Workshop 4 Solutions

Introduction

This problem is about understanding and using four-momenta in special relativity. In general, the four-momentum of a particle can be written as

$$p^\mu = \left(\tfrac{E}{c}, \quad \underline{p}\right) . \tag{1}$$

Recalling the dispersion relation,

$$E^2 = |\underline{p}|^2 c^2 + m^2 c^4 \ , \tag{2}$$

we see that the square of the four-momentum of a particle, $p^2 \equiv p^\mu p_\mu$, depends only on its mass and is in fact given by $p^2 = m^2c^2$. It is clear that this must be a Lorentz scalar since there are no free indices. Note that a particle which satisfies this dispersion relation is said to be *on-shell*.

We can see that for massless particles, such as the photon, p^2 is zero. In general, the square of the four-momentum of some particles, $(p_1 + p_2 + \ldots + p_n)^2$, is known as the invariant mass of the system.

For massive particles, we can write the spatial momentum as $\underline{p} = \gamma(v)m\underline{v}$. Inserting this into Eq. (2) gives us the useful result for massive particles:

$$E = \gamma mc^2 \ . \tag{3}$$

Now, we proceed to solve the problems.

Question 1

There are several ways to solve this problem.

Method 1

An intuitive way to solve this problem is to consider the reference frame where the total three-momentum of the electrons is zero. This is known as the centre-of-momentum frame, or COM frame. This is also commonly called the centre-of-mass frame. When solving problems in relativity, it is always possible to choose the reference frame which makes the problem most convenient to solve. In this frame, the four-momenta of the particles are given by,

$$\text{Photon}: \qquad p_1 = \left(\tfrac{hf}{c}, \quad \tfrac{hf}{c}\underline{n}\right) , \tag{4}$$

$$\text{Electron 1}: \qquad p_2 = \left(\gamma mc, \quad \gamma m\underline{v}\right) , \tag{5}$$

$$\text{Electron 2}: \qquad p_3 = \left(\gamma mc, \quad -\gamma m\underline{v}\right) . \tag{6}$$

Since four-momentum must be conserved, we necessarily require that $p_1 = p_2 + p_3$ for this process to be allowed. However, since in any reference frame the photon cannot be at rest, it is impossible for the spatial components to be conserved, and so the process cannot take place.

Method 2

We can solve this problem more formally by assuming that we have four-momentum conservation, and then showing that this leads to a contradiction. Using the properties of four-momenta discussed in the introduction, we have

$$\begin{aligned} p_1 &= p_2 + p_3 & (7)\\ \implies p_1^2 &= (p_2 + p_3)^2 & (8)\\ \implies 0 &= p_2^2 + p_3^2 + 2p_2 \cdot p_3 & (9)\\ \implies 0 &= 2m^2c^2 + 2p_2 \cdot p_3\ . & (10) \end{aligned}$$

We now just need to evaluate the dot product of p_2 and p_3. We have

$$\begin{aligned} -m^2c^2 &= p_2 \cdot p_3 & (11)\\ &= \frac{E_2E_3}{c^2} - |\underline{p_2}||\underline{p_3}|\cos\theta & (12)\\ &= \frac{E_2E_3}{c^2}\left(1 - \frac{|\underline{v_2}||\underline{v_3}|}{c^2}\cos\theta\right) \qquad \text{using } |\underline{p}| = \frac{E|\underline{v}|}{c^2} & (13)\\ &\geq \frac{E_2E_3}{c^2}\left(1 - \frac{|\underline{v_2}|}{c}\frac{|\underline{v_3}|}{c}\right) \qquad \text{since } \cos\theta \leq 1 & (14)\\ &= \frac{E_2E_3}{c^2}(1 - \beta_2\beta_3) & (15)\\ &> 0 \qquad \text{since } \beta < 1\ . & (16) \end{aligned}$$

This is clearly a contradiction: $-m^2c^2$, which has to be negative, cannot be greater than zero. Therefore, a single photon cannot decay into an electron-positron pair.

Question 2

In this problem, two particles (a proton with four-momentum p_1, and a neutron with four-momentum p_2, with equal masses) collide and produce some new particles with four-momenta q_i, where i ranges from 1 to 4.

(a) In the centre-of-mass frame the net spatial momentum of the particles is zero. Since there are only two particles in the initial state this implies that the proton and the neutron (which are of equal mass) have the same speed (but opposite velocities). Using the results from the introduction, this allows us to write,

$$p_1^{\mathrm{cm}} = \big(\gamma(v_{\mathrm{cm}})mc, \quad \gamma(v_{\mathrm{cm}})m\underline{v}_{\mathrm{cm}}\big)\,, \tag{17}$$
$$p_2^{\mathrm{cm}} = \big(\gamma(v_{\mathrm{cm}})mc, \quad -\gamma(v_{\mathrm{cm}})m\underline{v}_{\mathrm{cm}}\big)\,. \tag{18}$$

In the rest frame of the neutron we can write down,

$$p_1^{\mathrm{L}} = \big(\gamma(v_{\mathrm{L}})mc, \quad \gamma(v_{\mathrm{L}})m\underline{v}_{\mathrm{L}}\big)\,, \tag{19}$$
$$p_2^{\mathrm{L}} = \big(mc, \quad 0\big)\,. \tag{20}$$

(b) We can relate the different γ factors in these two frames by using the invariance of the Minkowski scalar product. This could be done in multiple ways—for instance, by comparing the invariant mass of the whole system in the two frames. However, a simpler way to do it is to calculate the scalar product between p_1 and p_2, and use the fact that this must be equal in both frames:

$$p_1^{\mathrm{cm}} \cdot p_2^{\mathrm{cm}} = p_1^{\mathrm{L}} \cdot p_2^{\mathrm{L}} \tag{21}$$
$$\implies \gamma(v_{\mathrm{cm}})^2 m^2 c^2 + \gamma(v_{\mathrm{cm}})^2 m^2 v_{\mathrm{cm}}^2 = \gamma(v_{\mathrm{L}}) m^2 c^2 \tag{22}$$
$$\implies \gamma(v_{\mathrm{L}}) = \gamma(v_{\mathrm{cm}})^2 \left(1 + \frac{v_{\mathrm{cm}}^2}{c^2}\right)\,. \tag{23}$$

(c) We know that four-momentum has to be conserved, so we can write,

$$p_1 + p_2 = q_1 + q_2 + q_3 + q_4\,. \tag{24}$$

We can move q_1 over to the left-hand side, and then square the result:

$$p_1 + p_2 - q_1 = q_2 + q_3 + q_4 \tag{25}$$
$$\implies 2(p_1 \cdot p_2 - p_1 \cdot q_1 - p_2 \cdot q_1) + 3m^2c^2 = 2(q_2 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4) + 3m^2c^2\,. \tag{26}$$

The masses are all equal, so the mass terms cancel, leaving only the cross terms,

$$p_1 \cdot p_2 - p_1 \cdot q_1 - p_2 \cdot q_1 = q_2 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4\,. \tag{27}$$

(d) The smallest possible energy of the incoming system corresponds to the case where all the outgoing particles are stationary, and hence have only rest mass energy. In this scenario we can write,

$$q_1 = q_2 = q_3 = q_4 = \big(mc, \quad 0\big)\,. \tag{28}$$

To find $\gamma(v_{\mathrm{cm}})$ in this situation, we can use the result derived in part (c). From Eq. (28) we can see that,

$$q_2 \cdot q_3 = q_2 \cdot q_4 = q_3 \cdot q_4 = m^2c^2\,, \tag{29}$$

and using Eqs. (17) and (18) we have,

$$p_1 \cdot q_2 = p_2 \cdot q_1 = \gamma(v_{\rm cm}) m^2 c^2 \ . \tag{30}$$

Plugging these into the result from part (c), we get,

$$\gamma(v_{\rm cm})^2 m^2 c^2 + \gamma(v_{\rm cm})^2 m^2 v_{\rm cm}^2 - 2\gamma(v_{\rm cm}) m^2 c^2 = 3m^2 c^2 \ . \tag{31}$$

This equation can now be solved. Since $v_{\rm cm}^2$ appears separately in the equation, it cannot be solved as simply a quadratic in $\gamma(v_{\rm cm})$, since this depends on $v_{\rm cm}$. Instead, the explicit expression for $\gamma(v_{\rm cm})$ can be substituted in, and the equation solved for $v_{\rm cm}$. The result of doing this is,

$$v_{\rm cm} = \frac{\sqrt{3}}{2} c \quad \Longrightarrow \quad \gamma(v_{\rm cm}) = 2 \ . \tag{32}$$

Using Eq. (23), we find,

$$\gamma(v_{\rm L}) = 7 \ . \tag{33}$$