ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 8 + 9 + 10 + 11

9 Conducting and Superconducting materials

9.1 Extensive and Intensive Properties of Conductors

$$V = IR \text{ – Ohm's Law} \qquad (9\text{-}1)$$

Let's rewrite Ohm's law using the intensive variables $\underline{\boldsymbol{E}}$ (= V/L) the electric field and $\underline{\boldsymbol{J}}$ (= I/A) the current density.

$$\frac{V}{L} = \frac{I}{A}(RA/L) \qquad (9\text{-}2)$$

Which gives:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{J}}\rho_n \text{ – Definition of resistivity} \qquad (9\text{-}3)$$

where the resistivity, ρ_n, is given by $\rho_n = \frac{RA}{L}$.

9.1.1 Current Densities and Charge Densities in Conductors

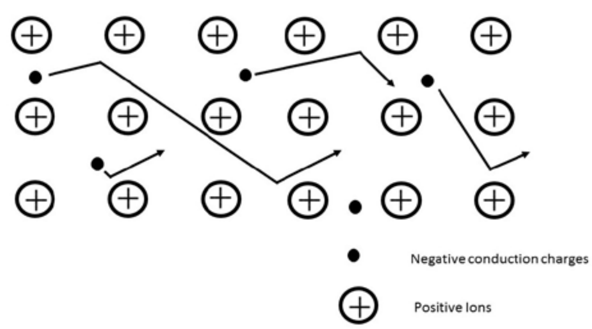
$$\underline{\boldsymbol{J}} = \sigma_n\underline{\boldsymbol{E}} \text{ – Definition of conductivity} \qquad (9\text{-}4)$$

where the conductivity, σ_n, is given by $\sigma_n = 1/\rho_n$.

Using Ohm's law ($\underline{\boldsymbol{E}} = \underline{\boldsymbol{J}}\rho_n$), Maxwell I ($\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0}$) and the continuity equation ($\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} = -\frac{\partial\rho}{\partial t}$), we can use a dimensionality argument to find that the characteristic life-time, τ , is:

$$\tau = \rho_n\varepsilon_0 \qquad (9\text{-}5)$$

9.2 Drude Model



The Drude model - The electric field accelerates the charges which then collide with the scattering sites.

In the Drude model, the charge-carriers accelerate for an average time τ - they then scatter and instanteously stop. The fraction of charge-carriers that scatter in a time ∂t is $\frac{\partial t}{\tau}$ and the remaining fraction that don't scatter is $(1-\frac{\partial t}{\tau})$. Hence the momentum of the charge carriers at time $t+\partial t$ is given by:

$$\mathrm{p}(\mathrm{t}+\partial \mathrm{t})=\left(1-\frac{\partial \mathrm{t}}{\tau}\right)\mathrm{p}(\mathrm{t})+\left(1-\frac{\partial \mathrm{t}}{\tau}\right)\mathrm{F}(\mathrm{t})\,\partial \mathrm{t}+\left\{\mathrm{F}(\mathrm{t})\,\partial \mathrm{t}\frac{\partial \mathrm{t}}{\tau}\right\} \tag{9-6}$$

Writing $\partial p = p(t+\partial t) - p(t)$, gives:

$$\frac{\partial \mathrm{p}}{\partial \mathrm{t}}=-\frac{\mathrm{p}(\mathrm{t})}{\tau}+\mathrm{F}(\mathrm{t})+\mathrm{O}(\partial \mathrm{t}) \tag{9-7}$$

$$\underline{\boldsymbol{F}}=\mathrm{q}(\underline{\boldsymbol{E}}+\underline{\boldsymbol{v}}\times\underline{\boldsymbol{B}}) \tag{9-8}$$

$|B|=\left|\frac{E}{c}\right|$ so $\underline{\boldsymbol{v}}\times\underline{\boldsymbol{B}} \ll \underline{\boldsymbol{E}}$**.** Using F = qE and $p = mv$ gives:

$$\mathrm{m}\frac{\mathrm{dv}}{\mathrm{dt}}=\mathrm{qE}-\frac{\mathrm{mv}}{\tau}-\text{ Equation of motion for carriers} \tag{9-9}$$

where the $\frac{\mathrm{mv}}{\tau}$ term can also be interpreted as a frictional damping term.

$$\underline{\boldsymbol{J}}=\mathrm{Nq}\underline{\boldsymbol{v}} \tag{9-10}$$

where N : number of charge carriers per volume, q : charge on each carrier and $\underline{\boldsymbol{v}}$: velocity of the carriers.

$$\mathrm{m}\frac{\mathrm{d}\underline{\boldsymbol{J}}}{\mathrm{dt}}=\mathrm{N}q^2\underline{\boldsymbol{E}}-\frac{\mathrm{m}}{\tau}\underline{\boldsymbol{J}} \tag{9-11}$$

We now assume that we can rewrite the equation of motion in terms of complex variables where

$$\mathrm{m}\frac{\mathrm{d}\underline{\tilde{\boldsymbol{J}}}}{\mathrm{dt}}=\mathrm{N}q^2\underline{\tilde{\boldsymbol{E}}}-\frac{\mathrm{m}}{\tau}\underline{\tilde{\boldsymbol{J}}} \tag{9-12}$$

The trial solutions are both the form:

$$\underline{\tilde{\boldsymbol{J}}}\text{ and }\underline{\tilde{\boldsymbol{E}}}\ \propto\ \exp-\mathrm{i}\omega\mathrm{t} \tag{9-13}$$

$$\underline{\tilde{\boldsymbol{J}}}=\frac{\mathrm{N}q^2}{\mathrm{m}(\tau^{-1}-\mathrm{i}\omega)}\cdot\underline{\tilde{\boldsymbol{E}}} \tag{9-14}$$

Comparing this expression to the definition for conductivity : $\underline{\boldsymbol{J}}=\sigma_n\underline{\boldsymbol{E}}$ **,** gives the complex conductivity as:

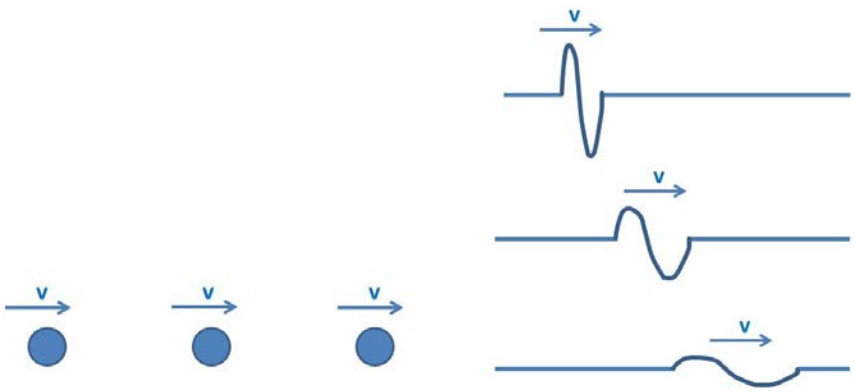
$$\tilde{\sigma}_{\mathrm{n}}=\frac{\mathrm{Nq}^2}{\mathrm{m}(\tau^{-1}-\mathrm{i}\omega)} \tag{9-15}$$

$$\underline{\boldsymbol{E}}=\mathrm{real}\big(\underline{\tilde{\boldsymbol{E}}}\big)=\mathrm{real}\big(\underline{\boldsymbol{E}}_0\exp-\mathrm{i}\omega\mathrm{t}\big)=\underline{\boldsymbol{E}}_0\cos(\omega t)\ . \tag{9-16}$$

$$\underline{\boldsymbol{J}} = \text{real}\left(\underline{\tilde{\boldsymbol{J}}}\right) = \text{real}\left(\frac{\text{N}q^2}{\text{m}(\tau^{-1} - \text{i}\omega)}\underline{\tilde{\boldsymbol{E}}}\right) = \text{real}\left(\frac{\text{N}q^2\tau(1 + \text{i}\omega\tau)}{\text{m}(1 + (\omega\tau)^2)}\underline{\tilde{\boldsymbol{E}}}\right) \qquad \text{9-17}$$

$$= \underline{\boldsymbol{E}}_0 \frac{\text{N}q^2\tau}{\text{m}(1+(\omega\tau)^2)}\{\cos(\omega t) + \omega\tau\sin(\omega t)\}\,.$$

9.3 Dispersive and ballistic motion of waves



LHS: A ball moving ballistically – its shape does not change as it moves. RHS: A wave moving dispersively along a wire – it spreads out as it moves.

Consider a single complex infinite plane wave with a single wave-vector and frequency propagating in the x-direction. The wave-vector k points in the x-direction and we have

$$\phi(x,t) = \phi_o \exp \text{i}(k \cdot x - \omega t) \qquad \text{9-18}$$

The full disturbance of a complex wavepacket (or profile) is

$$\Phi(x,t) = \int_{-\infty}^{\infty} \phi(k)\exp\text{i}(k \cdot x - \omega(k)t)\,\text{d}k\ . \qquad \text{9-19}$$

$$\phi(k) = \text{A}\exp[-\sigma(k - \text{k}_o)^2] \qquad \text{9-20}$$

$$\omega = \text{f}(k) - \text{ Definition of a Dispersion Relation} \qquad \text{9-21}$$

Using Taylor's Theorem

$$\omega = \omega_o + \alpha(k - \text{k}_o) + \frac{\beta}{2}(k - \text{k}_o)^2 + \ldots\ , \qquad \text{9-22}$$

where $\omega_o = \omega(\text{k}_o)$ is the angular frequency of the most important component wave and

$$\alpha = \left(\frac{\partial\omega}{\partial k}\right)_{k=\text{k}_o} \text{ and } \beta = \left(\frac{\partial^2\omega}{\partial k^2}\right)_{k=\text{k}_o} \qquad \text{9-23}$$

Substituting into Equation 9-19, the general solution for the wave is

$$\Phi(x,t) = \int_{-\infty}^{\infty} \mathrm{A}\exp[-\sigma(k-\mathrm{k_o})^2]\exp\mathrm{i}(k\cdot x - [\omega_o + \alpha(k-\mathrm{k_o}) + \frac{\beta}{2}(k - \mathrm{k_o})^2 + \ldots]t)\,\mathrm{d}k \quad 9\text{-}24$$

The second order solution to Equation 9-24 is

$$\Phi(x,t) = \mathrm{A}\left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}}.\exp\mathrm{i}(\mathrm{k_o}\cdot x - \omega_o t).\exp[-\frac{\sigma\pi^2(x-\alpha t)^2}{\sigma^2+(\pi\beta t)^2}]\,. \quad 9\text{-}25$$

$$\exp\left[-\frac{\sigma\pi^2(\lambda_{\mathrm{FWHM}}/2)^2}{\sigma^2+(\pi\beta t)^2}\right] = \frac{1}{2}, \quad 9\text{-}26$$

which can be rearranged to give

$$\lambda_{\mathrm{FWHM}} = 2\left\{\frac{ln2(\sigma^2+(\pi\beta t)^2)}{\sigma\pi^2}\right\}^{\frac{1}{2}}. \quad 9\text{-}27$$

<u>Important general results that you will need to remember are:</u>

1:

$$\mathrm{v_{phase}} = f\lambda = \frac{\omega}{k_{real}} \quad 9\text{-}28$$

2:

$$\mathrm{v_{group}} = \alpha = \left.\frac{\partial\omega}{\partial k_{real}}\right|_{k=\mathrm{k_o}} \quad 9\text{-}29$$

3:

$$\frac{\partial^2\omega}{\partial k_{real}^2} = \beta = 0 \quad 9\text{-}30$$

$$\frac{\partial^2\omega}{\partial k_{real}^2} = \beta \neq 0 \Rightarrow \text{dispersive} \quad 9\text{-}31$$

9.4 Electromagnetic waves propagating through metals (short scattering time)

$$\sigma_{\mathrm{n}} = \frac{\mathrm{Nq^2\tau}}{\mathrm{m}} \quad 9\text{-}32$$

Taking the curl of Maxwell III gives:

$$\underline{\boldsymbol{\nabla}}\times\underline{\boldsymbol{\nabla}}\times\underline{\boldsymbol{E}} = -\frac{\partial}{\partial \mathrm{t}}\underline{\boldsymbol{\nabla}}\times\underline{\boldsymbol{B}} \quad 9\text{-}33$$

Substituting Ohm's law ($\underline{\boldsymbol{J}} = \sigma_n \underline{\boldsymbol{E}}$) into Maxwell IV gives:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\underline{\boldsymbol{E}}}}{\partial \mathrm{t}} = \mu_0 \sigma_N \underline{\boldsymbol{E}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\underline{\boldsymbol{E}}}}{\partial \mathrm{t}} \tag{9-34}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} == -\mu_0 \sigma_N \frac{\partial \underline{\underline{\boldsymbol{E}}}}{\partial \mathrm{t}} - \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\underline{\boldsymbol{E}}}}{\partial t^2} \tag{9-35}$$

Using the vector identity

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) = \underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}) - \nabla^2 \underline{\boldsymbol{E}} \tag{9-36}$$

And Maxwell I for a conducting medium, $\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = 0$, gives

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \sigma_N \frac{\partial \underline{\underline{\boldsymbol{E}}}}{\partial \mathrm{t}} + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\underline{\boldsymbol{E}}}}{\partial t^2} \tag{9-37}$$

This equation is of the form of a travelling wave travelling along the x-direction. We assume a trial solution is of the :

$$\underline{\tilde{\boldsymbol{E}}}(\mathrm{r,t}) = \underline{\boldsymbol{E}_o} \exp \mathrm{i}(kx - \omega \mathrm{t}) \tag{9-38}$$

$$\mathrm{k}^2 = \mu_0 \varepsilon_0 \omega^2 + \mathrm{i}\omega \mu_0 \sigma_N \tag{9-39}$$

i) For a highly insulating material, $\sigma_n \to 0$:

$$\mu_0 \varepsilon_0 \omega^2 \gg \mu_0 \sigma_N \omega \tag{9-40}$$

$$\mathrm{k} = \sqrt{\mu_0 \varepsilon_0}\,\omega \quad - \text{ dispersion relation for an insulator} \tag{9-41}$$

The phase velocity is $v_{phase} = f\lambda = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$

The propagation is ballistic ($\frac{\partial^2 \omega}{\partial k^2}$=0)

ii)

$$\mathrm{k}^2 = \mathrm{i}\omega \mu_0 \sigma_n \tag{9-42}$$

Given:

$$\sqrt{\mathrm{i}} = \frac{1 + \mathrm{i}}{\sqrt{2}} \tag{9-43}$$

$$\mathrm{k} = (1 + \mathrm{i})\sqrt{\frac{\omega \mu_0 \sigma_n}{2}} = \mathrm{k}_{real} + i\mathrm{k}_{imaginary} \tag{9-44}$$

where $k_{real} = \sqrt{\frac{\omega \mu_0 \sigma_n}{2}}$ and $k_{imaginary} = \sqrt{\frac{\omega \mu_0 \sigma_n}{2}}$.

The solution for the E-field inside the good conductor is found by simply substituting k back into the plane wave equation so that for a good conductor:

$$\underline{\widetilde{\boldsymbol{E}}} = \underline{\boldsymbol{E}}_{\mathrm{o}} \exp\left[\mathrm{i}\left(\left(\mathrm{k}_{\mathrm{real}} + i\mathrm{k}_{\mathrm{imaginary}}\right)x - \omega t\right)\right]$$

$$= \underline{\boldsymbol{E}}_{\mathrm{o}} \exp\mathrm{i}(\mathrm{k}_{\mathrm{real}}\mathrm{x} - \omega \mathrm{t})\exp(-\mathrm{k}_{\mathrm{imaginary}}x) \quad \text{9-45}$$

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_{\mathrm{o}} \cos\left(\frac{x}{\delta} - \omega \mathrm{t}\right)\exp\left(-\frac{x}{\delta}\right) \quad \text{9-46}$$

$$\delta = \sqrt{2/\omega\mu_0\sigma_{\mathrm{n}}} \approx 4\ \mathrm{nm} \quad \text{9-47}$$

9.5 Electromagnetic waves propagating through low density plasmas (long scattering time)

$$\sigma_{\mathrm{n}} = \lim_{\omega \gg \tau^{-1}} \frac{\mathrm{Nq}^2}{\mathrm{m}(\tau^{-1} - \mathrm{i}\omega)} = \lim_{\omega \gg \tau^{-1}} \frac{-\mathrm{Nq}^2}{\mathrm{m_e i}\omega} = +\frac{\mathrm{iNq}^2}{\mathrm{m_e}\omega} \quad \text{9-48}$$

$$\mathrm{k}^2 = \mu\varepsilon\omega^2 + \mathrm{i}\omega\mu\sigma_{\mathrm{N}} \quad \text{9-49}$$

Since the plasma is not contained in a magnetic or dielectric media (i.e. $\varepsilon_{\mathrm{r}} = 1$ and $\mu_{\mathrm{r}} = 1$), the dispersion relation can be written,

$$\mathrm{k}^2 = \mu_0\varepsilon_0\omega^2 + \mathrm{i}\omega\mu_0\frac{\mathrm{iNe}^2}{\mathrm{m_e}\omega} = \mu_0\varepsilon_0\omega^2 - \frac{\mu_0\mathrm{Ne}^2}{\mathrm{m_e}} \quad \text{9-50}$$

$$\mathrm{k}^2 = \frac{\omega^2}{\mathrm{c}^2}\left\{1 - \left(\frac{\omega_p}{\omega}\right)^2\right\} \quad \text{9-51}$$

where the (angular) plasma frequency is given by,

$$\omega_{\mathrm{p}} = \left(\frac{\mathrm{Ne}^2}{\mathrm{m_e}\varepsilon_{\mathrm{o}}}\right)^{\frac{1}{2}} \quad \text{9-52}$$

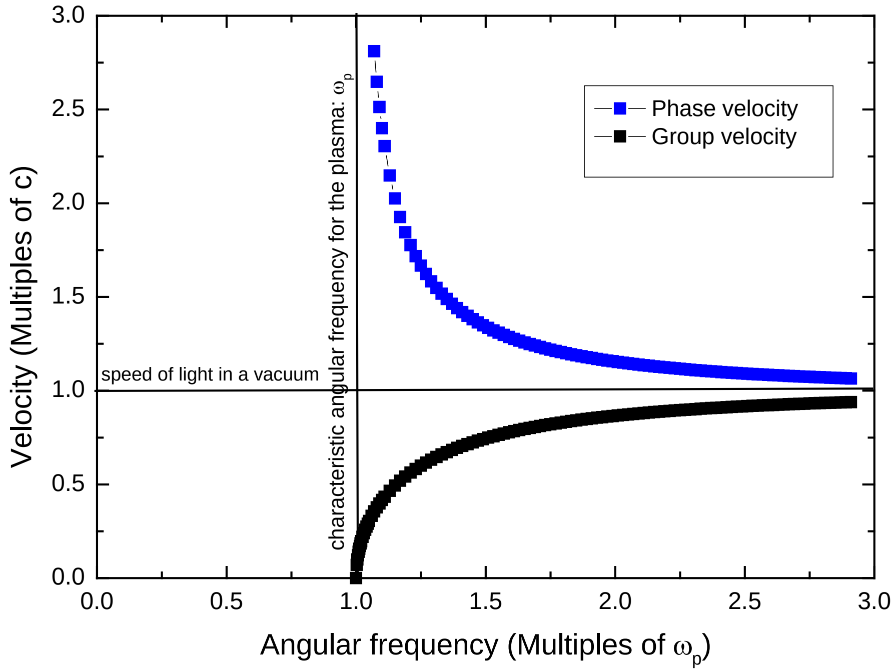
The dispersion relation has two distinct regimes:

i) $\omega > \omega_{\mathrm{p}}$; k is real and the electromagnetic waves propagate without attenuation through the plasma.

ii) $\omega < \omega_{\mathrm{p}}$; k is imaginary and there is an attenuated propagation of the electromagnetic waves.

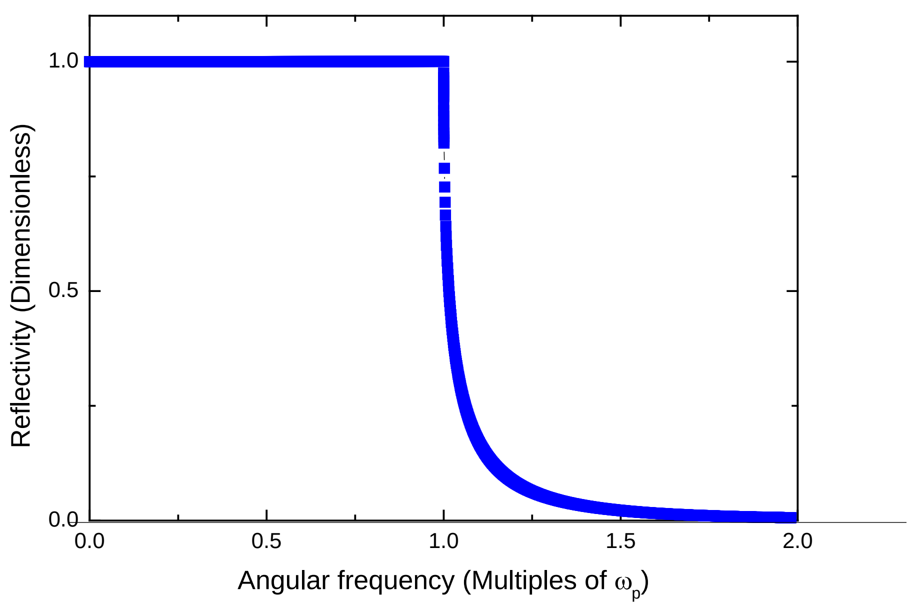
$$\mathrm{v_{phase}} = \frac{\mathrm{c}}{\sqrt{1 - \left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2}} \quad (> c \ \text{ for } \ \omega > \omega_{\mathrm{p}}) \quad \text{9-53}$$

$$\mathrm{v_{group}} = \mathrm{c}\sqrt{1 - \left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2} \quad (< c \ \text{ for } \ \omega > \omega_{\mathrm{p}}) \quad \text{9-54}$$



The velocity of an electromagnetic wave in a plasma as a function of angular frequency. Electromagnetic waves propagating through the plasma are attenuated at angular frequencies below the angular plasma frequency (ω_p).

9.5.1 Reflectance of metals



The typical reflectance versus angular frequency of an electromagnetic wave for a metal.

Elemental metal:	Li	Na	K	Rb
λ (Calculated from $\lambda = 2\pi c/\omega_p$) :	1550	2090	2870	3220
λ (Experimental data) :	1550	2100	3150	3400

9.6 Superconducting Materials

a) Zero resistivity

$$\mathrm{m}\underline{\boldsymbol{a}} = \mathrm{m}\frac{\partial \underline{\boldsymbol{v}}}{\partial \mathrm{t}} \ . \qquad \text{9-55}$$

Using $\underline{\boldsymbol{F}} = \mathrm{q}\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{J}} = \mathrm{Nq}\underline{\boldsymbol{v}}$, we find the first London equation,

$$\underline{\boldsymbol{E}} = \mu_0\lambda_L^2\frac{\partial \underline{\boldsymbol{J}}}{\partial \mathrm{t}} \quad (1^{\mathrm{st}}\text{ London Equation}), \qquad \text{9-56}$$

where the London penetration depth , λ_L, is given by $\lambda_L = \left(\frac{\mathrm{m_e}}{\mu_0 \mathrm{N_s} \mathrm{e}^2}\right)^{\frac{1}{2}}$ and $\mathrm{N_s}$ is the density of superelectrons.

b) The Meissner state – exclusion of magnetic flux

The 2nd London equation is given by,

$$\underline{\boldsymbol{B}} = -\mu_0\lambda_L^2\underline{\nabla}\times\underline{\boldsymbol{J}} \ . \ (2^{\mathrm{nd}}\text{ London Equation}). \qquad \text{9-57}$$

Substituting Maxwell's 4th equation for $\underline{\boldsymbol{J}}$, where $(\partial\underline{\boldsymbol{E}}/\partial \mathrm{t} = 0)$, into the second London equation gives

$$\underline{\boldsymbol{B}} = -\lambda_L^2\underline{\nabla}\times\underline{\nabla}\times\underline{\boldsymbol{B}} \qquad \text{9-58}$$

Using the vector relation:

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{\boldsymbol{B}}) = \underline{\nabla}\left(\underline{\nabla}\cdot\underline{\boldsymbol{B}}\right) - \nabla^2\underline{\boldsymbol{B}} \qquad \text{9-59}$$

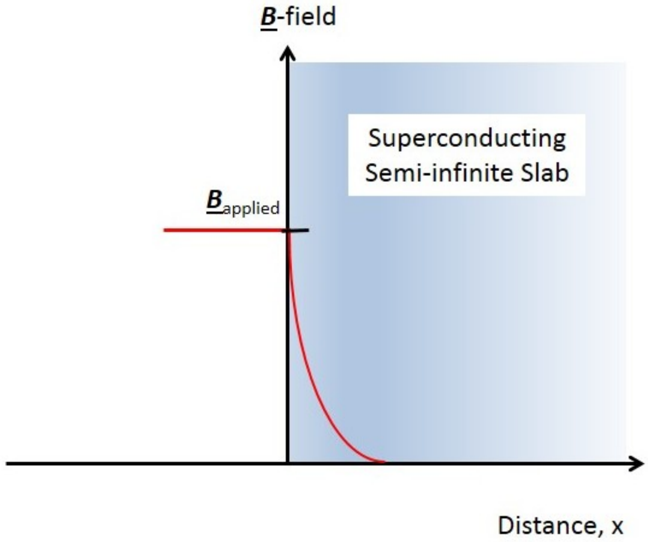
gives

$$\nabla^2\underline{\boldsymbol{B}} = \frac{1}{\lambda_L^2}\underline{\boldsymbol{B}} \qquad \text{9-60}$$

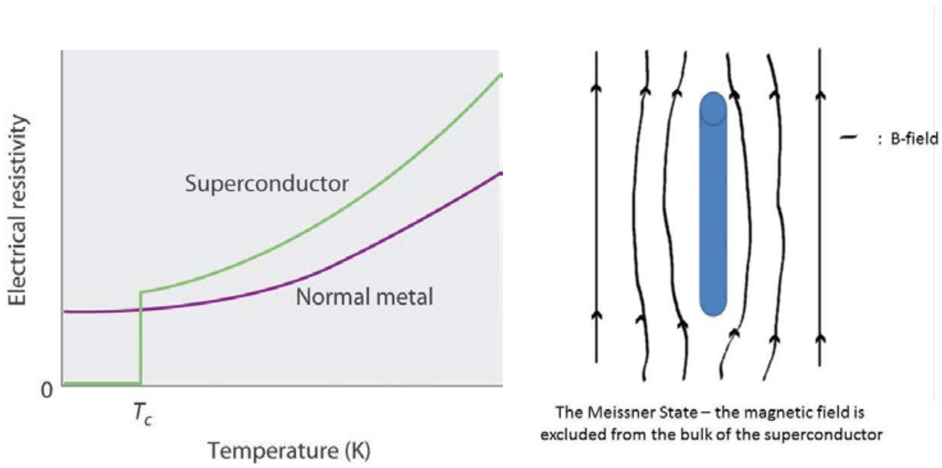
For a semi-infinite slab, the equation has solutions of the form:

$$\underline{\boldsymbol{B}}(x) = \underline{\mathbf{B}}_0\exp(-x/\lambda_L) \quad \text{for} \quad x > 0 \ , \qquad \text{9-61}$$

which describes the Meissner effect.



The magnetic field profile for a semi-infinite slab of superconductor in an applied field ($\underline{B}_{\text{applied}}$). The field decays exponentially at the surface to zero over a characteristic distance λ_L .



The two fundamental properties of superconductors. (LHS) At the critical temperature, T_C, the material becomes superconducting and the resistivity drops to zero. (RHS) The Meissner state in low fields. The magnetic field is excluded from the bulk of all metallic superconductors in low applied fields.