Mathematical Methods in Physics

Weekly Problems 4. Solution

4.1

We can use the formula $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ to write

$$f(x) = 1 + \sin(3x + \pi/5) = 1 + \sin(3x)\cos(\pi/5) + \cos(3x)\sin(\pi/5),$$

which is already in the right form for a Fourier series. So the Fourier coefficients can be read directly from this equation. They are:

$a_0 = 2, \quad b_3 = \cos(\pi/5)\ , \qquad a_3 = \sin(\pi/5)\ , \qquad a_r = b_r = 0 \quad \text{for } r \neq 3\ .$ $\boxed{\text{2 marks}}$

4.2

The function is neither even nor odd, so we need to compute both the sine and cosine coefficients.

$$\begin{aligned}
a_0 &= \int_0^2 x^2\,dx = \frac{8}{3}\,, \qquad \boxed{\text{1 mark}}\\
a_r &= \int_0^2 x^2\cos(\pi r\,x)\;dx = \left(\frac{x^2}{\pi r}\sin(\pi r\,x)\Big|_0^2 - \frac{2}{\pi r}\int_0^2 x\sin(\pi r\,x)dx\right)\\
&= -\frac{2}{r\pi}\left(-\frac{x}{\pi r}\cos(\pi r\,x)\Big|_0^2 \int_0^2 \frac{1}{\pi r}\cos(\pi r\,x)dx\right)\\
&= -\frac{2}{r\pi}\left(-\frac{2}{\pi r} + \frac{1}{(\pi r)^2}\cos(\pi r\,x)\Big|_0^2\right) = \frac{4}{\pi^2 r^2}\,, \qquad \boxed{\text{1 mark}}\\
b_r &= \int_0^2 x^2\sin(\pi r\,x)\;dx = \left(-\frac{x^2}{\pi r}\cos(\pi r\,x)\Big|_0^2 + \frac{2}{\pi r}\int_0^2 x\cos(\pi r\,x)dx\right)\\
&= -\frac{4}{\pi r} + \frac{2}{\pi r}\left(\frac{x}{\pi r}\sin(\pi r\,x)\Big|_0^2 - \frac{1}{(\pi r)^2}\int_0^2 \sin(\pi r\,x)dx\right)\\
&= -\frac{4}{\pi r} + \frac{2}{r\pi}\left(\frac{1}{(\pi r)^3}\cos(\pi r\,x)\Big|_0^2\right) = -\frac{4}{\pi r}\,. \qquad \boxed{\text{1 mark}}
\end{aligned}$$

So we have

$$f(x) = \frac{4}{3} + \sum_{r=1}^{\infty}\left(\frac{4}{\pi^2 r^2}\cos(\pi r\,x) - \frac{4}{\pi r}\sin(\pi r\,x)\right)\,. \qquad \boxed{\text{1 mark}}$$

4.3

a) By differentiating expression (1) on both side we get

$$-\frac{\sin(x/2)}{2} = -\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r r}{1-4r^2}\sin(rx).$$

By rearranging we obtain

$$g(x) = \sin(x/2) = \frac{8}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r r}{1-4r^2}\sin(rx). \qquad \boxed{\text{2 marks}}$$

You can verify that this is the correct expression by calculating directly the Fourier series of the function $g(x)$.

b) From the Fourier series of $h(x)$ at $x = 0$ we get

$$h(0) = 1 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{(1-4r^2)}\,.$$

Solving for the sum gives

$$\sum_{r=1}^{\infty}\frac{(-1)^r}{(1-4r^2)} = \frac{\pi}{4} - \frac{1}{2}\,. \qquad \boxed{\text{2 marks}}$$