ELECTROMAGNETISM – Workshop 4th Set (Qns) Basic Laws of Electricity and Magnetism

Professor D P Hampshire – 2nd Year Physics Lecture Course

The material for this workshop is split into just two parts. Part I: contains worked examples. Please make sure you can answer worked examples in Part I from scratch without reference to the worked solutions. Part II gives some additional unseen questions.

1 Directions of fields and forces using cross products

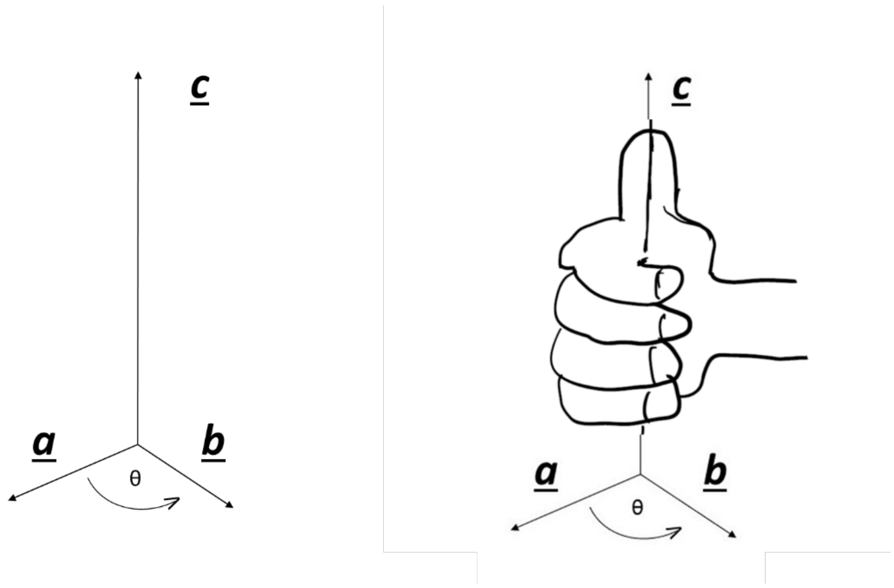


Figure 1 : The relationship between the vectors $\underline{\mathbf{a}}$, $\underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$ given $\underline{\mathbf{a}} \times \underline{\boldsymbol{b}} = |\underline{\mathbf{a}}||\underline{\boldsymbol{b}}|\sin(\theta)\,\hat{\boldsymbol{c}} = \underline{\boldsymbol{c}}$ and the right-hand screw rule convention required for the cross product.

One can be asked to calculate the direction of the magnetic field produced by a current or the direction of the force on a current flowing in an applied magnetic field. The internet has a vast selection of rules that describe the relative directions of parameters related through the cross product. Confusingly many of the rules on the internet provide the same parameters using both left and right hand versions of the rules. I recommend that you learn the definition of the cross product and the vector form of the relevant equation. The cross product can be written:

$$\underline{\mathbf{a}} \times \underline{\boldsymbol{b}} = |\underline{\mathbf{a}}||\underline{\boldsymbol{b}}| \sin(\theta)\, \hat{\boldsymbol{c}} = \underline{\boldsymbol{c}} \; - \text{Definition of the cross product}. \qquad \text{1-1}$$

The cross product of $\underline{\mathbf{a}}$ and $\underline{\boldsymbol{b}}$ is defined as a vector $\underline{\boldsymbol{c}}$ that has a magnitude equal to the product of the magnitude of vector $\underline{\mathbf{a}}$, the magnitude of vector $\underline{\boldsymbol{b}}$, the sine of the angle between the vectors and a unit vector, $\hat{\boldsymbol{c}}$, in the direction orthogonal to $\underline{\mathbf{a}}$ and $\underline{\boldsymbol{b}}$ given by the right-hand screw rule as shown in Figure 1. If one opens the palm of your right hand and point you fingers in the direction of vector $\underline{\mathbf{a}}$, and close your fingers to point in the direction $\underline{\boldsymbol{b}}$, then the right hand screw rule stipulates that your thumb points in the direction of $\hat{\boldsymbol{c}}$. You can check you are using the right-hand screw rule correctly by using the Lorentz force equation to work out the direction of the force on a current flowing in the x-direction experiencing an applied magnetic field in the z-direction – the force is in the minus y-direction. You can also, although it is a little more challenging, calculate the direction of the field circulating a straight wire using the Biot-Savart law (Equation 1-1).

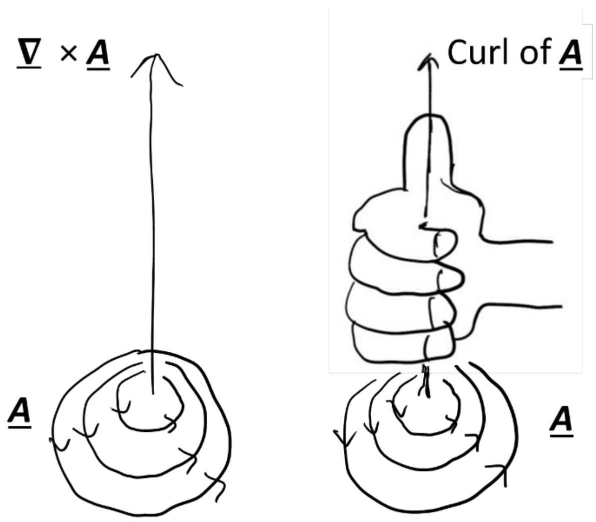


Figure 2 : The curl of a vector field. The curl of a vector field $\underline{\boldsymbol{A}}$ that is circulating with a rotation given by the direction of the fingers of the right-hand is given by the direction that the right-hand thumb points.

One can use the Biot-Savart law to calculate the direction of circulation for the magnetic field produced by a current flowing in an infinitely long straight wire along with the right-hand screw rule convention for the cross-product. Equivalently, Figure 2 shows how the field (represented by the direction the fingers point) circulates about the current (that flows in the direction that the thumb points) where we have considered Ampere's law written down in differential form (cf Maxwell's 4$^{\text{th}}$ equation)

$$\underline{\nabla} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} \;. \qquad \text{1-2}$$

2 Worked examples

2.1 Questions

It is imperative that, as soon as possible, you can write down the basic laws of classical Physics from memory in one of their basic forms. It is then usually straightforward to use dimensionality arguments or simple aide memoires to help you write down their other forms:

1. Write down Coulomb's law and/or (its integral form) Gauss's law.

2. Write down the (integral) form of Gauss' law that says there are no magnetic monopoles.

3. Write down Faraday's law.

4. Write down Ampère's law.

5. Outline an experiment that tests the laws cited in Qns. 1 - 4.

6. Write down Maxwell's four equations in differential form.

7. Write down the general expression for the force on a charge moving in an electric field and a magnetic field.

2.2 Answers

1.
Coulomb's law considers two charges labelled 1 and 2. The force on charge 2, $\underline{\boldsymbol{F}}_2$, from charge 1 is:

$$\underline{\boldsymbol{F}}_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}_{1\to2} \text{ Coulomb's law} \qquad 2\text{-}1$$

where r is the distance between the charges and charges of the opposite signs attract each other.
Gauss' law can be written as:

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \frac{\sum \mathrm{q}}{\varepsilon_\mathrm{o}} = \frac{1}{\varepsilon_\mathrm{o}}\int \rho \mathrm{d}V \qquad 2\text{-}2$$

2.
A description of magnetic fields in the form of Gauss' law is :

$$\oint \underline{\boldsymbol{B}} \,.\, \mathrm{d}\underline{\boldsymbol{S}} = 0 \qquad 2\text{-}3$$

3.
A form for Faraday's Law is

$$V = \frac{\partial \phi_B}{\partial \mathrm{t}} \qquad 2\text{-}4$$

4.

Ampère found that two parallel straight wires carrying currents I_1 and I_2 in the same direction lead to a force $\underline{\boldsymbol{F}}_2$ on a length L of the wire carrying current I_2 where:

$$\underline{\boldsymbol{F}}_2 = -\frac{\mu_0 I_1 I_2 L}{2\pi d}\hat{\mathbf{r}}_{1\to 2} \qquad 2\text{-}5$$

where d is the separation between the wires and the negative sign occurs because the force is attractive if the current flows in the same direction in both wires.
We can then introduce Ampère's Law for the magnetic field produced by a flowing current:

$$\oint \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{l}} = \mu_o I \qquad 2\text{-}6$$

where *I* is the current that passes through the surface described by the path integral.

5.
a) Measure the spatial variation of the forces between two charges and/or show that the flux of the electric field over any closed surface is equal to the sum of the charges within the surface divided by ε_0.
b) Measure the spatial variation of the magnetic field produced by say a magnetic material or a flowing current and show that the flux of the magnetic field over any closed surface is zero.
c)

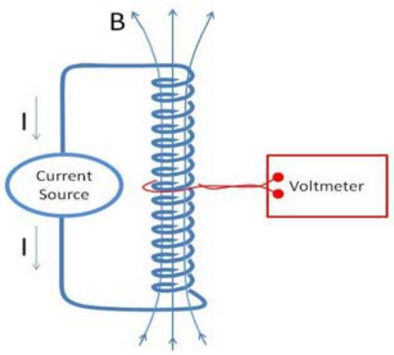


Figure 3 : A loop wrapped around a coil and with both it's ends attached to a voltmeter.

Pass an ac. current through a coil as shown above and measure the voltage induced in a single turn wrapped around the coil.
d)

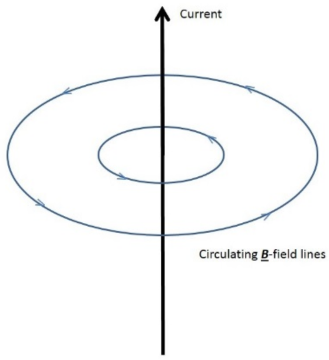


Figure 4 : The magnetic field produced by a current flowing in a straight wire.

Measure the magnetic field circulating a straight wire carrying a current.

6.
Maxwell's 4 equations:

From Coulomb's law:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \quad \text{(MI)} \qquad 2\text{-}7$$

Given no magnetic monopoles have been observed:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0 \quad \text{(MII)} \qquad 2\text{-}8$$

From Faraday's law of induction:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \quad \text{(MIII)} \qquad 2\text{-}9$$

From Ampère's law:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \quad \text{(MIV)} \qquad 2\text{-}10$$

where the symbol $\underline{\boldsymbol{\nabla}}$ denotes the vector operator 'del':

$$\underline{\boldsymbol{\nabla}} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \qquad 2\text{-}11$$

$\underline{\boldsymbol{E}}$: electric field (V m^{-1}). $\underline{\boldsymbol{B}}$: magnetic field – or flux density (T). ρ total volumetric charge density (C m^{-3}). $\underline{\boldsymbol{J}}$: total current density (A m^{-2})

7.
Force on a moving charge in a magnetic and electric field:

$$\underline{\boldsymbol{F}} = q(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \qquad 2\text{-}12$$

3 Unseen problems

Maxwell's first and third equations tell us that there are two ways in which we can generate a electric field – either from a stationary charge (Coulomb's Law) or a changing magnetic field (Faraday's law).

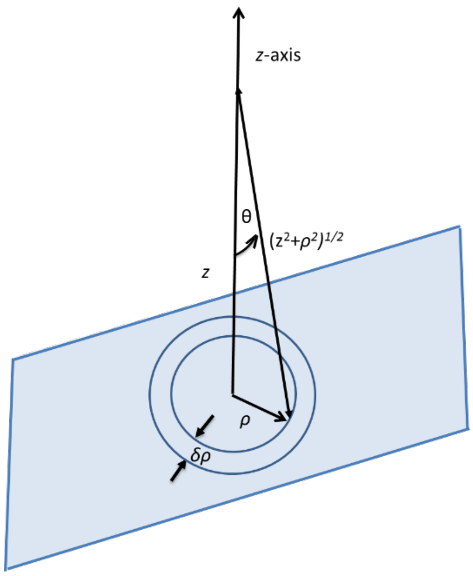


Figure 5 : A circular elemental strip of charge that is part of an infinitely large plane of charge density σ.

1.

Charge is uniformly distributed over an infinitely large plane centred at z = 0 with surface charge density σ.

a) Using Gauss' law and show that the E-field at any distance above the plane is $\underline{\boldsymbol{E}} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}.$

b) Find the same result as in (a) using calculus and Coulomb's law directly. **Hint:** You may use the figure shown. Note that the area element dA in cylindrical coordinates can be taken as $dA = 2\pi\rho d\rho$ and make sure to remember that $\underline{\boldsymbol{E}}$-fields add as vectors not as scalars.

c) Write down the expression for the $\underline{\boldsymbol{E}}$-field between two capacitor plates.

2.

Four charges: +3e, +4e, +e and -8e are enclosed by a closed spherical surface of radius R. What is the net flux crossing the spherical surface?

3.

Consider a volume charge density which has cylindrical symmetry and which in cylindrical coordinates is given by $12\exp(-3\rho^2)$ where ρ is the radial distance from the central axis. Use Gauss' law to find how the electric field varies with ρ.

4.

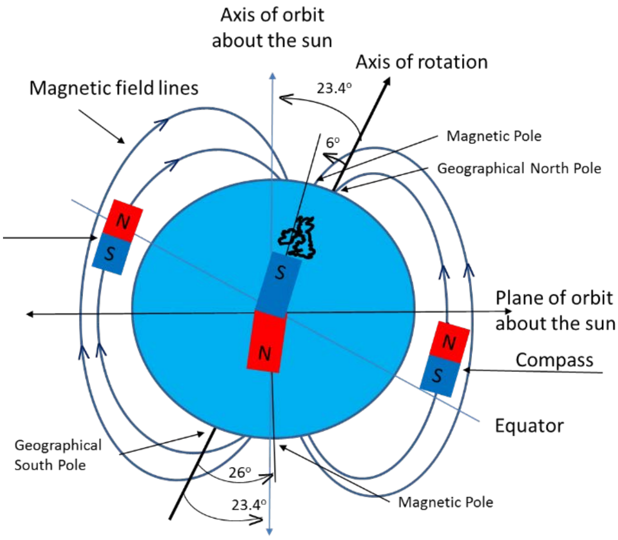


Figure 6 : The magnetic field lines produced around the Earth. To first order the magnetic field by a moment at the centre of the Earth of magnitude 7 x 10^{22} Am^2. On the Earth's surface, the magnetic field points from the Southern Hemisphere to the Northern Hemisphere. The Earth's field is not a perfect dipole moment so the magnetic pole in the Northern hemisphere is not opposite that in the Southern hemisphere. Both magnetic poles, defined by the location where a scientist's compass points down, typically move half a degree or so in latitude (i.e. North-South) every 5 – 10 years. By convention, the North pole of a compass (also known as a bar magnet) is geographically "North-seeking" and is therefore attracted to the south pole of the Earth's magnetic field as shown.

a) Draw the magnetic field lines around the magnetic material used in a compass (i.e. a bar magnet).
b) Find the direction that the moment points in the magnetic material used in a compass.
c) Why does a compass point along the Earth's magnetic field lines?
d) A bar magnet is pushed into a loop of copper wire as shown in the Figure 7. State and justify your answer for whether the current flows clockwise or anticlockwise from the perspective of someone sitting on the moving bar magnet.

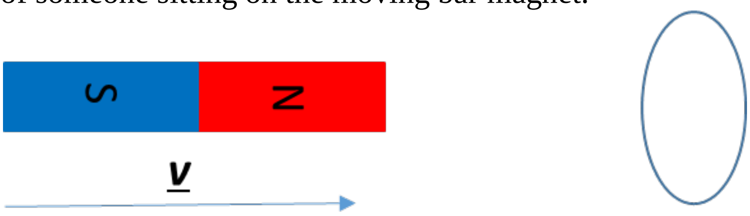


Figure 7 : A bar magnet moving towards a loop of wire.

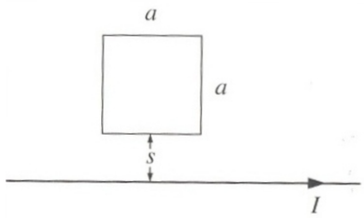
5.

A conducting disk (with radius b) is rotated about its axis at angular velocity ω on a non-conducting spindle (with radius a) in the presence of a uniform magnetic field (***B***) parallel to the axis. Assume that the electrons in the disk are free carriers. If the applied field ***B*** = 1.5 T and the radii for the spindle and disk are 3 cm and 40 cm respectively what is the voltage difference between the rim and spindle of the disk.
If the free carriers were the positive charges rather than the electrons, would the sign nof the voltage change?

6.

A long straight wire carries a current I. A square loop (side a) lies near to the wire as shown in the figure. The near side is at the distance s from the wire.

a) Calculate the magnetic flux through the loop.



b) If the loop is pulled directly away from the wire, at speed v, find the induced EMF generated in the loop, what direction does it flow?

c) Evaluate the value of the EMF, given that $a = 5$ cm, $I = 20$ A, $v = 1$ cm s^{-1} and $s = 2$ cm.

d) What if the loop is pulled to the right at speed v, instead of away?

Maxwell's second equation tells us something about how magnetic fields spread through space.

7.

Show that a magnetic field specified by $B_x = B_o x \sin\omega t, B_y = 0, B_z = 0$ cannot exist.

Maxwell's fourth equation tells us that there are two ways in which we can generate a magnetic field – either from a current flowing (Ampère's Law) or from a displacement current flowing (associated with a changing electric field).

$$\underline{\nabla} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}}_{total} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \qquad 3\text{-}1$$

Let's start by solving some problems where there are just currents associated with moving charges and Ampère's Law applies:

8.

Find the magnetic field at the radius 50 cm from an infinitely straight wire carrying the current of 0.5A.

9.

Find the magnetic field within a solenoid 0.20 cm long wrapped with 1500 turns when the current through it is 0.5 A.

10.

The xy plane contains an infinite sheet with current per unit length flowing in the positive x direction given by $\underline{\boldsymbol{K}} = K\hat{\mathbf{i}}$.

a) What is the direction of $\underline{\boldsymbol{B}}$ at a point below the sheet?

b) Show that the magnetic field at any point above the sheet i.e. $z > 0$ is given by $-\frac{1}{2}\mu_o K\hat{\mathbf{j}}$.

11.

Produce some imaginative questions that require the person answering them to provide numerical answers for:

i) The force on stationary charge in the presence of other stationary charges;

ii) The force on a moving charge in the presence of current flowing in an array of conductors;

iii) The force on a wire carrying current in the presence of other currents flowing. Show the most imaginative of your questions and answers to the demonstrators!