ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 16+17

13 Radiation and communication – Poynting vector

13.1 The Poynting Vector

The direction of propagation for an electromagnetic waves is $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}$. Poynting suggested that the instantaneous power per unit area for electromagnetic waves is given by $\underline{\boldsymbol{N}}$ where,

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \qquad \text{13-1}$$

[units E = $\mathrm{Vm^{-1}}$, H = $\mathrm{Am^{-1}}$, N = $\mathrm{VAm^{-2}} = \mathrm{Wm^{-2}}$].

General proof:

$$P_{radiation} = \int \underline{\boldsymbol{N}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}\right) \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad \text{13-2}$$

$$P_{radiation} = \int \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{N}}\mathrm{dV} = \int \underline{\boldsymbol{\nabla}} \cdot \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}\right)\mathrm{dV} \qquad \text{13-3}$$

$$\underline{\boldsymbol{\nabla}} \cdot \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}\right) = \underline{\boldsymbol{H}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}\right) - \underline{\boldsymbol{E}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}}\right) \qquad \text{13-4}$$

$$P_{radiation} = \int \underline{\boldsymbol{H}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}\right)\mathrm{dV} - \int \underline{\boldsymbol{E}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}}\right)\mathrm{dV} \qquad \text{13-5}$$

From Maxwell's equations for an infinite linear-isotropic-homogeneous medium we have:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} = -\mu_o \mu_r \frac{\partial \underline{\boldsymbol{H}}}{\partial \mathrm{t}} \qquad \text{13-6}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}} = \underline{\boldsymbol{J}}_{\mathrm{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial \mathrm{t}} = \underline{\boldsymbol{J}}_{\mathrm{free}} + \varepsilon_r \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad \text{13-7}$$

and

$$P_{radiation} = \int \underline{\boldsymbol{H}} \cdot \left(-\mu_o \mu_r \frac{\partial \underline{\boldsymbol{H}}}{\partial \mathrm{t}}\right)\mathrm{dV} - \int \underline{\boldsymbol{E}} \cdot \left(\underline{\boldsymbol{J}}_{\mathrm{free}} + \varepsilon_r \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}}\right)\mathrm{dV} \qquad \text{13-8}$$

$$P_{radiation} = -\frac{\partial}{\partial \mathrm{t}} \int \left(\frac{1}{2}\mu_r \mu_0 \mathrm{H}^2 + \frac{1}{2}\varepsilon_r \varepsilon_0 \mathrm{E}^2\right)\mathrm{dV} - \int \underline{\boldsymbol{E}} \cdot \underline{\boldsymbol{J}}_{\mathrm{free}}\mathrm{dV} \qquad \text{13-9}$$

The derivation is a self-consistent statement of the conservation of energy so ***N*** correctly represents the instantaneous power per unit area.

13.2 Maxwell's equations rewritten using potentials

For time independent problems, we use;

$$\underline{\boldsymbol{E}} = -\underline{\boldsymbol{\nabla}}V \qquad 13\text{-}10$$

where V is the electrostatic potential.

In general, the electric potential, V , is defined through,

$$\underline{\boldsymbol{E}} = -\underline{\boldsymbol{\nabla}}V - \frac{\partial \underline{\boldsymbol{A}}}{\partial t} \text{ (Definition of electric potential, V).} \qquad 13\text{-}11$$

Similarly, the magnetic vector potential, $\underline{\boldsymbol{A}}$, is defined as,

$$\underline{\boldsymbol{B}} = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}} \text{ (Defintion of magnetic vector potential)} \qquad 13\text{-}12$$

Consider Maxwell II,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0 \text{ since } \underline{\boldsymbol{\nabla}} \cdot (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}) = 0 \qquad 13\text{-}13$$

Consider Maxwell III:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \underline{\boldsymbol{\nabla}} \times \left(-\underline{\boldsymbol{\nabla}}V - \frac{\partial \underline{\boldsymbol{A}}}{\partial t}\right) = -\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}}V - \frac{\partial}{\partial t}\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}$$

$$= -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \text{ since } \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}}V = 0 - \text{ note similarity with vector algebra.} \qquad 13\text{-}14$$

Hence, if we rewrite $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{E}}$ in terms of $\underline{\boldsymbol{A}}$ and V, Maxwell II and III are *automatically solved*. Maxwell's four equations are thus reduced to two equations:

$$-\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}}V - \frac{\partial}{\partial t}\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = \frac{\rho}{\varepsilon_o} - \text{ Maxwell I} \qquad 13\text{-}15$$

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}) = \mu_o \underline{\boldsymbol{J}} + \mu_o \varepsilon_o \frac{\partial}{\partial t}\left[-\underline{\boldsymbol{\nabla}}V - \frac{\partial \underline{\boldsymbol{A}}}{\partial t}\right] - \text{ Maxwell IV} \qquad 13\text{-}16$$

The Lorentz condition constrains $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}}$,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = -\mu_o \varepsilon_o \frac{\partial V}{\partial t} \qquad 13\text{-}17$$

Maxwell I becomes

$$-\nabla^2 V + \mu_o \varepsilon_o \frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\varepsilon_o} - \text{ Wave equation for V} \qquad 13\text{-}18$$

and Maxwell IV becomes

$$-\nabla^2 \underline{\boldsymbol{A}} + \mu_o \varepsilon_o \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu_o \underline{\boldsymbol{J}} \text{ - Wave equation for } \underline{\boldsymbol{A}} \qquad 13\text{-}19$$

13.3 Hertzian dipole/ oscillating electric dipole

$$I(t)\underline{\boldsymbol{\delta l}} = \omega \underline{\boldsymbol{p}}(t) \qquad 13\text{-}20$$

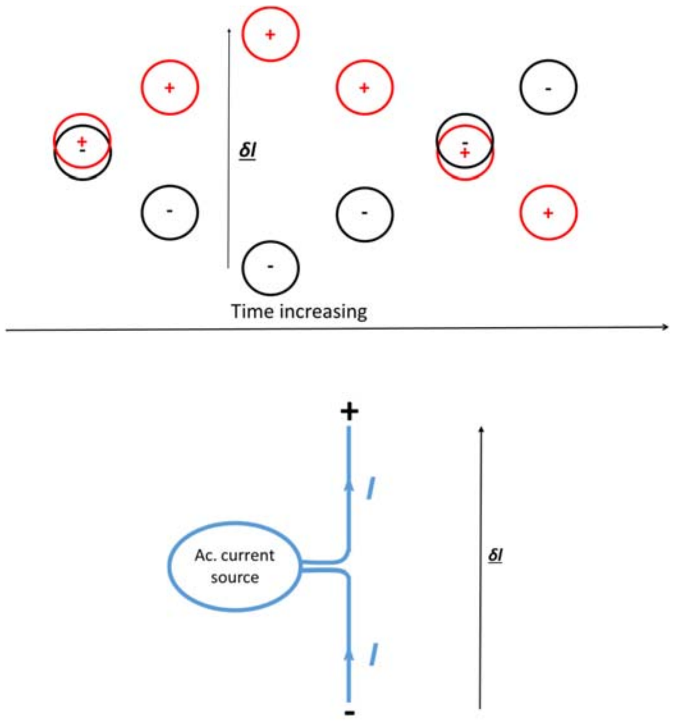


Figure 1 : A Hertzian dipole which is an alternating electric dipole moment. The dipole is equivalent to a length of wire in which a time dependent alternating current is flowing.

We can start to solve the wave equation for $\underline{\boldsymbol{A}}$ by considering (static) solutions for:

$$-\nabla^2 \underline{\boldsymbol{A}} = \mu_o \underline{\boldsymbol{J}} \qquad 13\text{-}21$$

From electrostatics, using $\underline{\boldsymbol{E}} = -\underline{\boldsymbol{\nabla}} V$, we know the solution for:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = -\nabla^2 V = \frac{\rho}{\varepsilon_o} \qquad 13\text{-}22$$

is

$$V = \frac{Q}{4\pi\varepsilon_o r} \text{ where } Q = \int \rho \, dV \qquad 13\text{-}23$$

By inspection, the static solution for $\underline{\boldsymbol{A}}$ is:

$$\underline{\boldsymbol{A}} = \frac{\mu_o I \underline{\boldsymbol{\delta l}}}{4\pi r} \qquad \text{where } I\underline{\boldsymbol{\delta l}} = \int \underline{\boldsymbol{J}}\, dV \qquad \text{13-24}$$

Recognising that if we set $\underline{\boldsymbol{J}}$ to be zero, we find oscillatory solutions for the complex wave equation for $\underline{\tilde{\boldsymbol{A}}}$. Hence the general solution to the complex wave equation for $\underline{\tilde{\boldsymbol{A}}}$ is

$$\underline{\tilde{\boldsymbol{A}}} = \frac{\mu_o I_0 \underline{\boldsymbol{\delta l}}}{4\pi r} e^{i(kr - wt)} \qquad \text{13-25}$$

where $\tilde{I}(t) = I_0 e^{-i(wt)}$. In spherical coordinates $\underline{\boldsymbol{\delta l}} = \delta\mathrm{l}\ (\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})$. Using $\underline{\tilde{\boldsymbol{B}}} = \underline{\boldsymbol{\nabla}} \times \underline{\tilde{\boldsymbol{A}}}$ so we have (leaving it as a homework problem)

$$\underline{\tilde{\boldsymbol{B}}}(r,\theta,\phi) = \left(\tilde{B}_{\mathrm{r}}\hat{\mathbf{r}} + \tilde{B}_{\theta}\hat{\boldsymbol{\theta}} + \tilde{B}_{\phi}\hat{\boldsymbol{\phi}}\right) = \frac{\mu_o I_0 \delta l}{4\pi r^2}(1 - ikr)\sin(\theta) e^{i(kr-\omega t)}\hat{\boldsymbol{\phi}} \qquad 13\text{-}26$$

$$\underline{\boldsymbol{B}}_{\mathrm{Near}} = \frac{\mu_o I_0 \delta l}{4\pi r^2}\sin(\theta)\cos(kr - \omega t)\hat{\boldsymbol{\phi}} \qquad 13\text{-}27$$

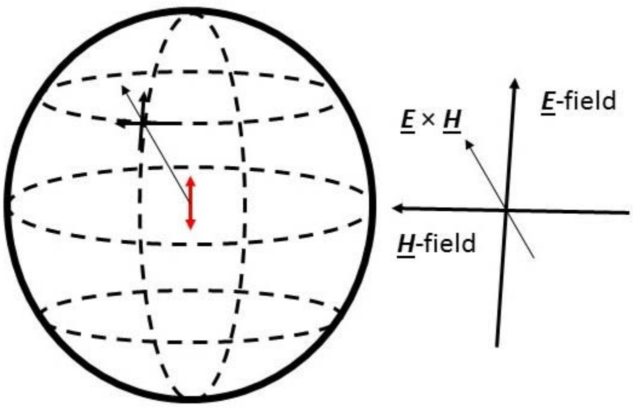
$$\underline{\boldsymbol{B}}_{\mathrm{Far}} = \frac{\mu_o I_0 \delta l}{4\pi r} k \sin(\theta)\sin(kr - \omega t)\hat{\boldsymbol{\phi}} \qquad 13\text{-}28$$

$$\underline{\tilde{\boldsymbol{E}}} = \frac{i}{\mu_0 \varepsilon_0 \omega}\,\underline{\boldsymbol{\nabla}} \times \underline{\tilde{\boldsymbol{B}}} \qquad 13\text{-}29$$

$$\underline{\tilde{\boldsymbol{E}}} = \frac{\mu_o I_0 \delta l}{2\pi}\frac{\omega(i + kr)\cos(\theta)}{k^2 r^3} e^{i(kr-\omega t)}\hat{\mathbf{r}} + \frac{\mu_o I_0 \delta l}{4\pi}\frac{\omega(i + kr - ik^2 r^2)\sin(\theta)}{k^2 r^3} e^{i(kr-\omega t)}\hat{\boldsymbol{\theta}} \qquad 13\text{-}30$$

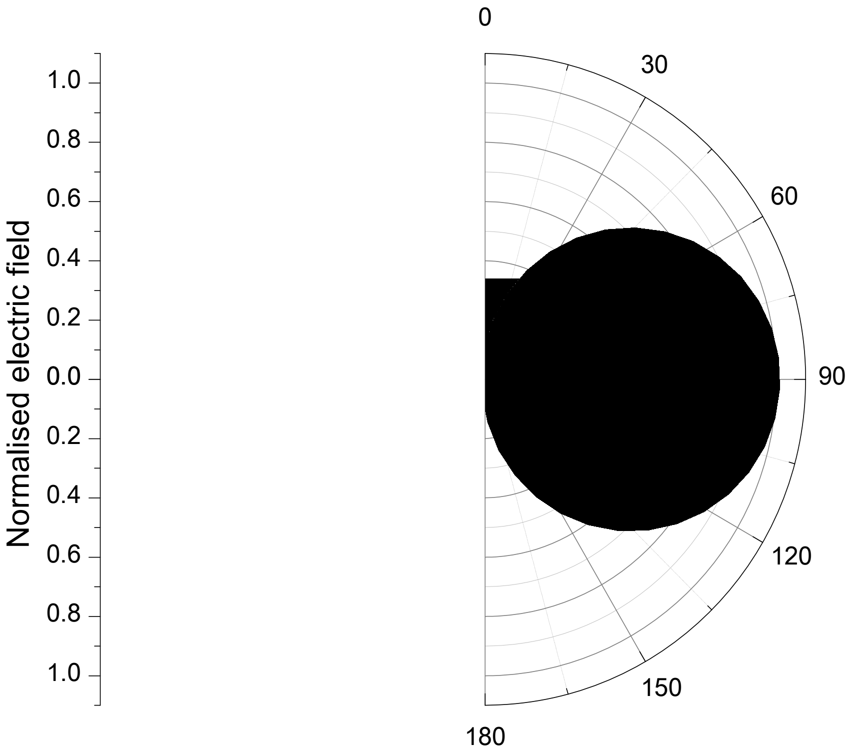
$$\underline{\boldsymbol{E}}_{\mathrm{Near}} = -\frac{\mu_o I_0 \delta l}{2\pi r^3}\frac{\omega}{k^2}\cos(\theta)\sin(kr - \omega t)\hat{\boldsymbol{r}} - \frac{\mu_o I_0 \delta l}{4\pi r^3}\frac{\omega}{k^2}\sin(\theta)\sin(kr - \omega t)\hat{\boldsymbol{\theta}} \qquad 13\text{-}31$$

$$\underline{\boldsymbol{E}}_{\mathrm{Far}} = \frac{\mu_o I_0 \delta l}{4\pi r}\omega\sin(\theta)\sin(kr - \omega t)\hat{\boldsymbol{\theta}} \qquad 13\text{-}32$$



E points up and down, B left to right horizontally

Figure 2 : The relative directions of the radiation fields of a Hertzian dipole. The red arrow at the centre is the Hertzian dipole. The radiating fields are tangential to the surface of a sphere centred on the dipole. For an isotropic medium the ***H***-field is parallel to the ***B***-field.



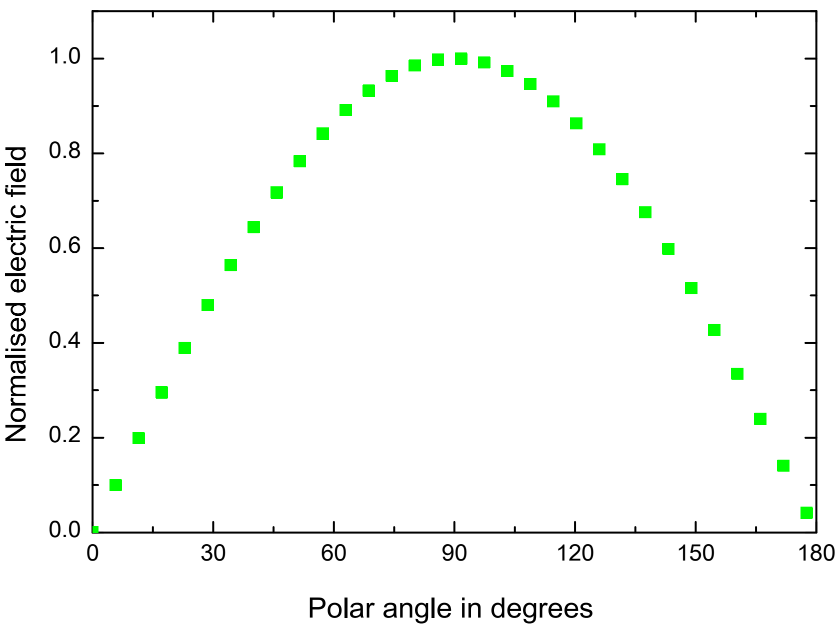
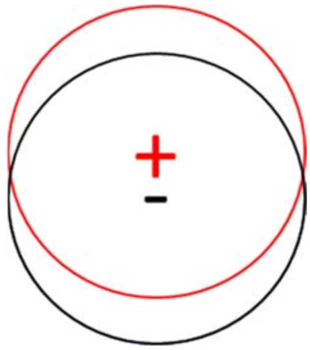
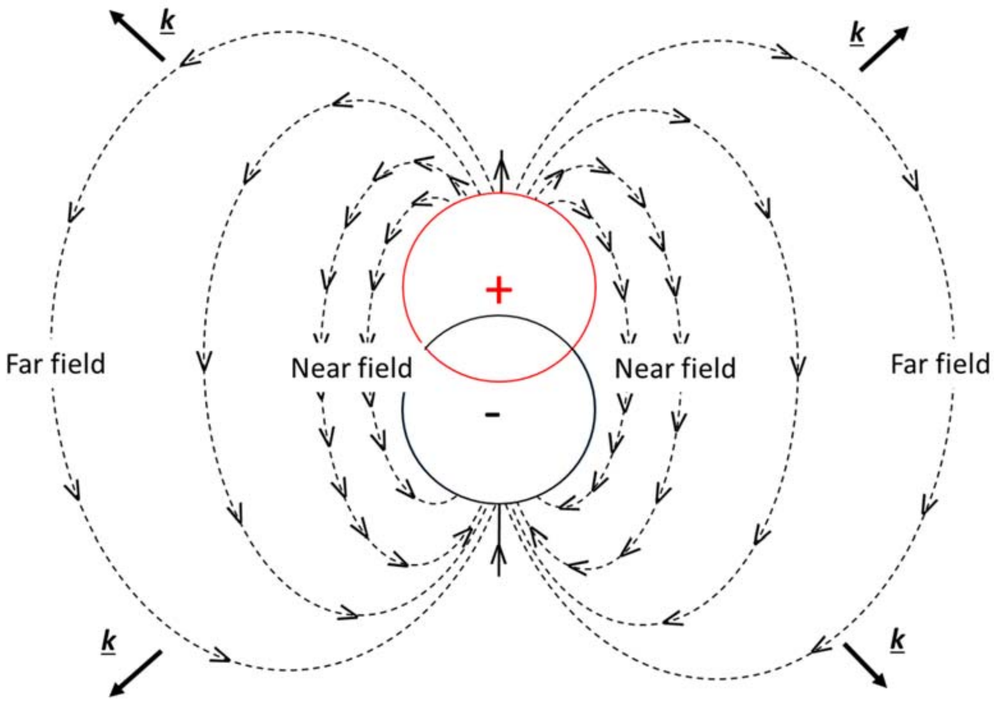


Figure 3 : The polar diagram for a Hertzian dipole. The solid curved line (i.e. circular) gives the relative strengths of the radiation field at different points on the surface of a sphere centred on the dipole. The dipole lies along the z-axis and the field is independent of the azimuthal angle. There is no radiation produced along the z-axis.

(i)



(ii)



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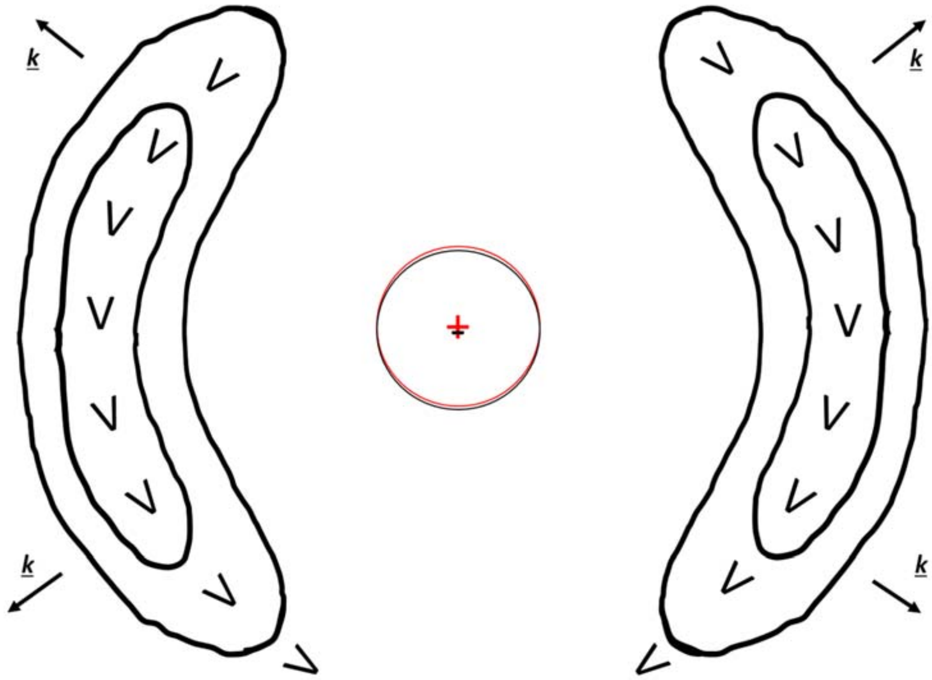


Figure 4 : The production of the first half of a wave in snapshots. We consider the initial state where the current is at its maximum value as the two charges pass each other. We chose not to draw all fields prior to the initial state. (i) The charges separate and there are E-fields (not shown) very close to the charges. (ii) The v-shaped chevrons show the direction of the electric field. For any snapshot in time, the charges behave like an extended dipole with both near field and far field terms contributing to the total E-field. (iii) The first half wave separates and radiates from the dipole. The solid lines are contours. The E-field points in the positive theta direction throughout the half-wave.

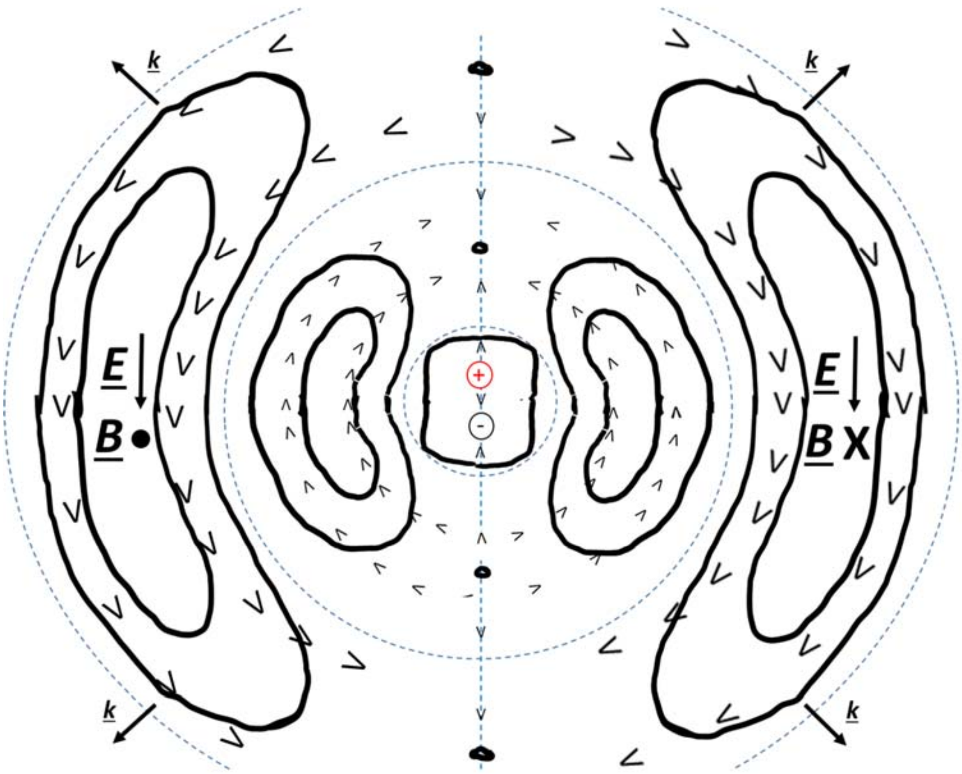
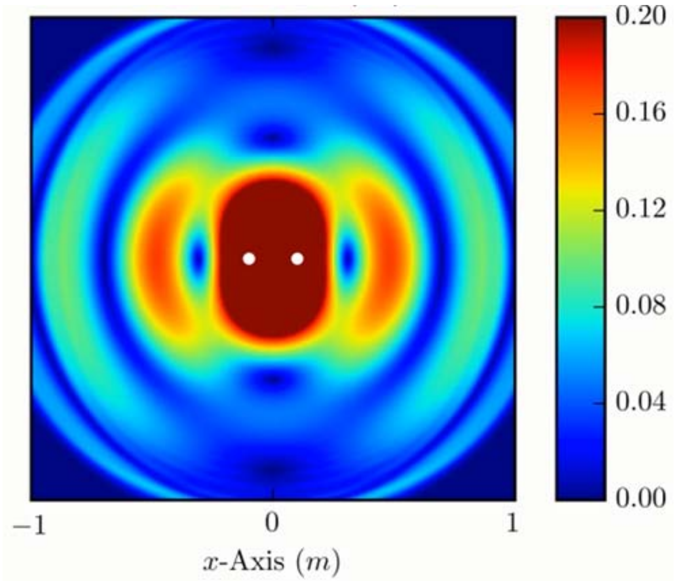


Figure 5 : The E-fields and B-fields in the far field. The complete wave is shown propagating away. The solid lines are contours. The E-field points in the direction of the v-shaped chevrons. Each magnetic field line is a circle with a dipole along its axis. The cross product E×B is directed away from the dipole (i.e. in the r-direction) at all points in space where and the right-hand screw rule for E×B gives the direction of B with respect to E.



A visualisation produced by Hooper:
http://community.dur.ac.uk/superconductivity.durham/Teaching%20Materials.html

13.3.1 The angular dependence of the power radiated by a Hertzian dipole

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \tag{13-33}$$

The angular dependence of time averaged power per unit area is:

$$\underline{\boldsymbol{N}}_{\text{time averaged}}(\theta) = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} = \frac{1}{2}\frac{\mu_0 c {I_0}^2 (\delta l)^2 \sin^2\theta}{4r^2\lambda^2}\underline{\hat{\boldsymbol{r}}} \tag{13-34}$$

13.3.2 The time averaged power (P_{Total}) radiated by a Hertzian dipole.

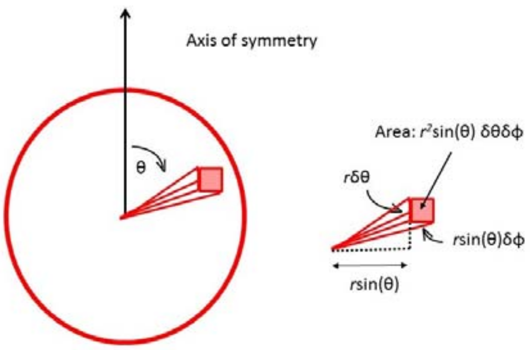


Figure 6 : An element of area in spherical coordinates.

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \tag{13-35}$$

$$P_{\text{Total}} = \int \frac{1}{2}\frac{\mu_0 c {I_0}^2 (\delta l)^2 \sin^2\theta}{4r^2\lambda^2} r^2 \sin\theta d\theta d\phi$$

$$= \frac{\mu_0 c {I_0}^2 (\delta l)^2}{8\lambda^2} \int_0^{\pi} \sin^3\theta d\theta \int_0^{2\pi} d\phi \tag{13-36}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots \frac{4}{3} \dots\dots\dots 2\pi$$

$$P_{\text{Total}} = \frac{\mu_0 c\pi}{3}\left(\frac{\delta l}{\lambda}\right)^2 {I_0}^2 \tag{13-37}$$

<u>Antennae:</u>

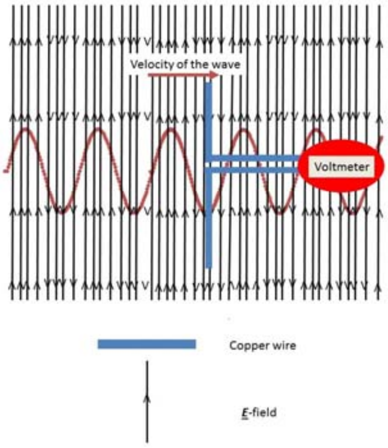


Figure 7 : An electric dipole antenna for detecting electromagnetic waves. The alternating electric field of the incoming wave produces an alternating current in the antenna.

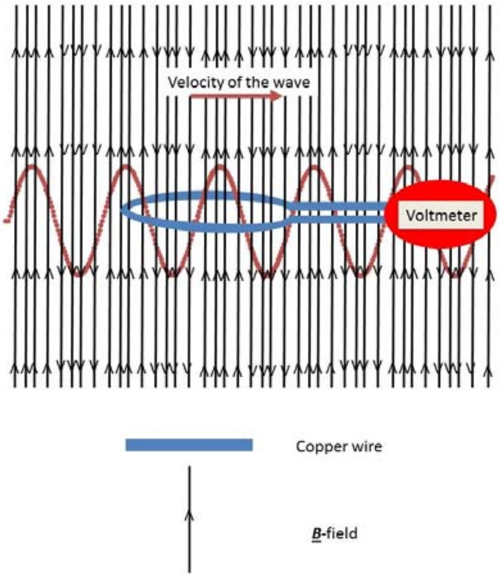


Figure 8 : A loop antenna for detecting electromagnetic radiation. The alternating magnetic flux through the loop due to the magnetic field of the radiation induces an alternating current in the loop.