University of Durham

EXAMINATION PAPER

May/June 2015   Examination code: PHYS3641WE01

ADVANCED PHYSICS 3

SECTION A. Soft Condensed Matter Physics
SECTION B. Optical Properties of Solids
SECTION C. Modern Atomic and Optical Physics

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes   **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**

- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8} \, \mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23} \, \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11} \, \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12} \, \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7} \, \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \, \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23} \, \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81 \, \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8} \, \mathrm{W\,m^{-2}\,K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. SOFT CONDENSED MATTER PHYSICS
Answer Question 1 and **either** Question 2 **or** Question 3.

1. a) Consider a liquid under shear. The liquid's viscosity η satisfies $\eta = \alpha \dot{e}^2$ where $\alpha > 0$ is a constant and $\dot{e}$ is the imposed shear rate. Qualitatively, what can be said about this liquid? Explain your reasoning with a graph. [4 marks]

   b) Two liquids A and B are mixed inside a container in ambient conditions. The mixture is homogenous. When cooled below a specific temperature T^*, the mixture spontaneously separates in two distinct phases. What can you say about the thermodynamics of the mixture? Give expressions for the enthalpy and entropy of mixing. [4 marks]

$$\left[\begin{array}{l} \text{Hint:} \quad \Delta G_{mix}^{RS} = \chi \phi_A \phi_B + k_\mathrm{B} T [\phi_A \ln(\phi_A) + \phi_B \ln(\phi_B)] \\ \text{where the symbols have their usual meanings} \end{array} \right]$$

   c) Surfactant molecules are dissolved in water. A typical surfactant molecule has a headgroup area $a_0 = 0.55$ nm^2, a single tail composed of an 80% extended chain of 16 carbon atoms. The length of a carbon-carbon link is 0.15 nm for an effective hydrocarbon volume of 1.0×10^{-2} nm^3. The surfactant concentration in water is Φ and the critical micelle concentration is given by Φ_{CMC}. Considering the ratio $\frac{v}{l_c a_0}$ where v is the total hydrocarbon volume of a surfactant molecular tail and l_c its length, determine what happens to the system when $\Phi > \Phi_{CMC}$. [4 marks]

   d) In a colloidal solution containing a simple monovalent salt, the average distance between neighbouring colloids is d. How would this distance change if the salt concentration ρ_0 was increased 4 times, assuming that no aggregation occurs? [4 marks]

$$\left[\text{Hint: } \kappa = \sqrt{\frac{2e^2 \rho_0 z^2}{\epsilon \epsilon_0 k_\mathrm{B} T}} \quad \begin{array}{l} \text{where } 1/\kappa \text{ is the Debye length and the} \\ \text{symbols have their usual meanings} \end{array} \right]$$

   e) Consider a polymer molecule of polymerisation degree N, and link length a in a solvent. The polymer is well described by the freely jointed chain model (theta condition). Assuming a diffusion coefficient D for the polymer, how long would it take for one molecule to travel a distance corresponding to its own size at room temperature, where $N = 1.0 \times 10^3$, $a = 60$ nm, and $D = 6.0 \times 10^{-3}$ cm^2 s^{-1} ? [4 marks].

2. A viscoelastic fluid is characterised by its stress relaxation function $G(t)$.

   a) Using a rheometer, we impose a periodic strain $e(t) = e_0 \cos(\omega t)$. Show that, after many cycles $(t \to \infty)$, the time evolution of the resulting stress $\sigma(t)$ can be written as

   $$\sigma(t) = G'e(t) + \frac{G''}{\omega}\dot{e}(t)$$

   where ω is the angular frequency, t the time, and G' and G'' are real numbers that do not depend on t and satisfy:

   $$G' + iG'' = i\omega \int_0^\infty G(s)e^{-i\omega s}ds$$

   Explain briefly the physical significance of G' and G'' (called respectively storage and loss moduli of the fluid) in light of this result. [10 marks]

   b) Assuming that the stress relaxation function $G(t)$ of the fluid is given by

   $$G(t) = G_0 \exp(-t/\tau)$$

   where G_0 and τ are positive constants that are characteristic of the fluid, calculate G' and G'' explicitly and sketch their evolution with the angular frequency ω. Is there a frequency at which both moduli are equal? [10 marks]

   $$\left[\text{Hint: } \sigma(t) = \int_0^t G(t - t')\dot{e}(t')dt'\right]$$

3. Consider a mixture of water and ethanol at room temperature. The volume fraction f of ethanol can vary between 0 (pure water) and 1 (pure ethanol). The mixture is homogenous for all f in ambient conditions.

a) Sketch the free energy of mixing of the water-ethanol solution as a function of f at room temperature. Assuming that the enthalpy of mixing is positive, what can be said about the enthalpy ΔH_{mix}, entropy ΔS_{mix} and free energy ΔG_{mix} of mixing? Explain your reasoning with a graph showing ΔH_{mix}, $T\Delta S_{mix}$ and ΔG_{mix} as a function of f. [8 marks]

b) We take a solution droplet of unknown composition f and place it on a flat surface. The angle formed by the droplet at the interface with the solid is θ and the surface tension of the solution is γ. Knowing the contact angles θ_1 and θ_2 formed respectively by pure water and pure ethanol droplets on the same surface, and their surface tensions γ_1 and γ_2, find an expression for f. Explain any assumption that you make and comment on their validity. [12 marks]

$$\left[\begin{array}{ll} \text{Hint:} & \gamma_{\mathrm{SL}} = \gamma_{\mathrm{S}} + \gamma_{\mathrm{L}} - W_{\mathrm{SL}} \text{ (Dupré)} \\ & W_{\mathrm{SL}} = \gamma_{\mathrm{L}}(\cos\theta + 1) \text{ (Young-Dupré)} \\ & \text{S and L stand for solid and liquid respectively.} \end{array} \right]$$

SECTION B. OPTICAL PROPERTIES OF SOLIDS
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) In travelling a distance of 2.2 μm through a material the intensity of light of frequency 3.5×10^{14} Hz is reduced by 90%. Determine the values of the absorption and extinction coefficients of the material. [4 marks]

    (b) An sp^3-type hybrid orbital is of the form

$$|sp^3\rangle = a|s\rangle + b|p_x\rangle + b|p_y\rangle + b|p_z\rangle$$

    and has energy $\epsilon(sp^3) = -8.70$ eV. Given that the individual s and p atomic orbitals have energies $\epsilon_s = -12.8$ eV, $\epsilon_p = -4.60$ eV find a detailed expression for the normalised $|sp^3\rangle$ orbital. [4 marks]

    (c) Make use of an appropriate Maxwell equation to demonstrate that when a standard plane wave with angular frequency ω,

$$\underline{E}(\underline{r}, t) = \underline{E}_o e^{i(\underline{k}.\underline{r} - \omega t)},$$

    where $\underline{r}$ and t represent position and time respectively, travels through a uniaxial dielectric material it is not the case that the wavevector $\underline{k}$ and polarisation direction $\underline{E}_o$ are orthogonal to each other. [4 marks]

    (d) Explain the significance of the labelling of the S_0, S_1 and T_1 states for a molecule such as ethylene (C_2H_4) and discuss how the nature of these states influences the possible optical transitions between them. [4 marks]

    (e) Describe an experiment which can be used to demonstrate the existence of bulk plasmons and explain why such plasmons cannot be directly excited by electromagnetic radiation. Provide a sketch indicating the form of results which can be obtained from such an experiment and explain the origin of any additional features which might be present. [4 marks]

5. When expressed as a sum of Lorentz oscillator terms the relative permittivity of a material as a function of angular frequency ω can be given by

$$\epsilon(\omega) = 1 + \frac{Ne^2}{m_\mathrm{e}\epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\gamma_j\omega)} \qquad j = 1, 2, 3......$$

Define the terms appearing in this expression. [4 marks]

Show that far from any absorption peaks we may rewrite this expression to give the so-called Sellmeier equation involving the refractive index, n, as a function of wavelength, λ:

$$n^2 = 1 + \sum_j \frac{A_j\lambda^2}{(\lambda^2 - \lambda_j^2)}.$$

Appropriate definitions should be given for A_j and λ_j. [7 marks]

Further, demonstrate that in the case of a single dominant first term in the summation that the above result can be used to obtain Cauchy's formula,

$$n = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + ...$$

(originally obtained as a purely empirical result) provided that $\lambda_1 \ll \lambda$. Obtain expressions for C_1 and C_2 in terms of A_1 and λ_1. [9 marks]

6. Explain how experimental measurements of the absorption coefficient as a function of photon energy for a semiconductor can be used to deduce whether it has a direct or indirect band gap. [6 marks]

An indirect gap (1.200 eV) semiconductor has wavevector $k = 1.600 \times 10^{10}$ m^{-1} at the conduction band minimum and the corresponding phonon energy at this wavevector is 30.00 meV. What is the value of the photon wavevector at the low temperature onset of optical absorption for this semiconductor? State any assumptions employed in obtaining this result. [4 marks]

Normally incident light of wavelength 1550 nm impinges on the semiconductor from a vacuum. Given that the value of the (power) reflection coefficient $R = 0.3100$ what is the refractive index of the semiconductor? [10 marks]

SECTION C. MODERN ATOMIC AND OPTICAL PHYSICS
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) *Transit-time* broadening and *natural* broadening are the two main factors that must be reduced in an atomic clock to minimise the clock frequency instability. Discuss briefly how these broadening mechanisms are reduced in the $^{27}\text{Al}^+$ ion quantum logic clock, and the influence of trapping time and atom number. [4 marks]

   (b) An atomic clock, where the upper state of the clock transition has a lifetime of 21.8 ns, has a clock uncertainty of 1.89×10^{-8}. Calculate the wavelength of the clock transition. [4 marks]

   (c) A stationary atom of mass $m = 1.42 \times 10^{-25}$ kg is exposed to a resonant laser of wavelength $\lambda = 780$ nm pointing along the z-axis. In a short period of time the atom completes one absorption and emission cycle, and the emitted photon has wave vector

   $$\underline{k}' = \frac{2\pi}{\lambda} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix},$$

   where the unit vector (x, y, z) specifies the direction in which the photon is emitted. Calculate the velocity vector of the atom after the absorption and emission cycle. [4 marks]

   (d) Explain briefly the factors that make the ground-state hyperfine transition in an alkali metal good for an atomic clock in comparison to the laser cooling transition in that atom. [4 marks]

   (e) Calculate the factor by which the fountain height, i.e. the maximum height an atom cloud travels after the first pass through the microwave cavity, of an atomic fountain clock would have to change in order to halve the clock uncertainty. [4 marks]

8. The wavefunctions of the 1s0, 2p0 and 2p1 states of the hydrogen atom are, respectively:

$$\psi_{1s0}(r, \theta, \phi) = Y_{0,0}R_{1,0} = \frac{1}{\sqrt{\pi}a_0^{3/2}}e^{-r/a_0};$$

$$\psi_{2p0}(r, \theta, \phi) = Y_{1,0}R_{2,1} = -\frac{1}{4\sqrt{2\pi}a_0^{3/2}}\frac{r}{a_0}e^{-r/(2a_0)}\cos\theta;$$

and

$$\psi_{2p1}(r, \theta, \phi) = Y_{1,+1}R_{2,1} = -\frac{1}{8\sqrt{\pi}a_0^{3/2}}\frac{r}{a_0}e^{-r/(2a_0)}\sin\theta e^{i\phi}.$$

(a) Give the functional forms of ψ_{1s0}, ψ_{2p0} and ψ_{2p1} along the x-axis and sketch them along the x-axis. [6 marks]

(b) Show graphically, and explain why the $\psi_{1s0} \to \psi_{2p0}$ transition is an electric-dipole allowed transition. [5 marks]

(c) Give the functional form of the time-dependent wavefunction of a 50 : 50 superposition of ψ_{1s0} and ψ_{2p1} in terms of ψ_{1s0} and ψ_{2p1} and the energies E_{1s0} and E_{2p1}. [3 marks]

(d) Derive the functional form of the time-dependent wavefunction $\psi(t)$ of a 50 : 50 superposition of ψ_{1s0} and ψ_{2p1} in terms of ψ_{1s0} and ψ_{2p1} at the first two times when the wavefunction is real (discounting a global phase). Sketch the wavefunction along the x-axis at those two times and explain why the hydrogen atom absorbs and emits photons during the $\psi_{1s0} \leftrightarrow \psi_{2p1}$ transition. [6 marks]

9. The only stable isotope of the alkali metal caesium is ^{133}Cs, which has a nuclear spin $I = 7/2$ and a ground state hyperfine splitting of A.

(a) Give the term symbol of the ground state of ^{133}Cs and list the possible values of total angular momentum quantum number F and magnetic sub-level quantum numbers m_F. [4 marks]

(b) The Zeeman shift, $\Delta E(F, m_F)$, of a given $|F, m_F\rangle$ state is given by

$$\frac{A}{2} \begin{pmatrix} 1 + mx & x\sqrt{(1+m)(1-m)} \\ x\sqrt{(1+m)(1-m)} & -1 - mx \end{pmatrix} \psi = \Delta E \psi,$$

where $m = 2m_F/(2I + 1)$, $x = 2\mu_B B/A$, μ_B is the Bohr magneton and B is the magnetic field. Show that the Zeeman shift of a given $|F, m_F\rangle$ state of ^{133}Cs is

$$\Delta E(F, m_F) = \pm\sqrt{\frac{A^2}{4} + \frac{A}{4}m_F\mu_B B + \mu_B^2 B^2}.$$

The $\pm$ refers to the upper $(+)$ and lower $(-)$ hyperfine levels. [7 marks]

(c) Using the expression for the Zeeman shift given in part (b) above, show for which m_F states the Zeeman shift is linear for all values of B. [4 marks]

(d) In an atomic clock, A is essentially measured as precisely as possible, so the measurement must be done under conditions where external magnetic fields are very low and using a transition that is as insensitive to external magnetic fields as possible. Show, using the expression for the Zeeman shift given in part (b) above, that the transition with the least sensitive Zeeman shift ΔE_{tr} is between the respective $m_F = 0$ magnetic sub-levels of the lower and upper hyperfine levels in the transition. [5 marks]

[Hint: for $ax \ll 1$, $(1 + ax)^{1/2} \approx 1 + ax/2$.]