Theoretical Physics 2019/20 — Solution of Problem QT2.6

(a) This first step is extremely simple, it just consists in replacing θ by ϵ, $\cos\epsilon$ by $1-\epsilon^2/2$ and $\sin\epsilon$ by ϵ in the matrices $\mathsf{R}_x(\theta)$ and $\mathsf{R}_y(\theta)$ given in the question. The result is

$$\mathsf{R}_x(\epsilon) \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\epsilon^2/2 & -\epsilon \\ 0 & \epsilon & 1-\epsilon^2/2 \end{pmatrix}, \quad \mathsf{R}_y(\epsilon) \approx \begin{pmatrix} 1-\epsilon^2/2 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\epsilon^2/2 \end{pmatrix}.$$

[3 marks]

(b) First, the proof of Eq. (3). (You can skip this safely and go straight to the solution of the questions if you prefer.)

We start by calculating the matrix product $\mathsf{R}_y(\epsilon)\mathsf{R}_x(\epsilon)$:

$$\begin{aligned}\mathsf{R}_y(\epsilon)\mathsf{R}_x(\epsilon) &\approx \begin{pmatrix} 1-\epsilon^2/2 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\epsilon^2/2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\epsilon^2/2 & -\epsilon \\ 0 & \epsilon & 1-\epsilon^2/2 \end{pmatrix} \\ &\approx \begin{pmatrix} 1-\epsilon^2/2 & \epsilon^2 & \epsilon-\epsilon^3/2 \\ 0 & 1-\epsilon^2/2 & -\epsilon \\ -\epsilon & \epsilon-\epsilon^3/2 & 1-2(\epsilon^2/2)+\epsilon^4/4 \end{pmatrix}\end{aligned}$$

Keeping only the terms of order 0, 1 or 2 in ϵ results in

$$\mathsf{R}_y(\epsilon)\mathsf{R}_x(\epsilon) \approx \begin{pmatrix} 1-\epsilon^2/2 & \epsilon^2 & \epsilon \\ 0 & 1-\epsilon^2/2 & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix}.$$

We now calculate the product of the four matrices, using the fact that the matrix $\mathsf{R}_y(-\epsilon)\mathsf{R}_x(-\epsilon)$ is simply the matrix $\mathsf{R}_y(\epsilon)\mathsf{R}_x(\epsilon)$ with ϵ changed into $-\epsilon$. For clarity, we first do the whole calculation and only then drop any term that is not of order 0, 1 or 2 in ϵ. It would have been as good (and more expedient) to drop these higher order terms as soon as they arise, without writing them at all.

$$\begin{aligned}\mathsf{R}_y(-\epsilon)\mathsf{R}_x(-\epsilon)\mathsf{R}_y(\epsilon)\mathsf{R}_x(\epsilon) &\approx \begin{pmatrix} 1-\epsilon^2/2 & \epsilon^2 & -\epsilon \\ 0 & 1-\epsilon^2/2 & \epsilon \\ \epsilon & -\epsilon & 1-\epsilon^2 \end{pmatrix}\begin{pmatrix} 1-\epsilon^2/2 & \epsilon^2 & \epsilon \\ 0 & 1-\epsilon^2/2 & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix} \\ &\approx \begin{pmatrix} 1-2(\epsilon^2/2)+\epsilon^4/4+\epsilon^2 & 2\epsilon^2-2(\epsilon^4/2)-\epsilon^2 & \epsilon-\epsilon^3/2-\epsilon^3-\epsilon+\epsilon^3 \\ -\epsilon^2 & 1-2(\epsilon^2/2)+\epsilon^4/4+\epsilon^2 & -\epsilon+\epsilon^3/2+\epsilon-\epsilon^3 \\ \epsilon-\epsilon^3/2-\epsilon+\epsilon^3 & \epsilon^3-\epsilon+\epsilon^3/2+\epsilon-\epsilon^3 & 2\epsilon^2+1-2\epsilon^2+\epsilon^4 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & \epsilon^2 & 0 \\ -\epsilon^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

We should compare this to $\mathsf{R}_z(-\epsilon^2)$. Replacing ϵ by $-\epsilon^2$ in Eq. (2) of the question and dropping the terms of order ϵ^4 yields

$$\mathsf{R}_z(-\epsilon^2) \approx \begin{pmatrix} 1 & \epsilon^2 & 0 \\ -\epsilon^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, to order 2 in ϵ, $\mathsf{R}_y(-\epsilon)\mathsf{R}_x(-\epsilon)\mathsf{R}_y(\epsilon)\mathsf{R}_x(\epsilon) = \mathsf{R}_z(-\epsilon^2)$.

Note: We can be sure that all the terms of order 0, 1 or 2 are taken into account in the above calculations because $\cos\theta$ differs from $1 - \theta^2/2$ only by terms in θ^4 and of higher order and $\sin\theta$ differs from θ only by terms in θ^3 and of higher order.

Now, the model solutions:

(i) The identity operator, $\hat{I}$, has no dimensions. Hence $(i/\hbar)\epsilon\hat{J}_n$ cannot have a physical dimension either. Since i is dimensionless and ϵ is an angle measured in rad, which has no physical dimension, $\hat{J}_n$ must have the same physical dimensions as $\hbar$, i.e., the dimensions of an energy times a time. Since an energy has the dimensions of a mass times the square of a velocity, $\hbar$ has the dimensions of a mass times a velocity times a length, i.e., the dimensions of an angular momentum (remember that $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ in Classical Mechanics). [3 marks]

(ii) We simply rewrite Eq. (4) of the question, replacing the rotation operators by their approximate expressions up to terms of order ϵ^2, and see what comes out. Let us start with $\hat{R}_y(\epsilon)\hat{R}_x(\epsilon)$:

$$\begin{aligned}\hat{R}_y(\epsilon)\hat{R}_x(\epsilon) &\approx \left[\hat{I} - \frac{i}{\hbar}\,\epsilon\,\hat{J}_y - \frac{1}{2\hbar^2}\,\epsilon^2\,\hat{J}_y^2\right]\left[\hat{I} - \frac{i}{\hbar}\,\epsilon\,\hat{J}_x - \frac{1}{2\hbar^2}\,\epsilon^2\,\hat{J}_x^2\right] \\ &\approx \hat{I} - \frac{i}{\hbar}\,\epsilon\left(\hat{J}_x + \hat{J}_y\right) - \frac{\epsilon^2}{\hbar^2}\left(\hat{J}_y\hat{J}_x + \hat{J}_x^2/2 + \hat{J}_y^2/2\right).\end{aligned}$$

Then

$$\begin{aligned}\hat{R}_y(-\epsilon)\hat{R}_x(-\epsilon)\hat{R}_y(\epsilon)\hat{R}_x(\epsilon) &\approx \left[\hat{I} + \frac{i}{\hbar}\,\epsilon\left(\hat{J}_x + \hat{J}_y\right) - \frac{\epsilon^2}{\hbar^2}\left(\hat{J}_y\hat{J}_x + \hat{J}_x^2/2 + \hat{J}_y^2/2\right)\right] \times \\ &\quad \left[\hat{I} - \frac{i}{\hbar}\,\epsilon\left(\hat{J}_x + \hat{J}_y\right) - \frac{\epsilon^2}{\hbar^2}\left(\hat{J}_y\hat{J}_x + \hat{J}_x^2/2 + \hat{J}_y^2/2\right)\right] \\ &\approx \hat{I} + \frac{\epsilon^2}{\hbar^2}\left[\left(\hat{J}_x + \hat{J}_y\right)^2 - 2\left(\hat{J}_y\hat{J}_x + \hat{J}_x^2/2 + \hat{J}_y^2/2\right)\right] \\ &\approx \hat{I} + \frac{\epsilon^2}{\hbar^2}\left[\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x\right].\end{aligned}$$

Now, up to order 2 in ϵ, $\hat{R}_z(-\epsilon^2) \approx \hat{I} + (i/\hbar)\,\epsilon^2\,\hat{J}_z$. Hence $(\epsilon^2/\hbar^2)[\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x] = (i/\hbar)\epsilon^2\hat{J}_z$, from which we see that, indeed, the operators $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ must be such that $\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x = i\hbar\hat{J}_z$. [4 marks]

(c) We know that, to first order in ϵ,

$$(1 - i\epsilon J_z/\hbar)\psi(x, y, z) = \psi(x'', y'', z''), \tag{1}$$

with (x'', y'', z'') and (x, y, z) related by the equations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon & 0 \\ \sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}. \tag{2}$$

First, it is clearly advantageous to work in terms of (x, y, z) only. With this in mind, let us replace x'', y'' and z'' by their expressions in terms of x, y and z. To this end, we need to invert Eq. (2). The hint tells us how to invert the matrix: simply change ϵ into $-\epsilon$ (which is kind of obvious: a rotation by θ followed by a rotation by $-\theta$ is the same as no rotation at all, hence the product of $\mathsf{R}_z(\theta)$ by $\mathsf{R}_z(-\theta)$ has got to be the unit matrix). Therefore

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\epsilon & \sin\epsilon & 0 \\ -\sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Since we work to first order in ϵ, we can change $\cos\epsilon$ by 1 (recall that $\cos\epsilon = 1 - \epsilon^2/2 + \cdots$) and $\sin\epsilon$ by ϵ. Thus, to first order in ϵ, $x'' = x + \epsilon y$, $y'' = y - \epsilon x$ and $z'' = z$. Moreover, also to first order in ϵ,

$$\psi(x'', y'', z'') = \psi(x, y, z) + \epsilon \left.\frac{\mathrm{d}\psi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \psi(x, y, z) + \epsilon \left.\left(\frac{\partial\psi}{\partial x''}\frac{\mathrm{d}x''}{\mathrm{d}\epsilon} + \frac{\partial\psi}{\partial y''}\frac{\mathrm{d}y''}{\mathrm{d}\epsilon} \right)\right|_{\epsilon=0}$$

Since $\mathrm{d}x''/\mathrm{d}\epsilon = y$ and $\mathrm{d}y''/\mathrm{d}\epsilon = -x$,

$$\psi(x'', y'', z'') = \psi(x, y, z) + \epsilon \left.\left(y\frac{\partial\psi}{\partial x''} - x\frac{\partial\psi}{\partial y''} \right)\right|_{\epsilon=0}$$

We also note that

$$\left.\frac{\partial}{\partial x''}\psi(x'', y'', z'')\right|_{\epsilon=0} = \frac{\partial}{\partial x}\psi(x, y, z), \qquad \left.\frac{\partial}{\partial y''}\psi(x'', y'', z'')\right|_{\epsilon=0} = \frac{\partial}{\partial y}\psi(x, y, z).$$

Therefore, to first order in ϵ,

$$\psi(x'', y'', z'') = \psi(x, y, z) + \epsilon \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) \psi(x, y, z)$$

Going back to Eq. (1), we see that

$$(1 - i\epsilon J_z/\hbar)\psi(x, y, z) = \psi(x, y, z) + \epsilon \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) \psi(x, y, z),$$

from which it is clear that

$$J_z = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right).$$

[No mark for this part, but congratulations if you have done it well.]