Mathematical Methods II
Weekly problem set 4

(1) Find all the singular points of the following equations

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \tfrac{m^2}{1-x^2}\right] y = 0 \quad \text{(associated Legendre equation)}$$
$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \quad \text{(Bessel equation)}$$

and classify them. (Do not forget to consider potential singularities at infinity). Here ℓ, m and ν are constants.

(2) The aim of this question it to find two power series solutions about $x = 0$ of the differential equation

$$(1-x^2)y'' - 3xy' + \lambda y = 0, \tag{1}$$

where λ is a constant and construct a general solution. Begin by checking that $x = 0$ is an ordinary point.

This series terminates at a particular value of λ, for order $n = N$. Deduce the value of λ for which the corresponding power series becomes a finite N-th order polynomial $y_N(x)$ - i.e. express λ in terms of N.

Construct two polynomial solutions, U_2 and U_3 that terminate at the x^2 and x^3 terms, respectively. (Please note that these are not two solutions to the same ODE as they will require different values for λ, and therefore cannot be combined to construct a general solution). Show by substitution that U_2 and U_3 satisfy their respective differential equations.