University of Durham

EXAMINATION PAPER

May/June 2011 Examination code: 042531/01

LEVEL 2 PHYSICS: THERMAL AND CONDENSED MATTER PHYSICS

SECTION A. THERMODYNAMICS & STATISTICAL MECHANICS
SECTION B. CRYSTALS, VIBRATIONS & X-RAYS
SECTION C. ELECTRONS IN SOLIDS

Time allowed : 2 hours and 30 minutes

Examination material provided : None

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 60% of the total marks for the paper. Answer **two** of the four optional questions, **not** both from the same section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Nuclear magneton:	$\mu_\mathrm{N} = 5.05 \times 10^{-27}\ \mathrm{J\,T^{-1}}$
Molar Gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$

SECTION A. THERMODYNAMICS & STATISTICAL MECHANICS
Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) If the magnetisation of a magnetic solid at temperature T in a magnetic field of strength B can be written as $M = M(B, T)$, write down an expression for the small change in magnetisation resulting from small changes in B and T. Give an expression for dM in a paramagnetic solid for which $M \propto B/T$. [4 marks]

   (b) Two identical copper blocks of initial temperature T_1 and T_2 are joined together and allowed to reach thermal equilibrium without any interaction with their surroundings. Write down the expression for the entropy change of the blocks in terms of the heat flow into or out of the blocks if each block has heat capacity C. Does the entropy of the universe increase, decrease or stay constant in this process? Show that your answer is true for any value of T_1 and T_2. [4 marks]

   (c) Sketch the operation of a Carnot engine in a pressure-volume diagram and also in a temperature-entropy diagram labelling the isotherms and adiabats and where heat enters or leaves the system. What does the area enclosed by each curve represent? [4 marks]

   (d) Write down an expression for the efficiency of a heat engine in terms of the heat transfer and work done. If the engine is reversible, by how much does the entropy change for each cycle? What is the ratio of the volume before and after an adiabatic process if $\gamma = 1.4$ and the pressure changes by a factor of 5? [4 marks]

   (e) The Helmholtz function is defined as $F = U - TS$. Show that

$$U = F - T\left(\frac{\partial F}{\partial T}\right)_V$$

   where all the symbols have their usual meaning. [4 marks]

2. (a) Define enthalpy, H, in terms of internal energy, U, pressure, P and volume V.

A refrigerator is cooled by Joule-Kelvin expansion in which a fluid evaporates after passing through a throttle that lowers the pressure from P_i to P_f. Enthalpy is conserved during this process. The change in temperature with pressure at constant enthalpy is given by the Joule-Kelvin coefficient, μ_{JK}. Use the First Law of Thermodynamics and the reciprocity theorem $((\partial x/\partial y)_z(\partial y/\partial z)_x(\partial z/\partial x)_y = -1)$ to show that this can be expressed as

$$\mu_{\mathrm{JK}} = -\frac{1}{C_P}\left(\frac{\partial H}{\partial P}\right)_T \qquad \text{where } C_P = \left(\frac{\partial H}{\partial T}\right)_P$$

is the heat capacity at constant pressure. [6 marks]

(b) Use the Second Law of Thermodynamics and the Maxwell relationship: $(\partial S/\partial P)_T = -(\partial V/\partial T)_P$ to show that

$$\left(\frac{\partial H}{\partial P}\right)_T = V - T\left(\frac{\partial V}{\partial T}\right)_P$$

and derive an expression for the temperature change as an integral over pressure.

What is the temperature change for an ideal gas? [6 marks].

(c) For a gas following the Van der Waals equation of state,

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

find an expression for the region of the P-V plane where cooling can occur and sketch the result. [8 marks]

3. (a) Write down the relationship between entropy and the number of microstates in a system of particles. A particular system contains N particles in two energy states denoted by $j = 1, 2$ with energies ϵ_j and population $n_j >> 1$. What is the number of microstates in this system? [5 marks]

   (b) After the system is put into contact with a heat reservoir at temperature T, the number of particles in state 1 becomes $n_1 + 1$ and the population of state 2 becomes $n_2 - 1$. What is the number of microstates in the system after this event? Show that the entropy has changed by $k_B \ln(n_2/n_1)$ in the limit of large populations. [6 marks]

   (c) What is the change in the entropy of the reservoir in terms of its temperature and energy if no external work is done? [5 marks]

   (d) Use the results from (b) and (c) to show that the system is consistent with the Maxwell-Boltzmann distribution. [4 marks]

SECTION B. CRYSTALS, VIBRATIONS AND X-RAYS

Question 4 is compulsory. Question 5 is optional.

4. (a) For a cubic unit cell, make sketches showing the planes $(1\bar{1}0)$ and (211). Place your sketches on a conventional set of Cartesian axes and label x, y and z. [4 marks]

   (b) A metal has a unit cell comprising a single atom basis on each lattice point of a tetragonal lattice. Sketch the unit cell, labelling the important angles and the lattice parameters a and c. For the case of $a = 0.180$ nm, and $c = 0.350$ nm, calculate the Bragg angle for scattering of x-rays of wavelength 0.1542 nm from the (020) planes. [4 marks]

   (c) Derive an expression for the absorption of x-rays by a medium of thickness t and mass absorption coefficient μ_m. What are the units of mass absorption coefficient? [4 marks]

   (d) State the dispersion relation for waves on an elastic string. For the case of a monatomic linear lattice of spacing a, the dispersion relation is

$$\omega = \frac{2v_0}{a} \sin \frac{ka}{2}$$

   where v_0 is a velocity, and ω and k have their usual meanings. State the conditions for which this dispersion relation is the same as for waves on an elastic string, and demonstrate that this is the case using the relation given above. [4 marks]

   (e) For a system of N oscillators in a volume V, waves of velocity v_0 have a maximum frequency of ω_D. The density of vibrational states is

$$g(\omega) = \frac{V}{2\pi^2} \frac{\omega^2}{v_0^3}.$$

   Find an expression for ω_D. [4 marks]

5. (a) Write an expression for the average energy of a set of particles obeying Boltzmann statistics each having energy $E = bz^2$, where b is a constant and z is a variable. Hence, show that the average energy per degree of freedom for each particle is $\frac{1}{2}k_\mathrm{B}T$, where k_B is Boltzmann's constant. You may use the standard integrals shown at the end of the question. [8 marks]

   (b) State the relationship between heat capacity at constant volume and the internal energy of a solid. [2 marks]

   (c) Use this and the equipartition theorem to derive a value for the heat capacity at constant volume for an ideal monatomic gas, explaining how you arrive at your answer. [3 marks]

   (d) Use the same principles to determine the classical heat capacity at constant volume for a solid. [2 marks]

   (e) What feature must be introduced into a model in order to account for the *temperature dependence* of heat capacity? [3 marks]

   (f) At room temperature, diamond has a heat capacity of 6 J K^{-1} mol^{-1} . What may be deduced about the Debye temperature of diamond in order to describe this behaviour? [2 marks]

$$\left[\int_{-\infty}^{+\infty} \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}},\ \int_{-\infty}^{+\infty} x^2\exp(-ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\mathrm{a}^3}}\right]$$

SECTION C. ELECTRONS IN SOLIDS

Question 6 is compulsory. Question 7 is optional.

6. (a) A monovalent metal has a resistivity of 1.21×10^{-8} Ω m and an electron density of 1.1×10^{28} m^{-3} at 40 K. Starting from the Drude conductivity formula,

$$\sigma = \frac{ne^2\tau}{m_\mathrm{e}}$$

where all symbols have their usual meaning, determine the classical value for mean free path of electrons at 40 K. [4 marks]

(b) Explain what is meant by the *independent electron approximation* as used in the determination of the electronic structure of solids. Why does this approximation work so well? [4 marks]

(c) Treating gold within the quantum free-electron (Sommerfeld) model, find the Fermi energy of the conduction electrons in this metal. The atomic mass of gold is 196.97 u and it has a density of 19300 kg m^{-3}. For the purposes of your calculation, assume that gold is monovalent. [1 $u = 1.661 \times 10^{-27}$ kg]. [4 marks]

(d) Define what is meant by the term *hole* as applied to the band-structure of a solid. List the physical properties possessed by holes. [4 marks]

(e) The dispersion relation of an electron in a one dimensional band of a solid, of lattice parameter a, is given by

$$E(k) = \alpha k^2 + cos(\beta k)$$

where α and β are constants. Derive an expression for the electron effective mass at the first Brillouin zone boundary. [4 marks]

7. (a) The number of electronic states in k-space between k and $k + \mathrm{d}k$ in a three dimensional quantum free-electron (Sommerfeld) metal is given by:

$$n(k)\mathrm{d}k = \frac{L^3}{\pi^2}k^2\mathrm{d}k$$

where L is the linear dimension of the free-electron gas and k is the electron wavevector. Use this equation to derive an expression for the density of electronic states, as a function of energy, $n(E)$, for a three-dimensional solid. [7 marks]

(b) Explain why both the magnetic susceptibility and heat capacity of a quantum free-electron metal are much smaller than predicted by the classical (Drude) model. [4 marks]

(c) The Pauli paramagnetic susceptibility of a quantum free-electron metal is given by:

$$\chi_P = \mu_0 \mu_B^2 \frac{n(E_\mathrm{F})}{V}$$

where $V = L^3$ is the volume of the metal and all other symbols have their usual meaning. The paramagnetic susceptibility of sodium is found to be 7.2×10^{-6}. Using the result from (a) and assuming that sodium may be treated as a free-electron metal, determine the Fermi energy and hence Fermi velocity of this solid. [9 marks]