University of Durham

EXAMINATION PAPER

May/June 2017 Examination code: PHYS2611-WE01

MATHEMATICAL METHODS IN PHYSICS

SECTION A. Mathematical Methods part 1
SECTION B. Mathematical Methods part 2

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}
Astronomical Unit:	AU = 1.50×10^{11} m
Parsec:	pc = 3.09×10^{16} m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. MATHEMATICAL METHODS PART 1

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Determine whether the given sets, together with the specified operations of addition and scalar multiplication, are vector spaces. For any that are not state an axiom that fails to hold.

   (i) The set of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in R^2 with $2x + 3y = 0$, with the usual vector addition and scalar multiplication.

   (ii) The set of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in R^2 with $x \geq y$, with the usual vector addition and scalar multiplication.

   [4 marks]

   (b) State whether the given sets of polynomials are linearly dependent or independent.

   (i) $\{1 + 2x + x^2, x - x^2, 1 + 3x^2\}$.

   (ii) $\{1 + x, x + x^2, 1 + x^2\}$.

   [4 marks]

   (c) (i) Show that if U_1 and U_2 are two unitary matrices then their product is also a unitary matrix.

   (ii) Show that the matrix $i(A - A^\dagger)$ is Hermitian.

   [4 marks]

   (d) Find a parametric representation of the following curves.

   (i) A straight line represented by

$$x + y = 3, \qquad 2x - 5y + z = 3.$$

   (ii) A curve represented by

$$(x - 1)^2 + (y + 2)^2 = 1, \qquad z = 1.$$

   [4 marks]

   (e) Given the following curve written in parametric form

$$\underline{r}(u) = \cos u\,\hat{\underline{i}} + \sin u\,\hat{\underline{j}} + u\,\hat{\underline{k}},$$

   find its unit tangent vector $\hat{\underline{t}}$ and its radius of curvature ρ, for which

$$\frac{\hat{\underline{n}}}{\rho} = \frac{d\hat{\underline{t}}}{ds} = \frac{d\hat{\underline{t}}}{du}\frac{du}{ds},$$

   where $\hat{\underline{t}}$, $\hat{\underline{n}}$ are the unit tangent vector and the unit normal vector to the curve, respectively, and s is the arc length. [4 marks]

(f) Calculate the following surface integral

$$I = \int_{\mathcal{S}} \underline{F} \cdot d\underline{S},$$

where $\underline{F} = y\,\hat{\underline{i}} + 2\,\hat{\underline{j}} + xz\,\hat{\underline{k}}$ and the surface $\mathcal{S}$ is represented parametrically by

$$\underline{r} = x\,\hat{\underline{i}} + x^2\,\hat{\underline{j}} + z\,\hat{\underline{k}}, \qquad 0 \le x \le 2, \quad 0 \le z \le 3.$$

[4 marks]

(g) The complex Fourier series of a periodic function $f(x)$ with period L is defined as follows

$$f(x) = \sum_{r=-\infty}^{\infty} c_r\, e^{2\pi i r x/L}, \qquad c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\, e^{-2\pi i r x/L}\, dx.$$

Find the complex Fourier series of

$$f(x) = e^x, \quad \text{for} \quad -\pi \le x \le \pi,$$

repeating with period 2π on the entire x-axis. [4 marks]

(h) The Laplace transform for a function $f(t)$ is defined as follows

$$\bar{f}(s) = \int_0^{\infty} f(t)\, e^{-ts} dt.$$

Calculate the Laplace transform for the function

$$f(t) = H(t-\pi) \times \cos t,$$

where $H(t)$ is the Heaviside step function defined as follows

$$H(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}.$$

[4 marks]

2. (a) Consider the following matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the eigenvalues of the matrix A and a corresponding set of orthonormal eigenvectors. [8 marks]

(b) Stokes' theorem for a vector field $\underline{F}$ on surface $\mathcal{S}$ enclosed by curve $\mathcal{C}$ states that

$$\int_{\mathcal{S}} (\underline{\nabla} \times \underline{F}) \cdot d\underline{S} = \int_{\mathcal{C}} \underline{F} \cdot d\underline{r}.$$

Verify Stokes' theorem explicitly by computing both sides of this expression for

$$\underline{F} = y\, e^{xy}\, \underline{\hat{i}} + (x+y)\, e^{xy}\, \underline{\hat{j}} + z\, e^{xy}\, \underline{\hat{k}}$$

over a rectangle $\mathcal{S}$ whose vertices $ABCD$ are $A = (0,1,0)$, $B = (1,1,0)$, $C = (1,3,0)$ and $D = (0,3,0)$. Note that the rectangle $\mathcal{S}$ lies in the plane $z = 0$. [12 marks]

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3. (a) (i) Consider the following vector fields

$$\underline{u} = x^2\,\hat{\underline{i}} + (y-z)\,\hat{\underline{j}} + xy\,\hat{\underline{k}}, \qquad \underline{v} = (x+y)^2\,\hat{\underline{i}} + z^2\,\hat{\underline{j}} + 2yz\,\hat{\underline{k}}.$$

Calculate

$$\underline{\nabla}\,(\underline{\nabla}\cdot\underline{v})\cdot\underline{u}.$$

[3 marks]

(ii) Suppose $\underline{F} = f(r)\,\underline{r}$ where f is a scalar function and r is the modulus of the position vector $\underline{r}$. Calculate $\underline{\nabla}\cdot\underline{F}$ and show that the function f for which $\underline{\nabla}\cdot\underline{F} = 0$ is c/r^3 where c is a constant. [7 marks]

[Hint: remember that $\underline{\nabla} f(r) = f'(r)\underline{\nabla} r$.]

(b) (i) Given the definition of the Fourier transform

$$\mathcal{F}[f(t)](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}dt,$$

compute the Fourier transform of the function $f(t)$ defined by straight lines joining point $(-b,0)$ to $(0,1)$ to $(b,0)$. Each point is identified by two coordinates which are $(t, f(t))$. The function $f(t)$ is zero outside the interval $-b \le t \le b$ where b is a positive constant. [6 marks]

(ii) Using the result in (i) and the property

$$\mathcal{F}[f(t+a)](\omega) = e^{ia\omega}\mathcal{F}[f(t)](\omega),$$

determine the Fourier transform of the function $g(t)$ defined by straight lines joining point $(0,0)$ to $(1/2,1)$ to $(1,0)$ to $(2,1)$ to $(3,0)$ and equal to zero outside the range specified. [4 marks]

SECTION B. MATHEMATICAL METHODS PART 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Find the general solution $y(x)$ of the ordinary differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{4}{x}.$$

[3 marks]

Verify your result by substitution into the equation above. [1 mark]

(b) Solve the second-order differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 4 + x + 10x^2$$

by finding the roots of the auxiliary equation and using the method of trial functions with undetermined coefficients. [4 marks]

(c) Write down the general expression for the n^{th} order linear homogeneous ordinary differential equation with constant coefficients. Assume that one root of the auxiliary equation has a k-fold degeneracy while other roots are distinct, and write the general solution of the differential equation. [4 marks]

(d) What are the regular singularities and the essential singular points of the second order linear ordinary differential equation? [2 marks]

Find the regular singular points of the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0.$$

[2 marks]

(e) Compute the Legendre polynomials $P_l(x)$ for $l = 1, 2, 3$ using the Rodrigues formula

$$P_0(x) = 1, \qquad P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$$

[2 marks]

Use your results to evaluate the integrals

$$\int_{-1}^{1} dx P_l(x) P_k(x), \quad \text{for} \quad l \neq k$$

for the four lowest polynomials P_0, P_1, P_2 and P_3. [2 marks]

(f) Check that $F(x, y) = -3y$ is a particular solution of the equation

$$y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} = 3x.$$

Find the general solution of the above equation. [4 marks]

(g) Find the general solutions for the one-dimensional wave equation, and for the two-dimensional Laplace equation. [4 marks]

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5. The goal of this question is to find the general solution $y(x)$ of the second-order differential equation

$$\frac{d^2y}{dx^2} - \frac{2}{(1-x)^2}\,y = 0\,.$$

   (a) Recast the equation above as a special case of the n^{th}-order Legendre linear equation

$$a_n(\alpha x+\beta)^n\frac{d^ny}{dx^n} + \ldots + a_1(\alpha x+\beta)\frac{dy}{dx} + a_0 y = 0$$

   and use the change of variables $\alpha x + \beta = e^t$ to reduce it to a differential equation with constant coefficients. [5 marks]

   (b) Find two particular solutions $y_1(x)$ and $y_2(x)$. [6 marks]

   (c) For the two particular solutions found in (b), prove their linear independence by computing the Wronskian and write down the general solution. [5 marks]

   (d) Briefly explain how you would go about solving the original equation for $y(x)$ using the series expansion around the point $x = 0$ as an alternative method. You do not need to determine the coefficients of the series. [4 marks]

6. The motion $x = x(t)$ of a simple pendulum is described by the equation of motion of a harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0,$$

where ω_0 is a constant frequency.

(a) What is the general solution of this homogeneous equation? Write down the initial conditions at time $t = 0$ which lead to the pendulum remaining stationary in equilibrium at $t > 0$. [3 marks]

(b) Assume that a time-dependent periodic force $F(t) = f_0 \sin(at)$ is applied to the pendulum hanging at rest. The parameters f_0 and a are constant. Write down the corresponding inhomogeneous equation of motion for the pendulum disturbed by this force. [2 marks]

Use the Laplace transform to find the solution of this equation and assume that $a^2 \neq \omega_0^2$. [8 marks]

[Hint: Note that the Laplace transform of $\sin(at)$ is $a/(s^2 + a^2)$.]

(c) Consider the situation where $a^2 = \omega_0^2$ and find the corresponding solution in this case. [5 marks]

[Hint: The inverse Laplace transform of $a^2/(s^2 + a^2)^2$ is $(1/2a)(\sin(at) - at\cos(at))$.]

What is the physical interpretation of the pendulum's behaviour in this case? [2 marks]