Mathematical Methods II
PDF 3

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Key Points

- Finding the complementary function of a 2^{nd} order ODE.
- Finding the auxillary equation of a 2^{nd} order ODE.
- Finding the particular integral of a 2^{nd} order ODE.
- Solving 2^{nd} order ODEs using trial functions/method of undetermined coefficients.

Definitions

- **Linear $\mathbf{n^{th}}$ order ODEs**: Linear equations of n^{th} order have the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

  - **Classification by coefficients**
    * **Constant coefficients** a_0, a_n, 2, 5/3...
    * **Variable coefficients** $a_0(x)$, $a_n(y)$, x, t...
  - **Classification by homogeneous/non-homogeneous**
    * **Homogeneous** when $f(x) = 0$.
    * **Non-homogeneous** when $f(x) \neq 0$.

- **Complementary function**: If the equation is homogeneous it can be solved by finding the 'complementary function', $y_c(x)$. If $y_1, y_2, ..., y_n$ are different solutions of an n^{th} order ODE and linearly independent (the determinent of the matrix of vectors $\neq 0$) then the general solution of the ODE is given by

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

  This is called the principle of linear superposition.

- **Auxiliary equation**: Also called the characteristic equation, determining the auxiliary equation is the first step in finding the complementary function. It is found by substituting $y = Ae^{\lambda x}$ into our equation as a trial solution. It is a polynomial equation in λ of order n, and reads

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0$$

  Note that the exponential terms cancel out after the derivatives have been taken. Exponentials make useful trial solutions since their derivitives are merely multiples of the function itself. In the case of a 2^{nd} order ODE this results in a quadratic in λ, which can be solved with relative ease.

  The auxiliary equation has n roots, say $\lambda_1, \lambda_2, ..., \lambda_n$. These roots can be real or complex and may repeat. Their nature will determine the form for the solution we must consider the three main cases. The following are general solutions for 2^{nd} order ODEs where $n = 2$:

  - **All roots are real and distinct** Solutions take the form of a sum of n linearly independent solutions.

$$y_c(x) = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$$

  - **Roots are real and equal** A repeated root means we have not found n linearly independent solutions. The second solution can be found by multiplying by x.

$$y_c(x) = (c_1 + c_2x)e^{\lambda_1 x}$$

    A triple root would require a third term in x^2 and so on.

  - **Roots are complex** If the auxiliary equation has a complex root $\alpha + i\beta$ then its complex conjugate $\alpha - i\beta$ is also a root. In this case solutions can be expressed in one of the following forms

$$y_c(x) = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}(d_1\cos\beta x + d_2\sin\beta x) = Ae^{\alpha x}\sin(\beta x + \phi)$$

- **Particular integral**: If the equation is non-homogeneous ($f(x) \neq 0$) extra work must be done in order to find the 'particular integral', $y_p(x)$ - this is any function that satisfies the equation. i.e. when $y_p(x)$ is substituted into the LHS it results in the given non-zero RHS.

  There is no general method for finding $y_p(x)$. For 2^{nd} order ODEs with *constant coefficients*, if $f(x)$ contains only polynomial, exponential, sine or cosine terms, we can test a trial function of similar form that contains undetermined parameters and substitute it into

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = 0.$$

  allowing us to determine the parameters. We can then deduce $y_p(x)$. Here are some standard trial functions:

  - If $f(x) = ae^{rx}$ try

$$y_p(x) = be^{rx}.$$

– If $f(x) = a_1 \sin rx + a_2 \cos rx$ (a_1 or a_2 may be 0) then try

$$y_p(x) = b_1 \sin rx + b_2 \cos rx.$$

– If $f(x) = a_0 + a_1 x + ... + a_N x^N$ (some a_m may be 0) then try

$$y_p(x) = b_0 + b_1 x + ... + b_N x^N.$$

– If $f(x)$ is a sum or product of any of the above try a $y_p(x)$ that is a sum or product of the corresponding trial functions.

Note: this method will fail if $y_p(x)$ contains any terms already included in $y_c(x)$. If this is the case multiply the trial function by the smallest power of x possible until it is no longer found in $y_c(x)$.

- **General solution**: The general solution to a homogeneous ODE is the complementary function

$$y(x) = y_c(x).$$

The general solution to a non-homogeneous ODE is given by the complementary function combined with the particular integral

$$y(x) = y_c(x) + y_p(x).$$

Note: If $f(x) \neq 0$, we assume that $f(x) = 0$ in order to find the complementary function, *before* finding the particular integral.

Solving linear 2$^{\text{nd}}$ order ODEs with constant coefficients

- **Linear 2$^{\text{nd}}$ order ODEs**: Linear 2$^{\text{nd}}$ order ODEs have the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

For 2$^{\text{nd}}$ order ODEs the complementary function is given by

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

- **Finding the complementary function**:
  - **Step 1** Find auxiliary equation
  - **Step 2** Find roots of the auxiliary equation
  - **Step 3** Find general solution

- **Homogeneous linear 2nd order ODEs**: $f(x) = 0$.

  Find the complementary function for the following equations:

  e.g. PDF3.1 Real roots

  $$y'' + 5y' + 6y = 0$$

  Find the auxiliary equation by substituting $y = Ae^{\lambda x}$

  $$\frac{d^2}{dx^2}(Ae^{\lambda x}) + 5\frac{d}{dx}(Ae^{\lambda x}) + 6(Ae^{\lambda x}) = 0$$

  $$Ae^{\lambda x}(\lambda^2 + 5\lambda + 6) = 0$$

  Given that $Ae^{\lambda x} = y = 0$ is the trivial case we can assume $Ae^{\lambda x} \neq 0$, hence

  $$\lambda^2 + 5\lambda + 6 = 0$$

  Find the roots $(\lambda + 3)(\lambda + 2) = 0$: $\lambda = -3, -2$. Distinct real roots! Solution is

  $$y(x) = c_1 e^{-3x} + c_2 e^{-2x}$$

  e.g. PDF3.2 Repeated roots

  $$y'' + 6y' + 9y = 0$$

  Find the auxiliary equation by substituting $y = Ae^{\lambda x}$

  $$\lambda^2 + 6\lambda + 9 = 0$$

  Find the roots $(\lambda + 3)^2 = 0$: $\lambda = -3, -3$. Repeated root! Solution is

  $$y(x) = (c_1 + c_2 x)e^{-3x}$$

  Not this:

  $$y(x) = (c_1 + c_2)e^{-3x} = c_3 e^{-3x}$$

  which leaves us with only one term. We need to multiply one term by x until it is linearly independent of the other term!

  e.g. PDF3.3 Complex roots

  $$y'' - 10y' + 26y = 0$$

  Find the auxiliary equation by substituting $y = Ae^{\lambda x}$

  $$\lambda^2 - 10\lambda + 26 = 0$$

  Find the roots

  $$\lambda = \frac{10 \pm \sqrt{100 - 4(26)}}{2}$$

  $$= \frac{10 \pm \sqrt{-4}}{2} = \frac{10 \pm 2i}{2} = 5 \pm i$$

  Complex root!

  $$\Rightarrow \alpha = 5, \beta = 1$$

  Solution is

  $$y(x) = e^{5x}(c_1 \cos x + c_2 \sin x)$$

- **Non-Homogeneous Linear 2nd order ODEs - method of trial functions/method of undetermined coefficients:** $f(x) \neq 0$.

  e.g. PDF3.4 Consider the following equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$$

Setting $RHS = 0$ we find it has a complementary equation of the form

$$y_c(x) = (c_1 + c_2x)e^x$$

Find a particular integral of this equation, and thus the general solution.

The solution is likely to have form e^x to match the RHS of the ODE, however e^x and xe^x are already included in the equation. Try $y_p(x) = bx^2e^x$. Sub into ODE to determine b

$$\frac{d^2}{dx^2}(bx^2e^x) - 2\frac{d}{dx}(bx^2e^x) + bx^2e^x = e^x$$

$$\frac{d}{dx}(2bxe^x + bx^2e^x) - 4bxe^x - 2bx^2e^x + bx^2e^x = e^x$$

$$2be^x + 2bxe^x + 2bxe^x + bx^2e^x - 4bxe^x - 2bx^2e^x + bx^2e^x = e^x$$

$$2b + 2bx + 2bx + bx^2 - 4bx - 2bx^2 + bx^2 = 1$$

$$2b = 1$$

$$b = \frac{1}{2}$$

Thus particular integral is

$$\frac{1}{2}x^2e^x$$

General solution given by $y = y_c + y_p$

$$y(x) = (c_1 + c_2x)e^x + \frac{1}{2}x^2e^x$$

Note: It is good practice to check your solutions work by substituting them back into the original equation and checking you obtain the RHS.