University of Durham

EXAMINATION PAPER

May/June 2012 Examination code: 042581/01

LEVEL 2 PHYSICS: FOUNDATIONS OF PHYSICS 2A

SECTION A. QUANTUM MECHANICS 2
SECTION B. ELECTROMAGNETISM

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\,\mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{26}\,\mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\,\mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. QUANTUM MECHANICS 2

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) The wavefunction $\psi(x) = Ax(1-x)$ for $0 \leq x \leq 1$ and is zero elsewhere. Write down the condition for the wavefunction to be normalised, and hence show that $A = \sqrt{30}$. Write down the appropriate form of the momentum operator, p, and calculate $\langle p \rangle$. [4 marks]

   (b) For the system in part (a), calculate $\langle xp \rangle$ and $\langle px \rangle$. Hence show that $\langle xp \rangle - \langle px \rangle = i\hbar$ [4 marks]

   (c) An electron in a time-independent potential, $V(x)$, is in one of the energy eigenstates $\psi_n(x)$ with corresponding energy E_n. Write down the time-dependent wavefunction, $\Psi(x,t)$, for this system. Show explicitly that $\langle x \rangle$ does not depend on time. [4 marks]

   (d) The wavefunction of an electron in a hydrogen atom is given by the superposition $\psi = \frac{1}{\sqrt{18}}(\psi_{100} + 3\psi_{200} + 2\psi_{211} + 2\psi_{321})$, where the ψ_{nlm} are the normalized energy eigenfunctions with quantum numbers n, l, m. What is the probability of finding the system with (i) $n = 1$? (ii) $n = 2$? (iii) $n = 3$? Use these probabilities to evaluate the expectation value of the energy, $\langle E \rangle$, given that $E_n = -13.6/n^2$ eV. Can any individual measurement of the energy give the value $\langle E \rangle$? [4 marks]

   (e) The normalised ground state wavefunction of a hydrogen atom is $\psi_{100} = (\pi a^3)^{-1/2} e^{-r/a}$, where a is the Bohr radius. Calculate the probability of finding the electron within dr of r, where $dr \ll r$. Show that the resulting radial probability density has a maximum at $r = a$. [4 marks]

   (f) The total angular momentum operator $\underline{J} = \underline{L} + \underline{S}$ where $\underline{L}$ and $\underline{S}$ are the orbital and spin angular momentum operators, respectively. Write down expressions for the eigenvalues of J^2, L^2 and S^2 in terms of their respective quantum numbers j, l and s. Calculate $J^2 = (\underline{L} + \underline{S})^2$, and hence obtain an expression for the eigenvalues of $\underline{L}.\underline{S}$ in terms of j, l and s. [4 marks]

   (g) Fine structure corrections to the energy levels in hydrogen give

$$E_{nj} \approx E_n \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + 1/2} - \frac{3}{4} \right) \right],$$

   where $E_n = -13.6/n^2$ eV, n is the principal quantum number, j is the total angular momentum quantum number and $\alpha = 1/137$ is the fine structure constant. Rearrange the expression for E_{nj} to obtain a formula for $(E_{nj} - E_n)/E_n$ and calculate this fractional difference in energy caused by fine structure effects for all the possible states with $n = 2$. [4 marks]

(h) Two states, ψ_a and ψ_b are degenerate, both having energy E^0, so any linear combination $\psi = \alpha\psi_a + \beta\psi_b$ also has energy E^0. A small perturbation, H', causes a small change in energy, and the first order approximation for this, E^1, is given by the solution of a matrix equation. Calculate E^1 for the particular case of the matrix equation

$$\begin{pmatrix} 1 & \kappa \\ \kappa & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E^1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where κ is real. [4 marks]

2. The Hamiltonian, H, of a one-dimensional harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}$$

where x is position and m and ω are real constants. At time $t = 0$ the oscillator is described by the normalized wavefunction

$$\psi(x) = \left(\frac{\beta^2}{\pi}\right)^{1/4} e^{-\beta^2 x^2/2}$$

where β is a positive real constant.

(a) Calculate the expectation value of the kinetic energy. [6 marks]

(b) Calculate the expectation value of the potential energy. [4 marks]

(c) Hence, determine the expectation value of the total energy. [2 marks]

(d) Show that $\beta = \sqrt{m\omega/\hbar}$ gives the minimum total energy, and evaluate this minimum energy. [4 marks]

(e) For what value of β is the wavefunction an eigenfunction of the Hamiltonian? [4 marks]

$$\left[\text{Note: } \int_{-\infty}^{+\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}, \qquad \int_{-\infty}^{+\infty} x^2 e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a^3}\right]$$

3. (a) A one-dimensional system described by a Hamiltonian, H^0, has non-degenerate eigenfunctions ψ_n^0 with energies E_n^0. A small perturbation, H', to the potential gives first-order corrections, E_n^1 and ψ_n^1, to the energies and eigenfunctions, respectively, related by $H^0\psi_n^1 + H'\psi_n^0 = E_n^0\psi_n^1 + E_n^1\psi_n^0$. Multiply this equation by ψ_n^{0*}, integrate over all space, and use the fact that H^0 is Hermitian to show that

$$E_n^1 = \int \psi_n^{0*} H' \psi_n^0 dx.$$

[6 marks]

(b) An electron trapped in a one dimensional infinite square well potential between $0 < x < L$ has eigenfunctions $\psi_n^0(x) = \sqrt{2/L}\sin(n\pi x/L)$ corresponding to energies $E_n^0 = n^2\pi^2\hbar^2/(2mL^2)$ where n is a positive integer. This system is subject to a perturbation $H' = a\delta(x - L/3)$, where δ is the Dirac delta function and a is a constant.

(i) Calculate the first-order corrections, E_n^1, to the energies E_n^0. Evaluate these explicitly for $n = 1, 2$ and 3. [5 marks]

(ii) The first-order corrections, ψ_n^1, to the eigenfunctions can be written as a linear sum of the unperturbed eigenfunctions, so $\psi_n^1 = \sum_m c_{nm}\psi_m$, where

$$c_{nm} = \frac{\int \psi_m^{0*} H' \psi_n^0 dx}{E_n^0 - E_m^0} \quad m \neq n$$

Calculate c_{12} and c_{13}, and hence estimate the first-order correction, ψ_1^1, to the ground state wavefunction. [5 marks]

(iii) The second-order corrections, E_n^2, to the unperturbed energies are given by $E_n^2 = \int \psi_n^{0*} H' \psi_n^1 dx$. Estimate E_1^2. [4 marks]

SECTION B. ELECTROMAGNETISM

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Using standard complex notation, the wavevector, $\underline{k_0}$, of a wave is given by $\underline{k_0} = (20 + 5i)\hat{\underline{n}}$ m^{-1}, where $\hat{\underline{n}}$ is a unit vector that points in the direction of propagation. What is the wavelength of the wave ? [4 marks]

(b) Write down the Maxwell equation that describes the electric field produced by a change in magnetic flux. Explain how this equation can be tested experimentally. [4 marks]

(c) A point charge of 2 μC is located at the position $\underline{r}$ where $\underline{r} = \hat{\underline{\imath}} + \hat{\underline{\jmath}}$ where $\hat{\underline{\imath}}$ and $\hat{\underline{\jmath}}$ are unit vectors in the x and y-directions respectively and $\underline{r}$ is in metres. A second charge of 1 μC is located at $\underline{r} = 3\hat{\underline{\imath}} + 3\hat{\underline{\jmath}}$. Calculate the magnitude of the force on the 1 μC charge. [4 marks]

(d) A 100 m long cylinder of metal has a voltage of 2 V applied along its length, which causes a current density of 5 A m^{-2} to flow. Calculate the electrical resistivity of the metal. [4 marks]

(e) Using Maxwell's equations, derive an expression for the velocity of light in a vacuum. [4 marks]

$$\left[\text{Hint } : \underline{\nabla} \times (\underline{\nabla} \times \underline{C}) = -\nabla^2\underline{C} + \underline{\nabla}(\underline{\nabla}.\underline{C}).\right]$$

(f) Define the term magnetisation. You should ensure that your answer includes all the units and definitions of all the terms you use. [4 marks]

(g) Briefly discuss the usefulness of Fresnel's equations. [4 marks]

5. (a) Explain what a dispersion relation is and discuss why such relations are useful. [3 marks]

(b) The dispersion relation for a plasma is given by

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right),$$

where

$$\omega_p^2 = \frac{ne^2}{m_{\mathrm{e}}\varepsilon_0},$$

n is the density of electrons in the plasma and ω_p is known as the plasma angular frequency. Provide a sketch of ω versus k and briefly discuss its important features. [4 marks]

(c) Recently there was disrupted communication at all frequencies up to 10^{11} Hz with an unmanned probe on Mars because a plasma cloud drifted between Mars and Earth. Calculate a value for the electron density in the plasma surrounding the probe. [5 marks]

(d) Sensitive measurements made in Durham suggested that at 10^{10} Hz the amplitude of the electromagnetic radio wave from the probe had fallen by about a factor 50 because of the plasma. Estimate the thickness of the plasma. [4 marks]

(e) Scientists in Paris unsuccessfully tried to detect signals at 10^8 Hz. Provide an explanation for why they failed. [4 marks]

6. (a) Briefly describe two different examples of a plasma and in each case explain how the plasma arises. [2 marks]

(b) In a high density plasma where the motion of only one type of carrier is considered important, the equation of motion for the carriers in an electric field, $\underline{E}$, where t is the time, is given by:

$$m_c \frac{d\underline{v}}{dt} = q\underline{E} - \frac{m_c \underline{v}}{\tau},$$

where m_c is the mass, q is the charge, $\underline{v}$ is the velocity, and τ is the scattering time for the carriers. Show that the electrical conductivity, σ_N, at angular frequency ω can be written in the form:

$$\sigma_N = \frac{Nq^2}{m_c(\tau^{-1} - i\omega)},$$

where N is the number of carriers per unit volume. [3 marks]

(c) A new metal has been discovered in which the electrons are the important carriers, the electron charge carrier concentration is 10^{30} m^{-3} and the electrical conductivity at low frequencies is 10^9 Ω^{-1}m^{-1}. Calculate a value for τ. [5 marks]

(d) Hence, determine at what frequency you would expect the electrical conductivity to decrease by a factor of 5 relative to the low frequency value. [6 marks]

(e) Calculate the phase difference between the current and the voltage along this new metal at this frequency. [4 marks]