Mathematical Methods II
PDF 8

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Key Points

- Legendre's differential equation
- Legendre polynomials

Special Functions

- **Special Functions:** Some 2^{nd} order ODEs appear so frequently in physics and engineering that they have been given names. The solutions to these equations, which obey particularly commonly occuring boundary conditions, have been studied extensively. One such example is Legendre functions.

- **Legendre's Differential Equation:** Legendre's differential equation has the form

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

where ℓ is a real constant. Any solution to this equation is called a Legendre Function. It has 3 regular singular points at $x = -1, 1, \infty$ and occurs in numerous physical applications, particularly in problems with axial symmetry that involve the ∇^2 operator (Laplace operator) when they are expressed in spherical polar coordinates.

Aside: The Laplace operator - This operator gives the divergence of the gradient of a function.

$$\nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

i.e. For a 2-variable problem the del (or nabla) operator gives the gradient ('grad')

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

and the Laplace operator gives the divergence ('div') of the gradient

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- **Legendre Recurrance Relation:** $x = 0$ is an ordinary point of Legendre's differential equation, so we can find two linearly independent series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

with derivatives,

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting these into the ODE,

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Expanding the first bracket and rewriting the sum,

$$\sum_{n=0}^{\infty} \left[n(n-1)a_n x^{n-2} - n(n-1)a_n x^n - 2n a_n x^n + \ell(\ell+1)a_n x^n\right] = 0.$$

Shifting the index to x^n and simplifying, factorising by a_n and a_{n+2},

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - \left[n(n+1) - \ell(\ell+1)\right] a_n\right] x^n = 0.$$

And so we find the recurrance relation,

$$a_{n+2} = \frac{\left[n(n+1) - \ell(\ell+1)\right]}{(n+1)(n+2)} a_n.$$

We can see that two solutions can be found by examining the even and odd values of n separately. Choosing $a_0 = 1$ and $a_1 = 0$ we get the solution,

$$y_1(x) = 1 - \ell(\ell+1)\frac{x^2}{2!} + (\ell-2)\ell(\ell+1)(\ell+3)\frac{x^4}{4!} - ...$$

Choosing $a_0 = 0$ and $a_1 = 1$ we find the solution,

$$y_2(x) = x - (\ell-1)(\ell+2)\frac{x^3}{3!} + (\ell-3)(\ell-1)(\ell+2)(\ell+4)\frac{x^5}{5!} - ...$$

The ratio test shows that both of these series converge for $|x| < 1$ hence their radius of convergence to the nearest singular point is $r = 1$. Since the series are made of odd and even powers, they cannot be proportional to each other and are therefore linearly independent. So the general solution to the Legendre differential equation for $|x| < 1$ is given by,

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

- **Legendre Functions for Integer ℓ:** Our counting variable, n, is of course an integer. In many physical applications of Legendre's DE ℓ is also an integer. If we increase n until $n = \ell$ then our recurrance relation at that point becomes,

$$a_{\ell+2} = \frac{[\ell(\ell+1) - \ell(\ell+1)]}{(\ell+1)(\ell+2)} a_\ell = 0,$$

i.e. the series will terminate after a_ℓ, as $a_{\ell+2}$ and subsequent terms will be zero. This will give us a polynomial solution to the equation of order $n = \ell$. These are written $P_\ell(x)$ and are valid for all finite values of x.

By convention, we normalise $P_\ell(x)$ such that $P_\ell(1) = 1$, consequently $P_\ell(-1) = (-1)^\ell$ (i.e. negative x will give positive $P_\ell(x)$ for even ℓ and negative $P_\ell(x)$ for odd ℓ). The first few Legendre polynomials are given by

$$P_0(x) = 1 \qquad P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

For example,

$$P_2(1) = \frac{1}{2}(3 - 1) = 1 \qquad P_3(1) = \frac{1}{2}(5 - 3) = 1$$
$$P_2(-1) = \frac{1}{2}(3 - 1) = 1 \qquad P_3(-1) = \frac{1}{2}(-5 + 3) = -1$$

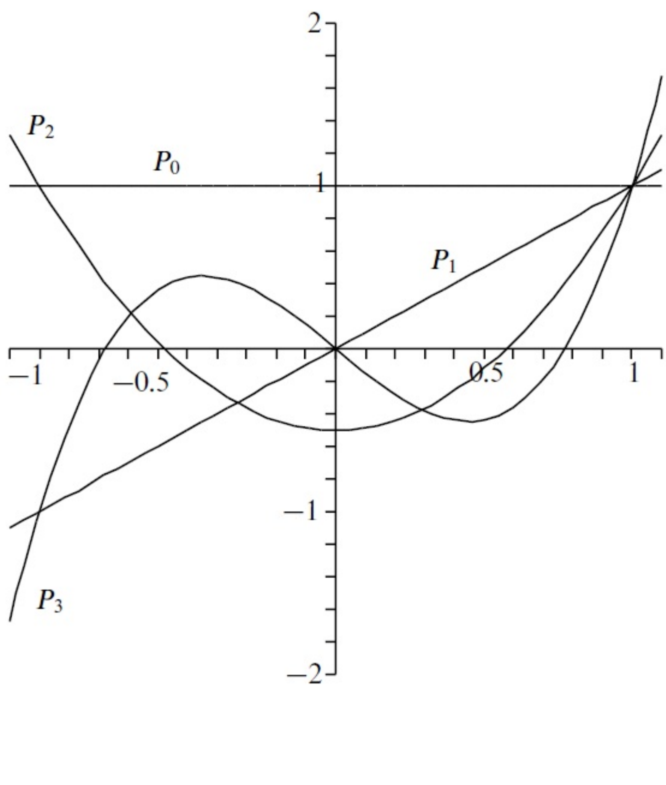

Figure 18.1 The first four Legendre polynomials.

So when ℓ is an odd or even integer, the y_1 series will terminate, giving the corresponding Legendre polynomial $P_\ell(x)$ as a solution to the Legendre DE. i.e. $y_1(x) = P_\ell(x)$.

To demonstate that the polynomials are solutions, lets try substituting one into the equation. When $\ell = 2$ the polynomial solution and its derivatives are

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_2'(x) = 3x$$

$$P_2''(x) = 3$$

Subbing this in we get

$$\begin{aligned}
&(1-x^2)y'' - 2xy' + \ell(\ell+1)y \\
&= (1-x^2)\times 3 - 2x \times 3x + 2(3)\frac{1}{2}(3x^2-1) \\
&= 3 - 3x^2 - 6x^2 + 9x^2 - 3 = 0
\end{aligned}$$

as expected.

In order to obtain a general solution we would still need $y_2(x)$. However, the second series will not terminate and only converge for $|x| < 1$. A second set of polynomials $Q_\ell(x)$ can be defined, referred to as *Legendre functions of the second kind.* Finding $Q_\ell(x)$ for the general solution is more complicated; we will restrict ourselves to functions of the first kind for the time being (see Pg 579 of Riley for more information on $Q_\ell(x)$).

- **Rodrigues' Formula:** Conveniently, there is a formula that allows one to quickly determine a Legendre polynomial for a given ℓ

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

e.g. PDF8.1 (a) State the name of the following type of equation and give its general form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 42y = 0$$

This is a Legendre equation, with general form

$$(1-x^2)y'' - 2xy' + vy = 0$$

where $v = \ell(\ell+1)$, and ℓ is a constant.

(b) State a polynomial solution to the equation with undetermined coefficients

We know that $v = 42 = \ell(\ell+1) = 6(7) \Rightarrow \ell = 6$. So for an even ℓ with a value of 6 the solution must be of the form

$$P_6(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6$$

where a_n are constants.

e.g. PDF8.2 Use Rodrigues' formula to determine a polynomial solution to the following equation.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 12y = 0$$

We know that $v = 12 = \ell(\ell+1) = 3(4) \Rightarrow \ell = 3$. So

$$\begin{aligned}
P_3(x) &= \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \\
&= \frac{1}{(2^3)(3!)}\frac{d^3}{dx^3}(x^2-1)^3 \\
&= \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) \\
&= \frac{1}{48}(120x^3 - 72x) \\
&= \frac{1}{2}(5x^3 - 3x)
\end{aligned}$$

- **Associated Legendre Differential Equation:** The associated Legendre's differential equation has the form

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]y = 0$$

and has 3 regular singular points at $x = -1, 1, \infty$. It is a more general form of Legendre's DE and reduces to that equation when $m = 0$. In physical applications the value of m is restricted such that $-\ell \le m \le \ell$.

The polynomials solutions for the associated Legendre DE are related to the those from of the non-associated version via the following

$$P_\ell^m(x) = (1-x^2)^{m/2}\frac{d^m P_\ell}{dx^m} \qquad \text{for } m > 0$$