Quantum Theory - Worksheet 8

Problem 1

(a) Consider two operators, $\hat{A}$ and $\hat{B}$. Explain why it is generally *not* the case that $\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A}+\hat{B})$ if $\hat{A}$ and $\hat{B}$ do *not* commute. (Note: $\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A}+\hat{B})$ if $\hat{A}$ commutes with $\hat{B}$; however, you are not asked to prove this.)

(b) Recall that the evolution operator $\hat{U}(t,t_0)$ for a time-*in*dependent Hamiltonian $\hat{H}$ is given by the equation

$$\hat{U}(t,t_0) = \exp[-i\hat{H}(t-t_0)/\hbar]. \tag{1}$$

This equation can also be written as

$$\hat{U}(t,t_0) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}\,\mathrm{d}t\right). \tag{2}$$

One might think that Eq. (2) generalizes to

$$\hat{U}(t,t_0) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t)\,\mathrm{d}t\right) \tag{3}$$

in the case of a time-dependent Hamiltonian. Show, to the contrary, that Eq. (3) does not give the evolution operator for $\hat{H}(t)$ if there is a time t_1 for which the operators

$$\int_{t_0}^{t_1}\hat{H}(t)\,\mathrm{d}t \quad \text{and} \quad \int_{t_1}^{t}\hat{H}(t)\,\mathrm{d}t$$

do not commute. [Hints: (1) Remember that $\hat{U}(t,t_0) = \hat{U}(t,t_1)\hat{U}(t_1,t_0)$. (2) Don't be confused by the notation: In Eq. (1), $\hat{H}(t-t_0)$ means $\hat{H}$ multiplied by $(t-t_0)$, while in Eq. (3) the (t) indicates that the Hamiltonian is a function of t.]

Problem 2

Consider a 1D system in a state $|\psi\rangle$ represented by a wave function $\psi(x)$ in the position representation and by a wave function $\phi(p)$ in the momentum representation. From general results,

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)\exp(-ipx/\hbar)\,\mathrm{d}x,$$
$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p)\exp(ipx/\hbar)\,\mathrm{d}p.$$

(a) The position and momentum operators $\hat{Q}$ and $\hat{P}$ acting on ket vectors are represented by different operators in these two representations, i.e., by the operators x_p and p_p in the position representation and by the operators x_m and p_m in the momentum representation. Thus x_p is the operator which, when acting on a wave function $\psi(x)$, multiplies it by x, and p_m is the operator which, when acting on a wave function $\phi(p)$, multiplies it by p: $x_\mathrm{p}\psi(x) = x\psi(x)$ and $p_\mathrm{m}\phi(p) = p\phi(p)$. Likewise, the ket vector $\hat{P}|\psi\rangle$ is represented by the function $p_\mathrm{p}\psi(x)$ in the position representation and by the function $p_\mathrm{m}\phi(p)$ in the momentum representation. Now, note that

$$\begin{aligned}
p_\mathrm{p}\psi(x) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}[p_\mathrm{m}\phi(p)]\exp(ipx/\hbar)\,\mathrm{d}p \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}p\,\phi(p)\exp(ipx/\hbar)\,\mathrm{d}p \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p)\,\frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}\exp(ipx/\hbar)\,\mathrm{d}p \\
&= \frac{1}{\sqrt{2\pi\hbar}}\frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}\phi(p)\exp(ipx/\hbar)\,\mathrm{d}p \\
&= -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x).
\end{aligned}$$

Hence, the momentum operator in the position representation, p_p, is $-i\hbar\,\mathrm{d}/\mathrm{d}x$. Show that the position operator in the momentum representation, x_m, is $i\hbar\,\mathrm{d}/\mathrm{d}p$.

(b) Show that

$$\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)\right]\exp(-ipx/\hbar)\,\mathrm{d}x = -\frac{1}{\hbar^2}\,p^2\phi(p).$$

[Hint: Integrate by parts, assuming that $\psi(x)$ and its derivatives go to zero for $x\to\pm\infty$.]

Hence, show that if

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x), \tag{4}$$

then

$$\frac{p^2}{2m}\phi(p) + \int_{-\infty}^{\infty}\tilde{V}(p-p')\,\phi(p')\,\mathrm{d}p' = E\phi(p)$$

with

$$\tilde{V}(p-p') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}V(x)\,\exp[-i(p-p')x/\hbar]\,\mathrm{d}x.$$

This result shows that a Schrödinger equation such as Eq. (4), which is a differential equation, becomes an integral equation in the momentum representation (i.e., an equation relating the dependent variable, $\phi(p)$ here, to an integral involving that function).

Problem 3

Let $\hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger$, where $\hat{U}$ is a unitary operator.

(a) Show that the operator $\hat{A}'$ is Hermitian if the operator $\hat{A}$ is Hermitian.

(b) Show that the operators $\hat{A}'$ and $\hat{A}$ have the same eigenvalues.