Mathematical Methods II
Lecture 15

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5/3/2019

Key Points

- General solutions to PDEs
- Homogeneity in PDEs
- Solving inhomogeneous problems

General Solutions to PDEs (ctud)

- **1st order PDEs:** Recalling the general form for a 1st order PDE

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y),$$

we have looked at solutions for when A or B are zero, and for when C and R are zero. The latter method needs modifying if $C \neq 0$. i.e. when

$$Au_x + Bu_y + Cu = 0.$$

Instead of looking for a solution of the form $u(x,y) = f(p)$ we look for $u(x,y) = h(x,y)f(p)$, where $h(x,y)$ is any solution to the PDE.

Let's see how this affects the derivatives. Remember

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x} = f' p_x$$

When $C = 0$	When $C \neq 0$
$u(x,y) = f(p)$	$u(x,y) = h(x,y)f(p)$
$u_x = f'p_x$	$u_x = h_x f + hf'p_x$
$u_y = f'p_y$	$u_y = h_y f + hf'p_y$

Subbing these into our PDE

$$A\left(h_x f + hf'p_x\right) + B\left(h_y f + hf'p_y\right) + Chf = 0.$$

Collecting terms in f and hf'

$$(Ah_x + Bh_y + Ch)f + (Ap_x + Bp_y)hf' = 0$$

Notice that the first bracket has exactly the form of the original PDE. Compare

$$u_x \leftrightarrow h_x \qquad\qquad u_y \leftrightarrow h_y \qquad\qquad u \leftrightarrow h$$

Since we assumed that $h(x, y)$ was a solution to the PDE, the left-hand term must equal zero to satisfy the RHS of the PDE. This leaves

$$(Ap_x + Bp_y)hf' = 0.$$

So our assumed solution of $h(x, y)f(p)$ has allowed us to eliminate the Cu term, leaving only derivative terms in p. Now, just as we did earlier we find that

1) We can assume a non-trivial $h(x, y)$, and for non-trivial p we can assume $df(p)/dp \neq 0$, so the bracket is zero giving

$$A(x, y)\frac{\partial p}{\partial x} + B(x, y)\frac{\partial p}{\partial y} = 0.$$

2) If we require that p is constant then the total derivative of p is zero

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0.$$

Ultimately we find the same relation as earlier between the coefficients and the derivatives by comparing the two cases,

$$\frac{dx}{A(x, y)} = \frac{dy}{B(x, y)}.$$

Integrating this relationship and finding the constant of integration will give us p. However this time the solution will be of the form

$$u(x, y) = h(x, y)f(p)$$

where $h(x, y)$ is any non-trivial solution to the PDE.

e.g. 15.1 1^{st}order PDE with 2 derivatives and a term in u: Find the general solution of

$$x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} - 2u = 0$$

Identifying $A = x$ and $B = 2$

$$\frac{dx}{A(x, y)} = \frac{dy}{B(x, y)}$$
$$\frac{dx}{x} = \frac{dy}{2}$$

Integrating we find

$$x = ke^{y/2}$$

Noting that the constant we seek is $p = k = xe^{-y/2}$ we find our general solution is given by

$$u(x, y) = h(x, y)f(xe^{-y/2})$$

where $f(p)$ is an arbitrary function of p and $h(x, y)$ is any solution to the PDE. For example, $h = e^y$, which satisfies the PDE, gives

$$u(x, y) = e^y f(xe^{-y/2}).$$

Alternatively, $h = x^2$ gives

$$u(x, y) = x^2 g(xe^{-y/2}),$$

making sure to distinguish the functions f and g with different labels in particular examples, as they are likely not identical, since $h(x, y)$ is different in each solution. In fact they can be related directly. In this case $g(p) = f(p)/p^2$

$$x^2 g(p) = x^2 \frac{f(p)}{p^2} = \frac{x^2}{(xe^{-y/2})^2} f(p) = e^y f(p).$$

So, both solutions work, $u = e^y f(p) = x^2 g(p)$.

- **Homogeneity in PDEs:** Homogeneity in PDEs can actually refer to one of two things; the equation or the problem.

  The *equation* is said to be homogeneous if a solution $u(x, y)$ can be found such that $\lambda u(x, y)$ is also a solution for any constant λ. i.e. multiples of the solution are also solutions.

  The *problem* is said to be homogeneous if in addition to the above, the boundary conditions satisfied by $u(x, y)$ are also satisfied by $\lambda u(x, y)$. These would then be called *homogeneous boundary conditions.*

  For example, the equation we looked at in e.g.14.2

  $$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0,$$

  is homogeneous. If we find a solution u, then $2u$, $3.5u$, etc will still satisfy the equation, resulting in $RHS = 0$. The constant factor can be factored out making the value of λ irrelevant.

  $$x\frac{\partial \lambda u}{\partial x} - 2y\frac{\partial \lambda u}{\partial y} = 0$$
  $$x\lambda u_x - 2y\lambda u_y = 0$$
  $$\lambda(xu_x - 2yu_y) = 0$$

  Since we know the bracket is zero, as it is the original PDE, it doesn't matter what value λ has as long as it is constant. Example 14.2 presented several solutions that satisfied the second BC of $u = 4$ at (1,1), so we could claim the problem is homogeneous and that BC is homogeneous too.

However the general equation for a 1st order PDE

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y)$$

is not homogeneous as changing u will result in a different value for R.

$$A\lambda u_x + B\lambda u_y + C\lambda u = \lambda(Au_x + Bu_y + Cu) = \lambda R(x,y) \neq R \quad \text{except for } \lambda = 1.$$

It can be made homogeneous by setting $R = 0$ (recall solving homogeneous ODEs with $RHS = 0$).

$$A\lambda u_x + B\lambda u_y + C\lambda u = \lambda(Au_x + Bu_y + Cu) = \lambda R(x,y) = 0 \quad \text{for all } \lambda.$$

- **Solving inhomogeneous PDEs:** The reason we're interested in the homogeneity of PDEs is that linear PDEs have a close parallel to the complementary functions and particular integrals of linear ODEs.

  That is to say the general solution to an inhomogeneous PDE can be written as the sum of the solution to the homogeneous problem ($RHS = 0$) and any particular solution.

e.g. 15.2 An inhomogeneous problem: Find the general solution of

$$yu_x - xu_y = 3x.$$

Hence find the most general particular solution (i) which satisfies $u(x,0) = x^2$ and (ii) which has the value $u(x,y) = 2$ at the point (1,0).

This equation has independent variables that are not associated with u or its derivatives and can be placed on the RHS, hence it is inhomogeneous. First let's solve the homogeneous problem by finding a solution u(x,y)=f(p).

$$\frac{dx}{y} = \frac{dy}{-x} \Rightarrow -xdx = ydy \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c$$

Let's say $p = 2c$ to avoid fractions, so the homogeneous solution is

$$u(x,y) = f(x^2 + y^2)$$

which is the equivalent of our complementary function for linear ODEs. We can find a nice easy particular solution by inspection,

$$u(x,y) = -3y$$

which when subbed into the PDE gives

$$y(-3y)_x - x(-3y)_y = y(0) - x(-3) = 3x$$

.

So our general solution is therefore

$$u(x, y) = f(x^2 + y^2) - 3y.$$

Next we want to solve the boundary problems. We're going to do that by picking a specific form for f(p) suitable for the given BCs.

(i) $\underline{u(x, 0) = x^2}$: This requires that

$$u(x, 0) = f(x^2 + 0) - 3(0) = f(x^2) = x^2.$$

Thinking about this as $f(z) = z$ can help us arrive at a solution, where $z = p|_{BC}$ (i.e. z is p after the BCs have been applied, so when forming the solution we write p instead of z). The most general particular solution that satisfies this is

$$u(x, y) = x^2 + y^2 - 3y$$

which clearly gives $u = x^2$ when $y = 0$. $z = x^2$ in this case, and since x is not specified by the BC and can take any value, the function is fully determined for any arbitrary value of x and does not require an additional g function term - there is only one way to write down the function.

(ii) $\underline{u(x, y) = 2 \text{ at the point } (1,0)}$: This requires that

$$u(1, 0) = f(x^2 + y^2)|_{x=1,y=0} = f(1) = 2.$$

We could interpret this as $f(z) = 2$ or $f(z) = 2z$. These give the respective particular solutions

$$u(x, y) = 2 - 3y$$
$$u(x, y) = 2(x^2 + y^2) - 3y$$

and the respective general solutions

$$u(x, y) = g(x^2 + y^2) + 2 - 3y$$

$$u(x, y) = g(x^2 + y^2) + 2(x^2 + y^2) - 3y$$

where g is an arbitrary function subject to the condition $g(1) = 0$. The g terms are required since the function is not fully specified; i.e. there is more than one way to write it down.

Let's quickly test the particular solutions to convince ourselves they work.

For (i) $u(x, y) = x^2 + y^2 - 3y$

$$u_x = 2x \qquad\qquad u_y = 2y - 3$$

Substituting these into the PDE

$$yu_x - xu_y = y(2x) - x(2y - 3) = 2xy - 2xy + 3x = 3x$$

as required by the RHS of the PDE.

For (ii) $u(x, y) = 2 - 3y$

$$u_x = 0 \qquad\qquad u_y = -3$$

Substituting these into the PDE

$$yu_x - xu_y = y(0) - x(-3) = 3x$$

as required by the RHS of the PDE.

Let's also try an arbitrary $g(p) = \sin(x^2 + y^2)$ for a (ii) general solution.

$$u(x, y) = \sin(x^2 + y^2) + 2 - 3y$$

$$u_x = 2x\cos(x^2 + y^2) \qquad\qquad u_y = 2y\cos(x^2 + y^2) - 3$$

Substituting these into the PDE

$$\begin{aligned} yu_x - xu_y &= y[2x\cos(x^2 + y^2)] - x[2y\cos(x^2 + y^2) - 3] \\ &= 2xy\cos(x^2 + y^2) - 2xy\cos(x^2 + y^2) + 3x \\ &= 3x \end{aligned}$$

as required by the RHS of the PDE.