Mathematical Methods in Physics

Workshop 6

6.1

Calculate the Fourier transform of

$$f(t) = e^{-\alpha t^2},$$

where α is a positive constant. In order to solve this integral use the technique of 'completing the square', i.e. use the fact that $a^2 + 2ab = (a+b)^2 - b^2$.

[*Hint: Use the result* $\int_{-\infty}^{\infty} e^{-\beta x^2} dx = \sqrt{\pi/\beta}$]

6.2

Calculate the integral

$$\int_{-4}^{4} \left(x f(x^4) \delta(x^2 - 2) + g(2x) \delta(\sin(x)) \right) dx.$$

6.3

Compute the Laplace transform of the function $f(t) = \sqrt{t}$ explicitly.

[*Hint: Perform changes of variables - for instance start by setting* $t = u^2$ *- and use the result* $\int_{0}^{\infty} x^2 e^{-x^2} dx = \sqrt{\pi}/4$]

6.4

The Laguerre polynomials $L_n(at)$ can be written in terms of the following generating function

$$L_n(at) = \frac{e^{at}}{n!} \frac{d^n}{dt^n} \left(t^n \, e^{-at} \right).$$

They appear, for instance, in quantum mechanics in the radial solution of the Schrödinger equation for the Hydrogen atom. Use the properties of the Laplace transforms (the exponential multiplication and the formula for derivatives - see Lecture Summary), together with the result $\mathcal{L}[t^n](s) = n!/s^{n+1}$, for $s > 0$, to show that their Laplace transform is

$$\mathcal{L}[L_n(at)](s) = \frac{(s-a)^n}{s^{n+1}}, \quad s > 0.$$

6.5

Evaluate the function

$$f(t) = \int_0^\infty \frac{\sin tx}{x} dx$$

using Laplace transforms.

(i) First consider the case $t > 0$. Take the Laplace transform of the function $f(t)$ and change the order of integration, which in this case is justified, i.e. integrate first with respect to t and then with respect to x. remember that the sin-function can be written as the sum of two exponentials. In order to obtain $f(t)$ given $\mathcal{L}[f(t)](s)$, notice that $\mathcal{L}[t^n](s) = n!/s^{n+1}$, for $s > 0$.

(ii) Evaluate $f(t)$ when $t = 0$ and when $t < 0$ bearing in mind that the sine function is odd.