Quantum Theory - Worksheet 5

This worksheet contains more problems than you might be able to pass through in 50 minutes. Try to complete Problems 1 and 2 during the workshop. Use Problem 3 for extra practice. Problem 4 is for interest.

Problem 1

Suppose that $\hat{A}$ is a Hermitian operator acting in a 2-dimensional Hilbert space and that this operator is represented by the matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

in a certain orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle\}$.

(a) What are the eigenvalues of this matrix?

(b) Why can you be sure that λ_1 and λ_2 are the eigenvalues of the operator $\hat{A}$?

(c) Is it always the case that a Hermitian operator is represented by a diagonal matrix in an orthonormal basis of eigenvectors of that operator?

Problem 2

As mentioned in a lecture, the δ-"function" $\delta(x-x_0)$ can be represented by a Fourier integral as follows:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[ik(x-x_0)]\,\mathrm{d}k = \delta(x-x_0). \qquad (1)$$

Recall that by definition of the δ-"function", $\delta(x-x_0)$ is such that

$$\int_{-\infty}^{\infty} \delta(x-x_0)\, f(x)\,\mathrm{d}x = f(x_0) \qquad (2)$$

for any function $f(x)$ continuous at $x = x_0$.

(a) Starting from Eq. (2), show that

$$\int_{-\infty}^{\infty} \delta(x-x_0)\,\mathrm{d}x = 1.$$

Also, show that

$$\int_a^b \delta(x-x_0)\, f(x)\,\mathrm{d}x = \begin{cases} f(x_0) & x_0 \in (a,b), \\ 0 & x_0 \notin [a,b]. \end{cases}$$

(The left-hand side of this last equation is not defined if $x_0 = a$ or $x_0 = b$.) [Hint: In Eq. (2), take $f(x)$ to be zero for $x < a$ and $x > b$.]

(b) Can $\delta(x)$ have a physical dimension?

(c) One can show that if $F(x)$ is a differentiable function, then

$$\delta[F(x)] = \sum_n \frac{1}{|F'(x_n)|}\,\delta(x-x_n), \qquad (3)$$

where the x_n's are the zeros of $F(x)$ (i.e., the values of x at which $F(x) = 0$) and

$$F'(x_n) = \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=x_n}.$$

$\delta[F(x)]$ has no mathematical meaning if it happens that $F(x)$ and $F'(x)$ are simultaneously zero. For example, $\delta(x^2)$ has no meaning.

(i) Show that $\delta(x-x_0) = \delta(x_0 - x)$.

(ii) Show that $\delta(\alpha x) = \delta(x)/|\alpha|$, where α is a non-zero real constant.

(iii) Show that

$$\delta(E-E_0) = \frac{m}{\hbar^2 k_0}\,[\delta(k-k_0) + \delta(k+k_0)],$$

where $E > 0$, $E_0 > 0$, $E = \hbar^2 k^2/(2m)$ and $k_0 = (2mE_0)^{1/2}/\hbar$.

(iv) Let $p = \hbar k$ and $p' = \hbar k'$. (p and p' are two momenta, k and k' are the corresponding wave numbers.) How is $\delta(p-p')$ related to $\delta(k-k')$?

(d) Consider the functions $\phi_p(x)$ defined as $\phi_p(x) = C\exp(ipx/\hbar)$, where p is real and C is real and positive. Find the constant C such that

$$\int_{-\infty}^{\infty} \phi_p^*(x)\,\phi_{p'}(x)\,\mathrm{d}x = \delta(p'-p).$$

[Hint: Use Eq. (1) above, changing the notation as appropriate.]

(e) The δ "function" is not a function but a more general type of mapping called a distribution. Its properties can be derived in a mathematically rigorous way. However, they can also be derived nonrigorously by formal manipulations of integrals. Adopting the latter approach, and using Eq. (1) [*not* Eq. (2)], show that

$$\int_{-\infty}^{\infty} \delta(x-q_1)\,\delta(q_2-x)\,\mathrm{d}x = \delta(q_2-q_1).$$

Problem 3

The following is stated on page 88 of the notes in regards to any Hermitian operator $\hat{A}$ acting in a finite-dimensional vector space and representing a certain physical quantity (an observable): "If $\hat{A}$ has p distinct eigenvalues λ_1, λ_2,..., λ_p, one can show that

$$\langle\psi|\hat{A}|\psi\rangle = \sum_{n=1}^{p} \lambda_n \operatorname{Pr}(\lambda_n; |\psi\rangle),$$

where $\operatorname{Pr}(\lambda_n; |\psi\rangle)$ is the probability that the value λ_n is found if the corresponding physical quantity is measured on a system in the state $|\psi\rangle$." Show that this equation is correct, starting from the spectral decomposition of $\hat{A}$. [To make is easier, first consider the simpler case where all the eigenvalues of $\hat{A}$ are non-degenerate before considering the more complicated case of degenerate eigenvalues.]

Problem 4

The ladder operators for a linear harmonic oscillator can be written as follows:

$$a_- = (2\hbar m\omega)^{-1/2}\left(m\omega x + \hbar\frac{\mathrm{d}}{\mathrm{d}x}\right),$$

$$a_+ = a_-^\dagger = (2\hbar m\omega)^{-1/2}\left(m\omega x - \hbar\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

(a) Check that the function $\phi_\alpha(x)$ defined below is a solution of the equation $a_-\phi_\alpha(x) = \alpha\phi_\alpha(x)$ for any value of the complex constant α:

$$\phi_\alpha(x) = C\,\exp\left(-\left[\left(\frac{m\omega}{2\hbar}\right)^{1/2} x - \alpha\right]^2\right),$$

where C is an arbitrary constant.

(b) Are there real or complex values of α for which $\phi_\alpha(x)$ is not square-integrable on $(-\infty,\infty)$? Hint: If ξ is real and α is complex,

$$\left|\exp\left[\pm(\xi-\alpha)^2\right]\right|^2 = \exp\left[\pm 2\,(\xi - \operatorname{Re}\alpha)^2\right]\,\exp\left[\mp 2\,(\operatorname{Im}\alpha)^2\right].$$

(c) Pretend that you do not know about $\phi_\alpha(x)$; instead, find the eigenfunctions of a_- by solving the eigenvalue equation $a_-\phi(x) = \alpha\phi(x)$ as a differential equation.

(d) Show that $a_-^\dagger$ has no square-integrable eigenfunctions.