Mathematical Methods II

Craig Testrow

Contents

1 Introduction to 1$^{\text{st}}$ Order Differential Equations

Key Points

- Recall various standard notations.
- Recall fundamental definitions related to differential equations.
- Solving 1$^{\text{st}}$ order ODEs by seperation of variables.
- Detemining if an equation is exact.
- Solving 1$^{\text{st}}$ order ODEs using integrating factors.

Notation

- Recall the differences in 'change' notation:
  - Δ: Relatively large change
  - δ: Relatively small change
  - d: Infinitesimal change (i.e. in the limit $x \to 0$), total derivative
  - ∂: Infinitesimal change (i.e. in the limit $x \to 0$), partial derivative
- Recall the different forms of derivative notation:
  - Total derivatives: The following notations are frequently encountered.
    * Leibniz's notation: d^2y/dx^2
    * Lagrange's notation: $f''(x)$
    * Euler's notation: D^2f
    * Newton's notation: $\ddot{y}$
  - Partial derivatives: The following notations are equivalent

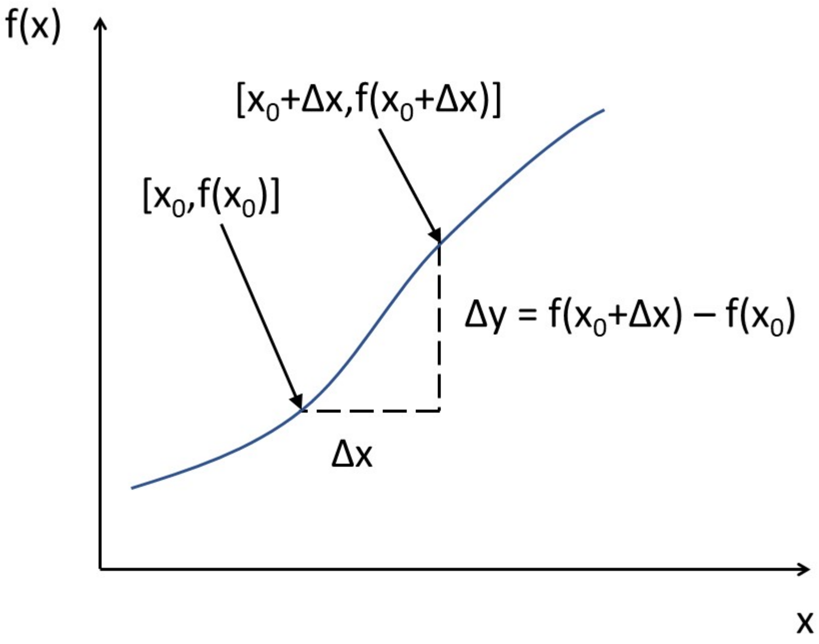
$$\frac{\partial f}{\partial x} = f_x$$

$$\frac{\partial f}{\partial x \partial y} = f_{xy} = \partial_{xy} f$$

Definitions

- **Differential equation**: an equation involving derivatives of a function or functions.
- **Derivative**: Shows a rate of change of a variable(s) w.r.t another variable(s).

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



- **Solution**: A solution to a differential equation is an equation that contains no derivatives, typically given in a form such as $y = f(x)$. A given DE may have more than one solution.
- **Classification by type**
  - **ODE**: Ordinary Differential Equation. Contains one or more dependent variables differentiated w.r.t one independent variable. e.g.

$$\frac{d^2u}{dx^2} + \frac{du}{dx} + u = e^2$$

    where u is the *dependent* variable and x is the *independent* variable.

  - **PDE**: Partial Differential Equation. Contains one or more dependent variables differentiated w.r.t two or more independent variables. e.g.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = e^{xy}$$

    where u is the *dependent* variable and x and y are the *independent* variables.

  - Recall the relationship between partial and total derivatives. For $u(x, y)$ we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

- **Classification by order/degree**
  - **Order of a DE**: Determined by the highest derivative
  - **Degree of a DE**: Determined by the exponent of the highest derivative (all exponents should be made integers first).

e.g. the equation

$$\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{dy}{dx}\right)^2 + y = 0$$

is of 3$^{\text{rd}}$ order and 4$^{\text{th}}$ degree.

- **Classification as linear/non-linear**
  - **Linear DEs**: The dependent variables and their derivatives are of 1$^{\text{st}}$ degree and each coefficient depends only on the independent variable.

    i.e. Linear equations are of the form

    $$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

    e.g.

    $$\frac{d^2y}{dx^2} + y = \sin x$$

  - **Non-linear DEs**: The dependent variables and their derivatives are *not* of 1$^{\text{st}}$ degree e.g.

    $$\frac{d^2y}{dx^2} + y^2 = \sin x$$

Solving 1$^{\text{st}}$ order ODEs

Not all equations can be solved. In some cases it is possible to prove that an equation is unsolvable. Additionally, there exists no single method for solving all ODEs; thus we require a variety of techniques so we can apply the most appropriate "tool" for the job.

Generally, ODEs become trickier to solve as they increase in order. To begin with we will examine how to solve 1$^{\text{st}}$ order ODEs.

- **Separable ODE**: If an equation of two variables and their derivatives can be arranged such that one variable appears only on the LHS of the equation and the other appears only on the RHS it allows us to integrate them independently. Suppose

  $$f(x,y) = \frac{dy}{dx} = u(x)v(y)$$

  This equation is separable, since we can write it as follows

  $$\frac{dy}{v(y)} = u(x)dx$$

  This can be solved by integrating both sides

  $$\int \frac{1}{v(y)}dy = \int u(x)dx$$

  e.g. PDF1.1 Solve the following equation, given that $y(0) = 3$

  $$\frac{dy}{dx} = \frac{y\cos x}{2}.$$

Separate the functions and integrate

$$\int \frac{2}{y} dy = \int \cos x dx$$

$$2 \ln y = \sin x + c_1$$

$$\ln y = \frac{\sin x}{2} + c_2$$

$$y = c_3 e^{\sin x/2}$$

Use the initial condition to find c_3

$$c_3 = \frac{y}{e^{\sin x/2}} = \frac{3}{1} = 3$$

Hence the solution is

$$y = 3e^{\sin x/2}$$

- **Exact ODE**: Exact and inexact ODEs are common in physics, and are of particular relevance in thermodynamics.

  Exact ODEs describe state/point functions - functions describing quantities in a system that do not depend on the path taken, such as internal energy, enthalpy and entropy. These values quantitatively describe the equilibrium state of a thermodynamic system, regardless of how the system arrived in that state.

  Inexact ODEs describe path/process functions - functions describing quantities in a system that do depend on the path taken, such as mechanical work and heat. These values quantitatively describe the transition between equilibrium states of a thermodynamic system.

  . The following form is useful in determining whether an ODE is exact.

$$A(x, y)dx + B(x, y)dy = 0.$$

  Where A and B are arbitrary functions of x and y. An ODE is exact if the following condition is met

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.$$

  Otherwise it is inexact. If the ODE is exact a function $u(x, y)$ exists such that

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = Adx + Bdy = 0$$

  We can solve $\partial u/\partial x = A$ and $\partial u/\partial y = B$ to find $u(x, y)$. Since $du = 0$, $u(x, y) = c$, a constant.

e.g. PDF1.2 Determine whether the following equation is exact, and solve it

$$2xy\frac{dy}{dx} + y^2 + x^2 = 0.$$

Start by $\times dx$ to give the standard form

$$(y^2 + x^2)dx + (2xy)dy = 0$$

So $A = y^2 + x^2$ and $B = 2xy$. Check if the equation is exact.

$$\frac{\partial A}{\partial y} = 2y$$

$$\frac{\partial B}{\partial x} = 2y$$

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

Exact! Now, since $A = \partial u/\partial x$ and $B = \partial u/\partial y$, integrate either A w.r.t x or B w.r.t y to find $u(x, y)$. Here the latter appears easier.

$$u(x, y) = \int 2xydy = xy^2 + F(x)$$

To find $F(x)$ we examine the unused function for A

$$A = \frac{\partial u}{\partial x} = y^2 + F^{'}(x) = y^2 + x^2$$

Thus

$$F^{'}(x) = x^2$$

$$F(x) = \int x^2 dx = \frac{x^3}{3} + c_1$$

So, substituting back into u

$$u(x, y) = xy^2 + \frac{x^3}{3} + c_1 = c_2$$

Since $du = 0$, $u = c_2$, a constant. Hence the solution, where $c_3 = c_2 - c_1$, is

$$xy^2 + \frac{x^3}{3} = c_3$$

$$y = \sqrt{\frac{c_3 - \frac{x^3}{3}}{x}}$$

- **Inexact ODEs and integrating factors**: If an equation is not exact it can be made exact by multiplying it with an integrating factor, μ. Using the same form as earlier

$$A(x, y)dx + B(x, y)dy = 0,$$

an equation is inexact if

$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x},$$

and can be made exact by multiplying by μ,

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x},$$

giving us the corrected form of

$$\mu A(x, y)dx + \mu B(x, y)dy = 0.$$

If $\mu = \mu(x, y)$ then there is no general method (other than possibly inspection) to find μ. But if $\mu = \mu(x)$ or $\mu = \mu(y)$ then we can test for it. Let's assume $\mu = \mu(x)$, then the last equation becomes,

$$\mu\frac{\partial A}{\partial y} = \mu\frac{\partial B}{\partial x} + B\frac{d\mu}{dx}.$$

We can rearrange this to give

$$B\frac{d\mu}{dx} = \mu\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right),$$

$$\frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)dx = f(x)dx,$$

where $f(x)$ is a function of x alone, as required by our initial condition. Integrating this equation gives us our integrating factor

$$\ln\mu = \int f(x)dx,$$

$$\mu = \exp\left(\int f(x)dx\right),$$

where

$$f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right).$$

Similarly for $\mu(y)$ where $g(y)$ is a function of y alone

$$\mu = \exp\left(\int g(y)dy\right),$$

where

$$g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right).$$

$f(x)$ and $g(y)$ can be used to test if the integrating factor is a factor of x or y alone.

e.g. PDF1.3 Show that the following equation is not exact. Determine an appropriate integrating factor and hence, solve the equation.

$$(xy + y^2 + y)dx + (x + 2y)dy = 0$$

First, show that the equation is not exact. Here $A = xy + y^2 + y$ and $B = x + 2y$.

$$\frac{\partial A}{\partial y} = x + 2y + 1$$

$$\frac{\partial B}{\partial x} = 1$$

$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

Not exact! Let's assume there is an integrating factor containing only x and rewrite our equation to include the factor

$$\mu(x)(xy + y^2 + y)dx + \mu(x)(x + 2y)dy = 0$$

Let $C = \mu(x)(xy + y^2 + y)$ and $D = \mu(x)(x + 2y)$. Assuming our integrating factor works then it should be true that

$$\frac{\partial C}{\partial y} = \frac{\partial D}{\partial x}$$

So differentiate C w.r.t y and D w.r.t. x.

$$C_y = \mu(x)(x + 2y + 1)$$

$$D_x = \mu'(x)(x + 2y) + \mu(x)$$

We can equate these equations and solve them by using the separable equation method.

$$\mu(x)(x + 2y + 1) = \mu'(x)(x + 2y) + \mu(x)$$

$$\mu(x)(x + 2y) = \mu'(x)(x + 2y)$$

$$\mu(x) = \mu'(x) \Rightarrow \mu(x) = e^x$$

So our equation, modified with out integrating factor, is now

$$e^x(xy + y^2 + y)dx + e^x(x + 2y)dy = 0$$

Test if it is exact

$$\frac{\partial C}{\partial y} = xe^x + 2ye^x + e^x$$

$$\frac{\partial D}{\partial x} = e^x(x + 2y) + e^x$$

$$\frac{\partial C}{\partial y} = \frac{\partial D}{\partial x}$$

Exact! The solution can now be found as previously shown, by finding u(x,y), giving

$$(xy + y^2)e^x = k$$

where k is a constant.

2 More 1^{st} Order Differential Equations

Key Points

- Solving 1^{st} order linear ODEs.
- Solving Bernoulli type non-linear ODEs.
- Solving homogeneous 1^{st} order ODEs.
- Solving isobaric 1^{st} order ODEs.

Solving 1^{st} order ODEs (cont.)

- **Linear 1^{st} order ODEs**: Linear 1^{st} order ODEs are a special type of inexact ODE and have the form,
$$a(x)\frac{dy}{dx} + b(x)y = c(x).$$
If we divide by $a(x)$ we arrive at the canonical form,
$$\frac{dy}{dx} + p(x)y = q(x),$$
where $p(x) = b(x)/a(x)$ and $q(x) = c(x)/a(x)$. If an ODE can be rearranged into this form there is a standard method for solving it using an integrating factor. Applying the integrating factor $\mu = \mu(x)$ gives
$$\mu\frac{dy}{dx} + \mu py = \mu q,$$
$$\frac{d}{dx}(\mu y) = \mu q.$$
Integrating gives,
$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)dx,$$
which is our solution for y. We can use the equalities above to determine our required integrating factor. It must satisfy the following,
$$\frac{d}{dx}(\mu y) = \frac{d\mu}{dx}y + \mu\frac{dy}{dx} = \mu\frac{dy}{dx} + \mu p(x)y,$$
which gives,
$$\frac{d\mu}{dx} = \mu p(x).$$
Thus μ is given by,
$$\mu(x) = exp\left(\int p(x)dx\right).$$

e.g. PDF2.1 Solve the following 1$^{\text{st}}$ order ODE

$$\frac{dy}{dx} + \left(\frac{1+x}{x}\right)y = \frac{e^x}{x}$$

Determine $p(x)$ and $q(x)$

$$p(x) = \frac{1+x}{x}$$
$$q(x) = \frac{e^x}{x}$$

Find the integrating factor, $\mu(x) = e^{\int p(x)dx}$

$$\begin{aligned}\mu(x) &= e^{\int [(1+x)/x]dx} \\ &= e^{\int [(1/x)+1]dx} \\ &= e^{(\ln x + x)} = e^{\ln x}e^x \\ &= xe^x\end{aligned}$$

Write down the equation including the integrating factor and solve

$$\frac{d}{dx}(xe^x y) = (xe^x)\frac{e^x}{x}$$
$$xe^x y = \int e^{2x}dx$$
$$y = \frac{1}{xe^x}\left[\frac{e^{2x}}{2} + c\right]$$
$$y = \frac{e^x}{2x} + \frac{c}{xe^x}$$

- **Bernoulli ODEs**: Bernoulli equations are of the general form

$$\frac{dy}{dx} + b(x)y = c(x)y^n$$

where $n \neq 0$,$n \neq 1$. This is a non-linear equation, but we can reduce it to a linear equation by using the substitution

$$z = y^{1-n}.$$

Once this has been done, we may solve the equation with an integrating factor, just as with other 1$^{\text{st}}$ order linear ODEs.

e.g. PDF2.2 Solve the following Bernoulli type ODE

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$$

Here, $n = 4$. Rearrange the equation for convenient -ve exponents by $\div y^4$

$$\frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3}y^{-3} = e^x$$

Use the substitution $z = y^{-3}$, work out dz/dx

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$\frac{1}{y^4}\frac{dy}{dx} = -\frac{1}{3}\frac{dz}{dx}$$

Substitute these back into the equation and rearrange into a linear form

$$-\frac{1}{3}\frac{dz}{dx} + \frac{1}{3}z = e^x$$

$$\frac{dz}{dx} - z = -3e^x$$

This can now be solved as a linear equation. Note that $p(x) = -1$ and $q(x) = -3e^x$. Find the integrating factor

$$\mu(x) = e^{\int -1 dx} = e^{-x}$$

Solve the equation

$$(e^{-x}z) = e^{-x}(-3e^x)$$

$$z = -\frac{1}{e^{-x}}\int 3dx$$

$$= -e^x(3x + c)$$

Resubstitute $z = y^{-3}$

$$y = (ce^x - 3xe^x)^{-1/3}$$

- **Homogeneous ODEs**: Homogeneous ODEs have the form

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)} = F\left(\frac{y}{x}\right)$$

where A and B are homogeneous functions of x and y and F is a function of y over x. Homogeneous functions obey the rule

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

That is, if you multiply each argument by a common factor λ, the value of the function will be multiplied by that factor raised to the n^{th} power for all real λ. n is called the degree of homogeneity.

If an ODE is homogeneous we can solve it by making the following substitutions

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

This allows us to separate the variables x and v and solve the equation.

e.g. PDF2.3 Solve the homogeneous equation

$$(y^2 + xy)dx - x^2 dy = 0$$

First, rearrange it into the standard form

$$x^2 dy = (y^2 + xy)dx$$

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}$$

Test for homogeneity

$$f(\lambda x, \lambda y) = \frac{(\lambda y)^2 + (\lambda x)(\lambda y)}{(\lambda x)^2} = \frac{\lambda^2 (y^2 + xy)}{\lambda^2 (x^2)} = f(x, y)$$

therefore, this ODE is homogeneous. Now apply the substitutions $y = vx$ and $dy/dx = v + x dv/dx$ to the ODE.

$$v + x\frac{dv}{dx} = \frac{\left((vx)^2 + x(vx)\right)}{x^2} = \frac{v^2 x^2 + vx^2}{x^2} = v^2 + v$$

$$x\frac{dv}{dx} = v^2 + v - v = v^2$$

Separate the variables and integrate

$$\int \frac{1}{v^2} dv = \int \frac{1}{x} dx$$

Following the working and subsituting v out we arrive at a solution

$$x = Ae^{-x/y}$$

Note: Our primary interest in homogeneity is related to 2^{nd} order linear ODEs of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

These equations are homogeneous when $f(x) = 0$ and non-homogeneous when $f(x) \neq 0$.

- **Isobaric ODEs**: Isobaric ODEs are a generalisation of homogeneous ODEs with the form

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)} = F\left(\frac{y}{x}\right)$$

The difference is that isobaric functions satisfy the following equality

$$f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x, y).$$

We can take advantage of dimensional consistency to solve isobaric ODEs. If we assign a relative weight of m to the exponents of y and dy, and a relative weight of 1 to the exponents of x and dx we are able to make the following substitutions

$$y = vx^m.$$

$$\frac{dy}{dx} = mvx^{m-1} + x^m \frac{dv}{dx}$$

This will make the ODE separable, and thus solvable.

e.g. PDF2.4 Solve the following isobaric equation,

$$x(1-2x^2y)\frac{dy}{dx} + y = 3x^2y^2$$

First arrange into dy and dx terms.

$$xdy - 2x^3ydy + ydx - 3x^2y^2dx = 0$$

Now assign the weights ($y, dy \to m$ and $x, dx \to 1$). Constants are assigned a weight of zero, essentially meaning they can be ignored. This gives the following sums for the 3 terms in the equation

$$xdy \to 1 + m$$

$$2x^3ydy \to 0 + 3 + m + m = 3 + 2m$$

$$ydx \to m + 1$$

$$3x^2y^2dx \to 0 + 2 + 2m + 1 = 3 + 2m$$

We equate the 4 sums and look for a value of m that satifies the equalities.

$$m + 1 = m + 1 = 3 + 2m = 3 + 2m$$

This is satisfied if

$$m + 1 = 3 + 2m$$

Hence

$$m = -2$$

We can now substitute $y = vx^{-2}$ and $\frac{dy}{dx} = \frac{dv}{dx}x^{-2} - 2vx^{-3}$ into the equation, separate it, integrate and solve it.

$$x(1-2x^2vx^{-2})\left(\frac{dv}{dx}x^{-2} - 2vx^{-3}\right) + vx^{-2} - 3x^2(vx^{-2})^2 = 0$$

$$x^{-1}\frac{dv}{dx} - 2vx^{-2} - 2vx^{-1}\frac{dv}{dx} + 4v^2x^{-2} + vx^{-2} - 3v^2x^{-2} = 0$$

$$\frac{dv}{dx} - vx^{-1} - 2v\frac{dv}{dx} + v^2x^{-1} = 0$$

$$(1-2v)\frac{dv}{dx} + (v^2 - v)x^{-1} = 0$$

$$\frac{(1-2v)}{v(1-v)}dv = x^{-1}dx$$

If we integrate and subsitute y back in we find the solution

$$4xy(1-x^2y) = 1$$

3 The Complementary Function and Particular Integrals

Key Points

- Finding the complementary function of a 2^{nd} order ODE.
- Finding the auxillary equation of a 2^{nd} order ODE.
- Finding the particular integral of a 2^{nd} order ODE.
- Solving 2^{nd} order ODEs using trial functions/method of undetermined coefficients.

Definitions

- **Linear $\mathbf{n^{th}}$ order ODEs**: Linear equations of n^{th} order have the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

  - **Classification by coefficients**
    - **Constant coefficients** a_0, a_n, 2, 5/3...
    - **Variable coefficients** $a_0(x)$, $a_n(y)$, x, t...
  - **Classification by homogeneous/non-homogeneous**
    - **Homogeneous** when $f(x) = 0$.
    - **Non-homogeneous** when $f(x) \neq 0$.

- **Complementary function**: If the equation is homogeneous it can be solved by finding the 'complementary function', $y_c(x)$. If $y_1, y_2, ..., y_n$ are different solutions of an n^{th} order ODE and linearly independent (the determinent of the matrix of vectors $\neq 0$) then the general solution of the ODE is given by

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x)$$

  This is called the principle of linear superposition.

- **Auxiliary equation**: Also called the characteristic equation, determining the auxiliary equation is the first step in finding the complementary function. It is found by substituting $y = Ae^{\lambda x}$ into our equation as a trial solution. It is a polynomial equation in λ of order n, and reads

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0$$

  Note that the exponential terms cancel out after the derivatives have been taken. Exponentials make useful trial solutions since their derivitives are merely multiples of the function itself. In the case of a 2^{nd} order ODE this results in a quadratic in λ, which can be solved with relative ease.

  The auxiliary equation has n roots, say $\lambda_1, \lambda_2, ..., \lambda_n$. These roots can be real or complex and may repeat. Their nature will determine the form for the solution we must consider the three main cases. The following are general solutions for 2^{nd} order ODEs where $n = 2$:

- **All roots are real and distinct** Solutions take the form of a sum of n linearly independent solutions.
$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
- **Roots are real and equal** A repeated root means we have not found n linearly independent solutions. The second solution can be found by multiplying by x.
$$y_c(x) = (c_1 + c_2 x)e^{\lambda_1 x}$$
A triple root would require a third term in x^2 and so on.
- **Roots are complex** If the auxiliary equation has a complex root $\alpha + i\beta$ then its complex conjugate $\alpha - i\beta$ is also a root. In this case solutions can be expressed in one of the following forms
$$y_c(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$
$$= e^{\alpha x}(d_1 \cos \beta x + d_2 \sin \beta x) = Ae^{\alpha x} \sin(\beta x + \phi)$$

- **Particular integral**: If the equation is non-homogeneous ($f(x) \neq 0$) extra work must be done in order to find the 'particular integral', $y_p(x)$ - this is any function that satisfies the equation. i.e. when $y_p(x)$ is substituted into the LHS it results in the given non-zero RHS.

  There is no general method for finding $y_p(x)$. For 2^{nd} order ODEs with *constant coefficients*, if $f(x)$ contains only polynomial, exponential, sine or cosine terms, we can test a trial function of similar form that contains undetermined parameters and substitute it into
$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0.$$
  allowing us to determine the parameters. We can then deduce $y_p(x)$. Here are some standard trial functions:

  - If $f(x) = ae^{rx}$ try
$$y_p(x) = be^{rx}.$$
  - If $f(x) = a_1 \sin rx + a_2 \cos rx$ (a_1 or a_2 may be 0) then try
$$y_p(x) = b_1 \sin rx + b_2 \cos rx.$$
  - If $f(x) = a_0 + a_1 x + ... + a_N x^N$ (some a_m may be 0) then try
$$y_p(x) = b_0 + b_1 x + ... + b_N x^N.$$
  - If $f(x)$ is a sum or product of any of the above try a $y_p(x)$ that is a sum or product of the corresponding trial functions.

    Note: this method will fail if $y_p(x)$ contains any terms already included in $y_c(x)$. If this is the case multiply the trial function by the smallest power of x possible until it is no longer found in $y_c(x)$.

- **General solution**: The general solution to a homogeneous ODE is the complementary function
$$y(x) = y_c(x).$$
The general solution to a non-homogeneous ODE is given by the complementary function combined with the particular integral
$$y(x) = y_c(x) + y_p(x).$$
Note: If $f(x) \neq 0$, we assume that $f(x) = 0$ in order to find the complementary function, *before* finding the particular integral.

Solving linear 2$^{\text{nd}}$ order ODEs with constant coefficients

- **Linear 2$^{\text{nd}}$ order ODEs**: Linear 2$^{\text{nd}}$ order ODEs have the form
$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$
For 2$^{\text{nd}}$ order ODEs the complementary function is given by
$$y_c(x) = c_1y_1(x) + c_2y_2(x)$$
- **Finding the complementary function**:
  - **Step 1** Find auxiliary equation
  - **Step 2** Find roots of the auxiliary equation
  - **Step 3** Find general solution
- **Homogeneous linear 2nd order ODEs**: $f(x) = 0$.

  Find the complementary function for the following equations:

  e.g. PDF3.1 Real roots
$$y'' + 5y' + 6y = 0$$
Find the auxiliary equation by substituting $y = Ae^{\lambda x}$
$$\frac{d^2}{dx^2}(Ae^{\lambda x}) + 5\frac{d}{dx}(Ae^{\lambda x}) + 6(Ae^{\lambda x}) = 0$$
$$Ae^{\lambda x}(\lambda^2 + 5\lambda + 6) = 0$$
Given that $Ae^{\lambda x} = y = 0$ is the trivial case we can assume $Ae^{\lambda x} \neq 0$, hence
$$\lambda^2 + 5\lambda + 6 = 0$$
Find the roots $(\lambda + 3)(\lambda + 2) = 0$: $\lambda = -3, -2$. Distinct real roots! Solution is
$$y(x) = c_1e^{-3x} + c_2e^{-2x}$$

e.g. PDF3.2 Repeated roots

$$y'' + 6y' + 9y = 0$$

Find the auxiliary equation by substituting $y = Ae^{\lambda x}$

$$\lambda^2 + 6\lambda + 9 = 0$$

Find the roots $(\lambda + 3)^2 = 0$: $\lambda = -3, -3$. Repeated root! Solution is

$$y(x) = (c_1 + c_2 x)e^{-3x}$$

Not this:

$$y(x) = (c_1 + c_2)e^{-3x} = c_3 e^{-3x}$$

which leaves us with only one term. We need to multiply one term by x until it is linearly independent of the other term!

e.g. PDF3.3 Complex roots

$$y'' - 10y' + 26y = 0$$

Find the auxiliary equation by substituting $y = Ae^{\lambda x}$

$$\lambda^2 - 10\lambda + 26 = 0$$

Find the roots

$$\lambda = \frac{10 \pm \sqrt{100 - 4(26)}}{2}$$

$$= \frac{10 \pm \sqrt{-4}}{2} = \frac{10 \pm 2i}{2} = 5 \pm i$$

Complex root!

$$\Rightarrow \alpha = 5, \beta = 1$$

Solution is

$$y(x) = e^{5x}(c_1 \cos x + c_2 \sin x)$$

- **Non-Homogeneous Linear 2nd order ODEs - method of trial functions/method of undetermined coefficients:** $f(x) \neq 0$.

e.g. PDF3.4 Consider the following equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$$

Setting $RHS = 0$ we find it has a complementary equation of the form

$$y_c(x) = (c_1 + c_2 x)e^x$$

Find a particular integral of this equation, and thus the general solution.

The solution is likely to have form e^x to match the RHS of the ODE, however e^x and xe^x are already included in the equation. Try $y_p(x) = bx^2e^x$. Sub into ODE to determine b

$$\frac{d^2}{dx^2}(bx^2e^x) - 2\frac{d}{dx}(bx^2e^x) + bx^2e^x = e^x$$

$$\frac{d}{dx}(2bxe^x + bx^2e^x) - 4bxe^x - 2bx^2e^x + bx^2e^x = e^x$$

$$2be^x + 2bxe^x + 2bxe^x + bx^2e^x - 4bxe^x - 2bx^2e^x + bx^2e^x = e^x$$

$$2b + 2bx + 2bx + bx^2 - 4bx - 2bx^2 + bx^2 = 1$$

$$2b = 1$$

$$b = \frac{1}{2}$$

Thus particular integral is

$$\frac{1}{2}x^2e^x$$

General solution given by $y = y_c + y_p$

$$y(x) = (c_1 + c_2x)e^x + \frac{1}{2}x^2e^x$$

Note: It is good practice to check your solutions work by substituting them back into the original equation and checking you obtain the RHS.

4 Laplace Transforms, and Legendre and Euler Equations

Key Points

- Solving 2$^{\text{nd}}$ order ODEs using Laplace transforms
- Solving Legendre equations
- Solving Euler equations

Solving linear 2$^{\text{nd}}$ order ODEs with constant coefficients (cont.)

- **Laplace transform method:** Laplace transforms are useful for solving linear ODEs with constant coefficients. Taking the Laplace tranform of such an equation tranforms it into a purely algebraic equation. Once this equation has been solved an inverse Laplace transform can be applied to obtain the general solution of the ODE. Laplace can be used as an alternative to the trial function method.

  The Laplace transform of a function $f(x)$ is defined as

  $$\bar{f}(s) \equiv \int_0^\infty e^{-sx} f(x) dx.$$

  The Laplace transform of the n$^{\text{th}}$ derivative of $f(x)$ is given by

  $$\bar{f^{(n)}}(s) = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f^{'}(0) - ... - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

  $\bar{f}(s)$ is the Laplace transform of the required solution; we solve for this. The primes and bracketed superscripts denote differentiation w.r.t x. e.g. The transform for a 2$^{\text{nd}}$ order derivative is given by

  $$\bar{f^{(n)}}(s) = s^2 \bar{f}(s) - s f(0) - f^{'}(0).$$

  These equations, combined with the table of Laplace transforms allows us to solve linear ODEs with constant coefficients.

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\lvert a\rvert$
$\cosh at$	$s/(s^2-a^2)$	$\lvert a\rvert$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0) = \begin{cases} 1 & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$	e^{-st_0}/s	0

Table 13.1 Standard Laplace transforms. The transforms are valid for $s > s_0$.

e.g. PDF4.1 Find the Laplace transform of $f(x) = c$.

Start with the definition

$$\bar{f}(s) = \int_0^\infty ce^{-sx}dx$$

$$= \left[-\frac{c}{s}\frac{1}{e^{sx}} \right]_0^\infty = -\frac{c}{s}\frac{1}{e^\infty} + \frac{c}{s}\frac{1}{e^0} = \frac{c}{s}$$

Note that the Laplace transform of a product of functions is not equal to the product of the transformed functions, i.e.

$$\mathcal{L}[fg] \neq \mathcal{L}[f]\mathcal{L}[g]$$

The relationship becomes complicated when both functions are variable. We will restrict ourselves for now to the case of a constant multiplied by a function, which for constant a has the simple result

$$\mathcal{L}[ag] = a\mathcal{L}[g]$$

e.g. PDF4.2 Solve the following ODE using the Laplace transform method

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{-x}$$

subject to the boundary conditions $y(0) = 2, y'(0) = 1$.

First, note the required substitutions

$$\mathcal{L}\left[\frac{d^2y}{dx^2}\right](s) = s^2\bar{y}(s) - sy(0) - y^{'}(0)$$

$$\mathcal{L}\left[\frac{dy}{dx}\right](s) = s\bar{y}(s) - y(0)$$

$$\mathcal{L}[y](s) = \bar{y}(s)$$

$$\mathcal{L}\left[e^{-x}\right](s) = \frac{1}{s+1}$$

Now, take the Laplace transform of the equation

$$s^2\bar{y}(s) - sy(0) - y^{'}(0) - 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{2}{s+1}$$

Collecting terms and subbing in our boundary conditions, this reduces to

$$(s^2 - 3s + 2)\bar{y}(s) - 2s + 5 = \frac{2}{s+1}$$

Now solve for $\bar{y}(s)$ using partial fractions

$$(s^2 - 3s + 2)\bar{y}(s) = \frac{2 + (2s-5)(s+1)}{s+1} = \frac{2s^2 - 3s - 3}{s+1}$$

$$\bar{y}(s) = \frac{2s^2 - 3s - 3}{(s+1)(s-1)(s-2)} = \frac{1}{3(s+1)} + \frac{2}{s-1} - \frac{1}{3(s-2)}$$

Finally, we can reverse the Laplace transform to find the general solution to the ODE

$$y(x) = \frac{1}{3}e^{-x} + 2e^{x} - \frac{1}{3}e^{2x}$$

Recall

Factor in denominator	**Term in partial fraction decomposition**
$ax + b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Source: http://tutorial.math.lamar.edu/Classes/Alg/PartialFractions.aspx

Solving linear ODEs with variable coefficients

- **Legendre linear equations:** Legendre's linear equation has the form

$$a_n(\alpha x+\beta)^n\frac{d^n y}{dx^n}+...+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0y=f(x)$$

where α, β and a_n are constants. The defining feature of this type of ODE is that the coefficient function of each derivative has a power of x equal to the order of the derivative (assuming $\alpha \neq 0$).

The 2$^{\text{nd}}$ order Legendre is therefore

$$a_2(\alpha x+\beta)^2\frac{d^2 y}{dx^2}+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0y=f(x)$$

and may be solved by making the substitution $\alpha x+\beta=e^t$. This gives

$$\frac{dy}{dx}=\frac{dt}{dx}\frac{dy}{dt}=\frac{\alpha}{\alpha x+\beta}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{\alpha^2}{(\alpha x+\beta)^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

The key point here is that making this substitution allows us to transform a difficult to solve ODE with variable coefficients (when in terms of y and x) into an easy to solve ODE with constant coefficients (when in terms of y and t).

e.g. PDF4.3 Solve the following equation using the substitute $x+2=e^t$

$$(x+2)^2y''+(x+2)y'+y=3x$$

Note that the equivalence between the following two operations

$$\frac{d}{dx}=\frac{dt}{dx}\frac{d}{dt}$$

Substitution: $x+2=e^t$. Find the derivatives. Note that $t=\ln|x+2|$ and $dt/dx=1/(x+2)$

$$y'=\frac{dt}{dx}\frac{dy}{dt}=\frac{1}{x+2}\dot{y}$$

$$\begin{aligned}y''=\frac{d}{dx}\frac{dy}{dx}&=-\frac{1}{(x+2)^2}\dot{y}+\frac{1}{x+2}\frac{dt}{dx}\frac{d\dot{y}}{dt}\\&=\frac{1}{(x+2)^2}\left[\ddot{y}-\dot{y}\right]\end{aligned}$$

Sub into ODE, $(x+2)$ terms cancel, and $RHS=3x=3(e^t-2)$

$$\ddot{y}-\dot{y}+\dot{y}+y=\ddot{y}+y=3(e^t-2)$$

Find auxiliary equation

$$\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$$

Complementary equation

$$y_c(t) = c_1 e^{it} + c_2 e^{-it}$$

Particular equation $3e^t - 6$ so try $Ae^t + B$

$$\frac{d^2}{dt^2}(Ae^t + B) + Ae^t + B = 3e^t - 6$$

$$Ae^t + Ae^t + B = 2Ae^t + B = 3e^t - 6$$

$\Rightarrow A = \frac{3}{2}$ and $B = -6$

$$y_p(t) = \frac{3}{2}e^t - 6$$

$$y(t) = c_1 e^{it} + c_2 e^{-it} + \frac{3}{2}e^t - 6$$

Change variable t to x, recall $t = \ln|x+2|$

$$y(x) = c_1(x+2)^i + c_2(x+2)^{-i} + \frac{3}{2}(x+2) - 6$$

$$= c_1(x+2)^i + c_2(x+2)^{-i} + \frac{3}{2}x - 3$$

- **Euler linear equations:** Euler's equation is a special case of Legendre's equation, where $\alpha = 1$ and $\beta = 0$

$$a_n x^n \frac{d^n y}{dx^n} + ... + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

The 2^{nd} order Euler is therefore

$$a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

which may be solved by making the substitution $x = e^t$, though if $f(x) = 0$ substituting $y = x^\lambda$ leads to a simple algebraic equation in λ.

e.g. PDF4.4 Solve the following equation using the substitute $x = e^t$

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

Work out the derivatives and note $t = \ln|x|$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt} = \frac{1}{e^t}\dot{y}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{dt}{dx}\frac{d\dot{y}}{dt} = -\frac{1}{e^{2t}}\frac{dy}{dt} + \frac{1}{e^{2t}}\frac{d\dot{y}}{dt} = \frac{1}{e^{2t}}\left(\ddot{y} - \dot{y}\right)$$

Sub into the equation and cancel e^t terms

$$e^{2t}\frac{1}{e^{2t}}\left(\ddot{y} - \dot{y}\right) + e^t \frac{1}{e^t}\dot{y} - 4y = 0$$

$$(\ddot{y} - \dot{y}) + \dot{y} - 4y = 0$$
$$\ddot{y} - 4y = 0$$

Find the auxiliary equation

$$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

Thus

$$y = c_1 e^{2t} + c_2 e^{-2t} = c_1 x^2 + c_2 x^{-2}$$

e.g. PDF4.5 Solve the following equation using the substitute $y = x^\lambda$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

Make the substitution

$$\lambda(\lambda - 1)x^\lambda + \lambda x^\lambda - 4x^\lambda = 0$$
$$(\lambda^2 - 4)x^\lambda = 0$$

Since $x^\lambda = 0$ is only true in the trivial case where $x = 0$ we can conclude that

$$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

Thus

$$y = c_1 x^2 + c_2 x^{-2}$$

5 The Wronskian Method

Key Points

- Calculating the Wronskian
- Method of variation of parameters/Wronskian method

Solving linear ODEs with variable coefficients

- **Wronskian:** The Wronskian of a set of functions is the determinant of a square matrix of those functions and their derivatives. It can be used to determine if the functions are linearly independent. The Wronskian for an n^{th} order ODE has the following form

$$W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & \dots \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix}$$

$W \neq 0$ between specified limits when all of the functions are linearly independent and $W = 0$ when they are not. Recall that the general solutions to an n^{th} order ODE must be constructed from n linearly independent solutions. A zero W would indicate that you need to seek further independent solutions.

The Wronskian for a 2^{nd} order ODE can be expressed as

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

e.g. PDF5.1 Check that the following solutions to a 2^{nd} order ODE are linearly independent.

$$y_1 = e^x$$
$$y_2 = e^{2x}$$

Calculate the Wronskian

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x.2e^{2x} - e^x.e^{2x} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

Hence y_1 and y_2 are linearly independent.

e.g. PDF5.2 Check that the following solutions to a 2^{nd} order ODE are not linearly independent.

$$y_1 = x^2 + 1$$
$$y_2 = 2x^2 + 2$$

Calculate the Wronskian

$$W = \begin{vmatrix} x^2+1 & 2x^2+2 \\ 2x & 4x \end{vmatrix} = (x^2+1)4x - 2x(2x^2+2) = 4x^3+4x-4x^3-4x = 0$$

Hence y_1 and y_2 are not linearly independent.

*** The following is for interest and not examinable ***

- **Wronskian method or method of variation of parameters (background):** This method can be used to determine the particular solution of an ODE and works with constant or variable coefficients.

  Consider the general form of an n^{th} order linear ODE

  $$a_n(x)\frac{d^n y}{dx^n} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

  The complementary function of this ODE is

  $$y_c(x) = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x)$$

  which is the general solution to the homogeneous part $[f(x) = 0]$. Let's assume that the particular integral y_p can be expressed in a similar form to y_c, but with the constant coefficients replaced with functions of x to ensure linear independence (recall that if y_p contains a term that already exists in y_c, we multiply it by powers of x until there is no longer a duplicated term).

  $$y_p(x) = k_1(x)y_1(x) + k_2(x)y_2(x) + ... + k_n(x)y_n(x)$$

  This no longer satisfies the complementary equation, but if $f(x) \neq 0$ and we pick suitable $k_i(x)$ terms it could be made equal to $f(x)$, providing us with a particular integral.

  We have n arbitrary functions, meaning we need n constraints on our system to determine what they are. The simplest way to obtain them is to differentiate y_p $n-1$ times

  $$y_p' = k_1y_1' + k_2y_2' + ... + k_ny_n' + [k_1'y_1 + k_2'y_2 + ... + k_n'y_n].$$

  We are free to choose our constraints as we wish, so let's say the bracketed term is equal to zero, leaving us with

  $$y_p' = k_1y_1' + k_2y_2' + ... + k_ny_n'.$$

  Which we can differentiate and discard the bracketed term again

  $$y_p'' = k_1y_1'' + k_2y_2'' + ... + k_ny_n''.$$

  We repeat this process until we have $(n-1)$ equations. We still want one more, in which we will not set the bracketed term to zero. The m^{th} derivative is given by

  $$y_p^{(m)} = k_1y_1^{(m)} + k_2y_2^{(m)} + ... + k_ny_n^{(m)}.$$

If we differentiate one more time to obtain the n^{th} derivative

$$y_p^{(n)} = k_1 y_1^{(n)} + k_2 y_2^{(n)} + ... + k_n y_n^{(n)} + \left[k_n y_n^{(n)} + k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + ... + k_n' y_n^{(n-1)} \right].$$

Now we have n equations, of which the first $n-1$ are of the same form. Let's substitute them back into our ODE

$$\sum_{j=1}^{n} k_j \left[a_n y_j^{(n)} + ... + a_1 y_j' + a_0 y_j \right] + a_n \left[k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + ... + k_n' y_n^{(n-1)} \right] = f(x).$$

But since the functions y_j are solutions of the complementary equation, we have (for all j)

$$a_n y_j^{(n)} + ... + a_1 y_1' + a_0 y_j = 0.$$

So the sum term disappears, leaving

$$a_n \left[k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + ... + k_n' y_n^{(n-1)} \right] = f(x).$$

So, we are left with a set of n linearly independent equations

$$\begin{aligned}
k_1'(x) y_1(x) + k_2'(x) y_2(x) + ... + k_n'(x) y_n(x) &= 0 \\
k_1'(x) y_1'(x) + k_2'(x) y_2'(x) + ... + k_n'(x) y_n'(x) &= 0 \\
&... \\
k_1'(x) y_1^{(n-2)}(x) + k_2'(x) y_2^{(n-2)}(x) + ... + k_n'(x) y_n^{(n-2)}(x) &= 0 \\
k_1'(x) y_1^{(n-1)}(x) + k_2'(x) y_2^{(n-1)}(x) + ... + k_n'(x) y_n^{(n-1)}(x) &= \frac{f(x)}{a_n(x)}
\end{aligned}$$

Since these equations are linearly independent the determinant of their coefficients must be equal to a non-zero Wronskian.

*** *** *** ***

Aside - Cramer's rule: In order to proceed we will need to make use of Cramer's rule for a 2x2 matrix. Given a system of linear equations this rule allows you to solve for just one of the variables without having to solve the whole system of equations. Take the following linear equations

$$ax + by = m$$

$$cx + dy = n$$

Find three determinants. D, D_x and D_y.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$D_x = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$$

$$D_y = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$$

The solutions are given by $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$

- **Wronskian method or method of variation of parameters (derivation for 2^{nd} order ODE):**

  Note: in order to use the Wronskian method to solve an ODE we need to know the complete y_c. This is more difficult to determine for ODEs with variable coefficients than ODEs with constant coefficients. In practice we will focus on examples with constant coefficients in order to learn how the method works. In the workshops you will encounter a method called 'reduction of order'; it allows you to find y_2 if you already know y_1 and is one method that could be used if you ever came across an ODE with variable coefficients and wanted to apply the Wronskian method.

  Consider a 2^{nd} order ODE

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$$

This equation has the homogeneous solution

$$y_c = c_1 y_1 + c_2 y_2.$$

We want to find two functions $k_1(x)$ and $k_2(x)$ such that

$$y_p(x) = k_1(x)y_1(x) + k_2(x)y_2(x) = f(x).$$

i.e. we want y_p to be constructed from the same solutions as y_c. This will only work if the coefficients k_1 and k_2 are functions of x, rather than constant, or we will not find linearly independent solutions.

We start by finding y_p' and y_p'' and making an important assumption,

$$k_1' y_1 + k_2' y_2 = 0.$$

This simplifies our derivatives and helps us find k_1 and k_2. Notice that this is the first of the set of n equations we derive in the background section above.

$$y_p' = k_1 y_1' + k_1' y_1 + k_2 y_2' + k_2' y_2 = k_1 y_1' + k_2 y_2'$$

$$y_p'' = k_1 y_1'' + k_1' y_1' + k_2 y_2'' + k_2' y_2'$$

Now sub these into our ODE

$$[k_1 y_1'' + k_1' y_1' + k_2 y_2'' + k_2' y_2'] + p(x)\,[k_1 y_1' + k_2 y_2'] + q(x)\,[k_1 y_1 + k_2 y_2] = f(x).$$

Rearranging

$$(k_1' y_1' + k_2' y_2') + k_1\,[y_1'' + p(x)y_1' + q(x)y_1] + k_2\,[y_2'' + p(x)y_2' + q(x)y_2] = f(x)$$

Now, since y_1 and y_2 are solutions to y_c, the homogeneous equation, the terms in square brackets must equal zero, and so

$$k_1'y_1' + k_2'y_2' = f(x)$$

This equation combined with the assumption we made can be used to solve the equation using Cramer's rule. Our determinants take the following forms

$$k_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-f(x)}{W(y_1, y_2)}y_2 \rightarrow k_1 = -\int \frac{y_2 f(x)}{W(y_1, y_2)}dx$$

$$k_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{f(x)}{W(y_1, y_2)}y_1 \rightarrow k_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)}dx$$

This process allows us to use the Wronskian to determine the coefficients of the solutions in y_p, allowing us to combine it with y_c and reach a general solution to an inhomogeneous linear ODE.

Note that this method still works if the coefficient of the 2^{nd} derivitive term is not equal to 1, with a small modification. Assume our 2^{nd} order ODE is

$$ay'' + by' + cy = f(x)$$

Then our k_1' and k_2' become

$$k_1' = \frac{-f(x)}{aW(x)}y_2$$
$$k_2' = \frac{f(x)}{aW(x)}y_1$$

Note that $W(y_1, y_2) \equiv W(x)$. This is not necessary if the equation is arranged into the standard form shown above where the 2^{nd} derivitive term has no coefficient.

- **Wronskian method or method of variation of parameters (example):**

  e.g. PDF5.3 Solve the following equation using the Wronskian method

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

First, find y_c. The auxiliary equation is given by

$$\lambda^2 - 4\lambda + 4 = 0 \rightarrow \lambda = 2, 2$$

So,

$$y_c = (c_1 + xc_2)e^{2x}$$

This allows us to identify $y_1 = e^{2x}$ and $y_2 = xe^{2x}$. Now let's find the Wronskian,

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{2x}(e^{2x} + 2xe^{2x}) - 2e^{2x}.xe^{2x} = e^{4x}$$

Note: This will still work if we swap y_1 and y_2, the sign of W will flip but it all works out in the end!

Now we have everything we need to determine the coefficients of y_p

$$k_1' = \frac{-f(x)}{W(x)} y_2 = \frac{-(x+1)e^{2x}}{e^{4x}} x e^{2x} = -x^2 - x$$

$$k_2' = \frac{f(x)}{W(x)} y_1 = \frac{(x+1)e^{2x}}{e^{4x}} e^{2x} = x + 1$$

Integrating w.r.t x

$$k_1 = -\frac{x^3}{3} - \frac{x^2}{2} + c_3$$

$$k_2 = \frac{x^2}{2} + x + c_4$$

Our particular solution is therefore

$$\begin{aligned} y_p = k_1 y_1 + k_2 y_2 &= \left(-\frac{x^3}{3} - \frac{x^2}{2} + c_3\right) e^{2x} + \left(\frac{x^2}{2} + x + c_4\right) x e^{2x} \\ &= e^{2x}\left(\frac{x^3}{2} - \frac{x^3}{3} + x^2 - \frac{x^2}{2} + c_4 x + c_3\right) \\ &= e^{2x}\left(\frac{x^3}{6} + \frac{x^2}{2} + c_4 x + c_3\right) \end{aligned}$$

Therefore our general solution is

$$y = (c_1 + x c_2)e^{2x} + y_p = e^{2x}\left(\frac{x^3}{6} + \frac{x^2}{2} + c_5 x + c_6\right)$$

Note: The above example is solved in the traditional way, by finding $y = y_c + y_p$, to match the logic of previous examples. This is fine, but a curious result arises: the particular solution contains two terms that were already in the complementary function. This is a natural result of determining y_p based on the solutions to y_c, i.e. y_1 and y_2. In fact using this method, the general solution $y = y_p$ is equivalent to $y = y_c + y_p$, because y_p will naturally contain y_c. There is not a contradiction here, but a subtle point.

My advice to you is to short-cut the last line of the solution, which yields the same result and do it this way in future problems to avoid confusion. Instead of $y = y_c + y_p$ just state $y = y_p$ when solving problems with this method. So, in the case of e.g. 5.4, we would state that

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$$y_p = k_1(x) e^{2x} + k_2(x) x e^{2x}$$

and instead of

$$y = y_c + y_p = e^{2x}\left(\frac{x^3}{6} + \frac{x^2}{2} + c_5 x + c_6\right)$$

we write

$$y = y_p = e^{2x}\left(\frac{x^3}{6} + \frac{x^2}{2} + c_4 x + c_3\right)$$

as the general solution. It should be clear that these are equivalent, given that the constant terms are abitrary.

Perhaps an example will clarify this. Consider a problem that has a homogeneous solution

$$y_c = Ae^x + Be^{2x}$$

Using the method of variation of parameters we are going to solve

$$y_p = C(x)e^x + D(x)e^{2x}$$

Now, imagine our solution to y_p yielded the results

$$C(x) = x + E$$

$$D(x) = x + F$$

Giving a general solution

$$y = (x + E)e^x + (x + F)e^{2x}$$

If we can expand this out we note that the constant coefficients give us an equivalent expression to the complementary function, with undetermined constant coefficients. Compare

$$Ee^x + Fe^{2x} \equiv Ae^x + Be^{2x}$$

If we desired to represent the solution in the traditional form we could deconstruct this solution so y_c and y_p are displayed separately, i.e.

$$y = \left[Ee^x + Fe^{2x}\right] + \left[xe^x + xe^{2x}\right] = [y_c] + [y_p]$$

which should look more familiar compared to earlier problems, but this is unecessary.

6 Power Series, and Ordinary/Singular Points

Key Points

- Power series
- Finding singular points

Series solutions to linear ODEs

- **Introduction:** Up to now we have examined a handful of techniques designed to solve homogeneous and inhomogeneous linear ODEs with constant or variable coefficients, by writing the solutions as elementary functions or integrals. However, in general those with variable coefficients cannot be solved this way.

  An alternative approach is to obtain solutions in the form of convergent series, which can be evaluated numerically. It is worth noting that there is no distinct boundary between series solutions and using elementary functions, as those functions can be written as convergent series themselves (i.e. using the relevant Taylor series). In fact, some series are so common that they are given their own names, e.g. $\sin x$, $\cos x$ and e^x.

- **Aside: Power series -** A power series is an infinite polynomial and has the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

  were a_n are constants. They occur often in physics and are useful as for $|x| < 1$ the later terms become small and may be discarded. e.g. if $x = 0.1$ for the series

$$S(x) = 1 + x + x^2 + x^3 + ...$$

  then the value of the series is 1 for one term, 1.1 for two terms, 1.11 for three, etc. If you make a measurement with an accuracy limited to 2 decimal places then any terms after x^2 can be ignored.

  When dealing with infinite series it is important to consider if the series will converge or diverge at a given point, as this will affect which method we use to solve the equation at this point. Taylor series solutions work well for 'ordinary points' but 'singular points' require more advanced techniques (such as the Frobenius series).

  - **Convergence:** The property of an infinite series to approach a limit (fixed value) as the series progresses - the sum of the series. In order for a series to converge on a value the individual terms in a series must themselves converge on zero (this condition is necessary, but does not guarantee a series will converge). e.g. The reciprocals of factorials are a convergent series

$$\frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + ... = e \text{ (Euler's number)}$$

- **Divergence:** If an infinite series does not converge, it usually diverges towards ∞. e.g. The reciprocals of positive integers are a divergent series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+...=\infty$$

Alternatively, the divergence may take the form of an infinitely oscillating sequence, such as $\Sigma(-1)^n$, which bounces between 0 and -1.

- **Testing convergence:** One way of testing whether a series converges is to apply D'Alembert's ratio test. This ratio is a simple way of comparing two consecutive terms in a series in the limit as $n \to \infty$. Assume for each term in a series (u_n, where all $u_n > 0$ and $0 \leq n \leq \infty$), there exists a value r (the radius of convergence) such that

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = r \begin{cases} \text{If } r < 1 \text{ power series converges.} \\ \text{If } r = 1 \text{ power series may diverge or converge - inconclusive.} \\ \text{If } r > 1 \text{ power series diverges.} \end{cases}$$

e.g. For the above reciprocals factorial series:

$$r = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)!}{1/n!}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0.$$

We find as $n \to \infty$, $r \to 0$, hence the series converges.

e.g. For the reciprocals of positive integers series:

$$r = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)}{1/n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1.$$

We find as $n \to \infty$, $r \to 1$. In this case we would need to use another method to test the convergence of the series.

Consider the general power series

$$P(x) = a_0 + a_1 x + a_2 x^2 + ...$$

D'Alembert's ratio test tells us that $P(x)$ converges if

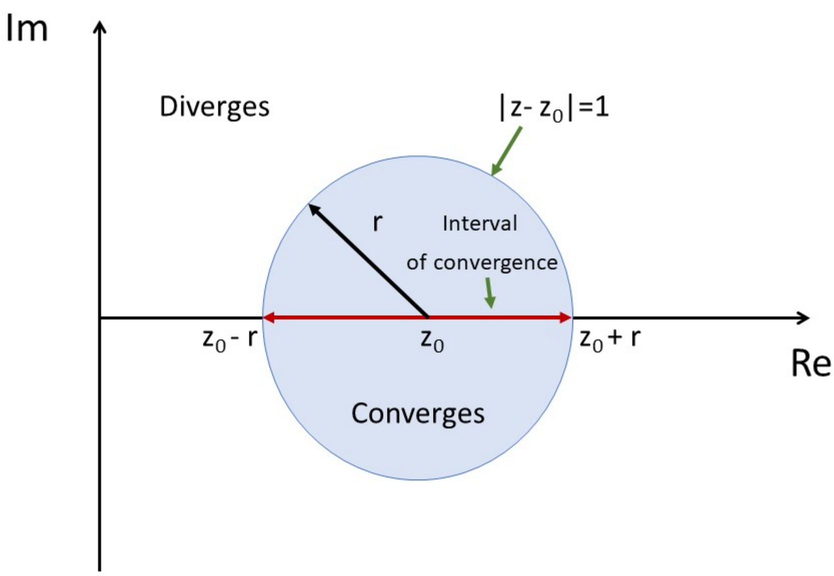
$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}|x|}{a_n}\right| = |x| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$$

Note that there is an x term, because each $(n+1)^{\text{th}}$ term will have a power of x one greater than the n^{th} term, cancelling in the fraction to a single x. So the convergence of $P(x)$ depends on the value of x; there will be a range of values of x for which $P(x)$ will converge, an *interval of convergence.* Section 4.3 of Riley explains how to determine this interval.

– **Radius of convergence:** This is the radius of the largest disc in the complex plane in which the series will converge (i.e. all points within the radius do converge, some outside of the radius may). For a real function with real variables we can describe an interval of convergence (1-dimensional), say $-1 \leq x \leq 1$. But if the function is complex then the interval is best described in 2 dimensions, hence as the radius of the largest disc of convergence.

For a radius of convergence r, centred on the point $z = z_0$

$$|z - z_0| \begin{cases} < r \text{ series converges.} \\ > r \text{ series diverges.} \end{cases}$$



– **Definition of a power series:** Power series are best defined in complex terms

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where z_0 is a complex constant, the centre of the disc of convergence, c_n is the n^{th} complex coefficient (which may be a variable function in z) and z is a complex variable.

- **Complex variables:** Let's look at the canonical form of a 2^{nd} order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0$$

Up to now we have assumed that $y(x)$ is a real function of a real variable x, but this is not always the case. We can easily generalise the homogeneous linear ODE to include functions of a complex variable z

$$y'' + p(z)y' + q(z)y = 0$$

where $y = y(z)$ and $y' = dy/dz$. We may treat differentiation w.r.t z just as we would w.r.t x.

- **Ordinary and singular points:** Let's take some point $z = z_0$. We are interested in evaluating the nature of $p(z)$ and $q(z)$ at this point. Expressed as complex power series the functions $p(z)$ and $q(z)$ have the following forms

$$p(z) = \sum_{n=0}^{\infty} p_n(z - z_0)^n, \qquad q(z) = \sum_{n=0}^{\infty} q_n(z - z_0)^n.$$

If $p(z)$ and $q(z)$ both converge to a finite value at $z = z_0$ it is an ordinary point. If $p(z)$ or $q(z)$ or both diverge at $z = z_0$ then it is a singular point.

If the ODE has a singular point it may still have a finite solution at that point (regular singular point), or it may not have a solution at that point (irregular singular point). To test if a singular point is regular or not we check if the following conditions converge or diverge,

$$(z - z_0)p(z) \quad \text{and} \quad (z - z_0)^2 q(z).$$

If both of these conditions converge to a finite value we have a regular singular point. If one or both of these conditions diverge then we have an irregular singular point.

e.g. PDF6.1 Legendre's equation has the form

$$(1 - z^2)y'' - 2zy' + \ell(\ell + 1)y = 0$$

where ℓ is a constant. Show that $z = 0$ is an ordinary point and $z = \pm 1$ are regular singular points of this equation.

First, let's rearrange the equation into the canonical form by dividing by the coefficient of the 2$^{\text{nd}}$ order derivative

$$y'' - \frac{2z}{1 - z^2}y' + \frac{\ell(\ell + 1)}{(1 - z^2)}y = 0$$

Now we identify $p(z)$ and $q(z)$

$$p(z) = \frac{-2z}{1 - z^2} = \frac{-2z}{(1 + z)(1 - z)}$$

$$q(z) = \frac{\ell(\ell + 1)}{1 - z^2} = \frac{\ell(\ell + 1)}{(1 + z)(1 - z)}$$

Now we can check the nature of the points at the first boundary condition, $z = 0$

$$p(0) = \frac{-2 \times 0}{(1 + 0)(1 - 0)} = \frac{0}{1} = 0$$

$$q(0) = \frac{\ell(\ell+1)}{(1+0)(1-0)} = \frac{\ell(\ell+1)}{1} = const$$

Both $p(0)$ and $q(0)$ converge, hence $z = 0$ is an ordinary point.

Now we can check the nature of the points at the second boundary condition, $z = 1$

$$p(1) = \frac{-2 \times 1}{(1+1)(1-1)} = \frac{-2}{0} = -\infty$$

$$q(1) = \frac{\ell(\ell+1)}{(1+1)(1-1)} = \frac{\ell(\ell+1)}{0} = \infty$$

Both $p(1)$ and $q(1)$ diverge, hence $z = 1$ is a singular point.

Now we can check for the nature of the singular point at $z = 1$ using our test conditions.

$$(z-1)p(1) = \frac{-2z(z-1)}{(1+z)(1-z)} = \frac{2z}{(1+z)} = \frac{2 \times 1}{(1+1)} = 1$$

$$(z-1)^2 q(1) = \frac{\ell(\ell+1)(z-1)^2}{(1+z)(1-z)} = \frac{-\ell(\ell+1)(z-1)}{(1+z)} = \frac{-\ell(\ell+1)(1-1)}{(1+1)} = 0$$

Both $(z-1)p(1)$ and $(z-1)^2 q(1)$ converge, hence $z = 1$ is a regular singular point.

To finish, repeat the steps for the final boundary condition, $z = -1$.

$$p(-1) = \frac{-2 \times -1}{(1-1)(1+1)} = \frac{2}{0} = \infty$$

$$q(-1) = \frac{\ell(\ell+1)}{(1-1)(1+1)} = \frac{\ell(\ell+1)}{0} = \infty$$

Both $p(-1)$ and $q(-1)$ diverge, hence $z = -1$ is a singular point.

$$(z+1)p(-1) = \frac{-2z(z+1)}{(1+z)(1-z)} = \frac{-2z}{(1-z)} = 1$$

$$(z+1)^2 q(-1) = \frac{\ell(\ell+1)(z+1)^2}{(1+z)(1-z)} = \frac{\ell(\ell+1)(z+1)}{(1-z)} = 0$$

Both $(z+1)p(-1)$ and $(z+1)^2 q(-1)$ converge, hence $z = -1$ is a regular singular point.

Note: with the Legendre equation it was easy to sub in values for z and arrive at sound conclusions regarding the nature of the points. In general this will not be true.

The general approach is to examine what happens in the limit $z \to z_0$. The weekly problems examine the Bessel equation. If you try to sub in values for one of the singularities you will get $0 \div 0$, which is undefined. However, the $\lim_{z \to 0}$ will give an answer of 1.

- **Singular points at infinity:** For testing singular points at infinity you can make a change of variable $x = 1/\omega$ and take the limit where $\omega \to 0$.

7 Taylor Series Solutions at Ordinary Points

Key Points

- Taylor series solutions

Series solutions to linear ODEs (ctud)

- **Series solutions:** Series solutions are a relatively straightforward way to assess the solution of an ODE at a given point. They can be truncated to give approximate local solutions to the ODE. Or they can be taken to the n^{th} degree in an effort to seek a general solution.

  Last time we looked at determining the nature of points of an ODE. We found that a given point $z = z_0$ can be an ordinary, regular singular or irregular singular point. Once we know the nature of a point we can decide how to approach a series solution for the ODE at that point.

  For ordinary points we can find the Taylor series of the ODE at that point. Singular points require a more general approach, where we generate a Frobenius series. We will focus on solving ODEs around ordinary points.

- **Taylor series:** The Taylor series is a series expansion of a function about a point. The 1D Taylor expansion of a real function $f(x)$ about a point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

  This expression can be considered as a statement that any real function of x can be represented as the sum of an infinite polynomial, as long as in a given range of x $f(x)$ is a continuous, single-valued function with continuous derivatives up to the n^{th} order (where $n = \infty$).

  The reason we have $x-a$ terms rather than just x terms is that it generalises the function, allowing easy access to information about the behaviour of the function near some point a, a distance from x, by testing the limits of the function as $x \to a$. We often simplify things by setting $a = 0$, producing what is known as a Maclaurin series.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n + ...$$

  The main advantage of a Taylor series is that it allows you to easily calculate the values of even highly complex functions. Here is $\ln x$ represented as a Taylor series

$$\ln x = \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} + \mathcal{O}(x^4)$$

$\mathcal{O}(x^n)$ means 'terms with orders of x to the power n and higher'. Notice that we cannot express $\ln x$ as a Maclaurin series, since we would have to divide by $a = 0$. The series is often expressed as $\ln|1+x|$ or $\ln|1-x|$ instead.

Here is e^x represented as a Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we test $e^1 = 2.718$ to 3 dp, with 5 terms from the series we get

$$e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708$$

- **Series solutions at ordinary points:** Recall the general form of the 2$^{\text{nd}}$ order complex homogeneous linear ODE

$$y'' + p(z)y' + q(z)y = 0.$$

We can express a solution to this equation as a Taylor series

$$y(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

If we reframe our coordinates and take z_0 as the origin ($z_0 = 0$) then we can simplify this equation, producing a Maclaurin series

$$y(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} a_n z^n.$$

Remember that this series will converge for $|z| < r$, where r is the radius of convergence, which is now simply the distance from $z = 0$ to the nearest singular point.

Since every solution has a finite value at an ordinary point it is always possible to obtain two independent solutions from which we can construct a general solution to the complex homogeneous linear ODE. Since we are dealing primarily with 2$^{\text{nd}}$ order ODEs it would be useful to know what the derivatives of the series solution w.r.t z are

$$y' = \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

To get the RHS terms we are just adding 1 or 2 to each n term that appears, for the first and second derivatives respectively. We are *shifting the index*. Note that it would seem appropriate to start the left hand sums from $n = 1$ and $n = 2$ respectively, but since the first terms are 0 when $n = 0$ we can start from there.

e.g. PDF7.1 Find the series solutions about $z = 0$ of

$$y''(z) + y(z) = 0$$

Here, we can tell by inspection that $z = 0$ is an ordinary point ($p = 0$, $q = 1$) we can go on to find two independent solutions by making the substitutions

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n + \sum_{n=0}^{\infty} a_n z^n = 0$$

Which we can rewrite as

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] z^n = 0$$

For this equation to work we require that each coefficient of z (the square bracket) is equal to zero. If they were were not, as long as $z \neq 0$ (the trivial case) the result will not match the zero RHS.

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad \text{for } n \geq 0.$$

This is a two-term recurrance relation that allows us to readily calculate the even coefficients if we start from a_0, or odd coefficients if we start from a_1. This in turn allows us to find two independent solutions of the ODE. We can set either $a_0 = 0$ or $a_1 = 0$.

Let's set $a_0 = 1$ and let $a_1 = 0$. So

$$a_0 = 1$$

$$a_1 = 0$$

$$a_2 = -\frac{a_0}{(0+2)(0+1)} = -\frac{1}{2!}$$

$$a_3 = 0$$

$$a_4 = -\frac{a_2}{(2+2)(2+1)} = -\frac{(-1/2)}{12} = \frac{1}{24} = \frac{1}{4!}$$

$$a_5 = 0$$

Similarly, setting $a_0 = 0$ and letting $a_1 = 1$ gives

$$a_0 = 0$$

$$a_1 = 1$$

$$a_2 = 0$$

$$a_3 = -\frac{a_1}{(1+2)(1+1)} = -\frac{1}{3!}$$

$$a_4 = 0$$

$$a_5 = -\frac{a_3}{(3+2)(3+1)} = \frac{1}{5!}$$

Since $y = \sum_{n=0}^{\infty} a_n z^n$, this gives the solutions

$$y_1(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = cosz$$

$$y_2(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = sinz$$

We can now say that our general solution to the ODE is

$$y = c_1 y_1 + c_2 y_2 = c_1 \cos z + c_2 \sin z$$

We were able to express this solution in a *closed form* (i.e. in terms of elementary functions) - this is not usually the case!

e.g. PDF7.2 Find the series solutions about $z = 0$ of

$$y''(z) - \frac{2}{(1-z)^2} y(z) = 0$$

Again, by inspection we can tell that $z = 0$ is an ordinary point, so we can find two independent solutions by substituting

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}$$

If we sub these into the ODE and mulitply by an expanded $(1-z)^2$ to remove the fraction, we get

$$(1 - 2z + z^2) \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} a_n z^n = 0.$$

Since we have used the negative index term substitutions, when we multiply out the brackets we won't have any terms higher than z^n

$$\sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} n(n-1) a_n z^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n z^n - 2 \sum_{n=0}^{\infty} a_n z^n = 0$$

Now we need to shift the index of each term, so we have only terms in z^n

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - 2 \sum_{n=0}^{\infty} n(n+1) a_{n+1} z^n + \sum_{n=0}^{\infty} (n^2 - n - 2) a_n z^n = 0$$

Reducing this to a single sum we can write

$$\sum_{n=0}^{\infty} (n+1)[(n+2) a_{n+2} - 2n a_{n+1} + (n-2) a_n] z^n = 0$$

Just like the previous example we require that the coefficient of z^n must be zero at each n

$$(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n = 0 \qquad \text{for } n \geq 0.$$

This means we can determine a_2 in terms of a_0 and a_1, and so on for $n \geq 2$. This is a three-term recurrence relation. Three-term recurrence relations and higher are generally a nuisance to solve, however this one has two simple solutions. First lets choose $a_n = a_0$ for all n. If we test this with $a_0 = 1$ we see that it satisfies the condition for the coefficient

$$(n+2) \times 1 - 2n \times 1 + (n-2) \times 1 = 2n - 2n + 2 - 2 = 0$$

So since $a_n = a_0 = 1$ for all n, recalling $y = \sum_{n=0}^{\infty} a_n z^n$, we can write the first solution as

$$y_1(z) = 1 + z + z^2 + z^3 + ... = \frac{1}{1-z}$$

i.e. the sum of an infinite geometric series (for $-1 \leq z \leq 1$, since it would otherwise diverge). The second solution can be found if $a_1 = -2a_0$, $a_2 = a_0$ and $a_n = 0$ for $n > 2$. If again we set $a_0 = 1$, we find that

$$y_2(z) = 1 - 2z + z^2 = (1-z)^2$$

which is a polynomial solution to the ODE. Thus our general solution is

$$y(z) = \frac{c_1}{1-z} + c_2(1-z)^2$$

Just as a check, let's test if our solutions are independent using the Wronskian.

$$W = y_1 y_2' - y_1' y_2 = \frac{1}{1-z}[-2(1-z)] - \frac{1}{(1-z)^2}(10z)^2 = -3$$

$W \neq 0$, so y_1 and y_2 are linearly independent.

8 Legendre Functions

Key Points

- Legendre's differential equation
- Legendre polynomials

Special Functions

- **Special Functions:** Some 2$^{\text{nd}}$ order ODEs appear so frequently in physics and engineering that they have been given names. The solutions to these equations, which obey particularly commonly occuring boundary conditions, have been studied extensively. One such example is Legendre functions.

- **Legendre's Differential Equation:** Legendre's differential equation has the form

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

where ℓ is a real constant. Any solution to this equation is called a Legendre Function. It has 3 regular singular points at $x = -1, 1, \infty$ and occurs in numerous physical applications, particularly in problems with axial symmetry that involve the ∇^2 operator (Laplace operator) when they are expressed in spherical polar coordinates.

Aside: The Laplace operator - This operator gives the divergence of the gradient of a function.

$$\nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

i.e. For a 2-variable problem the del (or nabla) operator gives the gradient ('grad')

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

and the Laplace operator gives the divergence ('div') of the gradient

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- **Legendre Recurrance Relation:** $x = 0$ is an ordinary point of Legendre's differential equation, so we can find two linearly independent series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

with derivatives,

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting these into the ODE,

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Expanding the first bracket and rewriting the sum,

$$\sum_{n=0}^{\infty}\left[n(n-1)a_n x^{n-2} - n(n-1)a_n x^n - 2n a_n x^n + \ell(\ell+1)a_n x^n\right] = 0.$$

Shifting the index to x^n and simplifying, factorising by a_n and a_{n+2},

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - \left[n(n+1) - \ell(\ell+1)\right]a_n\right]x^n = 0.$$

And so we find the recurrance relation,

$$a_{n+2} = \frac{\left[n(n+1) - \ell(\ell+1)\right]}{(n+1)(n+2)}a_n.$$

We can see that two solutions can be found by examining the even and odd values of n separately. Choosing $a_0 = 1$ and $a_1 = 0$ we get the solution,

$$y_1(x) = 1 - \ell(\ell+1)\frac{x^2}{2!} + (\ell-2)\ell(\ell+1)(\ell+3)\frac{x^4}{4!} - ...$$

Choosing $a_0 = 0$ and $a_1 = 1$ we find the solution,

$$y_2(x) = x - (\ell-1)(\ell+2)\frac{x^3}{3!} + (\ell-3)(\ell-1)(\ell+2)(\ell+4)\frac{x^5}{5!} - ...$$

The ratio test shows that both of these series converge for $|x| < 1$ hence their radius of convergence to the nearest singular point is $r = 1$. Since the series are made of odd and even powers, they cannot be proportional to each other and are therefore linearly independent. So the general solution to the Legendre differential equation for $|x| < 1$ is given by,

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

- **Legendre Functions for Integer ℓ:** Our counting variable, n, is of course an integer. In many physical applications of Legendre's DE ℓ is also an integer. If we increase n until $n = \ell$ then our recurrance relation at that point becomes,

$$a_{\ell+2} = \frac{\left[\ell(\ell+1) - \ell(\ell+1)\right]}{(\ell+1)(\ell+2)}a_\ell = 0,$$

i.e. the series will terminate after a_ℓ, as $a_{\ell+2}$ and subsequent terms will be zero. This will give us a polynomial solution to the equation of order $n = \ell$. These are written $P_\ell(x)$ and are valid for all finite values of x.

By convention, we normalise $P_\ell(x)$ such that $P_\ell(1) = 1$, consequently $P_\ell(-1) = (-1)^\ell$ (i.e. negative x will give positive $P_\ell(x)$ for even ℓ and negative $P_\ell(x)$ for odd ℓ). The first few Legendre polynomials are given by

$$P_0(x) = 1 \qquad P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

For example,

$$P_2(1) = \frac{1}{2}(3 - 1) = 1 \qquad P_3(1) = \frac{1}{2}(5 - 3) = 1$$
$$P_2(-1) = \frac{1}{2}(3 - 1) = 1 \qquad P_3(-1) = \frac{1}{2}(-5 + 3) = -1$$

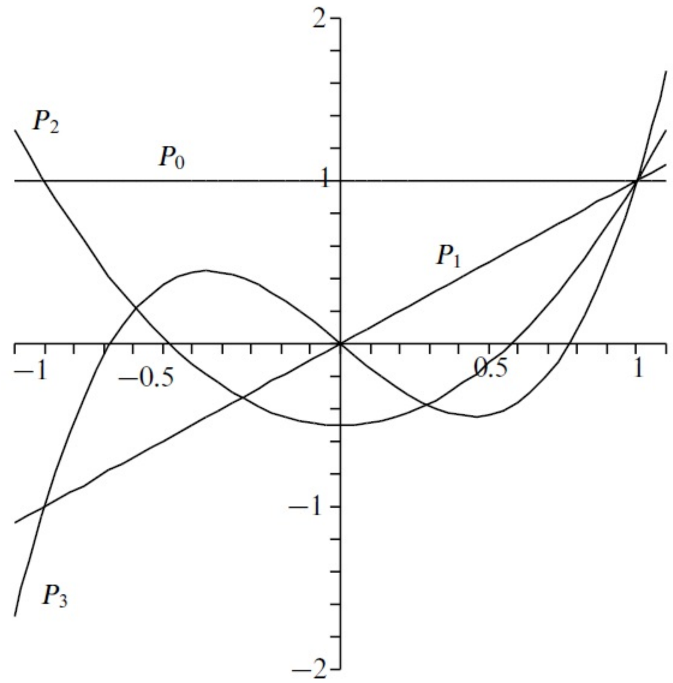


Figure 18.1 The first four Legendre polynomials.

So when ℓ is an odd or even integer, the y_1 series will terminate, giving the corresponding Legendre polynomial $P_\ell(x)$ as a solution to the Legendre DE. i.e. $y_1(x) = P_\ell(x)$.

To demonstate that the polynomials are solutions, lets try substituting one into the equation. When $\ell = 2$ the polynomial solution and its derivatives are

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_2'(x) = 3x$$

$$P_2''(x) = 3$$

Subbing this in we get

$$\begin{aligned}
&(1-x^2)y'' - 2xy' + \ell(\ell+1)y \\
&= (1-x^2)\times 3 - 2x \times 3x + 2(3)\frac{1}{2}(3x^2-1) \\
&= 3 - 3x^2 - 6x^2 + 9x^2 - 3 = 0
\end{aligned}$$

as expected.

In order to obtain a general solution we would still need $y_2(x)$. However, the second series will not terminate and only converge for $|x| < 1$. A second set of polynomials $Q_\ell(x)$ can be defined, referred to as *Legendre functions of the second kind*. Finding $Q_\ell(x)$ for the general solution is more complicated; we will restrict ourselves to functions of the first kind for the time being (see Pg 579 of Riley for more information on $Q_\ell(x)$).

- **Rodrigues' Formula:** Conveniently, there is a formula that allows one to quickly determine a Legendre polynomial for a given ℓ

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

e.g. PDF8.1 (a) State the name of the following type of equation and give its general form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 42y = 0$$

This is a Legendre equation, with general form

$$(1-x^2)y'' - 2xy' + vy = 0$$

where $v = \ell(\ell+1)$, and ℓ is a constant.

(b) State a polynomial solution to the equation with undetermined coefficients

We know that $v = 42 = \ell(\ell+1) = 6(7) \Rightarrow \ell = 6$. So for an even ℓ with a value of 6 the solution must be of the form

$$P_6(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6$$

where a_n are constants.

e.g. PDF8.2 Use Rodrigues' formula to determine a polynomial solution to the following equation.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 12y = 0$$

We know that $v = 12 = \ell(\ell+1) = 3(4) \Rightarrow \ell = 3$. So

$$\begin{aligned}
P_3(x) &= \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \\
&= \frac{1}{(2^3)(3!)}\frac{d^3}{dx^3}(x^2-1)^3 \\
&= \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) \\
&= \frac{1}{48}(120x^3 - 72x) \\
&= \frac{1}{2}(5x^3 - 3x)
\end{aligned}$$

- **Associated Legendre Differential Equation:** The associated Legendre's differential equation has the form

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] y = 0$$

and has 3 regular singular points at $x = -1, 1, \infty$. It is a more general form of Legendre's DE and reduces to that equation when $m = 0$. In physical applications the value of m is restricted such that $-\ell \leq m \leq \ell$.

The polynomials solutions for the associated Legendre DE are related to the those from of the non-associated version via the following

$$P_\ell^m(x) = (1-x^2)^{m/2}\frac{d^m P_\ell}{dx^m} \qquad \text{for } m > 0$$

9 Revision with Summary Table

Key Points

- Revision Lecture

Revision

- **Common terms in y_c and y_p:** Let's say you have the following solution to a homogeneous ODE $y_c = c_1 x + c_2$. Let's also say the inhomogeneous RHS is x^2. So you decide your $y_p = ax^2 + bx + c$. But as you want to avoid terms already found in y_c you multiply by x^2. Now $y_p = ax^4 + bx^3 + cx^2$, *not* $y_p = ax^4 + bx^3 + cx^2 + dx + e$. This is the equation you would use if you started with a quartic RHS. The d and e terms won't invalidate the solution, they will just add to the c_1 and c_2 terms in y_c. These terms are unecessary and just cause more work if included.

- **Legendre:** Please bare in mind Adrien-Marie Legendre, like many famous mathematicians, lent his name to more than one equation or technique. Legendre linear equations are not the same as the Legendre differential equation.

  2$^{\text{nd}}$ order Legendre linear equation (from L4),

$$a_2(\alpha x + \beta)^2 y'' + a_1(\alpha x + \beta)y' + a_0 y = f(x).$$

  Legendre's differential equation (from L8),

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0.$$

- **Singular points at ∞:** Show that Legendre's equation has a regular singular point at $|z| \to \infty$.

$$(1 - z^2)y'' - 2zy' + \ell(\ell + 1)y = 0$$

  Let $w = 1/z$. We need to eliminate z from the derivatives, expressing them in terms of w

$$\frac{dy}{dz} = \frac{dw}{dz}\frac{dy}{dw} = \frac{d}{dz}\left(\frac{1}{z}\right).\frac{dy}{dw} = -\frac{1}{z^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

$$\begin{aligned}\frac{d^2y}{dz^2} &= \frac{dw}{dz}\frac{d}{dw}\left(\frac{dy}{dz}\right),\\ &= \frac{-1}{z^2}\frac{d}{dw}\left(-w^2\frac{dy}{dw}\right),\\ &= -w^2\left(-2w\frac{dy}{dw} - w^2\frac{d^2y}{dw^2}\right),\\ &= 2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}.\end{aligned}$$

  Sub into the ODE,

$$\left(1 - \frac{1}{w^2}\right)\left(2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}\right) + 2\left(\frac{1}{w}\right)\left(w^2\frac{dy}{dw}\right) + \ell(\ell + 1)y = 0.$$

Expanding and simplifying,

$$\left(2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}\right) - \left(2w\frac{dy}{dw} + w^2\frac{d^2y}{dw^2}\right) + 2w\frac{dy}{dw} + \ell(\ell+1)y = 0,$$

which leaves us with a 2$^{\text{nd}}$ order ODE in terms of w,

$$(w^4 - w^2)\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw} + \ell(\ell+1)y = 0.$$

Dividing by $(w^4 - w^2)$ we find

$$p(w) = \frac{2w}{w^2-1}, \qquad q(w) = \frac{\ell(\ell+1)}{w^4-w^2}$$

Examining $p(0)$ and $q(0)$ we find,

$$\lim_{w\to 0} p(w) = \frac{2w}{w^2-1} \to 0, \qquad \lim_{w\to 0} q(w) = \frac{\ell(\ell+1)}{w^4-w^2} \to \infty,$$

$p(0) = 0$ but $q(0)$ diverges, so $|z| \to \infty$ is a singular point.
Testing $(w - w_0)p = wp$ and $(w - w_0)^2 q = w^2 q$ we find,

$$\lim_{w\to 0} wp(w) = \frac{2w^3}{w^2-1} \to 0, \qquad \lim_{w\to 0} w^2 q(w) = \frac{\ell(\ell+1)w^2}{w^4-w^2} = \frac{\ell(\ell+1)}{w^2-1} \to -\ell(\ell+1),$$

both converge at $w = 0$, so $|z| \to \infty$ is a regular singular point.

Name of ODE / method	Form/Condition	Order	Coeff.	Notes
Separable	$dy/dx = u(x)v(y)$	1	Var	Integrate independently
Exact	$du = A(x,y)dx$ $+B(x,y)dy = 0$ Test if $\partial A/\partial y = \partial B/\partial x$ $\partial u/\partial x = A, \partial u/\partial y = B$	1	Var	Find $u(x,y) = C$ by integrating A or B, use other to find $F(x \text{ or } y)$ from integral.
Integrating factor	$\mu(x,y)A(x,y)dx+$ $\mu(x,y)B(x,y)dy = 0$	1	Var	For inexact eqns
Homogeneous	$A(x,y)dx = B(x,y)dy$ $f(\lambda x, \lambda y) = \lambda^n f(x,y)$. Sub $y = vx$	1	Var	
Isobaric	$A(x,y)dx = B(x,y)dy$ $f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x,y)$. Sub $y = vx^m$	1	Var	Set powers of: $x, dx = 1, y, dy = m$
Linear 1st order	$dy/dx + p(x)y = q(x)$ $y = 1/\mu(x) \int \mu(x)q(x)dx$ $\mu(x) = e^{\int p(x)dx}$	1	Var	
Bernoulli	$dy/dx + b(x)y = c(x)y^n$ $z = y^{1-n}$	1	Var	Solve as linear 1st order
Linear nth order	$a_n(x)d^n y/dx^n$ $+a_{n-1}(x)d^{n-1}y/dx^{n-1} + ...$ $+a_1(x)dy/dx + a_0(x)y = f(x)$	n	Var	
Linear 2nd order	$y'' + p(z)y' + q(z)y = f(z)$	2	Var	
Complementary function (linear superposition)	$y_c = c_1 y_1(x) + c_2 y_2(x)$	2+	Const	Solve as RHS=0. y_1 and y_2 must be linearly independent
Auxiliary equation	Sub $y = Ae^{\lambda x}$ Real: $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ Repeat: $(c_1 + c_2 x)e^{\lambda_1 x}$ Complex: $c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$	2+	Const	Identify roots
Particular integral / trial functions	$y_p = be^{rx}$ or $b_1 sinrx + b_2 cosrx$ or $b_0 + b_1 x + ... + b_N x^N$	2+	Const	To find $RHS \neq 0$
General solution	$y = y_c + y_p$	2+	Const	
Laplace transform	$f(s) \equiv \int_0^\infty e^{-sx} f(x)dx$ $\bar{f^n}(s) = s^n \bar{f}(s) - s^{n-1}f(0)$ $-s^{n-2}f'(0) - ...$ $-sf^{(n-2)}(0) - f^{(n-1)}(0)$	2+	Const	

Name of ODE / method	**Form/Condition**	**Order**	**Coeff.**	**Notes**
Legendre linear eqns	$a_n(\alpha x+\beta)^n\frac{d^ny}{dx^n}+...$ $+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0y=f(x)$ Sub $\alpha x+\beta=e^t$	n	Var	Make coeffs. const. with sub.
Euler linear eqns	$a_nx^n\frac{d^ny}{dx^n}+...$ $+a_1x\frac{dy}{dx}+a_0y=f(x)$ Sub $x=e^t$	n	Var	Make coeffs. const. with sub.
Wronskian	$W=y_1y_2'-y_1'y_2$	2+	Var	Check for linear independence
Wronskian method / variation of parameters	$y_p(x)=$ $k_1(x)y_1(x)+k_2(x)y_2(x)$ $k_1'=\frac{-f(x)}{W(x)}y_2$ $k_2'=\frac{f(x)}{W(x)}y_1$	2+	Var	Find y_c as usual. $y=y_p$ as y_c is implicit in y_p
Ordinary and singular points	p and q finite $\rightarrow$ ordinary p or q infinite $\rightarrow$ singular $(z-z_0)p$ and $(z-z_0)^2q$ finite $\rightarrow$ regular singular $(z-z_0)p$ or $(z-z_0)^2q$ infinite $\rightarrow$ irregular singular	2+	Var	
Taylor series	$y(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ $=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ $y'=\sum_{n=0}^{\infty}na_nz^{n-1}$ $y''=\sum_{n=0}^{\infty}n(n-1)a_nz^{n-2}$	2+	Var	Requires ordinary point. Shift index by adding to n terms. Determine recurrance relation(s) for a_n.
Legendre's DE	$(1-x^2)y''-2xy'$ $+\ell(\ell+1)y=0$ $P_\ell(x)=\frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$	2	Var	Determine ℓ, solve with Rodrigues' formula

10 Introduction to PDEs: Separation of Variables

Key Points

- Partial differential equations
- Separation of variables (PDE)

Partial Differential Equations

- **PDEs:** A PDE is an equation relating an unknown function (the dependent variable) of two or more independent variables to its partial derivatives w.r.t those variables; most commonly those independent variables are space and time. We will largely be restricting ourselves to linear PDEs of order 2 and degree 1. Many of the ideas that we have explored with ODEs carry over into solving PDEs, so expect some parallels in the techniques used.

  One way of viewing the difference between an ODE and a PDE is that for an ODE we must assume that each variable has some dependence on the independent variable; whereas in a PDE, dependence is explicit.

  Let $u = u(x, y)$ be our dependent variable, where x and y are independent variables (i.e. one does not depend on the other). This generic function of two (or potentially more) variables is our function of interest. We would generally like to solve for this, just as we did for $y(x)$ for most of our ODE work.

  A reminder about notation: as partial derivatives can have more than one independent variable we must take care over notation, and be sure to distinguish between partial (e.g. ∂x) and total (e.g. dx) derivatives.

$$u_x = \frac{\partial u}{\partial x} \qquad\qquad u_y = \frac{\partial u}{\partial y}$$
$$u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad\qquad u_{yy} = \frac{\partial^2 u}{\partial y^2}$$
$$u_{xy} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = u_{yx}$$

  Remember, since x and y are independent of one another, when partially differentiating one w.r.t to the other it is treated as a constant.

$$y_x = \frac{\partial y}{\partial x} = 0 \qquad\qquad x_y = \frac{\partial x}{\partial y} = 0$$

  For 2$^{\text{nd}}$ order and higher partial derivatives of more than one variable, the order of the derivatives does not matter.

  e.g. PDF10.1 Show that $\partial^2 u/\partial x \partial y = \partial^2 u/\partial y \partial x$ if $u(x, y) = x^2 y^3$.

$$u_x = \frac{\partial u}{\partial x} = 2xy^3 \qquad\qquad u_y = \frac{\partial u}{\partial y} = 3x^2y^2$$
$$u_{yx} = \frac{\partial^2 u}{\partial y \partial x} = 6xy^2 \qquad\qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = 6xy^2$$

- **Total derivative:** As a reminder, we can express the total derivative in terms of partial derivatives
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$
So for example, du/dx
$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$
and du/dt
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

- **General form and classification of 2^{nd} order PDEs:** The general form of a 2^{nd} order PDE with 2 variables and constant coefficients is
$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y).$$
There is an interesting parallel here between 2^{nd} order PDEs and conics. When analysing conics one looks at a quadratic equation and based on its terms one can determine if the equation is elliptic, parabolic or hyperbolic. e.g. For conical quadratic equations
$$\begin{aligned} x^2 - y^2 = 1 &\qquad \text{is hyperbolic.} \\ x^2 - y = 0 &\qquad \text{is parabolic.} \end{aligned}$$
A similar approach works for 2^{nd} order PDEs. In order to be 2^{nd} order the coefficients of the 2^{nd} order terms, a,b and c cannot all be zero. We can define the discriminant of these coefficients to be
$$D(x, y) = b^2 - 4ac.$$
The type of the PDE can then easily be assessed.
  - If $D < 0$ then the equation is *elliptic*. e.g. the Laplace equation.
  - If $D = 0$ then the equation is *parabolic*. e.g. the heat equation.
  - If $D > 0$ then the equation is *hyperbolic*. e.g. the wave equation.

  In general, elliptic equations describe processes in equilibrium, while the parabolic and hyperbolic equations model processes that change over time.

  Some examples of common PDEs in physics are:
  - The 1D wave equation
  $$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
  where $u(x, t)$ is a function of the displacement of the wave and α is a constant.
  - The 2D Laplace equation
  $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
  where $u(x, y)$ is a twice differentiable real-valued function.

– The 3D heat equation
$$\frac{\partial u}{\partial t} = \alpha\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$
where $u(x, y, z, t)$ is the temperature and α is a the thermal diffusivity.

You may notice that the 'dimension' referred to in the name of these physical equations usually refers to spatial dimensions, excluding time.

Just as with ODEs, a good test of a solution $u(x, y)$ is to substitute it and its derivatives into a PDE as a check.

e.g. PDF10.2 Show that $u(x, t) = \sin\alpha t \sin x$ is a solution to the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

We need to show that the LHS matches the right. Find the derivatives

$$\begin{aligned} u_t &= \alpha\cos\alpha t \sin x & \qquad u_x &= \sin\alpha t\cos x \\ u_{tt} &= -\alpha^2 \sin\alpha t \sin x & \qquad u_{xx} &= -\sin\alpha t \sin x \end{aligned}$$

So, $LHS = \alpha^2 RHS$, which satisfies the equation.

- **Separation of Variables:** Just as with ODEs, if the variables of a PDE can be split up then we can solve it relatively easily.

First, let's assume u can be expressed as the product of two independent functions

$$u(x, t) = X(x)T(t)$$

Hence the partial derivatives of u are

$$\begin{aligned} u_x &= X'T & \qquad u_t &= XT' \\ u_{xx} &= X''T & \qquad u_{tt} &= XT'' \end{aligned}$$

Now if we substitute these derivatives into a PDE and separate the variables we will notice something interesting - the two sides of the equation *must be constant!*

e.g. PDF10.3 Consider the 1D heat equation

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$$

where k is a real consant. Show that it can be separated into two independent ODEs.

First, let's sub. our independent functions and their derivatives into the PDE

$$XT' = k^2 X''T.$$

Next, separate the variables

$$\frac{X''}{X} = \frac{1}{k^2}\frac{T'}{T}.$$

Now, here's the trick. Notice that the LHS is only dependent on x and the RHS is only dependent on t. Well, if both sides are dependent on different variables, but are always equal then we can draw only one conclusion - despite appearances, they must each be constant. In a 1D problem such as this, they must in fact be equal to the same constant - *the separation constant.*

$$\frac{X''}{X} = \frac{1}{k^2}\frac{T'}{T} = \mu.$$

This allows us to form two, independent ODEs, which can be solved separately.

$$\begin{aligned} \frac{X''}{X} &= \mu \\ X'' - \mu X &= 0 \end{aligned} \qquad\qquad \begin{aligned} \frac{1}{k^2}\frac{T'}{T} &= \mu \\ T' - k^2\mu T &= 0 \end{aligned}$$

Once we have solutions for X and T, their product will give us the solution for u.

$$\begin{aligned} \lambda_1^2 - \mu &= 0 \\ \lambda_1 &= \pm\sqrt{\mu} \\ X &= c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x} \end{aligned} \qquad\qquad \begin{aligned} \lambda_2 - k^2\mu &= 0 \\ \lambda_2 &= k^2\mu \\ T &= c_3 e^{k^2\mu t} \end{aligned}$$

Hence the general solution is

$$u(x,t) = XT = \left(c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}\right) e^{k^2\mu t}$$

Try substituting this solution back into the PDE to confirm that it viable.

The logic behind this method holds for higher dimensions PDEs, but with the added complication that although $LHS = RHS = const$, if there are multiple terms on one side of the equation each may have its own constant.

e.g. PDF10.4 Consider the 3D heat equation.

$$\frac{\partial u}{\partial t} = k^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right].$$

This time $u = X(x)Y(y)Z(z)T(t)$. Now after substituting in the terms for X, Y, Z and T we get

$$XYZT' = k^2\left[X''YZT + XY''ZT + XYZ''T\right].$$

After separating the variables by dividing by $k^2(XYZT)$ we are left with

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{k^2}\frac{T'}{T} = \mu$$

So once more we have $LHS = RHS = \mu$. But the LHS has multiple terms, they may each have a different value. Let's say then that

$$l + m + n = \mu$$

where

$$\frac{X''}{X} = l \qquad \frac{Y''}{Y} = m \qquad \frac{Z''}{Z} = n \qquad \frac{T'}{T} = k^2\mu$$

Each of these ODEs can be solved separately giving

$$X = c_1 e^{\sqrt{l}x} + c_2 e^{-\sqrt{l}x} \quad Y = c_3 e^{\sqrt{m}y} + c_4 e^{-\sqrt{m}y} \quad Z = c_5 e^{\sqrt{n}z} + c_6 e^{-\sqrt{n}z} \quad T = c_7 e^{k^2 \mu t}$$

Hence the general solution is

$$u(x, y, z, t) = XYZT = \left(c_1 e^{\sqrt{l}x} + c_2 e^{-\sqrt{l}x}\right)\left(c_3 e^{\sqrt{m}y} + c_4 e^{-\sqrt{m}y}\right)\left(c_5 e^{\sqrt{n}z} + c_6 e^{-\sqrt{n}z}\right) e^{k^2 \mu t}$$

It should be noted that separating the variables of a PDE in order to solve it is generally not possible. However, this method may be used on several common PDEs that occur in physics and engineering.

11 Separation of Variables with Boundary Conditions

Key Points

- Separation of variables (PDE)
- Applying boundary conditions

Separation of Variables for PDE's (ctud)

- **Applying boundary conditions:** Using the method of separation of variables we can determine a general solution to separable PDE's. If we wanted a complete solution to a given real world problem then we would need to find and apply boundary conditions when solving the PDE.

  In order to solve an equation of two variables for the given boundary conditions we need to use the following procedure. Stage 1 is the same as solving without BC's:

  - **Stage (1)** Produce separate ODE's.
    i) Assume a separable solution of the form $u(x,t) = X(x)T(t)$.
    ii) Find the derivatives of u.
    iii) Sub the derivatives of u into the PDE.
    iv) Separate the variables $X(x)$ and $T(t)$ onto the LHS and RHS.
    v) Introduce a separation constant μ and produce an X equation and a T equation.
  - **Stage (2)** Consider the boundary conditions $u(x_1, t_1)$ and $u(x_2, t_2)$. Since $u = XT$, we can use the boundary conditions on u to establish BC's for X and/or T.
    For example:
    If $u(0,t) = 0$ then $X(0)T(t) = 0$.
    If $u(x,0) = 0$ then $X(x)T(0) = 0$.
    Since $X(x) = 0$ and $T(t) = 0$ are trivial solutions (i.e. the temperature would be 0 for any given position or time) then we can conclude that $X(0) = 0$ and $T(0) = 0$.
  - **Stage (3)** We currently have no information on the value of μ. In order to ensure our solution is truly general we must have solutions for all the possible values of μ. Hence, we consider the 3 possible cases for μ
    i) $\mu = 0$.
    ii) $\mu < 0$.
    iii) $\mu > 0$.

    Solve for X or T at the boundary conditions. If $X(x) = 0$ or $T(t) = 0$ then there's no need to try the other function as this is a trivial solution. If there is a non-zero solution, solve for the other function.
  - **Stage (4)** Add up the solutions of interest for u for all the values of μ, use initial conditions to simplify the equation and work out the coefficients using the formulae for Fourier series coefficients.

Using the method of separation of variables we have previously sought a general solution to the 1D heat equation. The PDE was

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature at a given point and time. And the solution we arrived at was

$$u(x,t) = XT = \left(c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}\right) e^{k^2 \mu t}$$

If we had boundary conditions for a given problem we could obtain a more useful, more specific solution to our problem. Different boundary conditions can describe different physical scenarios. For instance one might consider two types of heat equation, depending on the given boundary conditions.

Consider a long, thin metal rod. Let's state some boundary conditions that describe what is happening at two points on the rod.

1) Zero temperature end points: $u(0,t) = 0$, $u(L,t) = 0$, $t > 0$.
2) Insulated end points: $u_x(x,0) = 0$, $u_x(L,t) = 0$, $t > 0$.

In the first scenario we are given information about two points on the rod separated by a distance L ($x = 0$ and $x = L$) and told the temperature at each one is zero, for some time after $t = 0$. We will solve the equation using these boundary conditions in this lecture.

In the second scenario we are told that the point x experiences no temperature change when $t = 0$ and a point a distance L away experiences no temperature change at any later time t. i.e. no heat flows across the boundary. You will solve the equation using these boundary conditions in a weekly problem set.

e.g PDF11.1 Solve the 1D heat equation for $k = 2$, using the boundary conditions for zero temperature end points: $u(0,t) = 0$, $u(1,t) = 0$, $t > 0$ and the initial condition $u(x,0) = x/4$ for $0 < x < 1$.

Stage (1) Based on our work from last time, we see that the 1D heat equation

$$\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$$

has solutions that can be determined from the following two ODE's.

$$X'' - \mu X = 0 \qquad\qquad T' - 4\mu T = 0$$

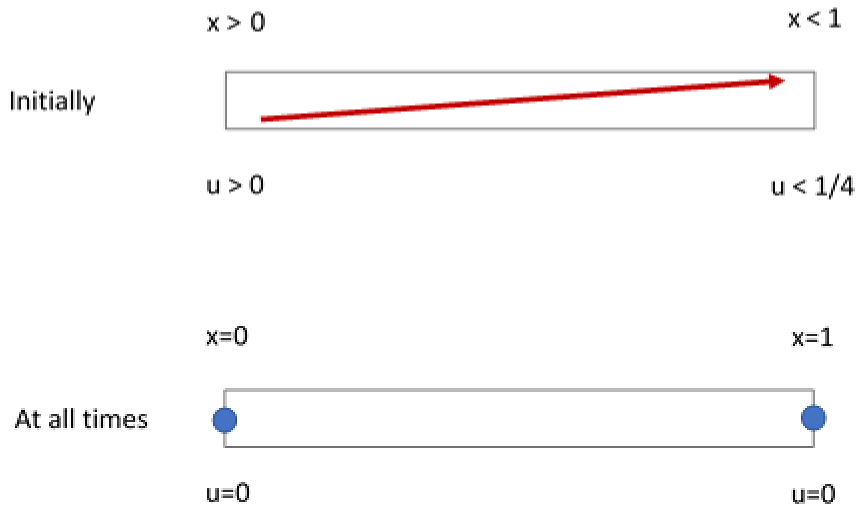
Stage (2) Our boundary conditions allow us to determine the following

$$u(0,t) = 0\text{: } X(0)T(t) = 0$$

$$u(1,t) = 0\text{: } X(1)T(t) = 0$$

The solution where $T(t) = 0$ is trivial and of no interest; it represents the scenario were the temperature never changes. So we can say that $X(0) = 0$ and $X(1) = 0$.

Stage (3) Let's consider the three possible cases of μ.



(1) $\mu = 0$: Beginning with the equation for X

$$X'' = 0$$

Examining the roots or by inspection we realise that the solution to this equation is

$$X(x) = ax + b,$$

as differentiating this equation twice will certainly result in zero, while differentiating fewer times may not, making it the optimal choice. Now apply the boundary conditions for X

$$X(0) = 0 \Rightarrow b = 0 \Rightarrow X(x) = ax,$$

$$X(1) = 0 \Rightarrow a = 0 \Rightarrow X(x) = 0.$$

Therefore when $\mu = 0$, $X(x) = 0$ for any given x and hence $u(x,t) = 0$ for any given x, so this solution is trivial and of no interest.

(2) $\mu > 0$: Let's say that $\mu = r^2$ so we can be sure it is a positive constant and avoid fractional powers in our solutions. Beginning with the equation for X

$$X'' - r^2 X = 0.$$

Examining the real roots we realise that this equation has the auxiliary and solution

$$\lambda^2 = r^2,$$

$$X(x) = ae^{rx} + be^{-rx}.$$

Now apply the boundary conditions for X

$$X(0) = 0 \Rightarrow a + b = 0 \Rightarrow b = -a,$$

Reducing our equation for X to

$$X(x) = a\left(e^{rx} - e^{-rx}\right).$$

Using our second BC

$$X(1) = 0 \Rightarrow a\left(e^{r} - e^{-r}\right) = 0 \Rightarrow a = 0 \Rightarrow b = 0.$$

Therefore when $\mu > 0$, $X(x) = 0$ for any given x and hence $u(x,t) = 0$ for any given x, so this solution is of no interest.

(3) $\mu < 0$: Let's say that $\mu = -r^2$ so we can be sure it is a negative constant and avoid fractional powers in our solutions. Beginning with the equation for X

$$X'' + r^2 X = 0$$

Examining the complex roots we realise that this equation has the auxiliary and solution

$$\lambda^2 = -r^2,$$

$$X(x) = a\cos rx + b\sin rx$$

Now apply the boundary conditions for X

$$X(0) = 0 \Rightarrow a = 0 \Rightarrow X(x) = b\sin rx,$$

$$X(1) = 0 \Rightarrow b\sin r = 0.$$

Since $\sin r$ is zero for some values of r, b can be non-zero. This means that when $\mu < 0$, $X(x)$ is non-zero for some values of x. So we have a solution of interest. If we let $\sin r = 0$ then $r = n\pi$ for $n = 1, 2, 3....$ Hence

$$X_n(x) = b_n \sin n\pi x.$$

Note: Consider that if we did not have boundary conditions (i.e. we were examining the simpler case of an infinitely long metal rod) then r could be anything. The boundary conditions limit us to integer muliples of half waves that fit within the boundaries, as our choice of r MUST give $u = 0$ at $x = 0$ and $x = 1$.

Since we have a non-trivial solution for X, it is worth looking at T. Following the same process, applying $\mu = -r^2$ the ODE for T becomes

$$T' + 4r^2 T = 0.$$

This ODE has the solution

$$\lambda = -4r^2$$

$$T(t) = ae^{-4r^2 t}$$

We have already established that $r = n\pi$, so subbing this in the equation becomes

$$T_n(t) = a_n e^{-4n^2\pi^2 t}$$

And so, given that $u = XT$, our solution for $u(x,t)$ is

$$u(x,t) = b_n \sin(n\pi x) a_n e^{-4n^2\pi^2 t} = C_n \sin(n\pi x) e^{-4n^2\pi^2 t}$$

where $C_n = a_n b_n$.

The physical interpretation of this solution is a temperature distribution between $x = 0$ and $x = 1$ (with the shape of half a wave for $n = 1$) that decays exponentially with time.

Stage (4) Now we have evaluated u for all μ we can add the solutions together using the superposition principle. The solutions are found by solving a set of differing ODEs, but are all solutions to the original *linear* PDE. As the solutions for $\mu < 0$ and $\mu > 0$ are valid for all values of μ below and above zero respectively, we sum them up to infinity

$$u(x,t) = u_{\mu=0} + \sum u_{\mu<0} + \sum u_{\mu>0}$$

In this instance, we only have one viable solution ($\mu < 0$), so this is straight forward.

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) e^{-4n^2\pi^2 t}$$

We were given an initial condition (i.e. when $t = 0$) that can be useful here. Let's apply the condition $u(x,0) = x/4$

$$u(x,0) = \frac{x}{4} = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

Notice that the time dependent term reduced to a value of 1. So this leaves us with a constant times a sin term. This may look familiar; it is a Fourier series! We can use Fourier series formulae to determine the value of the coefficient (C_n) of such a solution. Recall that a full fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{L}$$

where $f(x)$ is the function of interest, L is half the period of the function and a_0, a_n and b_n are Fourier coefficients. These coefficients each have their own formulae

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx = \frac{2}{L} \int_{0}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{\pi n x}{L} dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{\pi n x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{\pi n x}{L} dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{\pi n x}{L} dx$$

The coefficients can equivalently be defined in terms of the period $P = 2L$,

$$a_0 = \frac{2}{P} \int_{-P/2}^{P/2} f(x) dx = \frac{2}{P} \int_{0}^{P} f(x) dx$$

$$a_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \cos \frac{2\pi n x}{P} dx = \frac{2}{P} \int_{0}^{P} f(x) \cos \frac{2\pi n x}{P} dx$$

$$b_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \sin \frac{2\pi n x}{P} dx = \frac{2}{P} \int_{0}^{P} f(x) \sin \frac{2\pi n x}{P} dx$$

Note on variation of fourier series: Sometimes the half from $a_0/2$ in $f(x)$ gets switched to the a_0 coefficient formula.

We will use $L = 1$, as our function would be periodic between -1 and 1, hence its half-period ranges from 0 to 1,

$$\begin{aligned}
C_n &= \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx \\
&= \frac{2}{1} \int_{0}^{1} \frac{x}{4} \sin \frac{n\pi x}{1} dx \\
&= \frac{1}{2} \left[\frac{-x}{n\pi} \cos n\pi x \right]_0^1 + \frac{1}{n\pi} \int_0^1 \cos n\pi x dx \\
&= \frac{1}{2} \left[\frac{-x}{n\pi} \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x \right]_0^1 \\
&= -\frac{\cos n\pi}{2n\pi} \\
&= -\frac{(-1)^n}{2n\pi}.
\end{aligned}$$

Hence, the final solution is

$$u(x,t) = -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) e^{-4n^2\pi^2 t}$$

12 Sinusoidal and Hyperbolic Solutions

Key Points

- Separation of variables (PDE)
- Sinusoidal vs hyperbolic solutions

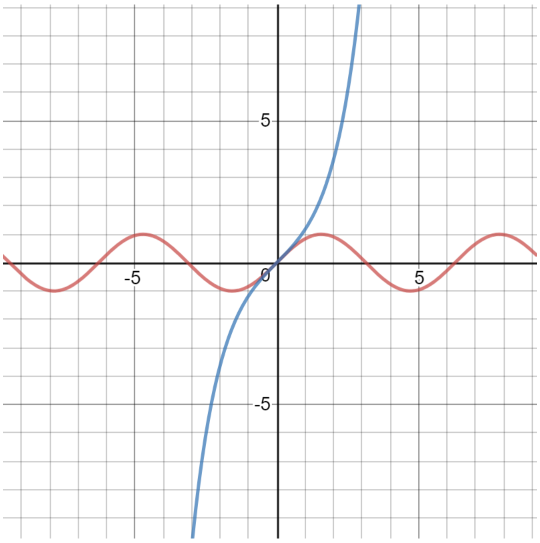
Separation of Variables for PDE's (ctud)

- **Checking BC's that tend to ∞:** We have already shown that imposing different boundary conditions on our equations can describe different physical scenarios. Checking BC's in the limit where a variable tends to infinity can also be very information, telling us where the solution exists and what its nature may be.

  For instance, consider the exponential forms of the sine and hyperbolic sine functions

  $$sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad\qquad sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

  The forms of these equations are very similar. Sometimes we may have difficulty telling immediately if a solution to an equation is sinusoidal or hyperbolic, especially if our solutions are complicated. Here are $\sin(x)$ (red) and $\sinh(x)$ (blue):

  Of course, knowing the nature of the solution is important. If we explore the value of our solution in the limit as one of our variables tends to infinity we can determine if it is sinusoidal or hyperbolic.

  e.g. PDF12.1 Use the method of separation of variables to obtain a solution for the 1D diffusion equation

  $$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

  that tends to zero for all x as $t \to \infty$. k is the diffusivity, a positive, real constant. Assume a separation constant of μ. Determine whether the solution in this limit is sinusoidal or hyperbolic.

We have two independent variables, x and t so we assume a solution of the form

$$u(x,t) = X(x)T(t).$$

Substituting X, T and their derivatives into the PDE and dividing by kXT we obtain

$$\frac{X''}{X} = \frac{T'}{kT} = \mu$$

This leads to two ODEs

$$X'' - \mu X = 0 \qquad\qquad T' - \mu k T = 0$$

which have solutions

$$X = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x} \qquad\qquad T = Ce^{\mu k t}$$

giving a combined solution of

$$u = XT = \left(Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}\right)e^{\mu k t}.$$

With this general solution in place we can apply the boundary conditions. We want $u \to 0$ as $t \to \infty$. Hence $\mu k < 0$. This means the time term will be moved to the denominator and as time goes on the denominator of the solution will increase and u will tend to zero for all x, which would not be the case if $\mu k = 0$ or $\mu k > 0$.

Since k is real and positive we know that μ is a real, negative number and that the solution to u is sinusoidal in x, as we have complex exponents from square rooting $\mu < 0$. The equation is therefore not hyperbolic.

e.g. PDF12.2 Use the method of separation of variables to obtain for the 1D diffusion equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

a solution that tends to zero for all x as $t \to \infty$. k is the diffusivity, a positive, real constant. This time assume the separation constant is $-\mu^2$. Determine whether the solution in this limit is sinusoidal or hyperbolic.

We have two independent variables, x and t so we assume a solution of the form

$$u(x,t) = X(x)T(t).$$

Substituting X, T and their derivatives into the PDE and dividing by kXT we obtain

$$\frac{X''}{X} = \frac{T'}{kT} = -\mu^2$$

Choosing to square μ helps because experience tells us we are going to square-root it when examining the roots of a 2$^{\text{nd}}$ order ODE. Choosing to make it negative helps as we will immediately produce a sin and cos based solution to the 2$^{\text{nd}}$ order ODE. Both of these decisions are based on foreknowledge; they are not necessary, but make solving this particular problem a little easier. This leads to two ODEs

$$X'' + \mu^2 X = 0 \qquad\qquad T' + \mu^2 kT = 0$$

which have solutions

$$X = A\cos\mu x + B\sin\mu x \qquad\qquad T = Ce^{-\mu^2 kt}$$

giving a combined solution of

$$u = XT = \frac{A\cos\mu x + B\sin\mu x}{e^{\mu^2 kt}}.$$

With this general solution in place we can apply the boundary conditions. We want $u \to 0$ as $t \to \infty$. Hence $\mu^2 k > 0$. This means as time goes on the denominator of the solution will increase and u will tend to zero for all x, which would not be the case if $\mu^2 k = 0$ or $\mu^2 k < 0$.

Since k is real and positive we know that μ is a real non-zero number and that the solution to u is sinusoidal in x, not hyperbolic. This time there is not need to consult the exponential forms for sin and sinh, the nature of the solution is obvious from inspection.

e.g. PDF12.3 MM2 2014 Q4(f): Solve the partial differential equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + 4\frac{\partial f(x,y)}{\partial y} = 0$$

where x, y are the Cartesian coordinates. [2 marks]

Determine whether or not your solution is physical if we impose the condition that $f(x,y)$ tends to zero when $y \to \infty$. [2 marks]

Assume a solution of $f = XY$, giving

$$\frac{X''}{X} = -4\frac{Y'}{Y} = m$$

If $m > 0$, i.e. m is positive

$$X = Ae^{\sqrt{m}x} + Be^{-\sqrt{m}x} \qquad\qquad Y = Ce^{-my/4}$$

Giving a hyperbolic solution (due to real exponents)

$$f(x,y) = \left(Ae^{\sqrt{m}x} + Be^{-\sqrt{m}x}\right)e^{-my/4}$$

If $m = -n$ where $n > 0$. i.e. m is negative

$$X = Ae^{i\sqrt{n}x} + Be^{-i\sqrt{n}x} \qquad\qquad Y = Ce^{ny/4}$$

Giving a sinusoidal solution (due to imaginary exponents)

$$f(x,y) = \left(Ae^{i\sqrt{n}x} + Be^{-i\sqrt{n}x}\right)e^{ny/4}$$

[2 marks]

Checking our boundary conditions we see that only one solution satisfies the condition that $f(x, y) \to 0$ when $y \to \infty$; the solution for $m > 0$, due to the negative exponent in the y term. [2 marks]

e.g. PDF12.4 MM2 2015 Q5: Consider a vibrating string.

(a) What would be the equation of motion describing the string's oscillations in a plane, with only one polarisation state, if they were undamped with time? [2 marks]
Solution
$$k^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

(b) In this context, what is the simplest interpretation of the following equation (u is a function of time and space and c, k are two constants):
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial t}$$
[2 marks]
Solution The first order du/dt term implies there is dissipation with time.

(c) What is the best method to solve this equation? [2 marks]
Solution Separation of variables with $u = X(x)T(t)$

(d) Show that if u vanishes at $x = 0$ and $x = L$ the spatial dependence of the function u is given by $u \propto \sin(Ax)$ where A is a constant that you need to determine. [6 marks]
Solution
$$T''X = c^2 TX'' - kT'X$$
$$\frac{T''}{T} + k\frac{T'}{T} = c^2 \frac{X''}{X} = \pm\mu^2$$
Meaning that
$$c^2 \frac{X''}{X} = \pm\mu^2$$
Consider $-\mu^2$:
$$X'' + \frac{\mu^2}{c^2} X = 0$$
$$\lambda^2 + \frac{\mu^2}{c^2} = 0$$
$$\lambda^2 = -\frac{\mu^2}{c^2}$$
$$\lambda = \pm i\frac{\mu}{c}$$
Complex roots imply the solution is sinusoidal
$$X = d_1 e^{i\mu x/c} + d_2 e^{-i\mu x/c} = a \sin\left(\frac{\mu}{c} x\right) + b \cos\left(\frac{\mu}{c} x\right)$$

Aside: To explain why we include both a sin and a cos term try adding sin and cos in exponential form

$$\begin{aligned}
\sin x + \cos x &= \frac{e^{ix} - e^{-ix}}{2i} + \frac{e^{ix} + e^{-ix}}{2} \\
&= \frac{-ie^{ix} + ie^{-ix}}{2} + \frac{e^{ix} + e^{-ix}}{2} \\
&= \frac{1}{2}\left(e^{ix} - ie^{ix} + e^{-ix} + ie^{-ix}\right) \\
&= \frac{1-i}{2}e^{ix} + \frac{1+i}{2}e^{-ix} \\
&= d_1 e^{ix} + d_2 e^{-ix}
\end{aligned}$$

Consider $-\mu^2$:

$$\begin{aligned}
X'' - \frac{\mu^2}{c^2}X &= 0 \\
\lambda^2 - \frac{\mu^2}{c^2} &= 0 \\
\lambda^2 &= \frac{\mu^2}{c^2} \\
\lambda &= \pm\frac{\mu}{c}
\end{aligned}$$

Complex roots imply the solution is hyperbolic

$$X = d_1 e^{\mu x/c} + d_2 e^{-\mu x/c} = a\sinh\left(\frac{\mu}{c}x\right) + b\cosh\left(\frac{\mu}{c}x\right)$$

Applying the boundary condition $u(0,t) = 0$ we see that $b = 0$ as both $\cos(0) = \cosh(0) = 1$.

Applying $u(L,t) = 0$ we realise that the solution must be $\sin(Ax)$, not $\sinh(Ax)$ as $\sin(AL)$ can have a value of zero at a distance L but $\sinh(AL) \neq 0$.

So the solution must be

$$X = a\sin\left(\frac{\mu}{c}x\right)$$

Giving the constant $A = \mu/c = 2\pi/L$, guaranteeing $X = a\sin(0) = 0$ at the BC's. So $u \propto \sin(Ax)$ as required.

(e) Find the time dependence of u [4 marks]

Solution Using

$$\begin{aligned}
\frac{T''}{T} + k\frac{T'}{T} &= -\mu^2 \\
T'' + kT' + \mu^2 T &= 0
\end{aligned}$$

Examining the roots we find

$$\lambda^2 + k\lambda + \mu^2 = 0$$

$$\lambda_{\pm} = \frac{-k \pm \sqrt{k^2 - 4\mu^2}}{2}$$

Therefore

$$T = me^{\lambda_+ t} + ne^{\lambda_- t}$$

(f) Assume that the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - k\frac{\partial u}{\partial t}$$

is a correct description of the vibrating string. Discuss the form of the time evolution of the vibration as a function of k. [4 marks]

Solution The vibrations are thus described by

$$u = a \sin\left(\frac{\mu}{c}x\right)\left[me^{\lambda_+ t} + ne^{\lambda_- t}\right]$$

with

$$\lambda_{\pm} = \frac{-k \pm \sqrt{k^2 - 4\mu^2}}{2}$$

When $k = 2\mu$, $\lambda_{\pm} < 0$ and the vibration is exponentially damped with time.

When $k > 2\mu$, the term in $e^{\lambda_- t}$ decreases with time while the other exponential vanishes so the solution is damped.

When $k < 2\mu$, the exponentials contain an imaginary part but the oscillations are damped.

At last when $k = 0$, there is no damping, just oscillations.

13 General Solutions to 1^{st} Order PDEs

Key Points

- General solutions to PDEs
- 1st order PDEs

General Solutions to PDEs

- **General form of solution:** In general PDE's are not separable and cannot be solved using the method of separation of variables. We require a more general method of solving them.

  For a given PDE we can try to produce a general solution by seeking a function made from a combination of the independent variables in the PDE. For the function $u(x, y)$ we would seek a solution $u(p)$ where $p = p(x, y)$. That is to say we can express a general solution $u(x, y)$ as a function of a specific combination of x and y. e.g. a PDE might have the solution
  $$u(x, y) = f(x^2 + y^2).$$
  In this example we are stating that the solution to our PDE is a function of $p = x^2 + y^2$. This may not seem terribly useful or enlightening, but when combined with boundary conditions we can be more specific and arrive at a particular solution to a given PDE.

- **1^{st} order PDEs (1 derivative):** Initially, let's consider 1^{st} order PDEs containing two independent variables. The most general form for this type of PDE is
  $$A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} + C(x, y)u = R(x, y)$$
  where $A(x, y)$, $B(x, y)$, $C(x, y)$ and $R(x, y)$ are given functions. If either $A = 0$ or $B = 0$ then the equation can simply be solved as a 1^{st} order linear ODE. Recall that the form of this type of equation is
  $$\frac{\partial u}{\partial x} + P(x, y)u = Q(x, y) \qquad\qquad \frac{\partial u}{\partial y} + P(x, y)u = Q(x, y)$$
  and that it can be solved using an integrating factor (that depends on which independent variable is being used in the derivatives).
  $$\mu(x, y) = e^{\int P(x,y)dx} \qquad\qquad \mu(x, y) = e^{\int P(x,y)dy}$$
  with a solution given by
  $$u = \frac{1}{\mu(x, y)}\int \mu(x, y)Q(x, y)dx \qquad\qquad u = \frac{1}{\mu(x, y)}\int \mu(x, y)Q(x, y)dy.$$

e.g. PDF13.1 1^{st}order linear PDE: Find the general solution $u(x, y)$ of

$$x\frac{\partial u}{\partial x} + 3u = x^2.$$

There are no y derivatives here, but since u is a function of x and y, this is still a 1^{st} order linear PDE rather than simplfying to an ODE. However it may be solved in the same way.

We can begin by dividing by x to arrive at the canonical form

$$\frac{\partial u}{\partial x} + \frac{3u}{x} = x.$$

This gives us the form of a 1^{st} order linear equation. After identifying $P(x) = 3/x$ and $Q(x) = x$ this can be solved by using an integrating factor

$$\mu(x) = e^{\int P(x)dx} = e^{\int (3/x)dx} = e^{3\ln x} = x^3$$

Multiplying through by μ we find

$$\mu\frac{\partial u}{\partial x} + \mu\frac{3u}{x} = \mu x$$

Recalling that the integrating factor is used so that we can apply the following relation

$$\frac{\partial}{\partial x}(\mu u) = \mu\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial x}u = \mu\frac{\partial u}{\partial x} + \mu P u$$

where

$$\frac{\partial \mu}{\partial x} = \mu P \qquad \text{since} \qquad \frac{\partial}{\partial x}e^{\int P dx} = P(x)e^{\int P dx}.$$

Applying this relation to the LHS, the equation becomes

$$\frac{\partial}{\partial x}(\mu u) = \mu x$$

$$\frac{\partial}{\partial x}(x^3 u) = x^4$$

Integrating

$$x^3 u = \frac{x^5}{5} + f(y)$$

where $f(y)$ is some unknown function of y instead of a constant, as y would be considered a constant when partially differentiating u w.r.t x. Hence

$$u(x, y) = \frac{x^2}{5} + \frac{f(y)}{x^3}$$

In our solution we have been able to specify the relation u has to x, but $f(y)$ remains unknown without boundary conditions to give us more information. This would be considered the most general solution to this problem.

- **1st order PDEs (2 derivatives):** The method above is fine for PDE's containing partial derivatives w.r.t only one independent variable. However this method will not work for an equation that contains both $\partial/\partial x$ and $\partial/\partial y$. Let's consider a special case, where $C(x,y) = R(x,y) = 0$, i.e.

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = 0$$

Again, let's look for a solution of the form $u(x,y) = f(p)$, where p is some unknown combination of x and y. Consider the derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial x} = \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f(p)}{\partial y} = \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y} = \frac{df(p)}{dp}\frac{\partial p}{\partial y}$$

Let's sub these into our PDE

$$A(x,y)\frac{df(p)}{dp}\frac{\partial p}{\partial x} + B(x,y)\frac{df(p)}{dp}\frac{\partial p}{\partial y} = 0$$

$$\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]\frac{df(p)}{dp} = 0$$

Let's consider two restrictions on p:

1) Let's choose a p such that the square bracket $= 0$,

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0,$$

$$\frac{\partial p}{\partial x}\Big/\frac{\partial p}{\partial y} = -\frac{B}{A}.$$

This has the advantage of removing any reference to the form of $f(p)$, so we don't need to know what it is yet (we intend to work it out - it is our solution).

2) Let's also require that p is constant, meaning that the total derivative of p is zero

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0,$$

$$\frac{\partial p}{\partial x}\Big/\frac{\partial p}{\partial y} = -\frac{dy}{dx}.$$

Notice that these two equations are very similar. In fact, they become the same if we require that

$$dx = A(x,y) \qquad\qquad dy = B(x,y)$$

Hence,

$$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)}.$$

p is a constant that relates x to y for our given PDE and so is the constant of integration of this equation. Hence p can be found by integrating this expression and setting the constant of integration (or a multiple of the constant) equal to p.

e.g. PDF13.2 1^{st}order PDE with 2 derivatives and boundary conditions: For

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0,$$

find (i) the solution that takes the value $u = 2y + 1$ on the line $x = 1$, and (ii) a solution that has the value 4 at the point (1,1).

So, we're seeking a solution of the form $u(x, y) = f(p)$. We now know that $u(x, y)$ will be constant along lines of (x, y) that satisfy

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}$$
$$\frac{dx}{x} = \frac{dy}{-2y}.$$

Integrating we get

$$\begin{aligned}
\int \frac{1}{x}dx &= -\frac{1}{2}\int \frac{1}{y}dy \\
\ln x &= -\frac{1}{2}\ln y + \ln c \\
\ln x &= \ln cy^{-1/2} \\
x &= e^{\ln cy^{-1/2}} \\
x &= Cy^{-1/2} \\
C &= xy^{1/2}
\end{aligned}$$

This constant gives us p, some function of x and y. Simply to avoid fractional powers in this case we shall say $p = C^2$, hence

$$p = x^2y$$

Thus the general solution to the PDE is

$$u(x, y) = f(x^2y)$$

where f is an arbitrary function. Before going on to apply our boundary conditions, let's try one or two functions as a test.

– $u = x^2y$: Our derivatives are

$$u_x = 2xy \qquad\qquad u_y = x^2$$

Subbing into the PDE

$$xu_x - 2yu_y = x(2xy) - 2y(x^2) = 2x^2y - 2x^2y = 0$$

we get $RHS = 0$ as required by the PDE.

– $u = \sin(x^2 y)$: Our derivatives are

$$u_x = 2xy\cos(x^2 y) \qquad\qquad u_y = x^2\cos(x^2 y)$$

Subbing into the PDE

$$xu_x - 2yu_y = x[2xy\cos(x^2 y)] - 2y[x^2\cos(x^2 y)] = 2x^2 y\cos(x^2 y) - 2x^2 y\cos(x^2 y) = 0$$

we get $RHS = 0$ as required by the PDE.

We have our general solution. Let's find our particular solutions by applying the respective boundary conditions.

(i) $\underline{u = 2y + 1 \text{ on the line } x = 1}$: Test the general solution at the BC,

$$f(x^2 y)|_{x=1} = f(y) = 2y + 1.$$

Comparing the argument of the function to the RHS we find

$$f(z) = 2z + 1,$$

where $z = p|_{x=1} = y$. The most general solution at this BC is therefore given by replacing the z (which is p defined only at the BC) with p (which is defined for all x and y)

$$u = f(p) = 2p + 1$$

i.e.

$$u(x, y) = f(x^2 y) = 2(x^2 y) + 1.$$

Note that we do not need to add a $g(p)$ term to this solution because it is fully specified, since at the boundary $z = y$, y is undetermined (i.e. z, and hence p, can take any arbitrary value). This means we can find a solution that applies for all values of y and therefore all values of p, without the need for an additional function. Effectively, there is only one way to write down the solution, an additional g term would be zero for all y and therefore all p.

(ii) $\underline{u = 4 \text{ at the point } (1,1)}$: Test the general solution at the BC,

$$f(x^2 y)|_{x=1,y=1} = f(1) = 4.$$

Comparing the argument of the function to the RHS we find that there are many possible interpretations of this relation for $z = p|_{x=1,y=1} = 1$, including,

$$\begin{aligned} f_1(z) &= z + 3, \\ f_2(z) &= 4z, \\ f_3(z) &= 4. \end{aligned}$$

If we replace the z's with p's as before we find

$$\begin{aligned} f_1(p) &= f_1(x^2 y) = x^2 y + 3, \\ f_2(p) &= f_2(x^2 y) = 4x^2 y, \\ f_3(p) &= f_3(x^2 y) = 4. \end{aligned}$$

Clearly we have a lot of freedom to choose a solution, since our BC here only has to apply to a single point rather than a continuum of points along a line or a curve. Our value for z is a constant ($z = 1$); it has no relation to our independent variables and so $f(z)$ (and $f(p)$) can only be determined for a single value of z (or p). This means there are many forms the solution could take. Thus, our corresponding general solutions require an additional arbitrary term $g(p)$, which can account (albeit in an unknown way) for p at all values of x and y. They are

$$\begin{aligned} u_1(x, y) &= g_1(x^2y) + x^2y + 3, \\ u_2(x, y) &= g_2(x^2y) + 4x^2y, \\ u_3(x, y) &= g_3(x^2y) + 4, \end{aligned}$$

where $g_n(x^2y) = g_n(p)$ is an arbitrary function subject to the condition $g_n(1) = 0$. Any one of these is acceptable as a general solution to the boundary value problem. Clearly g can have various forms and still obey the boundary conditions. For the first general solution u_1, substituting the following three forms of $g_1(p)$ into it will return the 3 particular solutions shown above, in order,

$$\begin{array}{ll} u_1 \text{ requires} & g_1(x^2y) = 0 \\ u_2 \text{ requires} & g_1(x^2y) = 3(x^2y) - 3 \\ u_3 \text{ requires} & g_1(x^2y) = 1 - x^2y \end{array}$$

Check these forms for g are acceptable by subbing them into the general solutions and testing that $u(1, 1) = 4$.

Applying boundary conditions can require a little thought. Remember, if you're not totally convinced that your solution works then check it by

i) subbing the solution into the PDE,

ii) subbing the boundary conditions into the solution.

A solution must satisfy both the original PDE and the given BCs to be a valid solution.

14 Homogeneity in PDE Solutions

Key Points

- General solutions to PDEs
- Homogeneity in PDEs
- Solving inhomogeneous problems

General Solutions to PDEs (ctud)

- **1st order PDEs:** Recalling the general form for a 1st order PDE

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y),$$

we have looked at solutions for when A or B are zero, and for when C and R are zero. The latter method needs modifying if $C \neq 0$. i.e. when

$$Au_x + Bu_y + Cu = 0.$$

Instead of looking for a solution of the form $u(x,y) = f(p)$ we look for $u(x,y) = h(x,y)f(p)$, where $h(x,y)$ is any solution to the PDE.

Let's see how this affects the derivatives. Remember

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x} = f'p_x$$

When $C = 0$	When $C \neq 0$
$u(x,y) = f(p)$	$u(x,y) = h(x,y)f(p)$
$u_x = f'p_x$	$u_x = h_x f + hf'p_x$
$u_y = f'p_y$	$u_y = h_y f + hf'p_y$

Subbing these into our PDE

$$A\left(h_x f + hf'p_x\right) + B\left(h_y f + hf'p_y\right) + Chf = 0.$$

Collecting terms in f and hf'

$$(Ah_x + Bh_y + Ch)f + (Ap_x + Bp_y)hf' = 0$$

Notice that the first bracket has exactly the form of the original PDE. Compare

$$u_x \leftrightarrow h_x \qquad u_y \leftrightarrow h_y \qquad u \leftrightarrow h$$

Since we assumed that $h(x,y)$ was a solution to the PDE, the left-hand term must equal zero to satisfy the RHS of the PDE. This leaves

$$(Ap_x + Bp_y)hf' = 0.$$

So our assumed solution of $h(x,y)f(p)$ has allowed us to eliminate the Cu term, leaving only derivative terms in p. Now, just as we did for the case where C=0, we place restrictions on h and p,

1) Let's assume a non-trivial $h(x, y)$ (i.e. $h \neq 0$) and choose a $p(x, y)$ such that the bracket $= 0$,

$$A(x, y)\frac{\partial p}{\partial x} + B(x, y)\frac{\partial p}{\partial y} = 0.$$

Again, we have the advantage of removing any reference to the form of $f(p)$, so we don't need to know what it is yet.

2) Let's also require that p is constant, meaning that the total derivative of p is zero

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0.$$

Ultimately we find the same relation as earlier between the coefficients and the derivatives by comparing these two equations,

$$\frac{dy}{dx} = \frac{B(x, y)}{A(x, y)}.$$

Integrating this relationship and finding the constant of integration will give us p. However this time the solution will be of the form

$$u(x, y) = h(x, y)f(p)$$

where $h(x, y)$ is any non-trivial solution to the PDE.

e.g. PDF14.1 1$^{\text{st}}$order PDE with 2 derivatives and a term in u: Find the general solution of

$$x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} - 2u = 0$$

Identifying $A = x$ and $B = 2$

$$\frac{dx}{A(x, y)} = \frac{dy}{B(x, y)}$$
$$\frac{dx}{x} = \frac{dy}{2}$$

Integrating we find

$$x = ke^{y/2}$$

Noting that the constant we seek is $p = k = xe^{-y/2}$ we find our general solution is given by

$$u(x, y) = h(x, y)f(xe^{-y/2})$$

where $f(p)$ is an arbitrary function of p and $h(x, y)$ is any solution to the PDE. For example, $h_1 = e^y$, which satisfies the PDE, gives

$$u(x, y) = e^y f(xe^{-y/2}).$$

Alternatively, $h_2 = x^2$ gives

$$u(x, y) = x^2 g(xe^{-y/2}),$$

making sure to distinguish the functions f and g with different labels in particular examples, as they are likely not identical, since $h(x, y)$ is different in each solution. In fact they can be related directly. In this case $g(p) = f(p)/p^2$

$$x^2 g(p) = x^2 \frac{f(p)}{p^2} = \frac{x^2}{\left(xe^{-y/2}\right)^2} f(p) = e^y f(p).$$

So, both solutions work, $u = e^y f(p) = x^2 g(p)$.

- **Homogeneity in PDEs:** Homogeneity in PDEs can actually refer to one of two things; the equation or the problem.

  The *equation* is said to be homogeneous if a solution $u(x, y)$ can be found such that $\lambda u(x, y)$ is also a solution for any constant λ. i.e. multiples of the solution are also solutions.

  The *problem* is said to be homogeneous if in addition to the above, the boundary conditions satisfied by $u(x, y)$ are also satisfied by $\lambda u(x, y)$. These would then be called *homogeneous boundary conditions.*

  For example, the equation we looked at in e.g. PDF13.2

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0,$$

  is homogeneous. We found a solution $u = 2x^2y + 1$ previously, and the since the equation is homogeneous $2u$, $3.5u$, etc will still satisfy the equation, resulting in $RHS = 0$. The constant factor can be factored out making the value of λ irrelevant.

$$x\frac{\partial \lambda u}{\partial x} - 2y\frac{\partial \lambda u}{\partial y} = 0$$

$$x\lambda u_x - 2y\lambda u_y = 0$$

$$\lambda(xu_x - 2yu_y) = 0$$

  Since we know the bracket is zero, as it is the original PDE, it doesn't matter what value λ has as long as it is constant.

  However, in e.g. PDF13.2 we also tried to satisfy the boundary condition $u = 2y + 1$ on the line $x = 1$. If we double u it becomes $u = 4x^2y + 2$, which has the value $u = 4y + 2$ on the line $x = 1$ and no longer satisfies the BC.

  However, the general equation for a 1st order PDE

$$A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} + C(x, y)u = R(x, y),$$

  is not homogeneous as changing u will result in a different value for R.

$$A\lambda u_x + B\lambda u_y + C\lambda u = \lambda(Au_x + Bu_y + Cu) = \lambda R(x, y) \neq R \qquad \text{except for } \lambda = 1.$$

It can be made homogeneous by setting $R = 0$ (recall solving homogeneous ODEs with $RHS = 0$).

$$A\lambda u_x + B\lambda u_y + C\lambda u = \lambda(Au_x + Bu_y + Cu) = \lambda R(x, y) = 0 \qquad \text{for all } \lambda.$$

- **Solving inhomogeneous PDEs:** The reason we're interested in the homogeneity of PDEs is that linear PDEs have a close parallel to the complementary functions and particular integrals of linear ODEs.

  That is to say the general solution to an inhomogeneous PDE can be written as the sum of the solution to the homogeneous problem ($RHS = 0$) and any particular solution.

e.g. PDF14.2 An inhomogeneous PDE: Find the general solution of

$$yu_x - xu_y = 3x.$$

Hence find the most general particular solutions which satisfy the separate boundary conditions (i) $u(x, 0) = x^2$ and (ii) $u(x, y) = 2$ at the point (1,0).

This equation has independent variables that are not associated with u or its derivatives and can be placed on the RHS, hence it is inhomogeneous. First let's solve the homogeneous problem by finding a solution $u(x, y) = f(p)$.

$$\frac{dx}{y} = \frac{dy}{-x} \Rightarrow -xdx = ydy \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c$$

Let's say $p = 2c$ to avoid fractions, so the homogeneous solution is

$$u(x, y) = f(x^2 + y^2)$$

which is the equivalent of our complementary function for linear ODEs. We can find a nice easy particular solution by inspection,

$$u(x, y) = -3y$$

which when subbed into the PDE gives

$$y(-3y)_x - x(-3y)_y = y(0) - x(-3) = 3x$$

.

So our general solution is therefore

$$u(x, y) = f(x^2 + y^2) - 3y.$$

Next we want to solve the boundary problems. We're going to do that by picking a specific form for $f(p)$ suitable for the given BCs.

(i) $\underline{u(x,0) = x^2}$: This requires that

$$u(x,0) = f(x^2 + 0) - 3(0) = f(x^2) = x^2.$$

Thinking about this as $f(z) = z$ can help us arrive at a solution, where $z = p|_{BC}$ (i.e. z is p after the BCs have been applied, so when forming the solution we write p instead of z). The most general particular solution that satisfies this is

$$u(x,y) = x^2 + y^2 - 3y$$

which clearly gives $u = x^2$ when $y = 0$. $z = x^2$ in this case, and since x is not specified by the BC and can take any value, the function is fully determined for any arbitrary value of x and does not require an additional g function term - there is only one way to write down the function.

(ii) $\underline{u(x,y) = 2 \text{ at the point (1,0)}}$: This requires that

$$u(1,0) = f(x^2 + y^2) - 3y|_{x=1,y=0} = f(1) = 2.$$

We could interpret this as $f_1(z) = 2$ or $f_2(z) = 2z$. These give the respective particular solutions

$$u_1(x,y) = 2 - 3y$$
$$u_2(x,y) = 2(x^2 + y^2) - 3y$$

and the respective general solutions

$$u_1(x,y) = g_1(x^2 + y^2) + 2 - 3y$$

$$u_2(x,y) = g_2(x^2 + y^2) + 2(x^2 + y^2) - 3y$$

where g is an arbitrary function subject to the condition $g(1) = 0$. The g terms are required since the function is not fully specified; i.e. there is more than one way to write it down.

Let's quickly test the particular solutions to convince ourselves they work.

For (i) $u(x,y) = x^2 + y^2 - 3y$

$$u_x = 2x \qquad\qquad u_y = 2y - 3$$

Substituting these into the PDE

$$yu_x - xu_y = y(2x) - x(2y - 3) = 2xy - 2xy + 3x = 3x$$

as required by the RHS of the PDE.

For (ii) $u(x,y) = 2 - 3y$

$$u_x = 0 \qquad\qquad u_y = -3$$

Substituting these into the PDE

$$yu_x - xu_y = y(0) - x(-3) = 3x$$

as required by the RHS of the PDE.

Let's also try an arbitrary $g(p) = \sin(x^2 + y^2)$ for a (ii) general solution.

$$u(x, y) = \sin(x^2 + y^2) + 2 - 3y$$

$$u_x = 2x\cos(x^2 + y^2) \qquad\qquad u_y = 2y\cos(x^2 + y^2) - 3$$

Substituting these into the PDE

$$\begin{aligned} yu_x - xu_y &= y[2x\cos(x^2 + y^2)] - x[2y\cos(x^2 + y^2) - 3] \\ &= 2xy\cos(x^2 + y^2) - 2xy\cos(x^2 + y^2) + 3x \\ &= 3x \end{aligned}$$

as required by the RHS of the PDE.

15 General Solutions to 2^{nd} Order PDEs

Key Points

- General solutions to PDEs
- 2nd order PDEs

General Solutions to PDEs (ctud)

- **1^{st} order PDEs:** Recalling the general form for a 1^{st} order PDE

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y),$$

  we have looked at solutions for:

  i) $A = 0$ or $B = 0$,
  ii) $C = 0$ and $R = 0$,
  iii) $R = 0$,
  iv) no terms are zero.

  The first scenario something of a special case. The second and third scenarios are variations on solving homogeneous PDEs with and without a term in u, which can be extended to inhomogeneous problems in the fourth scenario, as discussed in Lecture 15. The techniques used up to now allow us to come up with a solution to essentially any 1^{st} order PDE with 2 independent variables.

- **2^{nd} order PDEs:** Let's move on to higher orders. A given PDE could potentially be of any order and contain any number of independent variables. However, we are going to restrict ourselves to 2^{nd} order PDEs with just two independent variables. This is still a very broad class of equations so we are going to place another greatly simplifying restriction on our PDEs: our coefficients will be constant. The general equation for a 2^{nd} order PDEs with two independent variables and constant coefficients is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x,y),$$

  or in Lagrange notation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = R(x,y).$$

  As we have already noted, a number of important physical relationships fall into this category, such as:

  – the 1D wave equation,
  – the 1D diffusion/heat equation,
  – the 1D Schrodinger equation,

– the 2D Laplace equation.

But we're going to simplify again and only consider equations where $D = E = F = 0$, i.e.

$$Au_{xx} + Bu_{xy} + Cu_{yy} = R(x, y),$$

starting initially with the homogeneous problem, where $R = 0$ also. Notice this equation only contains 2^{nd} order terms, so the following technique will not work for equations that include additional 1^{st} order terms or terms in u. In other words, this method will work for the wave equation and Laplace equation, but not the diffusion equation or Schrodinger equation.

Let's begin as we did previously, by looking for a solution of the form $u(x, y) = f(p)$ where $p = p(x, y)$. For 1^{st} order PDEs we were able to produce an expression that could be factorised for $f(p)$, i.e.

$$\left(A\frac{\partial p}{\partial x} + B\frac{\partial p}{\partial y}\right)\frac{df(p)}{dp} = 0.$$

This allowed us to eliminate all references to the particular form of $f(p)$. We insisted p must be chosen such that the bracket equals 0 and that p must be constant. This restricted the form of p, hence restricting $f(p)$ and ultimately $u(x, y)$.

We would like to do something similar this time. Since we are dealing only with 2^{nd} order terms and assuming a solution of $u(x, y) = f(p)$, by analogy with the earlier technique we would like to arrive at an equation that has a common term of $d^2 f(p)/dp^2$. Let's consider our derivatives again,

$$\begin{aligned} u(x, y) &= f(p) & \qquad u(x, y) &= f(p) \\ \frac{\partial u}{\partial x} &= \frac{df(p)}{dp}\frac{\partial p}{\partial x} & \qquad \frac{\partial u}{\partial y} &= \frac{df(p)}{dp}\frac{\partial p}{\partial y} \end{aligned}$$

These are our 1^{st} order derivatives. Remember, our goal is to eliminate references to $f(p)$ from our PDE so that we don't need to consider its particular form. If we differentiate the above derivatives to obtain 2^{nd} order derivatives, the product rule is going to leave us with terms in both 1^{st} order and 2^{nd} order derivatives of $f(p)$.

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{df}{dp}\frac{\partial^2 p}{\partial x^2} + \frac{d^2 f}{dp^2}\frac{\partial p}{\partial x}.\frac{\partial p}{\partial x} & \qquad \frac{\partial^2 u}{\partial y^2} &= \frac{df}{dp}\frac{\partial^2 p}{\partial y^2} + \frac{d^2 f}{dp^2}\frac{\partial p}{\partial y}.\frac{\partial p}{\partial y} \\ &= \frac{df}{dp}\frac{\partial^2 p}{\partial x^2} + \frac{d^2 f}{dp^2}\left(\frac{\partial p}{\partial x}\right)^2 & \qquad &= \frac{df}{dp}\frac{\partial^2 p}{\partial y^2} + \frac{d^2 f}{dp^2}\left(\frac{\partial p}{\partial y}\right)^2 \end{aligned}$$

Hence, factorising $f(p)$ out via a single term that we can set equal to zero will not be possible. This can be remedied if we require that

$$\frac{\partial p}{\partial x} = a \quad \text{and} \quad \frac{\partial p}{\partial y} = b \quad \Rightarrow \quad \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y^2} = 0,$$

where a and b are constants, giving

$$\frac{\partial^2 u}{\partial x^2} = \frac{df}{dp}.0 + \frac{d^2 f}{dp^2}a^2 \qquad\qquad \frac{\partial^2 u}{\partial y^2} = \frac{df}{dp}.0 + \frac{d^2 f}{dp^2}b^2$$
$$= a^2\frac{d^2 f}{dp^2} \qquad\qquad = b^2\frac{d^2 f}{dp^2}$$

In addition, if the partial derivatives of p are constant then integrating the total derivative leads us directly to the form of p

$$\begin{aligned} dp &= \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \\ dp &= adx + bdy \end{aligned}$$

By integrating the partial derivatives separately and comparing the results we can determine p,

$$\begin{aligned} p &= \int \frac{\partial p}{\partial x}dx = \int adx = ax + g(y) \\ p &= \int \frac{\partial p}{\partial y}dy = \int bdy = by + h(x) \\ p &= ax + by. \end{aligned}$$

i.e. p must be a linear function of x and y, so our solution will be of the form

$$u = f(ax + by).$$

Let's quickly look at the effect this has on the derivatives, using the x derivatives as an example

$$\frac{\partial u}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x} = a\frac{df(p)}{dp}$$
$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial}{\partial x}\frac{df(p)}{dp} = a\frac{d^2 f(p)}{dp^2}\frac{\partial p}{\partial x} = a^2\frac{d^2 f(p)}{dp^2}$$

So if we assume a solution of this form then we can evaluate our 1st and 2nd order derivatives ready to be substituted into the equation

$$\frac{\partial u}{\partial x} = a\frac{df(p)}{dp} \qquad\qquad \frac{\partial u}{\partial y} = b\frac{df(p)}{dp}$$

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{d^2 f(p)}{dp^2} \qquad \frac{\partial^2 u}{\partial x \partial y} = ab\frac{d^2 f(p)}{dp^2} \qquad \frac{\partial^2 u}{\partial y^2} = b^2\frac{d^2 f(p)}{dp^2}$$

Subbing these into the PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$$Aa^2\frac{d^2 f(p)}{dp^2} + Bab\frac{d^2 f(p)}{dp^2} + Cb^2\frac{d^2 f(p)}{dp^2} = 0$$

$$\left(Aa^2 + Bab + Cb^2\right)\frac{d^2 f(p)}{dp^2} = 0.$$

This is the form we were looking for. We can ignore the solution to this equation where $d^2f(p)/dp^2 = 0$. Since the solution to any 2^{nd} order derivative that equals zero is just a first order polynomial and $u = f(ax+by)$, then this leads us to the solution $u = kx+ly+m$. This is a trivial solution, since it means that each individual 2^{nd} order derivative in our PDE is zero, which will obviously give a $RHS = 0$.

Instead we can find a solution independent of the form of $f(p)$ if we require that the constants a and b satisfy the equation

$$Aa^2 + Bab + Cb^2 = 0.$$

This is a quadratic equation, which we can solve. First divide by a^2

$$A + B\frac{b}{a} + C\frac{b^2}{a^2} = 0 \quad \rightarrow \quad A + B\lambda + C\lambda^2 = 0$$

Thus the factor b/a is given by the two solutions of

$$\frac{b}{a} = \lambda = \frac{-B \pm (B^2 - 4AC)^{1/2}}{2C}$$

If we call these two ratios λ_1 and λ_2 then any functions of the two variables p_1 and p_2

$$p_1 = x + \lambda_1 y \qquad\qquad p_2 = x + \lambda_2 y$$

will be solutions of the original PDE. Absorbing the a term into the λ terms is fine, since only the relative weighting of x and y in p is important. The particular form of the solution can be adjusted based on the given problem. You might consider then that the we have arrived at a solution of the form $u = f(p)$ where $p = x + (b/a)y$, which is equivalent to our assumption of $p = ax + by$.

Finally, as p_1 and p_2 are in general not the same, the general solution is given by

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

where f and g are arbitrary functions. This is a very simple solution, as in order to find find u we merely need to examine the coefficients of the derivatives.

e.g. PDF15.1 The general solution to the 1D wave equation: Find the general solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

First, we identify the coefficients as $A = 1$, $B = 0$ and $C = -1/c^2$. λ_1 and λ_2 are solutions to the following equation

$$A + B\lambda + C\lambda^2 = 0,$$

$$1 - \frac{\lambda^2}{c^2} = 0.$$

So λ_1 and λ_2 are given by

$$\lambda = \pm c.$$

Or using the quadratic formula

$$\begin{aligned}
\lambda &= \frac{-B \pm (B^2 - 4AC)^{1/2}}{2C} \\
&= \frac{\pm(4/c^2)^{1/2}}{-2/c^2} \\
&= \frac{\pm 2/c}{-2/c^2} \\
&= \pm c
\end{aligned}$$

Hence, $\lambda_1 = c$ and $\lambda_2 = -c$, thus the solution contains the forms of

$$p_1 = x + \lambda_1 t = x + ct, \qquad\qquad p_2 = x + \lambda_2 t = x - ct,$$

giving a general solution of

$$u(x, y) = f(x + ct) + g(x - ct),$$

where f and g are arbitary functions.

- **Limitations of the method:** Above we mentioned that this method will work for the wave equation and Laplace equation, but not the diffusion equation or the Schrodinger equation. We can see that this is because the latter have 1$^{\text{st}}$ order derivatives as well as 2$^{\text{nd}}$ order. So $f(p)$ cannot be eliminated. e.g. Let's examine the wave equation and diffusion equation to find an equation of the form

$$\left(Aa^2 + Bab + Cb^2\right) \frac{d^2 f(p)}{dp^2} = 0.$$

By substituting the derivatives in terms of a and b we found earlier into these equations, we find

The wave equation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} &= 0, \\
\left(a^2 - \frac{1}{c^2} b^2\right) \frac{d^2 f(p)}{dp^2} &= 0,
\end{aligned}$$

The diffusion equation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} - \frac{1}{k^2}\frac{\partial u}{\partial t} &= 0, \\
a^2 \frac{d^2 f(p)}{dp^2} - \frac{1}{k^2} b \frac{df(p)}{dp} &= 0.
\end{aligned}$$

Clearly $f(p)$ can only be eliminated for the wave equation, by assuming the bracket is equal to zero. Hence, this method will not work for the diffusion equation.

16 Inhomogeneous 2$^{\text{nd}}$ Order PDEs with Boundary Conditions

Key Points

- General solutions to PDEs
- 2$^{\text{nd}}$ order PDE with boundary conditions
- Inhomogeneous 2$^{\text{nd}}$ order PDEs

General Solutions to PDEs (ctud)

- **2nd order PDE with boundary conditions:** As before, we need to know how to apply boundary conditions to a problem to ensure it fits the physical description of the system we're studying.

  e.g. PDF16.1 2nd order PDE with boundary conditions: Solve

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

  subject to the boundary conditions $u(0, y) = 0$ and $u(x, 1) = x^2$.

  Starting by identifying the three coefficients, $A = 1$, $B = 2$, $C = 1$, we construct the quadratic equation

$$1 + 2\lambda + \lambda^2 = 0,$$
$$(1 + \lambda)(1 + \lambda) = 0,$$
$$\lambda = -1, -1.$$

  We have a repeated root, so using the same trick we applied to ODEs to ensure linear indepedence, we'll multiply one of our solutions by x, giving a general solution

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

$$u(x, y) = f(x - y) + xg(x - y).$$

  Note: It is worth pointing out here that f and g are functions of the same argument, but not necessarily the same function (one could be polynomial, the other sinusoidal for instance). The x guarantees linear independence either way, but in particular for when they are the same function.
  Now we'll consider the boundary conditions. This time both are to be applied together, unlike previous examples where they represented independent scenarios.

  (i) $\underline{u(0, y) = 0}$: This condition means,

$$u(x, y)\Big|_{x=0} = f(x - y) + xg(x - y)\Big|_{x=0} = 0.$$

One solution to this is the case where the $f(p)$ term and $xg(p)$ term are independently equal to zero

$$f(x-y)=0 \qquad\qquad xg(x-y)=0$$

However, we know that $x=0$ at this boundary condition, so the function $g(p)$ need not be zero to satisfy it and this is therefore not a trivial solution; thus the solution can be written as

$$u(x,y)=xg(x-y),$$

which obeys the condition $u=0$ when $x=0$ for all y.

(ii) $\underline{u(x,1)=x^2}$: This requires that

$$u(x,1)=xg(x-1)=x^2$$

$$\begin{aligned} g(x-1)&=x\\ g(z)&=z+1\\ g(p)&=p+1 \end{aligned}$$

where $z=x-1=p|_{y=1}$. Hence the particular solution, accounting for the BCs is

$$u(x,y)=xg(p)=x(x-y+1).$$

- **2nd order inhomogeneous PDE:** Finally, let's look at a problem with $RHS \neq 0$.

e.g. PDF16.2 2nd order inhomogeneous PDE: Find the general solution of

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=6(x+y).$$

Starting by identifying the three coefficients, $A=1$, $B=0$, $C=1$, we construct the quadratic equation,

$$\begin{aligned} 1+\lambda^2&=0\\ \lambda^2&=-1\\ \lambda&=\pm i \end{aligned}$$

Hence the complementary function is

$$u(x,y)=f(x+\lambda_1 y)+g(x+\lambda_2 y)$$

$$u(x,y)=f(x+iy)+g(x-iy).$$

As with other simple particular integrals, we can find this by inspection. Any solution that satisfies the RHS will do. Consider that the RHS has an x term and a y term, and we're dealing with 2^{nd} order derivatives, thus logically we should start with cubed terms. Trying $u(x,y)=(x^3+y^3)$ gives

$$\begin{aligned} u_x&=3x^2 & \qquad u_y&=3y^2\\ u_{xx}&=6x & \qquad u_{yy}&=6y \end{aligned}$$

$$u_{xx} + u_{yy} = 6(x + y)$$

Hence, the particular solution satisfies the equation. An alternative approach is to integrate the RHS terms independently, a number of times equal to the order of the derivatives on the LHS, in this case twice

$$\int\int 6x dx dx = \int 3x^2 + c_1 dx = x^3 + c_1 x + c_2$$

$$\int\int 6y dy dy = \int 3y^2 + d_1 dy = y^3 + d_1 y + d_2$$

These two expressions could be added together to form our particular solution, but by examining the LHS of our PDE we can see that the first order terms in x and y and the constant terms are not required (they will disappear when the derivatives of our particular solution are evaluated); they can be discarded, leaving us with $x^3 + y^3$ as we found by inspection.

Hence, the general solution is

$$u(x, y) = f(x + iy) + g(x - iy) + x^3 + y^3.$$

- **Extra examples:** The following are extra examples of finding the general solution to PDEs.

e.g. PDF16.3 2nd order inhomogeneous PDE (with differential BC): Solve

$$6u_{xx} - 5u_{xy} + u_{yy} = 14$$

subject to $u = 2x + 1$ and $u_y = 4 - 6x$, both on the line $y = 0$.

Identifying the coefficients as $A = 6$, $B = -5$, $C = 1$, we solve the homogeneous problem

$$6 - 5\lambda + \lambda^2 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, 3$$

Hence the complementary function is given by

$$u(x, y) = f(x + 2y) + g(x + 3y).$$

Examining the RHS we note that we're likely looking for a particular solution with a term in x^2 (since u_{xx} will give constant term) and y^2 (since u_{yy} will give a constant term). So try $u = x^2 + y^2$. We find

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 2$$

Subbing in,

$$6(2) - 5(0) + 2 = 14$$

The solution works. So our general solution is given by

$$u(x, y) = f(x + 2y) + g(x + 3y) + x^2 + y^2.$$

Now let's consider our BCs. Firstly, $u = 2x + 1$ at $y = 0$ gives

$$u(x, 0) = f(x) + g(x) + x^2 = 2x + 1$$

Note that at $y = 0$, the arbitrary functions f and g are only functions of x. We are going to differentiate this solution w.r.t x

$$\frac{d}{dx}u(x, 0) = f'(x) + g'(x) + 2x = 2,$$

because we'll need this when applying our second BC, which involves a bit more work since it's a differential BC. Before applying the second BC let's consider our y derivative,

$$\frac{\partial u}{\partial y} = \frac{df}{dp_1}\frac{\partial p_1}{\partial y} + \frac{dg}{dp_2}\frac{\partial p_2}{\partial y} + 2y,$$

$$\frac{\partial u}{\partial y} = 2\frac{df}{dp_1} + 3\frac{dg}{dp_2} + 2y,$$

But using our first BC we established that f and g are functions of x alone at $y = 0$, i.e. $p_1 = p_2 = x$, so

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 2\frac{df}{dx} + 3\frac{dg}{dx} = 2f'(x) + 3g'(x) = 4 - 6x.$$

Now we have two equations, and two unknown functions, so we can solve them simultaneously

$$f'(x) + g'(x) + 2x = 2, \qquad\qquad 2f'(x) + 3g'(x) = 4 - 6x.$$

Rearranging our 1$^{\text{st}}$ equation

$$f'(x) = 2 - 2x - g'(x).$$

Subbing this into the 2$^{\text{nd}}$ equation

$$2[2 - 2x - g'(x)] + 3g'(x) = 4 - 4x + g'(x) = 4 - 6x,$$

$$g'(x) = -2x.$$

Hence

$$f'(x) = 2 - 2x - x = 2.$$

Integrating g

$$g'(x) = -2x,$$

$$g(x) = -x^2 + c.$$

Using our expression for $u(x, 0)$

$$u(x, 0) = f(x) + g(x) + x^2 = f(x) - x^2 + c + x^2 = 2x + 1$$

$$f(x) = 2x + 1 - c$$

Looking at the function arguments, where $z_n = p_n|_{y=0}$

$$f(z_1) = 2z_2 + 1 - c \qquad g(z_2) = -z_2^2 + c$$
$$f(x+2y) = 2(x+2y) + 1 - c \qquad g(x+3y) = -(x+3y)^2 + c$$

So, putting it all together for the particular solution at the BCs

$$\begin{aligned} u(x,y) &= f\,(x+2y) + g\,(x+3y) + x^2 + y^2 \\ &= [2(x+2y) + 1 - c] + [-(x+3y)^2 + c] + x^2 + y^2 \\ &= 2x + 4y + 1 - c - 9y^2 - 6xy - x^2 + c + x^2 + y^2 \\ &= 1 + 2x + 4y - 6xy - 8y^2 \end{aligned}$$

e.g. PDF16.4 2nd order inhomogeneous PDE with BCs: Solve the equation

$$ck\frac{\partial^2 u}{\partial x^2} + (c^2 + k^2)\frac{\partial^2 u}{\partial x \partial y} + ck\frac{\partial^2 u}{\partial y^2} = 6ck(2x^2 + y)$$

given that $u(x,y) = 2 + (c/k)^2([c/k] - 1)$ where the line $x = 1$ interects the line $y = c/k$. Identifying the coefficients as $A = ck$, $B = c^2 + k^2$, $C = ck$, we solve the homogeneous problem

$$ck + (c^2 + k^2)\lambda + ck\lambda^2 = 0$$
$$(c\lambda + k)(k\lambda + c) = 0$$
$$\lambda = -\frac{k}{c}, -\frac{c}{k}$$

Hence the complementary function is given by

$$u(x,y) = f\left(x - \frac{k}{c}y\right) + g\left(x - \frac{c}{k}y\right).$$

Examining the RHS we note that we're likely looking for a particular solution with a term in x^4 (since u_{xx} will give a term in x^2) and y^3 (since u_{yy} will give a term in y). So try $u = x^4 + y^3$. We find

$$\frac{\partial u}{\partial x} = 4x^3 \qquad \frac{\partial u}{\partial y} = 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 \qquad \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 6y$$

Subbing in,

$$ck(12x^2) + (c^2 + k^2)(0) + ck(6y) = 6ck(2x^2 + y)$$

The solution works. So our general solution is given by

$$u(x,y) = f\left(x - \frac{k}{c}y\right) + g\left(x - \frac{c}{k}y\right) + x^4 + y^3.$$

To apply our boundary conditions consider that

$$f\left(x - \frac{k}{c}y\right)\Big|_{x=1, y=c/k} = f(0) = f(p_1)$$

A function of zero can be (though is not guaranteed to be) equal to zero. So if we assume $f(0) = 0$ then we can write

$$u(x, y) = f(p_1) + g\left(x - \frac{c}{k}y\right) + x^4 + y^3.$$

where $f(p_1) = 0$ when $p_1 = 0$. Now, since

$$g\left(x - \frac{c}{k}y\right)\Big|_{x=1, y=c/k} = g\left(1 - \frac{c^2}{k^2}\right)$$

we can try

$$g(p_2) = 1 - \frac{c^2}{k^2}.$$

This gives

$$u(x, y)|_{x=1, y=c/k} = \left(1 - \frac{c^2}{k^2}\right) + 1^4 + \frac{c^3}{k^3} = 2 + \left(\frac{c}{k}\right)^2\left(\frac{c}{k} - 1\right)$$

which obeys the boundary conditions. Hence the most general solution that accommodates the boundary conditions is given by

$$u(x, y) = x - \frac{c}{k}y + x^4 + y^3 + f(p_1)$$

$$u(x, y) = x(x^3 + 1) + y\left(y^2 - \frac{c}{k}\right) + f(p_1)$$

where $f(p_1) = 0$ when $p_1 = 0$.
Note: Remember, our boundary condition is a point, where the lines $x = 1$ and $y = c/k$ meet. Hence, it makes sense that we cannot eliminate $f(p_1)$ entirely, the function is only specificed for that point with these BCs.

e.g. PDF16.5 MM2 May 2018 Q4(f): Consider the partial differential equation

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = 0.$$

Find the most general solution such that $u = 5$ on the parabola $y = x^2$. [4 marks] Start by assuming a solution of $u(x, y) = f(p)$ where $p = p(x, y)$. Identify $A = 1$, $B = 2x$.

$$\frac{dx}{1} = \frac{dy}{2x}$$

$$\int 2x dx = \int dy$$

$$x^2 = y + c$$

$$c = x^2 - y$$

So $p = c = x^2 - y$. Hence we have a solution

$$u(x, y) = f(x^2 - y)$$

Applying the boundary condition we require $u = 5$ at $y = x^2$, hence since

$$f(x^2 - y)|_{y=x^2} = f(0) = 5,$$

we obtain the solution

$$u(x, y) = g(x^2 - y) + 5$$

where the function g is subject to the condition $g(0) = 0$.

17 Bonus: Green's Function Method (Not examinable)

Key Points

- Green's function method

Introduction to Green's function method

- **Introduction:** This method is similar to variation of parameters, in that you must first know the complementary function before finding the particular integral and general solution.

  The difference is that once the Green's function for the LHS of your ODE has been found for given boundary conditions, then the solution for any RHS (i.e. any $f(x)$) can be written down immediately, albeit in the form of an integral.

  There are different ways of approaching the Green's function method: the superposition of eigenfunctions, as an application to PDEs or based on the properties of the Dirac delta function. We will use this last approach.

- **Dirac delta function $\delta(z)$:** The Dirac delta function is often associated with Fourier transforms and was originally created by the physicist Paul Dirac to model the density of point masses/charges.

  It has the unusual property of being zero everywhere except at the origin, where it is infinite. You might think of it like an instantaneous pulse. It is also contrained to give a value of 1 when integrated. It is primarily used as a part of a product when integrating, so having a value of 1 makes those integrals easier to handle.

$$\delta(z) = \begin{cases} +\infty & \text{for } z = 0, \\ 0 & \text{for } z \neq 0. \end{cases}$$

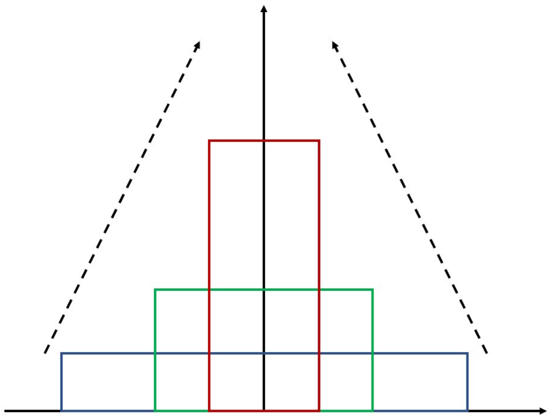
$$\int_{-\infty}^{\infty} \delta(z) dz = 1$$

  and

$$\int_{a}^{b} \delta(z - x) dz = 1$$

  where $a \leq x \leq b$.

  The Dirac delta is not a function in the traditional sense, so its unusual properties can be difficult to conceptualise. You might wonder how a function of zero width can have a non-zero area when integrated. Consider a short, wide rectangle of area 1 unit. Let's make the rectangle thinner and taller, maintaining the same area of 1 unit. It should be clear that in the limit where the width tends to zero, the height tends to infinity and we can maintain our area of 1 unit. This is an example of a step function.



We tend to use $\delta(z)$ to determine the value of a function $f(z)$ in the limits $a \leq x \leq b$. If we integrate the product of $\delta(z)$ and our function of interest

$$\int_a^b \delta(z-x) f(z) dz$$

we note that $\delta(z-x)$ is zero unless $z = x$, and therefore the integral only has a value when $f(z) = f(x)$, which does not depend on z, thus

$$\int_a^b \delta(z-x) f(x) dz = f(x) \int_a^b \delta(z-x) dz = f(x).$$

Hence for $a \leq x \leq b$,

$$\int_a^b \delta(z-x) f(z) dz = \begin{cases} f(x) & \text{when } z = x, \\ 0 & \text{when } z \neq x. \end{cases}$$

We will use this last property directly in the Green function method.

- **Green's function method:** Returning to Green's function, let's consider the n^{th} order linear ODE equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

For brevity we're going to represent the LHS by $Ly(x)$ (a linear differential operator acting on $y(x)$), so

$$Ly(x) = f(x)$$

Let's suppose there is a function $G(x,z)$ (a Green's function) which, subject to boundary conditions in the range $a \leq x \leq b$, gives the general solution

$$y(x) = \int_a^b G(x,z) f(z) dz$$

where z is the integration variable. If we apply our linear differential operator L to both sides of this equation and invoke the product rule we obtain

$$Ly(x) = \int_a^b [LG(x,z)] f(z) dz = f(x),$$

Remember, L is a linear differential operator in x and only $G(x, z)$ here is dependent on x. Making use of a standard property of Dirac functions shown above, we realise that

$$\int_a^b \delta(z-x) f(z) dz = f(x),$$

for $z = x$. This means that for any arbitrary $f(x)$, where $a \leq x \leq b$, we require that

$$LG(x, z) = \delta(z-x).$$

That is to say, the Green's function $G(x, z)$ must satisfy the original ODE which gives a RHS that is equal to the delta function, instead of the inhomogeneous function. i.e. compare the following

$$Ly(x) = f(x)$$
$$LG(x, z) = \delta(z-x)$$

Important: This means that as long as $x \neq z$ we are effectively solving a homogeneous equation for G where the $RHS = 0$. i.e. for $x < z$ and $x > z$ we can express G in terms of homogeneous solutions. Then, since G depends on L but not on $f(x)$, once we have G we can write down a solution to $Ly(x) = f(x)$ for any arbitrary $f(x)$!

This last equation is a restriction on $G(x, z)$. There are two more restrictions we must place on it for the arguments made here to hold up.

– The first is that the general solution $y(x)$ obeys the given boundary conditions. If our boundary conditions are homogeneous this condition is easily fulfilled if we demand that $G(x, z)$ also obeys the boundary conditions when it is only considered to be a function of x.

  To clarify, homogeneous boundary conditions mean that $y(x)$ and its derivatives are equal to zero at specified points. So if $y(a) = 0$ and $y(b) = 0$ then $G(a, z) = G(b, z) = 0$.

  Inhomogeneous boundary conditions, where $y(a) = \alpha$, $y(b) = \beta$ or $y(0) = y'(0) = \gamma$ and $\alpha, \beta, \gamma \neq 0$, are slightly more complicated. We need to make a change of variable to force the boundary conditions to be homogeneous. We can achieve $u(a) = 0$, $u(b) = 0$ or $u(0) = u'(0) = 0$ with the substitution

$$u = y - h(x)$$

  where $h(x)$ is an $(n-1)^{\text{th}}$ order polynomial that obeys the boundary conditions. For a 2^{nd} order ODE the change of variable would require a 1^{st} order polynomial

$$u = y - (mx + c).$$

  If our boundary conditions are $y(a) = \alpha$ and $y(b) = \beta$: $y = mx+c$ is the straight line through the points (a, α), (b, β), with $m = (\alpha-\beta)/(a-b)$ and $c = (\beta a - \alpha b)/(a-b)$.

  If our boundary conditions are $y(0) = y'(0) = \gamma$: $y = mx + c$ is the straight line through $(0, \gamma)$ with $m = c = \gamma$.

– The second extra restriction concerns the continuity or discontinuity of $G(x, z)$ and its derivatives at $z = x$. The statement that $LG(x, z) = \delta(z - x)$ means that adding together all the derivatives of G up to order n gives an infinite value at the point $z = x$.

To understand what this means we examine the ODE,

$$a_n(x)\frac{d^nG}{dx^n} + a_{n-1}(x)\frac{d^{n-1}G}{dx^{n-1}} + ... + a_0(x)G = \delta(z - x) = \infty \quad \text{at } z = x.$$

As this is a sum with infinite value, at least one of the terms of the ODE must be infinite; specifically, one of the derivatives must be infinite, as the coefficients of the ODE are required to be finite and continuous. In fact it must be the n^{th} derivative, because if it was any derivative $< n^{\text{th}}$ the series would terminate as you cannot differentiate a function with infinite value.

If we examine the integral of the ODE in the small interval $[z - \epsilon, z + \epsilon]$ around z,

$$\lim_{\epsilon\to 0}\int_{z-\epsilon}^{z+\epsilon}\left[a_n(x)\frac{d^nG}{dx^n} + a_{n-1}(x)\frac{d^{n-1}G}{dx^{n-1}} + ... + a_0(x)G\right]dx = \lim_{\epsilon\to 0}\int_{z-\epsilon}^{z+\epsilon}\delta(z - x)dx = 1$$

we realise, since G and its derivatives are continuous they equate to zero over this very small interval as $\epsilon \to 0$ (effectively we're integrating a normal function over an interval of zero)

$$\lim_{\epsilon\to 0}\int_{z-\epsilon}^{z+\epsilon} a_{n-1}(x)\frac{d^{n-1}G}{dx^{n-1}}dx + ... + \lim_{\epsilon\to 0}\int_{z-\epsilon}^{z+\epsilon} a_0(x)Gdx = 0.$$

The exception is the n^{th} derivative, which we have established is not continuous, and so

$$\lim_{\epsilon\to 0}\int_{z-\epsilon}^{z+\epsilon} a_n(x)\frac{d^nG}{dx^n}dx = \lim_{\epsilon\to 0}\int_{z-\epsilon}^{z+\epsilon}\delta(z - x)dx = 1.$$

Integrating by parts we find,

$$\lim_{\epsilon\to 0}\left[a_n(x)\frac{d^{n-1}G}{dx^{n-1}}\right]_{z-\epsilon}^{z+\epsilon} = 1.$$

This means the $(n - 1)^{\text{th}}$ derivative is discontinuous at $1/a_n$ as

$$a_n(z)\left[\left.\frac{d^{n-1}G}{dx^{n-1}}\right|_{x=z+} - \left.\frac{d^{n-1}G}{dx^{n-1}}\right|_{x=z-}\right] = 1,$$

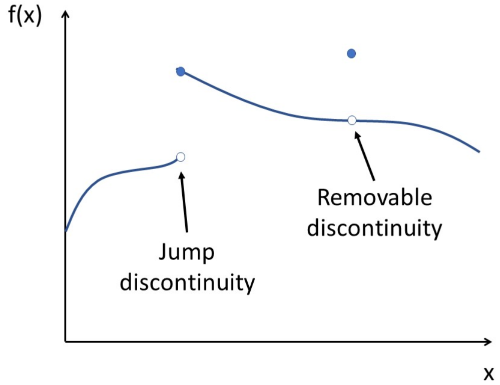
$$\left.\frac{d^{n-1}G}{dx^{n-1}}\right|_{x=z+} - \left.\frac{d^{n-1}G}{dx^{n-1}}\right|_{x=z-} = \frac{1}{a_n(z)}.$$

So we have restricted G as follows: $G(x, z)$ and its derivatives up to $(n - 2)$ are continuous at $z = x$, but $d^{n-1}G/dx^{n-1}$ has a discontinuity of $1/a_n(z)$ at $z = x$.

Aside: Continuity - A function $f(x)$ is continuous at an internal point $x = x_0$ (a point somewhere between your boundary conditions) if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

i.e. if you check the limit as x tends towards x_0, does it have the same value as the function evaluated at x_0? If so the function is continuous at that point, if not it is discontinous.



To summarise, the properties of Green's function $G(x,z)$ for an n^{th} order linear ODE are as follows:

- $G(x,z)$ obeys the original ODE but with $RHS = \delta(z-x)$, instead of $RHS = f(x)$.
- When considered as a function only of x, $G(x,z)$ obeys the specified (homogeneous) boundary conditions of $y(x)$.
- The derivatives of $G(x,z)$ w.r.t x up to order $(n-2)$ are continous at $z=x$, but order $(n-1)$ has a discontinuity of $1/a_n(z)$ at $z=x$.

e.g. PDF17.1 Use Green's function to solve

$$\frac{d^2y}{dx^2} + y = \csc x = \frac{1}{sinx}$$

subject to the boundary conditions $y(0) = y(\pi/2) = 0$.

Green's function $G(x,z)$ must satisfy

$$\frac{d^2G(x,z)}{dx^2} + G(x,z) = \delta(z-x)$$

Recalling that

$$\delta(z) = 0 \text{ for } z \neq 0$$

we can state that as long as $x \neq z$ then $\delta(z-x) = 0$. In which case we can begin by finding the general solution to the homogeneous equation - the complementary function, $G(x,z)$. Effectively we want to solve

$$G'' + G = 0 \qquad \text{for } x < z \text{ and } x > z,$$

$$G'' = -G$$

We want a solution to an equation where the 2^{nd} derivative is equal to minus the original function - i.e. $\sin x$, $\cos x$ or some combination of the two. But since this is a 2^{nd} order ODE, its 1^{st} derivative is required to have a discontinuity at $z = x$, y_c must consist of different superpositions on either side of $z = x$, so

$$G(x, z) = \begin{cases} A(z)\sin x + B(z)\cos x & \text{for } x < z, \\ C(z)\sin x + D(z)\cos x & \text{for } x > z. \end{cases}$$

Restriction 1: $G(x, z)$ should obey the homogeneous boundary conditions. Let's see if we can restrict $G(x, z)$ using our boundary conditions to help determine the functions. Important: Our BC's give values for $f(x_1)$ and $f(x_2)$. We will insist that $x_1 < z$ and $x_2 > z$, so they lie on different sides of the discontinuity. If we have a $f(x_1)$ and $f'(x_1)$ both BCs clearly apply to either $x > z$ or $x < z$; i.e. we only examine one side of the discontinuity, not both.

$G(0, z) = 0$ implies that in one of our equations we only need a sin term, so $B(z) = 0$. $G(\pi/2, z) = 0$ implies that in our other equation we only need a cos term, so $C(z) = 0$. This gives

$$G(x, z) = \begin{cases} A(z)\sin x & \text{for } x < z, \\ D(z)\cos x & \text{for } x > z, \end{cases}$$

ensuring we have maintained our discontinuity, as the two solutions hold different values at $z = x$.

Restriction 2: $G(x, z)$ is continuous at $z = x$ and dG/dx has a discontinuity of $1/a_2(z) = 1$. If $G(x, z)$ is continuous at $z = x$, then its two solutions must be equal at that point, so

$$D(z)\cos z = A(z)\sin z$$

$$D(z)\cos z - A(z)\sin z = 0$$

If dG/dx has a discontinuity of $1/a_2(z) = 1$ at $z = x$, then if we differentiate and set the result equal to the value of the discontinuity, we obtain

$$-D(z)\sin z - A(z)\cos z = 1.$$

Solving these equations for $A(z)$ and $D(z)$ we find

$$A(z) = -\cos z$$

$$D(z) = -\sin z$$

Thus we have

$$G(x, z) = \begin{cases} -\cos z \sin x & \text{for } x < z \\ -\sin z \cos x & \text{for } x > z. \end{cases}$$

Now we can obtain the general result that obeys the boundary conditions $y(0) = y(\pi/2) = 0$

$$y(x) = \int_0^{\pi/2} G(x, z) \csc z dz$$

$$= -\cos x \int_0^x \sin z \csc z dz - \sin x \int_x^{\pi/2} \cos z \csc z dz$$
$$= -\cos x \int_0^x 1 dz - \sin x \int_x^{\pi/2} \frac{\cos z}{\sin z} dz$$
$$= -x \cos x + \sin x \ln(\sin x)$$

- **e.g. PDF17.2** Step by step guide to Green's function method:

  Solve
  $$y'' = x$$
  subject to the conditions $f(0) = f'(0) = 0$.

  (1) Write down two solutions to the homogeneous equation. i.e. These solutions should be able to give $RHS = 0$ when substituted into the equation.
  $$G(x,z) = \begin{cases} ax + b & \text{for } x < z \\ cx + d & \text{for } x > z. \end{cases}$$

  (2) Apply the boundary conditions. Conditions should be applied as $z_1 < z < z_2$. If both conditions given can only apply to one half of the discontunity then you can eliminate one equation. We have $f(0) = f'(0) = 0$, which implies that we only have either z_1 or z_2. Let's say that $z_1 = 0$ is our lower bound. $f(0) = 0$ would therefore mean that $b = 0$, as $a \times 0 + 0 = 0$. And $f'(0) = 0$ means that $a = 0$, as differentiating leaves us with just $a = 0$. Hence
  $$G(x,z) = \begin{cases} 0 & = G_1, \text{ for } x_1 \leq x \leq z \\ cx + d & = G_2, \text{ for } z \leq x \leq x_2. \end{cases}$$
  We can write $\leq$ instead of $<$ as G is continuous at $z = x$ even though its derivative is not.

  (3) Enforce the condition that $G_2 - G_1 = 0$ at $z = x$. The order is important; function after z - function before z. This comes from the integral definitions of the restraining conditions.
  $$(cz + d) - (0) = 0$$
  $$cz + d = 0$$

  (4) Enforce the condition that $G_2' - G_1' = 1$ at $z = x$.
  $$(c) - (0) = 1$$
  $$c = 1$$
  Solving for d gives $d = -z$. Hence
  $$G(x,z) = \begin{cases} 0 & = G_1, \text{ for } x_1 \leq x \leq z \\ z - x & = G_2, \text{ for } z \leq x \leq x_2. \end{cases}$$

(5) Integrate to find $y(x)$. But be aware that we are integrating w.r.t z, not x, so the limits of the integrals appear to switch when compared to the inequalities above. Basically, set up as above, the top equation is always your higher integral in z and the bottom equation is always your lower integral in z.

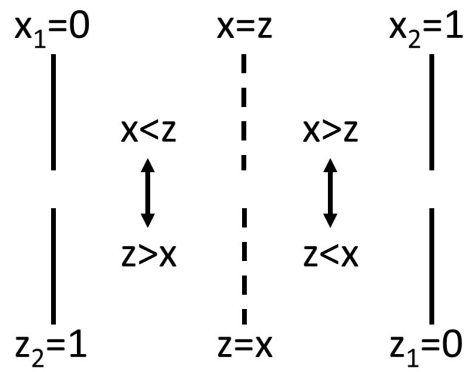
$f(z)$ is your particular integral at $z = x$. i.e. the RHS of your ODE, but in terms of z.

$$\begin{aligned}
y(x) &= \int_{z_1}^{z_2} G(x,z)f(z)dz \\
&= \int_{z_1}^{z=x} (z-x)f(z)dz + \int_{z=x}^{z_2} 0 \times f(z)dz \\
&= \int_0^x (z-x)zdz = \int_0^x xzdz - \int_0^x z^2dz \\
&= \left[\frac{xz^2}{2}\right]_0^x - \left[\frac{z^3}{3}\right]_0^x \\
&= \frac{x^3}{2} - \frac{x^3}{3} = \frac{x^3}{6}
\end{aligned}$$

If we test this solution,

$$\begin{aligned}
y(x) &= \frac{x^3}{6} \\
y'(x) &= \frac{x^2}{2} \\
y''(x) &= x
\end{aligned}$$

which agrees with our ODE.



Queries 1

- (1) When applying Restriction 1 $G(0,z) = 0$ allows us to restrict the first equation to $A(z)\sin(x)$. The other boundary condition of $G(\pi/2, 0) = 0$ allows us to restrict the second equation to $D(z)\cos(x)$.
- (2) The Green's function method works within the given boundary conditions, which is why you must integrate from BC_1 to x and x to BC_2. Outside of this range we cannot be sure our solution will work.

- (3) The (n-1)th derivative (1st derivative in the case of a 2nd order ODE) is discontinuous at $1/a_n$ (so $1/a_2$ for 2nd order, the coefficient of the 2nd derivative). Showing why this is true involves examining the behaviour of this derivative in proximity to z. Simply put $a_m G^{(n-2)}$ and lower derivatives tend to zero and $a_n G^{(n-1)}$ tends to one. So $G^{(n-1)}$ tends to $1/a_n$.

Queries 2

- (4) Our Green's function solutions can only be assumed to work for the given boundary conditions. In fact, in work shop 3 you will attempt to solve the same equation as in the lecture, but with different boundary conditions and hopefully arrive at a different solution for G. So, yes, it is only valid between our upper and lower limits. The infinite limit in the definition is an indication that this technique can have infinite limits, but any given problem typically does not.
- (5) We have different equations either side of $x = a$ since we require our 1st derivative to have a discontinuity at $x = a$ (for 2nd order, $(n-1)$th derivative in general).
- (6) Regarding boundary conditions, if given two values for x we would aim to integrate from lower BC to $z = x$ $(x > z)$ and $z = x$ to upper BC $(x < z)$. If, as in the example I mentioned above, we have only one value of x to work with [e.g $x = 0$: $f(0) = f'(0) = 0$] then we would generally aim to eliminate one of the two equations entirely. This should be possible as the Green's method requires homogeneous BCs, i.e. they should be $BC = 0$. If they are not, we adjust them until they are.
- (7) Careful to get the boundaries the right way around. If your boundaries are $0 < x < 1$ then $x > z$ (i.e. $z < x$) it the integral from 0 to $z = x$ and $x < z$ $(z > x)$ is the integral from $z = x$ to 1, because we're integrating w.r.t z, not x. Check the examples to convince yourself it is this way around.

Queries 3: The application of boundary conditions - Perhaps if we think about it like this:

Green's function is a function of x and z, and something interesting happens when they are equal. So we decide to describe Green's function to the left of this point and to its right separately - hence two equations. One valid for $x < z$ and one valid for $x > z$.

Now we have 2 generic equations that fit the ODE but they're not much use on their own. We need more information - enter the boundary conditions. They tell us how the function (or its derivatives) behave at certain values of x. Let's imagine two scenarios:

i) The boundary conditions are at different points. (e.g. $x = 0$ and $x = 1$) Naturally in this situation we simply say that one of the boundaries is on the left $(x < z)$ and one is on the right $(x > z)$ of our interesting point $z = x$. This is great news. We have 2 equations. One for the left domain, one for the right domain, and a piece of information about each to start eliminating unnecessary terms.

ii) The boundary conditions are at the same point. (Lets say both $f(x)$ and $f'(x)$ at $x = 0$) Oh dear. We have two pieces of information, but we can't decide that $x = 0$ is both on the left of $z = x$ and on its right. We're not Schrodinger! We are forced to conclude that we only have information about one side of our interesting point $z = x$. The good news is that we can still use the information. We can apply both conditions to the domain in which they both exist, let's say $x < z$.

So in answer to your initial question about the homework: since we have BC's at $x = 0$ and $x = 1$ we apply one to the equation that is valid between 0 and $z = x$ and the other to the equation valid for $z = x$ to 1, placing our weird point $z = x$ somewhere between the two.

Today I made the BCs both apply to one point, $x = 0$. So they can only possibly apply to one of the equations, because $x = 0$ cannot be in the range of both equations. They have different sets of x values; one set is all values higher than $z = x$ the other all values lower than $z = x$. It doesn't matter what z is, $x = 0$ can't be on both sides.