Mathematical Methods in Physics

Weekly Problems 2. Solution

2.1

a) Linearly dependent. Three 2-component vectors cannot be linearly independent. In fact, $\mathbb{R}^2$ has dimension two. You can see that the third vector is twice the second minus the first.

b) Linearly independent.

c) Linearly dependent. The last matrix in the set is the sum of the previous two matrices.

d) Linearly independent. In fact

$$\alpha_1(1+x) + \alpha_2(x+x^2) + \alpha_3(1+x^2) = 0$$

Then

$$(\alpha_1+\alpha_3) + (\alpha_1+\alpha_2)x + (\alpha_2+\alpha_3)x^2 = 0 \quad \longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

$\boxed{\text{3 marks}}$

2.2

a) Use the hint, that is suppose A^{-1} exists. Then multiply the expression $A^2 = A$ by A^{-1}. You get $A = I$, which contradicts the assumption that $A \neq I$. The inverse does not exists and $|A| = 0$. $\boxed{\text{1 marks}}$

b) Use the hint. Then the expression provided becomes $(I+A)(I-A/2) = I + A - A/2 - A/2 = I$. Therefore $I - A/2$ must be the inverse of $I + A$. $\boxed{\text{1 marks}}$

2.3

The starting point is:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right)$$

Perform the following row operations

$$R_2 \longrightarrow R_2 - 2R_1, \quad R_3 \longrightarrow R_3 - R_1, \quad \longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right),$$

$$R_2 \longrightarrow -R_2/2, \quad \longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right),$$

$$R_3 \longrightarrow R_3 - R_2, \quad \longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right),$$

$$R_1 \longrightarrow R_1 - 2R_2, \quad \longrightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 5 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right),$$

$$R_1 \longrightarrow R_1 - 5R_3, \quad \longrightarrow R_2 \longrightarrow R_2 + 3R_3, \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 9 & -3/2 & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right).$$

$\boxed{\text{3 marks}}$

2.4

$$\begin{aligned} \mathrm{Tr}\,(ABC) &= \sum_i (ABC)_{ii} \equiv (ABC)_{ii} = A_{il}B_{lk}C_{ki} \equiv \sum_i \sum_l \sum_k A_{il}B_{lk}C_{ki} \\ &= \sum_i \sum_l \sum_k C_{ki}A_{il}B_{lk} \equiv C_{ki}A_{il}B_{lk} = (CAB)_{kk} \equiv \sum_k (CAB)_{kk} = \mathrm{Tr}\,(CAB). \end{aligned}$$

$\boxed{\text{2 marks}}$