Mathematical Methods II
Weekly problem set 6

(1) Consider the 2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

where k is a real consant. Show that it can be separated into two independent ODEs and form a general solution to the PDE.

Solution Sub in the derivatives

$$XYT'' = c^2 \left[X''YT + XY''T \right]$$

Separate by dividing by $c^2 XYT$

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2} \frac{T''}{T} = \mu$$

Let's say

$$l + m = \mu$$

where

$$\frac{X''}{X} = l \qquad \frac{Y''}{Y} = m \qquad \frac{T''}{T} = c^2 \mu$$

Each of these ODEs can be solved separately giving

$$X = c_1 e^{\sqrt{l}x} + c_2 e^{-\sqrt{l}x} \quad Y = c_3 e^{\sqrt{m}y} + c_4 e^{-\sqrt{m}y} \quad T = c_5 e^{c\sqrt{\mu}t} + c_6 e^{-c\sqrt{\mu}t}$$

Hence the general solution is

$$u(x, y, t) = XYT = \left(c_1 e^{\sqrt{l}x} + c_2 e^{-\sqrt{l}x} \right) \left(c_3 e^{\sqrt{m}y} + c_4 e^{-\sqrt{m}y} \right) \left(c_5 e^{c\sqrt{\mu}t} + c_6 e^{-c\sqrt{\mu}t} \right)$$

(2) Solve the 1D heat equation for a long, thin metal rod of length L

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

using the boundary conditions for insulated end points, $u_x(0, t) = 0$, $u_x(L, t) = 0$, $t > 0$ and the initial condition $u(x, 0) = x$ for $0 < x < L$.

Solution Stage 1: Obtain two ODE's. Start by subbing in the u=XT derivatives

$$XT' = \alpha^2 X'' T.$$

Separate the variables

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = \mu.$$

$$\frac{X''}{X} = \mu \qquad\qquad \frac{1}{\alpha^2}\frac{T'}{T} = \mu$$

$$X'' - \mu X = 0 \qquad\qquad T' - \alpha^2 \mu T = 0$$

Stage 2: Consider the boundary conditions on u_x to restrict X' and T'

$$u_x(0,t) = 0\text{: } X'(0)T(t) = 0 \Rightarrow X'(0) = 0$$

$$u_x(L,t) = 0\text{: } X(L)T(t) = 0 \Rightarrow X'(L) = 0$$

Stage 3: Consider the 3 cases of μ

(1) $\mu = 0$: Beginning with the equation for X

$$X'' = 0$$

Examining the roots or by inspection we realise that the solution to this equation is

$$X(x) = ax + b.$$

Differentiate X to obtain X'

$$X'(x) = a.$$

Now apply the boundary conditions for X

$$X'(0) = 0 \Rightarrow a = 0 \Rightarrow X'(x) = 0,$$

$$X'(L) = 0 \Rightarrow \text{No effect on b.} \Rightarrow X(x) = b.$$

Since we have a non-trivial solution for X, it is worth looking at T. Following the same process, applying $\mu = 0$ the ODE for T becomes

$$T' = 0$$

Which has a solution

$$T = c$$

So the solution for u is

$$u(x,t) = bc = d$$

(2) $\mu > 0$: Let's say that $\mu = r^2$ so we can be sure it is a positive constant and avoid fractional powers in our solutions. Beginning with the equation for X

$$X'' - r^2 X = 0$$

Examining the roots we realise that this equation has the solution

$$X(x) = ae^{rx} + be^{-rx}$$

Differentiating

$$X'(x) = rae^{rx} - rbe^{-rx}$$

Now apply the boundary conditions for X'

$$X'(0) = 0 \Rightarrow r(a-b) = 0 \Rightarrow a = b \text{ since } r \neq 0,$$

Using our second BC

$$X'(L) = 0 \Rightarrow ra\left(e^{rL} - e^{-rL}\right) = 0 \Rightarrow a = 0 \Rightarrow X(x) = 0.$$

Therefore when $\mu > 0$, $X(x) = 0$ for any given x and hence $u(x,t) = 0$ for any given x, so this solution is of no interest.

(3) $\mu < 0$: Let's say that $\mu = -r^2$ so we can be sure it is a negative constant and avoid fractional powers in our solutions. Beginning with the equation for X

$$X'' + r^2 X = 0$$

Examining the roots we realise that this equation has the solution

$$X(x) = a\cos rx + b\sin rx$$

Differentiating

$$X'(x) = -ra\sin rx + rb\cos rx$$

Now apply the boundary conditions for X'

$$X'(0) = 0 \Rightarrow rb = 0 \Rightarrow b = 0 \Rightarrow X'(x) = -ra\sin rx,$$

$$X'(L) = 0 \Rightarrow -ra\sin rL = 0 \Rightarrow rL = n\pi.$$

So $r = n\pi/L$ for $n = 1, 2, 3...$, giving

$$X_n(x) = A_n \cos\frac{n\pi x}{L}$$

Since we have a non-trivial solution for X, it is worth looking at T. Following the same process, applying $\mu = 0$ the ODE for T becomes

$$T' + \alpha r^2 T = 0.$$

This ODE has the solution

$$T(t) = ce^{-\alpha^2 r^2 t}$$

We have already established that $r = n\pi/L$, so subbing this in the equation becomes

$$T_n(t) = c_n e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

And so, given that $u = XT$, our solution for $u(x,t)$ is

$$u(x,t) = A_n \cos\frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

where $A_n = a_n c_n$.

Stage 4: Sum the 3 cases of μ

$$u(x,t) = d + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

Apply the initial condition

$$u(x,0) = x = d + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$

Use the Fourier formulae to determine the coefficients

$$D = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\int_0^L x dx = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{\pi n x}{L}dx = \frac{2}{L}\int_0^L x\cos\frac{\pi n x}{L}dx$$

$$= \frac{2L}{\pi^2}\int_0^L \left[\frac{\cos \pi n}{n^2} - \frac{1}{n^2}\right] = \frac{2L}{\pi^2}\int_0^L \left[\cos\frac{(-1)^n - 1}{n^2}\right]$$

Thus the final solution is

$$u(x,t) = \frac{L}{2} + \frac{2L}{\pi^2}\int_0^L \left[\cos\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$