Mathematical Methods in Physics

Weekly Problems 5. Solution

5.1

a) Set $a + t = t'$, then

$$\begin{aligned}
\mathcal{F}[f(t+a)](\omega) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t+a)\, e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t')\, e^{-i(t'-a)\omega} dt' \\
&= e^{ia\omega} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t')\, e^{-it'\omega} dt' = e^{ia\omega} \hat{f}\,(\omega) \qquad \boxed{\text{1 mark}}
\end{aligned}$$

b)

$$\mathcal{F}[e^{at}\, f(t)](\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)\, e^{-i\omega t + at} dt = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)\, e^{-it(\omega + ia)} dt = \hat{f}\,(\omega + ia) \quad \boxed{\text{1 mark}}$$

5.2

By definition

$$\begin{aligned}
\mathcal{F}[f(t)](\omega) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left(\frac{\delta(a-t) + \delta(a+t)}{1+t^2} \right) e^{-i\omega t} dt \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\delta(a-t)}{1+t^2}\, e^{-i\omega t} dt + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\delta(a+t)}{1+t^2}\, e^{-i\omega t} dt.
\end{aligned}$$

Then

$$\mathcal{F}[f(t)](\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{ia\omega}}{1+a^2} \right) + \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ia\omega}}{1+a^2} \right) = \sqrt{\frac{2}{\pi}}\, \frac{\cos(\omega a)}{(1+a^2)}\,. \qquad \boxed{\text{3 mark}}$$

5.3

$$
\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)\, e^{-it\omega} dt = \frac{1}{\sqrt{2\pi}a^2} \int_{-a}^{0} (a+t)\, e^{-it\omega} dt + \frac{1}{\sqrt{2\pi}a^2} \int_{0}^{a} (a-t)\, e^{-it\omega} dt \\
&= \frac{1}{\sqrt{2\pi}a^2} \left((a+t)\, \frac{e^{-it\omega}}{-i\omega}\bigg|_{-a}^{0} + \int_{-a}^{0} \frac{e^{-it\omega}}{i\omega} \right) + \frac{1}{\sqrt{2\pi}a^2} \left((a-t)\, \frac{e^{-it\omega}}{-i\omega}\bigg|_{0}^{a} + \int_{0}^{a} (-1) \frac{e^{-it\omega}}{i\omega} \right) \\
&= \frac{1}{\sqrt{2\pi}a^2} \left(-a\frac{1}{i\omega} + \frac{1-e^{it\omega}}{\omega^2} + a\frac{1}{i\omega} - \frac{e^{-it\omega}-1}{\omega^2} \right) = \sqrt{\frac{2}{\pi}} \frac{(1-\cos(\omega a))}{\omega^2 a^2}\,. \qquad \boxed{\text{2 marks}}
\end{aligned}
$$

In the limit $a \to 0$ we have

$$
\cos a\omega \simeq 1 - (\omega a)^2/2
$$

so that the Fourier transform becomes

$$
\lim_{a\to 0} \hat{f}(\omega) = \lim_{a\to 0} \sqrt{\frac{2}{\pi}} \frac{(1-\cos(\omega a))}{(\omega a)^2} = \lim_{a\to 0} \sqrt{\frac{2}{\pi}} \frac{(1-(1-(\omega a)^2/2+\ldots))}{(\omega a)^2} = \frac{1}{\sqrt{2\pi}},
$$

which is the Fourier transform of the Dirac δ-function (see Example 2 in Lecture 10).

$\boxed{\text{3 mark}}$

You may have noticed that the area bounded by the function f and the t-axis is a triangle of area 1 (the base is $2a$ and the height $1/a$). In the limit of small a you get the infinitely high and infinitely narrow pulse that represents the Dirac δ-function.