Relativistic Electrodynamics

Solutions to problem sheet 4

1. For this problem, $\theta = \pi/2$, $\underline{\hat{R}} = \underline{\hat{y}}$, $R = d$, so (with $c^2 = 1/(\epsilon_0\mu_0)$)

[2 marks]

$$\underline{E} = -\frac{q}{4\pi\epsilon_0}\frac{\gamma}{d^2}\,\underline{\hat{y}}, \quad \underline{B} = -\frac{q}{4\pi\epsilon_0}\frac{v}{c^2}\frac{\gamma}{d^2}\,\underline{\hat{z}},$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$, where v is the speed of the charge.

2.

$$\underline{F} = q(\underline{E} + (-v\underline{\hat{x}}) \times \underline{B}) = -\frac{q^2}{4\pi\epsilon_0}\frac{\gamma}{d^2}\left(\underline{\hat{y}} - \frac{v^2}{c^2}\,\underline{\hat{x}} \times \underline{\hat{z}}\right) = -\frac{q^2}{4\pi\epsilon_0}\frac{\gamma}{d^2}\left(1 + \frac{v^2}{c^2}\right)\underline{\hat{y}}.$$

[1 mark]

3. We note $-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \underline{E}^2 - c^2\underline{B}^2$. **[1 mark]**

So

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = -(E^2 - c^2B^2) = -\left(\frac{q}{4\pi\epsilon_0 d^2}\right)^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) = -\left(\frac{q}{4\pi\epsilon_0 d^2}\right)^2.$$

[1 mark]

4. The velocity of the charge $-q$ in the frame where $+q$ is at rest is given by

[1 mark]

$$v_{S'} = \frac{v + v}{1 + v^2/c^2} = \frac{2v}{1 + v^2/c^2}.$$

5. The γ-factor of $-q$ in the frame S' is then given by

$$\gamma_{S'} = \frac{1}{\sqrt{1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}}} = \frac{(1 + v^2/c^2)^2}{\sqrt{1 - 2\frac{v^2}{c^2} + \frac{v^4}{c^4}}} = \frac{(1 + v^2/c^2)}{(1 - v^2/c^2)} = \gamma^2(1 + v^2/c^2),$$

where γ and v are those found in the frame on the figure in the problem (and used in the first part of the problem).

We then find for the fields

[2 marks]

$$\underline{E}' = -\frac{q}{4\pi\epsilon_0}\frac{1}{d^2}\gamma^2\left(1 + \frac{v^2}{c^2}\right)\underline{\hat{y}}, \qquad \underline{B}' = -\frac{q}{4\pi\epsilon_0}\frac{2v}{c^2}\frac{\gamma^2}{d^2}\underline{\hat{z}}.$$

Here we used that the boost is along the x-axis, and thus there is no change in the coordinates along the y- and z-axes (i.e. $\underline{\hat{y}}_{S'} = \underline{\hat{y}}$).

6. Since in this frame $+q$ is at rest, there is no magnetic force, and the full Lorentz force is

[1 mark]

$$\underline{F} = q\underline{E} = -\frac{q^2}{4\pi\epsilon_0}\frac{\gamma^2}{d^2}\left(1 + \frac{v^2}{c^2}\right)\underline{\hat{y}}.$$

[indeed, we see that $F_S = \frac{1}{\gamma}F_{S'}$, which works since the boost between the frames is perpendicular to the force]

7. This is a Lorentz invariant, and so is given by minus the answer in **3**. Alternatively,

$$E^2 - c^2B^2 = \left(\frac{q}{4\pi\epsilon_0 d^2}\right)^2\gamma^4\left(1 + \frac{2v^2}{c^2} + \frac{v^4}{c^4} - 4\frac{v^2}{c^2}\right) = \left(\frac{q}{4\pi\epsilon_0 d^2}\right)^2\gamma^4/\gamma^4 = \left(\frac{q}{4\pi\epsilon_0 d^2}\right)^2$$

[1 mark]