(a) The system has 6 degrees of freedom, like any rigid body in 3D space. **[2 marks,B]**
If the rods have length l and the coordinates of the point masses i are denoted by $\underline{r}_i$, then one relevant constraint is $|\underline{r}_1 - \underline{r}_2| = l$. **[2 marks,B]**

(b) The auxiliary equation is the quadratic in λ for the assumed solution $x = e^{\lambda t}$. This is $\lambda^2 + b\lambda + k/m = 0$. **[2 marks,B]**
The critically damped solution has two equal roots, so $b = 2\sqrt{k/m}$. **[2 marks,B]**

(c) The transient solution oscillates with the frequency of the undriven oscillator and has an exponentially decaying amplitude. The steady-state solution persists and oscillates with the driving frequency. **[4 marks,B]**

(d) The normal modes represent simple harmonic oscillations for which all parts of the system oscillate with the same frequency. **[2 marks, B]**
The motion will only be simple harmonic and described as a superposition of normal modes if the Lagrangian can be accurately described by Taylor-expanding the Lagrangian to second order around an equilibrium configuration. Hence the oscillations must be small. **[2 marks, B]**

(e) The Lagrangian $L = T - V$, and $p = \partial L/\partial \dot{z} = m(\dot{z} + v)$. Hence $H = p^2/(2m) - pv + mgz$. **[3 marks,U]**
This is not the total energy, $E = T + V$, because of the additional $-pv$ term. **[1 mark,U]**

(f) $q = Q^2$ and $P = 2pQ$, hence $Q = \sqrt{q}$ and $P = 2p\sqrt{q}$. **[2 marks, U]**
The Poisson bracket $\{Q, P\} = 1/(2\sqrt{q}).(2\sqrt{q}) - 0 = 1$, hence F produces a canonical transformation. **[2 marks, U]**

(g) The Euler force is an inertial/fictitious/pseudo force. Such forces arise when considering dynamics in a non-inertial, i.e. accelerating, reference frame. **[2 marks,B]**
Its magnitude is proportional to the rate of change of the angular velocity of the rotating frame, which, for the Earth, is usually negligibly small. **[2 marks,B]**

(h) A diagonal 3×3 matrix with two of the terms being $I_0 + MR^2$ and the other being I_0. **[4 marks,U]**

(a) (i) [2 marks total] **(Unseen)**

$$T = \frac{1}{2}\left(m_1\dot{y}_1^2 + m_2\dot{y}_2^2 + m_3\dot{y}_3^2\right)$$

and

$$V = g(m_1y_1 + m_2y_2 + m_3y_3).$$

[2 marks]

(ii) [5 marks total] **(Unseen)** If m_1 moves up by δ, then both m_2 and m_3 will move down by δ. Therefore, $2y_1 + y_2 + y_3 = c$, a constant, is a suitable constraint equation. **[2 marks]**
Using $L = T - V$, eliminating y_3 using the constraint equation, and noting that the constant c can be ignored in the Lagrangian leads to the required equation. **[3 marks]**

(b) [7 marks total] **(Unseen)**
The E-L equations for y_1 and y_2 yield

$$(m_1 + 4m_3)\ddot{y}_1 + 2m_3\ddot{y}_2 + g(m_1 - 2m_3) = 0$$

and

$$2m_3\ddot{y}_1 + (m_2 + m_3)\ddot{y}_2 + g(m_2 - m_3) = 0$$

respectively. **[4 marks]**
Rearranging the second of these gives

$$\ddot{y}_2 = \frac{-g(m_2 - m_3) - 2m_3\ddot{y}_1}{m_2 + m_3}.$$

[1 mark]

Substituting this into the first equation gives

$$\ddot{y}_1[(m_1 + 4m_3)(m_2 + m_3) - 4m_3^2] + g[(m_1 - 2m_3)(m_2 + m_3) - 2m_3(m_2 - m_3)] = 0,$$

$$\ddot{y}_1[m_1m_2 + m_3(m_1 + 4m_2)] + g[m_1m_2 + m_1m_3 - 4m_2m_3] = 0.$$

From which the required result follows. **[2 marks]**

(c) (i) [2 marks total] **(Unseen)**
$\ddot{y}_1 = 0$, so m_1 remains at rest. From symmetry, both m_2 and m_3 also remain at rest. **[2 marks]**

(ii) [4 marks total] **(Unseen)**
$\ddot{y}_1 = -g/7$, so m_1 accelerates downwards, meaning that pulley P_2 accelerates upwards. Relative to the pulley, m_2 accelerates downwards and m_3 accelerates upwards. **[2 marks]**
Overall, m_2 will be accelerating downwards, because if the tension in the upper string is less than $4mg$, as it must be given the motion of m_1, then the centre of mass of the (P_2, m_2, m_3) part of the system will be accelerating downwards. **[2 marks]**

(a) [6 marks total] **(Unseen)**
This is just motion with a constant acceleration, so $\dot{z} = v_0 - gt$ and $z = v_0 t - gt^2/2$, where z is radially outwards from the surface. **[2 marks]**
Setting $\dot{z} = 0$ gives $t_{\max}$ **[1 mark]**
and then one can evaluate $z(t_{\max})$ to give the required result for h. **[1 mark]**
The Coriolis acceleration acts perpendicular to the motion so does not, to first order, affect the radial motion. The centrifugal acceleration is very small compared with the gravitational acceleration, i.e. $\omega^2 R_{\text{Earth}} \ll g$, so can be ignored. **[2 marks]**

(b) [2 marks total] **(Unseen)**
The Coriolis acceleration acts to the west for the upward-bound stone and the east for the falling stone. (Saying it acts to the right would be ambiguous, so would not merit any marks.) **[2 marks]**

(c) [8 marks total] **(Unseen)**
The velocity is predominantly in the radial direction, so the impact of the Coriolis acceleration can be written as

$$\ddot{x} = -2\omega\dot{z}\cos\lambda,$$

where x points locally east. **[2 marks]**
The initial conditions are $x(0) = \dot{x}(0) = 0$. **[2 marks]**
Integrating twice gives

$$\dot{x} = -2\omega\cos\lambda(\sqrt{2gh}\,t - gt^2/2)$$

and

$$x = -2\omega\cos\lambda(\sqrt{gh/2}\,t^2 - gt^3/6).$$

[2 marks]
Evaluating the x position at the landing time of $t = 2t_{\max} = 2\sqrt{2h/g}$ gives

$$\begin{aligned} x(2t_{\max}) &= -2\omega\cos\lambda\left[\sqrt{\frac{gh}{2}}.4.\frac{2h}{g} - \frac{g}{6}.8\left(\frac{2h}{g}\right)^{\frac{3}{2}}\right] \\ &= -\frac{4}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}}. \end{aligned}$$

The landing point is $(4/3)\omega\cos\lambda\sqrt{8h^3/g}$ west of the launch point. **[2 marks]**

(c) [4 marks total] **(Unseen)**
The separation in (c) is 4 times as large as that for the dropped stone, and in the opposite direction. [Marks will be given for an accurate comparison with whatever result the student found for part (c).] **[2 marks]**
By conservation of orbital angular momentum, the dropped stone must land ahead (east) of the drop point. When the stone is thrown radially outwards it is placed on an orbit with no angular momentum, and the Earth turns beneath it. (The longer flight time and accumulated westerly velocity of the launched stone are responsible for the factor of 4.) [Marks will be given for relevant comments independent of the result the student found for part (c).] **[2 marks]**

Academic year 2017/18 — Theoretical Physics 2 — May/June 2018 paper — Question 4
[All bookwork except where indicated otherwise]

(a) (i) For example, the position operator, $\hat{x}$. This operator is linear because

$$x[c_1\phi(x) + c_2\psi(x)] = c_1 x\phi(x) + c_2 x\psi(x)$$

for any two wave functions $\phi(x)$ and $\psi(x)$ and any two complex numbers c_1 and c_2. [2 marks]

(ii) The adjoint of an operator $\hat{A}$ is the operator $\hat{A}^\dagger$ such that $\langle\phi|\hat{A}^\dagger|\psi\rangle = \langle\psi|\hat{A}|\phi\rangle^*$ for any pair of vectors $|\phi\rangle$ and $|\psi\rangle$. [2 marks]

(b) For example, take the Hamiltonian, $\hat{H}$. The eigenvalues of this operator are the energies E_n for which there exists a vector $|n\rangle$ such that $\hat{H}|n\rangle = E_n|n\rangle$. [2 marks] Given a state vector $|\psi\rangle$, the expectation value of $\hat{H}$ in that state is (normally) not a single eigenenergy but rather the average of the eigenenergies E_n one could obtain in a measurement on that system, each E_n being weighted in the average by the probability of obtaining it. This expectation value can be written as $\langle\psi|\hat{H}|\psi\rangle$. [2 marks]

(c) (i) $\hat{U}^{-1} = \hat{U}^\dagger$, where $\hat{U}^{-1}$ is the inverse of $\hat{U}$ and $\hat{U}^\dagger$ is its adjoint. [2 marks]
(ii) With $\hat{I}$ denoting the unit operator, $\langle\phi'|\psi'\rangle = \langle\phi|\hat{A}^\dagger\hat{A}|\psi\rangle = \langle\phi|\hat{A}^{-1}\hat{A}|\psi\rangle = \langle\phi|\hat{I}|\psi\rangle = \langle\phi|\psi\rangle$. [2 marks]

(d) State vectors are time-dependent in the Schrödinger picture and time-independent in the Heisenberg picture, whereas operators which are time-independent in the Schrödinger picture correspond to time-dependent operators in the Heisenberg picture. [2 marks] If $\hat{A}$ is a time-independent operator in the Schrödinger picture, then its counterpart in the Heisenberg picture is $\hat{U}^\dagger(t,t_0)\hat{A}\hat{U}(t,t_0)$, where $\hat{U}(t,t_0)$ is the time evolution operator for evolution from a given time t_0 to time t. [2 marks]

(e) $O_{aa} = \langle a|\hat{O}|a\rangle = \langle a|b\rangle = 0$, $O_{ab} = \langle a|\hat{O}|b\rangle = \langle a|a\rangle = 1$, $O_{ba} = \langle b|\hat{O}|a\rangle = \langle b|b\rangle = 1$, $O_{bb} = \langle b|\hat{O}|b\rangle = \langle b|a\rangle = 0$. In this basis, the operator $\hat{O}$ is thus represented by the 2×2 matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are the numbers λ such that

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0,$$

i.e., $\lambda = 1$ and $\lambda = -1$. [4 marks]

(f) [Unseen but close to an example students will have seen] Since the eigenvalues of $\hat{\underline{J}}^2$ are $\hbar^2 J(J+1)$ with $J \geq 0$, J is necessarily equal to 2 here. In a joint eigenstate of $\hat{\underline{J}}^2$ and $\hat{J}_z$, the possible eigenvalues of $\hat{J}_z$ are $\hbar M$ with $M = -J, -J+1, \ldots, J$. Since $J = 2$ here, the possible eigenvalues of $\hat{J}_z$ are found by taking $M = -2$, -1, 0, 1 or 2. [4 marks]

(g) M is necessarily equal to $m^{(1)} + m^{(2)}$, hence $M = 0$ here. Moreover, $|j^{(1)} - j^{(2)}| \leq J \leq j^{(1)} + j^{(2)}$. [4 marks]

Academic year 2017/18 — Theoretical Physics 2 — May/June 2018 paper — Question 5
[All unseen unless indicated otherwise.]

(a)(i) [Bookwork] $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$. [2 marks]

(a)(ii) This component of $\underline{\hat{L}}$ is a combination of the position operators for the y- and z-directions, ($\hat{y}$ and $\hat{z}$) and the corresponding momentum operators ($\hat{p}_y$ and $\hat{p}_z$). Hence one would use the operators $\hat{a}_y$ and $\hat{a}_y^\dagger$ and also the operators $\hat{a}_z$ and $\hat{a}_z^\dagger$ such that $\hat{z} = \alpha(\hat{a}_z + \hat{a}_z^\dagger)$ and $\hat{p}_z = (\hbar/2i\alpha)\,(\hat{a}_z - \hat{a}_z^\dagger)$. [3 marks][There is no requirement that $\hat{a}_z$ and $\hat{a}_z^\dagger$ be formally defined, the definition in terms of $\hat{z}$ and $\hat{p}_z$ are given in the above solution for completeness only.]

(b)(i) $\hat{b}_1^\dagger = (\hat{a}_x^\dagger - i\hat{a}_y^\dagger)/\sqrt{2}$. [2 marks] Hence

$$
\begin{aligned}
[\hat{b}_1, \hat{b}_1^\dagger] &= (1/2)(\hat{a}_x + i\hat{a}_y)(\hat{a}_x^\dagger - i\hat{a}_y^\dagger) - (1/2)(\hat{a}_x^\dagger - i\hat{a}_y^\dagger)(\hat{a}_x + i\hat{a}_y) \\
&= (1/2)(\hat{a}_x\hat{a}_x^\dagger + i\hat{a}_y\hat{a}_x^\dagger - i\hat{a}_x\hat{a}_y^\dagger + \hat{a}_y\hat{a}_y^\dagger) - (1/2)(\hat{a}_x^\dagger\hat{a}_x - i\hat{a}_y^\dagger\hat{a}_x + i\hat{a}_x^\dagger\hat{a}_y + \hat{a}_y^\dagger\hat{a}_y) \\
&= (1/2)(\hat{a}_x\hat{a}_x^\dagger - \hat{a}_x^\dagger\hat{a}_x) + (i/2)(\hat{a}_y\hat{a}_x^\dagger - \hat{a}_x^\dagger\hat{a}_y) - (i/2)(\hat{a}_x\hat{a}_y^\dagger - \hat{a}_y^\dagger\hat{a}_x) + (1/2)(\hat{a}_y\hat{a}_y^\dagger - \hat{a}_y^\dagger\hat{a}_y).
\end{aligned}
$$

[2 marks] Therefore $[\hat{b}_1, \hat{b}_1^\dagger] = 1$ since $[\hat{a}_x, \hat{a}_x^\dagger] = [\hat{a}_y, \hat{a}_y^\dagger] = 1$ and the x-operators commute with the y-operators. [4 marks]

(b)(ii) $\hat{b}_1 + \hat{b}_2 = 2\hat{a}_x/\sqrt{2}$, thus $\hat{a}_x = (\hat{b}_1 + \hat{b}_2)/\sqrt{2}$. Similarly, $\hat{b}_1 - \hat{b}_2 = 2i\hat{a}_y/\sqrt{2}$, thus $\hat{a}_y = (\hat{b}_1 - \hat{b}_2)/(i\sqrt{2})$ and $\hat{a}_y^\dagger = -(\hat{b}_1^\dagger - \hat{b}_2^\dagger)/(i\sqrt{2})$. [3 marks]

(b)(iii) Since the $\hat{b}_1$, $\hat{b}_1^\dagger$, $\hat{b}_2$ and $\hat{b}_2^\dagger$ are raising and lowering operators, the spectrum of the operators $\hat{b}_1^\dagger\hat{b}_1$ and $\hat{b}_2^\dagger\hat{b}_2$ is the set of all non-negative integers. [3 marks] Therefore the spectrum of $\hat{L}_z/\hbar$ is the set of all integers. [1 mark]

Academic year 2017/18 — Theoretical Physics 2 — May/June 2018 paper — Question 6

(a) [Unseen]

$$\langle\chi|\chi\rangle = \begin{pmatrix}\sqrt{3}/2 & -i/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 \\ i/2\end{pmatrix} = 3/4 + 1/4 = 1.$$

[3 marks]

(b)(i) [Bookwork] There are two possibilities: $\hbar/2$ and $-\hbar/2$. [3 marks]

(b)(ii) [Unseen] The probability of finding $\hbar/2$ is $|\sqrt{3}/2|^2 = 3/4$. The probability of finding $-\hbar/2$ is $|i/2|^2 = 1/4$. [4 marks]

(b)(iii) [Unseen]

$$\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix}\sqrt{3}/2 & -i/2\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 \\ i/2\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\sqrt{3}/2 & -i/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 \\ -i/2\end{pmatrix} = \frac{\hbar}{2}\left(\frac{3}{4} - \frac{1}{4}\right) = \frac{\hbar}{4}.$$

Thus $\langle\chi|\hat{S}_z|\chi\rangle^2 = \hbar^2/16$.

$$\langle\chi|\hat{S}_z^2|\chi\rangle = \frac{\hbar^2}{4}\begin{pmatrix}\sqrt{3}/2 & -i/2\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 \\ i/2\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}\sqrt{3}/2 & -i/2\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 \\ i/2\end{pmatrix}.$$

Hence $\langle\chi|\hat{S}_z^2|\chi\rangle = \hbar^2\langle\chi|\chi\rangle/4 = \hbar^2/4$ and the uncertainty is $\hbar[1/4 - 1/16]^{1/2} = \sqrt{3}\hbar/4$.

[Marking scheme: 3 marks for the method, 1 mark for a correct final answer.]

(c)[Unseen] There are only two possible outcomes in this measurement, $\hbar/2$ and $-\hbar/2$. Therefore requiring that the outcome is a positive value with probability 1 amounts to saying that $|\chi\rangle$ is an eigenstate of this component of the spin operator corresponding to the eigenvalue $\hbar/2$. [3 marks] I.e., that

$$\begin{pmatrix}\cos\theta & \exp(-i\phi)\sin\theta \\ \exp(i\phi)\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 \\ i/2\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 \\ i/2\end{pmatrix}.$$

Here $\exp(i\phi) = i$ and and $\exp(-i\phi) = -i$ since ϕ is taken to be $\pi/2$. [1 mark] The eigenvalue equation thus reduces to the following system of linear equations:

$$\begin{aligned}(\sqrt{3}/2)\cos\theta + (1/2)\sin\theta &= \sqrt{3}/2 \\ i(\sqrt{3}/2)\sin\theta - i(1/2)\cos\theta &= i/2.\end{aligned}$$

This last equation can be simplified to $\cos\theta = \sqrt{3}\sin\theta - 1$. Substituting in the first equation of the system one finds $(\sqrt{3}/2)(\sqrt{3})\sin\theta - (\sqrt{3}/2) + (1/2)\sin\theta = \sqrt{3}/2$, which gives $2\sin\theta = \sqrt{3}$, thus $\sin\theta = \sqrt{3}/2$. Thus $\cos\theta = 3/2 - 1 = 1/2$, which means that θ must be $\pi/3$. [2 marks]