Relativistic Electrodynamics, Workshop 3

Relativistic Addition of 3-Velocities

In this exercise you will derive the relativistic rule for combining velocities. In Galilean mechanics the rule is simple: The velocity measured by observer A, $\underline{v}_A$, compared to that measured by observer B, $\underline{v}_B$, is $\underline{v}_A = \underline{v}_B + \underline{v}_{B,A}$, where $\underline{v}_{B,A}$ is the velocity of observer B measured by observer A. This obviously cannot hold in special relativity, since faster-than-light velocities could be obtained this way.

Consider two frames of inertia S and S', with S' moving along the direction of x-axis with velocity u, as measured by an observer in S. An observer at rest in S' measures the 3-velocity of a trajectory to be v'.

1. Use the relation between the 0-component of the 4-velocities v^μ, v'^μ to derive the following relation

$$\gamma(v) = \gamma(u)\ \gamma(v') \left(1 + \frac{v'_x u}{c^2}\right) \tag{1}$$

2. Find the 3-velocity, v, measured by an observer in S, by explicitly constructing the 4-velocity measured in S' and transform this into S. The result is

$$v_x = \frac{u + v'_x}{1 + uv'_x/c^2} \quad v_y = \frac{v'_y}{\gamma(u)\,(1 + uv'_x/c^2)} \quad v_z = \frac{v'_z}{\gamma(u)\,(1 + uv'_x/c^2)}. \tag{2}$$

3. Show that for small velocities $|u|, |\underline{v}'| \ll c$ this reduces to the simple Galilean result.

Note that for collinear velocities, i.e. $\underline{v}$ aligned with $\underline{u}$ (the x-axis), the result is symmetric under exchanging u and v'_x, just as the Galilean result is symmetric. However, when $\underline{v}$ is not along the boost-axis, then the result is not symmetric, i.e. in general the velocity of a trajectory is different whether it is measured as $\underline{v}_1$ by an observer moving relative to you with velocity $\underline{v}_2$, as opposed to a measurement of $\underline{v}_2$ by an observer moving relative to you with velocity $\underline{v}_1$.

Solution

The Lorentz transformation going from S' to S is given by

$$\Lambda_{S'\to S}{}^{\mu}_{\nu} = \begin{pmatrix} \gamma(u) & \gamma(u)u/c & 0 & 0 \\ \gamma(u)u/c & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

The four-velocities are given by $v^\mu = \gamma(v)(c, \underline{v})$ etc. Therefore, the relationship between v^0 and v'^0 is

$$\gamma(v)c = \gamma(u)\ \gamma(v')\,(c + v'_x u/c) = \gamma(u)\ \gamma(v')\left(1 + v'_x u/c^2\right) c, \tag{4}$$

and therefore

$$\gamma(v) = \gamma(u)\ \gamma(v')\left(1 + v'_x u/c^2\right). \tag{5}$$

The transformation of the x-component of the 3-velocities we find by studying the Lorentz-transformation of the 1-component of the 4-vector:

$$\gamma(v)c_x = \gamma(v')\gamma(u)(uc/c + v'_x) = \gamma(v')\gamma(u)(v'_x + u) \tag{6}$$

$$\therefore v_x = \frac{v'_x + u}{1 + v'_x u/c^2}, \tag{7}$$

using Eq. (5).

For the y and z components we find

$$\gamma(v)v_y = \gamma(v')\ v'_y \tag{8}$$

$$v_y = \frac{v'_y}{\gamma(u)(1 + v'_x u/c^2)}, \tag{9}$$

and similarly for v_z.

Expanding for $u/c \ll 1, v'_x/c \ll 1$ and keeping just the first order in u and v'_x we find

$$\underline{v} = \underline{u} + \underline{v}' \quad \text{non-relativistic limit!!} \tag{10}$$