GA 4

(1) Consider a spherical density distribution, and denote by $M(< R)$ the mass enclosed in a sphere of radius R. The speed, $V_c(R)$, of a particle on a circular orbit with radius R in such a density distribution, is given by

$$V_c^2(R) = \frac{GM(< R)}{R}\,. \tag{1}$$

Demonstrate that a constant circular velocity, $V_c = V_{c,0}$, implies that the density distribution is $\rho(R) = V_{c,0}^2/(4\pi G R^2)$. [2 marks]

Answer: Rewriting the previous equation for constant $V_c = V_{c,0}$ yields

$$V_{c,0}^2\, R = GM(< R)\,. \tag{2}$$

Taking the derivative of both sides with respect to R gives for constant V_c

$$V_{c,0}^2 = G\frac{dM}{dR} = G\, 4\pi\rho R^2\,, \tag{3}$$

since $dM/dR = 4\pi\rho R^2$ for a spherical mass distribution. Note that this does **not** follow from taking the derivative with respect to R of $M = (4\pi/3)\rho\, R^3$, which only holds if ρ is *constant* (which is not the case here). Therefore

$$\rho(R) = \frac{V_{c,0}^2}{4\pi G R^2}\,. \tag{4}$$

[2 marks]

(2) The surface density, $\sigma(R)$, of stars in a spiral galaxy falls with distance R to the centre as

$$\sigma(R) = \sigma_d\, \exp(-R/R_d)\,.$$

Here, σ_d is the central surface density, and R_d is the scale-length. Both are constants. Demonstrate that the disc mass enclosed in a sphere of radius R is

$$M_d(< R) = 2\pi\, \sigma_d\, R_d^2\, (1 - (1 + x)\exp(-x))\,,$$

where $x \equiv R/R_d$. [2 marks]

Answer: Using cylindrical co-ordinates, the enclosed mass is

$$\begin{aligned} M_d(< R) &= \int_0^R \sigma(R) 2\pi\, R dR \\ &= 2\pi\sigma_d\, R_d^2 \int_0^x x \exp(-x)\, dx \\ &= 2\pi\, \sigma_d\, R_d^2\, (1-(1+x)\exp(-x))\,, \end{aligned}$$

using $x \equiv R/R_d$. [2 marks]

The relation between enclosed mass and circular velocity given in Eq.(1) is only approximately valid for a disc. However you may assume it does hold in what follows.

(3) Take $R_d = 3$ kpc, $M_d(R \to \infty) = 2 \times 10^{10}\ M_\odot$, and assume that the circular velocity of disc and dark halo combined is V_c=220 km s^{-1} at distance $R = 100$ kpc. Evaluate V_c at $R = 5$ kpc and $R = 10$ kpc. [3 marks]

Answer: The rotation speed due to dark halo and disc combined, is
$V_c^2(R) = V_{c,0}^2 + V_d^2$ [1 mark]
where $V_d^2 = GM_d(< R)/R$.
Note: this is most easily seen from Eq. (1): clearly we need to add the enclosed masses of all components in a mass distribution to obtain the *total* enclosed mass. In this particular case, this means adding disc and halo enclosed masses. In terms of circular speed, it means we need to add the corresponding circular speeds *in quadrature*, according to Eq. (1).
Inserting $R = 100$ kpc in the expression for M_d, and using the previous expression relating V_d and $M_d(< R)$, yields $V_d(R = 100\text{ kpc})$. From this and the value of $V_c(R = 100\text{ kpc})$ we can determine $V_{c,0}$. [1 mark]
Evaluating V_c for $R = 5$ and 10 kpc yields 237 and 234 km s^{-1}, respectively. [1 mark]

(4) This galaxy is at a distance of $d = 10$ Mpc, and its disc is tilted by 30 degrees (with 90 degrees corresponding to the case where the galaxy is face on). An observer uses a radio telescope to detect gas in the disc moving with the circular velocity using the HI 21-cm line. Sketch the detected wavelength of the line as function of angle from the centre of the galaxy. [3 marks]

Answer: The velocity v detected along the sight-line at a given radius R is $\cos(\theta)\, V_d(R)$, where $\theta = 30^\circ$ is the disc's inclination angle (so that for $\theta = 90^\circ$, $v = 0$ since then the galaxy spins in the plane of the sky. [1 mark]

The observed wavelength follows from the Doppler shift, $\Delta\lambda/\lambda = v/c$. The relation between angle ϕ from the centre of the galaxy and radius R is $\tan(\phi) \approx \phi = R/d$. [1 mark]

The measured wavelength then depends on angle as shown in Fig. 1. 1 mark for approximate sketch that show wavelengths smaller than 21 cm on one side, and larger than 21 cm on other side of galaxy centre. Note: we used the results from the previous section to computed $V_c(R)$, the circular speed as a function of R, multiplying this with $\cos(\theta)$ to convert the circular speed to the line-of-sight speed. We then used $R = \phi\, d$ to relate angle to R.

[1 pc $= 3.09 \times 10^{16}$ m, $M_\odot = 2.0 \times 10^{30}$ kg, $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}]

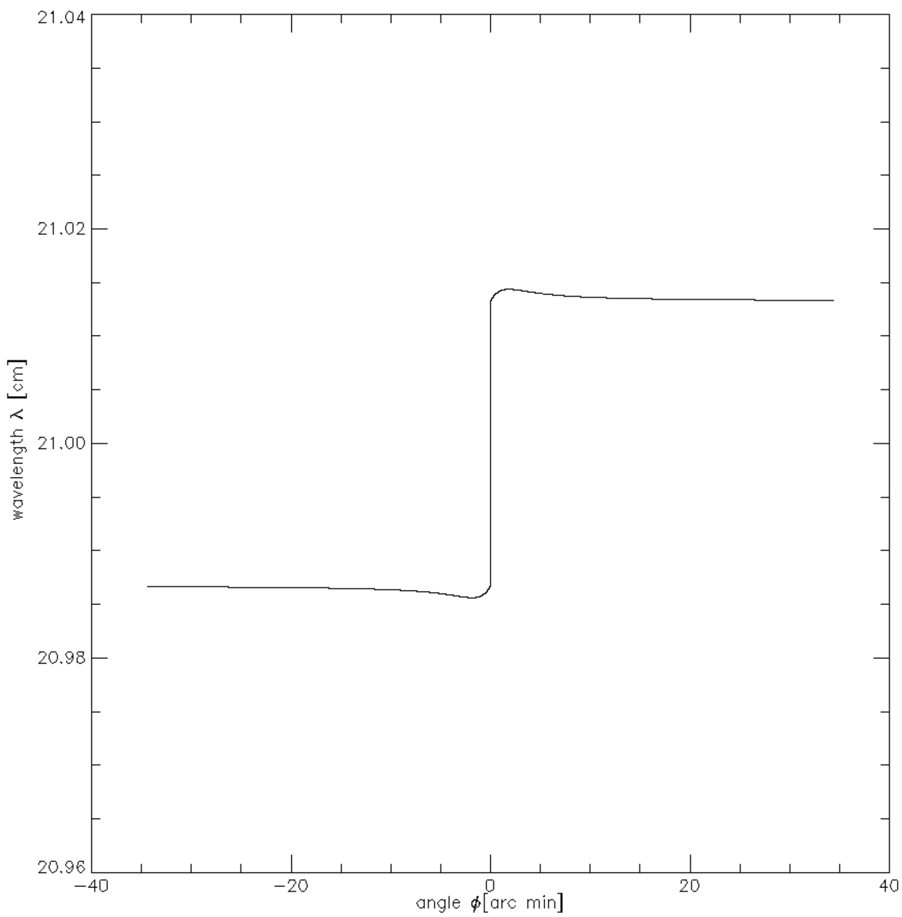

Figure 1: Wave-length of the 21-cm emission line as function of the angle from the centre of the galaxy.