Mathematical Methods - Part 1

Vector Algebra

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Levi-Cevita Symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{if } (i,j,k) \in \{(3,2,1),(1,3,2),(2,1,3)\} \\ 0 & \text{otherwise} \end{cases}$$

Scalar Product

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j = a_i b_i$$

Vector Product

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$$

Scalar Triple Product

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \epsilon_{ijk} a_i b_j c_k$$

Vector Triple Product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Equation of a Line

For

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equation of a line:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$$

Where $\mathbf{a}$ is a position vector for a point on the line, $\mathbf{b}$ is a vector parallel to the line, and λ is an arbitrary scalar.

The equation can also be written as:

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$$

Equation of a Plane

Equation of a plane:

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0$$

Where $\mathbf{a}$ is the position vector of a point on the plane and $\hat{\mathbf{n}}$ is the normal vector to the plane.

Or, alternatively:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

Where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are position vectors of points on the plane.

Vector Spaces

Linear Vector Space

A set of vectors forms a vector space, V, if there are two operations defined on the elements of the set called addition and multiplication by scalars, which obey the following simple rules (the axioms of the vector space):

1. If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$. If $\mathbf{v} \in V$ then $\alpha\mathbf{v} \in V$ where $\alpha \in \mathbb{C}$. The vector space is closed with respect to addition and scalar multiplication.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$
3. There exists a neutral element, $\mathbf{0}$, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$.
4. There exists an inverse element, $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$.
5. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
6. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$, $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$
7. $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$

Where α, β are scalars and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. If $\alpha, \beta \in \mathbb{R}$ then V is called a real vector space. If $\alpha, \beta \in \mathbb{C}$ then V is called a complex vector space.

Linear Combinations

The set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \alpha_i\mathbf{v}_i$$

Is called the "span" of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.

Linearly Independent Vectors

k vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, are said to be linearly independent if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = 0$$

is satisfied if and only if all $\alpha_i = 0$. Otherwise the vectors are said to be linearly dependent.

Basis

A basis is a set of vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$, which are linearly independent and satisfy $V = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$. Then

1. The numbers of vector in a basis is called the dimension of the space V, $\dim V$.
2. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis of the vector space V, then any vector $\mathbf{v}$ in V can be written as linear combination of the basis vectors. The coefficients of basis vectors are called the components of $\mathbf{v}$ with respect to the basis.

Inner Product

The inner product between vectors in V, $\langle\mathbf{u}|\mathbf{v}\rangle$, satisfies:

1. $\langle\mathbf{u}|\mathbf{v}\rangle = \langle\mathbf{v}|\mathbf{u}\rangle^*$
2. $\langle\mathbf{u}|(\lambda\mathbf{v} + \mu\mathbf{w})\rangle = \lambda\langle\mathbf{u}|\mathbf{v}\rangle + \mu\langle\mathbf{u}|\mathbf{w}\rangle$
3. $\langle\mathbf{u}|\mathbf{u}\rangle > 0$ if $\mathbf{u} \neq 0$

The length of a vector is $|\mathbf{u}| = \sqrt{\langle \mathbf{u}|\mathbf{u}\rangle}$ and two vectors are orthogonal is $\langle \mathbf{u}|\mathbf{w}\rangle = 0$.

Matrices

Linear Operators

An object, $\mathcal{A}$, is a linear operator if acts on the vectors $\mathbf{v}$ and $\mathbf{u}$ like so:

$$\mathcal{A}(\alpha\mathbf{v} + \beta\mathbf{u}) = \alpha(\mathcal{A}\mathbf{v}) + \beta(\mathcal{A}\mathbf{u})$$

Matrix Operations

1. Addition: $(A+B)_{ij} = A_{ij} + B_{ij}$
2. Scalar multiplication: $(\alpha A)_{ij} = \alpha A_{ij}$
3. Multiplication of matrices: $(AB)_{ij} = A_{ik}B_{kj}$
4. Transposition: $(A^T)_{ij} = A_{ji}$ with $(ABC\dots F)^T = F^T \dots C^T B^T A^T$
5. Complex conjugation: $(A^*)_{ij} = (A_{ij})^*$
6. Hermitian conjugation (adjoint): $(A^\dagger)_{ij} = (A_{ji})^*$ with $(ABC\dots F)^\dagger = F^\dagger \dots C^\dagger B^\dagger A^\dagger$

Determinant of a Square Matrix

For a square matrix, A, the determinant is:

$$|A| = A_{jk}C_{jk}$$

For any row j or

$$|A| = A_{kj}C_{kj}$$

For any column j.

Where

$$C_{mn} = (-1)^{m+n}|A_{mn}|$$

$|A_{mn}|$ is determinant of the matrix obtained by removing the mth row and nth column from the matrix A.

Properties:

1. $|ABC\dots F| = |A||B||C|\dots|F|$
2. $|A^T| = |A|,\ |A^*| = |A|^*,\ |A^\dagger| = |A|^*,\ |A^{-1}| = |A|^{-1}$
3. If the rows or columns are linearly dependent then $|A| = 0$
4. If B is obtained from A by multiplying the elements of any row (or column) by a factor α, then $|B| = \alpha|A|$
5. If B is obtained from A by interchanging two rows (or columns), then $|B| = -|A|$
6. If B is obtained from A by adding k times one row (or column) to the other row (or column), then $|A| = |B|$

Elementary Row Operations

1. Multiply any row by a non-zero scalar
2. Interchange any two rows
3. Add some multiple of row to any other row

Inverse of a Square Matrix

The inverse of a matrix A is:

$$A^{-1} = \frac{C^T}{|A|}$$

Where $A^{-1}A = AA^{-1} = I$

Properties:

1. $(ABC\dots F)^{-1} = F^{-1}\dots C^{-1}B^{-1}A^{-1}$
2. $(A^T)^{-1} = (A^{-1})^T,\ (A^\dagger)^{-1} = (A^{-1})^\dagger$

Trace of a Square Matrix

The trace of a matrix A is:

$$\text{Tr } A = \sum_k A_{kk} \equiv A_{kk}$$

Properties:

1. Trace is a linear operation
2. $\text{Tr } A^T = \text{Tr } A$; $\text{Tr } A^\dagger = (\text{Tr } A)^*$
3. $\text{Tr } (AB) = (AB)_{ii} = A_{ij}B_{ji} = B_{ji}A_{ij} = (BA)_{jj} = \text{Tr } (BA)$
4. $\text{Tr } (ABC) = \text{Tr } (BCA) = \text{Tr } (CAB)$ - cyclical order

Special Matrices

Symmetric Matrices: $A^T = A$

Antisymmetric or Skew Matrices: $A^T = -A$

Any matrix can be written as the sum of a symmetric matrix and an antisymmetric matrix.

Hermitian matrices: $A^\dagger = A$

Antihermitian matrices: $A^\dagger = -A$

Any matrix can be written as the sum of a hermitian matrix and an antihermitian matrix.

Unitary matrices: $A^\dagger = A^{-1}$

Unitary matrices preserve the length of vectors.

Orthogonal matrices: $A^T = A^{-1}$

Orthogonal matrices also preserve the length of vectors.

The Eigenvalue Problem

Eigenvalues

For the equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

1. λ is the eigenvalue, $\mathbf{x}$ is the corresponding eigenvector
2. The determinant, $|A - \lambda I|$, is called the characteristic polynomial or degree n
3. The eigenvalue equation has a nontrivial solution if and only if $|A - \lambda I| = 0$
4. The eigenvalues of the matrix A are the roots of the characteristic polynomial
5. The eigenvectors associated to the eigenvalue μ are the vectors $\mathbf{x}$ such that

$$(A - \mu I)\mathbf{x} = 0$$

Eigenvectors associated to different eigenvalues are linearly independent.

If a matrix has an eigenvalue equal to zero, then the matrix is singular since its determinant is zero.

Eigenvalues of Special Matrices

1. Hermitian and symmetric matrices have real eigenvalues
2. Antihermitian and antisymmetric matrices have eigenvalues which are purely imaginary or zero
3. Unitary and orthogonal matrices have eigenvalues whose absolute values are equal to one
4. The eigenvectors of all special matrices are linearly independent. In addition, they can always be chosen in such a way that they form a mutually orthogonal set.

Similar Matrices

Two matrices A and B can be said to be to similar if there exists a matrix S such that

$$B = S^{-1}AS$$

The two matrices represent the same linear operator in different basis.

Diagonalising a Matrix

If the new basis is a set of eigenvectors of A then $B \equiv D$ is diagonal with

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & \lambda_n \end{pmatrix}, \ S = \begin{pmatrix} | & | & \dots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}$$

Where λ_i are the eigenvalues and $\mathbf{x}_i$ are the corresponding eigenvectors.

Applications:

1. nth power of A:

$$A^n = AA\dots A = (SDS^{-1})(SDS^{-1})\dots(SDS^{-1}) = SD^nS^{-1}$$

2. Exponential of A:

$$e^A = e^{SDS^{-1}} = \sum_{n=0}^{\infty} \frac{(SDS^{-1})^n}{n!} = Se^DS^{-1}$$

Fourier Series

Fourier Series of a Periodic Function

The Fourier series of a period function $f(x)$ with period L is given by:

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right) \right]$$

The Fourier coefficients a_0, a_r and b_r are:

$$a_0 = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\mathrm{d}x$$

$$a_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)$$

$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi rx}{L}\right)$$

Properties of the Fourier Series

1. If the function $f(x)$ is even, $b_r = 0$ for all r
2. If the function $f(x)$ is odd, $a_r = 0$ for all r

Complex Fourier Series

The Fourier series can be written in terms of complex coefficients as:

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{i2\pi rx/L}$$

With:

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-2\pi irx/L}\mathrm{d}x$$

Integral Transforms

Fourier Transforms

The Fourier transform of the function $f(t)$ is:

$$\mathcal{F}[f(t)](\omega) \equiv \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t$$

This integral exists if $f(x)$ has a finite number of finite discontinuities and $\int_{-\infty}^{\infty} |f(t)|\mathrm{d}t$ is finite.

If f is continuous, the inverse is:

$$\mathcal{F}^{-1}[\hat{f}(\omega)](t) = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\mathrm{d}\omega$$

Properties:

$$\mathcal{F}[f(at)](\omega) = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}[f(t+a)](\omega) = e^{ia\omega}\hat{f}(\omega)$$

$$\mathcal{F}[e^{\alpha t}f(t)](\omega) = \hat{f}(\omega + i\alpha)$$

$$\mathcal{F}[f'(t)](\omega) = (i\omega)\hat{f}(\omega)$$

For a derivative of order n we have:

$$\mathcal{F}[f^{(n)}(t)](\omega) = (i\omega)^n\hat{f}(\omega)$$

The convolution of two functions f and g over the interval $(-\infty, \infty)$ is a function h defined as:

$$h(y) = \int_{-\infty}^{\infty} f(x)g(y-x)\mathrm{d}x \equiv (f*g)(y) = (g*f)(y)$$

The convolution theorem for Fourier transforms is:

$$\mathcal{F}[h(z)](\omega) = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$$

Laplace Transforms

The Laplace transform of the function $f(t)$ is:

$$\mathcal{L}[f(t)](s) \equiv \bar{f}(s) = \int_{0}^{\infty} f(t)e^{-st}\mathrm{d}t$$

Where s is taken to be real.

Properties:

$$\mathcal{L}[H(t-a)f(t-a)](s) = e^{-sa}\bar{f}(s)$$

$$\mathcal{L}[e^{at}f(t)](s) = \bar{f}(s-a)$$

$$\mathcal{L}[f(at)](s) = \frac{1}{|a|}\bar{f}\left(\frac{s}{a}\right)$$

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{\mathrm{d}^n \bar{f}(s)}{\mathrm{d}s^n}$$

$$\mathcal{L}[f'(t)](s) = -f(0) + s\bar{f}(s)$$

For a derivative of order n:

$$\mathcal{L}[f^{(n)}(t)](s) = s^n \bar{f}(s) - s^{n-1}f(0) - s^{n-2}f^{(1)} - \cdots - f^{(n-1)}(0)$$

For integration:

$$\mathcal{L}\left[\int_0^t f(u)\mathrm{d}u\right] = \frac{\bar{f}(s)}{s}$$

The convolution theorem for a Laplace transform:

If the functions f and g have Laplace transforms $\bar{f}$ and $\bar{g}$, then:

$$\mathcal{L}[(f*g)](s) = \mathcal{L}[(g*f)](s) = \mathcal{L}\left[\int_0^t f(u)g(t-u)\mathrm{d}u\right](s) = \bar{f}(s)\bar{g}(s)$$

The inverse of a Laplace transform: $\mathcal{L}^{-1}[\bar{f}(s)] = f(t)$.

The Dirac Delta Function

Dirac Delta Function

The defining properties are:

$$\delta(x-a) = 0$$

For $x \neq a$. And:

$$\int_\alpha^\beta f(x)\delta(x-a)\mathrm{d}x = \begin{cases} (a) & \alpha < a < \beta \\ 0 & \text{otherwise} \end{cases}$$

Integral representation of the Dirac delta function:

$$\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}\mathrm{d}\omega$$

Properties:

$$\delta(x) = \delta(-x)$$

$$\delta(g(x)) = \sum_a \frac{\delta(x-a)}{|g'(a)|}$$

Where a are the roots of the function $g(x)$ i.e. $g(a) = 0$ and $g'(a) \neq 0$.

$$\int_{-\infty}^{\infty} f(x)\delta'(x-a)\mathrm{d}x = -f'(a)$$

Heaviside Step Function

The Heaviside is defined as:

$$H'(x) = \delta(x)$$

Where:

$$H(x) = \begin{cases} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Vector Calculus

The del operator

The linear vector differential operator del in cartesian coordinates as:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The Gradient

The gradient of a scalar field ϕ:

$$\text{grad } \phi = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

It has the property:

$$\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$$

Divergence

The divergence of a vector field $\mathbf{a}$:

$$\text{div } \mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

Properties:

$$\nabla(\phi\mathbf{a}) = \nabla\phi \cdot \mathbf{a} + \phi(\nabla \cdot \mathbf{a})$$

$$\nabla(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

Curl

The curl of a vector field $\mathbf{a}$:

$$\text{curl } \mathbf{a} = \nabla \times \mathbf{a} = \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

Properties:

$$\nabla \times (\phi\mathbf{a}) = (\nabla\phi) \times \mathbf{a} + \phi(\nabla \times \mathbf{a})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\nabla \cdot \mathbf{b})\mathbf{a}$$

Laplacian

The divergence of the gradient of a scalar field is called the Laplacian:

$$\nabla \cdot (\nabla\phi) = \nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Other Second Order Derivatives

$$\nabla \times (\nabla \phi) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

Integrals

Line Integrals

The line integral of a vector field $\mathbf{a}(\mathbf{r})$ along the curve C is:

$$\int_C \mathbf{a}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{u_{\min}}^{u_{\max}} \mathbf{a}(\mathbf{r}(u)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} \mathrm{d}u$$

Where C is a smooth oriented curve defined by the equation $\mathbf{r}(u)$ with endpoints $\mathbf{A} = \mathbf{r}(u_{\min})$ and $\mathbf{B} = \mathbf{r}(u_{\max})$.

A region D is **simply connected** if every closed path within D can be shrunk to a point without leaving the region.

Surface Integrals

The surface integrals of vector functions $\mathbf{a}(\mathbf{r})$ over a smooth surface S, defined by $\mathbf{r}(u, v)$ with orientation given by the normal $\hat{\mathbf{n}}$ is:

$$\int_S \mathbf{a}(\mathbf{r}) \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{a}(\mathbf{r}) \cdot \hat{\mathbf{n}} \mathrm{d}S = \int_{u_{\min}}^{u_{\max}} \int_{v_{\min}}^{v_{\max}} \mathbf{a}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \mathrm{d}u \mathrm{d}v$$

Volume integrals

Volume integrals of a function $\phi(\mathbf{r})$ over a volume V described by $\mathbf{r}(u, v, w)$ is:

$$\int_V \phi(\mathbf{r}) \mathrm{d}V = \int_{u_{\min}}^{u_{\max}} \int_{v_{\min}}^{v_{\max}} \int_{w_{\min}}^{w_{\max}} \phi(\mathbf{r}(u, v, w)) \left| \frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w} \right) \right| \mathrm{d}u \mathrm{d}v \mathrm{d}w$$

Divergence theorem (Gauss' theorem)

$$\iiint_V (\nabla \cdot \mathbf{a}) \mathrm{d}V = \iint_S \mathbf{a} \cdot \mathrm{d}\mathbf{S}$$

Stokes' theorem

$$\iint_S (\nabla \times \mathbf{a}) \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{a} \cdot \mathrm{d}\mathbf{r}$$

Orthogonal Curvilinear Coordinates

Cylindrical Polar Coordinates

$$\mathbf{r}(\rho, \phi, z) = \rho \cos \phi \mathbf{i} + \rho \sin \phi \mathbf{j} + z \mathbf{k}$$

$$\hat{\mathbf{e}}_\rho = \cos \rho \mathbf{i} + \sin \phi \mathbf{j}, \quad \hat{\mathbf{e}}_\phi = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}, \quad \hat{\mathbf{e}}_\rho = \mathbf{k}$$

$$\mathrm{d}\mathbf{S} = \begin{cases} \mathbf{e}_\rho \rho \mathrm{d}\phi \mathrm{d}z & \rho = \text{const} \\ \hat{\mathbf{e}}_\phi \mathrm{d}\rho \mathrm{d}z & \phi = \text{const} \\ \hat{\mathbf{e}}_z \rho \mathrm{d}\rho \mathrm{d}\phi & z = \text{const} \end{cases}$$

$$\mathrm{d}V = \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z$$

Spherical Polar Coordinates

$$\mathbf{r}(r, \theta, \phi) = r \sin \theta \cos \phi \mathbf{i} + r \sin \theta \sin \phi \mathbf{j} + r \cos \theta \mathbf{k}$$

$$\hat{\mathbf{e}}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k} \quad \hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k} \quad \hat{\mathbf{e}}_r = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$$

$$\mathrm{d}\mathbf{S} = \begin{cases} \mathbf{e}_r r^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi & r = \text{const} \\ \hat{\mathbf{e}}_\theta r\sin\theta \mathrm{d}r\mathrm{d}\phi & \theta = \text{const} \\ \hat{\mathbf{e}}_\phi r\mathrm{d}r\mathrm{d}\theta & \phi = \text{const} \end{cases}$$

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

Mathematical Methods - Part 2

1st Order ODEs

Separable ODE

Form:

$$f(x,y) = \frac{\mathrm{d}s}{\mathrm{d}t} = u(x)v(y)$$

Solution:

$$\int \frac{1}{v(y)}\mathrm{d}y = \int u(x)\mathrm{d}x$$

Exact ODE

Form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0$$

Exactness condition:

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

If exact, solution is:

$$u(x,y) = \int A(x,y)\mathrm{d}x \quad \text{or} \quad u(x,y) = \int B(x,y)\mathrm{d}y$$

Since

$$A(x,y) = \frac{\partial u}{\partial x} \quad \text{and} \quad B(x,y) = \frac{\partial u}{\partial y}$$

Also since $du = 0$, $u =$ constant

Inexact ODE

Form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0$$

Inexactness condition:

$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

Use integrating factor, μ, and assume that it is exact:

$$\mu A(x,y)\mathrm{d}x + \mu B(x,y)\mathrm{d}y = 0$$

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}$$

Integrating factor can be either $\mu = \mu(x)$ or $\mu = \mu(y)$.

Linear 1st Order ODE

Form:

$$a(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b(x)y = c(x)$$

Or:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

Use integrating factor:

$$\mu(x) = \exp\left(\int P\mathrm{d}x\right)$$

Sub in and solve:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x)y\right) = \mu(x)q(x) \quad \Rightarrow \quad y = \frac{1}{\mu(x)}\int \mu(x)q(x)\mathrm{d}x$$

Bernoulli ODE

Form ($n \notin \{0, 1\}$):

$$\frac{\mathrm{d}y}{\mathrm{d}x} + b(x)y = c(x)y^n$$

Substitution:

$$z = y^{1-n}$$

Homogenous ODE

Form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

Homogenous functions follow the rule:

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

Substitutions:

$$y = vx$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Isobaric ODE

Form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

Isobaric functions follow the rule:

$$f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x,y)$$

Assign the weights $y, \mathrm{d}y \to m$ and $x, \mathrm{d}x \to 1$ and solve for m.

Use substitution:

$$y = vx^m$$

2nd Order ODEs

Linear 2nd Order ODE with Constant Coefficients

Form:

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 = f(x)$$

Find auxiliary equation and solve for λ:

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

Find complementary function.

For two solutions of λ, λ_1 and λ_2:

$$y_c = A\exp(\lambda_1 x) + B\exp(\lambda_2 x)$$

Or for a repeated root, λ_1:

$$y_c = (A + Bx)\exp(\lambda_1 x)$$

Then find particular solution, y_p, using a trial function and subbing into original ODE.

General solution is:

$$y = y_c + y_p$$

Laplace Transform Method for Linear 2nd Order ODEs with Constant Coefficients

Laplace Transform is defined as:

$$\mathcal{L}[f(x)](s) = \bar{f}(s) \equiv \int_0^\infty e^{-sx} f(x)\mathrm{d}x$$

For a second order derivative:

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right] = s^2 \mathcal{L}[f] - sf(0) - \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=0}$$

And for first order derivative:

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = s\mathcal{L}[f] - f(0)$$

Sub these into ODE, apply boundary conditions and apply inverse Laplace transform using the standard transforms table.

Note that the Laplace transform and its inverse are linear:

$$\mathcal{L}[af(x) + bg(x)] = a\mathcal{L}[f(x)] + b\mathcal{L}[g(x)]$$

$$\mathcal{L}^{-1}[af(x) + bg(x)] = a\mathcal{L}^{-1}[f(x)] + b\mathcal{L}^{-1}[g(x)]$$

Where $a, b \in \mathbb{C}$.

Legendre Linear Equations

Form:

$$a_2(\alpha x+\beta)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1(\alpha x+\beta)\frac{\mathrm{d}y}{\mathrm{d}x}+a_0y=f(x)$$

Use substitution $\alpha x+\beta=e^t$ which gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\alpha}{\alpha x+\beta}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\alpha^2}{(\alpha x+\beta)^2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-\frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

Euler Linear Equations

Form:

$$a_2x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1x\frac{\mathrm{d}y}{\mathrm{d}x}+a_0y=f(x)$$

Make substitution $x=e^t$ or, if $f(x)=0$, you may use $y=x^\lambda$.

Wronskian Method for Linear 2nd Order ODEs

The Wronskian for an nth order ODE is defined as:

$$W(y_1,y_1,\dots,y_n)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \vdots \\ \vdots & & \ddots & \\ y_1^{(n-1)} & \cdots & & y_n^{(n-1)} \end{vmatrix}$$

Hence for a 2nd order ODE the Wronskian is:

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Where y_1,y_2 are solutions to the ODE.

We can then find a general solution to the ODE using:

$$y=y_p=k_1y_1+k_2y_2$$

Where

$$k_1=-\int\frac{y_1f(x)}{W(y_1,y_2)}\mathrm{d}x$$

$$k_2=\int\frac{y_2f(x)}{W(y_1,y_2)}\mathrm{d}x$$

Ordinary and Singular Points

For the 2nd order ODE where $y=y(z)$:

$$y''+p(z)y'+q(z)y=0$$

The point $z=z_0$ is an ordinary point if both

$$\lim_{z\to z_0}p(z)$$

and

$$\lim_{z\to z_0}q(z)$$

exist and are finite (i.e. $p(z)$ and $q(z)$ converge at z_0)

If $p(z)$ and/or $q(z)$ diverge at $z = z_0$, it is a singular point.

The point $z = z_0$ is a regular singular point if both

$$\lim_{z \to z_0} (z - z_0) p(z)$$

and

$$\lim_{z \to z_0} (z - z_0)^2 q(z)$$

exist and are finite (i.e. $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ converge at z_0).

If $(z - z_0)p(z)$ and/or $(z - z_0)^2 q(z)$ diverge at $z = z_0$, it is an irregular singular point.

When checking $z_0 = \infty$, use the substitutions $w = \frac{1}{z}$, $w_0 = 0$.

Series Solutions at Ordinary Points

To solve the 2nd order ODE where $y = y(z)$:

$$y'' + p(z)y' + q(z)y = 0$$

Substitute:

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

To get:

$$\sum_{n=0}^{\infty} \Big[(n+2)(n+1)a_{n+2} + p(z)(n+1)a_{n+1} + q(z)a_n \Big] z^n = 0$$

Then for the case where $z^n \neq 0$:

$$(n+2)(n+1)a_{n+2} + p(z)(n+1)a_{n+1} + q(z)a_n = 0$$

Then rearrange to get a recurrence relation.

Legendre's Differential Equation

Form:

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \ell(\ell+1)y = 0$$

Solutions are Legendre Polynomials, $P_\ell(x)$.

Rodrigues' Formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2 - 1)^\ell$$

Associated Legendre Differential Equation

Form:

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]y = 0$$

Solution:

$$P_\ell^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m P_\ell}{\mathrm{d}x^m}$$

For $m > 0$.

Partial Differential Equations

Separation of Variables

For PDE of the form:

$$au_{xx} + bu_{tt} + cu_x + du_t + eu = 0$$

Assume solution to PDE can be expressed as:

$$u(x,t) = X(x)T(t)$$

Substitute in and rearrange:

$$a\frac{X''}{X} + c\frac{X'}{X} + e = -b\frac{T''}{T} - d\frac{T'}{T}$$

Separate equations into two ODEs and solve:

$$a\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + c\frac{\mathrm{d}X}{\mathrm{d}x} + (e-\mu)X = 0$$

$$b\frac{\mathrm{d}^2T}{\mathrm{d}t^2} + d\frac{\mathrm{d}T}{\mathrm{d}t} + \mu T = 0$$

Separation of a 3D PDE

Assume solution to PDE can be expressed as:

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t)$$

Taking an example PDE, the heat equation:

$$\frac{\partial u}{\partial t} = k^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Substituting in the solution:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{k^2}\frac{T'}{T} = \mu$$

Separating:

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - lX = 0$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} - mY = 0$$

$$\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} - nZ = 0$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \mu k^2 T = 0$$

Where $l + m + n = \mu$.

General Solutions to 1st Order PDEs - $A = 0$ or $B = 0$

The most general form of this type of PDE is:

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y)$$

Where $A(x,y)$, $B(x,y)$, $C(x,y)$ and $R(x,y)$ are given functions. If either $A = 0$ or $B = 0$ then the equation can simply be solved as 1st order linear ODE.

The forms of these types of equations are:

$$\frac{\partial u}{\partial x} + P(x,y)u = Q(x,y)$$

Or:

$$\frac{\partial u}{\partial y} + P(x,y)u = Q(x,y)$$

Which can be solved using an integrating factor, $\mu(x,y)$.

General Solutions to 1st Order PDEs - $C = R = 0$

The form of this PDE is:

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = 0$$

Look for solution of the form $u(x,y) = f(p)$, where $p = p(x,y)$.

Subbing into the PDE gives:

$$\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]\frac{\mathrm{d}f(p)}{\mathrm{d}p} = 0$$

1. For non-trivial p, $\mathrm{d}f(p)/\mathrm{d}p \neq 0$ so:

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}$$

2. If p is constant then the total derivative is zero:

$$\mathrm{d}p = \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y = 0$$

We can then say:

$$\mathrm{d}x = A(x,y) \qquad \mathrm{d}y = B(x,y)$$

$$\frac{\mathrm{d}x}{A(x,y)} = 1 \qquad \frac{\mathrm{d}y}{B(x,y)} = 1$$

Hence:

$$\frac{\mathrm{d}x}{A(x,y)} = \frac{\mathrm{d}y}{B(x,y)}$$

Then by setting p as the constant of integration of this equation we get:

$$u(x,y) = f(p)$$

Where f is an arbitrary function.

General Solutions to 1st Order PDEs - $R = 0$

Form:

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu = 0$$

Look for solutions of the form $u(x, y) = h(x, y)f(p)$ where $h(x, y)$ is any solution to the PDE.

We get:

$$\left(A\frac{\partial p}{\partial x} + B\frac{\partial p}{\partial y}\right) h\frac{\mathrm{d}f}{\mathrm{d}p} = 0$$

Assuming non-trivial $h(x, y)$ and non-trivial p, where $\mathrm{d}f/\mathrm{d}p \neq 0$, we find:

$$\frac{\mathrm{d}x}{A(x, y)} = \frac{\mathrm{d}y}{B(x, y)}$$

Setting the constant of integration to be p, we get:

$$u(x, y) = h(x, y)f(p)$$