Foundations of Physics 2B/3C 2018/2019

Thermodynamics – Summer Examinations 2019

Short Questions, Final

(Parts of answer in **bold** what is expected from students, rest explanations)

a) The total derivative of G is

$$\boldsymbol{dG = dU + mdB + Bdm - TdS - SdT}.$$

Substituting for the first law, with appropriate work, $dU = TdS - mdB$, gives

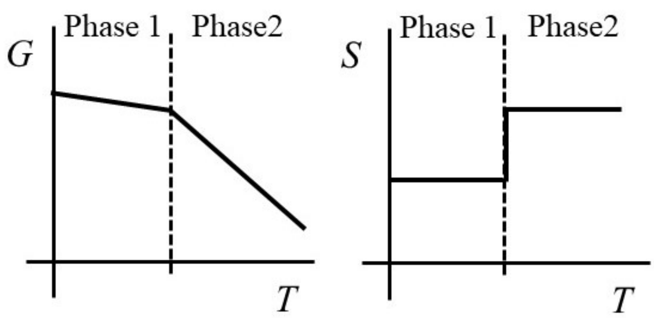
$$\boldsymbol{dG = Bdm - SdT}.$$

[1 mark – application]

From the above, we find $\boldsymbol{-S = \left(\frac{\partial G}{\partial T}\right)_m}$**,** so if there is an entropy change with phase change we have,

$$\boldsymbol{S_f - S_i = -\left[\left(\frac{\partial G_f}{\partial T}\right)_m - \left(\frac{\partial G_i}{\partial T}\right)_m\right]}.$$

In other words, the derivative of G is discontinuous. [1 mark – application]



Plots are shown left, and as we are told that **increasing temperature causes the magnetic moment alignment to break down**, we have **entropy increasing with temperature.**

[2 marks – application and analysis]

b) The first condition arises because the **particle number is constant,** $\boldsymbol{N = \sum_j n_j}$, so changing the distribution of particles between the states doesn't affect their total number.
The second condition arises because the **system has a fixed total energy,** $\boldsymbol{E = \sum_j \varepsilon_j n_j}$. Each state has fixed energy, so the derivative is as given in (ii)
The final condition comes from the fact that **at equilibrium, the system is in the most probable macrostate.**

[2 marks – comprehension]

Using Langrange multipliers (when three terms are zero, so must be their sum) and then simplifying the equation, because it must hold for each state, j, so the term in brackets must be identically zero

$$\boldsymbol{\alpha \sum_j dn_j + \beta \sum_j \varepsilon_j dn_j + \sum_j (\ln n_j) dn_j = 0 \Rightarrow \sum_j dn_j\, (\alpha + \beta\varepsilon_j + \ln n_j) = 0}.$$

So, $(\alpha + \beta\varepsilon_j + \ln n_j) = 0$. Rearranging gives,

$$\boldsymbol{n_j = \exp(-\alpha - \beta e_j) = A \exp(-\beta\varepsilon_j)},$$

which is none other than Boltzmann statistics, with $\beta = 1/k_BT$.

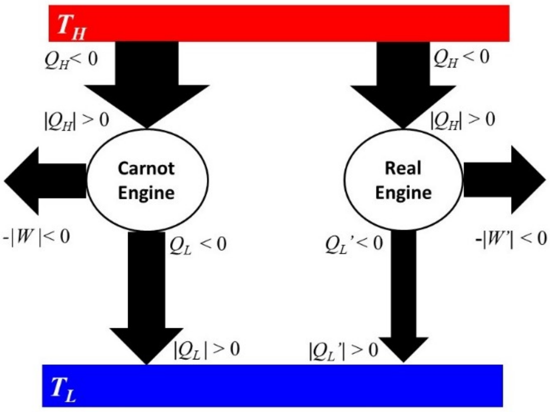
[2 marks – comprehension]

c) For a Carnot cycle, we have from the first law, $\boldsymbol{Q_H - |Q_L| = -W}$. Also, the Carnot relation is

$$\boldsymbol{\frac{|Q_L|}{Q_H} = \frac{T_L}{T_H}} \quad \Rightarrow \quad \frac{Q_H}{T_H} - \frac{|Q_L|}{T_L} = 0$$

$$\Rightarrow \quad \boldsymbol{\frac{Q_H}{T_H} + \frac{Q_L}{T_L} = 0},$$

since heat $Q_L < 0$ is rejected by the engine it. The above is none other than $\oint \frac{\delta Q}{T} = 0$ for the cycle.



[1 mark – application]

For the real engine, it still operates in a cycle having $dU = 0 \Rightarrow \delta Q = \delta W'$. If it operates between the two given temperatures, but does less work than a Carnot (reversible) engine, we have, $|\boldsymbol{W'}| < |\boldsymbol{W}|$, and more heat rejected. So,

$$\boldsymbol{Q_H - |Q_L'| = W'} \text{ **but** } \boldsymbol{|Q_L'| > |Q_L|} \quad \Rightarrow \quad \boldsymbol{|W'| < |W|}.$$

[2 marks – application]

Comparing with above, if $\boldsymbol{Q_H, T_H}$ **and** $\boldsymbol{T_L}$ remain unchanged

$$\boldsymbol{\frac{Q_H}{T} + \frac{|Q_L'|}{T_L} < 0}.$$

In other words, $\oint \delta Q/T \leq 0$. The Clausius inequality holds! [1 mark – application]

d) Upon marking it was discovered that the question was ambiguous – it wasn't explicitly clear whether the blocks came into thermal equilibrium with themselves whilst isolated, or into thermal equilibrium with the environment. Therefore both methods were given credit.

The energy lost by the hotter block is gained by the colder block as the equilibrium temperature, T_E is attained. Using heat capacities$-\int_{T_H}^{T_E} \boldsymbol{mc_1 dT} = \int_{T_L}^{T_E} \boldsymbol{mc_2 dT} \quad \Rightarrow$ $\boldsymbol{-c_1(T_E - T_H) = c_2(T_E - T_L)}$

$$T_E(c_1 + c_2) = c_1T_H + c_2T_L \quad \Rightarrow$$

$$T_E = \frac{500c_1 + 300c_2}{c_1 + c_2} = \frac{(5 * 500 + 300)c_2}{6c_2} = \mathbf{467\ K}$$

[1 mark – application]

Entropy change is given by $\Delta S = \int \delta Q/T$. For each block we can use the heat capacity to represent the differential heats, $\delta Q_1 = mc_1dT$

$$\boldsymbol{\Delta S_H = \int_{T_H}^{T_E} \frac{mc_1 dT}{T} = mc_1 \ln\left(\frac{T_E}{T_H}\right) = \ln\left(\frac{467}{500}\right) = -0.0690 mc_1 \,\mathrm{J\,K^{-1}}}.$$

$$\boldsymbol{\Delta S_L} = \int_{T_L}^{T_E} \frac{m_2 c_2 dT}{T} = mc_2 \ln\left(\frac{467}{300}\right) = \boldsymbol{0.442 mc_2 \,\mathrm{J\,K^{-1}}}.$$

(Temperatures must be in Kelvin). The Universe entropy change is thus $\boldsymbol{\Delta S_U = \Delta S_H + \Delta S_L = 0.0969\, mc_2 \mathrm{J\,K^{-1}}}$.

[2 marks – application]

Now $I = T_0 \Delta S_u$, hence

$$\boldsymbol{T_0 * 0.0969 mc_2 = 24.2 mc_2 \quad \Rightarrow \quad T_0 = 250\ \mathrm{K}.}$$

[1 mark – application]

Or

Entropy change is given by $\Delta S = \int \delta Q/T$, and the environment is a reservoir at T_E. The overall entropy change is given by

$$\Delta S_U = \int_{T_H}^{T_E} \frac{mc_1 dT}{T} + \int_{T_L}^{T_E} \frac{mc_2 dT}{T} + \Delta S_{Air}$$

[1 mark]

The air's entropy change is $\Delta S_{Air} = \frac{\Delta Q_{Air}}{T_E}$, where ΔQ_{Air} is the energy taken in by the air as the blocks come into equilibrium (So equal and opposite to the block's energy losses/gains)

$$\Delta Q_{Air} = -(mc_1(T_E - 500) + mc_2(T_E - 300))$$

[1 mark]

Now $I = T_E \Delta S_u$, hence

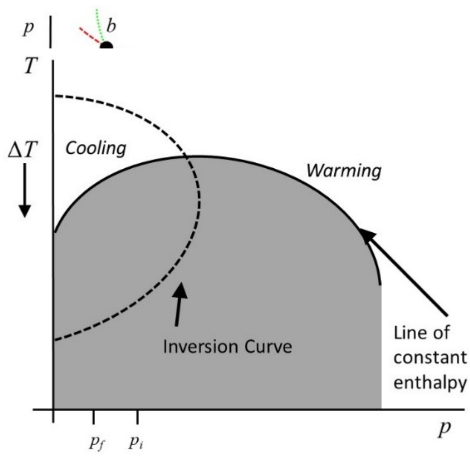
$$24.2 mc_2 = T_E \left[5mc_2 \ln\left(\frac{T_E}{500}\right) + mc_2 \ln\left(\frac{T_E}{300}\right) + \frac{mc_2}{T_E}(2800 - 6T_E)\right] T_E$$

$$24.2 = T_E \ln\left(\frac{T_E^6}{500^5 \times 300}\right) + 2800 - 6T_E.$$

[2 marks]

This could in principal be solved iteratively. If this is done, we find the minimum irreversibility is 44.8, when the two blocks overall entropy change is zero, i.e. $T_E = \sqrt[6]{500^5 \times 300} = 459$ K.

e) **Adiabatic cooling** begins with an **isothermal compression**, (ab) removing heat from the gas. As the gas is then **adiabatically expanded** (bd) **it comes back to the starting pressure** at point c but **because adiatbats are steeper than isotherms**, this new point at p_0 is a is **on a lower temperature isotherm**. The process repeats, but as $T \to 0$, the isotherms get



closer together, so cooling becomes more difficult.

[2 marks – comprehension]

When ***Joule-Kelvin*** cooling is used at constant enthalpy, . In the expansion, $p_i \to p_f < p_i$, and on any given line of constant enthalpy, we see the curve decreases below the inversion point, so $T_f < T_i$ here, leading to cooling.

[2 marks – comprehension]

Foundations of Physics 2B/3C 2018/2019

Thermodynamics – Summer Examinations 2019

Long Question, Final

(Parts of answer in **bold** what is expected from students, rest explanations

a) The derivative is

$$\boldsymbol{\left(\frac{\partial p}{\partial V}\right)_T = -\frac{RT}{(V-b)^2} + \frac{2a}{V^3}} = \frac{2a(V-b)^2 - RTV^3}{V^3(V-b)^2}.$$

[2 marks - application]

$$\boldsymbol{\kappa_T = -\frac{1}{V} \times \frac{V^3(V-b)^2}{2a(V-b)^2 - RTV^3} = \frac{V^2(V-b)^2}{RTV^3 - 2a(V-b)^2}}$$

[2 marks – application]

At the critical point

$$\boldsymbol{\left(\frac{\partial p}{\partial V}\right)_T = 0 \quad \Rightarrow \quad RT = \frac{2a(V-b)^2}{V^3}}.$$

$$\boldsymbol{\left(\frac{\partial^2 p}{\partial V^2}\right)_T = \frac{2RT}{(V-b)^3} - \frac{6a}{V^4} = 0 \quad \Rightarrow \quad RT = \frac{3a(V-b)^3}{V^4}}.$$

Eliminating RT above two expressions, will give us V_c

$$\frac{3a(V-b)^3}{V^4} = \frac{2a(V-b)^2}{V^3} \quad \Rightarrow \quad \boldsymbol{\frac{3}{2}(V-b) = V \quad \Rightarrow \quad V_C = 3b.}$$

[2 marks - application]

At $\boldsymbol{V_n = \frac{2}{3}, V_c = 2b}$,

$$\boldsymbol{\kappa_T = \frac{4b^2 \times b^2}{R \times (8a/27Rb) \times 8b^3 - 2a \times b^2}} = \frac{4b^4}{\frac{64ab^2}{27} - 2ab^2} = \boldsymbol{\frac{4b^2}{10a/27}}.$$

Positive as expected, so on critical isotherm must have $\left(\frac{\partial p}{\partial V}\right)_T < 0$ and hence increasing the pressure leads to a reduction in volume in line with expectations. (It is for temperatures lower than this that an increase in pressure leads to an increase in volume causing $\left(\frac{\partial p}{\partial V}\right)_T > 0$ and hence a negative compressibility!)

As $V \to V_c$

$$\boldsymbol{\lim_{V \to V_c} \kappa_T = \frac{9b^2 \times 4b^2}{R \times (8a/27Rb) \times 27b^3 - 2a \times 4b^2} = \frac{36b^4}{8ab^2 - 8ab^2}} \to \infty.$$

[2 marks - analysis]

b) The total differential of the Helmholtz function is, and using the first law, $dU = TdS - pdV$

$$\boldsymbol{dF = TdS - pdV - TdS - SdT = -pdV - SdT}.$$

The pressure is given by $\boldsymbol{p = -\left(\frac{\partial F}{\partial V}\right)_T}$, so hence

$$\left(\frac{\partial F}{\partial V}\right)_T = -\frac{RT}{(V-b)} + \frac{a}{V^2}$$

$$\boldsymbol{p = \frac{RT}{V-b} - \frac{a}{V^2}},$$

and the above could represent the Helmholtz function of the van der Waals gas.

[2 marks – application]

From above $p = -\left(\frac{\partial F}{\partial V}\right)_T$. Hence

$$\boldsymbol{G = F - V(-p)} = U - TS + pV.$$

[1 mark – application]

$$\boldsymbol{G = -RT\ln(V-b) - \frac{a}{V} + f(T) - V\left(-\frac{RT}{V-b} + \frac{a}{V^2}\right)}$$

$$\boldsymbol{= -RT\ln(V-b) + \frac{VRT}{(V-b)} - \frac{2a}{V} + f(T).}$$

[2 marks – application]

c) At **constant entropy**, we have

$$\boldsymbol{-C_V dT = T\left(\frac{\partial p}{\partial T}\right)_V dV}$$

[1 mark – application]

Now, for the virial expansion

$$\boldsymbol{\left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V} + \frac{bR}{V^2} + \frac{cR}{V^3}}$$

$$-C_V dT = T\left(\frac{R}{V} + \frac{bR}{V^2} + \frac{cR}{V^3}\right) dV$$

[1 mark – application]

Separating variables and integrating,

$$\boldsymbol{-\frac{C_V}{R}\int \frac{1}{T}dT = \int\left(\frac{1}{V} + \frac{b}{V^2} + \frac{c}{V^3}\right)dV} \quad \Rightarrow \quad -\frac{C_V}{R}\ln T + \text{const} = \ln V - \frac{b}{V} - \frac{c}{2V^2}$$

$$\textbf{Konst}\ \boldsymbol{T^{-C_{V/R}} = \exp\left(\ln V - \frac{b}{V} - \frac{c}{2V^2}\right).}$$

[2 marks – application]

As V becomes large, we find that $\boldsymbol{\ln V}$ **will tend to zero less rapidly than** $\boldsymbol{1/V, 1/V^2}$, so we can approximate the right hand side as

$$\textbf{Konst}\ \boldsymbol{T^{-C_V/R} = \exp(\ln V)}$$

[1 mark – analysis]

Hence, for large V we have that $\boldsymbol{V = \textbf{Konst}\ T^{-C_V/R}}$**,** which can be rewritten as $V^{R/C_V}T = k$, where k is a further constant. Now

$$\boldsymbol{\frac{R}{C_V} = \frac{C_p - C_V}{C_V} = \gamma - 1.}$$

[1 mark – analysis]

For an **ideal gas, an adiabatic has** $\boldsymbol{pV^\gamma = \textbf{const}}$, and since $pV = RT$, this can be written $\boldsymbol{RTV^{\gamma-1} = \textbf{const}.}$ Therefore, for our Virial expansion, when the **volume is small, the adiabatic lines approach those of an ideal gas.** [1 mark – analysis]

L2 Foundation 2B Optics 2019-20
Short questions

(a) The electric field amplitude of two light fields can be written as $\mathcal{E}_1 = \mathcal{E}_0 e^{i(k_x x + k_z z)}$ and $\mathcal{E}_2 = \mathcal{E}_0 e^{i(-k_x x + k_z z)}$, respectively. Derive an expression for the total intensity given by $\mathcal{I} = \frac{1}{2} c\epsilon_0 |\mathcal{E}_1 + \mathcal{E}_2|^2$. Write your answer in terms of the intensity of each field individually, $\mathcal{I}_0 = \frac{1}{2} c\epsilon_0 \mathcal{E}_0^2$. [4 marks]

[APPLICATION]
$$\begin{aligned} \mathcal{I} &= \tfrac{1}{2} c\epsilon_0 |\mathcal{E}_1 + \mathcal{E}_2|^2 = \tfrac{1}{2} c\epsilon_0 \mathcal{E}_0^2 |e^{i(k_x x + k_z z)} + e^{i(-k_x x + k_z z)}|^2 \ ,^{[1]} \\ &= \tfrac{1}{2} c\epsilon_0 \mathcal{E}_0^{2\,[1]} |2\cos k_x x|^{2\,[1]} = \mathcal{I}_0 4\cos^2 k_x x \ .^{[1]} \end{aligned}$$

(b) An opaque screen containing four small holes in a line along the horizontal axis is illuminated normally by monochromatic light. The far-field intensity along the horizontal axis is found to contain regions with high and low intensity. Draw phasors diagrams corresponding to two distinct positions where the intensity is observed to be zero. [4 marks] [CONCEPT] See Fig. 1 below.

(c) The electric field along the x axis in the $z = 0$ plane for a paraxial spherical wave propagating along the z axis with source point at $(0, 0, -f)$ is $\mathcal{E} = [\mathcal{E}_0/(ikf)]e^{ikx^2/2f}$. A thin lens with focal length f is placed in the $z = 0$ plane. Derive an expression for the field in a plane a distance z downstream of the lens. State any assumptions you make. [4 marks] [ANALYSIS] The effect of the lens is to imprint a phase $e^{-ikx^2/2f}$ so immediately after the lens we have,

$$\mathcal{E} = \frac{\mathcal{E}_0}{ikf} e^{ikx^2/2f} e^{-ikx^2/2f\,[1]} = \frac{\mathcal{E}_0}{ikf}{}^{[1]} \ ,$$

which is a plane wave. At a distance z, $\mathcal{E} = [\mathcal{E}_0/(ikf)]e^{ikz}$.[1] The assumption is that the lens has infinite spatial extent.[1]

(d) Calculate the Rayleigh range of a laser guide star with wavelength $\lambda = 589$ nm, and initial beam size, on Earth, of $w_0 = 1.00$ m. Estimate the size of the laser guide star in the Atmospheric sodium layer 90 km above the Earth's surface. [4 marks] [APPLICATION] The Rayleigh range is given by

$$z_R = \frac{\pi w_0^2}{\lambda} = \frac{3.14 \cdot 1.00^2}{589 \times 10^{-9}} = 5.33 \times 10^{6\,[1]} \mathrm{m}^{[1]} \ .$$

As the distance to the sodium layer is much less than the Rayleigh range the laser beam size is approximately the same,[1] i.e. the guide star will have a transverse side of approximately 1 m.[1]

(e) Sketch the Fraunhofer diffraction pattern for an aperture with the shape of the letter X. [4 marks] [APPLICATION] See Fig. 1(e).

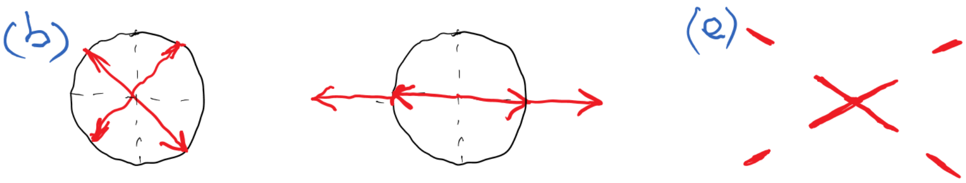


Figure 1: (b) 4 phasors[1] in both cases.[1] In one case they form a cross[1] in the other they form two opposed pairs.[1] (e) Sketch of far-field diffraction pattern for letter X. sinc-squared[1] with cos-squared fringes[1] in horizontal and sinc-squared[1] at angle of about 30°.[1]

L2 Foundation 2B Optics 2019-20
Long question: Solution

(a) (i) Draw a right-handed coordinate system with the z axis going from upper left to lower right. [2 marks]

[APPLICATION] If y is vertical[1] then x point into page, see Fig. 1(a).[1]

(ii) On your sketch add the electric vectors for right-circularly polarized light at $t = 0$ and positions $z = 0$, $\lambda/4$, $\lambda/2$, $3\lambda/4$, and λ. [4 marks]

[APPLICATION] If we begin at $z = 0$ with $\boldsymbol{\mathcal{E}}$ then at $\lambda/4$ it is along $+y$,[1] then $-x$ at $\lambda/2$,[1] $-y$ at $3\lambda/4$,[1] and x at λ.[1]

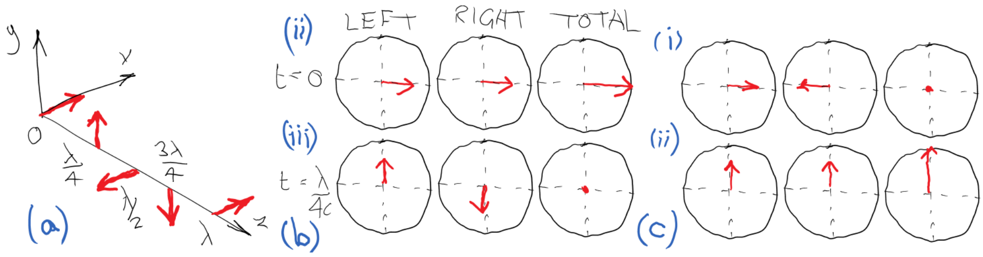


Figure 1: Sketches for parts (a) (i) and (ii), (b) (ii) and (iii) and (c) (i) and (ii).

(b) A linearly polarized laser beam propagating along z enters a sugar solution at $z = 0$.

(i) If no light scattering is observed along the x axis at $z = 0$, which is the initial polarization state of the laser? [2 marks] [APPLICATION] Linearly polarized[1] along x.[1]

(ii) Draw a sketch to show the left and right-handed components of the polarization vector at $z = 0$ for $t = 0$. [2 marks] [ANALYSIS] Both[1] left and right-handed components point in $+x$ direction.[1] The total field is also shown for information. See Fig. 1(b).

(iii) Repeat the above sketch at $z = 0$ for $t = \lambda/(4c)$. [3 marks]

[ANALYSIS] $t = \lambda/(4c)$ corresponds to $\omega t = \pi/2$[1] so left and right vectors have rotated by $+\pi/2$ (anti-clockwise) and $-\pi/2$ (clockwise), respectively. Left-hand component points in $+y$.[1] Right-handed components point in $-y$ direction.[1]

(c) At a distance $z = \Lambda/2$, no scattering is observed along the y axis.

(i) Sketch the left- and right-hand components at $z = 0$ for $t = 0$. Assume that $n_{\rm L}k\Lambda/2 = 2m\pi$, where $N_{\rm L}$ is the refractive index for left-circular polarized light and m is an integer. [3 marks]

[ANALYSIS] As optical path is integer,[1] left-hand component will point in $+x$ again at $t = 0$.[1] As there is no scattered light along y, polarization is along y and the total x component (left+ right) must be zero, which requires that the right-handed component must point in the $-x$ direction.[1]

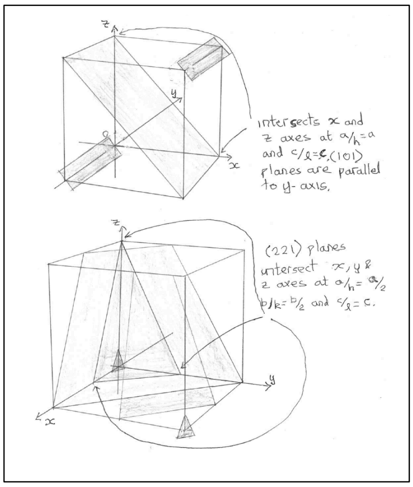
(ii) Repeat the above sketch at $z = 0$ for $t = \lambda/(4c)$. [2 marks]

[ANALYSIS] Using the anti-clockwise (clockwise) rotation of left (right) by $\pi/2$ we find that both[1] left and right-handed components point in $+y$ direction.[1]

(d) Explain, briefly, how you could distinguish whether the sugar solution contains left- or right-handed molecules. [2 marks]

[ANALYSIS] If we look at the scattered light at $z = \Lambda/4$,[1] it will be along the $\phi = -45°$ ($\phi = +45°$) diagonal[1] for left- (right-) handed molecules, respectively.

Examination Questions May/June 2019
Foundations of Physics 2B, Condensed Matter Physics, Q5 (5 parts):
SOLUTIONS

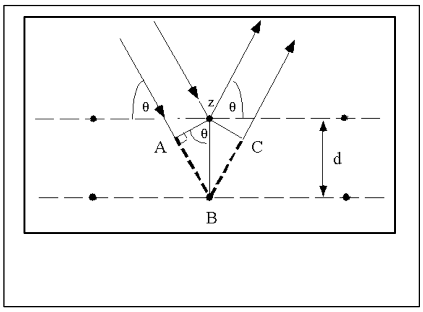


Synopsis / Problem

a. Sketch shown in diagram.
The spacing of each of the families of planes is given by:

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

where a is the lattice constant (0.5 nm)
Substituting in the appropriate ($h\ k\ l$) values gives: 0.35 nm and 0.17 nm **2 marks for diagram and 1 mark for each answer.**
[4 marks]



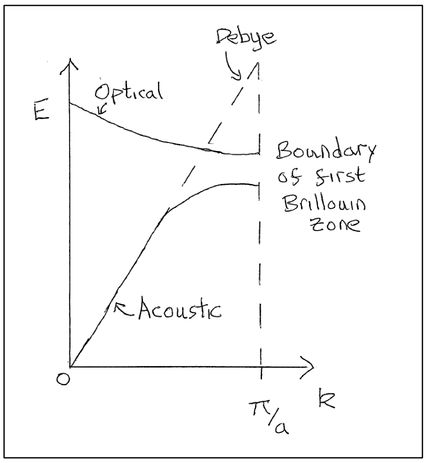
Synopsis

b. Bragg's law arises from considering the path length difference in the reflections from adjacent planes. **[1]** In the diagram the path length difference is A-B-C, when this corresponds to constructive interference a peak appears. This leads to the Bragg law

$$2d \sin\theta = n\lambda$$

where d is the plane spacing, λ is the x-ray wavelength and n is the order (usually only consider n = 1). **[1]**

Bragg scattering involves the partial reflection of incident x-ray waves from parallel planes of atoms in a crystalline solid. Each plane of atoms reflects a proportion or the intensity of the incident x-ray wave – related to the electron density. Bragg law models crystal as parallel planes. **[1]** **[1 mark for sketch]**
[4 marks]



Synopsis

c. The sketch shows the phonon dispersion curves for the optical and acoustic branches in the first Brillouin zone – up to *π/a*. The acoustic branch is described by the Debye approximation having a constant velocity (the velocity of sound waves at long wavelengths). **[4 marks]**

Conceptual

d. The Wiedemann-Franz law states that the ratio of the electrical to thermal conductivities in metals is proportional to temperature with a common constant for all metals. The Drude model treats electrons as classical particles with mass m_e, charge –e. Using the classical equipartition theorem to determine the thermal conductivity and the Drude electrical conductivity it predicts that the ratio of electrical to thermal conductivities is

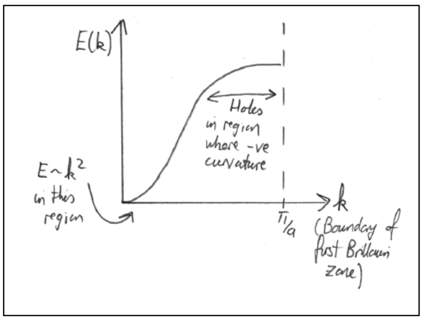
$$\frac{\kappa}{\sigma} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 T.$$

The reason this works reasonably well is that this expression does not include the Drude scatting time τ which the Drude model significantly overestimates (as a consequence of treating electrons as classical particles). **[4 marks]**

Conceptual **e.** The sketch shows general shape **(1 mark for correct shape)**. The effective mass is determined from the inverse curvature of the *E(k)* dispersion curve using

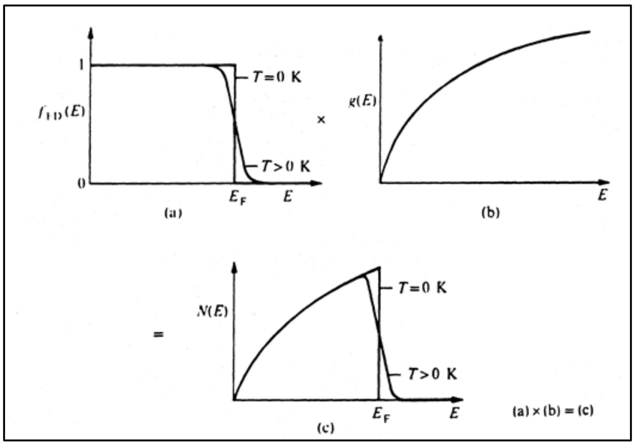
$$m_e^* = \hbar^2 \left(\frac{d^2E}{dk^2}\right)^{-1}.$$

The effective mass is positive at low *k* values increasinf as *k* increases. It becomes negative at larger *k* as *k* approaches the first Brillouin zone boundary. It is in this region that holes occur (corresponding to regions of negative effective mass). **[4 marks]**



Examination Questions May/June 2019
Foundations of Physics 2B, Condensed Matter Physics, Q6
SOLUTION

Analysis **a)** The energy density of states function gives the number of available electron energy states per unit energy range which can be occupied. **[1]** The Fermi-Dirac function gives the probability of these states being occupied. **[1]** The *Fermi energy* is the highest occupied electron state in a system when it is in the ground state (equivalent to 0 K). **[1]** Below the Fermi energy all energy states are filled, above the Fermi energy all energy states are empty. [1] These functions together give the total electron energy distribution as shown in the sketch. **[3 for sketch]**



[7 marks in total]

Application **b)** The Hall coefficient, R_H, gives both the charge and density of the charge carriers.

$$R_{\mathrm{H}} = -\frac{1}{ne}$$

where *e* is the electronic charge and *n* is the charge carrier density. **[1]** The negative value of the Hall coefficient shows that the sample contains negatively charged carriers i.e. electrons. We can obtain the free electron density of the sample by rearranging the above equation

$$n = \frac{1}{R_{\mathrm{H}}e} = \frac{1}{9.00\times10^{-11}\times1.6\times10^{-19}} = 6.9\times10^{28}\ \mathrm{m}^{-3}.\ \ [\mathbf{2}]$$

Application **c)** The Fermi energy is related to the electron density by $E_{\mathrm{F}} = \frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{\frac{2}{3}}$. **[1]** The Fermi energy gives the highest energy of the occupied free electron states i.e. the energy of the states on the Fermi sphere with a wavevector corresponding to the Fermi wavevector. This gives

$$E_{\mathrm{F}} = \frac{\hbar^2}{2m_e}(3\pi^2 n)^{\frac{2}{3}} = \frac{(1.054\times10^{-34})^2}{2\times9.11\times10^{-31}}(3\times(3.14)^2\times6.9\times10^{28})^{\frac{2}{3}} = 9.8\times10^{-19}\ \mathrm{J}$$
$$= 6.1\ \mathrm{eV}\ \ [\mathbf{1}]$$

The Fermi velocity is obtained from the Fermi wavevector

$$k_F = (3\pi^2 n)^{\frac{1}{3}} = (3\times3.14^2\times6.9\times10^{28})^{\frac{1}{3}} = 1.3\ \times10^{10}\ \mathrm{m}^{-1}\ \ [2]$$

And

$$v_F = \frac{\hbar k_F}{m_e} = \frac{1.054\times10^{-34}\times1.3\times10^{10}}{9.11\times10^{-31}} = 1.5\ \times10^{6}\ \mathrm{m\ s^{-1}}\ \ [2]$$

Comprehension **d)** The thermal energy at 300 K can be estimated by $E = k_B T$ where k_B is Boltzmann's constant and T = 300 K. This gives

$$E = k_B T = 1.38\times10^{-23}\times300 = 4.1\times10^{-21}\ \mathrm{J} = 26\times10^{-3}\ \mathrm{eV}\ \ [1]$$

The ratio between the Fermi energy and the thermal energy is therefore

$$\frac{E_F}{E} = \frac{7.3\times10^{-19}}{4.1\times10^{-21}} = 1.8\times10^{2}$$

For classical particles this gives a thermal velocity that can be estimated from the thermal energy being related to the kinetic energy $k_B T = \frac{1}{2}m_e v^2$ rearranging gives

$$v = \sqrt{\frac{2k_{\mathrm{B}}T}{m_e}} = \sqrt{\frac{2\times1.38\times10^{-23}\times300}{9.11\times10^{-31}}} = 9.5\times10^{4}\ \mathrm{m\ s^{-1}}\ [1]$$

This again gives a ratio between the Fermi velocity and the thermal velocity of

$$v_F/v = \frac{1.5 \times 10^6}{9.4 \times 10^4} = 16$$

The far greater energy of quantum free electrons compared to that expected due to thermal energy is a consequence of the Fermi-Dirac distribution which pushes the Fermi energy to very high energies compared to classical thermal energies. This in turn means that the Fermi velocities of particles at the Fermi energy is considerably higher than that expected from classical momentum. **[2 marks]**