Relativistic Electrodynamics, Workshop 5

1. Suppose $\phi = 0$ and $\underline{A} = A_0 \ \sin(kx - \omega t)\underline{\hat{y}}$, where A_0, ω, and k are constants. Find $\underline{E}$ and $\underline{B}$, and check that they satisfy Maxwell's Equations in vacuum. What relation must be imposed between ω and k?

2. Find the fields, and the charge and current distributions, corresponding to
$$\phi(\underline{r}, t) \ = \ 0, \qquad \underline{A}(\underline{r}, t) \ = \ -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\underline{\hat{r}}. \tag{1}$$

3. Use the gauge function
$$\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r} \tag{2}$$
to transform the potentials in the previous problem, and comment on the results.

4. Which of the potentials in Question 2 and 3 satisfy the condition of the Coulomb gauge? Which satisfy the Lorenz gauge condition? Note that these are not mutually exclusive.

5. Show that it is always possible to find a solution to the Maxwell's Equations for the potentials, which also satisfy the Lorenz gauge condition
$$\underline{\nabla} \cdot \underline{A} = -\mu_0\varepsilon_0\frac{\partial \phi}{\partial t}. \tag{3}$$
I.e. show that if one can find a solution to the Maxwell's equations for the fields, then one can further find one that satisfy the Lorentz gauge condition.

   Is it always possible to pick $\phi = 0$? How about $\underline{A} = 0$?

6. Confirm that the retarded potentials satisfy the Lorenz gauge condition. Start by showing that
$$\underline{\nabla} \cdot \left(\frac{\underline{j}}{R}\right) = \frac{1}{R}\left(\underline{\nabla} \cdot \underline{j}\right) + \frac{1}{R}\left(\underline{\nabla}' \cdot \underline{j}\right) - \underline{\nabla}' \cdot \left(\frac{\underline{j}}{R}\right), \tag{4}$$
where $\underline{\nabla}$ denotes derivatives with respect to $\underline{r}$, and $\underline{\nabla}'$ denotes derivatives with respect to $\underline{r}'$. $R = |\underline{r} - r'|$. Next, note that the current density $\underline{j}(\underline{r}', t_r)$ depends on $\underline{r}'$ both explicitly and through R, whereas it depends on $\underline{r}$ only through R. Confirm that
$$\nabla \cdot \underline{j} = -\frac{1}{c}\left(\frac{\partial}{\partial t}\underline{j}\right) \cdot (\underline{\nabla} r), \qquad \nabla' \cdot \underline{j} = -\frac{\partial}{\partial t}\rho - \frac{1}{c}\left(\frac{\partial}{\partial t}\underline{j}\right) \cdot (\underline{\nabla}' r). \tag{5}$$
Finally, use this to calculate the divergence of the vector potential $\underline{A}$.