ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 1+2+3 (Rev 4)_

1 Fabulous Science.

1.1 Maxwell's Equations and Classical Physics

FEYNMAN claims there are 7 equations that describe all of classical Physics

Maxwell's 4 equations:

From Coulomb's law:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \quad \text{(MI)} \qquad \textbf{1-1}$$

Given no magnetic monopoles have been observed:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{B}} = 0 \quad \text{(MII)} \qquad \textbf{1-2}$$

From Faraday's law of induction:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \frac{-\partial \underline{\boldsymbol{B}}}{\partial t} \quad \text{(MIII)} \qquad \textbf{1-3}$$

From Ampere's law:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \quad \text{(MIV)} \qquad \textbf{1-4}$$

where the symbol $\underline{\boldsymbol{\nabla}}$ denotes the vector operator 'del':

$$\underline{\boldsymbol{\nabla}} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \qquad \textbf{1-5}$$

$\underline{\boldsymbol{E}}$: electric field (V m^{-1})
$\underline{\boldsymbol{B}}$: magnetic field – or flux density (T)
ρ total charge density (C m^{-3})
$\underline{\boldsymbol{J}}$: total current density (A m^{-2})

c = 3×10^8 m s^{-1}, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$, $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}

Newton's law of motion:

$$\underline{\boldsymbol{F}} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \text{ where } \underline{\boldsymbol{p}} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \textbf{1-6}$$

Newton's law of Gravity:

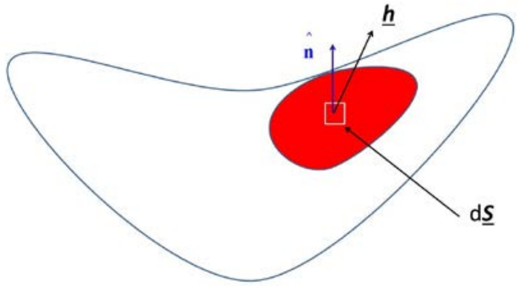
$$\underline{\boldsymbol{F}}_2 = -\frac{Gm_1 \mathrm{m}_2}{\mathrm{r}^2}\hat{\mathbf{r}}_{1\to 2} \qquad \textbf{1-7}$$

Force on a moving charge in a magnetic and electric field:

$$\underline{\boldsymbol{F}} = \mathrm{q}(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \qquad \textbf{1-8}$$

2 Vector Fields

2.1 The flux of a vector field



An arbitrary three-dimensional 'closed surface' is shown which is the surface of a single contiguous volume. The shaded surface is characterised as an 'open surface' because it does not enclose a volume.

$$\phi = \int \underline{\boldsymbol{h}}\,.\,\widehat{\boldsymbol{n}}\;\mathrm{d}S = \int \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{2-1}$$

If the area is a closed surface we can help the reader and make it explicit by writing a closed loop on the integral sign:

$$\phi = \oint \underline{\boldsymbol{h}}\;.\,\widehat{\boldsymbol{n}}\;\mathrm{d}S = \oint \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{2-2}$$

2.2 Gauss' (divergence) theorem

$$\oint \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} = \int \underline{\boldsymbol{\nabla}}\,.\,\underline{\boldsymbol{h}}\,\mathrm{d}V \qquad \textbf{2-3}$$

where $\underline{\boldsymbol{h}}$ is any arbitrary vector field

2.3 Stoke's (curl) theorem

$$\oint \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{l}} = \int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{h}})\mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{2-4}$$

where $\underline{\boldsymbol{h}}$ is any arbitrary vector field.

2.4 Differential vector identities

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}) = \underline{\boldsymbol{\nabla}}\big(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}}\big) - \nabla^2 \underline{\boldsymbol{A}} \qquad \textbf{2-5}$$

$$\underline{\boldsymbol{\nabla}} \cdot \big(\underline{\boldsymbol{A}} \times \underline{\boldsymbol{B}}\big) = \underline{\boldsymbol{B}} \cdot \big(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}\big) - \underline{\boldsymbol{A}} \cdot \big(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}\big) \qquad \textbf{2-6}$$

3 Maxwell I (From Coulomb's Law)

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \qquad \textbf{3-1}$$

where ρ is the total charge density.

3.1 Coulomb's law for interacting charges and Gauss' law

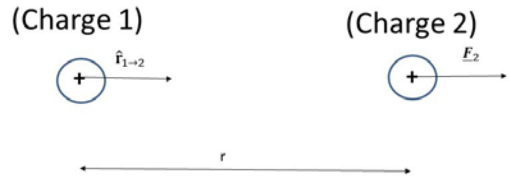


Figure 1 : Two positive charges interacting.

$$\underline{\boldsymbol{F}}_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}_{1\to2} \quad \textbf{3-2}$$

$$\underline{\boldsymbol{F}}_2 = q_2\underline{\boldsymbol{E}}_1 \quad \textbf{3-3}$$

$$\underline{\boldsymbol{E}}_1 = \frac{q_1}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}_{1\to2} \quad \textbf{3-4}$$

3.2 Deriving Gauss's law and Maxwell I from Coulomb's law

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \frac{\sum q}{\varepsilon_o} = \frac{1}{\varepsilon_o}\int \rho \mathrm{d}V \quad \textbf{3-5}$$

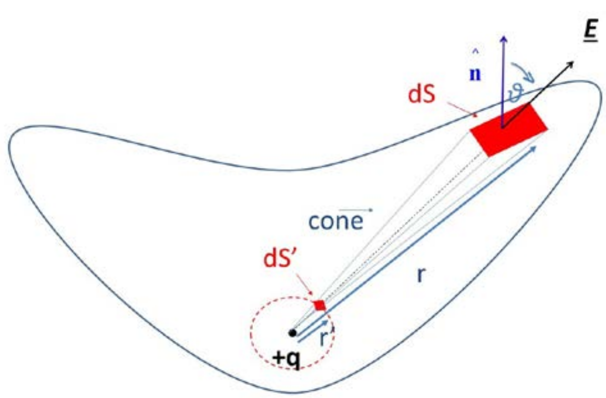


Figure 2 : The cone first passes through a sphere (S') centred about the charge and then through the surface of arbitrary shape (S). The areas bounded by the cone at the sphere and at the arbitrary surface are given by $\mathrm{d}\underline{\boldsymbol{S}}'$ and $\mathrm{d}\underline{\boldsymbol{S}}$ respectively.

We now use geometrical arguments and superposition of $\underline{\boldsymbol{E}}$-fields to find Gauss' law. Gauss' law is equivalent to Coulomb's law but can be applied to a collection of charges. Consider first the specific case of a sphere surrounding the charge at its centre. The electric field ($\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}')$) has constant magnitude over the surface of the sphere and is everywhere parallel to $\mathrm{d}\underline{\boldsymbol{S}}'$, (i.e. $\hat{\boldsymbol{n}}\,\mathrm{d}S'$) so we have**:**

$$\oint \underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}').\,\mathrm{d}\underline{\boldsymbol{S}}' = \left|E(r')\right| \oint \mathrm{d}S' \qquad \textbf{3-6}$$

$$= \frac{q}{4\pi\epsilon_0 (r')^2} 4\pi (r')^2 = \frac{\mathrm{q}}{\varepsilon_\mathrm{o}}$$

Consider now the flux through the elemental area d$\underline{\boldsymbol{S}}$ which is part of the surface of the arbitrary shape surrounding the charge. We have:

$$\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}).\hat{\boldsymbol{n}}\,\mathrm{d}S = |E(r)||\hat{\boldsymbol{n}}|\cos(\theta)\,\mathrm{d}S \qquad \textbf{3-7}$$

Since dS is a projection of dS' (because they are both bounded by the cone), we can relate them using:

$$\frac{dS'}{\pi(r')^2} = \frac{dS}{\pi r^2}\cos(\theta) \qquad \textbf{3-8}$$

Substituting Equation 3-8 into Equation 3-9 , we have :

$$\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}).\hat{\boldsymbol{n}}\,\mathrm{d}S = E(r)\frac{r^2}{(r')^2}dS' = \frac{\mathrm{q}}{4\pi\varepsilon_0 r^2}\frac{r^2}{(r')^2}\,dS' = \frac{\mathrm{q}}{4\pi\varepsilon_0 (r')^2}dS' \qquad \textbf{3-9}$$

Equation 3-9 shows that the flux through the surface dS is the same as the flux through dS'. Integrating both sides of Equation 3-9 gives:

$$\oint \underline{\boldsymbol{E}} \cdot \boldsymbol{d}\underline{\boldsymbol{S}} = \frac{\mathrm{q}}{\varepsilon_\mathrm{o}} \qquad \textbf{3-10}$$

Applying superposition to a collection of charges inside the arbitrary shaped surface:

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \frac{\sum q}{\varepsilon_\mathrm{o}} = \frac{1}{\varepsilon_\mathrm{o}}\int \rho \mathrm{d}V - \text{ Gauss'Law} \qquad \textbf{3-11}$$

Then, using the divergence (Gauss') theorem we can rewrite Gauss' law as:

$$\int \underline{\boldsymbol{\nabla}}\,.\,\underline{\boldsymbol{E}}\,\mathrm{d}V = \frac{1}{\varepsilon_\mathrm{o}}\int \rho \mathrm{d}V \qquad \textbf{3-12}$$

Since the volume integrals are equal for any arbitrary volume (no matter how big or small), the integrands must be equal so:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \quad \text{Maxwell's 1st equation} \qquad \textbf{3-13}$$

3.3 Superposition of Fields

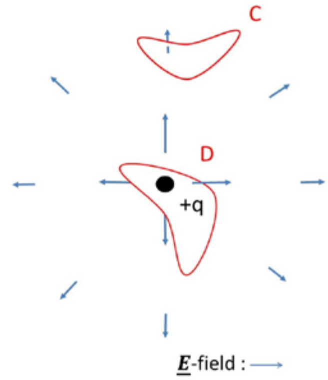


Figure 3 : An electric vector field found near a point source charge. Surface C does not enclose any charge so the net flux through the surface is zero. Surface C has zero net flux passing through it. Surface D has a net flux passing through it of magnitude $\frac{q}{\varepsilon_o}$.

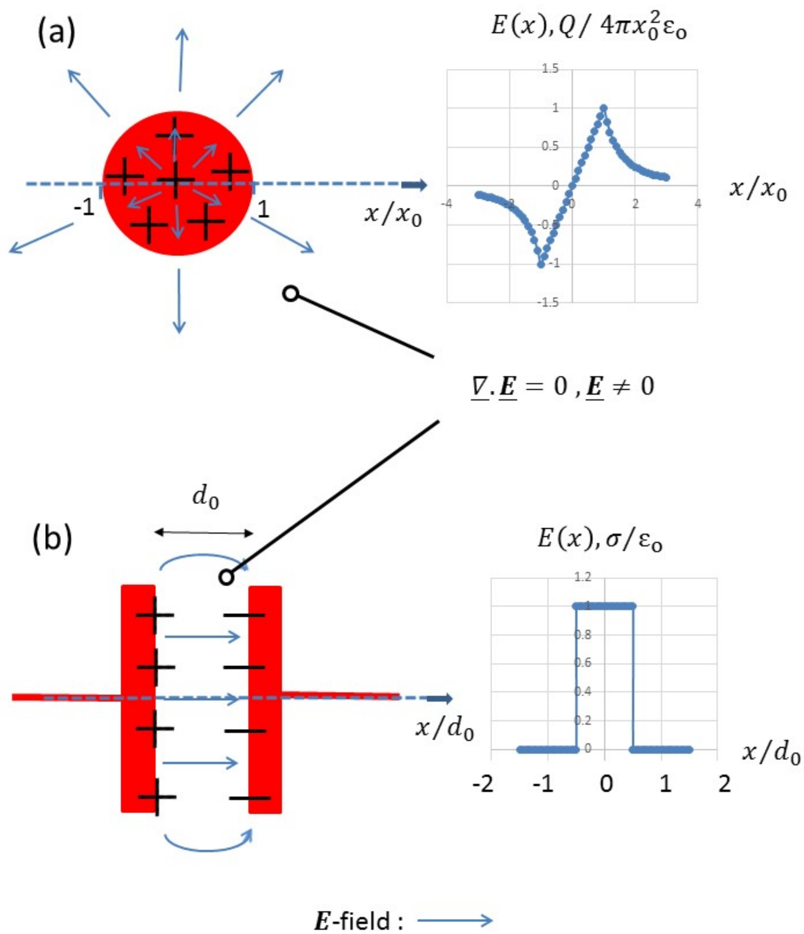


Figure 4 : Two charge configurations – a spherical volume of charge density and two charged capacitor plates.

The $\underline{\boldsymbol{E}}$-field resulting from any charge distribution gives $\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = 0$ in the local regions where there is no charge.

In the regions where there are no charges:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad \textbf{3-14}$$

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}^2\mathrm{E_r}) + \frac{1}{\mathrm{rsin\theta}}\frac{\partial}{\partial \theta}(\mathrm{E_\theta sin\theta}) + \frac{1}{\mathrm{rsin\theta}}\frac{\partial}{\partial \phi}(\mathrm{E}_\phi) = 0 \qquad \textbf{3-15}$$

4 Maxwell II (No magnetic monopoles)

4.1 Ampere's Law

$$\underline{\boldsymbol{F}}_2 = -\frac{\mu_0 \mathrm{I}_1 \mathrm{I}_2 L}{2\pi \mathrm{d}}\hat{\mathbf{r}}_{1\rightarrow 2} \qquad \textbf{3-16}$$

$$\underline{\boldsymbol{F}} = \mathrm{q}(\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \qquad \textbf{3-17}$$

$$\mathrm{I} = \mathrm{Q}_L \mathrm{v} \qquad \textbf{3-18}$$

where Q_L is the charge per unit length and v is the velocity of the charges or equivalently, for a length L,

$$\mathrm{I} = \frac{\sum q}{L}\mathrm{v} \qquad \textbf{3-19}$$

Where $\sum q$ isa the sum of all the charges in a length L .

$$\oint \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_0 \mathrm{I} \quad \text{Ampere's Law} \qquad \textbf{3-20}$$

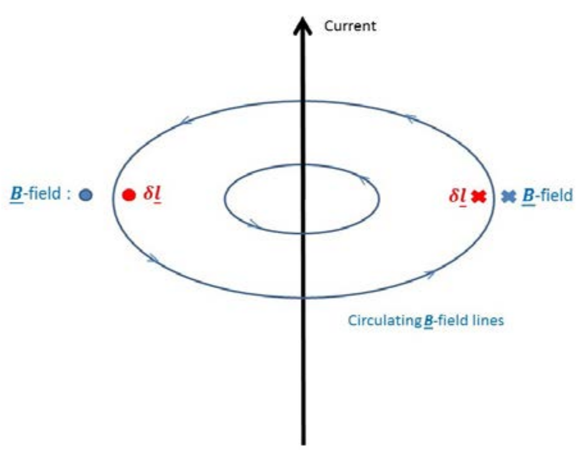


Figure 5 : The $\underline{\boldsymbol{B}}$-field outside a straight wire carrying a current. Note the $\underline{\boldsymbol{B}}$-field circulates the current

We simplify the vector integration in Ampere's law by taking advantage of symmetry to account for the magnitude and direction of $\underline{\boldsymbol{B}}$ so the (LHS) integral is a simple scalar integral and for a straight wire:

$$\mathrm{B} \times 2\pi\mathrm{r} = \mu_o \mathrm{I} \qquad \textbf{3-21}$$

$$\underline{\boldsymbol{B}} = \frac{\mu_o \mathrm{I}}{2\pi\mathrm{r}} \hat{\boldsymbol{\varphi}} \qquad \textbf{3-22}$$

4.2 Maxwell II from the Biot-Savart law

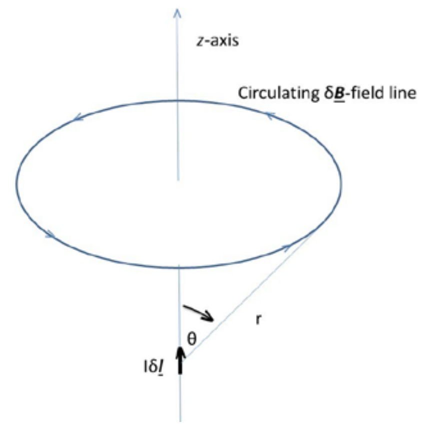


Figure 6 : The configuration associated with the Biot-Savart law. The small length of a current element (Id$\underline{\boldsymbol{l}}$) produces a small circulating magnetic field (δ$\underline{\boldsymbol{B}}$).

Ampère′s law can also be written in the form that describes the small magnetic field (δ$\underline{\boldsymbol{B}}$) produced by a small length of a current element (Id$\underline{\boldsymbol{l}}$) where

$$\delta\underline{\boldsymbol{B}} = \frac{\mu_0 I}{4\pi r^2}\mathrm{d}\underline{\boldsymbol{l}} \times \hat{\boldsymbol{r}} = \frac{\mu_0 I}{4\pi r^2}\sin\Theta\, \mathrm{dl}\, \widehat{\boldsymbol{\phi}} \text{ - the Biot Savart law.} \qquad 3\text{-}23$$

Given that the divergence of δ$\underline{\boldsymbol{B}}$ is a physical quantity, it's value does not depend on the coordinate system it is calculated in. More specifically, it doesn't matter whether $\underline{\boldsymbol{\nabla}}.\delta\underline{\boldsymbol{B}}$ is calculated in (Cartesian coordinates - not shown) spherical polar coordinates where

$$\underline{\boldsymbol{\nabla}}.\delta\underline{\boldsymbol{B}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\delta B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\delta B_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\delta B_\phi), \qquad 3\text{-}24$$

or in cylindrical polar coordinates where

$$\underline{\boldsymbol{\nabla}}.\delta\underline{\boldsymbol{B}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho(\delta B_\rho)) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\delta B_\phi) + \frac{\partial}{\partial \mathrm{z}}(\delta B_z) = 0\,. \qquad 3\text{-}25$$

In both cases, inserting equation **3-23** into either equation **3-24** or **3-25** shows that δB_r and δB_θ (spherical polar coordinates) or δB_ρ and δB_z (cylindrical coordinates) are both zero and that because δB_ϕ has no phi dependence, $\underline{\boldsymbol{\nabla}}.\,\delta\underline{\boldsymbol{B}} = 0$. It then follows from superposition that $\sum \delta\underline{\boldsymbol{B}} = \underline{\boldsymbol{B}}$, and hence for any configuration of currents

$$\underline{\boldsymbol{\nabla}}.\,\underline{\boldsymbol{B}} = \underline{\boldsymbol{\nabla}}.\sum \delta\underline{\boldsymbol{B}} = \sum \underline{\boldsymbol{\nabla}}.\,\delta\underline{\boldsymbol{B}} = 0. \qquad \text{3-26}$$

Equation 3-26 is Maxwell's 2nd equation given by

$$\underline{\boldsymbol{\nabla}}.\,\underline{\boldsymbol{B}} = 0 - \text{Maxwell's } 2^{nd} \text{ Equation}\,. \qquad \text{3-27}$$

The spatial variation of magnetic fields

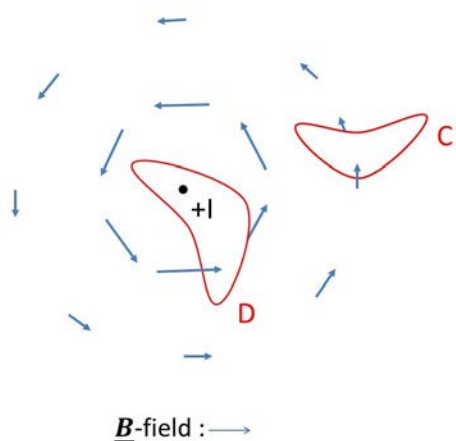


Figure 7 : The magnetic vector field, $\underline{\boldsymbol{B}}$, surrounding a current, I, flowing through a long straight wire coming out of the board. Although surface C encloses a source and surface D does not, the net flux through both surfaces is zero for this circulating vector field.

$$\int \underline{\boldsymbol{\nabla}}\,.\,\underline{\boldsymbol{B}}\,\mathrm{d}V = 0 \qquad \textbf{3-28}$$

$$\oint \underline{\boldsymbol{B}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} = 0 \qquad \textbf{3-29}$$

