Mathematical Methods II
Lecture 17

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Key Points

- General solutions to PDEs
- 2^{nd} order PDE with boundary conditions
- Inhomogeneous 2^{nd} order PDEs

General Solutions to PDEs (ctud)

- **2nd order PDE with boundary conditions:** As before, we need to know how to apply boundary conditions to a problem to ensure it fits the physical description of the system we're studying.

  e.g. 17.1 2nd order PDE with boundary conditions: Solve

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

  subject to the boundary conditions $u(0, y) = 0$ and $u(x, 1) = x^2$.

  Starting by identifying the three coefficients, A=1, B=2, C=1, we construct the quadratic equation,

$$1 + 2\lambda + \lambda^2 = 0$$

$$(1 + \lambda)(1 + \lambda) = 0$$

$$\lambda = -1$$

  We have a repeated root, so using the same trick we applied to ODEs to ensure linear indepedence, we'll multiply one of our solution by x, giving a general solution

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

$$u(x, y) = f(x - y) + xg(x - y).$$

  Now we'll consider the boundary conditions. This time both are to be applied together, unlike previous examples where they represented independent scenarios.

(i) $\underline{u(0,y) = 0}$: Since we begin with the assumption that $u(x,y) = f(p)$, this implies $u(0,y) = f(p) = 0$. Since the $g(p)$ term is multiplied by $x = 0$ and it is possible that $f \neq g$, it may be that $g(p) \neq 0$. So our solution becomes

$$u(x,y) = xg(x-y).$$

(ii) $\underline{u(x,1) = x^2}$: This requires that

$$u(x,1) = xg(x-1) = x^2$$

$$g(x-1) = x$$

$$g(z) = z+1 \quad \Rightarrow \quad g(p) = p+1$$

where z is p with the BCs applied. Hence the particular solution, accounting for the BCs is

$$u(x,y) = xg(p) = x(x-y+1)$$

- **2nd order inhomogeneous PDE:** Finally, let's look at a problem with $RHS \neq 0$.

e.g. 17.2 2nd order inhomogeneous PDE: Find the general solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x+y).$$

Starting by identifying the three coefficients, $A = 1$, $B = 0$, $C = 1$, we construct the quadratic equation,

$$1 + \lambda^2 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

Hence the complementary function is

$$u(x,y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

$$u(x,y) = f(x+iy) + g(x-iy).$$

As with other simple particular integrals, we can find this by inspection. Any solution that satisfies the RHS will do. Consider that the RHS has an x term and a y term, and we're dealing with 2^{nd} order derivatives, thus logically we should start with cubed terms. Trying $u(x,y) = (x^3 + y^3)$ gives

$$u_x = 3x^2 \qquad\qquad u_y = 3y^2$$

$$u_{xx} = 6x \qquad\qquad u_{yy} = 6y$$

$$u_{xx} + u_{yy} = 6(x+y)$$

Hence, the particular solution satisfies the equation. An alternative approach is to integrate the RHS terms independently, a number of times equal to the order of the derivatives on the LHS, in this case twice

$$\int\int 6x dx dx = \int 3x^2 + c_1 dx = x^3 + c_1 x + c_2$$

$$\int\int 6y dy dy = \int 3y^2 + d_1 dy = y^3 + d_1 y + d_2$$

These two expressions could be added together to form our particular solution, but by examining the LHS of our PDE we can see that the first order terms in x and y and the constant terms are not required (they will disappear when the derivatives of our particular solution are evaluated); they can be discarded, leaving us with $x^3 + y^3$ as we found by inspection.

Hence, the general solution is

$$u(x, y) = f(x + iy) + g(x - iy) + x^3 + y^3.$$

- **Extra examples:** The following are extra examples of finding the general solution to PDEs.

  e.g. 17.3 2nd order inhomogeneous PDE (with differential BC): Solve

  $$6u_{xx} - 5u_{xy} + u_{yy} = 14$$

  subject to $u = 2x + 1$ and $u_y = 4 - 6x$, both on the line $y = 0$.

  Identifying the coefficients as $A = 6$, $B = -5$, $C = 1$, we solve the homogeneous problem

  $$6 - 5\lambda + \lambda^2 = 0$$

  $$(\lambda - 2)(\lambda - 3) = 0$$

  $$\lambda = 2, 3$$

  Hence the complementary function is given by

  $$u(x, y) = f(x + 2y) + g(x + 3y).$$

  Examining the RHS we note that we're likely looking for a particular solution with a term in x^2 (since u_{xx} will give constant term) and y^2 (since u_{yy} will give a constant term). So try $u = x^2 + y^2$. We find

  $$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y$$

  $$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 2$$

  Subbing in,

  $$6(2) - 5(0) + 2 = 14$$

The solution works. So our general solution is given by

$$u(x,y) = f\,(x+2y) + g\,(x+3y) + x^2 + y^2.$$

Now let's consider our BCs. Firstly, $u = 2x+1$ at $y = 0$ gives

$$u(x,0) = f(x) + g(x) + x^2 = 2x + 1$$

Note that at $y = 0$, the arbitrary functions f and g are only functions of x. We are going to differentiate this solution w.r.t x

$$\frac{d}{dx}u(x,0) = f'(x) + g'(x) + 2x = 2,$$

because we'll need this when applying our second BC, which involves a bit more work since it's a differential BC. Before applying the second BC let's consider our y derivative,

$$\frac{\partial u}{\partial y} = \frac{df}{dp_1}\frac{\partial p_1}{\partial y} + \frac{dg}{dp_2}\frac{\partial p_2}{\partial y} + 2y,$$

$$\frac{\partial u}{\partial y} = 2\frac{df}{dp_1} + 3\frac{dg}{dp_2} + 2y,$$

But using our first BC we established that f and g are functions of x alone at $y = 0$, i.e. $p_1 = p_2 = x$, so

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 2\frac{df}{dx} + 3\frac{dg}{dx} = 2f'(x) + 3g'(x) = 4 - 6x,$$

Now we have two equations, and two unknown functions, so we can solve them simultaneously

$$f'(x) + g'(x) + 2x = 2, \qquad\qquad 2f'(x) + 3g'(x) = 4 - 6x.$$

Rearranging our 1$^{\text{st}}$ equation

$$f'(x) = 2 - 2x - g'(x).$$

Subbing this into the 2$^{\text{nd}}$ equation

$$2[2 - 2x - g'(x)] + 3g'(x) = 4 - 4x + g'(x) = 4 - 6x,$$

$$g'(x) = -2x.$$

Hence

$$f'(x) = 2 - 2x - x = 2.$$

Integrating g

$$g'(x) = -2x,$$

$$g(x) = -x^2 + c.$$

Using our expression for $u(x,0)$

$$u(x,0) = f(x) + g(x) + x^2 = f(x) - x^2 + c + x^2 = 2x + 1$$

$$f(x) = 2x + 1 - c$$

Looking at the function arguments, where $z_n = p_n|_{y=0}$

$$f(z_1) = 2z_2 + 1 - c \qquad\qquad g(z_2) = -z_2^2 + c$$
$$f(x+2y) = 2(x+2y) + 1 - c \qquad\qquad g(x+3y) = -(x+3y)^2 + c$$

So, putting it all together for the particular solution at the BCs

$$\begin{aligned} u(x,y) &= f\,(x+2y) + g\,(x+3y) + x^2 + y^2 \\ &= [2(x+2y) + 1 - c] + [-(x+3y)^2 + c] + x^2 + y^2 \\ &= 2x + 4y + 1 - c - 9y^2 - 6xy - x^2 + c + x^2 + y^2 \\ &= 1 + 2x + 4y - 6xy - 8y^2 \end{aligned}$$

e.g. 17.4 2nd order inhomogeneous PDE with BCs: Solve the equation

$$ck\frac{\partial^2 u}{\partial x^2} + (c^2+k^2)\frac{\partial^2 u}{\partial x \partial y} + ck\frac{\partial^2 u}{\partial y^2} = 6ck(2x^2 + y)$$

given that $u(x,y) = 2 + (c/k)^2([c/k] - 1)$ where the line $x = 1$ interects the line $y = c/k$. Identifying the coefficients as $A = ck$, $B = c^2 + k^2$, $C = ck$, we solve the homogeneous problem

$$ck + (c^2+k^2)\lambda + ck\lambda^2 = 0$$
$$(c\lambda + k)(k\lambda + c) = 0$$
$$\lambda = -\frac{k}{c}, -\frac{c}{k}$$

Hence the complementary function is given by

$$u(x,y) = f\left(x - \frac{k}{c}y\right) + g\left(x - \frac{c}{k}y\right).$$

Examining the RHS we note that we're likely looking for a particular solution with a term in x^4 (since u_{xx} will give a term in x^2) and y^3 (since u_{yy} will give a term in y). So try $u = x^4 + y^3$. We find

$$\frac{\partial u}{\partial x} = 4x^3 \qquad\qquad \frac{\partial u}{\partial y} = 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 \qquad\qquad \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad\qquad \frac{\partial^2 u}{\partial y^2} = 6y$$

Subbing in,

$$ck(12x^2) + (c^2+k^2)(0) + ck(6y) = 6ck(2x^2 + y)$$

The solution works. So our general solution is given by

$$u(x,y) = f\left(x - \frac{k}{c}y\right) + g\left(x - \frac{c}{k}y\right) + x^4 + y^3.$$

To apply our boundary conditions consider that

$$f\left(x-\frac{k}{c}y\right)\bigg|_{x=1,y=c/k}=f(0)=f(p_1)$$

A function of zero can be (though is not guaranteed to be) equal to zero. So if we assume $f(0)=0$ then we can write

$$u(x,y)=f(p_1)+g\left(x-\frac{c}{k}y\right)+x^4+y^3.$$

where $f(p_1)=0$ when $p_1=0$. Now, since

$$g\left(x-\frac{c}{k}y\right)\bigg|_{x=1,y=c/k}=g\left(1-\frac{c^2}{k^2}\right)$$

we can try

$$g(p_2)=1-\frac{c^2}{k^2}.$$

This gives

$$u(x,y)|_{x=1,y=c/k}=\left(1-\frac{c^2}{k^2}\right)+1^4+\frac{c^3}{k^3}=2+\left(\frac{c}{k}\right)^2\left(\frac{c}{k}-1\right)$$

which obeys the boundary conditions. Hence the most general solution that accommodates the boundary conditions is given by

$$u(x,y)=x-\frac{c}{k}y+x^4+y^3+f(p_1)$$

$$u(x,y)=x(x^3+1)+y\left(y^2-\frac{c}{k}\right)+f(p_1)$$

where $f(p_1)=0$ when $p_1=0$.

e.g. 17.5 MM2 May 2018 Q4(f): Consider the partial differential equation

$$\frac{\partial u}{\partial x}+2x\frac{\partial u}{\partial y}=0.$$

Find the most general solution such that $u=5$ on the parabola $y=x^2$. [4 marks] Start by assuming a solution of u(x,y)=f(p) where p=p(x,y). Identify A=1, B=2x.

$$\frac{dx}{1}=\frac{dy}{2x}$$

$$\int 2x dx=\int dy$$

$$x^2=y+c$$

$$c=x^2-y$$

So $p = c = x^2 - y$. Hence we have a solution

$$u(x, y) = f(x^2 - y)$$

Applying the boundary condition we require $u = 5$ at $y = x^2$, hence since

$$f(x^2 - y)|_{y=x^2} = f(0) = 5,$$

we obtain the solution

$$u(x, y) = g(x^2 - y) + 5$$

where the function g is subject to the condition $g(0) = 0$.