University of Durham

EXAMINATION PAPER

May/June 2015 Examination code: PHYS2581WE01

FOUNDATIONS OF PHYSICS 2A

SECTION A. Quantum Mechanics 2
SECTION B. Electromagnetism

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU = 1.50×10^{11} m
Parsec:	pc = 3.09×10^{16} m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A. QUANTUM MECHANICS 2

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Sketch (without any derivation) the potential and the first three bound states (assuming these all exist) of the one dimensional systems below:

   i) infinite square well, $V = 0$ for $-a < x < a$, $V = \infty$ elsewhere

   ii) finite square well, $V = 0$ for $-a < x < a$, $V = V_0$ elsewhere

   iii) inverse gaussian, $V = V_0(1 - e^{-x^2/2a^2})$ for $-\infty < x < \infty$

   Are there infinite numbers of bound states in all these potentials? Explain your answer. [4 marks]

   (b) An electron in an infinite square well where $V = 0$ for $0 < x < 1$ has an initial wave function $\psi(x, 0) = Ax(1 - x)$. Find A. [4 marks]

   (c) An electron is in a superposition state of energy eigenfunctions ψ_n each with energy $E_n = n^2 E_1$ such that $\psi = (\psi_1 + 3\psi_3 + 5\psi_5)/\sqrt{35}$. What is the probability that a measurement of energy gives the value E_5? Write down a sum for $\langle E\rangle$ in terms of E_1 and evaluate it. Can any single measurement of energy give $\langle E\rangle$? Explain your answer. [4 marks]

   (d) Write down the expression for the momentum operator, p, in one dimension. Hence show that $[H, p]\psi = i\hbar \frac{dV}{dx}\psi$ where $H = p^2/2m+V(x)$. [4 marks]

   (e) For any operator Q, the Schroedinger equation implies

   $$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle [H, Q]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

   Use this, together with (d) above, to calculate $d\langle p\rangle/dt$. Relate your answer to the expectation from classical mechanics for the rate of change of momentum (Ehrenfest theorem). [4 marks]

   [Hint: these operators do not depend explicitly on time so $\partial Q/\partial t = 0$.]

   (f) The ground state of hydrogen has wavefunction $\psi_{100}(r\theta\phi) = (\pi a^3)^{-1/2} e^{-r/a}$ where all the symbols have their usual meanings. Derive an expression for $\langle r\rangle$. Compare this with the most probable value of r. [4 marks]

   $$\left[\text{Hint} : \int_0^\infty x^p e^{-qx} dx = \frac{p!}{q^{p+1}}\right]$$

   (g) The three dimensional infinite square well with $V = 0$ for $0 < x < L$, $0 < y < L$, $0 < z < L$ has energies

   $$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$$

   Write down the energies of the first 6 energy levels and give the degeneracy of each level. [4 marks]

   (h) The one dimensional harmonic oscillator has a ground state wavefunction $\psi_0 = (\alpha/\pi)^{1/4} e^{-\alpha x^2/2}$ with energy $E_0 = \hbar\omega/2$. This is perturbed by $H' = V_0\delta(x - 0)$. Calculate the first order correction to the energy from non-degenerate perturbation theory $E_0^1 =< \psi_0|H'\psi_0 >$. Is degeneracy generally seen in one dimensional systems? Explain your answer in terms of how degeneracy can be produced in two dimensional systems. [4 marks]

2. The common eigenfunctions of the angular momentum operators L^2 and L_z are the spherical harmonics, Y_{lm}, such that $L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$ and $L_z Y_{lm} = m\hbar Y_{lm}$.

   The associated ladder operators are $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$, where $L_\pm Y_{lm} = A_{lm} Y_{lm\pm 1}$ and $A_{lm} = \hbar\sqrt{l(l+1) - m(m \pm 1)}$

   (a) Use the definitions above to explicitly show that $[L^2, L_z]Y_{lm} = 0$. If $L^2 Y_{lm}$ is measured to be $30\hbar^2$, what are the possible values of a measurement of L_z? [3 marks]

   (b) Show that $L_x = \frac{1}{2}(L_+ + L_-)$. Use this, together with the orthonormal properties of wavefunctions, to calculate $\langle L_x \rangle$ for any spherical harmonic Y_{lm}. [3 marks]

   (c) The eigenfunctions of L_x are $\psi = aY_{11} + bY_{10} + cY_{1-1}$. Use the definitions of the ladder operators to calculate $L_-\psi$ and $L_+\psi$. Hence find the values of the coefficients a, b, c in order that $L_x\psi = \frac{1}{2}(L_+ + L_-)\psi = q\hbar\psi$. Solve this explictly to find the normalised eigenfunctions of L_x for $q = 1, 0, -1$. [11 marks]

   (d) The ground state wavefunction of hydrogen is $\psi_{100} = R_{10}Y_{00}$ where R_{nl} is the radial wavefunction. What is the result of a measurement of L_z? What is the result of a measurement of L_x? Is this deterministic? Explain why this is a special case using the definition of $L^2 = L_x^2 + L_y^2 + L_z^2$. [3 marks]

3. The $n = 2$ level in hydrogen is degenerate with multiple levels giving the same energy, E^0. This is perturbed by an external electric field, $E_{ext} \propto \epsilon$, along the z-axis (Stark effect) giving $H' \propto \epsilon$. In degenerate first order perturbation theory this only affects $\psi^0_{210} = \psi^0_1$ and $\psi^0_{200} = \psi^0_2$. The first order correction to the energy, E^1, is given by the solution of a matrix equation in $W_{ij} =< \psi^0_i | H' \psi^0_j >$ for $i, j = 1, 2$, operating on a wavefunction $\psi = \alpha\psi^0_1 + \beta\psi^0_2$. In this particular case it gives

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & -\epsilon \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E^1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

(a) Solve the equation to find E^1 in terms of ϵ, and hence estimate the perturbed energies. What are the corresponding wavefunctions which follow the perturbation? [5 marks]

(b) The two levels, ψ^0_1 and ψ^0_2, are not quite degenerate in the unperturbed hydrogen atom due to fine structure introducing a small splitting, $\Delta \ll E^0$, such that $H^0\psi^0_1 = E^0_1\psi^0_1 = (E^0 + \Delta)\psi^0_1$ and $H^0\psi^0_2 = E^0_2\psi^0_2 = (E^0 - \Delta)\psi^0_2$ (where we neglect the small shift in E^0 caused by the fine structure). For this quasi-degenerate case, non-degenerate perturbation theory shows that the perturbed wavefunctions can be approximated as $\psi_i = \alpha\psi^0_1 + \beta\psi^0_2$. The total Hamiltonian $(H^0 + H')\psi_i = E_i\psi_i$ where E_i are the perturbed energies and it is assumed that $\epsilon \ll E^0$. Set $i = 1$, multiply by ψ^{0*}_j and integrate over all space using Dirac notation or otherwise. Show that this gives

$$\alpha E^0_1 \delta_{j1} + \beta E^0_2 \delta_{j2} + \alpha W_{j1} + \beta W_{j2} = E_1 \alpha \delta_{j1} + E_1 \beta \delta_{j2}.$$

where $\delta_{jn} = 1$ for $n = j$ and 0 otherwise. [6 marks]

(c) Evaluate the equation above for $j = 1$ and 2, substituting for W_{ij} from (a) and E^0_i from (b). Solve the resulting two simultaneous equations for E_1. There should be two solutions as choosing $i = 2$ leads to exactly the same simultaneous equations except for E_2. Hence write down E_2. [9 marks]

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SECTION B. ELECTROMAGNETISM

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) The general dispersion relation for an electromagnetic wave propagating in an infinitely large, non-magnetic, conducting medium is given by

$$k^2 = \mu_0 \epsilon \omega^2 + i\omega\mu_0\sigma_N,$$

where k is the wavevector, ω is the angular frequency, σ_N is the electrical conductivity and $\epsilon = \epsilon_r\epsilon_0$ where ϵ_r is the relative permittivity. A medium has a relative permittivity of 20 and an electrical conductivity of $2 \times 10^8\ \Omega^{-1}\mathrm{m}^{-1}$. Can this medium be considered a good conductor at a frequency of 10^{11} Hz? [4 marks]

(b) Briefly describe how a radio aerial works. [4 marks]

(c) A point charge of 5 μC is located at the position $\underline{r}$ where $\underline{r} = 2\hat{\underline{i}} + 3\hat{\underline{j}} + 4\hat{\underline{k}}$ and $\hat{\underline{i}}$, $\hat{\underline{j}}$ and $\hat{\underline{k}}$ are unit vectors in the x, y and z-directions respectively and $\underline{r}$ is in metres. A second charge of 7 μC is located at $\underline{r} = 2\hat{\underline{i}} + 3\hat{\underline{j}} + 8\hat{\underline{k}}$. Calculate the force on the 5 μC charge. [4 marks]

(d) Briefly describe what waveguides are and give two examples of their use. [4 marks]

(e) A very long cylinder of magnetic material has a magnetic field of 1 T applied along its axis by a magnet. This produces a net field along the axis of the material of 6 T. Calculate the susceptibility of the material. [4 marks]

(f) Briefly describe how the transmission and reflection coefficients associated with an electromagnetic wave crossing the boundary between two media are derived. [4 marks]

(g) Write down the Maxwell equation that describes the magnetic field produced by a flowing current and a changing electric field. Explain how this equation can be tested experimentally. [4 marks]

5. (a) Write down the definition of the Poynting vector. [2 marks]

(b) An electromagnetic wave, with magnetic and electric fields given by

$$\underline{B} = \underline{B}_0 \exp[i(\underline{k} \cdot \underline{r} - \omega t)], \ \underline{E} = \underline{E}_0 \exp[i(\underline{k} \cdot \underline{r} - \omega t)],$$

where $\underline{k}$, ω, $\underline{B}_0$ and $\underline{E}_0$ are constants, is travelling through an isotropic dielectric with a relative permittivity of 100. Show, using Maxwell's equations, that the direction of propagation of the wave, the magnetic field and the electric field are all orthogonal to each other. [4 marks]

(c) The Poynting vector of the wave described above points in the $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ direction, $\underline{B}_0$ has a magnitude of 3 μT and points in the $\begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}$ direction, and ω is 3×10^{15} rad s^{-1}. Calculate the vectors $\underline{k}$, $\underline{B}_0$ and $\underline{E}_0$. [8 marks]

(d) Calculate the Poynting vector for the wave considered above. [3 marks]

(e) A scientist fires the wave described above at a sheet of material that completely absorbs it. Calculate the heat generated in the material per second given that the projected area of the material in the direction of propagation is 5 m^2. [3 marks]

6. (a) Consider a long insulating solid rod with a circular cross-sectional area of radius a, and a charge density ρ. Show that the spatial dependence of the electric field produced outside the rod is given by

$$\underline{E} = \frac{\rho a^2}{2\epsilon_0 r}\hat{r}.$$

[4 marks]

(b) Although the outer surface of the charged rod shows no sign of porosity, the rod manufacturer thinks the rod has a few small isolated spherical cavities embedded in it which have a radius b and are off-centred from the axis of the rod by c, where $a > b + c$. Given the rod lies in the horizontal direction, find an expression for the spatial dependence of the electric field outside the rod, vertically above one small isolated cavity that lies vertically above the axis. [6 marks]

(c) A scientist is trying to detect the small cavities by measuring the electric field at the surface of the rod along its length. Find an expression for the maximum change in the electric field that the scientist can expect to measure due to one isolated cavity that can be anywhere in the rod. [5 marks]

(d) Whenever the scientist thinks he has detected a cavity, he stops measuring the rod along its length and measures the electric field above the cavity while he continuously and slowly rotates the rod about its axis. Calculate the maximum percentage variation in the electric field, with respect to the maximum value, the scientist can expect at the surface of the rod while it rotates, in the limit that the cavities are very small. [5 marks]

