Mathematical Methods II
PDF 11

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Key Points

- Separation of variables (PDE)
- Applying boundary conditions

Separation of Variables for PDE's (ctud)

- **Applying boundary conditions:** Using the method of separation of variables we can determine a general solution to separable PDE's. If we wanted a complete solution to a given real world problem then we would need to find and apply boundary conditions when solving the PDE.

  In order to solve an equation of two variables for the given boundary conditions we need to use the following procedure. Stage 1 is the same as solving without BC's:

  - **Stage (1)** Produce separate ODE's.
    i) Assume a separable solution of the form $u(x,t) = X(x)T(t)$.
    ii) Find the derivatives of u.
    iii) Sub the derivatives of u into the PDE.
    iv) Separate the variables $X(x)$ and $T(t)$ onto the LHS and RHS.
    v) Introduce a separation constant μ and produce an X equation and a T equation.
  - **Stage (2)** Consider the boundary conditions $u(x_1, t_1)$ and $u(x_2, t_2)$. Since $u = XT$, we can use the boundary conditions on u to establish BC's for X and/or T.
    For example:
    If $u(0,t) = 0$ then $X(0)T(t) = 0$.
    If $u(x,0) = 0$ then $X(x)T(0) = 0$.
    Since $X(x) = 0$ and $T(t) = 0$ are trivial solutions (i.e. the temperature would be 0 for any given position or time) then we can conclude that $X(0) = 0$ and $T(0) = 0$.
  - **Stage (3)** We currently have no information on the value of μ. In order to ensure our solution is truly general we must have solutions for all the possible values of μ. Hence, we consider the 3 possible cases for μ
    i) $\mu = 0$.

ii) $\mu < 0$.

iii) $\mu > 0$.

Solve for X or T at the boundary conditions. If $X(x) = 0$ or $T(t) = 0$ then there's no need to try the other function as this is a trivial solution. If there is a non-zero solution, solve for the other function.

- **Stage (4)** Add up the solutions of interest for u for all the values of μ, use initial conditions to simplify the equation and work out the coefficients using the formulae for Fourier series coefficients.

Using the method of separation of variables we have previously sought a general solution to the 1D heat equation. The PDE was

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature at a given point and time. And the solution we arrived at was

$$u(x,t) = XT = \left(c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}\right) e^{k^2 \mu t}$$

If we had boundary conditions for a given problem we could obtain a more useful, more specific solution to our problem. Different boundary conditions can describe different physical scenarios. For instance one might consider two types of heat equation, depending on the given boundary conditions.

Consider a long, thin metal rod. Let's state some boundary conditions that describe what is happening at two points on the rod.

1) Zero temperature end points: $u(0,t) = 0$, $u(L,t) = 0$, $t > 0$.
2) Insulated end points: $u_x(x,0) = 0$, $u_x(L,t) = 0$, $t > 0$.

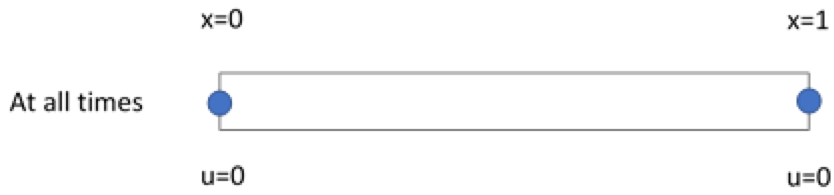
In the first scenario we are given information about two points on the rod separated by a distance L ($x = 0$ and $x = L$) and told the temperature at each one is zero, for some time after $t = 0$. We will solve the equation using these boundary conditions in this lecture.

In the second scenario we are told that the point x experiences no temperature change when $t = 0$ and a point a distance L away experiences no temperature change at any later time t. i.e. no heat flows across the boundary. You will solve the equation using these boundary conditions in a weekly problem set.

e.g PDF11.1 Solve the 1D heat equation for $k = 2$, using the boundary conditions for zero temperature end points: $u(0,t) = 0$, $u(1,t) = 0$, $t > 0$ and the initial condition $u(x,0) = x/4$ for $0 < x < 1$.

Stage (1) Based on our work from last time, we see that the 1D heat equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$



has solutions that can be determined from the following two ODE's.

$$X'' - \mu X = 0 \qquad\qquad T' - 4\mu T = 0$$

Stage (2) Our boundary conditions allow us to determine the following

$$u(0,t) = 0\text{: } X(0)T(t) = 0$$

$$u(1,t) = 0\text{: } X(1)T(t) = 0$$

The solution where $T(t) = 0$ is trivial and of no interest; it represents the scenario were the temperature never changes. So we can say that $X(0) = 0$ and $X(1) = 0$.

Stage (3) Let's consider the three possible cases of μ.

(1) $\mu = 0$: Beginning with the equation for X

$$X'' = 0$$

Examining the roots or by inspection we realise that the solution to this equation is

$$X(x) = ax + b,$$

as differentiating this equation twice will certainly result in zero, while differentiating fewer times may not, making it the optimal choice. Now apply the boundary conditions for X

$$X(0) = 0 \Rightarrow b = 0 \Rightarrow X(x) = ax,$$

$$X(1) = 0 \Rightarrow a = 0 \Rightarrow X(x) = 0.$$

Therefore when $\mu = 0$, $X(x) = 0$ for any given x and hence $u(x,t) = 0$ for any given x, so this solution is trivial and of no interest.

(2) $\mu > 0$: Let's say that $\mu = r^2$ so we can be sure it is a positive constant and avoid fractional powers in our solutions. Beginning with the equation for X

$$X'' - r^2 X = 0.$$

Examining the real roots we realise that this equation has the auxiliary and solution

$$\lambda^2 = r^2,$$

$$X(x) = ae^{rx} + be^{-rx}.$$

Now apply the boundary conditions for X

$$X(0) = 0 \Rightarrow a + b = 0 \Rightarrow b = -a,$$

Reducing our equation for X to

$$X(x) = a\left(e^{rx} - e^{-rx}\right).$$

Using our second BC

$$X(1) = 0 \Rightarrow a\left(e^{r} - e^{-r}\right) = 0 \Rightarrow a = 0 \Rightarrow b = 0.$$

Therefore when $\mu > 0$, $X(x) = 0$ for any given x and hence $u(x,t) = 0$ for any given x, so this solution is of no interest.

(3) $\mu < 0$: Let's say that $\mu = -r^2$ so we can be sure it is a negative constant and avoid fractional powers in our solutions. Beginning with the equation for X

$$X'' + r^2 X = 0$$

Examining the complex roots we realise that this equation has the auxiliary and solution

$$\lambda^2 = -r^2,$$

$$X(x) = a\cos rx + b\sin rx$$

Now apply the boundary conditions for X

$$X(0) = 0 \Rightarrow a = 0 \Rightarrow X(x) = b\sin rx,$$

$$X(1) = 0 \Rightarrow b\sin r = 0.$$

Since $\sin r$ is zero for some values of r, b can be non-zero. This means that when $\mu < 0$, $X(x)$ is non-zero for some values of x. So we have a solution of interest. If we let $\sin r = 0$ then $r = n\pi$ for $n = 1, 2, 3....$ Hence

$$X_n(x) = b_n \sin n\pi x.$$

Note: Consider that if we did not have boundary conditions (i.e. we were examining the simpler case of an infinitely long metal rod) then r could be anything. The boundary conditions limit us to integer muliples of half waves that fit within the boundaries, as our choice of r MUST give $u = 0$ at $x = 0$ and $x = 1$.

Since we have a non-trivial solution for X, it is worth looking at T. Following the same process, applying $\mu = -r^2$ the ODE for T becomes

$$T' + 4r^2 T = 0.$$

This ODE has the solution

$$\lambda = -4r^2$$

$$T(t) = ae^{-4r^2 t}$$

We have already established that $r = n\pi$, so subbing this in the equation becomes

$$T_n(t) = a_n e^{-4n^2\pi^2 t}$$

And so, given that $u = XT$, our solution for $u(x,t)$ is

$$u(x,t) = b_n \sin(n\pi x) a_n e^{-4n^2\pi^2 t} = C_n \sin(n\pi x) e^{-4n^2\pi^2 t}$$

where $C_n = a_n b_n$.
The physical interpretation of this solution is a temperature distribution between $x = 0$ and $x = 1$ (with the shape of half a wave for $n = 1$) that decays exponentially with time.

Stage (4) Now we have evaluated u for all μ we can add the solutions together using the superposition principle. The solutions are found by solving a set of differing ODEs, but are all solutions to the original *linear* PDE. As the solutions for $\mu < 0$ and $\mu > 0$ are valid for all values of μ below and above zero respectively, we sum them up to infinity

$$u(x,t) = u_{\mu=0} + \sum u_{\mu<0} + \sum u_{\mu>0}$$

In this instance, we only have one viable solution ($\mu < 0$), so this is straight forward.

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) e^{-4n^2\pi^2 t}$$

We were given an initial condition (i.e. when $t = 0$) that can be useful here. Let's apply the condition $u(x,0) = x/4$

$$u(x,0) = \frac{x}{4} = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

Notice that the time dependent term reduced to a value of 1. So this leaves us with a constant times a sin term. This may look familiar; it is a Fourier series! We can use Fourier series formulae to determine the value of the coefficient (C_n) of such a solution. Recall that a full fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{L}$$

where $f(x)$ is the function of interest, L is half the period of the function and a_0, a_n and b_n are Fourier coefficients. These coefficients each have their own formulae

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{\pi n x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}dx = \frac{2}{L}\int_0^L f(x)\sin\frac{\pi n x}{L}dx$$

The coefficients can equivalently be defined in terms of the period $P = 2L$,

$$a_0 = \frac{2}{P}\int_{-P/2}^{P/2} f(x)dx = \frac{2}{P}\int_0^P f(x)dx$$

$$a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\frac{2\pi n x}{P}dx = \frac{2}{P}\int_0^P f(x)\cos\frac{2\pi n x}{P}dx$$

$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\frac{2\pi n x}{P}dx = \frac{2}{P}\int_0^P f(x)\sin\frac{2\pi n x}{P}dx$$

Note on variation of fourier series: Sometimes the half from $a_0/2$ in $f(x)$ gets switched to the a_0 coefficient formula.

We will use $L = 1$, as our function would be periodic between -1 and 1, hence its half-period ranges from 0 to 1,

$$\begin{aligned}
C_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \\
&= \frac{2}{1}\int_0^1 \frac{x}{4}\sin\frac{n\pi x}{1}dx \\
&= \frac{1}{2}\left[\frac{-x}{n\pi}\cos n\pi x\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x dx \\
&= \frac{1}{2}\left[\frac{-x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1 \\
&= -\frac{\cos n\pi}{2n\pi} \\
&= -\frac{(-1)^n}{2n\pi}.
\end{aligned}$$

Hence, the final solution is

$$u(x,t) = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x)e^{-4n^2\pi^2 t}$$