ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 4+5+6 (Rev3)

5 Maxwell III (From Faraday's Law)

5.1 Faraday's fabulous experiments

5.1.1 Inductive Electromotive force

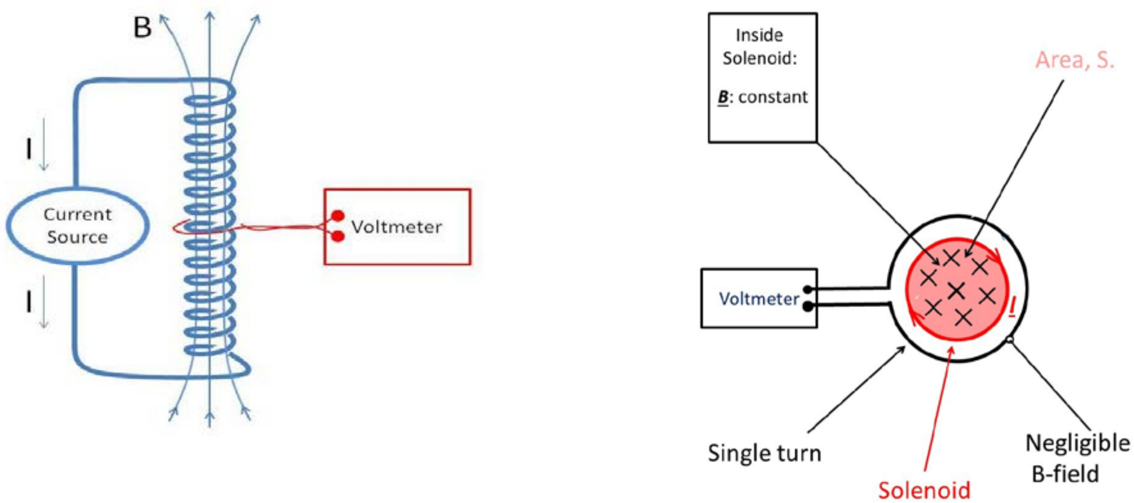

Figure 1 : LHS: A current source producing a constant magnetic field inside a long solenoid. A single loop passes round the outside of the solenoid and is attached to a voltmeter. There is negligible magnetic field outside the solenoid. The solenoid has a cross sectional area, *S*. Under these simple conditions, the voltage measured is zero. RHS: A plan view of the LHS.

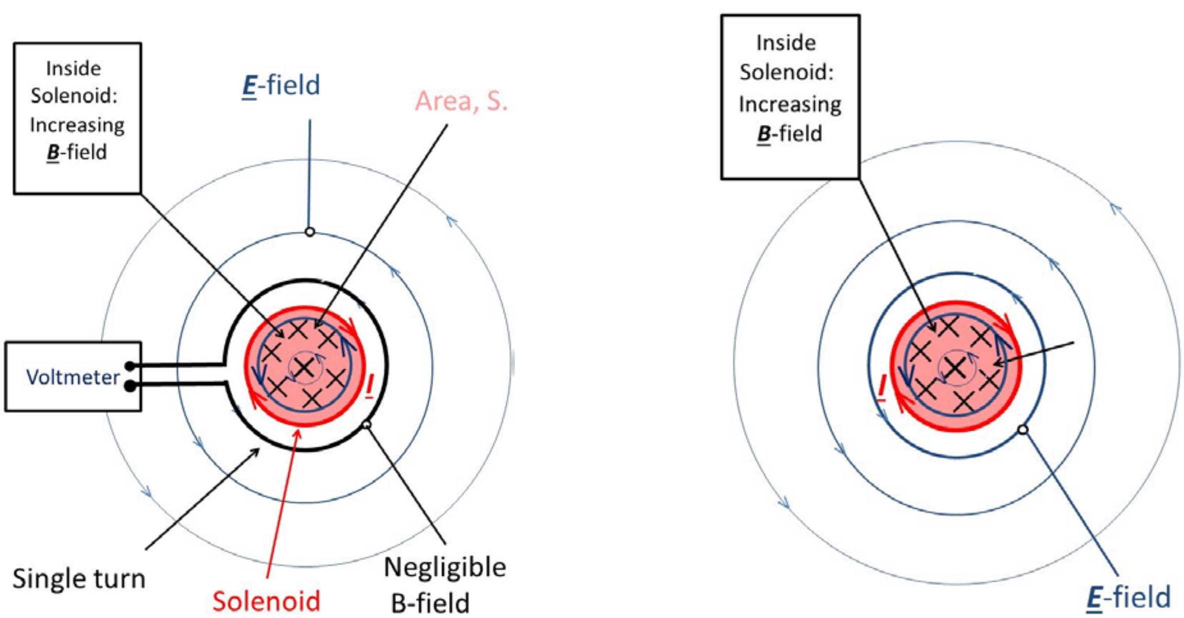

Figure 2 : LHS: A plan view of current source producing a steadily increasing magnetic field inside a long solenoid. A single loop passes round the outside of the solenoid and is attached to a voltmeter which measures a non-zero voltage. RHS: For clarity, the $\underline{\boldsymbol{E}}$-fields and $\underline{\boldsymbol{B}}$-fields (alone) are shown. Of course the fields are produced whether or not the voltmeter is present.

$$\mathrm{V} = \frac{\partial \underline{B}}{\partial t} \cdot \underline{\mathrm{S}} = \frac{\partial \phi_{\mathrm{B}}}{\partial \mathrm{t}} \quad \text{Faraday's Law of Induction} \qquad \textbf{5-1}$$

5.1.2 Motional Electromotive force

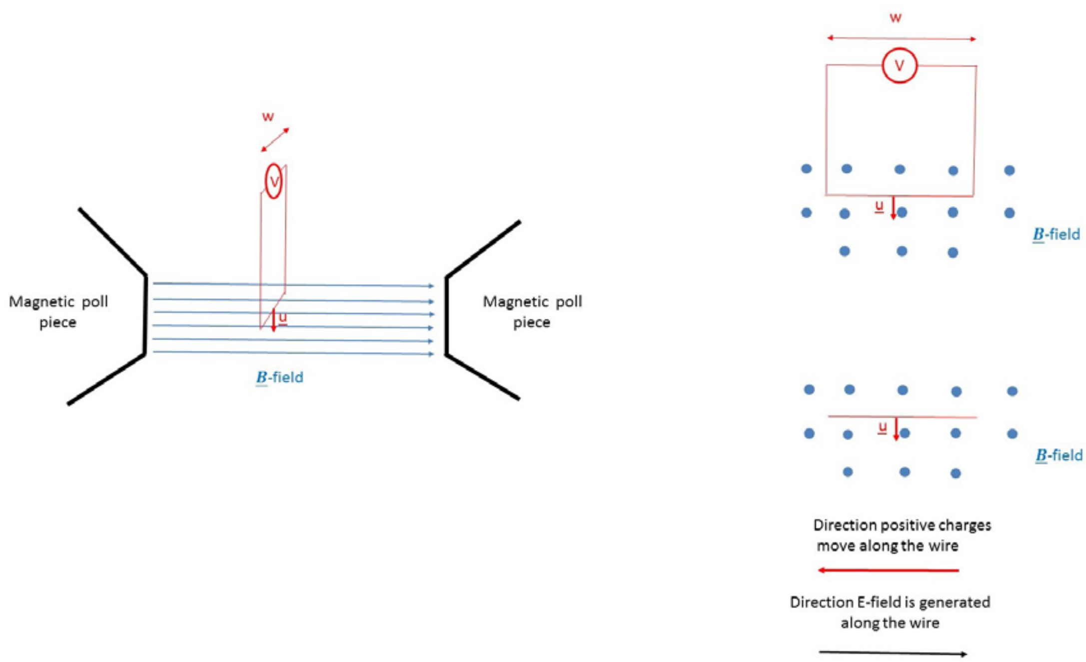

Figure 3 : LHS: A loop of wire moving with a velocity, $\underline{\mathrm{u}}$, down through the homogenous field produced between two magnetic pole-pieces. The loop has a width w and is attached to a voltmeter. RHS (upper): A side view of the experiment looking from the RHS pole-piece along the field lines. RHS (lower): The forces on the (conventional) positive charges in the horizontal bottom part of the loop of wire.

$$V = \frac{\partial \phi_{\mathrm{B}}}{\partial \mathrm{t}} = B\frac{\mathrm{dS}}{\mathrm{dt}} = \mathrm{Buw} \qquad \textbf{5-2}$$

$$\underline{\boldsymbol{F}} = \mathrm{q}\left(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}\right) \qquad \textbf{5-3}$$

$$\underline{\boldsymbol{E}} = -\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}} \qquad \textbf{5-4}$$

$$\mathrm{V} = -\mathrm{wE} = Buw \qquad \textbf{5-5}$$

5.2 Using Faraday's Law to Find Maxwell III

$$\xi_t = -\frac{\partial \phi_B}{\partial t} \quad \text{Faraday's Law} \tag{5-6}$$

$\Uparrow$

Lenz's Law: the emf tries to oppose the change in flux.

$$\xi_t = \oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = -\mathrm{V} \tag{5-7}$$

$$\xi = \oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = -\frac{\partial \phi_B}{\partial t} = -\frac{\partial}{\partial t}\int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \tag{5-8}$$

Stokes' Theorem gives:

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\boldsymbol{l} = \int \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}.\,\mathrm{d}\underline{\boldsymbol{S}} \tag{5-9}$$

Therefore:

$$\int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = -\frac{\partial}{\partial t}\int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \tag{5-10}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \quad \text{Maxwell's 3rd Equation} \tag{5-11}$$

6 Maxwell IV (From Ampere's Law)

6.1 Maxwell's Correction to Ampere's Law

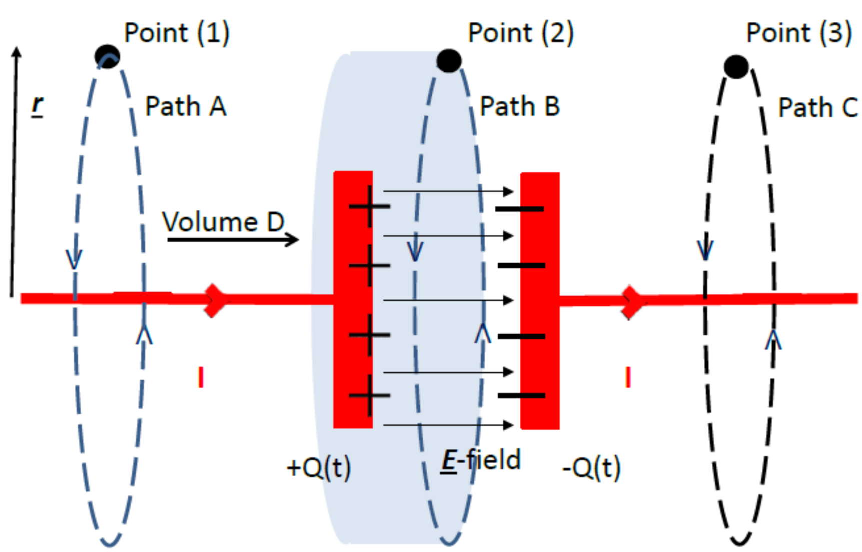

Figure 4 : There are paths A, B and C shown. The surfaces bounded by paths A and C each have a current I that threads through them. Path B has no current threading through it. Only an increasing E-field threads through the flat circular surface bounded by path B. The volume D, which is lightly coloured, includes one of the capacitor plates. There is a charge, +Q, on each of the capacitor plates.

$$\oint_{Path\ A} \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_o \mathrm{I} \quad \textbf{6-1}$$

$$\oint_{\text{Path A}} \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_o \oint_{\text{Path A}} \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \quad \textbf{6-2}$$

$$\oint_{\boldsymbol{D}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int_{D} \frac{\rho}{\varepsilon_o} \mathrm{dV} = \frac{Q}{\varepsilon_o} \quad \textbf{6-3}$$

$$\mathrm{I} = \frac{\partial Q}{\partial \mathrm{t}} \quad \textbf{6-4}$$

$$\oint_{\text{Closed surface.D}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int_{\text{Circular area bounded by path B}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \quad \textbf{6-5}$$

$$=> \frac{\partial}{\partial \mathrm{t}} \int_{\text{Circular area bounded by path B}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \frac{1}{\varepsilon_o} \frac{\partial \mathrm{Q}}{\partial \mathrm{t}} = \frac{1}{\varepsilon_o} \mathrm{I} \quad \textbf{6-6}$$

$$\oint \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_o \int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \mu_o \varepsilon_o \frac{\partial}{\partial \mathrm{t}} \int \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \quad \textbf{6-7}$$

$$\int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \mu_o \int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \mu_o \varepsilon_o \frac{\partial}{\partial \mathrm{t}} \int \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \quad \textbf{6-8}$$

$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}}$ – Maxwell's 4th Equation **6-9**

7 Transverse electromagnetic waves and charge conservation

7.1 Electromagnetic waves in a vacuum

If we assume that there are no charges or electrical currents

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = 0,$$
$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0,$$
$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}},$$
$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}}.$$

Taking the curl of Maxwell III

$$\underline{\nabla} \times \underline{\nabla} \times \underline{\boldsymbol{E}} = -\frac{\partial}{\partial t}\underline{\nabla} \times \underline{\boldsymbol{B}} \qquad \textbf{7-1}$$

.

Substituting Maxwell IV gives,

$$\underline{\nabla} \times \underline{\nabla} \times \underline{\boldsymbol{E}} = -\mu_0\varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad \textbf{7-2}$$

.

Using the vector identity

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\boldsymbol{E}}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{\boldsymbol{E}}) - \nabla^2 \underline{\boldsymbol{E}} \qquad \textbf{7-3}$$

(Nb: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$).
which gives:

$$\underbrace{\underline{\nabla}(\underline{\nabla} \cdot \underline{\boldsymbol{E}})}_{\text{zero by Maxwell I}} - \nabla^2 \underline{\boldsymbol{E}} = -\mu_0\varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad \textbf{7-4}$$

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0\varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad \textbf{7-5}$$

$$c = \sqrt{\frac{1}{\mu_0\varepsilon_0}} \approx 3.00 \times 10^8 \mathrm{ms}^{-1}.$$

$$v_{Maxwell} \approx 3.11 \times 10^8 \mathrm{ms}^{-1}$$

$$v_{Foucault} \approx 2.98 \times 10^8 \mathrm{ms}^{-1}$$

Taking the curl of Maxwell IV gives:

$$\nabla^2 \underline{\boldsymbol{B}} = \mu_0\varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{B}}}{\partial t^2} \qquad \textbf{7-6}$$

7.2 The vector nature of electromagnetic waves

Given that $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ are plane waves we can write them as the real or imaginary part of:

$$\underline{\boldsymbol{E}}(\mathrm{r,t}) = \underline{\boldsymbol{E}}_o \exp i(\underline{\mathbf{k}} \cdot \underline{\boldsymbol{r}} - \omega t) \qquad \textbf{7-7}$$

$$\underline{\boldsymbol{B}}(\mathrm{r,t}) = \underline{\boldsymbol{B}}_o \exp i(\underline{\mathbf{k}} \cdot \underline{\boldsymbol{r}} - \omega t) \qquad \textbf{7-8}$$

$\underline{\boldsymbol{E}}_o$ and $\underline{\boldsymbol{B}}_o$ are the polarization directions of $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ respectively and $\underline{\mathbf{k}}$ gives both the magnitude of the wavevector and the direction of propagation. There are non-trivial solutions (in vacuum) only if ω and $\underline{\boldsymbol{k}}$ are the same for $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$.

The solution to the wave equation (Equation 7-5) for electromagnetic waves can be written:

$$\underline{\mathbf{k}} \cdot \underline{\mathbf{k}} = k^2 = \omega^2\mu_0\varepsilon_0 \qquad \textbf{7-9}$$

,

$$\Rightarrow v_{phase} = \frac{\omega}{k} = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \qquad \textbf{7-10}$$

Substitute the plane wave solutions into Maxwell III

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = i\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} = i\omega\underline{\boldsymbol{B}} \qquad \textbf{7-11}$$

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}}_o = \omega\underline{\boldsymbol{B}}_o \qquad \textbf{7-12}$$

Similarly from Maxwell IV

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{B}}_o = -\mu_0\varepsilon_0\omega\underline{\boldsymbol{E}}_o \qquad \textbf{7-13}$$

For plane waves:

- $\underline{\boldsymbol{E}}_o$, $\underline{\boldsymbol{k}}$, $\underline{\boldsymbol{B}}_o$ are perpendicular to each other
- $\underline{\boldsymbol{B}}_o$ is a factor *c* smaller than $\underline{\boldsymbol{E}}_o$
- $\underline{\boldsymbol{E}}_o \times \underline{\boldsymbol{B}}_o$ is in the direction of $\underline{\boldsymbol{k}}$ and gives the direction of propagation.

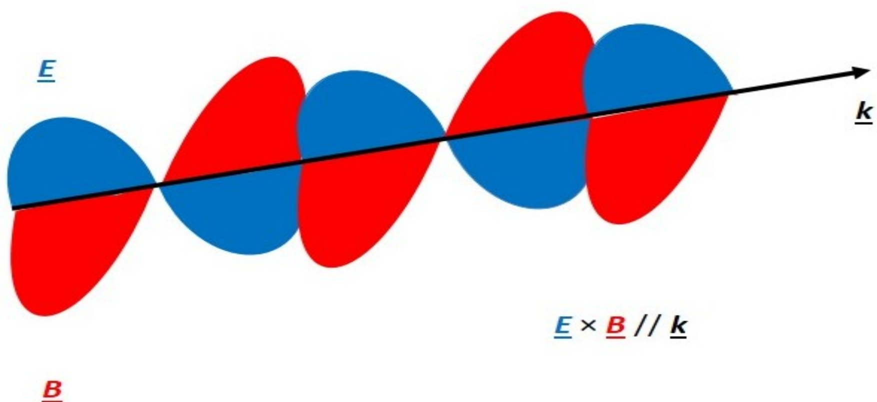

Figure 5 : An electromagnetic wave propagating through space showing the relative directions of $\underline{\boldsymbol{E}}$, $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{k}}$.

7.3 The continuity equation and charge conservation

Maxwell IV

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0\underline{\boldsymbol{J}} + \mu_0\varepsilon_0\frac{\partial \underline{\boldsymbol{E}}}{\partial t}.$$

Take the divergence of both sides,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \underline{\boldsymbol{\nabla}} \cdot \mu_0\underline{\boldsymbol{J}} + \underline{\boldsymbol{\nabla}} \cdot \mu_0\varepsilon_0\frac{\partial \underline{\boldsymbol{E}}}{\partial t} \qquad \textbf{8-14}$$

Use the vector identity

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = 0 \qquad \textbf{8-15}$$

(note the similarity with vector algebra ($\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} = 0$)

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} = -\varepsilon_0 \frac{\partial}{\partial \mathrm{t}} \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} \qquad \textbf{8-16}$$

.

Using

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0},$$

We find the differential continuity equation:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} = -\frac{\partial \rho}{\partial \mathrm{t}}, \qquad \textbf{8-17}$$

where ***J*** is current density (charge flowing per unit area per unit time) and ρ is the charge density.

Take the volume integral of both sides,

$$\int \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} \mathrm{dV} = -\int \frac{\partial \rho}{\partial \mathrm{t}} \mathrm{dV} \qquad \textbf{8-18}$$

We find the continuity equation in integral form:

$$\underbrace{\int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}}}_{\text{Charge flowing out through the surface every second}} = \underbrace{-\frac{\partial}{\partial \mathrm{t}} \int \rho \mathrm{dV}}_{\text{Change in total charge inside a volume every second}} \qquad \textbf{8-19}$$

Hence the continuity equation is equivalent to conservation of charge. ⇒ Maxwell's equations require conservation of charge.

8 Maxwell and Einstein

8.1 The collapse of Newtonian/Galilean physics

a) Observer P- stationary with respect to positive charges

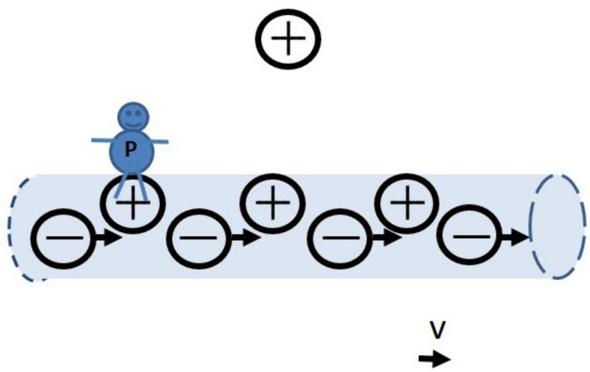

Figure 6 : Observer P loves alliteration and therefore sits on a positive charge. The positive test charge and the positive charges in the wire are stationary for Observer P. The negative charges in the wire move to the right with respect to observer P with a velocity v.

$$\underline{\boldsymbol{F}} = \mathrm{q}\left(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}\right) = 0$$

⇒ +ve test charge does not move

b) Observer N- stationary with respect to negative charges

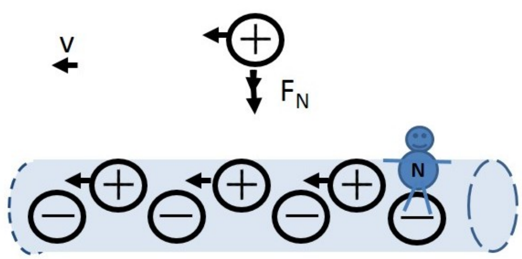

Figure 7 : Observer N also loves alliteration and therefore sits on a negative charge. The negative charges in the wire are stationary with respect to observer N. The positive test charge and the positive charges in the wire move to the left with velocity v.

$$\underline{\boldsymbol{F}} = \mathrm{q}(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \neq 0$$

⇒ the +ve test charge moves towards the wire

8.2 Einstein discovers relativity

- Assume that for observer N, the wire is charge neutral (i.e. in the rest frame, of the battery)

$$I = n_l e v \quad \textbf{7-20}$$

n_l: number of charges per unit length.

$$\mathrm{B} = \frac{\mu_o I}{2\pi r} = \frac{\mu_o n_l e v}{2\pi r} \quad \textbf{7-21}$$

r : distance from the wire to the test charge.

$$\underline{\boldsymbol{F}}_N = q\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}} \Rightarrow F_N = \frac{\mu_o q n_l e v^2}{2\pi r} \quad \textbf{7-22}$$

- For observer P (sitting on the +ve charge), the is a Lorentz contraction of the moving negative charge:

$$\lambda_{-ve} = -\gamma n_l e \quad \textbf{7-23}$$

where λ charge per unit length and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Observer P also says that in order that there is charge neutrality for observer N:

$$\lambda_{+ve} = \frac{n_l e}{\gamma} \quad \textbf{7-24}$$

$$\lambda_{TOTAL} = -\gamma n_l e \left(1 - \frac{1}{\gamma^2}\right) \quad \textbf{7-25}$$

$$\lambda_{TOTAL} = -\gamma n_l e \frac{v^2}{c^2} \quad \textbf{7-26}$$

From Gauss' law:

$$\Rightarrow E_P = \frac{-\lambda_{TOTAL}}{2\pi\varepsilon_o r} = \frac{\gamma \mu_o n_l e v^2}{2\pi r} \quad \textbf{7-27}$$

$$\Rightarrow F_P = qE_P = \gamma q v B = \gamma F_N \quad \textbf{7-28}$$

Observer N is in the rest frame, so

$$\delta\tau_N = \gamma \delta\tau_P \quad \textbf{7-29}$$

Since, $\underline{\boldsymbol{F}}_N = \frac{\delta \underline{\boldsymbol{P}}_N}{\delta\tau_N}$, $\underline{\boldsymbol{F}}_P = \frac{\delta \underline{\boldsymbol{P}}_P}{\delta\tau_P}$ where *P* = relativistic momentum.

$$\Rightarrow \delta\underline{\boldsymbol{P}}_P = \delta\underline{\boldsymbol{P}}_N \quad \textbf{7-30}$$

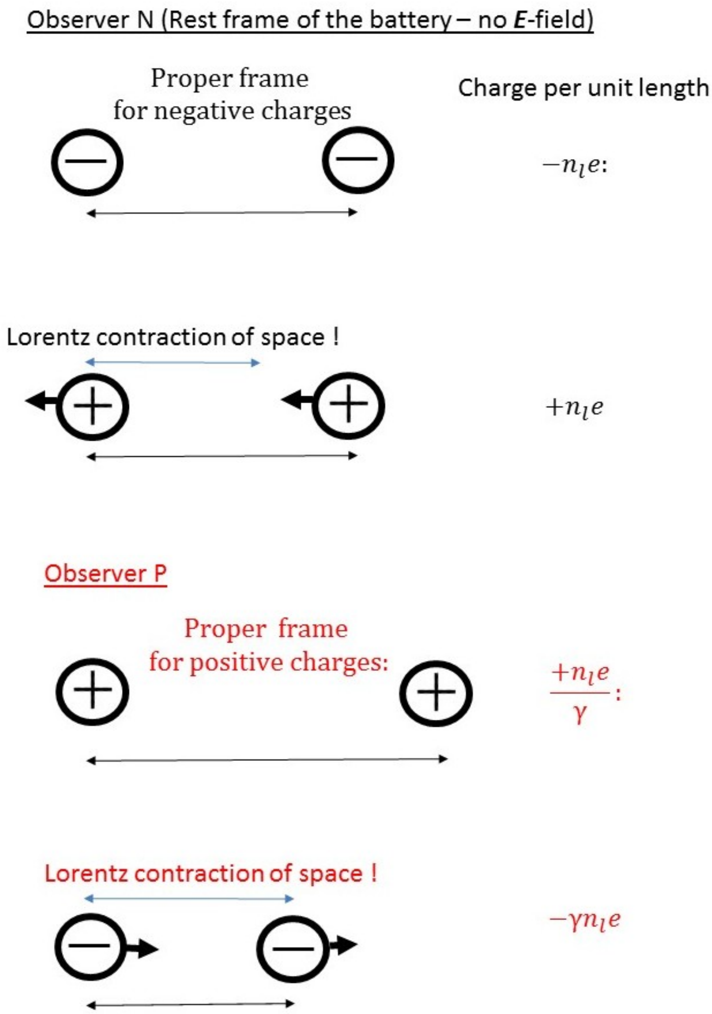

Figure 8 : A summary of the observations made by observer N (who is in the rest frame of the battery) and observer P.

- Coulomb's Law can be regarded as fundamental with magnetic forces and induced EMF as relativistic corrections.

Albert Einstein (1952)
"What lead me more or less directly to the special theory of relativity was the conviction that the electromotive force acting on a body in motion in a magnetic field was nothing other than the electric field"