University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 042631/01

LEVEL 2 PHYSICS: THEORETICAL PHYSICS 2

SECTION A. CLASSICAL MECHANICS
SECTION B. QUANTUM THEORY 2

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU = 1.50×10^{11} m
Parsec:	pc = 3.09×10^{16} m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. CLASSICAL MECHANICS

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) How many degrees of freedom are needed to describe the most general motion of most rigid bodies, and what sort of motion do they describe? Why is the limiting case of two point masses connected by a zero-thickness rigid rod an exception? [4 marks]

   (b) State Hamilton's principle, where the action integral is given by

   $$S = \int_{t_i}^{t_f} L dt,$$

   and L is the Lagrangian. How is it related to the Euler-Lagrange equations giving the correct equations of motion? [4 marks]

   (c) The second order linear differential equation describing the motion of a damped oscillator is

   $$\ddot{y} + \frac{\omega}{Q}\dot{y} + \omega^2 y = 0.$$

   Inserting a trial solution, $y = Ae^{\lambda t}$, leads to the auxiliary equation

   $$\lambda^2 + \frac{\omega}{Q}\lambda + \omega^2 = 0.$$

   Describe briefly the different types of solution for the oscillator motion, $y(t)$, making clear the range of quality factor, Q, for which they apply. [4 marks]

   (d) Normal coordinates and normal modes describe the small oscillations of a system of interacting bodies around an equilibrium configuration. Why is it in general important for the oscillations to be considered small? [4 marks]

   (e) How do the Lagrangian and Hamiltonian formulations differ in terms of the variables they use to describe a mechanical system? For a system with N degrees of freedom, how many differential equations and of what type describe the dynamics in the two different formulations? [4 marks]

   (f) Consider a system with one degree of freedom, described in the Hamiltonian formulation of classical mechanics in terms of the coordinate q, and the canonically conjugate momentum p. A canonical transformation is applied, such that the transformed Hamiltonian is described in terms of the transformed coordinate Q, and the transformed momentum P. Explain whether P will be canonically conjugate to Q, and how Poisson brackets may be used to check this. [4 marks]

   (g) Explain how the Coriolis force arises, and give an example of an effect due to the Coriolis force observed on Earth. [4 marks]

   (h) A held-shut book can be reasonably well described as a solid parallelepiped of uniform density. With the aid of a diagram, indicate the orientation of a non-square book's principal axes with respect to its centre of mass. State the principal axis theorem. [4 marks]

2. An Atwood machine consists of two point masses, m_1 and m_2, attached via a massless, inextensible rope into which a massless spring, of spring constant k, is inserted. The rope, which remains tight, is placed over a frictionless, massless pulley of radius r. When the spring is unstretched the combined length of the rope and spring is $l + \pi r$. You may assume that there is a uniform gravitational acceleration, g.

   (a) (i) By defining the zero of height and potential energy to be a distance $l/2$ below the pulley centre, and the extension of the spring as x, write down a constraint equation relating the heights of the two masses, y_1 and y_2, to the extension of the spring. [1 mark]

   (ii) Write down expressions for the kinetic (T) and potential (V) energies of the system using the constraint equation to eliminate x. Hence show that the Lagrangian for the system is

$$L = \frac{1}{2}(m_1\dot{y}_1^2 + m_2\dot{y}_2^2) - (m_1 y_1 + m_2 y_2)g - \frac{k}{2}(y_1 + y_2)^2.$$

   [4 marks]

   (b) Using the Euler-Lagrange equation for a generalised coordinate q:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

   determine second order coupled linear differential equations for the two generalised coordinates y_1 and y_2. [3 marks]

   (c) For the case $m_1 = m_2 = m$, use these coupled equations to determine the equilibrium extension of the spring. [4 marks]

   (d) (i) Use the matrix formulation of the equations of motion

$$(\hat{\upsilon} - \omega^2\hat{\tau})\underline{b} = 0,$$

   where

$$\tau_{jk} = \frac{1}{2}\frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k}\bigg|_{\dot{q}_j,\dot{q}_k=0}, \qquad \upsilon_{jk} = \frac{1}{2}\frac{\partial^2 V}{\partial q_j \partial q_k}\bigg|_{q_j,q_k=0},$$

   and the trial solution is of the form $\underline{q} = \underline{b}e^{i\omega t}$, to find the normal mode frequencies and normalised mode vectors of the system considered in (c). [6 marks]

   (ii) Describe the motions associated with the normal modes. [2 marks]

3. A victorious table tennis player celebrated by throwing their bat into the air. The subsequent rotational motion of the table tennis bat can be understood by approximating it as two connected cylinders - a long, thin cylinder for the handle, and a short, fat cylinder for the blade with which the ball is typically hit - and calculating the inertia tensor of this compound object. Assume that the blade has radius R, height $2h \ll R$ and mass M, while the handle has radius $u \ll R$, height R and mass M. The bat is constructed by attaching one end of the handle to the circular edge of the blade such that the axis of symmetry of the handle passes through the centre of mass of the blade.

   (a) For a uniform density cylinder of height $2h$, radius R and total mass M, calculate the inertia tensor,

   $$I_{\alpha\beta} = \int_{\text{volume}} dxdydz\rho(x,y,z)(|\underline{r}|^2\delta_{\alpha\beta} - r_\alpha r_\beta),$$

   where ρ is the mass density, $|\underline{r}|^2 = x^2 + y^2 + z^2$, and α and β run over the three Cartesian coordinates of vector $\underline{r}$ (i.e. $r_1 \equiv x$, $r_2 \equiv y$ and $r_3 \equiv z$). Show that, with the z axis lined up with the symmetry axis of the cylinder, the inertia tensor about the centre of mass can be written as

   $$\hat{I}_{\text{CoM}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{xx} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix},$$

   where $I_{xx} = M(R^2/4 + h^2/3)$ and $I_{zz} = MR^2/2$. [8 marks]

   For the remainder of the question, use 'body' coordinates such that axis 1 is along the axis of symmetry of the handle and axis 2 is parallel to the symmetry axis of the blade.

   (b) (i) Determine the position of the centre of mass of the bat, $\underline{R}_C$. [2 marks]

   (ii) Using the displaced axis theorem,

   $$\hat{I} = \hat{I}_{\text{CoM}} + M\hat{A},$$

   where $A_{\alpha\beta} = |\underline{R}_C|^2\delta_{\alpha\beta} - R_{C,\alpha}R_{C,\beta}$, determine the inertia tensors for the handle and the blade about the centre of mass of the bat. Hence find the inertia tensor for the bat with respect to rotations about its centre of mass. [5 marks]

   (c) (i) Rank the principal moments of inertia of the bat about the 1, 2 and 3 axes (I_1, I_2 and I_3 respectively) in order of increasing size. [2 marks]

   (ii) The initial angular velocity given to the bat is $\underline{\omega} = (0, \omega_2, \omega_3)$, where $\omega_2 \gg \omega_3 > 0$. Describe the subsequent rotational motion of the bat. [3 marks]

SECTION B. QUANTUM THEORY 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) When are the state vectors $|\phi_1\rangle$, $|\phi_2\rangle$, $\ldots$, $|\phi_n\rangle$ linearly independent? What is meant by the statement that they form an orthonormal set? [4 marks]

   (b) Name the objects which are used to represent physical observables in quantum mechanics. Write down equations which show how these objects can be used to represent (i) the outcome of a single experiment and (ii) the result of a large sample of measurements. [4 marks]

   (c) Write down the form of the time evolution operator for a time-independent Hamiltonian operator. Using this operator, express the solutions of the equation of motion for an arbitrary operator $\hat{A}$ without any explicit time dependence and a state vector $|\psi\rangle$ in the Schrödinger and in the Heisenberg picture. [4 marks]

   (d) Write the Hamiltonian of the one-dimensional harmonic oscillator in terms of creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, and give their commutation relations. [4 marks]

   (e) Why is it in general impossible to measure, at the same time and with no uncertainty, two components of a particle's angular momentum? Evaluate the minimal uncertainty. [4 marks]

   (f) If the operator of the total angular momentum squared $\hat{J}^2$ has the eigenvalue $12\hbar^2$, which eigenvalues are possible for $\hat{J}_z$? [4 marks]

   (g) Calculate the commutators $[\hat{L}_x, \hat{x}]$ and $[\hat{L}_x, \hat{y}]$. [4 marks]

5. Consider the harmonic oscillator in one dimension with Hamiltonian given by

$$\hat{H} = \frac{1}{2m}\,\hat{p}^2 + \hat{V}(x) = \frac{1}{2m}\,\hat{p}^2 + \frac{m\omega^2}{2}\,\hat{x}^2,$$

where m is the mass of the oscillator, ω is the frequency of oscillation, $\hat{p}$ is the momentum operator, $\hat{x}$ is displacement and $\hat{V}(x)$ is the potential.

(a) Express this Hamiltonian through the number operator $\hat{N}$. Why has $\hat{N}$ only real, non-negative eigenvalues? [4 marks]

What is the connection of the creation and annihilation operators with the position and momentum operators? [2 marks]

(b) In the following use the commutation properties between the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ and the number operator. If $|n\rangle$ is an energy eigenket with eigenvalue $E_n = \hbar\omega(n + 1/2)$, show that $\hat{a}|n\rangle$ and $\hat{a}^\dagger|n\rangle$ are eigenkets of $\hat{H}$. To which energy eigenvalues do these kets belong? [4 marks]

(c) Show, by explicit calculation, that

$$\phi(q) = \alpha\,(2q^2 - 1)\,\exp\left(-\frac{q^2}{2}\right),$$

where $q = x\sqrt{m\omega/\hbar}$, is an eigenfunction of the Hamiltonian above. [4 marks]
What is the energy eigenvalue corresponding to $\phi(x)$? [2 marks]

[Hint: Transform the variables in the Hamiltonian from $\hat{x}$ to $\hat{q}$ and from $\hat{p}_x$ to $\hat{p}_q$ and use the differential form of the momentum operator to calculate $\hat{H}\phi(q)$.]

(d) Using the findings of part (a), verify that the expectation values of the kinetic and potential energy in energy eigenstates of the harmonic oscillator satisfy

$$\langle n|\hat{T}|n\rangle = \langle n|\hat{V}|n\rangle\,,$$

by rewriting the kinetic energy, $\hat{T}$, and potential energy, $\hat{V}$, in terms of the creation and annihilation operators and their action on the energy eigenstates. [4 marks]

6. Consider a rotor with Hamiltonian

$$\hat{H} = \frac{1}{2J}\,\hat{L}^2.$$

(a) Give the commutator relations between the angular momentum operators $\hat{L}_{x,y,z}$ themselves and with $\hat{L}^2$. [2 marks]

(b) What are the eigenvalues and joint eigenfunctions of the operators $\hat{L}^2$ and $\hat{L}_z$? [2 marks]

(c) Write down the eigenvalues and eigenfunctions of the Hamiltonian above. Discuss potential degeneracies. [4 marks]

(d) At a given time $t_0 = 0$ the rotor is in the state

$$|\psi\rangle = \alpha\left[\cos^2\theta + \sin^2\theta\cos(2\phi)\right],$$

where α is a normalisation constant.

Express the state above through spherical harmonics and fix α through the normalisation requirement on $|\psi\rangle$. [6 marks]

(e) In the state $|\psi\rangle$ in (d), what are the probabilities for energy measurements to yield the values $3\hbar^2/J$, $\hbar^2/J$, and 0? [4 marks]

(f) What is the probability for a simultaneous measurement of $\hat{L}_z$ and $\hat{L}^2$ to result in $\{-2\hbar,\, 6\hbar^2\}$ if the system is in the state $|\psi\rangle$ of (d)? [2 marks]

Hint: At various stages, you will need some of the relations and definitions listed below.
Explicit representations:

$$\begin{aligned}
Y_{00}(\theta,\,\phi) &= \sqrt{\frac{1}{4\pi}}\\
Y_{1\pm1}(\theta,\,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{\pm i\phi}\\
Y_{10}(\theta,\,\phi) &= \sqrt{\frac{3}{4\pi}}\,\cos\theta\\
Y_{2\pm2}(\theta,\,\phi) &= \sqrt{\frac{15}{32\pi}}\,\sin^2\theta e^{\pm 2i\phi}\\
Y_{2\pm1}(\theta,\,\phi) &= \mp\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta e^{\pm i\phi}\\
Y_{20}(\theta,\,\phi) &= \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)
\end{aligned}$$

Normalisation:

$$\int_{-1}^{1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi Y_{lm}(\theta,\,\phi)Y^*_{l'm'}(\theta,\,\phi) = \delta_{ll'}\delta_{mm'}$$