Theoretical Physics 2019/20 — Solution of Problem QT2.1

(a) The axioms are given on page 23 of the notes, in Section 2.8.

1. It is indeed the case that $(f, g) = (g, f)^*$ for any two of these functions, $f(x)$ and $g(x)$, since

$$\int_{-1}^{1} f^*(x)\, g(x)\, \mathrm{d}x = \left[\int_{-1}^{1} g^*(x)\, f(x)\, \mathrm{d}x\right]^* .$$

2. This definition of the inner product is also consistent with the second axiom since it is indeed the case that for any functions $f_1(x)$, $f_2(x)$ and $g(x)$ of this form and for any complex numbers α and β,

$$\int_{-1}^{1} [\alpha f_1(x) + \beta f_2(x)]^*\, g(x)\, \mathrm{d}x = \alpha^* \int_{-1}^{1} f_1^*(x)\, g(x)\, \mathrm{d}x + \beta^* \int_{-1}^{1} f_2^*(x)\, g(x)\, \mathrm{d}x.$$

3. Also, for any function $f(x)$,

$$(f, f) = \int_{-1}^{1} f^*(x)\, f(x)\, \mathrm{d}x = \int_{-1}^{1} |f(x)|^2 \mathrm{d}x,$$

which cannot be negative since the integrand is always positive or zero. In fact, (f, f) is the area under the curve of equation $y = |f(x)|^2$. The only possibility for this area to be zero is that $f(x) = 0$ for all values of x — i.e., that $f(x)$ is the zero vector.

[3 marks for this part of the problem.]

(b) Functions are orthogonal if their inner product is zero. Check:

$$\begin{aligned}
(v_0, v_1) = \int_{-1}^{1} v_0^*(x) v_1(x)\, \mathrm{dx} &= \int_{-1}^{1} x\, \mathrm{d}x = \left.\frac{x^2}{2}\right|_{-1}^{1} = 0. \\
(v_0, v_2) = \int_{-1}^{1} v_0^*(x) v_2(x)\, \mathrm{dx} &= \int_{-1}^{1} x^2\, \mathrm{d}x = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}. \\
(v_0, v_3) = \int_{-1}^{1} v_0^*(x) v_3(x)\, \mathrm{dx} &= \int_{-1}^{1} x^3\, \mathrm{d}x = \left.\frac{x^4}{4}\right|_{-1}^{1} = 0. \\
(v_1, v_2) = \int_{-1}^{1} v_1^*(x) v_2(x)\, \mathrm{dx} &= \int_{-1}^{1} x^3\, \mathrm{d}x = \left.\frac{x^4}{4}\right|_{-1}^{1} = 0. \\
(v_1, v_3) = \int_{-1}^{1} v_1^*(x) v_3(x)\, \mathrm{dx} &= \int_{-1}^{1} x^4\, \mathrm{d}x = \left.\frac{x^5}{5}\right|_{-1}^{1} = \frac{2}{5}. \\
(v_2, v_3) = \int_{-1}^{1} v_2^*(x) v_3(x)\, \mathrm{dx} &= \int_{-1}^{1} x^5\, \mathrm{d}x = \left.\frac{x^6}{6}\right|_{-1}^{1} = 0.
\end{aligned}$$

Thus $v_0(x)$ and $v_2(x)$ are orthogonal to $v_1(x)$ and $v_3(x)$ (since their inner product is zero) but $v_0(x)$ is not orthogonal to $v_2(x)$ and $v_1(x)$ is not orthogonal to $v_3(x)$ (since their inner product is non-zero). [3 marks]

(c) We follow the recipe outlined in the notes, taking the vectors a, b, c to be the functions $v_0(x)$, $v_1(x)$ and $v_2(x)$ and the vectors a', b' and c' to be the functions $v_0(x)$, $v_1(x)$ and $w_2(x)$ — we can set $b' = b$ here since $v_1(x)$ is orthogonal to $v_0(x)$. Hence, Eq. (2.69) of the notes says that

$$w_2(x) = v_2(x) - [(v_0, v_2)/(v_0, v_0)]v_0(x) - [(v_1, v_2)/(v_1, v_1)]v_1(x).$$

Since $(v_1, v_2) = 0$ and since $v_0(x) \equiv 1$ and $v_2(x) \equiv x^2$,

$$w_2(x) = x^2 - [(v_0, v_2)/(v_0, v_0)].$$

We have found in (b) that $(v_0, v_2) = 2/3$. Moreover,

$$(v_0, v_0) = \int_{-1}^{1} |v_0(x)|^2 \,\mathrm{dx} = \int_{-1}^{1} \mathrm{d}x = \left.\frac{x}{1}\right|_{-1}^{1} = 2.$$

Hence

$$w_2(x) = x^2 - 1/3.$$

We iterate the process to obtain $w_3(x)$:

$$\begin{aligned} w_3(x) &= v_3(x) - [(v_0, v_3)/(v_0, v_0)]v_0(x) \\ &\qquad - [(v_1, v_3)/(v_1, v_1)]v_1(x) - [(v_2, v_3)/(v_2, v_2)]v_2(x) \\ &= x^3 - [(v_1, v_3)/(v_1, v_1)]\, x. \end{aligned}$$

We have found in (b) that $(v_1, v_3) = 2/5$. Moreover,

$$(v_1, v_1) = \int_{-1}^{1} |v_1(x)|^2 \,\mathrm{dx} = \int_{-1}^{1} x^2 \,\mathrm{d}x = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}.$$

Hence

$$w_3(x) = x^3 - 3x/5.$$

[2 marks for $w_2(x)$ and 2 marks for $w_3(x)$. You'll find something differing from the above by a constant factor if you have done Gram-Schmidt orthonormalisation rather than orthogonalisation, which is equally good.]

(d) Within normalization factors of, respectively, 3/2 and 5/2, the functions $w_2(x)$ and $w_3(x)$ found in (c) are nothing else than the Legendre polynomials $P_2(x)$ and $P_3(x)$. (Note that the normalization condition used for these polynomials is that their value is 1 at $x = 1$, which differs from the normalization condition usually chosen for wave functions. The Legendre polynomials are standard functions cropping up frequently in Quantum Mechanics and in many other areas of Physics and other sciences.) [No mark for this part of the problem.]