Theoretical Physics 2019/20 — Problem QT2.7

This problem concerns the ladder operators for a linear harmonic oscillator, $\hat{a}_-$ and $\hat{a}_+$. Recall that $\hat{a}_+ = \hat{a}_-^\dagger$. To avoid unnecessary subscripts, we will denote $\hat{a}_-$ by $\hat{a}$ and $\hat{a}_+$ by $\hat{a}^\dagger$. In the position representation, these two operators take on the following forms, with $p = -i\hbar\mathrm{d}/\mathrm{d}x$:

$$
\begin{aligned}
a &= (2\hbar m\omega)^{-1/2}\left(m\omega x + \hbar\frac{\mathrm{d}}{\mathrm{d}x}\right) = (2\hbar m\omega)^{-1/2}\,(m\omega x + ip) \\
a^\dagger &= (2\hbar m\omega)^{-1/2}\left(m\omega x - \hbar\frac{\mathrm{d}}{\mathrm{d}x}\right) = (2\hbar m\omega)^{-1/2}\,(m\omega x - ip),
\end{aligned}
$$

As seen previously, $[a, a^\dagger] = 1$, and, for any complex value of α, one can find a function $\phi_\alpha(x)$ such that $a\phi_\alpha(x) = \alpha\,\phi_\alpha(x)$. In the Dirac notation, $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\hat{a} = (2\hbar m\omega)^{-1/2}\,(m\omega\hat{x} + i\hat{p})$ and $\hat{a}^\dagger = (2\hbar m\omega)^{-1/2}\,(m\omega\hat{x} - i\hat{p})$, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators for the x- direction.

(a) Show that $\hat{x} = [2\hbar/(m\omega)]^{1/2}\,\hat{S}$ and $\hat{p} = (2\hbar m\omega)^{1/2}\,\hat{D}$, where $\hat{S} = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{D} = (\hat{a} - \hat{a}^\dagger)/(2i)$.

(b) Let us consider a linear harmonic oscillator and suppose that it is in the state $|\alpha\rangle$ at $t = 0$. Then, at that time, the expectation values of its position and its momentum are given by the equations $\langle x\rangle(t = 0) = \langle\alpha|\hat{x}|\alpha\rangle$ and $\langle p\rangle(t = 0) = \langle\alpha|\hat{p}|\alpha\rangle$. Given that $\langle\alpha|\hat{S}|\alpha\rangle = \mathrm{Re}\,\alpha$ and $\langle\alpha|\hat{D}|\alpha\rangle = \mathrm{Im}\,\alpha$ (see Question 2 of the Progress Test), show that

$$\langle\alpha|\hat{x}|\alpha\rangle = [2\hbar/(m\omega)]^{1/2}\,\mathrm{Re}\,\alpha \quad \text{and} \quad \langle\alpha|\hat{p}|\alpha\rangle = (2\hbar m\omega)^{1/2}\,\mathrm{Im}\,\alpha.$$

(c) Let us describe the time evolution of the system in the Heisenberg picture. Hence, we describe its state by the time-independent ket vector $|\alpha\rangle$ and we associate the position and the momentum with time-dependent Heisenberg operators $\hat{x}_\mathrm{H}(t)$ and $\hat{p}_\mathrm{H}(t)$ such that $\hat{x}_\mathrm{H}(t = 0) = \hat{x}$ and $\hat{p}_\mathrm{H}(t = 0) = \hat{p}$. Accordingly, we calculate the expectation values $\langle x\rangle(t)$ and $\langle p\rangle(t)$ as, respectively, $\langle\alpha|\hat{x}_\mathrm{H}(t)|\alpha\rangle$ and $\langle\alpha|\hat{p}_\mathrm{H}(t)|\alpha\rangle$.

  (i) The Heisenberg operator reducing to the Schrödinger operator $\hat{a}$ for $t \to 0$ is $\hat{a}_\mathrm{H}(t) = \hat{U}(0,t)\,\hat{a}\,\hat{U}^\dagger(0,t)$, where $\hat{U}$ is the time evolution operator. Similarly, the equation $\hat{H}_\mathrm{H}(t) = \hat{U}(0,t)\,\hat{H}\,\hat{U}^\dagger(0,t)$ relates the Hamiltonian operator in the Heisenberg picture, $\hat{H}_\mathrm{H}(t)$, to the Hamiltonian in the Schrödinger picture, $\hat{H}$. As seen in a lecture, $[\hat{a}_\mathrm{H}(t), \hat{H}_\mathrm{H}(t)] = \hat{U}(0,t)\,[\hat{a}, \hat{H}]\,\hat{U}^\dagger(0,t)$. Given that $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$, show that $[\hat{a}_\mathrm{H}(t), \hat{H}_\mathrm{H}(t)] = \hbar\omega\,\hat{a}_\mathrm{H}(t)$.

  (ii) The Heisenberg equation of motion for $\hat{a}_\mathrm{H}(t)$ is

$$\frac{\mathrm{d}\hat{a}_\mathrm{H}}{\mathrm{d}t} = \frac{1}{i\hbar}\,[\hat{a}_\mathrm{H}(t), \hat{H}_\mathrm{H}(t)].$$

  Given that $\hat{a}_\mathrm{H}(t = 0) = \hat{a}$, show that $\hat{a}_\mathrm{H}(t) = \hat{a}\,\exp(-i\omega t)$ and $\hat{a}_\mathrm{H}^\dagger(t) = \hat{a}^\dagger\,\exp(i\omega t)$.

  (iii) Hence, show that $\langle x\rangle(t) = A\cos(\omega t - \arg\alpha)$ and $\langle p\rangle(t) = -m\omega A\sin(\omega t - \arg\alpha)$, with $\alpha = |\alpha|\exp(i\arg\alpha)$ and $A = [2\hbar/(m\omega)]^{1/2}\,|\alpha|$. (Note the physical meaning of this result: in the state $|\alpha\rangle$, the expectation values of the position and the momentum of this quantum harmonic oscillator vary in time exactly in the same way as the position and the momentum of a classical harmonic oscillator.)