University of Durham

EXAMINATION PAPER

May/June 2012 Examination code: 042631/01

LEVEL 2 PHYSICS: THEORETICAL PHYSICS 2

SECTION A. CLASSICAL MECHANICS
SECTION B. QUANTUM THEORY 2

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. CLASSICAL MECHANICS

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Explain what a holonomic constraint is. How many degrees of freedom are possessed by a system of M point masses, moving in three-dimensional space, subject to j independent holonomic constraints? [4 marks]

   (b) For an underdamped, sinusoidally driven harmonic oscillator, explain what is meant by the steady-state solution and the transient solution. What are the oscillation frequencies of these two solutions? [4 marks]

   (c) Explain what an impulsive force is. How is this related to a Green's function? [4 marks]

   (d) The Hamiltonian of a mechanical system is defined through the following Legendre transformation of the Lagrangian

   $$H(p, q) = p\dot{q} - L(q, \dot{q}).$$

   Consider a Lagrangian of the form $L = \alpha \dot{q}^{\beta} - V(q)$, where α and β are constants. Determine the necessary condition on β if the Hamiltonian is to equal the total energy of the system. [4 marks]

   (e) Describe, briefly, Noether's theorem and give an example. [4 marks]

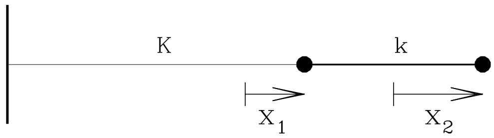
   (f) What is a canonical transformation? If the generating function used in a canonical transformation is the action, S, then what is the value of the transformed Hamiltonian and what is the name of the equation involving the original Hamiltonian that this produces? [4 marks]

   (g) What does the principal axis theorem state? How do the principal moments of inertia of a prolate symmetric ellipsoid compare to each other. [4 marks]

   (h) Euler's equations of motion for a rigid body are

   $$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= N_1, \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= N_2, \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= N_3. \end{aligned}$$

   Explain, briefly, which quantities the symbols in these equations represent, making clear what coordinate system is being used. [4 marks]



2. A diatomic molecule can be considered as two point masses, each of mass m, connected by a spring with a spring constant k. This molecule is attached to a surface through a chemical bond of spring constant K as shown in the diagram. Longitudinal displacements of the atoms from their equilibrium positions can be represented using the dynamical variables x_1 and x_2. Ignore transverse displacements and gravity.

   (a) What are normal modes? [2 marks]

   (b) (i) Write down the kinetic and potential energies of this system, and hence determine its Lagrangian in terms of x_1, x_2, $\dot{x}_1$ and $\dot{x}_2$. [3 marks]

   (ii) Using the Euler-Lagrange equation for a generalised coordinate q:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

   show that the equations of motion can be written in the form

$$m\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} K+k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

   [3 marks]

   (c) (i) Use a trial solution of the form $\underline{x} = \underline{b}e^{i\omega t}$ to find solutions for $m\omega^2$. [4 marks]

   (ii) Show that, for $K \ll k$, the normal mode frequencies satisfy $m\omega^2 \approx 2k$ or $K/2$. [2 marks]

   (iii) Find the corresponding mode vectors and describe the motion associated with each mode. [2 marks]

   (d) Consider the case when $K \to 0$, and describe the two modes in terms of the centre of mass and reduced mass. [4 marks]

3. A stone is dropped from a helicopter hovering a height h over a point on the equator. h is small compared with the radius of the Earth, R, so that the acceleration due to gravity can be taken as a constant, g. Ignore air resistance.

(a) (i) The operator relating time derivatives of a vector viewed in an inertial frame, S, to those measured in a frame B, rotating with angular velocity $\underline{\omega}$ (for example that associated with the rotation of the Earth), is

$$\left[\frac{d}{dt}\right]_{\text{in } B} = \left(\left[\frac{d}{dt}\right]_{\text{in } S} - \underline{\omega}\times\right).$$

Use this to show that the force measured in the rotating frame satisfies

$$m\ddot{\underline{r}} = \underline{F} - m\underline{\omega} \times (\underline{\omega} \times \underline{r}) - 2m\underline{\omega} \times \dot{\underline{r}} - m\dot{\underline{\omega}} \times \underline{r},$$

where $\underline{r}$ represents the position measured in frame B, and $\underline{F}$ is the force in the inertial frame. [3 marks]

(ii) In which directions do the three inertial forces point? [3 marks]

The rate at which the Earth's rotation is decreasing is very small, so assume $\dot{\omega} = 0$ for the rest of the question.

(b) Ignoring inertial forces, find expressions for the speed and height of the stone as a function of time, and the time taken before the stone hits the ground, t_0. What condition must be satisfied for the inertial forces to be unimportant in determining t_0? [4 marks]

(c) (i) Using the Coriolis force and the approximation that it always acts in the same direction, how far, in terms of t_0, does the stone land from the point beneath the helicopter, and in which direction? [4 marks]

(ii) Solve the same problem by using the conservation of angular momentum, and compare your answer with that from (c)(i). [6 marks]

SECTION B. QUANTUM THEORY 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Explain, through an example, the difference between the eigenvalue and the expectation value of an observable. [4 marks]

   (b) When are two observables compatible? What does this imply for their uncertainty relation? [4 marks]

   (c) Prove that two commuting hermitean operators have the same eigenkets. You may assume that all eigenvalues of both operators are non-degenerate. [4 marks]

   (d) What are the energy eigenkets and eigenvalues of the Hamiltonian given by

   $$\hat{H} = H_{11}|1\rangle\langle 1| + H_{12}|1\rangle\langle 2| + H_{21}|2\rangle\langle 1| + H_{22}|2\rangle\langle 2|\,,$$

   where all entries $H_{ij} \in \mathbf{R}$ are real numbers and where, in particular, $H_{12} = H_{21}$. [4 marks]

   (e) What is the Hamiltonian operator of the one-dimensional harmonic oscillator, expressed in position and momentum operators, and expressed in creation and annihilation operators? [4 marks]

   (f) Show, by proving the relation for $\{ijk\} = \{xyz\} = \{123\}$, that the three commutation relations

   $$[\hat{L}_i\,, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$$

   hold true. [4 marks]

   [Hint: Remember, the anti-symmetric Levi-Civita symbol $\epsilon_{ijk} = +1$ for cyclical and $\epsilon_{ijk} = -1$ for anti-cyclical permutations of ijk.]

   (g) What are the common eigenfunctions of $\underline{\hat{L}}^2$ and $\hat{L}_z$ and the corresponding eigenvalue equations? [4 marks]

5. Assume a spin-1/2 particle with mass m and charge e in a time-independent homogenous magnetic field $\underline{B} = B\underline{e}_z = (0,\, 0,\, B)$. This field induces a force on the particle given by

$$\underline{F} = -\underline{\nabla}(\underline{\mu} \cdot \underline{B})\,,$$

where the magnetic moment of the particle is given by

$$\underline{\mu} = -\frac{e}{mc}\,\underline{\hat{S}}\,.$$

(a) Show that the Hamiltonian operator of the system is given by

$$\hat{H} = \omega \hat{S}_z\,,$$

where the precession frequency $\omega = -eB/(mc)$ and $\hat{S}_z = (\hbar/2) \cdot \sigma_z$, with the Pauli-matrix

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[1 mark]

(b) What is the energy splitting between the two energy eigenstates of the system, and what are the corresponding energy eigenkets? [2 marks]

(c) What is the general form of Heisenberg's equation of motion for an operator in the Heisenberg picture for an arbitrary, time-independent Hamiltonian? [3 marks]

(d) Give the Heisenberg equations of motion for $\hat{S}_{x,y,z}$ and solve them. [6 marks]

[Hint: Remember that $\hat{S}_{x,y,z} = \frac{\hbar}{2}\,\sigma_{x,y,z}$, and that the commutation relations for the Pauli matrices are,

$$[\sigma_i,\, \sigma_j] = 2i\epsilon_{ijk}\,\sigma_k$$

]

(e) Using the previous results, calculate, in the Heisenberg picture, the time evolution of the expectation value $\langle \hat{S}_z \rangle_\alpha$ for a state α given by the system being in a pure spin-1/2 state oriented along the x-axis,

$$|s(t=0)\rangle = |s_x,\, +\rangle\,.$$

[4 marks]

(f) Construct the time evolution operator for time-independent Hamiltonians and apply it to the system, now in the Schrödinger picture. As initial condition for the state $|\beta(t)\rangle$ describing the system take a pure spin-1/2 state oriented along the positive z-axis. What is the time-evolution of the expectation value of S_z with respect to this state? [4 marks]

6. Consider a system consisting of two spin-1/2 objects with Hamiltonian

$$\hat{H}_S = \frac{\alpha}{\hbar}\,\left(\hat{S}_{z,(1)} + \hat{S}_{z,(2)}\right) + \frac{4\beta}{\hbar^2}\,\underline{\hat{S}}_{(1)} \cdot \underline{\hat{S}}_{(2)}\,,$$

where the various $\hat{S}_{(i)}$ are the corresponding spin operators for the particles $i = 1$, 2, and where α and β are real positive numbers, $\alpha,\, \beta \in \mathbf{R}^+$.

(a) What are the allowed values of the total spin S, and how do they relate to the multiplicity of the corresponding states? [3 marks]

(b) Write down the spin eigenkets

$$\left|\frac{1}{2}\frac{1}{2}SM_S\right\rangle \equiv \left|SM_S\right\rangle$$

of the combined system in terms of the eigenkets

$$\left|\frac{1}{2}\frac{1}{2}m_1m_2\right\rangle \equiv \left|m_1m_2\right\rangle = |m_1\rangle^{(1)}|m_2\rangle^{(2)}.$$

[5 marks]

[Hint: Remember that these new spin eigenkets are the simultaneous eigenkets of all relevant operators, i.e. $\underline{\hat{S}}^2_{(1)}$, $\underline{\hat{S}}^2_{(2)}$, $\underline{\hat{S}}^2$, and $\hat{S}_z$.]

(c) Express the Hamiltonian in terms of $\underline{\hat{S}}^2_{(1)}$, $\underline{\hat{S}}^2_{(2)}$, $\underline{\hat{S}}^2$, and $\underline{\hat{S}}_z$. [4 marks]

(d) What are the eigenvalues of the operators $\hat{S}_z$, $\underline{\hat{S}}^2$, and $\underline{\hat{S}}^2_{(i)}$, and which are the corresponding eigenstates? Calculate the resulting energy eigenvalues of the system and the energy-splitting between different states. What does this imply in the limit where $\alpha \ll \beta$? [6 marks]

(e) When are the energy-eigenstates degenerate? [2 marks]