Theoretical Physics 2019/20 — Problem QT2.4

Background reading: Problem 2 of Worksheet 3 and Section 7.5.1 of Professor Cole's notes for the Term 1 Quantum Mechanics course. Problem 1 of Worksheet 4 is also relevant but you do not need to look at it before you attempt the problem below.

In Quantum Optics, an electromagnetic field is treated as a quantum object whose state can be represented by a ket vector $|\psi\rangle$. In particular, a state in which the field contains exactly n photons of a same frequency, polarization and direction of propagation can be represented by a ket vector $|\psi_n\rangle$. These $|\psi_n\rangle$'s form an orthonormal basis for the relevant Hilbert space. Amongst the various operators which act on these kets figure an operator $\hat{a}$, called the annihilation operator, and its adjoint, $\hat{a}^\dagger$, called the creation operator. Their names refer to their action on the $|\psi_n\rangle$'s: $\hat{a}|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle$ $(n > 0)$ and $\hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$ $(n \geq 0)$, with $\hat{a}|\psi_0\rangle = 0$. Thus $\hat{a}$ and $\hat{a}^\dagger$ transform states containing a given number of photons into states containing, respectively, one fewer and one more photon. These two operators do not commute with each other; instead, $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + \hat{I}$, where $\hat{I}$ is the identity operator (i.e., $[\hat{a}, \hat{a}^\dagger] = \hat{I}$).

(a) Show that $\hat{n}|\psi_n\rangle = n|\psi_n\rangle$, where $\hat{n} = \hat{a}^\dagger\hat{a}$. (In view of this relation, $\hat{n}$ is called the number operator.)

(b) Show that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and that $\langle\alpha|\alpha\rangle = 1$ if

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|\psi_n\rangle, \tag{1}$$

where α is an arbitrary complex number. [Assume that $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. One defines $|\alpha\rangle$ for $\alpha = 0$ as the limit of $|\alpha\rangle$ for $|\alpha| \to 0$; thus $|0\rangle = |\psi_0\rangle$.]

(c) Show that $\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$ and $\langle\alpha|\hat{n}^2|\alpha\rangle = |\alpha|^4 + |\alpha|^2$. [Hint: Use $\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \langle\alpha|\hat{a}|\alpha\rangle^*$. Do not assume that $|\alpha\rangle$ is an eigenstate of $\hat{n}$ or that $|\alpha\rangle$ is an eigenvector of $\hat{a}^\dagger$ ($|\alpha\rangle$ is an eigenstate of $\hat{n}$ only if $\alpha = 0$, and $\hat{a}^\dagger$ has no eigenvectors).]

(d) Suppose that someone measures the number of photons present in a field prepared in the state $|\alpha\rangle$. Show that the probability $P(n)$ of finding exactly n photons follows a Poisson distribution of mean $|\alpha|^2$,

$$P(n) = \frac{|\alpha|^{2n}}{n!}\,\exp(-|\alpha|^2).$$

[Hint: Recall, from (a), that $|\psi_n\rangle$ is an eigenvector of $\hat{n}$ with eigenvalue n.]

(e) What do the results obtained in (c) say about the variance of this probability distribution?

Note: The ladder operator a_- introduced in the Term 1 course to obtain the energy levels of a harmonic oscillator has the same properties as the operator $\hat{a}$ considered above: As you have seen or will see in the 4th Quantum Theory workshop, $[a_-, a_-^\dagger] = 1$, $a_-\psi_{n+1}(x) = \sqrt{n+1}\,\psi_n(x)$ and $a_-^\dagger\psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x)$, where the $\psi_n(x)$'s are the energy eigenfunctions of the harmonic oscillator. Moreover, as mentioned in the Term 1 course, $a_-\psi_0(x) \equiv 0$ for the ground state wave function $\psi_0(x)$. Therefore there are close mathematical similarities between quantized electromagnetic fields and quantum harmonic oscillators, although the Physics context is very different.