University of Durham

EXAMINATION PAPER

May/June 2017 Examination code: PHYS3651-WE01

PLANETS AND COSMOLOGY 3

SECTION A. Cosmology
SECTION B. Planetary Systems

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. COSMOLOGY

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) In the spectrum of a galaxy, an emission feature of rest-wavelength $\lambda_0 = 500.7$ nm is observed at a wavelength of 505.0 nm. Calculate the redshift of the galaxy and its distance in Mpc if it follows Hubble flow for which $H_0 = 70$ km s^{-1} Mpc^{-1}. [4 marks]

   (b) A galaxy is observed at redshift $z = 3$. Consider the Friedmann equation for a matter-only Universe,

   $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

   where a is the expansion factor, ρ is mass-energy density and k is a constant. Calculate the fraction of the age of the Universe for which its light has been travelling to us if the density parameter $\Omega_0 = 1$ [4 marks]

   (c) A Universe comprises matter and radiation with current density parameters of $\Omega_{M,0} = 0.3$ and $\Omega_{\gamma,0} = 4 \times 10^{-5}$, respectively. Calculate the redshift at which $\Omega_M/\Omega_\gamma = 1000$ and the value of Ω_M at this epoch. [4 marks]

   (d) A disk galaxy rotates effectively as a solid body with constant angular velocity out to a radius of 10 kpc. At $r \geq 10$ kpc, the rotational velocity assumes a constant value, v_0. If the total mass of the galaxy within 100 kpc is 10^{12} $M_\odot$, calculate v_0 in km s^{-1} and the mass density at the centre of the galaxy's halo in $M_\odot$ pc^{-3}. [4 marks]

   (e) A dark energy fluid has an equation of state between pressure, p and density, ρ, given by $p = -\alpha\rho c^2$, where α is an unknown positive constant, and obeys the fluid equation,

   $$\dot{\rho} + \frac{3\dot{a}}{a}\left[\rho + \frac{p}{c^2}\right] = 0,$$

   where a is the expansion factor. The current ratio of dark energy to matter density parameters is $\Omega_{DE,0}/\Omega_{M,0} = 3$. Calculate α if $\Omega_{DE}/\Omega_M = 1$ at the epoch when the cosmic microwave background temperature was 50 per cent higher than its current value. [4 marks]

   (f) Some Grand Unified Theories predict that heavy particles such as magnetic monopoles were produced abundantly when the temperature of the Universe satisfied $k_{\rm B}T \sim 10^{15}$ GeV. Describe the relic particle abundance problem and how inflation solves it. [2 marks]

   Would this problem be solved if inflation ended at the time when the temperature of the Universe was $T = 3 \times 10^{28}$ K? [2 marks]

   (g) During Big Bang Nucleosynthesis (BBN), freeze-out occurred at $k_{\rm B}T \approx 0.8$ MeV. While in thermodynamical equilibrium, neutrons and protons had a number ratio given by

   $$\frac{N_n}{N_p} \approx \exp\left[-\frac{\Delta mc^2}{k_{\rm B}T}\right],$$

where $\Delta m = m_n - m_p \approx 1.3$ MeV$/c^2$ is the mass difference between a neutron and a proton. Assume that 75% of the neutrons at freeze-out ended up as neutrons in Helium-4 after BBN. What is the mass fraction of Helium-4 produced by BBN? [2 marks]

Will this mass fraction increase or decrease if the interaction rate between neutrons and protons is increased to maintain them in thermal equilibrium for longer (with everything else unchanged)? [2 marks]

(h) The present-day density parameter for massive neutrinos is given by

$$\Omega_{\nu,0} = \frac{\sum_i m_{\nu,i}c^2}{94\ \text{eV}}\left[\frac{H_0}{100\ \text{km s}^{-1}\text{Mpc}^{-1}}\right]^{-2}.$$

Taking $H_0 = 70$ km s^{-1}Mpc^{-1}, and assuming that we have three neutrino species with equal mass m_ν, find the value of m_ν (in units of eV$/c^2$) for which neutrinos can account for all of the dark matter today, i.e., $\Omega_{DM,0} = 0.26$. [2 marks]

If instead the dark matter is cold dark matter, with a present-day density parameter $\Omega_{CDM,0} = 0.26$, and the neutrino mass, again equal for all three neutrino species, satisfies $m_\nu \ll 0.1$ eV$/c^2$, then matter-radiation equality happens when the neutrino temperature is $k_{\text{B}}T_\nu \sim 0.6$ eV. In the all-neutrino hot dark matter model above, will matter-radiation equality happen at a temperature higher than, or lower than, or equal to, $k_{\text{B}}T_\nu \sim 0.6$ eV? Only a qualitative argument is needed. [2 marks]

2. (a) Consider a hypothetical radiation-dominated expanding Universe with density parameter $\Omega_0 = 3$. Starting from the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

calculate the factor by which the Universe will expand before the expansion comes to a halt. You may assume that radiation is the only contributor to the mass-energy density ρ. [5 marks]

(b) A laser signal is fired from Earth towards a galaxy, A, whose present-day observed redshift is z_A. Calculate z_A and the comoving distance to galaxy A, in units of c/H_0, if A receives the laser signal at the instant the expansion stops. You may assume that the comoving distance between two observers is given by

$$X_{\rm CM} = \int_{t_1}^{t_2} \frac{c\,dt}{a},$$

if light is emitted by one observer at time t_1 and received by the other at time t_2. [9 marks]

(c) Galaxy B lies along the same line-of-sight as galaxy A but is more distant, with current observed redshift z_B, and thus receives the laser signal after the Universe has started to collapse. Calculate the comoving distance from Earth to the Big Bang, in units of c/H_0, and hence the expansion factor at which B will receive the signal in the limit $z_B \to \infty$. [6 marks]

[Hint: You may use the following integral

$$\int \frac{dx}{\sqrt{\alpha^2 - x^2}} = \arcsin(x/\alpha),$$

where the arcsin function takes values between $\pm\pi/2$.]

3. (a) In a spatially flat universe filled with a cosmological constant (density parameter $\Omega_{\Lambda,0} = 0.7$ today), radiation ($\Omega_{R,0} = 7\times10^{-5}$) and non-relativistic matter ($\Omega_{M,0} = 1 - \Omega_{\Lambda,0} - \Omega_{R,0}$), what is the density ratio between the cosmological constant and radiation at $z = 10^{10}$? [4 marks]

Briefly describe the coincidence problem of the cosmological constant. [2 marks]

(b) Particle physicists Dvali, Gabadadze and Porrati proposed an alternative model, called the DGP model, in which there is no cosmological constant. In their model, the Friedmann equation, in a spatially flat matter-dominated universe, takes the following form

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{c}{r_c}\frac{\dot{a}}{a} = \frac{8\pi G}{3}\rho_M(a),$$

where c is the speed of light, r_c is a model parameter with the dimensions of length, $\rho_M(a)$ is the matter density at scale factor a, and $\dot{a}/a$ is the Hubble expansion rate. By dividing both sides of the above equation by H^2, find the value of r_c in units of Mpc, if $\Omega_M(a) = 8\pi G\rho_M(a)/(3H^2)$ in this model is defined in the usual way, $\Omega_{M,0} = \Omega_M(a = 1) = 0.3$, and the Hubble expansion rate today is $H_0 = 70$ km s^{-1}Mpc^{-1}. [4 marks]

(c) By taking the first time derivative of the Friedmann equation for the DGP model given in (b), and using the definition of the deceleration parameter, $q \equiv -(\ddot{a}/a)/(\dot{a}/a)^2$, show that in this model q can be written as

$$q(a) = \frac{2\Omega_M(a) - 1}{\Omega_M(a) + 1}.$$

[6 marks]

(d) Assuming that $\Omega_{M,0} = 0.3$ at the present day, explain why the expansion of the universe in this model changed from decelerating to accelerating in the past, and estimate the value of $\Omega_M(a)$ when the transition happened. [4 marks]

4. For a universe containing non-relativistic matter and a cosmological constant (but no radiation), and a spatial curvature described by k, the comoving distance of an object at redshift z from us is given by

$$r(z) = \frac{c}{H_0} \int_0^z \frac{1}{\sqrt{\Omega_{M,0}(1+z')^3 + \Omega_k(1+z')^2 + \Omega_{\Lambda,0}}} dz',$$

where $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$ are respectively the present-day values of the density parameters for non-relativistic matter and the cosmological constant, H_0 is the Hubble expansion rate today and $\Omega_k = -kc^2/3H_0^2$.

(a) Briefly describe what standard candles and standard rulers are, and why they are useful in cosmology. [4 marks]

(b) Two cosmologists are trying to find the cosmological parameters of their universe. They agree that the universe is spatially flat, and $H_0 = 70$ km s^{-1}Mpc^{-1}. However, they disagree about the matter content. Cosmologist A thinks that the universe has $\Omega_{M,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$, while cosmologist B believes $\Omega_{M,0} = 1$, $\Omega_{\Lambda,0} = 0$.

In the end, they decide to measure a quantity ξ defined as

$$\xi = \frac{1}{2(1+z)} \sqrt{\frac{L}{\pi f(z)}},$$

where L is the luminosity of a class of objects thought to be standard candles, and $f(z)$ is the flux measured from such objects at redshift z. Use the relation between luminosity and flux to show that ξ is indeed the comoving distance of the object. [2 marks]

They manage to measure ξ for many objects at various redshifts, so that they can calculate $\xi'(z) = d\xi(z)/dz$ from the data. They find that the value of ξ' at $z = 0.5$ is 3275 Mpc. Assuming that one of them has the right model, show that it must be A. [6 marks]

(c) B then argues that the objects they observe are not standard candles, and instead their luminosity L depends on redshift. Assuming that B has the correct cosmological parameters, use the relation between luminosity and flux to obtain an expression for $L(z)$ in terms of c, H_0, z and $f(z)$ only. [4 marks]

(d) To determine whether B is right about the cosmological parameters, you decide to do an independent observation. You have found an object at $z = 2$, which has an angular size of 0.087 ± 0.003 radian. You also know that the physical size of this object is 150 Mpc. Does your observation support B's argument? [4 marks]

SECTION B. PLANETARY SYSTEMS

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) What defining characteristics are shared by planets and dwarf planets, and which is the key distinction between them? [4 marks]

(b) The planet HD 4113b orbits its parent star in a highly eccentric orbit, with eccentricity $e = 0.903$ and period $T = 526$ days. Calculate its distance (in AU) from its parent star, HD 4113, at closest approach, given that HD 4113 has a mass of $0.99M_{\odot}$. [4 marks]

(c) Explain what the Roche limit is, and how it can be used to explain the presence of ring systems around the gas giant planets in our Solar System. [4 marks]

(d) The pressure p at a height h above the surface of a planet can be calculated as $p = p_0 \exp(-h/H)$ for an isothermal atmosphere, where p_0 is the surface pressure and H is the scale height of the atmosphere. It can be shown that $H = k_{\mathrm{B}}T/(g\mu_{\mathrm{a}}m_{\mathrm{H}})$, where T is the temperature of the atmosphere, g is the surface gravity of the planetary body, m_{H} is the mass of a hydrogen atom ($\simeq m_{\mathrm{p}}$), and μ_{a} is the mean mass of molecules in the atmosphere.

If the pressure at a height of 10.0 km above the surface of Saturn's moon Titan is 89.2 kPa, show that the main constituent of Titan's atmosphere is molecular nitrogen, N_2, given that atomic nitrogen has a mean mass of 14 atomic mass units and the following data for Titan.

Mass	1.35×10^{23} kg
Radius	2.58×10^{6} m
Surface pressure	1.47×10^{5} Pa
Surface Temperature	93.7 K

[4 marks]

(e) Explain the nature of the P and S waves produced in seismic events.

Measurements of seismic waves passing through a rock layer, with a depth of 50.0 km and an estimated uniform density $\rho = 4.10 \times 10^3$ kg m^{-3}, show that the first waves take 6.20 s to emerge, and a second burst of waves is seen a further 3.40 s later. Given that the velocities of primary and secondary waves are given by

$$v_{\mathrm{p}} = \sqrt{\frac{\kappa + (4/3)\mu}{\rho}} \quad \text{and} \quad v_{\mathrm{s}} = \sqrt{\frac{\mu}{\rho}}$$

respectively, determine the shear modulus μ and the bulk modulus κ for this rock layer. [4 marks]

(f) The Kepler spacecraft detects a transit of a Sun-like star by a small planet that lasts 14 hours. Assuming the planet is in a circular orbit, estimate the minimum radius of its orbit about the star, clearly stating the condition for this to be the minimum. [4 marks]

[Hint: the radius of the Sun $R_{\odot} = 6.96 \times 10^8$ m.]

(g) Outline how solar nebula theory explains the coplanar and prograde motions of the planets. [4 marks]

6. (a) Describe, with the aid of one or more diagrams, how a gravitational slingshot around a planet can be used to accelerate a spacecraft in the frame of the Sun. [5 marks]

(b) You are put in charge of the interplanetary travel for the return journey to Earth of a spacecraft orbiting Jupiter in the same orbit as, but not close to, its moon Ganymede. You are tasked with working out the quickest return journey, and decide to approximate it using only Hohmann orbits, which can be split two steps: (1) a transit from Ganymede's orbit to a much higher orbit around Jupiter, under the influence of Jupiter's gravity; and (2) a further transit from this orbit to the Earth, under the gravitational influence of the Sun. Given that the high orbit is circular and has a radius of 0.347 AU from Jupiter, perform the following tasks.

(i) Sketch the path for the return journey (NOT required to be to scale!) [4 marks]

(ii) For the second Hohmann transfer orbit, calculate the changes in velocity at each point they are required, noting the direction of the thrust. Remember that at the start of this transit you need to correct the motion from Jupiter's frame to that of the Sun. [6 marks]

(iii) Determine the ratio of mass of fuel to mass of spacecraft (excluding fuel) required to perform the manoeuvres in the second Hohmann transfer, if the spacecraft flight is powered by a rocket with exhaust velocity $v_{\rm ex} = 3.00$ km s^{-1}. The rocket equation may be written as $\Delta v = v_{\rm ex} \ln(m_0/m)$, for an initial rocket mass m_0, a final mass m, and a change in velocity Δv. Use your result to comment on the practicalities of interplanetary spaceflight using only rockets. [5 marks]

[Hint: the mass of Jupiter is 1.90×10^{27} kg and it has a circular orbit with radius 5.20 AU.]

7. (a) The mass-radius relation for a planet of mass M_{p} and radius R_{p}, composed of atoms with atomic number Z and atomic mass A is given by

$$2\beta\left(\frac{M_{\mathrm{p}}Z^5}{A^5}\right)^{\frac{1}{3}} = R_{\mathrm{p}}\left(\frac{\alpha Z^2}{A^{\frac{4}{3}}} + \gamma M_{\mathrm{p}}^{\frac{2}{3}}\right),$$

where α, β and γ are all constants. Derive a simple proportionality relationship between M_{p} and R_{p} in the limit of both small and large planetary masses. [4 marks]

(b) In a distant planetary system, a dwarf planet forms out of equal masses of rock and ice and has an initial rotation period of 10.3 hours. At this time the rock and ice are uniformly mixed, but radioactive heating leads to all of the ice melting. After millions of years the radioactive heating decreases and all of the water turns back into ice, which forms an icy mantle around a rocky core. Assume the rock and ice have densities independent of pressure of 3000 kg m^{-3} and 940 kg m^{-3} respectively, and the radius of the planet remains unchanged.

(i) Starting by noting that the volume of the dwarf planet is simply the sum of the volumes of its constituents, show that the radius of the core R_{c} after the water freezes back to ice is

$$R_{\mathrm{c}} = \left(\frac{\rho_{\mathrm{i}}}{\rho_{\mathrm{i}} + \rho_{\mathrm{c}}}\right)^{\frac{1}{3}} R_{\mathrm{p}},$$

where ρ_{i} and ρ_{c} are the densities of the icy mantle and the rocky core respectively, and R_{p} is the radius of the dwarf planet. [4 marks]

(ii) Hence, estimate the rotation period of the planet after the water freezes, in the absence of any tidal interactions. [6 marks]

[Hint: the moment of inertia of a spherical shell is given by

$$I = \frac{2}{5}M\left(\frac{a^5 - b^5}{a^3 - b^3}\right),$$

where the shell has mass M, outer radius a and inner radius b.]

(c) The rotation of the dwarf planet will have had an effect on its shape. Briefly describe qualitatively what this effect could have been, and how it might impact on the calculations in part (b). [6 marks]

8. (a) Describe, with the aid of one or more diagrams, how gravitational microlensing works, and how planets are detected by this method. [6 marks]

(b) The effective temperature of a planet, T_{e}, may be found as

$$T_{\mathrm{e}} = T_* \sqrt{\frac{R_*}{2a}} (1 - A)^{\frac{1}{4}},$$

for a stellar surface temperature T_*, stellar radius R_*, orbital semi-major axis a and albedo A.

A planet in the super-Earth class orbits a main sequence K star in a circular orbit with a period of 139 days. Calculate an upper limit on the albedo of the planet such that it lies within the habitable zone for the star. The K star has surface temperature $T_* = 4980$ K, radius $R_* = 0.716 R_\odot$ and mass $M_* = 0.63 M_\odot$. [5 marks]

[Hint: the radius of the Sun $R_\odot = 6.96 \times 10^8$ m.]

(c) Given the data in part (b), that the planet has a mass of 4.23×10^{25} kg and a radius of 1.52×10^7 m, and that it transits the K star equatorially, determine the following observational quantities for this planet. Comment on whether each quantity is readily accessible with current observatories.

(i) The transit depth as a fraction of the K star's luminosity. [3 marks]

(ii) The astrometric shift induced in the parent star by the planet as a result of its reflex motion, given that the size of this shift (θ, in arcseconds) may be calculated as

$$\theta = \frac{M_{\mathrm{p}} a}{M_* d},$$

where M_{p} is the mass of the planet, the semi-major axis of the planet a in is AU, and the distance $d = 12.5$ pc. [2 marks]

(iii) The maximum observed fractional shift in the wavelengths of stellar spectral features as a result of the K star's reflex motion, given that the semi-velocity amplitude of a system in a circular orbit may be calculated as

$$K = \left(\frac{2\pi G}{T}\right)^{\frac{1}{3}} \frac{M_{\mathrm{p}} \sin i}{(M_* + M_{\mathrm{p}})^{\frac{2}{3}}},$$

for an inclination to our line of sight i and orbital period T. [4 marks]

