ELECTROMAGNETISM

Level 2 Physics problems – Foundations of physics 2

Solution 7 Cycle 2 Version 1

Professor D P Hampshire – 2nd Year Physics Lecture Course

Information underlined or indicated by red text is required for marks to awarded. The mark scheme is a guide and solutions should not be considered to be unique. Marks are awarded for correct relevant Physics.

1. a) The Poynting vector $\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}$. It gives the instantaneous power transmitted per unit area. 1-1

1 mark for correct equation and statement 1-1

b) From MIII:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \frac{-\partial \underline{\boldsymbol{B}}}{\partial t}$$

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E_0}} = \omega \; \underline{\boldsymbol{B_0}} \qquad \text{1-2}$$

From MIV:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}$$

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{B_0}} = -\mu_0 \varepsilon_0 \omega \underline{\boldsymbol{E_0}} \qquad \text{1-3}$$

Hence,

$$\underline{\boldsymbol{k}} \perp \underline{\boldsymbol{E_0}} \perp \underline{\boldsymbol{B_0}} \qquad \text{1-4}$$

1 mark for derivation of 1-2, 1-3 to produce correct conclusion 1-4

c) In free space for $\underline{\boldsymbol{k}}$,

$$\frac{\omega}{k} = \mathrm{c}$$

$$k = \frac{3 \times 10^{15}}{3 \times 10^{8}} = 10^{7}\mathrm{m}^{-1} \qquad \text{1-5}$$

$\underline{\boldsymbol{k}}$ is in the same direction as $\underline{\boldsymbol{N}}$, so:

$$\hat{\boldsymbol{k}} = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \mathrm{m}^{-1}$$

Hence:

$$\underline{\boldsymbol{k}} = \frac{10^{7}}{\sqrt{14}} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \mathrm{m}^{-1} \qquad \text{1-6}$$

For $\underline{\boldsymbol{B_o}}$,

$$\underline{\boldsymbol{B_o}} = \frac{10^{-8}}{\sqrt{26}}\begin{pmatrix}3\\4\\-1\end{pmatrix}\mathrm{T} \tag{1-7}$$

Lastly $|\underline{\mathbf{E_o}}|$,

$$|\underline{\mathbf{E_o}}| = |\underline{\mathbf{B_o}}|\ \mathrm{c} = 3\ \mathrm{Vm^{-1}} \tag{1-8}$$

1 mark for producing correct answers 1-6, 1-7 and 1-8
[Qn 1: 3 marks total]

2. a) At normal incidence:

$$\frac{E_{0R}}{E_{0I}} = \frac{(n_1 - n_2)}{(n_1 + n_2)} \text{ and } \frac{E_{0T}}{E_{0I}} = \frac{2n_1}{(n_1 + n_2)} \tag{2-1}$$

$$R = \left(\frac{E_{0R}}{E_{0I}}\right)^2,\ T = \left(\frac{n_2}{n_1}\right)\left(\frac{E_{0T}}{E_{0I}}\right)^2,\ \ R + T = 1 \tag{2-2}$$

Using $R = 1/2$ we have $\frac{(n_1 - n_2)^2}{(n_1 + n_2)^2} = {}^1\!/_2$ 2-3

$$=> \left(\frac{n_1}{n_2}\right)^2 - 6\left(\frac{n_1}{n_2}\right) + 1 = 0 \tag{2-4}$$

$$=> \left(\frac{n_1}{n_2}\right) = \frac{6 \pm \sqrt{32}}{2} = 5.83 \textbf{ or } 0.1715 \tag{2-5}$$

1 mark for BOTH correct solutions 2-5

b) If $n_1 = \varepsilon_1^{1/2} = \sqrt{9} = 3$ 2-6

$$=> n_2 = 3/5.83 = 0.515 \text{ or } 3/0.1715 = 17.5. \tag{2-7}$$

However, $n_2 = 0.515$ not possible as we require n > 1 for a dielectric.
Only possible for a plasma. 2-8

1 mark for BOTH solutions 2-7. Not putting statement 2-8 does not lose any marks.
[Qn 2: 2 marks total]

3. a)

$$\underline{\boldsymbol{N}} = \frac{C_0 I_0^2}{r^2}\sin(\theta)\hat{\boldsymbol{r}} \tag{3-1}$$

$$P = \frac{1}{2}I_0^2 R_r = \int \underline{\boldsymbol{N}}.d\underline{\boldsymbol{S}} = \int \frac{C_0 I_0^2}{r^2}\sin(\theta).r^2\sin(\theta)d\theta d\phi \tag{3-2}$$

where $r^2\sin(\theta)d\theta d\phi$ is the Jacobian for this coordinate system.

$$= C_0 I_0^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^2(\theta) d\theta = C_0 I_0^2 2\pi \frac{\pi}{2}$$ 3-3

$$R_r = 2C_0 2\pi \frac{\pi}{2} = 197.4$$ 3-4

1 mark for correct answer 3-4

b) Beam width, *W*, is the angle subtended between the half-power directions 3-5
$\sin(\theta)=1/2$ when $\theta = 30^o$ or 150^o so $W = 120^o$

1 mark for correct answer 3-5

c) The Directivity, is;

$$D = \frac{C_0 I_0^2}{r^2}\sin(\pi/2) \Bigg/ \frac{\frac{1}{2} I_0^2 R_r}{4\pi r^2} = \frac{8\pi C_0}{R_r} = 1.27$$ 3-6

1 mark for correct answer 3-6
[Qn 3: 3 marks total]

4.

$$\underline{\widetilde{\boldsymbol{B}}} = \underline{\boldsymbol{\nabla}} \times \underline{\widetilde{\boldsymbol{A}}}(r,t) = \frac{\mu_0 I_0}{4\pi} \underline{\boldsymbol{\nabla}} \times \frac{e^{i(kr-wt)}}{r}\left(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}}\right) dl$$ 4-1

Ignoring the prefactors for now;

$$\underline{\boldsymbol{\nabla}} \times \frac{e^{+ikr}}{r}\left(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}}\right) dl = \begin{vmatrix} \frac{\hat{\mathbf{r}}}{\mathrm{r}^2 \sin\theta} & \frac{\hat{\boldsymbol{\theta}}}{\mathrm{r}\sin\theta} & \frac{\hat{\boldsymbol{\phi}}}{\mathrm{r}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \mathrm{A_r} & \mathrm{rA_\theta} & \mathrm{rA_\phi \sin\theta} \end{vmatrix}$$ 4-2

where $A_r = \frac{e^{+ikr}}{r}\cos\theta$, $A_\theta = -\sin\theta \frac{e^{+ikr}}{r}$ and $A_\phi = 0$. 4-3

$$= \frac{e^{+ikr}}{r^2 \sin\theta}\left\{0 \cdot \hat{\mathbf{r}} + 0 \cdot r\hat{\boldsymbol{\theta}} + r\sin\theta \hat{\boldsymbol{\phi}}\left[\frac{\partial}{\partial r}(-e^{+ikr}\sin\theta) - \frac{\partial}{\partial \theta}\left(\frac{e^{+ikr}}{r}\cos\theta\right)\right]\right\} dl$$ 4-4

Putting prefactors back in;

$$\underline{\widetilde{\boldsymbol{B}}} = \frac{\mu_0 I_0}{4\pi} e^{-i\omega t} dl \frac{1}{r^2 \sin\theta}\left\{r\sin\theta \hat{\boldsymbol{\phi}}\left[(-ike^{+ikr}\sin\theta) + \left(\frac{e^{+ikr}}{r}\sin\theta\right)\right]\right\}$$ 4-5

$$\underline{\widetilde{\boldsymbol{B}}} = \frac{\mu_0 I_0}{4\pi}\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right)\sin\theta e^{i(kr-wt)} dl \hat{\boldsymbol{\phi}}.$$ 4-6

Taking the real part of $\underline{\tilde{\boldsymbol{B}}}$ in the far field limit gives:

$$\underline{\boldsymbol{B}}_{\mathrm{Far}} = \frac{\mu_0 I_0 \delta l}{4\pi r} k \sin(\theta)\sin(kr - \omega t)\hat{\boldsymbol{\phi}}$$

We can either proceed as we did in the lectures, carrying all terms, and using $\underline{\tilde{\boldsymbol{B}}}$, or given we are looking for $\underline{\boldsymbol{E}}_{\mathrm{Far}}$, we can simply use Maxwell IV

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}_{\mathrm{Far}} = \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}_{\mathrm{Far}}}{\partial t} .$$

Using the determinant above to calculate $\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}_{\mathrm{Far}}$, **we find:**

$$\underline{\boldsymbol{E}}_{\mathrm{Far}} = \frac{\mu_0 I_0 \delta l}{4\pi} \frac{\omega}{r} \sin(\theta)\sin(kr - \omega t)\hat{\boldsymbol{\theta}}$$

1 mark for $\underline{\boldsymbol{B}}_{\mathrm{Far}}$ and 1 mark for $\underline{\boldsymbol{E}}_{\mathrm{Far}}$.
[Qn 4: 2 marks total]

Total for all questions 10 marks

Below are the solutions to some additional straightforward revision questions (no marks):

5. a)

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \quad \text{(MI)}$$
$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0 \quad \text{(MII)}$$
$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \quad \text{(MIII)}$$
$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \quad \text{(MIV)}$$

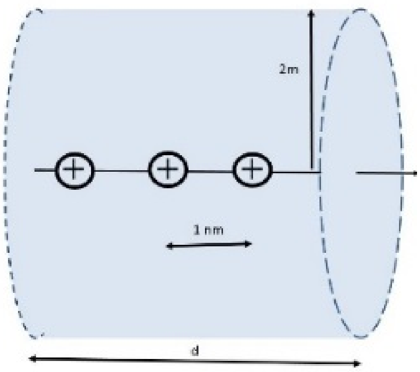
5-9

b) The skin effect – a high frequency magnetic field will <u>induce Eddy currents</u>. At <u>high frequencies</u>, <u>current flows at the surface</u> and the applied magnetic field are <u>'screened'</u> out from the interior. 5-10

c) Consider Gauss' law;

$$\oint \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{S}} = \frac{q_{\mathrm{enclo}}}{\varepsilon_0}$$

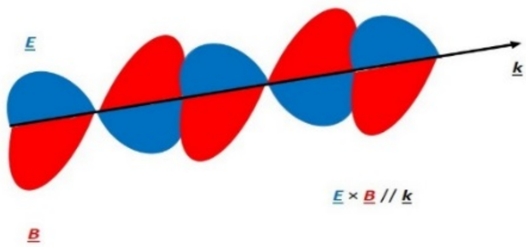
5-11



$$E \cdot 2\pi r d = \frac{\lambda d}{\varepsilon_o} = \frac{e \cdot d}{10^{-9} \cdot \varepsilon_o}$$ 5-12

$$E = 1.44\ \mathrm{Vm}^{-1}$$ 5-13

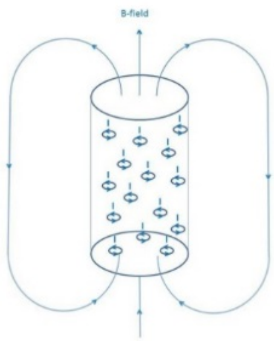
d)



5-14

$\underline{\boldsymbol{E}} \perp \underline{\boldsymbol{B}} \perp \underline{\boldsymbol{k}}$ and $(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}})$ is parallel to $\underline{\boldsymbol{k}}$ 5-15

e) Ampère's model – inside a magnetized material there are <u>current loops</u> flowing. The <u>currents flow on an atomic scale</u> and <u>add together</u> to produce a <u>macroscopic field</u>. 5-16



5-17

f) A plasma is a <u>net neutral</u> collection of <u>electrons</u> and <u>ions</u>.
Examples: the ionosphere, a fluorescent bulb, solar wind, welding arcs, the atmosphere of stars, aurorae… 5-18

g) In polar dielectrics there are <u>permanent dipole moments</u> that rotate when an $\underline{\boldsymbol{E}}$-field is applied.

In non-polar dielectrics, there is the <u>displacement of charges</u> to form dipoles. 5-19

6. Let;

$$\underline{\boldsymbol{E}}(\mathrm{r},\mathrm{t}) = \underline{\boldsymbol{E}}_o \exp i(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega \mathrm{t}) \qquad 6\text{-}7$$

Substitute into wave equation;

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad 6\text{-}8$$

Leads to,

$$\mathrm{k}^2 = \omega^2 \mu_0 \varepsilon_0 \qquad 6\text{-}9$$

Phase velocity;

$$\mathrm{v_{phase}} = \frac{\omega}{\mathrm{k}} = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \qquad 6\text{-}10$$

Group velocity;

$$\mathrm{v_{group}} = \frac{\partial \omega}{\partial \mathrm{k}} = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \qquad 6\text{-}11$$

7. a) Substituting $\underline{\boldsymbol{E}}$ into the wave equation; 7-1

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad 7\text{-}2$$

$$-\frac{25\pi^2}{a^2} - \frac{25\pi^2}{a^2} - k^2 = -\mu_0 \varepsilon_0 \omega^2 \qquad 7\text{-}3$$

$$k^2 = \mu_0 \varepsilon_0 \omega^2 - \frac{50\pi^2}{a^2} \qquad 7\text{-}4$$

For propagation, we require k is real, therefore $k^2 > 0$. 7-5

$$\omega^2 > \frac{50\pi^2}{\mu_0 \varepsilon_0 a^2} \qquad 7\text{-}6$$

Substituting in numbers,

$$\omega > 3.3 \times 10^{10} \text{ rad s}^{-1}$$
$$f > 5.3 \times 10^9 \text{ Hz} \qquad 7\text{-}7$$

b) Using the dispersion relation:

$$k^2 = \mu_0 \varepsilon_0 \omega^2 - A$$

Differentiated w.r.t k,

$$2\mathrm{k} = 2\mu_0\varepsilon_0\omega\frac{\partial\omega}{\partial k}$$ 7-8

Differentiated w.r.t k again,

$$1 = \mu_0\varepsilon_0\left(\left[\frac{\partial\omega}{\partial k}\right]^2 + \omega\left[\frac{\partial^2\omega}{\partial k^2}\right]\right)$$ 7-9

Substituting in 7-9 into 7-10 to eliminate $\frac{\partial\omega}{\partial k}$

$$1 = \frac{1}{\mu_0\varepsilon_0}\left[\frac{k}{\omega}\right]^2 + \mu_0\varepsilon_0\omega\left[\frac{\partial^2\omega}{\partial k^2}\right]$$ 7-10

Substituting in,

$$\frac{k^2}{\omega^2} = \left(\mu_0\varepsilon_0 - \frac{A}{\omega^2}\right)$$ 7-11

$$\left[\frac{\partial^2\omega}{\partial k^2}\right] = \frac{Ac^4}{\omega^3}$$

Pulse moves <u>Dispersively.</u> 7-12

7-13

c) The condition for minimum frequency (ω_{min}) is:

$$\omega_{min} > \frac{\sqrt{2}\pi c}{a}5$$

The constant number 5 can be replaced by any integer – if the integer is lower, the minimum frequency is lower. For example, a possible solution is:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0\sin(\frac{10\pi x}{a})\sin(\frac{10\pi z}{a})\exp i(ky - \omega t)$$ 7-14

The photon energy is not affected by the spatial profile. It is only affected by the frequency. If the frequency of the new wave is the same as the original wave so is the photon energy. 7-15
Answer I correct.

7-16

7-17