University of Durham

EXAMINATION PAPER

May/June 2011 Examination code: 043541/01 or 044131/01

LEVEL 3 PHYSICS: ASTROPHYSICS
LEVEL 4 PHYSICS: ASTROPHYSICS 4

SECTION A. PLANETARY SYSTEMS
SECTION B. COSMOLOGY

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron mass:	$m_e = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_p = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Bohr magneton:	$\mu_B = 9.27 \times 10^{-24}\,\mathrm{J\,T^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Nuclear magneton:	$\mu_N = 5.05 \times 10^{-27}\,\mathrm{J\,T^{-1}}$
Molar Gas Constant:	$R = 8.31 \times 10^{3}\,\mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's Constant:	$N_A = 6.02 \times 10^{26}\,\mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\,\mathrm{m\,s^{-2}}$
Stefan-Boltzmann Constant:	$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$

SECTION A. PLANETARY SYSTEMS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Draw a schematic diagram of the Solar System marking clearly the Sun and the locations of the Jovian and terrestrial planets, the Kuiper belt and the main asteroid belt. [4 marks]

   (b) A spacecraft in a circular orbit at an altitude of 200 km above the Earth, fires a rocket briefly, increasing its orbital velocity impulsively by 2500 m s^{-1} along its orbit as seen in the Earth's rest frame. Calculate the semi-major axis of the resulting Hohmann transfer orbit. [4 marks]

   [The radius of the Earth is 6371 km and its mass is 5.98×10^{24} kg.]

   (c) Explain with the aid of two diagrams how a gravity assist works, taking as an example a spacecraft sent from the inner solar system past Jupiter. [4 marks]

   (d) Modelling the Moon as a uniform density sphere, and assuming hydrostatic equilbrium, calculate the pressure at the centre of the Moon. [4 marks]

   [The radius of the Moon is 1738 km and its mass is 7.36×10^{22} kg.]

   (e) Taking the radius of the Earth's core as 2971 km, and the conductivity of the molten core as 7×10^5 Ohm^{-1} m^{-1}, give a reasoned estimate of the decay time of the Earth's magnetic field, were the dynamo to stop working. Give the answer in years. [4 marks]

   (f) An exoplanet with an albedo of 0.31 orbits a K dwarf star with a luminosity of a half of the Sun, on a circular orbit of radius 6.1×10^7 km. Estimate the equilibrium blackbody temperature of the planet. [4 marks]

   [The luminosity of the Sun is 3.9×10^{26} W.]

   (g) Estimate for a distant observer how long the Earth takes to transit the Sun across the centre of the disk. Ignore the size of the Earth. Give the answer in hours to three significant figures. [4 marks]

   [The radius of the Sun is 6.96×10^8 m, 1AU = 1.50×10^{11} m.]

2. (a) Write down the equation for Kepler's third law and define the meaning of each symbol.

Two unequal masses orbit each other in a mildly eccentric Keplerian orbit. Draw a diagram to show the orbital paths of each of the masses in the frame where the centre of mass is fixed. Mark on the diagram the simultaneous positions of the two masses at an instant approximately half-way in time between the nearest and furthest approaches of the two bodies to each other. [6 marks]

(b) Starting from the polar coordinate equation for an ellipse, $r = a(1-e^2)/(1+e\cos\theta)$, where a is the semi-major axis, e the eccentricity, show using Kepler's second law that the angular speed ω of a planet about the Sun, is given by

$$\omega = \omega_p(1 + e\cos\theta)^2(1 - e^2)^{-3/2},$$

where w_p is the average angular speed of the planet about the Sun. [6 marks]

[The area of an ellipse with semi-major axis a, and eccentricity e, is $\pi a^2\sqrt{1-e^2}$.]

(c) Mercury orbits the Sun in 87.9 days, and rotates about an axis perpendicular to its orbit with a period which is exactly two-thirds of the orbital period, and in the same sense as the orbit. The eccentricity, e, of Mercury's orbit is 0.2056.

Seen from the surface of Mercury, the Sun appears to move in the opposite direction in the sky from usual when Mercury is close to the Sun near perihelion. Use: (i) the Kepler equation, $M = E - e\sin E$, where M and E are the mean and eccentric anomalies; (ii) the relation $(1-e)^{1/2}\tan(\theta/2) = (1+e)^{1/2}\tan(E/2)$, where θ is the true anomaly; and (iii) the result from part (b); to determine for how long the Sun appears to move backwards around the time of perihelion for an observer on Mercury's surface. Give an answer in days accurate to two significant figures. State any assumptions. [8 marks]

3. (a) The approximate densities of many extrasolar planets are known. Explain how these densities were obtained, giving a brief description of the type of observations required. In addition to the observational errors, what are the main sources of uncertainty in the determination of these densities? [6 marks]

   (b) Taking the density of a small Earth-like exoplanet to be 3450 kg m^{-3}, and modelling it as consisting of at most three materials: iron, silicate rocks and water, with densities of 7500 kg m^{-3}, 3000 kg m^{-3} and 1000 kg m^{-3} respectively, deduce the minimum and maximum amounts of water, expressed as a fraction of the total mass of the planet. Assume the planet is no more iron rich than the Earth, where the mass in iron is half that of silicate rocks. Also assume all three materials to be incompressible. [6 marks]

   (c) Mercury has a much higher iron to silicate rock ratio than the Earth. Taking the moment of inertia factor for Mercury to be 0.3305 ± 0.0005, show using the data from part (b), that a silicate shell of thickness exactly 0.3 Mercurian radii outside of a pure iron core, gives the correct moment of inertia factor to within the errors. [6 marks]

   [The moment of inertia of a spherical mass distribution, radius R, is given by: $I = (8\pi/3) \int_0^R r^4 \rho(r)\, dr$, where $\rho(r)$ is density.]

   (d) The Earth has a moment of inertia factor that is almost identical to that of Mercury, and a mean density very close to the mean density of Mercury. Explain how this is possible given their significantly different compositions. [2 marks]

4. (a) With the aid of a diagram show the locations of the five Lagrange points with respect to two unequal masses orbiting each other in an orbit of zero eccentricity. Clearly mark on the diagram which is the greater and lesser of the two masses. Define what a Lagrange point is, and give an example to illustrate the relevance of Lagrange points to the dynamics of the Solar System. [6 marks]

   (b) A pair of satellites, each with a mass of 1000 kg are positioned at relative rest and close to each other and colinear with the Earth, which is 1.60×10^6 km away. Starting from the equation for the Roche radius r for a liquid drop of density ρ about a spherical planet of radius R and density ρ_p, given by

$$r = 2.44 \left(\frac{\rho_p}{\rho}\right)^{1/3} R,$$

   make a reasoned estimate for how far apart the satellites can be placed without being tidally pulled apart by the Earth (the Moon's small tidal field can be ignored). For the purposes of this estimate, consider the pair of satellites to be part of a cloud of equally spaced satellites in all three coordinate directions. State any assumptions. [6 marks]

   [Mass of the Earth is 5.98×10^{24} kg.]

   (c) The satellite pair are sufficiently far from the Earth that the tidal force from the Sun is also significant. Use Kepler's third law together with the definition of tidal force to calculate the maximum tidal force acting on the satellite pair. Give the answer for the strength of the tidal field in relative terms comparing to the case in part (b) where only the Earth was considered. State the relative positions of the satellites and the Earth with respect to the Earth-Sun line for the case when the combined tidal forces acting on the satellites due to the Earth and Sun is minimised. [8 marks]

SECTION B. COSMOLOGY

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) Give the definition of redshift.

   A spectrograph is being designed to study Hα emission from galaxies in the redshift range 2.5-3.5. The rest frame wavelength of Hα is 656 nm. What wavelength range must the spectrograph be sensitive to? [4 marks]

   (b) Give the definitions of the following terms:

   (i) the critical density;

   (ii) the density parameter;

   (iii) comoving distance;

   (iv) look-back time.

   [4 marks]

   (c) Starting from the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

   where a is the expansion factor, k is a constant and ρ is the density, show that

$$k = \frac{H_0^2}{c^2}(\Omega_0 - 1),$$

   where H_0 and Ω_0 are the present-day values of the Hubble parameter and density parameter. [4 marks]

(d) Assuming that there are three species of neutrino and that the temperature of the cosmic neutrino background is 2 K, what is the present-day energy density of neutrinos?

If the present-day density of non-relativistic dark matter $\rho_{\rm DM}$ is $2 \times 10^{-27}\,{\rm kg\,m^{-3}}$, at what redshift is the energy density in neutrinos equal to that in dark matter? [4 marks]

[Hint: the energy density of a thermal distribution of relativistic particles is $2g_*\sigma T^4/c$, where T is their temperature, g_* is the effective number of bosonic degrees of freedom and σ is the Stefan-Boltzmann constant.]

(e) Assume the universe is flat, with $\Omega_m = 1$, such that the comoving distance, r, is related to redshift z, by

$$r = \frac{2c}{H_0}\left(1 - (1+z)^{-1/2}\right)$$

where c is the speed of light and H_0 the present-day Hubble parameter. Assuming that $H_0 = 75\,{\rm km\,s^{-1}\,Mpc^{-1}}$, calculate the flux from a galaxy with luminosity $L = 10^{10} L_\odot$ observed at redshift $z = 1.4$. [4 marks]

[The solar luminosity, $L_\odot = 3.85 \times 10^{26}$ W; 1 Mpc $= 3.09 \times 10^{22}$ m.]

(f) State two problems with the standard Big Bang model for the origin of the universe, and briefly explain why a period of cosmic inflation may offer an explanation. [4 marks]

(g) State the definition of the deceleration parameter, q_0, in terms of the expansion factor, a, and its time derivatives.

Evaluate q_0 for the case of a critical density, matter dominated universe for which Friedmann's equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{a^3}.$$

[4 marks]

(h) The energy density of the early Universe is dominated by radiation and other relativistic particles, and is very close to the critical density. In addition, you may assume that the effective number of bosonic degrees of freedom g_* is constant.

By integrating the Friedmann equation, show that the age of the Universe, t, is given by

$$t = \kappa\, T^{-2}$$

where T is the temperature, and determine an expression for the constant κ. [4 marks]

6. (a) Briefly describe the steps required to use X-ray observations of clusters of galaxies in order to measure the matter density of the universe. [8 marks]

   (b) Small groups of galaxies also emit X-rays and can also be used to measure the mass density. The density profile of gas in a galaxy group can be approximated by:

$$\rho_g(r) = \rho_{g,0}\left(\frac{r_0}{r}\right)$$

   where $r_0 = 30\,\mathrm{kpc}$ and $\rho_{g,0} = 3 \times 10^{-23}\,\mathrm{kg\,m^{-3}}$. You may assume that the gas is all at temperature $T_0 = 3 \times 10^7\,\mathrm{K}$, and that the X-ray emission can be detected out to $r_{\mathrm{max}} = 100\,\mathrm{kpc}$.

   Use this information to estimate the total mass of the group within r_{max}. [6 marks]

   Calculate the mass of X-ray emitting gas within r_{max} and show that this is less than 10% of the total mass. [3 marks]

   [Hint: the X-ray emitting gas has an equation of state, $P = \rho_g k_{\mathrm{B}} T/\mu m_p$, where $\mu = 0.59$; $1\,\mathrm{pc} = 3.09 \times 10^{16}\,\mathrm{m}$.]

   (c) A recent determination of cosmological parameters from the WMAP satellite gives:

$$\Omega_{b,0} = 0.0455$$

$$\Omega_{m,0} = 0.272$$

   Compare the estimate from galaxy groups with those from WMAP, and suggest one way to account for the discrepancy. [3 marks]

7. (a) Explain how type Ia supernovae can be used to measure the geometry of the universe. Your answer should include an explanation of (i) the progenitors of type Ia supernovae, (ii) why they are good "standard candles", (iii) why the geometry of the universe affects the apparent brightness of supernovae, and (iv) give an outline of the recent results. [8 marks]

(b) Starting from the Acceleration Equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho + \frac{3P}{c^2}\right],$$

where a is the expansion factor, ρ is the density and P is the pressure, show that for a universe containing *only* radiation and a vacuum energy, the deceleration parameter,

$$q_0 = \Omega_{\gamma,0} - \Omega_{\rm vac,0}$$

where $\Omega_{\gamma,0}$ and $\Omega_{\rm vac,0}$ are the present-day density parameters of radiation and the vacuum energy density. [6 marks]

[Hint: the vacuum energy density has an associated pressure, $P_{\rm vac} = -\rho_{\rm vac}c^2$.]

(c) Consider a hypothetical universe that is flat (i.e., $k = 0$, where k is the spatial curvature parameter) contains only radiation and a vacuum energy, and has $q_0 = -0.5$. Determine $\Omega_{\gamma,0}$ and $\Omega_{\rm vac,0}$.

By considering the redshift dependence of $\Omega_{\rm vac}$, determine the redshift at which $\Omega_{\rm vac} = 0.1$. [6 marks]

8. (a) Briefly explain how the following three pieces of observational evidence support the *hot* big bang model as an explanation for the origin of the universe.

   (i) the Hubble flow of galaxies.

   (ii) the existence of a microwave background with a black-body spectrum.

   (iii) the primordial abundance of Helium and other light elements.

   [10 marks]

   (b) The production of super-symmetric particles has many similarities with the production of neutrons during the nucleosynthesis era. We will consider the production of a heavy super-symmetric particle, X, through the quark–antiquark reactions:

$$q + \bar{q} \iff X + \bar{X}$$

   Assume that the q particle is effectively massless and that the mass of X is $100\,\mathrm{GeV\,c^{-2}}$ (you may also assume that both are spin-$\frac{1}{2}$ particles). When the temperature, T, is sufficiently high, the reaction timescale is short compared to the expansion timescale of the Universe and the particles are in thermal equilibrium. Write down the abundance of X particles relative to q particles. [2 marks]

   Assuming that the reaction timescale, τ_X, is

$$\tau_X = G_X^{-1} \left(\frac{k_\mathrm{B} T}{1\,\mathrm{GeV}} \right)^{-5} \mathrm{s},$$

   where G_X is a dimensionless constant, show that the reaction will stop when the abundance of X particles relative to q particles is approximately

$$\frac{N_X}{N_q} = \exp\left(-G_X^{1/3}\right)$$

   [6 marks]

   If the ratio freezes out at $N_X/N_q = 10^{-12}$, use this information to estimate the constant G_X. [2 marks]

   [Hint: you may assume that the Universe is radiation dominated and has the critical density, so that the Hubble parameter is

$$H = 10^6 \left(\frac{k_\mathrm{B} T}{1\,\mathrm{GeV}} \right)^2 \mathrm{s}^{-1}$$

   at the temperatures relevant to this problem.]