Theoretical Physics 2019/20 — Solution of Problem QT2.3

(a) As mentioned near the beginning of the question, any spin state can be represented by a non-zero two-component column vector. However, any two-component column vector can be represented as a linear combination of χ_a and χ_b since these two vectors form a basis for the Hilbert space of two-component column vectors. Since a non-zero vector orthogonal to both χ_a and χ_b cannot be written as a linear combination of these two basis vectors, such a state does not exist.

[1 mark]

(b) Since c_a, c_b, χ_a and χ_b are constant, $\chi(t)$ depends on t only through the two exponentials. Therefore

$$\begin{aligned} i\hbar\frac{\partial\chi}{\partial t} &= i\hbar\,[c_a\,\chi_a(-iE_a/\hbar)\exp(-iE_at/\hbar) + c_b\,\chi_b(-iE_b/\hbar)\exp(-iE_bt/\hbar)] \\ &= c_a\,\chi_aE_a\exp(-iE_at/\hbar) + c_b\,\chi_bE_b\exp(-iE_bt/\hbar). \end{aligned} \tag{1}$$

Moreover,

$$\begin{aligned} H\,\chi(t) &= c_a\,H\chi_a\exp(-iE_at/\hbar) + c_b\,H\chi_b\exp(-iE_bt/\hbar) \\ &= c_a\,E_a\,\chi_a\exp(-iE_at/\hbar) + c_b\,E_b\,\chi_b\exp(-iE_bt/\hbar). \end{aligned} \tag{2}$$

Comparing Eqs. (1) and (2) shows that indeed

$$i\hbar\frac{\partial\chi}{\partial t} = H\chi(t).$$

[1 mark]

(c) We need to show that the norm of $\chi(t)$ is 1 when $|c_a|^2 + |c_b|^2 = 1$. The square of the norm of $\chi(t)$ is $(\chi(t), \chi(t))$. Now,

$$\begin{aligned} (\chi(t),\chi(t)) =& c_a^*c_a(\chi_a,\chi_a)\exp[i(E_a-E_a)t/\hbar] + c_a^*c_b(\chi_a,\chi_b)\exp[i(E_a-E_b)t/\hbar] \\ &+ c_b^*c_a(\chi_b,\chi_a)\exp[i(E_b-E_a)t/\hbar] + c_b^*c_b(\chi_b,\chi_b)\exp[i(E_b-E_b)t/\hbar]. \end{aligned} \tag{3}$$

Since $(\chi_a, \chi_a) = 1$ and $(\chi_a, \chi_b) = 0$ (remember that the question says that these two column vectors are orthonormal),

$$(\chi(t),\chi(t)) = c_a^*c_a + c_b^*c_b = |c_a|^2 + |c_b|^2.$$

Thus $\chi(t)$ is normalized if $|c_a|^2 + |c_b|^2 = 1$.

[1 mark]

(d) As defined by Eq. (3) of the question, $\chi(t)$ is a linear combination of the two vectors χ_a and χ_b. Since χ_a and χ_b are orthonormal, the coefficients of this superposition can be found simply by calculating the inner product of $\chi(t)$ with each of these two basis vectors: At $t = 0$, $\exp(-iE_at/\hbar) = \exp(-iE_bt/\hbar) = 1$, and therefore $\chi(t = 0) = c_a\chi_a + c_b\chi_b$. Thus $(\chi_a, \chi(t = 0)) = c_a(\chi_a, \chi_a) + c_b\,(\chi_a, \chi_b)$.

Since $(\chi_a, \chi_a) = 1$ and $(\chi_a, \chi_b) = 0$, $(\chi_a, \chi(t=0)) = c_a$. Likewise, $(\chi_b, \chi(t=0)) = c_b$. The question says that $\chi(t) = \chi_+$ at $t = 0$. Thus

$$c_a = (\chi_a, \chi_+) = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} -k & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{-k}{\sqrt{k^2+1}},$$
$$c_b = (\chi_b, \chi_+) = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} 1 & k \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{k^2+1}}.$$

(No complex conjugation is indicated in the row vectors because k is real since B_x, B_z and $|\mathbf{B}|$ are real.)

Check:

$$|c_a|^2 + |c_b|^2 = \frac{k^2}{k^2+1} + \frac{1}{k^2+1} = 1.$$

[3 marks for this part of the problem.]

(e) The probability of finding the electron in the spin state represented by χ_+ at time t is $|(\chi_+, \chi(t))|^2$,

Calculating $|(\chi_+, \chi(t))|^2$ is not particularly difficult: Using the fact that $(\chi_+, \chi_a) = (\chi_a, \chi_+)$ since (χ_+, χ_a) is real, and similarly for (χ_+, χ_b),

$$\begin{aligned}
(\chi_+, \chi(t)) &= c_a (\chi_+, \chi_a) \exp(-iE_a t/\hbar) + c_b\, (\chi_+, \chi_b) \exp(-iE_b t/\hbar) \\
&= \frac{-k}{\sqrt{k^2+1}} \frac{-k}{\sqrt{k^2+1}} \exp(-iE_a t/\hbar) + \frac{1}{\sqrt{k^2+1}} \frac{1}{\sqrt{k^2+1}} \exp(-iE_b t/\hbar) \\
&= \frac{k^2}{k^2+1} \exp(-i\mu|\mathbf{B}|t/\hbar) + \frac{1}{k^2+1} \exp(i\mu|\mathbf{B}|t/\hbar).
\end{aligned}$$

Recall that $k = B_x/(B_z + |\mathbf{B}|)$. Thus $k = \pm 1$ when $B_y = B_z = 0$ since $B_x = \pm|\mathbf{B}|$ in that case. Accordingly, the probability of finding the electron in the state χ_+ is

$$[(1/2)\exp(-i\mu|\mathbf{B}|t/\hbar) + (1/2)\exp(i\mu|\mathbf{B}|t/\hbar)]^2 = \cos^2(\mu|\mathbf{B}|t/\hbar).$$

[3 marks]

(f) Since $\chi(t)$ is normalized and χ_a and χ_b are orthonormal, these two probabilities are, respectively, $|c_a \exp(-iE_a t/\hbar|^2$ and $|c_b \exp(-iE_b t/\hbar|^2$ (i.e., the square of the modulus of the coefficients of χ_a and χ_b in $\chi(t)$. That is, $|c_a|^2$ and $|c_b|^2$ (recall that $|\exp(ix)| = 1$ if x is a real number). In order for $|c_a|^2$ to be equal to $|c_b|^2$, k^2 must be equal to 1, which means that the field must be oriented in the positive or the negative x-direction.

[1 mark for (f)]