Foundations 3A - QM
Worksheet 5

Problem 1

Solution

The TD Hamiltonian term is: $(q = -e)$

$$H'(\mathbf{r},t) = -q\,\mathbf{r}\cdot\boldsymbol{\mathcal{E}}(t) = e\,\frac{\mathcal{E}_0}{2}\,[\mathbf{r}\cdot\hat{\boldsymbol{\epsilon}}\exp(-i\omega t) + \mathbf{r}\cdot\hat{\boldsymbol{\epsilon}}^*\exp(i\omega t)]$$

$$H'(\mathbf{r},t) = e\,\frac{\mathcal{E}_0}{2\sqrt{2}}\,[(x-iy)\exp(-i\omega t) + (x+iy)\exp(i\omega t)]$$

For the transition to be nonzero, the following matrix element must be nonzero:

$$\langle n_a l_a m_a|H'|n_b l_b m_b\rangle = e\,\frac{\mathcal{E}_0}{2\sqrt{2}}\,\underbrace{\langle n_a l_a m_a|x-iy|n_b l_b m_b\rangle}\exp(-i\omega t) + e\,\frac{\mathcal{E}_0}{2\sqrt{2}}\,\underbrace{\langle n_a l_a m_a|x+iy|n_b l_b m_b\rangle}\exp(i\omega t)$$

So, either

$$\langle n_a l_a m_a|x-iy|n_b l_b m_b\rangle \neq 0, \quad \text{or} \quad \langle n_a l_a m_a|x+iy|n_b l_b m_b\rangle \neq 0$$

(See lecture notes) We have

$$\begin{aligned}
&\langle n_a l_a m_a|[L_z, x-iy]|n_b l_b m_b\rangle = -\hbar\,\langle n_a l_a m_a|x-iy|n_b l_b m_b\rangle \\
&\Rightarrow \hbar(m_a - m_b)\langle n_a l_a m_a|x-iy|n_b l_b m_b\rangle = -\hbar\,\langle n_a l_a m_a|x-iy|n_b l_b m_b\rangle \\
&\Rightarrow \hbar(m_a - m_b + 1)\langle n_a l_a m_a|x-iy|n_b l_b m_b\rangle = 0
\end{aligned}$$

Also,

$$\begin{aligned}
&\langle n_a l_a m_a|[L_z, x+iy]|n_b l_b m_b\rangle = \hbar\,\langle n_a l_a m_a|x+iy|n_b l_b m_b\rangle \\
&\Rightarrow \hbar(m_a - m_b)\langle n_a l_a m_a|x+iy|n_b l_b m_b\rangle = \hbar\,\langle n_a l_a m_a|x+iy|n_b l_b m_b\rangle \\
&\Rightarrow \hbar(m_a - m_b - 1)\langle n_a l_a m_a|x+iy|n_b l_b m_b\rangle = 0
\end{aligned}$$

Therefore, if $\langle n_a l_a m_a|x-iy|n_b l_b m_b\rangle \neq 0$, or $\langle n_a l_a m_a|x+iy|n_b l_b m_b\rangle \neq 0$, then we must have: $m_a - m_b = 1$ or $m_a - m_b = -1$.

Problem 2 (See Griffiths Example 1.12)

Solution [a]

We can use the same equation for $c_b^{(1)}(t)$ with $\omega = 0$. We can no longer neglect one of the terms:

$$c_a^{(1)}(t) = 1 \quad c_b^{(1)}(t) = -\frac{\mathcal{V}_{ba}}{2\hbar}\left[\frac{e^{i\,\omega_0\,t} - 1}{\omega_0} + \frac{e^{i\,\omega_0\,t} - 1}{\omega_0}\right] = -\frac{i\,2\,\mathcal{V}_{ba}}{\hbar}\,e^{i\,\omega_0\,t/2}\,\frac{\sin[\omega_0\,t/2]}{\omega_0}$$

$$P_{a\to b}(t) = \frac{4\,|\mathcal{V}_{ba}|^2}{\hbar^2}\,\frac{\sin^2[\omega_0\,t/2]}{\omega_0^2}$$

In the equations with the time-independent potential, the matrix element $\mathcal{V}_{ba}$ is multiplied by 2.

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Solution [b]:

$$j_i(k') = \frac{1}{V}\,\frac{\hbar\,k'}{m}$$

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Solution [c]
$\mathcal{V}_{if}$ must be multiplied by 2 and $\rho(E_f)$ is given above:

$$R_{i\to d\Omega} = \frac{2\pi|\mathcal{V}_{if}|^2}{\hbar}\rho(E_f)$$

with

$$\mathcal{V}_{if} \to \langle\psi_i|\mathcal{V}|\psi_f\rangle = \frac{1}{V}\int d^3\mathbf{r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}}\,\mathcal{V}(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}} \Rightarrow |\mathcal{V}_{if}|^2 = \frac{1}{V^2}\left|\int d^3\mathbf{r}\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\,\mathcal{V}(\mathbf{r})\right|^2$$

and

$$\rho(E_f) = V\,\frac{\sqrt{2\,m^3\,E_f}}{8\,\pi^3\,\hbar^3}\,d\Omega$$

So, we get:

$$R_{i\to d\Omega} = \frac{2\pi}{\hbar}\,\frac{1}{V}\left|\int d^3\mathbf{r}\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\,\mathcal{V}(\mathbf{r})\right|^2\,\frac{\sqrt{2\,m^3\,E_f}}{8\,\pi^3\,\hbar^3}\,d\Omega$$

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[d] **Extra Problem: Born Approximation** *Solution*

The probability to scatter from ψ_i to ψ_f is nonzero only for $E_i = E_f$, i.e. for $k' = k$.
Then, the probability current is:

$$j_i(k') = \frac{1}{V}\,\frac{\hbar\,k'}{m} = \frac{1}{V}\sqrt{\frac{2E_f}{m}}$$

The differential scattering cross section becomes:

$$\frac{d\sigma}{d\Omega} = \frac{R_{i\to d\Omega}}{J_i\,d\Omega} = \frac{2\,\pi}{\hbar}\,\frac{1}{V}\left|\int d^3\mathbf{r}\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\,\mathcal{V}(\mathbf{r})\right|^2 \times \frac{\sqrt{2\,m^3\,E_f}}{8\,\pi^3\,\hbar^3} \times V\sqrt{\frac{m}{2E_f}} = \left|\frac{m}{2\,\pi\,\hbar^2}\int d^3\mathbf{r}\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\,\mathcal{V}(\mathbf{r})\right|^2$$