i) For $l = 3$, $E_3 = 3(4)\hbar^2/(2I) = 6\hbar^2/I$ and $-3 < m < 3$, degeneracy 7, i.e. m=-3,-2,-1,0,1,2,3

For $l = 4$, $E_4 = 4(5)\hbar^2/(2I) = 10\hbar^2/I$, degeneracy 9, m=-4,-3,-2,-1,0,1,2,3,4

For $l = 5$, $E_4 = 5(6)\hbar^2/(2I) = 15\hbar^2/I$ and degeneracy 11, i.e. m=-5,-4-3,-2,-1,0,1,2,3,4,5

[1 mark]

ii) The probability of finding the electron in $d\Omega$ is

$$dP = Y^*_{3,-2}Y_{3,-2}\sin\theta\, d\theta d\phi = \left(\frac{105}{32\pi}\right)\sin^5\theta\cos^2\theta\, d\theta d\phi$$

$$\int_\phi dP = \int_{\phi=0}^{2\pi}\left(\frac{105}{32\pi}\right)\sin^5\theta\cos^2\theta\, d\theta d\phi = \left(\frac{105}{16}\right)\sin^5\theta\cos^2\theta\, d\theta$$

so the probability density per unit θ is $\boxed{(105/16)\sin^5\theta\cos^2\theta}$ [1 mark]

The maxima (and minima) occur where the derivative is zero.

$$d/d\theta(\sin^5\theta\cos^2\theta) = 5\sin^4\theta\cos\theta\cos^2\theta + \sin^5\theta\, 2\cos\theta(-\sin\theta) = 0$$

$$\Rightarrow 5\sin^4\theta\cos^3\theta - 2\cos\theta\sin^6\theta$$

Dividing by $\sin^4\theta\cos\theta$ (this factor produces zero gradient at $\cos\theta = 0$, $\theta = 90°$, but that corresponds to $dP = 0$ and is a minimum. Likewise for the sin factor at $\theta = 0$ and $180°$)

$$\Rightarrow 5\cos^2\theta = 2\sin^2\theta = 2(1-\cos^2\theta)$$

so $\cos\theta = \pm\sqrt{2/7}$ $\quad\Rightarrow \boxed{\theta = 57.69°, 122.31°}$ [2 marks]

$$\langle\theta\rangle = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{105}{32\pi}\right)\theta\sin^5\theta\cos^2\theta\, d\theta d\phi = \left(\frac{105}{32\pi}\right)2\pi\frac{8\pi}{105} = \frac{\pi}{2}$$

[1 mark]

$$\langle\cos\theta\rangle = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{105}{32\pi}\right)\sin^5\theta\cos^3\theta\, d\theta d\phi = 0$$

[1 mark]

prob electron in region $0 < \theta < \pi/3$ is

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/3}\left(\frac{105}{32\pi}\right)\sin^5\theta\cos^2\theta d\theta d\phi$$

$$= \frac{105}{32\pi}2\pi\, 0.0459 = 0.301$$

[1 mark]

iii) For an eigenfunction $L^2 Y_{lm} = cY_{lm}$ where c is a constant. $Y_{lm} = A\sin^2\theta\cos\theta e^{-2i\phi}$ where $A = (105/32\pi)^{1/2}$

$$\begin{aligned}
\mathbf{L}^2 Y_{lm} &= -A\hbar^2\left\{\frac{e^{-2i\phi}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(\sin^2\theta\cos\theta)}{\partial\theta}\right)+\frac{\sin^2\theta\cos\theta}{\sin^2\theta}\frac{\partial^2(e^{-2i\phi})}{\partial\phi^2}\right\}\\
&= -\hbar^2 A\left\{\frac{e^{-2i\phi}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta(2\cos^2\theta-\sin^2\theta)\right)+\cos\theta(-2i)^2e^{-2i\phi}\right\}\\
&= -\hbar^2 A\left\{\frac{e^{-2i\phi}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta(2-3\sin^2\theta)\right)+\cos\theta(-2i)^2e^{-2i\phi}\right\}\\
&= -\hbar^2 Ae^{-2i\phi}\left\{\frac{1}{\sin\theta}(4\sin\theta\cos\theta-12\sin^3\theta\cos\theta)-4\cos\theta\right\}\\
&= -\hbar^2 Ae^{-2i\phi}(-12\sin^2\theta\cos\theta) = 12\hbar^2\, Y_{3,-2}
\end{aligned}$$

so this is an eigenfunction of L^2 [2 marks]

$$L_z Y_{3,-2} = -i\hbar\frac{\partial}{\partial\phi}A\sin^2\theta\cos\theta e^{-2i\phi} = -iA\sin^2\theta\cos\theta(-2i\hbar)e^{-2i\phi} = -2\hbar\, Y_{3,-2}$$

so this is an eigenfunction of L_z with eigenvalue $-2\hbar$.

$$\begin{aligned}
L_x Y_{3,-2} &= i\hbar A\left(e^{-2i\phi}\sin\phi\,\frac{\partial(\sin^2\theta\cos\theta)}{\partial\theta}+\cot\theta\cos\phi\,(\sin^2\theta\cos\theta)\frac{\partial e^{-2i\phi}}{\partial\phi}\right)\\
&= i\hbar A\left(e^{-2i\phi}\sin\phi\,(2\sin\theta\cos^2\theta-\sin^3\theta)-2i\sin\theta\cos^2\theta\cos\phi\, e^{-2i\phi}\right)\\
&= i\hbar Ae^{-2i\phi}\left(\sin\phi\,\sin\theta(2-3\sin^2\theta)-2i\sin\theta\cos^2\theta\cos\phi\right)
\end{aligned}$$

Looking only at the ϕ dependence it is clear this is not proportional to $Y_{3,-2}$ so this is not an eigenfunction of L_x [1 mark]