# University of Durham

## EXAMINATION PAPER

May/June 2012 Examination code: 043522/01

**LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3 PAPER 1**

**SECTION A**. CLASSICAL MECHANICS
**SECTION B**. STATISTICAL PHYSICS
**SECTION C**. MODERN OPTICS

**Time allowed : 3 hours**

**Examination material provided : None**

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

**ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

---

**Information**

| | |
|---|---|
| Elementary charge: | $e = 1.60 \times 10^{-19}$ C |
| Speed of light: | $c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$ |
| Boltzmann constant: | $k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$ |
| Electron mass: | $m_\mathrm{e} = 9.11 \times 10^{-31}$ kg |
| Gravitational constant: | $G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$ |
| Proton mass: | $m_\mathrm{p} = 1.67 \times 10^{-27}$ kg |
| Planck constant: | $h = 6.63 \times 10^{-34}$ J s |
| Permittivity of free space: | $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$ |
| Magnetic constant: | $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$ |
| Molar gas constant: | $R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$ |
| Avogadro's constant: | $N_\mathrm{A} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$ |
| Gravitational acceleration at Earth's surface: | $g = 9.81\ \mathrm{m\,s^{-2}}$ |
| Stefan-Boltzmann constant: | $\sigma = 5.67 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$ |
| Astronomical Unit: | AU $= 1.50 \times 10^{11}$ m |
| Parsec: | pc $= 3.09 \times 10^{16}$ m |
| Solar Mass: | $M_\odot = 1.99 \times 10^{30}$ kg |
| Solar Luminosity: | $L_\odot = 3.84 \times 10^{26}$ W |

**SECTION A**. CLASSICAL MECHANICS

Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) State Hamilton's principle, where the action integral is given by

$$S = \int_{t_i}^{t_f} L dt,$$

and $L$ is the Lagrangian. How is it related to the fact that Euler-Lagrange equations give the correct equations of motion? [4 marks]

(b) Consider an underdamped, sinusoidally driven harmonic oscillator. Qualitatively describe the dynamics of the oscillator in terms of the steady-state solution and the transient solution, including the long-time behaviour and the oscillation frequency. [4 marks]

(c) Normal coordinates and normal modes describe the small oscillations of a system of interacting bodies around an equilibrium configuration. Why is it in general important for the oscillations to be considered small? [4 marks]

(d) Briefly describe Noether's theorem and give an example of when Noether's theorem applies. [4 marks]

(e) How many degrees of freedom are needed to describe the most general motion of most rigid bodies, and what sort of motion do they describe? Why is the limiting case of two point masses connected by a zero-thickness rigid rod an exception? [4 marks]

2. When considering the motion of a particle in 3 spatial dimensions, one may describe the Poisson bracket for two functions $F$ and $G$ of the Cartesian coordinates $q_x$, $q_y$, $q_z$ and their canonically conjugate momenta $p_x$, $p_y$, $p_z$ as

$$\{F, G\} = \sum_{k=x,y,z} \left( \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right).$$

(a) Consider a Lagrangian of the general form

$$L = \frac{m}{2}(\dot{q}_x^2 + \dot{q}_y^2 + \dot{q}_z^2) - V,$$

where $V$ is an unspecified potential energy function of the coordinates $q_x$, $q_y$, $q_z$.

(i) Show that the canonically conjugate momenta $p_x$, $p_y$, $p_z$ are then equal to the mechanical momenta $m\dot{q}_x$, $m\dot{q}_y$, $m\dot{q}_z$. [2 marks]

(ii) Determine the Poisson brackets $\{p_x, p_y\}$, $\{p_y, p_z\}$, $\{p_z, p_x\}$. [2 marks]

(b) Still considering the same general form of Lagrangian as in (a), the angular momentum vector is defined as $\underline{J} = m\underline{q} \times \dot{\underline{q}}$, where $\underline{q} = (q_x, q_y, q_z)$.

(i) Determine the components $J_x$, $J_y$, $J_z$ in terms of the coordinates and their canonically conjugate momenta. [2 marks]

(ii) Determine the Poisson brackets $\{J_x, J_y\}$, $\{J_y, J_z\}$, $\{J_z, J_x\}$. [6 marks]

(iii) Determine the nested Poisson bracket $\{J_y, \{J_y, \{J_y, J_x\}\}\}$. [4 marks]

(c) Consider the Lagrangian, expressed in spherical coordinates as

$$L = \frac{m}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2(\theta)] - V.$$

(i) Determine the canonically conjugate momenta $p_r$, $p_\theta$, $p_\phi$ to $r$, $\theta$, $\phi$. [2 marks]

(ii) Determine the Poisson brackets $\{r, p_r\}$, $\{\theta, p_\theta\}$, $\{\phi, p_\phi\}$. [2 marks]

3. In the study of ultracold atomic gases, one may consider the atoms to be confined within a rapidly rotating anisotropic harmonic potential, and treat the dynamics classically. Viewed within a constantly rotating frame defined by the angular velocity vector $\underline{\omega} = (0, 0, \omega)$, where the frame in which the lab coordinates are fixed (lab frame) is considered to be an inertial frame, we consider a single particle of mass $m$ observed to be within a harmonic potential
$$V = \frac{m}{2}(\nu_x^2 x^2 + \nu_y^2 y^2 + \nu_z^2 z^2),$$
where $\{\nu_x, \nu_y, \nu_z\}$ are the angular frequencies of oscillation.

   (a) With the aid of a diagram, show the direction of rotation of the potential $V$ within the lab (non-rotating) frame. Include the angular velocity vector in your diagram. [2 marks]

   (b) The Coriolis and centrifugal force terms are given by $-2m\underline{\omega} \times \dot{\underline{r}}$ and $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$, respectively.

      (i) We will not consider the Euler force. Explain why this is appropriate. [2 marks]

      (ii) Show that the force associated with the potential, as viewed in the rotating frame, is $\underline{F} = -m(\nu_x^2 x, \nu_y^2 y, \nu_z^2 z)$. [2 marks]

      (iii) Determine the equations of motion for the particle in the $x$, $y$, and $z$ directions. [5 marks]

      (iv) In the case where $\nu_x = \nu_y = \nu_0$, and using $\xi(t) = x(t) + iy(t)$, show that the solution to the combined equation of motion is
$$\xi(t) = e^{-i\omega t}[A\cos(\nu_0 t) + B\sin(\nu_0 t)],$$
where $A = \xi(0)$ and $B = \dot{\xi}(0)/\nu_0 + i(\omega/\nu_0)\xi(0)$. [7 marks]

   (c) Consider the particle to be initially at rest, and determine its subsequent motion in the $x, y$ plane for the case where $\omega = \nu_0$. [2 marks]

**SECTION B**. STATISTICAL PHYSICS

Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) By reference to the Ehrenfest Urn model, explain how apparent irreversibility can be observed in a reversible system. [4 marks]

   (b) Show that the statistical weight, $\Omega$, of a collection of $n_i$ fermions distributed among $g_i$ states is

$$\Omega = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}.$$

   [4 marks]

   (c) State which statistical distribution (Classical Boltzmann, Fermi-Dirac or Bose-Einstein) would be appropriate in describing the following systems, and explain why: (i) Dilute magnetic spins in a paramagnet such as cerium magnesium nitrate $Ce_2Mg_3(NO_3)_{12}.24\,H_2O$ at 300 K. (ii) Free electrons in a metal at a temperature of 300 K. [4 marks]

   (d) Calculate the heat capacities using the equipartition theorem of (i) a gas of $N$ helium atoms at high temperatures and (ii) a gas of $N$ diatomic molecules of chlorine ($Cl_2$) at high temperatures. [4 marks]

   (e) In copper metal there are $8.45 \times 10^{22}$ conduction electrons per $cm^3$, which behave approximately as a free electron gas. Calculate the magnitude of the Fermi wavevector and the Fermi energy (in eV) of the free electrons in copper. [4 marks]

5. A paramagnetic solid consists of a dilute collection of $N$ spin 1/2 ions. In a magnetic field $B$, each spin may be either parallel to the field and have an energy $-\mu_{\rm B}B$, or anti-parallel with energy $+\mu_{\rm B}B$. The single magnetic ion partition function of the system is given by

$$Z_1 = 2\cosh x \qquad \text{where} \qquad x = \frac{\mu_{\rm B}B}{k_{\rm B}T}$$

where $T$ is the spin temperature.
The Bohr magneton $\mu_{\rm B}$ is $9.27 \times 10^{-24}$ Joules per Tesla.

(a) (i) Show that the mean number of spins parallel to $B$ is given by

$$n_1 = \frac{Ne^x}{e^x + e^{-x}}.$$

[3 marks]

(ii) Find an expression for the spin temperature as a function of $n_1$. [3 marks]

(iii) The ions, whose magnetic moment is equal to the Bohr magneton, are confined within a magnetic field of 2.98 Tesla. Calculate the spin temperature of the solid when the ratio of spins parallel/anti-parallel ($n_1/n_2$) is $n_1/n_2 = 5$ and $n_1/n_2 = 0.2$, where $n_1 + n_2 = N$. [6 marks]

(b) Discuss whether a negative spin temperature is 'hotter' (i.e. higher energy) or 'colder' than a positive spin temperature [4 marks]

(c) Suggest how a negative spin temperature might be produced in a laboratory experiment. [4 marks]

6. (a) Show that the statistical weight $\Omega$ of a Bose gas composed of many particles is given by

$$\Omega = \prod_i \frac{(n_i + g_i)!}{n_i! g_i!},$$

where $n_i$ is the number of bosons with energy $\epsilon_i$, and $g_i$ is the degeneracy of the $i$th level. [6 marks]

(b) By calculating the entropy $S$ of a Bose gas, show that the occupation probabilities $f_i$ corresponding to the minimum value of $U - TS - \mu N$, are

$$f_i = \frac{1}{e^{(\epsilon_i - \mu)/k_\mathrm{B}T} - 1},$$

where $U = \Sigma_i n_i \epsilon_i$ is the total energy, $N = \Sigma_i n_i$ is the total number of particles, $\mu$ is the chemical potential and $T$ the temperature. [6 marks]

(c) At very low temperatures a Bose gas can condense to form a Bose-Einstein condensate. For an ideal gas this transition happens at the Bose temperature $T_\mathrm{B}$ given by

$$\frac{N}{V}\left(\frac{2\pi\hbar^2}{mk_\mathrm{B}T_\mathrm{B}}\right)^{3/2} = 2.612$$

where $N$ is the number of gas particles, $m$ is the particle mass and $V$ is the volume. Calculate an approximate value for the Bose temperature of $^4$He given that the experimental mass density of liquid helium at low temperatures is $\left(\frac{mN}{V}\right) = 0.146 \times 10^3$ kg m$^{-3}$. [4 marks]

(d) Experimentally, the lambda transition occurs at 2.18 K. Explain why the temperature calculated above differs from this. [4 marks]

**SECTION C**. MODERN OPTICS

Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively. Write down the Fourier transform of: (i) $f(2x)$, (ii) $f(x-a)$, (iii) $f \otimes g$ and (iv) $fg$. [4 marks]

   (b) A mask has a one-dimensional amplitude transmission profile given by $T(x) = 0.6 + 0.4\cos(2\pi x/d)$, where $d$ is a constant. The mask is illuminated normally with plane waves of wavelength $\lambda$. Sketch the form of the far-field intensity diffraction pattern. What are the relative intensities of the features? [4 marks]

   (c) Draw a phasor diagram corresponding to (i) a principal maximum, (ii) a subsidiary maximum, (iii) a minimum, and (iv) a point where the intensity is one ninth of the maximum, in the far–field diffraction pattern produced by three slits. [4 marks]

   (d) Sketch the far-field intensity distribution produced by diffraction of a uniform monochromatic light wave with diameter $a$ normally incident on (i) a square aperture with side $b$, and (ii) an opaque square with side $b$, for $a \gg b$. What is the functional form of the difference between the two field patterns? [4 marks]

   (e) A mask with an aperture with the shape of the letter A is illuminated normally by uniform monochromatic light. Sketch the far–field intensity distribution. Sketch the corresponding intensity pattern if the A is replaced by a triangular aperture of similar size to the upper half of the A. Comment on the symmetry in both cases. [4 marks]

8. Produce a labelled sketch of the $4f$ optical system used for spatial filtering. Give an expression for the output field in terms of the input field. [6 marks]

   If the input intensity is

$$I = I_0 \mathrm{e}^{-2(x^2+y^2)/w^2} \cos^2(2\pi x/a)$$

   where $w$ is the Gaussian beam radius and $a$ determines the fringe spacing, write down an expression for the field in the Fourier plane. [2 marks]

   Sketch the input intensity distribution along the $x$–axis for $w = 2a$. How many intensity maxima are there in the $x$ and $y$–directions between $\pm w$ (include maxima at the limiting values)? [4 marks]

   Sketch the intensity distribution in the Fourier plane along the $x$–axis for $w = 2a$. [3 marks]

   A square glass plate is placed in the Fourier plane. The thickness of the plate is such that it shifts the phase of the field by $\pi$. The plate is positioned to cover only the positive $xy$ quadrant. Sketch the output intensity distribution (i) in the $x$ direction for $y = \pm\epsilon$, where $\epsilon \ll w$, and (ii) along the $y$ axis. [5 marks]

9. The function Gauss$(x/a)$ is defined as $\mathrm{e}^{-x^2/a^2}$. Show that the Fourier transform of Gauss$(x/a)$ is $\pi^{1/2}a$ Gauss$(ka/2)$. [4 marks]

   Write an expression for the electric field of a cylindrically symmetric Gaussian laser beam at the position of the beam waist. [2 marks]

   A lens is used to focus a Gaussian beam. Find a relationship between the input beam waist $w_\mathrm{i}$ and the focused beam waist $w_\mathrm{f}$. [4 marks]

   If the lens has a finite diameter $D$ write an expression for the electric field in the focal plane. [4 marks]

   Sketch the intensity distribution in the two limiting cases: (i) $w_\mathrm{i} < D$ and (ii) $w_\mathrm{i} > D$. [4 marks]

   Sketch how the peak intensity at the focus varies as function of the ratio $w_\mathrm{i}/D$. [2 marks]

   [Hint: $\int_{-\infty}^{\infty} \exp\left(-\xi^2\right) \mathrm{d}\xi = \pi^{1/2}$]