University of Durham

EXAMINATION PAPER

May/June 2016 Examination code: PHYS3621-WE01

FOUNDATIONS OF PHYSICS 3A

SECTION A. Quantum Mechanics 3
SECTION B. Nuclear and Particle Physics

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

Continued Next Page

SECTION A. QUANTUM MECHANICS 3

Answer Question 1 and **at least one** of Questions 2 and 3.

1. (a) A time-independent Hamiltonian, H_0, is perturbed by a Hamiltonian $H'(t)$ such that $H'(t) \equiv 0$ for $t < 0$. The system for which $H_0 + H'(t)$ is the total Hamiltonian is in the eigenstate a of H_0 for $t < 0$. In first order perturbation theory, the probability $P_{ba}(t)$ that it is in the eigenstate b of H_0 at time $t > 0$ is given by the following equation, assuming that $\omega_{ba} \neq 0$:

$$P_{ba}(t) = \frac{1}{\hbar^2}\left|\int_0^t H'_{ba}(t')\exp(i\omega_{ba}t')dt'\right|^2,$$

where $\omega_{ba} = (E_b - E_a)/\hbar$, with E_a and E_b the eigenenergies of the states a and b, respectively. (i) Define the quantity $H'_{ba}(t')$ appearing in this equation in terms of eigenstates of H_0. (ii) Show that this equation reduces to

$$P_{ba}(t) = \frac{|H'_{ba}|^2}{\hbar^2\omega_{ba}^2}\Big(2 - 2\cos(\omega_{ba}t)\Big)$$

if $H'(t)$ is constant in time for $t > 0$ (and thus $H'_{ba}(t') \equiv H'_{ba}$ for $t > 0$). [4 marks]

(b) Explain the electric dipole approximation. [4 marks]

(c) (i) What are the eigenenergies of the Hamiltonian $\underline{L}^2/(2I)$, where $\underline{L}$ is the angular momentum operator and I is a constant moment of inertia?
(ii) The lowest excited state of an atom of sodium has a lifetime, τ, of about 16 ns, and can only decay to the ground state of the atom. What is the uncertainty on the corresponding Bohr transition frequency arising from the finite lifetime of the excited state? [4 marks]

(d) (i) In the approximation where the nucleus has an infinite mass, and spin-orbit coupling and other relativistic effects are neglected, the Hamiltonian of atomic hydrogen is

$$-\frac{\hbar^2}{2m_e}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{|\underline{r}|}.$$

Write down the corresponding expression for the Hamiltonian of helium.
(ii) The eigenfunctions of the latter Hamiltonian that are symmetric for the interchange of the spatial coordinates of the two electrons necessarily correspond to spin-singlet states. Why is this so? [4 marks]

(e) Consider a system of two electrons prepared in the triplet spin-state $\big(\alpha(1)\beta(2) + \beta(1)\alpha(2)\big)/\sqrt{2}$, where $\alpha(n)$ and $\beta(n)$ denote the states of spin up and of spin down of electron n ($n = 1, 2$). Let $S_{1z} = \hat{\underline{z}} \cdot \underline{S}_1$ and $S_{2z} = \hat{\underline{z}} \cdot \underline{S}_2$, where $\underline{S}_1$ is the spin operator for electron 1, $\underline{S}_2$ is the spin operator for electron 2, and $\hat{\underline{z}}$ is a unit vector in the z-direction. Show that this spin-state is an eigenstate of $S_{1z} + S_{2z}$ and find the corresponding eigenvalue. [4 marks]

(f) What is the Born-Oppenheimer approximation, in relation to the calculation of the wave functions and energy levels of molecules? [4 marks]

2. (a) Suppose that an atomic state b can decay only by spontaneous emission to a certain state a. How is the corresponding A-coefficient (A_{ab}) related to the lifetime of the state b? [3 marks]

(b) In the case of atomic hydrogen, and neglecting relativistic corrections, only the ground state ($1\mathrm{s}_{m=0}$) is lower in energy than the $2\mathrm{p}_{m=0}$ state and the $2\mathrm{p}_{m=1}$ state. Can either of these two excited states spontaneously decay to the ground state according to the electric dipole selection rules? (Justify your answer.) [3 marks]

(c) Show that

$$\begin{aligned}
&\left|\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, Y_{1,0}^*(\theta,\phi)\, x\, Y_{0,0}(\theta,\phi)\right|^2 = \\
&\left|\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, Y_{1,0}^*(\theta,\phi)\, y\, Y_{0,0}(\theta,\phi)\right|^2 = \\
&\qquad = \left|\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, Y_{1,1}^*(\theta,\phi)\, z\, Y_{0,0}(\theta,\phi)\right|^2 = 0
\end{aligned}$$

and that

$$\begin{aligned}
&\left|\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, Y_{1,1}^*(\theta,\phi)\, x\, Y_{0,0}(\theta,\phi)\right|^2 + \\
&\left|\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, Y_{1,1}^*(\theta,\phi)\, y\, Y_{0,0}(\theta,\phi)\right|^2 \\
&\qquad = \left|\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, Y_{1,0}^*(\theta,\phi)\, z\, Y_{0,0}(\theta,\phi)\right|^2,
\end{aligned}$$

given that $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. [8 marks]

$$\left[\begin{array}{ll}
\text{Hints:} & \\
(1) & Y_{0,0}(\theta,\phi) = \sqrt{\frac{1}{4\pi}}, \quad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}. \\
(2) & \int_0^\pi \sin^3\theta\, d\theta = \int_{-1}^1 (1-\cos^2\theta)\, d(\cos\theta).
\end{array}\right]$$

(d) Discuss whether the $2\mathrm{p}_{m=0}$ and $2\mathrm{p}_{m=1}$ states of atomic hydrogen have the same lifetime. [6 marks]

  Continued Next Page

3. The energy functional that the Rayleigh-Ritz variational method is based on is defined by the following general equation:

$$E[\phi] = \frac{\int \phi^*(\tau) H \phi(\tau)\, d\tau}{\int \phi^*(\tau)\phi(\tau)\, d\tau},$$

where H is the Hamiltonian and the integration extends over the full range of all the coordinates of the system.

Consider a one-dimensional system containing a single particle of coordinate x and mass m confined to the region $-L < x < L$ by infinite potential walls located at $x = -L$ and $x = L$. Suppose that inside this region, the particle has a potential energy $V(x)$, with

$$V(x) = \frac{\hbar^2}{mL^2}\left(\frac{x}{L}\right)^2.$$

(a) Show that $E[\phi] = (39/28)\,\hbar^2/(mL^2)$ if

$$\phi(x) = \begin{cases} 1 - (x/L)^2 & \text{for } -L \le x \le L, \\ 0 & \text{otherwise.} \end{cases}$$

[10 marks]

(b) Let E_0 be the exact ground state energy of this system. E_0 cannot be calculated analytically. Is E_0 certainly lower than $E[\phi]$, or could it be that $E_0 = E[\phi]$ or even that $E_0 > E[\phi]$? Justify your answer. [6 marks]

(c) Now, let

$$\chi(x) = \begin{cases} (x/L)[1 - (x/L)^2] & \text{for } -L \le x \le L, \\ 0 & \text{otherwise,} \end{cases}$$

and let E_1 be the exact energy of the first excited state of this system. E_1 cannot be calculated analytically either. Is $E_1 \le E[\chi]$, or could it be that $E_1 > E[\chi]$? Justify your answer but do not calculate $E[\chi]$. [4 marks]

  Continued Next Page

SECTION B. NUCLEAR AND PARTICLE PHYSICS

Answer Question 4 and **at least one** of Questions 5, 6, 7 and 8.

4. (a) Express the condition on the *nuclear* mass $m(Z, A)$ of a nucleus ${}^A_Z$X with proton number Z and mass number A for it to be allowed to decay through the following decays. Express the conditions in terms of nuclear masses (not atomic masses) and the electron mass m_e. Neglect electron binding energies. [4 marks]

   - β^--decay
   - β^+-decay
   - α-decay
   - electron capture

   (b) A nucleus with mass M_1 decays through α-decay to a nucleus of mass M_2. Show that the momentum of the α-particle is given by

$$|\vec{p}| = \sqrt{\frac{(M_1^2 + M_2^2 - M_\alpha^2)^2 - 4M_1^2 M_2^2}{4M_1^2}}\ ,$$

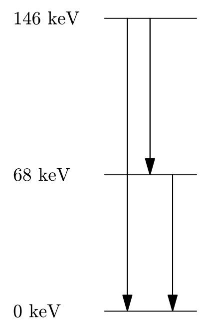
   where M_α is the mass of the α-particle. [4 marks]

   (c) What are the spin and parity predicted by the shell model for ${}^{39}_{19}$K? (The order of the lowest shells in the shell model for protons and neutrons is $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, ...) [4 marks]

   (d) ${}^{44}_{22}$Ti decays through β^+-decay to the two first excited states of ${}^{44}_{21}$Sc. ${}^{44}_{22}$Ti has $J^P = 0^+$ and the ground state of ${}^{44}_{21}$Sc has $J^P = 2^+$.

   The following gamma emission energies are observed:

Energy [keV]	multipolarity
146	M2
78	M1
68	E1



   No other gamma energies have been seen. Given that the probability for the β-decay into the two excited states is much larger than the probability to decay into the ground state, give the spin and parity of the excited states, explaining your reasoning. [4 marks]

(e) The semi-empirical mass formula for a nucleus with Z protons and N neutrons contains an asymmetry term

$$a_a \frac{(Z-N)^2}{4A}$$

where a_a is fitted to experimental data and $A = Z + N$. Explain the source of this term and explain qualitatively how the neutron-to-proton ratio in stable nuclei would change if there were two distinct types of neutron that have identical properties under the nuclear force. [4 marks]

(f) Give all possible values of the orbital angular momentum of the final state wave-function allowed by parity and angular momentum conservation for the following decays.

(i) $\rho^+ \to \pi^+\pi^0$
(ii) $\omega \to \mu^+\mu^-$
(iii) $\pi^0 \to e^+e^-$
(iv) $f_2 \to \rho^+\rho^-$

$\rho^{+,0,-}$, ω are vector mesons with $J^P = 1^-$ and the pions $\pi^{-,0,+}$ have $J^P = 0^-$. The f_2 meson has $J^P = 2^+$. [4 marks]

(g) Explain why uud and uuu baryons have different possible spins. [4 marks]

(h) Explain the effect of parity violation in the leptonic decay of the charged pions. Is the decay involving a muon more or less likely than that involving an electron? [4 marks]

(i) For each of the following reactions find which particle X has to be so that the reaction is allowed in the Standard Model and draw a possible Feynman diagram for it. [4 marks]

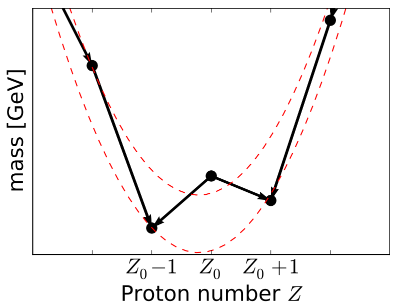
- $\mu^- \to e^-\bar{\nu}_e X$
- $e^+e^- \to tX$
- $\nu_e X \to \nu_\mu e^-$
- $X \to \nu_e\bar{\nu}_e$

5. The semi-empirical mass formula predicts the *nuclear* mass for a nucleus of even mass number A, proton number Z and neutron number N to be

$$\begin{aligned} m(Z,A) &= Zm_p + Nm_n - a_V A + a_S A^{\frac{2}{3}} + a_c \frac{Z^2}{A^{\frac{1}{3}}} + a_a \frac{(N-Z)^2}{4A} + \frac{\delta}{A^{\frac{1}{2}}} \\ \delta &= \begin{cases} -\delta_p & \text{for even } Z \text{ and } N \text{ (even-even nuclei)} \\ \delta_p & \text{for odd } Z \text{ and } N \text{ (odd-odd nuclei)} \end{cases}, \end{aligned}$$

with $m_p = 938.3\,\mathrm{MeV}/c^2$, $m_n = 939.6\,\mathrm{MeV}/c^2$, $a_V = 15.67\,\mathrm{MeV}/c^2$, $a_S = 17.23\,\mathrm{MeV}/c^2$, $a_c = 0.714\,\mathrm{MeV}/c^2$, $a_a = 93.15\,\mathrm{MeV}/c^2$ and $\delta_p = 11.2\,\mathrm{MeV}/c^2$.

In this question we consider the case where an isobar (set of nuclei with the same A) has two stable even-even nuclei with one odd-odd nucleus that can decay into either stable one, as illustrated below.



(a) Explain the source of the pairing term in the semi-empirical mass formula. [3 marks]

(b) Prove that

$$2A^{1/2}\delta_p - a_a - A^{2/3}a_c > 0 \ ,$$

is a necessary condition for the ordering of the nuclear masses

$$m(Z_0-1,A) < m(Z_0,A) > m(Z_0+1,A) \ ,$$

where Z_0 is the atomic number of the odd-odd nucleus. [10 marks]

[Hint: You can reduce the amount of algebra by considering the mass difference $m(Z_0,A) - m(Z_0+s,A)$ as a function of s.]

(c) When the even-even nuclei cannot decay through β-decay to the odd-odd nucleus the heaviest of the two can decay into the other through a so-called *double beta decay* where two protons (neutrons) are simultanously exchanged for two neutrons (protons), two positrons (electrons) and two neutrinos (anti-neutrinos). What is the maximum momentum an electron can have in such a decay in the rest frame of the parent nucleus? The parent nucleus has a mass M_1, the daughter nucleus a mass M_2 and the mass of the neutrino can be neglected. [7 marks]

6. The baryons which contain a single charm quark (c), at least one strange (s), and at most one up (u) or down (d) quark are: $\Xi_c^+(cus)$; $\Xi_c^0(cds)$; $\Omega_c^0(css)$. In this problem consider only $L=0$ states.

   a) What are the possible spins S_{qq} of the pair of non-charm quarks in each of these baryons? Use the spins of the pair of non-charm quarks to obtain the possible spins of the baryons and hence show that there are two Ω_c^0 states, one with $J=\frac{1}{2}$ and the other with $J=\frac{3}{2}$, and determine the possible $\Xi_c^{+,0}$ states. [4 marks]

   b) Show that for a ground-state baryon with total angular-momentum J

$$\sum_{\substack{i,j=1\\ i<j}}^{3} \vec{s}_i \cdot \vec{s}_j = \frac{1}{2}\left[J(J+1)-\frac{9}{4}\right]. \qquad \text{[2 marks]}$$

   c) The hyperfine mass splitting for baryons is

$$\Delta M_{ss} = \frac{16}{9}\pi\alpha_S\,|\psi(0)|^2 \sum_{\substack{i,j=1\\ i<j}}^{3} \frac{\vec{s}_i \cdot \vec{s}_j}{m_i m_j},$$

   where the sum is over the three quarks in the baryon, and $\vec{s}_i$ and m_i are the spin and constituent masses of each quark respectively. Here $|\psi(0)|^2$ is the probability that the two quarks are at the same point in space.

   i) Calculate the masses of the two Ω_c^0 states given that the constituent masses of the strange and charm quarks are $m_s = 538\,\text{MeV}$ and $m_c = 1625\,\text{MeV}$, respectively, and $\pi\alpha_S|\psi(0)|^2 = 2.4\times10^7\,\text{MeV}^3$. [10 marks]

   ii) The masses of the lightest Ξ_c baryons are 2468 MeV for Ξ_c^+ and 2471 MeV for Ξ_c^0. The mass of the kaons are 494 MeV for the K^{+-} and 498 MeV for K^0. Determine how the $J=\frac{3}{2}$ Ω_c^0 baryon is most likely to decay and draw a quark-line diagram for this process. [4 marks]

7. As for protons one can define parton distribution functions for mesons. Consider the parton distribution of the pion π^- whose valence quark composition is $\bar{u}d$. We consider the deep inelastic kinematics in the parton model where high energy electrons scatter off a constituent of the meson.

   (a) Show that if the scattered parton with a fraction η of the pion momentum is massless we have

   $$\eta = x \;, \qquad \text{with} \qquad x = \frac{Q^2}{2P \cdot q} \;, \qquad Q^2 = -q^2$$

   where P is the pion four-momentum and $q = p - p'$ is the momentum transfer between the initial-state and final-state electron momenta p and p'. [3 marks]

   We denote the probability of finding a $\bar{u}$ quark with momentum fraction x by $\bar{u}(x)$ and the probability of finding a d quark with $d(x)$. Since the dynamics of the meson bound state is dominated by the strong force, which does not distinguish between the $\bar{u}$ and d, we can use the approximation $\bar{u}(x) = d(x)$. In the following we neglect sea quarks and gluon parton distribution functions.

   (b) What constraint does the above assumption put on the functional form of $\bar{u}(x)$ and what are the values of the following integrals? [3 marks]

   $$I_1 = \int_0^1 \bar{u}(x)\, dx \;, \qquad I_2 = \int_0^1 x\bar{u}(x)\, dx \;.$$

   (c) In the high energy limit the differential cross section has the form

   $$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\,\frac{F_2(x,Q^2)}{x} + \frac{y^2}{2}\frac{2xF_1(x,Q^2)}{x}\right],$$

   where F_1 and F_2 are the form factors and $y = Q^2/(sx)$. Given that the cross section for the scattering of an electron from a massless spin-$\frac{1}{2}$ particle with charge Q_f is

   $$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{2\pi\alpha^2 Q_f^2}{Q^4}\left(1 + (1-y)^2\right),$$

   show that the form factors $F_1(x,Q^2)$ and $F_2(x,Q^2)$ are given by

   $$F_2(x,Q^2) = 2xF_1(x,Q^2) = \frac{5}{9}x\bar{u}(x) \;. \qquad [6 \text{ marks}]$$

   (d) Calculate the differential cross section $d\sigma/dQ^2$ if the parton distribution function is given by

   $$\bar{u}(x) = d(x) = x^\beta(1-x)^\beta \;.$$

   Express your result in terms of integrals

   $$I(\alpha,\beta) = \int_0^1 x^\alpha(1-x)^\beta\, dx \;. \qquad [8 \text{ marks}]$$

8. The τ lepton mass $m_\tau = 1.777$ GeV is heavy enough to produce mesons in its decay, in particular, the charged pion or kaon

$$\tau^- \to \pi^- + \nu_\tau \quad \text{and} \quad \tau^- \to K^- + \nu_\tau \,.$$

The charged pion π^- has mass $139.6\,\mathrm{MeV}/c^2$ and the charged kaon K^- has mass $493.7\,\mathrm{MeV}/c^2$.

(a) Draw the Feynman diagrams for both decays. [2 marks]

(b) What is the preferred direction of the meson with respect to the direction of the spin of the τ lepton? [4 marks]

(c) Consider the decay of a particle with mass M into a particle with mass m and a massless particle. Determine the energies of the outgoing particles and the absolute values of their momenta in the rest frame of the decaying particle. [6 marks]

(d) The matrix element squared for the decay of the τ lepton to a pseudo-scalar meson (PS) obeys the following relation

$$|\mathcal{M}_{\tau^-\to PS+\nu_\tau}|^2 \propto |V|^2 \, f_{PS}^2 \, |\underline{p}_{PS}|^2 \,,$$

where V is the relevant CKM matrix element (i.e. V_{us} for the decay into a kaon and V_{ud} for the decay into a pion), $PS = \pi^-, K^-$ and the decay constants are given by $f_\pi = 130$ MeV and $f_K = 160$ MeV. The branching ratios are proportional to the respective widths given by

$$\Gamma_{\tau^-\to PS+\nu_\tau} = \frac{|\underline{p}_{PS}|}{8\pi m_\tau^2} \cdot |\mathcal{M}_{\tau^-\to PS+\nu_\tau}|^2$$

and the branching ratios are given by $BR(\tau \to X) = \Gamma_{\tau\to X}/\Gamma$. The experimental values are:

$$\begin{aligned} BR(\tau^- \to \pi^- + \nu_\tau) &= 10.90 \pm 0.070\,\% \\ BR(\tau^- \to K^- + \nu_\tau) &= 0.691 \pm 0.023\,\% \,. \end{aligned}$$

Calculate $\sin\theta_c$ where θ_c is the Cabibbo angle: $V_{ud} \simeq \cos\theta_c$, $V_{us} \simeq \sin\theta_c$. [8 marks]

