# Lecture 9: The Role of Pressure in the Expanding Universe

## 9.1 The Fluid and Acceleration Equations

[Liddle sec:3.4 and 3.5]

Consider an expanding fluid

$$dE + PdV = TdS = dQ \tag{9.1}$$

In the case of the expanding universe $dQ = 0$ (heat cannot enter or leave the universe) and

$$E = \frac{4\pi r^3}{3}\rho c^2 = \rho c^2 V$$

where $\rho c^2$ is the total energy density in the universe. Hence

$$\frac{dE}{dt} = \rho c^2 \frac{dV}{dt} + V\frac{d\rho}{dt}c^2.$$

Substituting into (9.1)

$$\dot{\rho}V + \left(\rho + \frac{P}{c^2}\right)\frac{dV}{dt} = 0$$

or

$$\boxed{\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0.}$$

(9.2 : **The Fluid Equation**)

Another important equation known as the **acceleration equation** can then be derived by combining this with the Friedmann equation. (It is not an independent equation.) Differentiating the Friedmann equation we get

$$\frac{a^2 2\dot{a}\ddot{a} - \dot{a}^2 2a\dot{a}}{a^4} = \frac{8\pi G\dot{\rho}}{3} + \frac{2kc^2\dot{a}}{a^3}$$

if we substitute for $\dot{\rho}$ using the Fluid Equation and factor out $2\dot{a}/a$ we get

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{P}{c^2}\right) + \frac{kc^2}{a^2}$$

now adding to this the Friedmann equation we get

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)}$$

(9.3 : **The Acceleration Equation**)

The equation of state therefore effects the expansion of the universe. Note that in this analysis the pressure does not create a force that contributes to the expansion of the universe as there are no pressure gradients, but does alter the energy density of the universe through the $PdV$ work that is done.

Also note $\ddot{r}$ is always negative and the universe decelerates provided $P > -1/3\ \rho c^2$.

## 9.2 The Cosmological Constant

[Liddle sec:7]

What if the vacuum has a positive energy density $\epsilon_{\text{vac}} = \rho_{\text{vac}} c^2$? Note that in modern quantum field theory we do not view the vacuum to be simply nothing, but rather a place where virtual particle/anti-particle pairs are constantly being created and destroyed. In this picture it is not unreasonable to suppose that the energy of the vacuum state might not be precisely zero.

Using $dE + PdV = 0$ in this case we have

$$\rho_{\text{vac}} c^2 dV + P_{\text{vac}} dV = 0$$

and so

$$P_{\text{vac}} = -\rho_{\text{vac}} c^2.$$

Splitting the RHS of the acceleration equation into parts for the vacuum and for all other material we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) - \frac{4\pi G}{3}\left(\rho_{\text{vac}} - 3\rho_{\text{vac}}\right)$$

and hence

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3} \tag{9.4}$$

where $\Lambda \equiv 8\pi G \rho_{\text{vac}}$ is a constant.

$\Lambda$ is called the **Cosmological Constant**. In the same way we can split the density in the RHS of the Friedmann equation into parts for the vacuum and for all other material and find that $\Lambda$ enters the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho(t)}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3}. \tag{9.5}$$

The $\Lambda$-term dominates as $a$ gets very large and results in $a \propto \exp(\mathrm{H}t)$ and $\mathrm{H} = \sqrt{\Lambda/3}$. Such exponential growth is called **inflation**.

Observationally

$$\Omega_{\Lambda,0} \equiv \frac{\Lambda}{3\mathrm{H}_0^2} \equiv \frac{\rho_{\text{vac}}}{3\mathrm{H}_0^2/8\pi G} \approx 0.7,$$

i.e it appears to contribute significantly to current energy density of the universe and is of the same order of magnitude as the energy density in mass ($\Omega_M \simeq 0.3$).

Note, however, that the natural magnitude for any vacuum energy density is $M_{\text{planck}}/l^3_{\text{planck}}$, i.e.

$$\Omega_{\Lambda,0} = \frac{8\pi G}{3H_0^2}\left(\frac{\hbar c}{G}\right)^{1/2}\left(\frac{c^3}{G\hbar}\right)^{3/2} = 10^{122} \text{ !!}$$

The presence of non-zero $\Lambda$ clearly alters the expansion history of the Universe and results in an older Universe with greater look-back times, as shown in Fig.1.

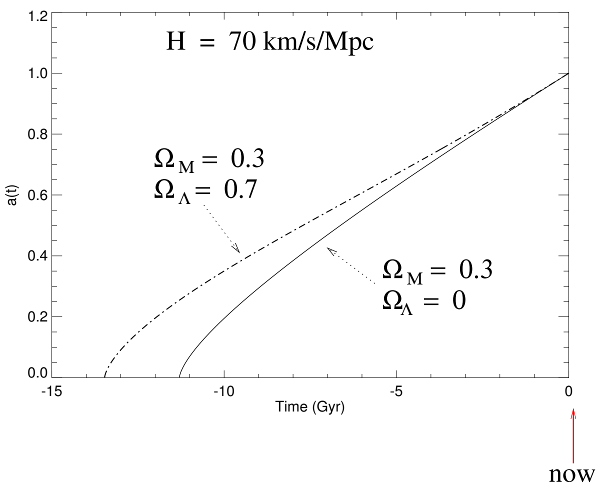


Figure 1: The expansion history of the Universe with and without $\Lambda$ (cf. Fig 4 of Lecture 3).

## 9.3 Quintessence

Recently physicists have speculated that even more exotic forms of **dark energy**, called quintessence, might exist. In these models the energy density of the vacuum is again non-zero, but now varies as the universe expands. For many of these models this can be described as an equation of state for the dark energy in which $w = P_{\rm DE}/(\rho_{\rm DE}\, c^2) < 0$

In such cases the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2} + \frac{8\pi G\rho_{\rm DE}(t)}{3}.$$

The dependence of $\rho_{\rm DE}(t)$ on time or expansion factor can be found by solving the fluid equation. This gives

$$\rho_{\rm DE}(t) = \rho_{\rm DE,0}\left(\frac{a}{a_0}\right)^{-3(1+w)}.$$

Thus we can rewrite the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2} + \frac{\Lambda_{\rm DE,0}}{3}\left(\frac{a}{a_0}\right)^{-3(1+w)},$$

where $\Lambda_{\rm DE,0} \equiv 8\pi G\rho_{\rm DE,0}$ is a constant. We see that the cosmological constant is just a special case of dark energy for which $w = -1$. As $w < 0$ the dark energy term will always come to dominate when $a$ gets sufficiently large.

[Note that this is very speculative! WMAP+ $\Rightarrow$ $w < -0.79$]

## Examples

9.1 Use the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}.$$

to find an expression for the Hubble parameter at late times in a universe dominated by vacuum energy density with $\rho_{\rm vac} = \Lambda/8\pi G$.

9.2 Use the Friedmann equation

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3},$$

where $\rho$ denotes non-relativistic matter, to derive an expression for $\Omega_\Lambda$ in terms of $\Lambda$ and $H$ at arbitrary time.

If, at the current epoch, $\Omega_{M,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$, evaluate the redshift at which $\Omega_M = \Omega_\Lambda$.

At what redshift did the cosmic expansion begin to accelerate?