University of Durham

EXAMINATION PAPER

May/June 2015 Examination code: PHYS3621WE01

FOUNDATIONS OF PHYSICS 3A

SECTION A. Quantum Mechanics 3
SECTION B. Nuclear and Particle Physics

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\,\mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\,\mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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SECTION A. QUANTUM MECHANICS 3

Answer Question 1 and **at least one** of Questions 2 and 3.

1. (a) (i) Explain why when t increases the function $F(t, \Delta)$, given by

$$F(t, \Delta) = \frac{2\sin^2(t\Delta/2)}{\Delta^2},$$

becomes much larger for $|\Delta| \approx 0$ than for $|\Delta| \gg 0$. (ii) How is this fact related to the occurrence of resonances in quantum systems submitted to a periodic perturbation? [4 marks]

(b) The $(n = 2, l = 1, m = 0)$ state of atomic hydrogen has a spontaneous decay lifetime of about 1.6 ns. This state can decay only to the ground state. (i) What is the A-coefficient for this transition? (ii) What is the natural width of the corresponding energy level, expressed in eV? [4 marks]

(c) Why is the $(n = 2, l = 0, m = 0)$ state of atomic hydrogen much longer lived than the $(n = 2, l = 1, m = 0)$ state in the absence of any external pertubation? [4 marks]

(d) (i) What are the values of the orbital quantum number L and of the spin quantum number S in the ^{5}D states of atomic oxygen? (ii) Why can't the total angular momentum quantum number J be zero in a state with $L = 3$ and $S = 1$? [4 marks]

(e) Why is the 1s^2 configuration of a two-electron atomic system (e.g., the ground state of helium) necessarily a spin singlet? [4 marks]

(f) (i) Write down a linear combination of two 1s atomic orbitals which can be taken as an approximate, unnormalized molecular orbital of gerade symmetry describing the ground electronic state of the molecular ion H_2^+. Define, in a few words, all the symbols appearing in your equation(s). (ii) Why does this linear combination describe a gerade state rather than an ungerade state? [4 marks]

2. A time-independent Hamiltonian, H_0, is perturbed by a time-dependent perturbation, $H'(t)$, such that $H'(t) \to 0$ for $t \to \pm\infty$. The system for which $H_0 + H'(t)$ is the total Hamiltonian is in the eigenstate a of H_0 for $t \to -\infty$. In first order perturbation theory, the probability P_{ba} that it is in the state b of H_0 at time $t \to \infty$ is given by the following equation, assuming that $\omega_{ba} \neq 0$:

$$P_{ba} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} H'_{ba}(t) \exp(i\omega_{ba} t)\, dt \right|^2 .$$

(a) Define the quantities $H'_{ba}(t)$ and ω_{ba} appearing in this equation in terms of the eigenstates and eigenenergies of H_0. [4 marks]

(b) An atom of hydrogen is submitted to a perturbation described by the Hamiltonian

$$H'(t) = A\, xz \exp(-t^2/\tau^2),$$

where A and τ are two constants and x and z are two of the three Cartesian coordinates of the electron. Initially ($t \to -\infty$), the atom is in the $1\mathrm{s}_{m=0}$ state.

(i) Show that to first order in $H'(t)$ the atom has a non-zero probability to be in the $3\mathrm{d}_{m=1}$ state at $t \to \infty$, and a zero probability to be in the $3\mathrm{d}_{m=2}$ state. [10 marks]

Hints:

(1) In terms of spherical polar coordinates,
$x = r \sin\theta\, [\exp(i\phi) + \exp(-i\phi)]/2,$
$z = r\cos\theta.$

(2) The wave functions of the $1\mathrm{s}_{m=0}$, $3\mathrm{d}_{m=1}$ and $3\mathrm{d}_{m=2}$ states are
$\psi_{1\mathrm{s}0}(r) = N_{1\mathrm{s}0} \exp(-r/a_0),$
$\psi_{3\mathrm{d}1}(r,\theta,\phi) = -N_{3\mathrm{d}1} (r/a_0)^2 \exp(-r/3a_0) \sin\theta\cos\theta \exp(i\phi),$
$\psi_{3\mathrm{d}2}(r,\theta,\phi) = \quad N_{3\mathrm{d}2} (r/a_0)^2 \exp(-r/3a_0) \sin^2\theta \exp(2i\phi),$
where a_0 is the Bohr radius and $N_{1\mathrm{s}0}$, $N_{3\mathrm{d}1}$ and $N_{3\mathrm{d}2}$ are three normalization constants.

(3) You may use the following results without proof:
$\int_0^{\pi} \sin^3\theta \cos^2\theta\, d\theta = 4/15,$
$\int_0^{\infty} r^6 \exp(-\alpha r)\, dr = 720/\alpha^7,$
$\int_{-\infty}^{\infty} \exp(-t^2/\tau^2 + i\omega t)\, dt = \pi^{1/2}\tau \exp(-\omega^2\tau^2/4).$

(ii) Propose and prove two selection rules for the transitions between states of atomic hydrogen effected by $H'(t)$, valid to first order of perturbation theory. [6 marks]

3. (a) Outline the principles of the Rayleigh-Ritz variational method for the calculation of the ground state energy of a quantum system. [5 marks]

(b) An electron, at rest, is exposed to a constant magnetic field $\underline{B}$ such that

$$\underline{B} = B\,(\sin\theta\,\underline{\hat{x}} + \cos\theta\,\underline{\hat{z}}),$$

where $\underline{\hat{x}}$ and $\underline{\hat{z}}$ are the unit vectors in, respectively, the x- and z-directions and θ is a certain angle. Correspondingly, the Hamiltonian of the electron is $g\mu_{\mathrm{B}}\underline{B}\cdot\underline{S}/\hbar$, where $\underline{S}$ is the spin operator, g is the gyromagnetic ratio of the electron, and μ_{B} is the Bohr magneton. Use the variational method to find an expression for the ground state energy of this system, taking $|\lambda\rangle$ defined by the following equation as a trial state vector:

$$|\lambda\rangle = \cos\lambda|\uparrow\rangle + \sin\lambda|\downarrow\rangle.$$

In this equation, $|\uparrow\rangle$ and $|\downarrow\rangle$ represent the states of spin up and spin down, respectively, and λ is a variational parameter. [11 marks]

$$\left[\begin{array}{l} \text{Hints:} \quad \cos 2\lambda = \cos^2\lambda - \sin^2\lambda, \quad \sin 2\lambda = 2\sin\lambda\cos\lambda, \\ \underline{B}\cdot\underline{S}/\hbar \equiv \dfrac{B}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad |\lambda\rangle \equiv \begin{pmatrix} \cos\lambda \\ \sin\lambda \end{pmatrix}. \end{array}\right]$$

(c) Discuss whether in the case studied in (b) a variational calculation based on the trial state vector $|\lambda\rangle$ can give the exact ground state energy. [4 marks]

SECTION B. NUCLEAR AND PARTICLE PHYSICS
Answer Question 4 and **at least one** of Questions 5, 6, 7 and 8.

4. (a) The $\pi_2^+(1670)$ ($J^P = 2^-$) meson mass is greater than twice the mass of the pion. Explain whether or not the $\pi_2^+(1670)$ can decay to a neutral pion π^0 ($J^P = 0^-$) and a positively charged pion π^+ ($J^P = 0^-$) via the strong force? [4 marks]

(b) An electron with energy E travelling along the z-direction collides with a stationary neutron, scattering the electron and producing the Δ^0 baryon, *i.e.* $e^- + n \to e^- + \Delta^0$. Show that the energy of the scattered electron is

$$E' = E\frac{1 + \frac{m_{\mathrm{n}}^2 - m_\Delta^2}{2Em_{\mathrm{n}}}}{1 + \frac{E}{m_{\mathrm{n}}}(1 - \cos\theta')},$$

where θ' is the angle through which the electron is scattered, m_{n} is the mass of the neutron and m_Δ is the mass of the Δ^0 baryon. [4 marks]

(c) The Yukawa potential $V(r) = -\left(\frac{g}{r}\right)\exp(-r/R) = -g\phi(r)$ can be interpreted as arising from the exchange of a particle described by the field ϕ. Show that ϕ satisfies the relativistic equation

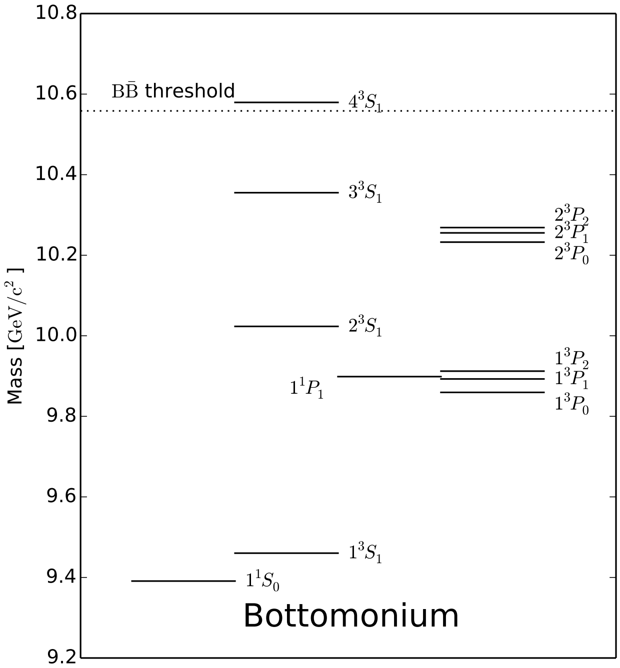
$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = m^2\phi$$

for some R. State the form of R. If the range of the nuclear force is 1.2 fm make a prediction for the mass of the exchange particle in MeV. [4 marks]
[Hint: $\hbar c = 197$ MeV fm]

(d) In the independent particle shell model of the nucleus the first seven energy levels in order of increasing energy are $1s_{\frac{1}{2}}$, $1p_{\frac{3}{2}}$, $1p_{\frac{1}{2}}$, $1d_{\frac{5}{2}}$, $2s_{\frac{1}{2}}$, $1d_{\frac{3}{2}}$ and $1f_{\frac{7}{2}}$. What values are predicted by the independent particle shell model for the spin, parity and magnetic moment of the $^{48}_{20}$Ca nucleus? [4 marks]

(e) Given that the binding energies of the sulphur isotopes $^{31}_{16}$S, $^{32}_{16}$S and $^{33}_{16}$S are 256.74, 271.78 and 280.42 MeV respectively, calculate the separation of the $2s_{\frac{1}{2}}$ and $1d_{\frac{3}{2}}$ energy levels for nuclei with atomic mass number $A \approx 32$. [4 marks]

(f) Which states in the following figure of the bottomonium spectrum have a positive parity? List all M1 transitions. Sketch the photon emission spectrum of the 2^3S_1 state. [4 marks]



(g) Explain why there are spin 3/2 and spin 1/2 ground-state baryons with uud quark content but only a spin 3/2 ground-state baryon with uuu content. [4 marks]

(h) Give the preferred direction of flight of the electron in muon decay with respect to the spin of the muon when the energy of the electron is maximal and give your reasoning. [4 marks]

(i) For each of the following reactions find which particle X has to be to make the reaction allowed in the Standard Model and draw a possible Feynman diagrams for it. [4 marks]

- $\tau^- \to e^- \bar{\nu}_e X$
- $\nu_e \mu^- \to \nu_\mu X$
- $uX \to e^+ e^-$
- $t \to bX$

5. The semi-empirical mass formula predicts that the nuclear binding energy for a nucleus with mass number A, atomic number Z and neutron number N is

$$B = a_V A - a_S A^{\frac{2}{3}} - a_c \frac{Z^2}{A^{\frac{1}{3}}} - a_a \frac{(N-Z)^2}{4A} - \frac{\delta}{A^{\frac{1}{2}}}$$

where $a_V = 15.19\,\text{MeV}$, $a_S = 16.85\,\text{MeV}$, $a_c = 0.72\,\text{MeV}$, $a_a = 41.74\,\text{MeV}$ and

$$\delta = \begin{cases} -11.8\,\text{MeV} & \text{for even } Z \text{ and } N \text{ (even} - \text{even nuclei)}, \\ 0\,\text{MeV} & \text{for odd } A \qquad\qquad \text{(odd} - \text{even nuclei)}, \\ 11.8\,\text{MeV} & \text{for odd } Z \text{ and } N \text{ (odd} - \text{odd nuclei)}. \end{cases}$$

a) For nuclei with odd A show that the most stable nucleus with mass number A has atomic number Z given by

$$\frac{Z}{A} = \frac{m_{\rm n} - m_{\rm p} + a_a}{2\left(a_a + a_c A^{\frac{2}{3}}\right)}.$$

[5 marks]

The following nuclei with $A = 7$ have been observed

$${}^7_1\text{H}, \quad {}^7_2\text{He}, \quad {}^7_3\text{Li}, \quad {}^7_4\text{Be}, \quad {}^7_5\text{B}.$$

b) Which of these isobars is stable? [2 marks]

c) For nuclei with a large excess of neutrons or protons, decay via the emission of a single neutron or proton is possible. Which, if any, of these isobars decay via either proton or neutron emission? [13 marks]

[Hint: $(m_{\rm n} - m_{\rm p}) = 1.294\,\text{MeV}$, $m_{\rm p} = 938.3\,\text{MeV}$.]

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6. The form factor for the differential cross section for the scattering of an electron from an extended charge distribution is

$$F(|\underline{q}|^2) = \int \exp\left(i\underline{q}\cdot\underline{x}\right) f(\underline{x})\mathrm{d}^3x,$$

where $\underline{q}$ is the change in the momentum of the electron and $f(\underline{x})$ is the charge distribution.

a) Show that for a spherically symmetric charge distribution

$$F(\underline{q}^2) = \frac{4\pi}{|\underline{q}|}\int f(r) r \sin\left(|\underline{q}|r\right)\mathrm{d}r.$$

[4 marks]

b) Consider the spherically symmetric charge distribution

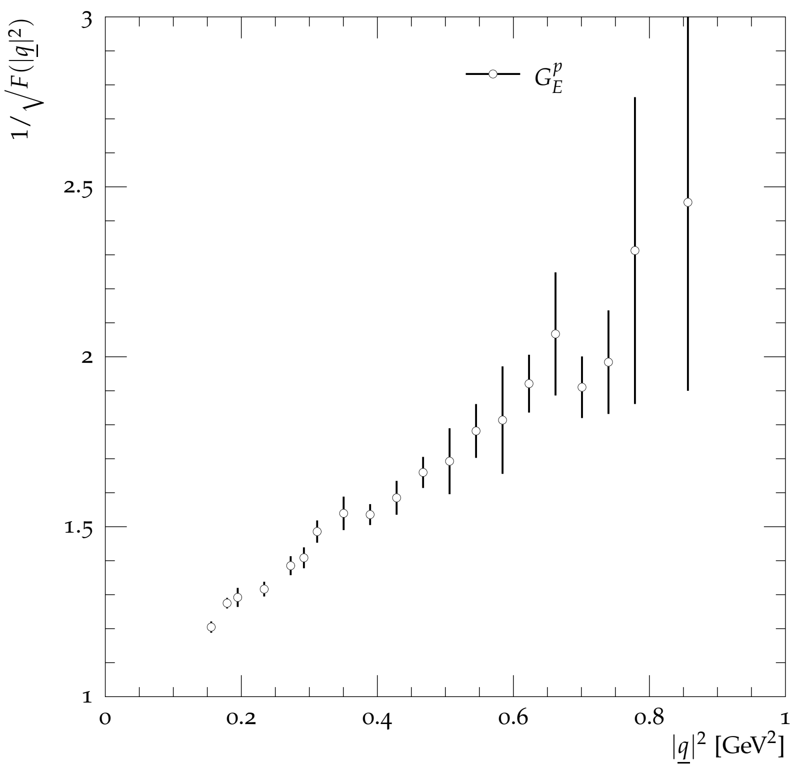
$$f(r) = b\exp(-ar),$$

where a and b are constants. Determine b in terms of a and show that the form factor is

$$F(|\underline{q}|^2) = \frac{1}{\left(1+\frac{|\underline{q}|^2}{a^2}\right)^2}.$$

[8 marks]

[Hint: For $I_n = \int_0^\infty \mathrm{d}r\, r^n \exp(-\alpha r)$, $I_n = \frac{n}{\alpha} I_{n-1}$ and $I_0 = \frac{1}{\alpha}$.]



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c) The electromagnetic form factor of the proton can be described using the result of b). Use the figure above which shows $1/\sqrt{|F(|\underline{q}|^2)|}$ for the proton to determine the constant a for the charge distribution of the proton. [4 marks]

d) Calculate the root mean square radius, $\sqrt{\langle r^2 \rangle}$ of the proton in fm. [4 marks]

[Hint: $\hbar c = 0.197\,\mathrm{GeV\,fm}$.]

7. In this problem we consider a bound state of four quarks, two s quarks and two $\bar{u}$ antiquarks $X(\bar{u}\bar{u}ss)$. Such particles (called tetraquarks) have not yet been observed, but they are currently searched for. We assume that the colour wave function for this bound state is anti-symmetric under the exchange of the two s quarks or under the exchange of the two $\bar{u}$ quarks. We further assume that the spatial part of the tetraquark wave function is in an $L = 0$ symmetric configuration.

   (a) What are the possible spins for this tetraquark? [4 marks]

   (b) Show that for a system of n spin-1/2 particles with ground-state spatial wave function and total angular momentum J we have

   $$\left\langle \sum_{i,j=1;i<j}^{n} \underline{s}_i \cdot \underline{s}_j \right\rangle = \frac{1}{2}\left(J(J+1) - \frac{3n}{4}\right)$$

   where $\underline{s}_i$ is the spin of the i-th particle. [4 marks]

   We assume a hyperfine mass splitting given by

   $$\Delta M = \frac{16}{9}\pi\alpha_S \left|\psi(0)\right|^2 \sum_{i,j=1;i<j}^{4} \frac{\left\langle \underline{s}_i \cdot \underline{s}_j \right\rangle}{m_i m_j} \,,$$

   where m_i is the constituent mass of the i-th quark. $|\psi(0)|^2$ is the probability that the two quarks are at the same position and α_S is the strong coupling.

   (c) Calculate the mass of the tetraquark for all possible total spins. Use $\alpha_s\pi\left|\psi(0)\right|^2 = 1.6 \times 10^7\,\text{MeV}^3$, $m_s = 538\,\text{MeV}$, $m_u = 363\,\text{MeV}$. [7 marks]

   (d) The tetraquark could decay into a pair of either pseudo-scalar ($m_{K^-} = 494\,\text{MeV}$) or vector kaons ($m_{K^{*-}} = 892\,\text{MeV}$), or one of each. Determine which decay channels are allowed (i.e. parity and angular momentum are conserved) for the state with the lowest spin. [5 marks]

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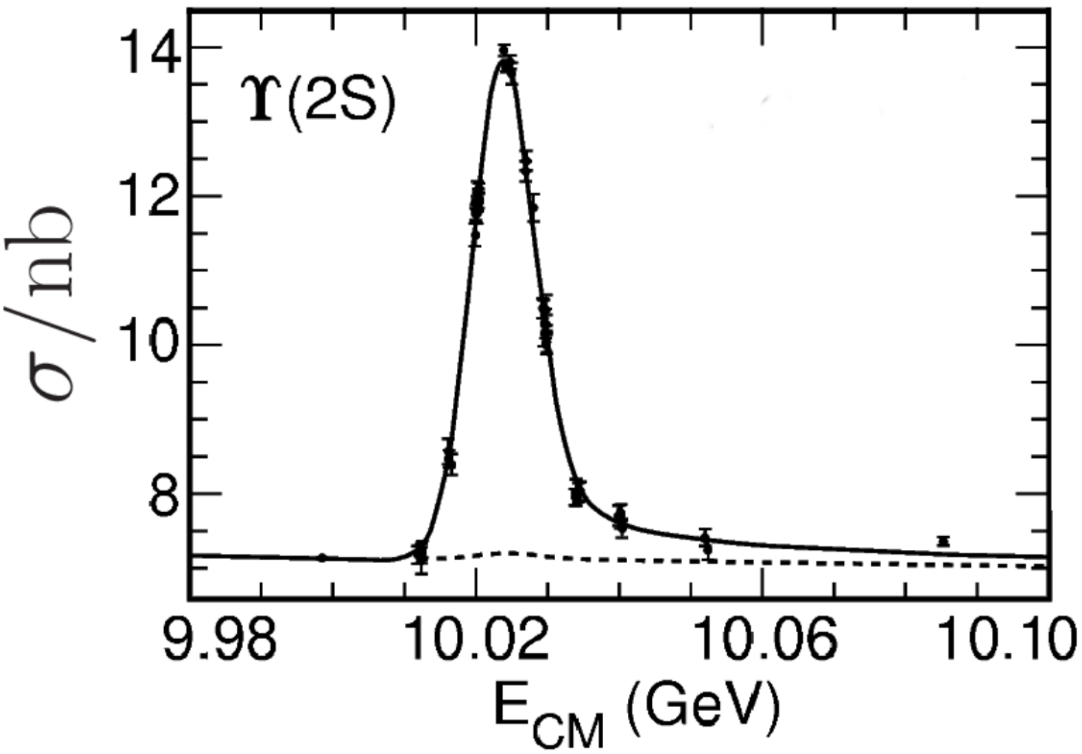
8. The cross section for the scattering of two particles with spins s_a and s_b via a resonance with spin J at centre-of-mass energy E is

$$\sigma(E) = \frac{\lambda^2(2J+1)}{4\pi(2s_a+1)(2s_b+1)} \frac{\Gamma_i\Gamma_f}{(E-M)^2+\frac{\Gamma^2}{4}},$$

where $\lambda = 4\pi/E$, M is the mass of the resonance, Γ is the total width of the resonance and $\Gamma_{i,f}$ are the partial widths for the decay of the resonance into the initial and final states, respectively.



a) Using the expansion for small ϵ

$$\int_{-\infty}^{\infty} \frac{\epsilon}{(x-x_0)^2+\epsilon^2} f(x)\,dx = \pi f(x_0) + \pi\epsilon f'(x_0) + \mathcal{O}(\epsilon^2)$$

show that for a function $f(E)$ that is not varying much around $E = M$ we have

$$\int_{E_{min}}^{E_{max}} f(E)\frac{1}{(E-M)^2+\frac{\Gamma^2}{4}}\,dE = \frac{2\pi}{\Gamma}f(M)\,,$$

provided $\Gamma \ll M$, and that the integral boundaries satisfy the conditions $\Gamma \ll (M - E_{min})$ and $\Gamma \ll (E_{max} - M)$. [4 marks]

Quantify the condition "not varying much" in terms of $f(M)$, $f'(M)$ and Γ. [2 marks]

b) Assuming that $\Gamma \ll M$ show that the cross section for the production of the $\Upsilon(2S)$ resonance (the bottomonium 2^3S_1 state) in e^+e^- collisions, followed by its decay into hadrons, integrated over all centre-of-mass energies, is

$$\sigma_{\text{int}} = \int_{E_{min}}^{E_{max}} \sigma(E)\mathrm{d}E = \frac{6\pi^2}{M^2_{\Upsilon(2S)}} \frac{\Gamma_{\Upsilon(2S)\to e^+e^-}\Gamma_{\Upsilon(2S)\to\text{hadrons}}}{\Gamma},$$

where $\Gamma_{\Upsilon(2S)\to e^+e-}$ and $\Gamma_{\Upsilon(2S)\to\text{hadrons}}$ are the partial widths for the decay of the $\Upsilon(2S)$ to e^+e^- and hadrons, respectively, and $M_{\Upsilon(2S)}$ is the mass of the $\Upsilon(2S)$. [4 marks]

c) For each cross section measurement with a fixed beam energy E the uncertainty in the beam energy produces a distribution in the centre-of-mass energies E' about the average centre-of-mass energy with a probability distribution $f(E - E')$. Show that the measured area under the resonance peak is the same as the true area under the peak, *i.e.*

$$\int \sigma_{\text{measured}}(E)\mathrm{d}E = \int \sigma(E)\mathrm{d}E. \quad \text{[4 marks]}$$

d) Assuming lepton universality ($\Gamma_{\Upsilon(2S)\to e^+e^-} = \Gamma_{\Upsilon(2S)\to\mu^+\mu^-} = \Gamma_{\Upsilon(2S)\to\tau^+\tau^-}$) and that the experimental measurement of the hadronic cross section includes all non-leptonic decays of the $\Upsilon(2S)$, use the plot of the measured hadronic cross section above and the measured branching ratio, $B(\Upsilon(2S) \to \mu^+\mu^-) = 2.03\%$, to estimate the mass and width of the $\Upsilon(2S)$ resonance. [6 marks]

[Hint: $(\hbar c)^2 = 0.389 \times 10^6\,\mathrm{GeV}^2\mathrm{nb}$. There is a significant non-resonant contribution to the cross section, in addition to the resonant production of the $\Upsilon(2S)$, which must be subtracted. The width of the distribution is dominated by the uncertainty in the beam energies and not by the fundamental width of the resonance.]