulus of the inner product of ψ_{12}^- with the vector ϕ_{12} , with

$$\phi_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ i \end{pmatrix}_2. \quad (7.27)$$

We note, at this stage, that it would not make sense to take the inner product of a column vector describing a state of electron 1 with one describing a state of electron 2, not more than in the first example it would have made sense to calculate the inner product of a function of x_A with a function of

x_B . Calculating $|(\phi_{12}, \psi_{12}^-)|^2$ is done by taking the inner products of column vectors pertaining to the *same* electron and combining the results:

$$\begin{aligned} (\phi_{12}, \psi_{12}^-) &= \frac{1}{2} \left(\left[(1 \ 0)_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right] \left[(1 \ -i)_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \right. \\ &\quad \left. - \left[(1 \ 0)_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \left[(1 \ -i)_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right) \\ &= (1 \times (-i) - 0 \times 1)/2 = -i/2. \end{aligned} \quad (7.28)$$

Hence $|(\phi_{12}, \psi_{12}^-)|^2 = 1/4$.

☞ You have come across still another example of bi-partite system in the Term 1 course, although it was not presented as such: the hydrogen atom. Ignoring relativistic effects, the quantum state of an atom of hydrogen-1 can be described by a wave function $\Psi(\mathbf{r}_{\text{pr}}, \mathbf{r}_{\text{el}}, t)$, where \mathbf{r}_{pr} and \mathbf{r}_{el} are, respectively, the position vector of the proton and the position vector of the electron. The Hamiltonian of this 2-particle system is

$$H_{\text{at}} = -\frac{\hbar^2}{2m_{\text{pr}}} \nabla_{\text{pr}}^2 - \frac{\hbar^2}{2m_{\text{el}}} \nabla_{\text{el}}^2 - \frac{e^2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}_{\text{pr}} - \mathbf{r}_{\text{el}}|}, \quad (7.29)$$

with m_{pr} the mass of the proton, m_{el} the mass of the electron, and ∇_{pr}^2 and ∇_{el}^2 the Laplace operators with respect to the coordinates of the proton and those of the electron:

$$\nabla_{\text{pr}}^2 = \frac{\partial^2}{\partial x_{\text{pr}}^2} + \frac{\partial^2}{\partial y_{\text{pr}}^2} + \frac{\partial^2}{\partial z_{\text{pr}}^2}, \quad \nabla_{\text{el}}^2 = \frac{\partial^2}{\partial x_{\text{el}}^2} + \frac{\partial^2}{\partial y_{\text{el}}^2} + \frac{\partial^2}{\partial z_{\text{el}}^2}. \quad (7.30)$$

However, instead of writing the wave functions and the Hamiltonian in terms of the coordinates of these two particles, we can also write them in terms of the coordinates of the centre of mass of the atom and of the coordinates of the electron with respect to the proton. Let us denote by \mathbf{r}_{CM} the position vector of the centre of mass and by \mathbf{r} the position of the electron relative to the proton:

$$\mathbf{r}_{\text{CM}} = \frac{m_{\text{pr}} \mathbf{r}_{\text{pr}} + m_{\text{el}} \mathbf{r}_{\text{el}}}{m_{\text{pr}} + m_{\text{el}}}, \quad \mathbf{r} = \mathbf{r}_{\text{el}} - \mathbf{r}_{\text{pr}}. \quad (7.31)$$

Using \mathbf{r}_{CM} and \mathbf{r} instead of \mathbf{r}_{pr} and \mathbf{r}_{el} is a transformation of the coordinates. It can be shown that this transformation separates H_{at} into the sum of a Hamiltonian H_{CM} depending only on the coordinates of the centre of mass and a Hamiltonian H depending only on the relative coordinates:

$$H_{\text{at}} = H_{\text{CM}} + H \quad (7.32)$$

with

$$H_{\text{CM}} = -\frac{\hbar^2}{2M} \nabla_{\text{CM}}^2, \quad H = -\frac{\hbar^2}{2\mu} \nabla^2 - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r}. \quad (7.33)$$

In these equations, ∇_{CM}^2 and ∇^2 are the Laplace operators with respect to \mathbf{r}_{CM} and to \mathbf{r} , M is the mass of the atom ($M = m_{\text{pr}} + m_{\text{el}}$) and μ is the reduced mass of the electron - proton system,

$$\mu = \frac{m_{\text{pr}} m_{\text{el}}}{m_{\text{pr}} + m_{\text{el}}}. \quad (7.34)$$

The operator H is the Hamiltonian you have studied in Term 1 when you obtained the energy levels of hydrogen. As written in Eq. (7.32), the Hamiltonian of the atom does not contain a term coupling the motion of the electron relative to the nucleus to the centre of mass motion. The atom can thus be in a separable state whose wave function is the product of a function of \mathbf{r}_{CM} and a function of \mathbf{r} . For such states, it makes sense to talk about the eigenenergies and eigenfunctions of the Hamiltonian H without reference to the motion of the atom as a whole. However, this is not the case when the atom is in an entangled state in which its internal state is not independent from its state of motion — e.g., in a state described by a wave function of the form $\Psi_{\text{CM}}(\mathbf{r}_{\text{CM}}, t)\Psi(\mathbf{r}, t) + \Phi_{\text{CM}}(\mathbf{r}_{\text{CM}}, t)\Phi(\mathbf{r}, t)$.

7.2 Tensor products

As shown by the examples discussed in the previous section, it is often the case that the system of interest is formed of distinct parts which need to be considered jointly. Consider a bipartite system consisting of two individual quantum systems, namely system A and system B . Suppose that the quantum states of system A are described by vectors belonging to a certain Hilbert space, \mathcal{H}_A say, and those of system B by vectors belonging to another Hilbert space, \mathcal{H}_B , say. Consider these two systems jointly, as a single quantum system. The quantum states of the joint system are then described by vectors belonging to still another Hilbert space, \mathcal{H}_{AB} , called the tensor product of \mathcal{H}_A and \mathcal{H}_B .

How \mathcal{H}_{AB} is related to \mathcal{H}_A and \mathcal{H}_B is not difficult to understand from the examples of the previous section. We will work with ket vectors here. Consider, for example, a ket vector $|\psi\rangle_A$ representing a state of system A and a ket vector $|\phi\rangle_B$ representing a state of system B . To these two ket vectors we can associate a vector $|\psi\rangle_A|\phi\rangle_B$ which represents a state of the joint system (a product state in which system A is in the state $|\psi\rangle_A$ and system B is in the state $|\phi\rangle_B$).

It should be noted that the symbol $|\psi\rangle_A|\phi\rangle_B$ does not represent a product in the usual sense of the word, even though what it represents is commonly referred to as the product of $|\psi\rangle_A$ and $|\phi\rangle_B$. Properly speaking, it denotes what is called the tensor product of $|\psi\rangle_A$ and $|\phi\rangle_B$. An alternative notation is $|\psi\rangle_A \otimes |\phi\rangle_B$, which makes it clear that we are not talking about an usual product (the symbol \otimes stands for the tensor product). We will use this notation throughout the rest

of this section, for clarity, but not later in the course. Both $|\psi_n\rangle_A|\phi_m\rangle_B$ and $|\psi_n\rangle_A \otimes |\phi_m\rangle_B$ represent the same thing, which is the pair $\{|\psi\rangle_A, |\phi\rangle_B\}$.

☞ As long as it is clear that $|\psi\rangle_A$ and $|\phi\rangle_B$ refer, respectively, to a state of system A and a state of system B , the order in which these two vectors appear in the product does not matter: $|\psi_n\rangle_A \otimes |\phi_m\rangle_B \equiv |\phi_m\rangle_B \otimes |\psi_n\rangle_A$.

In particular, consider an orthonormal set of vectors $\{|\psi_n\rangle_A, n = 1, \dots, N\}$ forming a basis for \mathcal{H}_A and an orthonormal set of vectors $\{|\phi_m\rangle_B, m = 1, \dots, M\}$ forming a basis for \mathcal{H}_B . (We assume that \mathcal{H}_A and \mathcal{H}_B are finite-dimensional here, for simplicity. The theory for infinite-dimensional spaces runs similarly.) Then the set of all the product states $\{|\psi_n\rangle_A \otimes |\phi_m\rangle_B, n = 1, \dots, N, m = 1, \dots, M\}$ forms a basis for the Hilbert space \mathcal{H}_{AB} . In fact, \mathcal{H}_{AB} can be defined as being the vector space spanned by these product states, equipped with the inner product defined below). Since there are $N \times M$ different products $|\psi_n\rangle_A \otimes |\phi_m\rangle_B$ if there are N different vectors $|\psi_n\rangle_A$ and M different vectors $|\phi_m\rangle_B$, \mathcal{H}_{AB} can be of a much larger dimension than \mathcal{H}_A and \mathcal{H}_B : if \mathcal{H}_A is N -dimensional and \mathcal{H}_B is M -dimensional, then \mathcal{H}_{AB} is $(N \times M)$ -dimensional.

Note that the vectors belonging to \mathcal{H}_{AB} include not only these product basis vectors but also all the linear combinations that can be made of them. The terminology introduced in the previous section applies generally: a state of the whole system is called a separable state (or a product state) if it can be represented by the tensor product of a vector of \mathcal{H}_A and a vector of \mathcal{H}_B , and the others are called entangled states.

Inner products and operators

The inner product of two vectors of \mathcal{H}_{AB} is defined in terms of the inner products for \mathcal{H}_A and \mathcal{H}_B : If $|\eta\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$ and $|\eta'\rangle_{AB} = |\psi'\rangle_A \otimes |\phi'\rangle_B$, then the inner product of $|\eta\rangle_{AB}$ and $|\eta'\rangle_{AB}$ is obtained by multiplying the inner product of $|\psi\rangle_A$ and $|\psi'\rangle_A$ by the inner product of $|\phi\rangle_B$ and $|\phi'\rangle_B$:

$${}_{AB}\langle\eta|\eta'\rangle_{AB} = {}_A\langle\psi|\psi'\rangle_A \times {}_B\langle\phi|\phi'\rangle_B. \quad (7.35)$$

(Both ${}_A\langle\psi|\psi'\rangle_A$ and ${}_B\langle\phi|\phi'\rangle_B$ are complex numbers, hence the right-hand side of this equation is the product of two complex numbers.)

For example, take $|\psi_n\rangle_A \otimes |\phi_m\rangle_B$ and $|\psi_{n'}\rangle_A \otimes |\phi_{m'}\rangle_B$, two of the basis vectors formed by the tensor products of the orthonormal vectors $|\psi_n\rangle_A$ with the orthonormal vectors $|\phi_m\rangle_B$. The inner product $|\psi_n\rangle_A \otimes |\phi_m\rangle_B$ and $|\psi_{n'}\rangle_A \otimes |\phi_{m'}\rangle_B$ is ${}_A\langle\psi_n|\psi_{n'}\rangle_{AB}\langle\phi_m|\phi_{m'}\rangle_B$, which is $\delta_{nn'}\delta_{mm'}$ since both the $|\psi_n\rangle_A$'s and the $|\phi_m\rangle_B$'s are orthonormal. Hence the vectors $|\psi_n\rangle_A \otimes |\phi_m\rangle_B$ are also orthonormal.

☞ Since $|\psi\rangle_A$ and $|\phi\rangle_B$ belong to different Hilbert spaces, their inner product is not defined: the symbol ${}_A\langle\psi|\phi\rangle_B$ has no mathematical meaning.

Operators acting on vectors belonging to \mathcal{H}_A or \mathcal{H}_B can also be made to act on the joint states of A and B . More specifically, take the case of an operator \hat{A}_A acting on the ket vector $|\psi\rangle_A$ and an operator \hat{B}_B acting on the ket vector $|\phi\rangle_B$. Then

$$\hat{A}_A|\psi\rangle_A \otimes |\phi\rangle_B = (\hat{A}_A|\psi\rangle_A) \otimes |\phi\rangle_B, \quad (7.36)$$

$$\hat{B}_B|\psi\rangle_A \otimes |\phi\rangle_B = |\psi\rangle_A \otimes (\hat{B}_B|\phi\rangle_B), \quad (7.37)$$

and also

$$(\hat{A}_A + \hat{B}_B)|\psi\rangle_A \otimes |\phi\rangle_B = (\hat{A}_A|\psi\rangle_A) \otimes |\phi\rangle_B + |\psi\rangle_A \otimes (\hat{B}_B|\phi\rangle_B). \quad (7.38)$$

The important point is that when acting on a vector of \mathcal{H}_{AB} , the operator \hat{A}_A acts only the vectors belonging to \mathcal{H}_A and the operator \hat{B}_B acts only the vectors belonging to \mathcal{H}_B .

☞ Tensor products of operators are also used in applications. By definition, the tensor product of \hat{A}_A and \hat{B}_B is the operator $\hat{A}_A \otimes \hat{B}_B$ such that

$$\hat{A}_A \otimes \hat{B}_B |\psi\rangle_A \otimes |\phi\rangle_B = (\hat{A}_A|\psi\rangle_A) \otimes (\hat{B}_B|\phi\rangle_B) \quad (7.39)$$

for any $|\psi\rangle_A$ the operator \hat{A}_A may act on and any $|\phi\rangle_B$ the operator \hat{B}_B may act on. Since \hat{A}_A and \hat{B}_B always act on different vectors, it is clear that $\hat{A}_A \otimes \hat{B}_B \equiv \hat{B}_B \otimes \hat{A}_A$.

For simplicity, the symbol \otimes is usually not specified: in the same way that $|\psi\rangle_A \otimes |\phi\rangle_B$ is often written $|\psi\rangle_A |\phi\rangle_B$, the operator $\hat{A}_A \otimes \hat{B}_B$ is often written $\hat{A}_A \hat{B}_B$.

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