(a) Using Cartesian coordinates and assuming that the motion is confined to the $x - z$ plane with the pivot at the origin, the constraints can be written as: $y = A$, a constant, and $x^2 + z^2 = l^2$, where l is the length of the rod. **[4 marks,U]**

(b) The Lagrangian can be written as a quadratic in both the generalised coordinate and velocity, ie $L = \dot{q}^2 + Aq^2$, where A is a constant. (Other constant factors are fine too.) **[2 marks,B]**
If $A > 0$ then the potential energy decreases when moving from equilibrium, representing unstable equilibrium. Conversely, $A < 0$ leads to oscillatory motion around a position of stable equilibrium. **[2 marks,B]**

(c) A Green's function describes the response of a system to the application of an impulsive force, i.e. the evolution of a dynamical system subsequent to the application of an impulsive force. **[2 marks,B]**
By considering the driving force to be made up of a sum of an infinite number of impulsive forces, the motion of a driven oscillator can be constructed using an integral of an infinite number of Green's functions. This can be written as a convolution. **[2 marks,B]**

(d) This system will have 2 degrees of freedom and hence 2 normal modes. One involves a coherent translation of both masses, mode vector $(1, 1)$. The second would have the two masses moving in antiphase, mode vector $(1, -1)$. **[4 marks,U]**

(e) A central force is one that acts along the direction between the point where the force originates and the location of the particle feeling the force. It has a magnitude that depends only on the distance between these two positions. **[2 marks, B]**
The translational invariance of the Lagrangian for the system implies that the centre of mass coordinate is ignorable and the linear momentum of the system is conserved. Hence the centre of mass moves like a free particle. **[2 marks, B]**

(f) $p = Qe^q$, $P = -e^q$, hence $Q = pe^{-q}$. **[2 marks, U]**
The Poisson bracket $\{Q, P\} = 0 - (e^{-q})(-e^q) = 1$, hence F produces a canonical transformation. **[2 marks, U]**

(g) The Coriolis force is an inertial force that appears due to considering dynamics in a non-inertial, rotating frame. $\underline{\omega}$ represents the angular velocity of the rotating frame and $\dot{\underline{r}}$ is the velocity of the mass with respect to the rotating frame. **[3 marks, B]**
The force acts to the right (North). **[1 mark, U]**

(h) 1, 2 and 3 represent directions along the principal axes of the rigid body. I_n is the principal moment of inertia associated with rotations around the nth principal axis. **[2 marks,B]**
ω_n is the component of the instantaneous angular velocity along the n axis, and $\dot{\omega}_n$ is its time derivative. N represents an applied external torque. **[2 marks,B]**

(a) [6 marks total] **(Unseen)**
Using $r = \lambda l$, the kinetic energy of the upper mass is $m(\lambda l\dot{\phi})^2/2$. **[1 mark]**
The position of the lower mass is $(r\sin\phi + l\sin\theta, -r\cos\phi - l\cos\theta)$. Differentiating with respect to time gives $\underline{\dot{r}} = (r\dot{\phi}\cos\phi + l\dot{\theta}\cos\theta, r\dot{\phi}\sin\phi + l\dot{\theta}\sin\theta)$. Hence,

$$\underline{\dot{r}}^2 = \lambda^2 l^2\dot{\phi}^2 + l^2\dot{\theta}^2 + 2\lambda l^2\dot{\phi}\dot{\theta}(\cos\phi\cos\theta + \sin\phi\sin\theta).$$

Therefore the total kinetic energy is

$$T = \frac{ml^2}{2}\left[2(\lambda\dot{\phi})^2 + \dot{\theta}^2 + 2\lambda\dot{\phi}\dot{\theta}(\cos\phi\cos\theta + \sin\phi\sin\theta)\right]$$

. **[3 marks]**
The potential energy can be written as $V = mg[\lambda l(1-\cos\phi) + \lambda l(1-\cos\phi) + l(1-\cos\theta)]$, where the second and third terms come from the lower mass. **[2 marks]**

(b) [7 marks total] **(Unseen)**
Using the small angle approximations, $T \approx \frac{ml^2}{2}\left[2(\lambda\dot{\phi})^2 + \dot{\theta}^2 + 2\lambda\dot{\phi}\dot{\theta}\right]$ and
$V \approx mgl(\lambda\phi^2 + \theta^2/2) = \frac{ml^2\omega_0^2}{2}(2\lambda\phi^2 + \theta^2)$. **[2 marks]**
The definitions of $\hat{\tau}$ and $\hat{\upsilon}$ then lead to

$$\hat{\tau} = \frac{ml^2}{2}\begin{pmatrix} 2\lambda^2 & \lambda \\ \lambda & 1 \end{pmatrix}$$

and

$$\hat{\upsilon} = \frac{ml^2}{2}\begin{pmatrix} 2\lambda\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{pmatrix}.$$

[3 marks]
Guessing a solution $\underline{q} = \underline{b}e^{i\omega t}$, differentiating twice and substituting into the matrix formulation of the Euler-Lagrange equation leads to

$$\frac{ml^2}{2}\begin{pmatrix} 2\lambda\omega_0^2 - 2\lambda^2\omega^2 & -\lambda\omega^2 \\ -\lambda\omega^2 & \omega_0^2 - \omega^2 \end{pmatrix}\underline{b} = 0,$$

. **[2 marks]**

(c) [7 marks total] **(Unseen)**
Non-trivial solutions to the generalised eigenvalue problem require

$$\begin{vmatrix} 2\lambda\omega_0^2 - 2\lambda^2\omega^2 & -\lambda\omega^2 \\ -\lambda\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0,$$

from which $2\lambda(\omega_0^2 - \lambda\omega^2)(\omega_0^2 - \omega^2) - \lambda^2\omega^4 = 0$. This yields normal mode frequencies satisfying $\lambda\omega^2 = \omega_0^2(1 + \lambda \pm \sqrt{1+\lambda^2})$. **[2 marks]**
When $\lambda = 3/4$, $\omega^2 = (4/3)\omega_0^2(7/4 \pm \sqrt{25/16}) = 4\omega_0^2 \quad \text{or} \quad 2\omega_0^2/3$. **[1 mark]**
Plugging these normal mode frequencies and $\lambda = 3/4$ back into the matrix equation in part (b) and considering the second row gives, for $\omega^2 = 4\omega_0^2$,

$$-3b_1 + (1-4)b_2 = 0, \quad \text{i.e.} \quad b_1 = -b_2.$$

Therefore the antiphase mode has equal amplitude oscillations of ϕ and θ. **[2 marks]**
For $\omega^2 = 2\omega_0^2/3$,

$$(-1/2)b_1 + (1-2/3)b_2 = 0, \quad \text{i.e.} \quad b_1 = 2b_2/3.$$

Therefore the in-phase mode has ϕ oscillating with two thirds the amplitude of θ. **[2 marks]**

(a) (i) [6 marks total] **(Unseen)**
The uniform density, $\rho = M/(8abc)$. **[1 mark]**
$I_{xx} = \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\rho(y^2+z^2)dxdydz = 2a\rho\int dz\left(\frac{2b^3}{3}+2bz^2\right) = \frac{M}{3}(b^2+c^2)$.
[2 marks]
$I_{xy} = \int\int\int_{-a}^{a}\rho(-xy)dxdydz = 0$. **[1 mark]**
Hence, the required matrix follows from symmetry, with all off-diagonal terms being zero. **[2 marks]**

(ii) [3 marks total] **(Unseen)** The centre of mass is $R_{\mathrm{C}} = (a,0,0)$. **[1 mark]**
Therefore the inertia tensor with respect to the centre of a face with sides $2b$ and $2c$ is

$$\hat{I} = \begin{pmatrix} M(b^2+c^2)/3 & 0 & 0 \\ 0 & M(c^2+a^2)/3+Ma^2 & 0 \\ 0 & 0 & M(a^2+b^2)/3+Ma^2 \end{pmatrix}.$$

[2 marks]

(b) [6 marks total] **(Unseen)**
From the matrix in part (a)i), the relevant moment of inertia for rotations about the centre of mass is $I = Ma^2/3$. The rotational kinetic energy, $T_{\mathrm{rot}} = I\dot{\theta}^2/2 = Ma^2\dot{\theta}^2/6$. **[2 marks]**
The centre of mass position is $\underline{r} = (a\sin\theta, ft^2/2 - a\cos\theta)$.
Differentiating with respect to time leads to $\underline{\dot{r}} = (a\dot{\theta}\cos\theta, ft + a\dot{\theta}\sin\theta)$.
The translational kinetic energy is then

$$T_{\mathrm{trans}} = M\underline{\dot{r}}^2/2 = \frac{M}{2}(a^2\dot{\theta}^2 + f^2t^2 + 2aft\dot{\theta}\sin\theta).$$

[3 marks]
The potential energy is constant, so the Lagrangian $L = T_{\mathrm{rot}} + T_{\mathrm{trans}}$, giving the required result. **[1 mark]**

(c) [5 marks total] **(Unseen)**
Substituting the Lagrangian into the Euler-Lagrange equation gives

$$\frac{d}{dt}\left(\frac{8a^2\dot{\theta}}{3} + 2aft\sin\theta\right) - 2aft\dot{\theta}\cos\theta = 0.$$

Hence

$$\frac{8a^2\ddot{\theta}}{3} + 2af\sin\theta + 2aft\dot{\theta}\cos\theta - 2aft\dot{\theta}\cos\theta = 0,$$

from which

$$\ddot{\theta} = -\frac{3f}{4a}\sin\theta.$$

[2 marks]
For small angles, $\sin\theta \approx \theta$, so $\ddot{\theta} = -\omega^2\theta$, where $\omega = \sqrt{3f/(4a)}$. The door takes one quarter of a period to close, i.e. a time $T = \pi/(2\omega) = \pi\sqrt{a/(3f)}$. **[3 marks]**

All the short questions are bookwork.

(a) Linear independence of the set $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle$ means that there are **no non-zero complex numbers** b_i such that

$$\sum_i b_i|\phi_i\rangle = \underline{0}$$

where $\underline{0}$ is the null state vector. [2 marks]
An orthonormal basis satisfies the condition

$$\langle\phi_i|\phi_j\rangle = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$, 0 otherwise is the Kroenecker symbol. [2 marks]

(b) Hermitian operators $\hat{A}$ have the defining property that

$$\langle\alpha|\hat{A}|\beta\rangle = \left[\langle\beta|\hat{A}|\alpha\rangle\right]^*$$

where $*$ denotes the complex conjugate. [2 marks]
[NB. The more concise statement $\hat{A} = \hat{A}^\dagger$ would also receive [2 marks]]

If $\hat{A}|\alpha_i\rangle = a_i|\alpha_i\rangle$ then

$$\langle\alpha_i|\hat{A}|\alpha_i\rangle = a_i\langle\alpha_i|\alpha_i\rangle$$

The left-hand side of this expression is real using the Hermiticity condition above. On the right-hand side $\langle\alpha_i|\alpha_i\rangle$ is real since it is an inner product of identical states, hence we conclude that the eigenvalues a_i are real. [2 marks]

(c) In Dirac notation the corresponding wavefunction is $\psi(x) = \langle x|\psi\rangle$. [1 mark]

Inserting a complete set of position eigenstates

$$\langle\phi|\psi\rangle = \int dx \langle\phi|x\rangle\langle x||\psi\rangle$$

[2 marks]

$$= \int dx \phi^*(x)\psi(x)$$

an overlap integral of the two wavefunctions. [1 mark]

(d) An operator $\hat{A}$ is unitary if $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} = \hat{1}$ where $\hat{A}^\dagger$ is the adjoint operator defined by

$$\langle\alpha|\hat{A}^\dagger|\beta\rangle = [\langle\beta|\hat{A}|\alpha\rangle]^*$$

and $\hat{1}$ is the unit operator. [2 marks]

We have $\langle\phi'|\psi'\rangle = \langle\phi'|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi'\rangle^* = \langle\psi|\hat{A}^\dagger\hat{A}|\phi\rangle^* = \langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle$. as required. [2 marks]

(e) We have the anticommutation relation $\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = \hat{1}$ If we require that $\hat{b}$ annihilates the vacuum state then $\hat{b}|0\rangle = 0$. Acting with the number operator $\hat{N} = \hat{b}^\dagger\hat{b}$ on $\hat{b}^\dagger|0\rangle$ yields

$$\hat{N}\hat{b}^\dagger|0\rangle = \hat{b}^\dagger\hat{b}\hat{b}^\dagger|0\rangle = \hat{b}^\dagger(\hat{1} - \hat{b}^\dagger\hat{b})|0\rangle = (\hat{b}^\dagger - \hat{b}^\dagger\hat{N})|0\rangle = \hat{b}^\dagger|0\rangle$$

So we see that $\hat{b}^\dagger|0\rangle = |1\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n = 1$. [1 mark]

If we attempt to create another quantum by acting again with $\hat{b}^\dagger$, however, we find

$$\hat{N}\hat{b}^\dagger|1\rangle = \hat{b}^\dagger(\hat{1} - \hat{b}^\dagger\hat{b})|1\rangle = \hat{b}^\dagger(\hat{1} - \hat{N})|1\rangle = 0$$

So the state $|1\rangle$ is annihilated by $\hat{b}^\dagger$, and the possible eigenvalues are $n = 0$ and $n = 1$. [2 marks]

This is exactly what is required by the Pauli exclusion principle for fermions. [1 mark]

(f) The Heisenberg picture evolution equation is

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}]$$

[2 marks]

If the operator corresponds to a conserved quantity one then finds $[\hat{H}, \hat{O}] = 0$, and the operator commutes with the Hamiltonian. [2 marks]

(g) The general expression is

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$$

where ϵ_{ijk} is the Levi-Civita symbol with $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$ and $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$. Corresponding to even and odd signature permutations of $\{1, 2, 3\}$ respectively. [3 marks]

So we find $[\hat{L}_x, \hat{L}_z] = i\epsilon_{132}\hat{L}_y = -i\hat{L}_y$. [1 mark]

Expressing $\psi(x, y, z)$ in spherical polar co-ordinates

$$x = r\sin\theta\cos\phi \ \ , \ \ y = r\sin\theta\sin\phi \ \ , \ \ z = r\cos\theta$$

where $r^2 = x^2 + y^2 + z^2$ we have

$$\psi(r, \theta, \phi) = N[\sin\theta(\cos\phi + \sin\phi) + \cos\theta]re^{-r^2/\alpha^2}$$

[4 marks bookwork]

We can write $\psi(r, \theta, \phi)$ as a product of two functions $\psi(r, \theta, \phi) = R(r)F(\theta, \phi)$ where $R(r) = Nre^{-r^2/\alpha^2}$ and

$$F(\theta, \phi) = \sum_{l,m} a_{lm}Y_{lm}(\theta, \phi) = \sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta$$

We then find

$$F(\theta, \phi) = \sqrt{\frac{8\pi}{3}}\left[\frac{1}{2}(Y_{1-1} - Y_{11}) - \frac{1}{2i}(Y_{1-1} + Y_{11})\right] + \sqrt{\frac{4\pi}{3}}Y_{10}$$

[4 marks unseen]

which can be simplified to

$$F(\theta, \phi) = \sqrt{\frac{2\pi}{3}}[(1 + i)Y_{1-1} - (1 - i)Y_{11} + \sqrt{2}Y_{10}]$$

[2 marks unseen]

We need to introduce a normalization constant N' so that $F(\theta, \phi) = N'[\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta]$ with

$$\int F^*(\theta, \phi)F(\theta, \phi)d\theta d\phi = N'^2\frac{2\pi}{3}(2 + 2 + 2) = 4\pi N'^2 = 1$$

So that we fix $N' = 1/\sqrt{4\pi}$. [4 marks bookwork]

Then

$$F(\theta, \phi) = \frac{1}{\sqrt{6}}[(1 + i)Y_{1-1} - (1 - i)Y_{11} + \sqrt{2}Y_{10}]$$

Then we can compute the probabilities

(a) corresponds to $\hat{L}^2 = 2\hbar^2$, $\hat{L}_z = 0$ and so we have $l = 1, m = 0$. Thus

$$P = |\langle 1, 0|F(\theta, \phi)\rangle|^2 = |\frac{\sqrt{2}}{\sqrt{6}}|^2 = \frac{1}{3}$$

[2 marks unseen]

(b) corresponds to $\hat{L}^2 = 2\hbar^2$, $\hat{L}_z = \hbar$ and so we have $l = 1, m = 1$.

$$P = |\langle 1, 1|F(\theta, \phi)\rangle|^2 = |-\frac{1-i}{\sqrt{6}}|^2 = \frac{1}{3}$$

[2 marks unseen]

(c) corresponds to $\hat{L}^2 = 2\hbar^2$, $\hat{L}_z = -\hbar$ and so we have $l = 1, m = -1$. Thus

$$P = |\langle 1, -1|F(\theta, \phi)\rangle|^2 = |\frac{1+i}{\sqrt{6}}|^2 = \frac{1}{3}\ .$$

[2 marks unseen]

The probabilities add to one as required.

(a) The Hamiltonian matrix between the states $|\pm\rangle$ will correspond to a $-\underline{\mu} \cdot \underline{B}$ interaction.

$$\hat{H} = \begin{pmatrix} -\mu B_z & -\mu B_x \\ -\mu B_x & \mu B_z \end{pmatrix}$$

This has secular equation

$$(-\mu B_z - E)(\mu B_z - E) - \mu^2 B_x^2 = 0$$

Which reduces to

$$E^2 - \mu^2 B_z^2 - \mu^2 B_x^2 = 0$$

So

$$E_{\pm} = \pm\mu\sqrt{B_x^2 + B_z^2} = \pm\mu|\underline{B}|$$

[4 marks unseen]

As expected for a $\underline{\mu} \cdot \underline{B}$ interaction.

Writing the eigenvectors as $|\psi\rangle = a|+\rangle + b|-\rangle$ We require

$$\begin{pmatrix} -\mu B_z \mp \mu|\underline{B}| & -\mu B_x \\ -\mu B_x & \mu B_z \mp \mu|\underline{B}| \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So that

$$a(-\mu B_z \mp \mu|\underline{B}|) - \mu B_x b = 0$$

and

$$-\mu B_x a + (\mu B_z \mp \mu|\underline{B}|)b = 0$$

[4 marks unseen]

For the $E = +\mu|\underline{B}|(\hbar)$ case we then find

$$a = \frac{-B_x}{B_z + |\underline{B}|} b$$

$$b = \frac{B_x}{B_z - |\underline{B}|} a$$

[2 marks unseen]

For the $E = -\mu|\underline{B}|(\hbar)$ case we then find

$$a = \frac{-B_x}{B_z - |\underline{B}|} b$$

$$b = \frac{B_x}{B_z + |\underline{B}|} a$$

[2 marks unseen]

(b) Defining

$$k = \frac{-B_x}{B_z + |\underline{B}|} \ , \ \frac{1}{k} = \frac{B_x}{B_z - |\underline{B}|}$$

We have orthonormal eigenvectors of the form

$$|\psi_+\rangle = \frac{1}{\sqrt{1+k^2}}(k|+\rangle + |-\rangle) \ , \ |\psi_-\rangle = \frac{1}{\sqrt{1+k^2}}(|+\rangle - k|-\rangle) \ ,$$

as required. [4 marks unseen]

(c) $|\psi(0)\rangle = |+\rangle = \langle\psi_+|+\rangle|\psi_+\rangle + \langle\psi_-|+\rangle|\psi_-\rangle$ We have

$$\langle\psi_+|+\rangle = \frac{k}{\sqrt{1+k^2}}$$

and

$$\langle\psi_-|+\rangle = \frac{1}{\sqrt{1+k^2}}$$

So

$$|\psi(t)\rangle = \frac{k}{(1+k^2)} \exp\left(-\frac{iE_+t}{\hbar}\right)(k|+\rangle + |-\rangle) + \frac{1}{(1+k^2)} \exp\left(-\frac{iE_-t}{\hbar}\right)(|+\rangle - k|-\rangle) \ ,$$

with $E_\pm = \pm\mu|\underline{B}|$. [4 marks unseen]