Cosmology: Problem Cos 1.4

(a) Calculate the present-day energy density in the $T = 2.73\ K$ cosmic microwave background radiation in units of J m^{-3} and the corresponding value of Ω_γ if $H_0 = 70$ km s^{-1} Mpc^{-1}.

(b) If the density of the Universe ρ includes contributions from non-relativistic matter and cosmological constant with present-day (i.e. when $a = 1$) density parameters $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, respectively, show that the Hubble parameter, H, at arbitrary expansion factor a is given by

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\Lambda,0}\left(1 - a^{-2}\right) + \Omega_{M,0}\left(a^{-3} - a^{-2}\right) + a^{-2},$$

where H_0 is the present-day value of the Hubble parameter.

(c) For Universes with $\Omega_{\Lambda,0} > 0$, use the result of (b) to discuss qualitatively whether the Universe will continue to expand to infinite size for the three separate cases of $\Omega_{M,0} < 1$, $\Omega_{M,0} = 1$ and $\Omega_{M,0} > 1$.

(d) At arbitrary expansion factor, show that $\Omega_{tot} = \Omega_\Lambda + \Omega_M$ is given by:

$$\Omega_{tot} = \frac{\Omega_{M,0} + \Omega_{\Lambda,0}a^3}{\Omega_{\Lambda,0}(a^3 - a) + a(1 - \Omega_{M,0}) + \Omega_{M,0}}.$$

For Universes with $0 < \Omega_{M,0} < 1$, evaluate Ω_{tot} in the limits $t \to 0$ and $t \to \infty$, for the two cases $\Omega_{\Lambda,0} = 0$ and $\Omega_{\Lambda,0} > 0$. [In the latter case, you may assume that $\Omega_{\Lambda,0}$ is sufficiently small that a big bang occurred.]