Mathematical Methods II
Workshop 5

The *associated Legendre equation* has the form

$$(1-x^2)\,\frac{d^2y}{dx^2} \;-\; 2x\,\frac{dy}{dx} \;+\; \left[l(l+1) - \frac{m^2}{1-x^2}\right]\,y \;=\; 0\,,$$

where l and m are integers such that $l \geq 0$ and $-l \leq m \leq l$.

(1) For $m = 0$ the equation is solved by the Legendre polynomials $P_l(x)$ which are determined by the Rodrigues formula

$$P_l(x) \;=\; \frac{1}{2^l l!}\,\frac{d^l}{dx^l}(x^2-1)^l\,.$$

Use it to compute $P_1(x)$, $P_2(x)$ and $P_3(x)$.
Check that each of them satisfies the appropriate Legendre equation.

(2) In the more general case, $-l \leq m \leq l$, the associated Legendre equation is solved by the *associated Legendre polynomials* $P_l^m(x)$ which are given by

$$P_l^m(x) \;=\; (1-x^2)^{m/2}\,\frac{d^m P_l}{dx^m} \qquad \text{for}\, m > 0\,,$$

and $P_l^{-m}(x)$ are found via

$$P_l^{-m}(x) \;=\; (-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m(x)\,.$$

Use this to compute all $P_l^m(x)$ for $l = 1, 2, 3$ and all integer values of m in the interval $-l \leq m \leq l$. Check that they satisfy the appropriate associated Legendre equation.

(3) The associate Legendre polynomials are orhtogonal,

$$\int_{-1}^{+1} P_l^m(x)\,P_{l'}^m(x)\,dx \;=\; 0$$

for $l \neq l'$. Check that this relation is satisfied for the ones you computed.