

# University of Durham

## EXAMINATION PAPER

May/June 2016

Examination code: PHYS3641-WE01

### ADVANCED PHYSICS 3

**SECTION A.** Soft Condensed Matter Physics

**SECTION B.** Optical Properties of Solids

**SECTION C.** Modern Atomic and Optical Physics

**Time allowed:** 3 hours

**Additional material provided:** None

**Materials permitted:** None

**Calculators permitted:** Yes   **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

**Visiting students may use dictionaries:** No

---

#### Instructions to candidates:

- Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

---

#### Information

A list of physical constants is provided on the next page.

**Information**

Elementary charge:	$e = 1.60 \times 10^{-19} \text{ C}$
Speed of light:	$c = 3.00 \times 10^8 \text{ m s}^{-1}$
Boltzmann constant:	$k_{\text{B}} = 1.38 \times 10^{-23} \text{ J K}^{-1}$
Electron mass:	$m_{\text{e}} = 9.11 \times 10^{-31} \text{ kg}$
Gravitational constant:	$G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
Proton mass:	$m_{\text{p}} = 1.67 \times 10^{-27} \text{ kg}$
Planck constant:	$h = 6.63 \times 10^{-34} \text{ J s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12} \text{ F m}^{-1}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}$
Molar gas constant:	$R = 8.31 \text{ J K}^{-1} \text{ mol}^{-1}$
Avogadro's constant:	$N_{\text{A}} = 6.02 \times 10^{23} \text{ mol}^{-1}$
Gravitational acceleration at Earth's surface:	$g = 9.81 \text{ m s}^{-2}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Astronomical Unit:	$\text{AU} = 1.50 \times 10^{11} \text{ m}$
Parsec:	$\text{pc} = 3.09 \times 10^{16} \text{ m}$
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30} \text{ kg}$
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26} \text{ W}$

## SECTION A. SOFT CONDENSED MATTER PHYSICS

Answer Question 1 and **either** Question 2 **or** Question 3.

1. a) We compare two viscoelastic materials whose stress relaxation functions are given by  $G(t) = \exp(-t/a)$  and  $G(t) = \exp(-at)$  respectively, where  $a$  is a positive constant. Determine which of the two materials should be considered the more elastic. Explain your answer with a graph showing the response of each material to a step strain at the time  $t = 0$ . [4 marks]
- b) A droplet of water in an oil bath comes into contact with a wall of the container. Knowing the interfacial energy  $\gamma$  between water and oil, and the work of adhesion  $W$  of water to the surface in oil, find the angle  $\theta$  formed by the droplet with the surface. [4 marks]

$$\left[ \begin{array}{l} \text{Hint: } W_{\text{LMS}} = \gamma_{\text{SM}} + \gamma_{\text{LM}} - \gamma_{\text{SL}} \text{ (Dupr )} \\ \gamma_{\text{LM}} \cos \theta = \gamma_{\text{SM}} - \gamma_{\text{SL}} \text{ (Young)} \\ \text{S, L and M stand for solid, liquid and surrounding respectively.} \end{array} \right]$$

- c) Salt is added to a colloidal solution containing particles with a broad range of sizes. Shortly after, aggregates containing mostly small colloids start to appear. Explain what has happened. [4 marks]
- d) Consider a homopolymer in solution. The polymer is in the theta condition. How will its end-to-end distance change if the number of monomers in the polymer is doubled? [4 marks]
- e) The free energy  $G$  of a polymer molecule in a brush with grafting density  $\sigma$  is given by:

$$G = k_{\text{B}}T \frac{h^2}{Na^2} + k_{\text{B}}Tv \left( 1 - \frac{2\chi}{k_{\text{B}}T} \right) \frac{\sigma N^2}{2ha^2}$$

where the first term of the equation describes the entropic contribution, the second term combines solvent interactions and excluded volume, and the variables have their usual meaning. Show that at equilibrium, the length  $h$  of the polymer is directly proportional to its degree of polymerisation, regardless of the length  $a$  of a monomer. [4 marks]

2. Consider a mixture of two liquids at a temperature  $T_0$ . The volume fraction of liquid 1 in the solution is given by  $c$  with  $0 < c < 1$ . The mixture is homogenous for all  $c$ .
- The solution is cooled down to a temperature  $T^* < T_0$ . Demixing into two specific composition  $c_1$  and  $c_2$  with respective free energy  $G_1$  and  $G_2$  is observed, but only for certain values of  $c$ . Explain what is happening and discuss the evolution of the free energy of the system in terms of entropy, enthalpy and convexity. Use a graph to help your explanation. [5 marks]
  - We consider a particular composition  $c^*$  with  $c_1 < c^* < c_2$ . Its initial free energy is  $G^*$ . We can define a constant  $\alpha$  with  $0 < \alpha < 1$  so that  $c^* = \alpha c_1 + (1 - \alpha)c_2$ . Calculate, as a function of  $\alpha, G^*, G_1$  and  $G_2$  how much free energy  $\Delta_{c^*}G$  the system gains if demixing occurs. Does  $\Delta_{c^*}G$  depend on temperature? [4 marks]
  - Practically, it is found that the system does not demix at  $c^*$  over the timescale of the experiment. Explain why. [4 marks]
  - Assume that only a small droplet of  $c^*$  occasionally demixes into  $c_1$  and  $c_2$ . What is the droplet interfacial energy  $\Gamma$  with its surrounding? Take  $\gamma_1$  and  $\gamma_2$  to be the interfacial energies between  $c^*$ ,  $c_1$  and  $c_2$  respectively, and that the droplet radius is  $r$ . [3 marks]
  - Assuming that all the free energies are given per volume, calculate the droplet radius  $R$  beyond which demixing of the whole system will spontaneously proceed. [4 marks]

3. We dissolve amphiphilic molecules into a liquid. Each amphiphilic molecule has a headgroup area  $a_0 = 0.5 \text{ nm}^2$  and a single tail composed of an 80% extended chain of 14 carbon atoms. The length of a carbon-carbon link is  $0.15 \text{ nm}$  for an effective hydrocarbon volume of  $1.0 \times 10^{-2} \text{ nm}^3$ .

- a) We progressively add the amphiphilic molecules to the solution at a constant temperature  $T$ . The concentration of molecules in the solution is  $\phi$ . Past a certain concentration  $\phi^*$  molecules start to self-assemble into aggregates. What is  $\phi^*$  called? Considering the ratio  $v/(l_c a_0)$  where  $v$  is the total hydrocarbon volume of the amphiphile molecular tail and  $l_c$  its length, determine what type of aggregate will form when  $\phi > \phi^*$ . [4 marks]
- b) The chemical potential  $\mu_M$  of a molecule in an aggregate of size  $M$  is given by:

$$\mu_M = \epsilon_M + \frac{k_B T}{M} \ln \left( \frac{X_M}{M} \right)$$

where  $X_M$  is the volume fraction of molecules in aggregates of size  $M$  and  $\epsilon_M$  the change of enthalpy experienced by a molecule when transferring from bulk solution into the aggregate. Here,  $\epsilon_M = \epsilon_N + \Lambda(M - N)^2$  with  $N$  the most probable number of molecules in an aggregate and  $\Lambda$  a positive constant. Show that at equilibrium,  $X_N$  can be written as:

$$X_N = N \left[ X_1 \exp \left( \frac{\Lambda(1 - N)^2}{k_B T} \right) \right]^N$$

Justify any assumption you make. [6 marks]

- c) We are in a situation where  $\Lambda$  is large and hence assume that the solution contains only dissolved monomers and aggregates of size  $N$  (i.e.  $\phi = X_1 + X_N$ ). Calculate  $\phi^*$ , and justify the assumption. [4 marks]
- d) We want to increase the fraction of molecules present in aggregates without adding any new molecules to the system. Find an expression describing how the temperature should change so that the ratio  $\frac{X_N}{X_1}$  doubles. Is it an increase or a decrease of temperature? [6 marks]

## SECTION B. OPTICAL PROPERTIES OF SOLIDS

Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) CdTe has a complex refractive index of  $\tilde{n} = 3.14 + 0.525i$  at a wavelength of 496 nm. Determine the value of the reflection coefficient when light of this wavelength impinges on a planar CdTe surface from air at normal incidence. [4 marks]
- (b) The static and high frequency dielectric constants of InAs are 14.9 and 12.3. If the TO phonon frequency is  $6.60 \times 10^{12}$  Hz find the upper and lower wavelengths of the Reststrahl band of this material. [4 marks]
- (c) Germanium is an indirect band gap semiconductor with a band gap between the conduction band minimum (at the  $L$  point in the Brillouin zone) and the valence band maximum (at the  $\Gamma$  point) of 0.66 eV at 291 K and 0.74 eV at 20 K. If the average  $L$  point phonon energy is taken as 20 meV, determine the onset of optical absorption at (i) 291 K and (ii) 20 K. Justify your answers. [4 marks]
- (d) Explain why spin-orbit coupling can allow radiative transitions between singlet and triplet states. [4 marks]
- (e) In response to an applied time-dependent electric field  $E = E_0 \cos(\omega t)$ , a particular material exhibits an induced polarisation which is well approximated by:

$$P = \epsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 \right),$$

where  $\chi^{(1)}$  and  $\chi^{(2)}$  are the first and second order susceptibilities respectively. Demonstrate why this leads to the phenomenon of second harmonic generation (SHG). [4 marks]

5. Aluminium (Al) has a free-electron density of  $1.81 \times 10^{29} \text{ m}^{-3}$  and a d.c. conductivity of  $3.54 \times 10^7 \text{ } \Omega^{-1}\text{m}^{-1}$ .

- (a) Calculate the plasma frequency for aluminium. [2 marks]
- (b) Sketch the form of the reflectivity of aluminium close to this frequency, stating any assumptions that you make. [4 marks]
- (c) Starting from the expression for the complex dielectric function within the Lorentz model:

$$\epsilon_r(\omega) = 1 - \frac{Ne^2}{m_0\epsilon_0} \frac{1}{(\omega^2 + i\gamma\omega)},$$

where  $\tau = 1/\gamma$  is the relaxation time,  $\omega$  the frequency of the incident light and  $N$  the free electron density, determine the real and imaginary parts of the dielectric function of aluminium for light of wavelength  $1.00 \text{ } \mu\text{m}$ . [9 marks]

- (d) For light of wavelength  $1.00 \text{ } \mu\text{m}$  the absorption coefficient of Al,  $\alpha$ , is  $1.59 \times 10^8 \text{ m}^{-1}$ . Neglecting reflection at the surfaces, what thickness of aluminium foil is required to attenuate normally incident light of this wavelength by 50%? [3 marks]
- (e) How would inclusion of the reflectivity of the aluminium foil affect the answer to part (d)? [2 marks]

6. (a) Explain what is meant by the term *exciton* as applied to the electronic excitation of a solid and briefly describe the difference between *Mott-Wannier* and *Frenkel* excitons. In which types of materials would these excitons be found? [4 marks]

(b) The semiconductor InP has a band gap of 1.42 eV (at 298 K), relative permittivity (at zero frequency),  $\epsilon_r(0)=12.4$ , hole effective mass at the top of the valence band of  $m_h^* = 0.200 m_e$  and electron effective mass at the bottom of the conduction band of  $m_e^* = 0.0770 m_e$ .

- (i) Determine the binding energy of the lowest lying excitonic state in InP. [4 marks]
- (ii) Under what conditions would a clear excitonic peak be seen in the optical absorption spectrum of InP? [2 marks]
- (iii) Using the same axes, sketch the optical absorption spectra of InP at 298 K which would be expected with and without the existence of excitons. [4 marks]
- (iv) Estimate the *Mott density* of the lowest lying exciton in InP. [4 marks]
- (v) What happens to the optical absorption strength in the spectral region close to the band edge as the Mott density is approached? [2 marks]

[The Bohr radius and the Rydberg energy of the hydrogen atom are:  $a_H = 5.29 \times 10^{-11}$  m and  $R_H = 13.6$  eV, respectively.]



### SECTION C. MODERN ATOMIC AND OPTICAL PHYSICS

Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) Explain the term “magic wavelength” in the context of the  $^{87}\text{Sr}$  optical-lattice clock and why the optical-lattice trap is required to be at this wavelength. [4 marks]
- (b) Explain the role of the  $^{25}\text{Mg}^+$  ion in the  $^{27}\text{Al}^+$  ion quantum logic clock. [4 marks]
- (c) The Einstein A-coefficient for an electric dipole transition in an atom is

$$A_{21} = \frac{g_1}{g_2} \frac{\omega_{21}^3 d^2}{3\pi\epsilon_0 \hbar c^3},$$

where  $g_1$  and  $g_2$  are the degeneracies of the lower and upper states of the transition,  $\omega_{21}$  is the angular frequency of the transition and  $d$  is the dipole moment of the atom. The  $6s\ ^2S_{1/2} \rightarrow 6p\ ^2P_{3/2}$  electric dipole transition in  $^{133}\text{Cs}$  occurs at  $\lambda_{21} = 852\text{ nm}$  and the lifetime of the  $6p$  state is  $\tau = 31\text{ ns}$ . Ignoring the hyperfine structure of the states, calculate  $d$ . [4 marks]

- (d) Focusing a  $P = 100\text{ W}$  laser beam traveling in the  $z$  direction makes a simple optical dipole trap. The trap has an intensity profile of

$$I(r, z) = \frac{2P}{\pi w(z)^2} \exp\left(-\frac{2r^2}{w(z)^2}\right),$$

where  $w(z)$  is the beam width and  $r$  is the distance along the radial direction. Calculate the size of the beam waist if the maximum intensity is  $1.00 \times 10^8\text{ W cm}^{-2}$ . [4 marks]

- (e) Explain the origins of Doppler broadening, transit-time broadening and natural broadening with respect to atomic transitions. [4 marks]

8. The table below gives orbital angular momentum eigenfunctions,  $Y_{l,m}$ , and radial eigenfunctions,  $R_{n,l}$ , for states of the hydrogen atom with quantum numbers  $n$ ,  $l$  and  $m$ , where  $\rho = r/na_0$ .

$n$	$l$	$m$	$Y_{l,m}$	$R_{n,l}$
1	0	0	$\sqrt{\frac{1}{4\pi}}$	$2\left(\frac{1}{a_0}\right)^{3/2} e^{-\rho}$
2	0	0	$\sqrt{\frac{1}{4\pi}}$	$2\left(\frac{1}{2a_0}\right)^{3/2} (1 - \rho) e^{-\rho}$
2	1	0	$\sqrt{\frac{3}{4\pi}} \cos \theta$	$\frac{2}{\sqrt{3}} \left(\frac{1}{2a_0}\right)^{3/2} \rho e^{-\rho}$
2	1	$\pm 1$	$\mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$	$\frac{2}{\sqrt{3}} \left(\frac{1}{2a_0}\right)^{3/2} \rho e^{-\rho}$

- (a) Give the electron configurations and term symbols,  $^{2s+1}L_J$ , for the ground state, first excited state and second excited state of the hydrogen atom. [4 marks]
- (b) Give the wavefunctions,  $\psi_{nlm}(r, \theta, \phi)$ , for the ground state and second excited state ( $m = 0$ ) of the hydrogen atom. [4 marks]
- (c) Show pictorially that the dipole moment of the ground state of the hydrogen atom is zero. [2 marks]
- (d) Showing your full working and evaluating the radial and angular parts of the function, confirm mathematically that the dipole moment of the ground state of the hydrogen atom is zero. [10 marks]

[Hint: Use spherical polar coordinates, where the Jacobian is  $r^2 \sin \theta$ . The integral  $\int r^3 e^{-br} dr = -e^{-br}(b^3 r^3 + 3b^2 r^2 + 6br + 6)/b^4$ , where  $b = 2/a_0$ , may be useful.]

9. An atom-beam clock consists of a beam of hydrogen atoms passing consecutively through two microwave cavities of length  $l = 3$  cm separated by a field-free region of length  $L = 30$  cm. The microwave generator is tuned and stabilised to the  $F = 0, m_F = 0 \rightarrow F = 1, m_F = 0$  (i.e.  $|0, 0\rangle \rightarrow |1, 0\rangle$ ) ground state hyperfine transition of the hydrogen atom at  $A = 1.42$  GHz. The probability of a hydrogen atom being in the  $|1, 0\rangle$  state after passing through the first microwave cavity is

$$|c_{|1,0\rangle}|^2 = \frac{\Omega^2 \tau_p^2}{4} \text{sinc}^2\left(\frac{W \tau_p}{2}\right),$$

where  $W = (\delta^2 + \Omega^2)^{1/2}$ ,  $\Omega$  is the Rabi frequency,  $\tau_p$  is the transit time through each cavity and  $\delta$  is the microwave frequency detuning from  $A$ .

The Hamiltonian describing the hyperfine energy levels of the ground state of the hydrogen atom in a magnetic field  $B$  is

$$H' = \begin{pmatrix} A/4 + \mu_B B & 0 & 0 & 0 \\ 0 & -A/4 - \mu_B B & A/2 & 0 \\ 0 & A/2 & -A/4 + \mu_B B & 0 \\ 0 & 0 & 0 & A/4 - \mu_B B \end{pmatrix}.$$

- (a) Show that the energies as a function of  $B$  of the  $|0, 0\rangle$  and  $|1, 0\rangle$  states are

$$E_{|0,0\rangle} = -\frac{A}{4} - \sqrt{\frac{A^2}{4} + \mu_B^2 B^2} \quad \text{and} \quad E_{|1,0\rangle} = -\frac{A}{4} + \sqrt{\frac{A^2}{4} + \mu_B^2 B^2},$$

then find an expression for the Zeeman shift of the transition between the two states. [10 marks]

- (b) For an ensemble of hydrogen atoms passing through the clock at  $300 \text{ m s}^{-1}$ , calculate the Rabi frequency required such that, when passing through the first cavity, the atoms experience a  $\pi$ -pulse that excites all of the atoms in the ensemble to the  $|1, 0\rangle$  state. [2 marks]
- (c) Calculate the minimum magnetic field that would have to be present for none of the atoms in (b) to be excited to the  $|1, 0\rangle$  state after passing through the first cavity. [6 marks]
- (d) Assuming that no magnetic fields are present and that the atoms pass through both cavities to the detector, calculate the clock uncertainty of the clock based on the width of the central Ramsey fringe. [2 marks]