University of Durham

EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2019	PHYS2631-WE01

Title: Theoretical Physics 2

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Attempt **all** questions. The short-answer questions at the start of each section carry 50% of the total marks for the paper. The remaining 50% of the marks are carried by the longer questions, which are equally weighted.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Classical Mechanics
Section B: Quantum Theory 2

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A: CLASSICAL MECHANICS

1. (a) What is an ignorable coordinate? How does the presence of an ignorable coordinate in a Lagrangian simplify efforts to find the equation of motion? [4 marks]

   (b) Using q as the displacement from equilibrium, write down the Lagrangian for a mechanical system with one degree of freedom in the vicinity of an equilibrium configuration. How does the Lagrangian differ between stable and unstable equilibrium configurations? [4 marks]

   (c) What does a Green's function represent in the context of driven oscillators? What would be the functional form of the causal Green's function for an underdamped linear oscillator as a function of $t - t'$, where t' is the instant at which the oscillator is struck? [4 marks]

   (d) What is a central force? For a system of two point masses interacting only via a central force, the Lagrangian is translationally invariant. Explain what this implies for the motion of the centre of mass. [4 marks]

   (e) The Hamiltonian of a single-particle system is $H(q, p) = ap^2 + bq^3$, where a and b are positive constants. Use Hamilton's equations, $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$ to derive an expression for the acceleration of the particle, $\ddot{q}$, in terms of the displacement q. Describe the subsequent motion of the particle if it is released from rest from a negative value of q. [4 marks]

   (f) Using the implicit transformation equations $q = -\partial F/\partial p$ and $P = -\partial F/\partial Q$, and the properties of the Poisson bracket of two arbitrary functions J and K, where

   $$\{J, K\} = \frac{\partial J}{\partial q}\frac{\partial K}{\partial p} - \frac{\partial J}{\partial p}\frac{\partial K}{\partial q},$$

   determine whether or not the generating function $F = p^2 Q$ produces a canonical transformation. [4 marks]

   (g) A rigid body is turning about the origin O, where Oxyz represent Cartesian axes. The components of velocity of the particle with coordinates $(1, 0, 0)$ are $(0, 7, 1)$. What can be inferred about the angular velocity vector of the rigid body? [4 marks]

   (h) The Euler force on a mass m is $\underline{F} = -m\dot{\underline{\omega}} \times \underline{r}$. What gives rise to this force and what are $\dot{\underline{\omega}}$ and $\underline{r}$? In which direction does the Euler force act in Durham? [4 marks]

2. Two plane pendulums of length l comprise light rods holding bobs of mass M and m, with angular displacements from vertical of θ and ϕ respectively, in a uniform gravitational field, g. The pendulum pivots are at the same height and separated by a horizontal distance d, which is also the natural length of a light spring, with spring constant k, connecting the two bobs.

   (a) For small angular displacements, show that the Lagrangian of the system can be written as

   $$L = \frac{l^2}{2}\left(M\dot{\theta}^2 + m\dot{\phi}^2\right) - \frac{gl}{2}\left(M\theta^2 + m\phi^2\right) - \frac{kl^2}{2}(\phi - \theta)^2 .$$

   [4 marks]

   (b) For small oscillations around equilibrium, an appropriate trial solution is $\underline{q} = \underline{b}\, e^{i\lambda t}$, where $\underline{q} = (\theta, \phi)$, λ is a normal mode frequency and $\underline{b}$ is the corresponding mode vector. Using the matrix formulation of the Euler-Lagrange equations, $\hat{\tau}\underline{\ddot{q}} + \hat{\upsilon}\underline{q} = 0$, with matrix elements given by

   $$\tau_{jk} = \frac{1}{2}\frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k} \qquad \text{and} \qquad \upsilon_{jk} = \frac{1}{2}\frac{\partial^2 V}{\partial q_j \partial q_k},$$

   show that the normal modes satisfy

   $$\frac{l^2}{2}\begin{pmatrix} M(\omega^2 - \lambda^2) + k & -k \\ -k & m(\omega^2 - \lambda^2) + k \end{pmatrix}\underline{b} = 0,$$

   where $\omega^2 = g/l$. [8 marks]

   (c) Solve this eigenvalue problem for the normal mode frequencies and their corresponding mode vectors. [7 marks]

The two masses are disconnected from their respective rods and placed on a smooth horizontal surface, while still being connected by the spring. The masses are set spinning around their common centre of mass, such that the system angular momentum is given by the constant J.

   (d) (i) This system can be described with an effective potential

   $$V_{\text{eff}}(r) = \frac{1}{2}\frac{J^2}{\mu r^2} + \frac{1}{2}k(r - d)^2,$$

   where $\mu = Mm/(M + m)$ is the reduced mass and r is the separation of the two masses. Sketch the function $V_{\text{eff}}(r)$ and find an algebraic equation the solution of which gives the separation for circular orbits of the two masses, r_c. [6 marks]

   (ii) Calculate the moment of inertia and the angular velocity of the system in the case of circular orbits. By balancing forces in a suitable rotating reference frame, find an algebraic equation for r_c. [5 marks]

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SECTION B: QUANTUM THEORY 2

3. (a) Define the term "Hilbert space" and give an example of a Hilbert space. [4 marks]

   (b) Define the term "Hermitian operator" and give an example of a Hermitian operator. [4 marks]

   (c) Let $\phi_p(x) = \exp(ipx/\hbar)/(2\pi\hbar)^{1/2}$, where x is a position and p a momentum (x can take any real value between $-\infty$ and ∞).

      (i) Why can one say that these functions are "normalized to a delta function in momentum space"? [1 mark]

      (ii) These functions are often regarded as being generalized eigenfunctions of the momentum operator, P_x. Why "generalized eigenfunction" rather than merely "eigenfunction"? [1 mark]

      (iii) Express P_x in terms of the position variable, x. [2 marks]

   (d) (i) Define the term "unitary operator". [2 marks]

      (ii) For what value of x is the matrix

$$\begin{pmatrix} 1/\sqrt{2} & x \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

      a unitary operator? [2 marks]

   (e) The Hamiltonian of a linear harmonic oscillator of angular frequency ω can be written in terms of ladder operators $\hat{a}$ and $\hat{a}^\dagger$ as $\hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$ such that $\hat{a}^\dagger\hat{a}|\phi_n\rangle = n|\phi_n\rangle$ when $|\phi_n\rangle$ is an eigenstate of the Hamiltonian with energy $\hbar\omega(n + 1/2)$.

      (i) What ket vector(s) does $\hat{a}$ map the eigenstate $|\phi_n\rangle$ to? [2 marks]

      (ii) The superscript $\dagger$ indicates a certain mathematical relationship between the operators $\hat{a}$ and $\hat{a}^\dagger$. What is the name of this relationship and how can it be defined formally? [2 marks]

   (f) Two systems, 1 and 2, in the angular momentum states characterized by the ket vectors $|j_1, m_1\rangle_1$ and $|j_2, m_2\rangle_2$, are combined to form a new system in the state $|j_1, j_2, J, M\rangle_{12}$ such that

$$|j_1, j_2, J, M\rangle_{12} = \sum_{m_1, m_2} \langle j_1, j_2, m_1, m_2 | J, M\rangle |j_1, m_1\rangle_1 |j_2, m_2\rangle_2,$$

   where the $\langle j_1, j_2, m_1, m_2|J, M\rangle$ are Clebsh-Gordan coefficients. Which operators do the quantum numbers j_1, m_1, J and M refer to and what are the corresponding eigenvalues? [4 marks]

   (g) Suppose that $j_1 = m_1 = 1$, $j_2 = m_2 = 3/2$ and $J = 5/2$. Find the value of the Clebsch-Gordan coefficient $\langle j_1, j_2, m_1, m_2|J, M\rangle$, (i) for $M = 3/2$, (ii) for $M = 5/2$. (Justify your answers.) [4 marks]

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4. (a) Let $|\phi_\alpha\rangle = (|1\rangle + |2\rangle + i|3\rangle)/\sqrt{3}$ and $|\phi_\beta\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$, where $\langle m|n\rangle = \delta_{mn}$, $m, n = 1, 2, 3$. Show that $\langle \phi_\alpha | \phi_\alpha\rangle = 1$ and that $\langle \phi_\alpha | \phi_\beta\rangle = 0$. [6 marks]

(b) Let $|\phi_\gamma\rangle = (|1\rangle + |2\rangle + x|3\rangle)/\sqrt{6}$, where x is such that the three ket vectors $|\phi_\alpha\rangle$, $|\phi_\beta\rangle$ and $|\phi_\gamma\rangle$ form an orthonormal basis for the subspace they span. Calculate x. [4 marks]

(c) Suppose that A and B are two physical quantities represented, respectively, by operators $\hat{A}$ and $\hat{B}$ acting in the subspace spanned by the ket vectors $|1\rangle$, $|2\rangle$ and $|3\rangle$. Either A or B is measured on a quantum system prepared in the state $|1\rangle$.

(i) Calculate the probability that the eigenvalue λ of $\hat{A}$ is found in a measurement of A if $\hat{A}|\phi_\beta\rangle = \lambda|\phi_\beta\rangle$ assuming this eigenvalue is non-degenerate. [4 marks]

(ii) Give the probability that the eigenvalue μ of $\hat{B}$ is found in a measurement of B if $\hat{B}|\phi_\alpha\rangle = \mu|\phi_\alpha\rangle$ and $\hat{B}|\phi_\gamma\rangle = \mu|\phi_\gamma\rangle$, assuming that this eigenvalue is doubly degenerate. [2 marks]

(d) Suppose that in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis the operator $\hat{A}$ is represented by the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and the Hamiltonian of the system by the matrix

$$\mathsf{H} = \begin{pmatrix} E_0 & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix},$$

where E_0 is a constant energy. Suppose, also, that the system is in the state $|\phi_\beta\rangle$ at time $t = 0$. Calculate the expectation value of A and the uncertainty ΔA on the value of A,

(i) at $t = 0$; [4 marks]

(ii) at a time $t \neq 0$. [2 marks]

(e) Given that $\hat{B}|\phi_\alpha\rangle = \mu|\phi_\alpha\rangle$ and $\hat{B}|\phi_\gamma\rangle = \mu|\phi_\gamma\rangle$, show that the uncertainty on the value of B is zero when the system is in the state $|3\rangle$. [4 marks]

(f) Write down the matrix representing the projector $|\phi_\alpha\rangle\langle\phi_\alpha|$, (i) in the basis $\{|1\rangle, |2\rangle, |3\rangle\}$, (ii) in the basis $\{|\phi_\alpha\rangle, |\phi_\beta\rangle, |\phi_\gamma\rangle\}$. [4 marks]