University of Durham

EXAMINATION PAPER

May/June 2011 Examination code: 043522/02

LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3 PAPER 2

SECTION A. QUANTUM AND ATOMIC PHYSICS
SECTION B. QUANTUM AND NUCLEAR PHYSICS
SECTION C. QUANTUM AND PARTICLE PHYSICS

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Nuclear magneton:	$\mu_{\mathrm{N}} = 5.05 \times 10^{-27}\ \mathrm{J\,T^{-1}}$
Molar Gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$

SECTION A. QUANTUM AND ATOMIC PHYSICS

Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) Describe briefly Doppler cooling. [4 marks]

   (b) The clock transition in a ^{133}Cs fountain clock is the ground state hyperfine transition $F = 3,\ m_F = 0 \rightarrow F' = 4,\ m'_F = 0$ and has an angular frequency of $\omega_0 = 2\pi \times 9.192631770 \times 10^9$ s^{-1}. The time between the end of the first pass and the beginning of the second pass through the microwave cavity is

   $$T = \left(\frac{8h}{g}\right)^{\frac{1}{2}}.$$

   For a fountain height $h = 0.5$ m, calculate the width of the central Ramsey fringe. Calculate the clock uncertainty given that the line centre can be determined to 1 part in 10^4. [4 marks]

   [Hint: Treat the numbers given as exact and use the full precision of your calculator. Use the standard gravitational acceleration value $g = 9.80665$ m s^{-2}.]

   (c) Draw a schematic of an atomic beam clock using the following components: a feedback servo; an atomic beam; a microwave cavity; an oscillation counter; an atom detector and a microwave radiation source. Use your schematic to explain how such a clock can be used to measure time. [4 marks]

   (d) The wavefunctions for the ground state and second excited state of a particle of mass m in an harmonic oscillator potential are

   $$\psi^0_{v=0} = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \text{ and}$$

   $$\psi^0_{v=2} = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2}} \left(\frac{2m\omega x^2}{\hbar} - 1\right) \exp\left(-\frac{m\omega x^2}{2\hbar}\right),$$

   respectively, where ω is the classical oscillation angular frequency. The particle is in the ground state and is suddenly exposed at $t' = t_0 = 0$ to an oscillating electric field

   $$H'(x, t') = -e\xi_0 x \cos(\omega t'),$$

   where ξ_0 is the peak electric field strength. Obtain the first-order probability that the particle will be in the second excited state some time $t' = t$ later, and explain your reasoning. [4 marks]

   (e) A stationary atom of mass m is exposed to a resonant laser beam of wavelength λ pointing along the z-axis. In a short period of time, the atom completes two absorption and spontaneous emission cycles. The first and second spontaneously emitted photons have wavevectors

$$\hat{k}_1 = \frac{2\pi}{\lambda}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{k}_2 = \frac{2\pi}{\lambda}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix},$$

respectively, where the vector (x, y, z) specifies the direction in which the photon is emitted. Show that the velocity of the atom after emission of the second photon is

$$\hat{v} = -\frac{h}{m\lambda}\begin{pmatrix} 1/\sqrt{2} \\ 1 \\ (1-2\sqrt{2})/\sqrt{2} \end{pmatrix}.$$

[4 marks]

2. For a hydrogen atom in its ground state, the four spin configuration basis vectors

$$|++\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |+-\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad ,$$

$$|-+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |--\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

form a group of three degenerate eigenstates of eigenvalue λ_1 and one other eigenstate of eigenvalue λ_2 due to the hyperfine interaction

$$H' = \frac{1}{4}\mathcal{A} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

(a) Determine λ_1 and λ_2 in terms of the hyperfine splitting $\mathcal{A}$. [9 marks]

(b) In an atomic clock, a microwave field drives a transition from a lower hyperfine level to an upper hyperfine level. The transition energy ΔE between the levels exhibits a variation in the presence of an external magnetic field $\mathcal{B}$. Derive the matrix H' for the hyperfine interaction in the presence of $\mathcal{B}$. [5 marks]

Derive an expression for the transition energy between the two hyperfine levels that vary non-linearly in $\mathcal{B}$. [6 marks]

[Hint: The z-axis spin matrix $\hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.]

3. For small vibrational quantum numbers, a diatomic molecule can be approximated as a quantum mechanical harmonic oscillator with a potential

$$V(r - r_{\mathrm{e}}) = \frac{1}{2}k(r - r_{\mathrm{e}})^2,$$

where k is the force constant of the chemical bond and $r - r_{\mathrm{e}}$ is the displacement from the equilibrium bond length r_{e}, i.e. when $r = r_{\mathrm{e}}$, $V(0) = 0$. The eigenenergies of the unperturbed vibrational energy eigenstates are

$$E_v^0 = \hbar\omega(v + 1/2),$$

where $v = 0, 1, 2, ...$ is the vibrational quantum number and ω is the classical angular frequency. Suppose a small perturbation was to cause the force constant to increase slightly from k to $k' = (1 + \beta)k$, where β is a positive number $\ll 1$, such that E_v^0 increases to give exact new eigenenergies $E_v = \hbar\omega(1 + \beta)^{1/2}(v + 1/2)$. The matrix elements of $(r - r_{\mathrm{e}})^2$ are

$$\langle\psi_w^0|(r - r_{\mathrm{e}})^2|\psi_v^0\rangle = \begin{cases} \frac{\hbar}{2m\omega}[(v+1)(v+2)]^{1/2} & w = v + 2, \\ \frac{\hbar}{2m\omega}[v(v-1)]^{1/2} & w = v - 2, \\ 0 & \text{otherwise, except } w = v, \end{cases}$$

where m is the mass. Use time-independent perturbation theory to calculate

(a) The first-order correction to the eigenenergies. [8 marks]

(b) The second-order correction to the eigenenergies. [9 marks]

(c) Compare the first and second order corrections to the eigenenergies with the exact new eigenenergies E_v and comment. [3 marks]

[Hint: $E_v^0 = 2\langle\psi_v^0|V(r - r_{\mathrm{e}})|\psi_v^0\rangle$ and $k = m\omega^2$]

SECTION B. QUANTUM AND NUCLEAR PHYSICS
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) List four methods of measuring nuclear radii. [4 marks]

   (b) The electrostatic energy of a nucleus of atomic number Z is $0.864Z(Z-1)/R$ MeV where R is its radius in fm. ${}_8O^{15}$ can decay to ${}_7N^{15}$ with the emission of a positron which has a maximum kinetic energy of 1.72 MeV. Obtain from this a value for the nuclear radius. [4 marks]

   $$[\ (m_n - m_p)c^2 = 1.294 \text{ MeV};\ m_e c^2 = 0.511 \text{ MeV}\ ]$$

   (c) Give four physical quantities that are conserved in low-energy nuclear reactions. [4 marks]

   (d) Predict the spin, parity, magnetic dipole and electric quadrupole moments of a ${}_{20}Ca^{40}$ nucleus. [4 marks]

   (e) A nucleon-nucleon system has isospin and total spin quantum numbers I and S and an orbital angular momentum quantum number $\ell = 0$. Use the Pauli exclusion principle to show that $S + I$ must be odd. [4 marks]

5. Express the magnetic dipole moment of a nucleus in terms of its total angular momentum, $\underline{J}$, and g-factor, g_J. Relate this to the total orbital and spin angular momenta of its nucleons and the corresponding g-factors, g_L and g_S. [4 marks]

   Show that

$$g_J = \frac{g_L + g_S}{2} + \frac{g_L - g_S}{2}\left(\frac{\ell(\ell+1) - s(s+1)}{j(j+1)}\right),$$

   where j, ℓ and s are the quantum numbers of the total, orbital and spin angular momenta. [4 marks]

   In the independent particle shell model the total angular momenta of even-odd nuclei are determined by the odd nucleon. Without a spin-orbit interaction the first four energy levels of protons in a nucleus, labelled by their orbital angular momenta, are $1s$, $1p$, $2s$ and $1d$. Draw the energy level diagram which results when these levels are split by the spin-orbit interaction giving the j quantum numbers of the levels and the numbers of protons that may occupy each level. From this calculate g_J for the ground states of ${}_8\mathrm{O}^{15}$, ${}_{11}\mathrm{Na}^{23}$ and ${}_{19}\mathrm{K}^{39}$. [12 marks]

[The spin g-factors of the proton and neutron are $g_s = 5.59$ and $g_s = -3.83$ respectively.]

6. The Schrödinger equation for the bound state of the deuteron in the neutron-proton centre-of-mass frame is

$$-\frac{\hbar^2}{M}\frac{d^2u(r)}{dr^2} + (V(r) + V_B)u(r) = 0,$$

where $u(r) = r\psi(r)$ is the radial wavefunction and $V_B = 2.2$ MeV is the deuteron binding energy. What does M represent? [2 marks]

Assume that the inter-nucleon potential can be described by a square well, of radius R, with a repulsive core, of radius R_c.

$$\begin{aligned} V(r) &= +\infty, & r < R_c \\ &= -V_0, & R_c < r < R \\ &= 0, & r > R. \end{aligned}$$

Give general mathematical forms of the solutions in the three regions and sketch the wave function as a function of the neutron-proton separation r. [8 marks]

Assuming that $R = 2.6$ fm, deduce from the wave function within the well,

(a) the minimum possible value of V_0, [7 marks]

(b) the maximum value that R_c can take if $V_0 = 25$ MeV. [3 marks]

[Use $Mc^2 = 939$ MeV and $\hbar c = 197$ MeV fm.]

SECTION C. QUANTUM AND PARTICLE PHYSICS
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) Explain the difference between mesons and baryons in the quark model. Specify the quark content of the following hadrons:

$$\text{proton},\ \pi^+,\ K^-.$$

[4 marks]

(b) Which fundamental interactions are described by the Standard Model? Which gauge bosons are associated to these interactions?
Which gauge boson is exchanged in the process

$$e^+e^- \to \bar{\nu}_\mu \nu_\mu \ ?$$

Draw the corresponding Feynman diagram. [4 marks]

(c) Draw labelled Feynman diagrams for the fundamental Standard Model interactions that are responsible for the following particle reactions:

$$\pi^+ \to \mu^+ \nu_\mu\,, \qquad e^- p \to e^- p\,,$$

What kind of hadronic quantities are being measured in each reaction?
[4 marks]

(d) Give the result for the following integrals over quark distribution functions in the proton

(i)

$$\int_0^1 dx\,[u(x) - \bar{u}(x)],$$

(ii)

$$\int_0^1 dx\,[d(x) - \bar{d}(x)],$$

(iii)

$$\int_0^1 dx\,[s(x) - \bar{s}(x)].$$

Explain why

$$\sum_{q=u,d,s} \int_0^1 dx\, x\,[q(x) + \bar{q}(x)] < 1\,.$$

[4 marks]

(e) Which symmetries of the Standard Model are broken by the Higgs vacuum expectation value, and which symmetries remain unbroken? What happens to the masses of the electro-weak gauge bosons? What does this imply for the corresponding interaction ranges? [4 marks]

8. The main decay mode of the top-quark is given by $t \to bW^+$. In the following, we can neglect the bottom mass m_b, $m_b/m_t \simeq 0$, where m_t is the top mass. The decay matrix element squared at tree level is given by

$$|\mathcal{M}_{t\to bW^+}|^2 \;=\; \frac{g_w^2}{4m_W^2}\, m_t^4\, |V_{tb}|^2\, (1 + x^2 - 2x^4)\,, \qquad (x = m_W/m_t)\,,$$

where m_W is the W^+ boson mass.

(a) Draw and label the tree-level Feynman diagram for the decay amplitude. Explain the appearance of the weak coupling constant g_w and the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{tb}. Write down the relation between the weak coupling constant g_w and the Fermi constant G_F. [5 marks]

(b) In the rest frame of the decaying top-quark, determine the energies and absolute values of momenta for the outgoing W^+-boson and b-quark (with $m_b \to 0$). Use $m_t = 175$ GeV, $m_W = 80$ GeV to obtain a numerical estimate. [6 marks]

(c) Write down the relation between the 2-body decay width and its matrix element squared. [2 marks]

(d) Show that for the following ratio of decay widths, the relation

$$R \;=\; \frac{\Gamma[t \to bW^+]}{\sum_{q=d,s,b} \Gamma[t \to qW^+]} = |V_{tb}|^2$$

holds. Which property of the CKM matrix in the Standard Model do you have to use?

The CDF experiment measures $R = 0.99 \pm 0.29$. What does this imply for the central value of $|V_{tb}|$? [4 marks]

(e) In the limit $m_W \ll m_t$, and $|V_{tb}| \simeq 1$, the width is approximated by

$$\Gamma(t \to bW^+) \;\simeq\; \frac{G_F m_t^3}{8\pi\sqrt{2}}.$$

Give a numerical estimate for Γ using $m_t \simeq 175$ GeV and $G_F = 1.66 \times 10^{-5}$ GeV^{-2}. Compare your result with the scale $\Lambda_{\text{strong}} \sim 300\,\text{MeV}$ appearing in the running strong coupling constant, and explain why the top quark does not form hadronic bound states. [3 marks]

9. An important observable in particle physics is the ratio of cross sections,

$$R \equiv \frac{\sigma[e^+e^- \to \text{hadrons}]}{\sigma[e^+e^- \to \mu^+\mu^-]} .$$

At leading order, its value is given by

$$R_0 = 3 \sum_{\text{quark}} e^2_{\text{quark}} ,$$

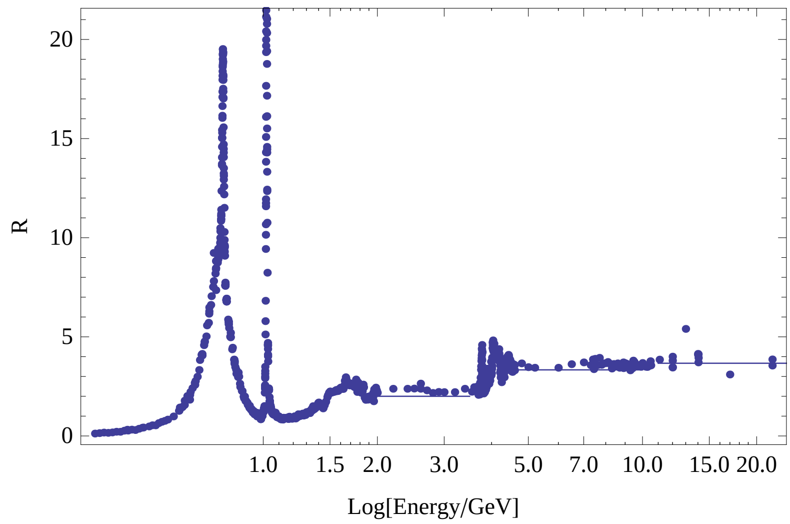
where e_{quark} is the electric charge in units of the elementary charge $|e|$.

(a) Draw the tree-level Feynman diagrams contributing to the two cross sections, and label the momenta for the incoming and outgoing elementary particles. Which gauge boson is exchanged (for not too high energies)? [4 marks]

(b) Give the definition of the Mandelstam variables s, t, u in terms of the particles' energies and momenta. Which Mandelstam variable corresponds to the energy and momentum transferred by the gauge boson? What is its relation to the center-of-mass energy of the electron and positron? [5 marks]

(c) Explain the given result for R_0: Where do the different terms come from? Why does R_0 depend on the center-of-mass energy E? Calculate the numerical value of R_0 for $E = (2, 5, 20)$ GeV. [6 marks]

(d) Compare the results for R_0 with the measurements sketched in the following figure: What is the physical origin of the spikes? [2 marks]



(e) Imagine a world where quarks come in 5 colours, up-type quarks have fractional electric charge $+3/5$, and the neutron consists of $(uuddd)$: What would be the fractional charge of the down-type quarks? How would the values for R_0 change compared to (c)? [3 marks]