University of Durham

EXAMINATION PAPER

May/June 2015 Examination code: PHYS2631WE01

THEORETICAL PHYSICS 2

SECTION A. Classical Mechanics
SECTION B. Quantum Theory 2

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

Continued Next Page

SECTION A. CLASSICAL MECHANICS

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) What is an ignorable coordinate? How can they help one to find the equation of motion of a mechanical system. [4 marks]

   (b) The displacement, x, of a damped oscillator of mass m satisfies the following differential equation:

   $$\ddot{x} + b\dot{x} + \frac{k}{m}x = 0,$$

   where b and k are constants. Find the auxiliary equation and determine an expression for b in the case that the oscillator is critically damped. [4 marks]

   (c) What does a Green's function represent? Explain, briefly, how they can be used to find the motion of a driven linear oscillator. [4 marks]

   (d) The Hamiltonian of a mechanical system is defined through the following Legendre transformation of the Lagrangian

   $$H(p, q) = p\dot{q} - L(q, \dot{q}).$$

   Consider a Lagrangian of the form $L = \alpha\dot{q}^{\beta} - V(q)$, where α and β are constants. Write down equations for H and the total energy E. Use these to determine the necessary condition on β for the Hamiltonian to equal the total energy of the system. [4 marks]

   (e) Describe, briefly, Noether's theorem and give an example. [4 marks]

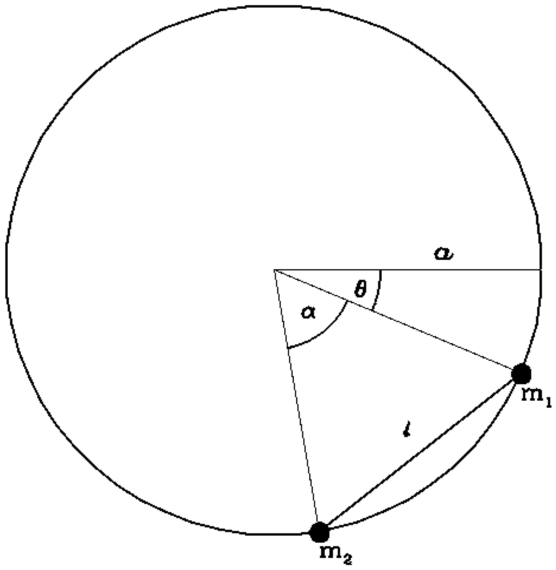
   (f) What type of force is the Euler force and how do such forces arise? Why is the Euler force usually neglected when considering motion on Earth? [4 marks]

   (g) Calculate the moment of inertia for a cube of uniform density, of mass M and side length $2a$, about an axis running through its centre of mass and the centres of two opposite faces. [4 marks]

   (h) What does the principal axis theorem state? How do the principal moments of inertia of an oblate symmetric ellipsoid compare with each other? [4 marks]

  Continued Next Page

2. Two particles of mass m_1 and m_2 are connected by a light, inextensible rod of length l. The particles are constrained to slide on a smooth circular wire hoop of radius a, which is held in a vertical plane in a uniform gravitational field of strength g. θ and $\theta + \alpha$ are the angular coordinates, measured downwards from the horizontal, of m_1 and m_2 with respect to the centre of the hoop (α is fixed).



(a) Find the Lagrangian of the system in terms of the generalised coordinate θ, and use the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

to show that

$$\ddot{\theta} = \frac{g[m_1 \cos\theta + m_2 \cos(\theta + \alpha)]}{a(m_1 + m_2)}.$$

[6 marks]

(b) Consider the case with $m_1 = m_2 = m$.

(i) What value of θ corresponds to the position of stable equilibrium? [1 mark]

(ii) Find the equation of motion for small oscillations about the position of stable equilibrium, θ_{eq}, by rewriting the differential equation for $\ddot{\theta}$ in terms of $\phi = \theta - \theta_{\text{eq}}$ and solving for $\phi(t)$. Give your answer in terms of $\phi(0)$ and $\dot{\phi}(0)$. [7 marks]

(iii) What is the solution when $\alpha = \pi$ and what motion does this represent? [3 marks]

(c) With what frequency does a single particle of mass $2m$, confined to the same circular hoop, undergo small oscillations about its equilibrium position? [3 marks]

3. The distant, rapidly rotating planet Cyclo is spherical with mass M, radius R and a constant angular velocity $\underline{\omega}$ pointing out through the north pole of the planet. Accelerations, $\ddot{\underline{r}}$, measured in this rotating frame are related to those determined in an inertial frame, $\underline{a}_S$, via the equation

$$\ddot{\underline{r}} = \underline{a}_S - \underline{\omega} \times (\underline{\omega} \times \underline{r}) - 2\underline{\omega} \times \dot{\underline{r}}\,,$$

where $\underline{r}$ represents the position measured in the rotating frame.

(a) (i) Given that the effective gravity at the equator is zero, find an expression for the size of the angular velocity, ω. [4 marks]

(ii) What are the sizes of the Coriolis and centrifugal accelerations at the north pole for a particle moving with speed v along the surface, and in which directions do they point? [3 marks]

A competition was held at the north pole, involving one archer from the settlement at each pole firing a single arrow at a circular target with diameter d a distance l away. The south pole competitor took aim as they would have done if they were at the south pole. For the rest of this question, ignore gravity and assume that $d/l \ll 1$ and that the arrows travel at a constant speed.

(b) (i) Show that the north pole archer aimed an angle $\omega l/v$ (viewed from above) to the left of the target in order to hit its centre. [5 marks]

(ii) By what distance and in which direction did the south pole archer's arrow miss the target centre? [2 marks]

The following year at the north pole, the south pole competitor cunningly substituted a target with diameter fd at a distance fl, where $0 < f < 1$. The north pole competitor, having only one eye like all Cyclopsians, could not perceive this change so did not alter their aim.

(c) Assuming that the southerner continued to aim as if at the south pole, albeit using their knowledge of the smaller target size and distance, find expressions for the distance by which each competitor misses the target centre and use these to infer the range of f for which the southern Cyclopsian's arrow lands nearer to the target centre. [6 marks]

SECTION B. QUANTUM THEORY 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) What condition is required for a set of state vectors $|\phi_1\rangle$, $|\phi_2\rangle$ $\ldots$, $|\phi_n\rangle$ to be linearly independent? If they also form an orthonormal set what property is satisfied? [4 marks]

   (b) What is the defining property of a Hermitian operator? Show that such operators have real eigenvalues. [4 marks]

   (c) Given a state vector $|\psi\rangle$, write down the corresponding wavefunction $\psi(x)$ in Dirac notation. By inserting a complete set of position eigenstates $|x\rangle$, obtain the inner product of state vectors $\langle\phi|\psi\rangle$ as an overlap integral of wavefunctions. [4 marks]

   (d) What is the defining property of a unitary operator? Show that if $\hat{A}$ is a unitary operator and if $\hat{A}|\phi\rangle = |\phi'\rangle$, $\hat{A}|\psi\rangle = |\psi'\rangle$, then $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$, so that inner products of states are preserved under a unitary operation. [4 marks]

   (e) Fermionic annihilation and creation operators satisfy the algebra

$$\{\hat{b}, \hat{b}^\dagger\} = \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = 1 \ .$$

   Show that the possible eigenvalues of the number operator $\hat{N} = \hat{b}^\dagger\hat{b}$ are $n = 0, 1$. Comment on the physical significance of this result. [4 marks]

   (f) Write down the time evolution equation for an operator $\hat{O}$ in the Heisenberg picture with Hamiltonian $\hat{H}$. If the operator corresponds to a conserved physical quantity what condition is satisfied? [4 marks]

   (g) Give the general expression for the commutator of angular momenta $[\hat{L}_i, \hat{L}_j]$, where $\{i, j, k\} = \{x, y, z\} = \{1, 2, 3\}$. Evaluate $[L_x, L_z]$. [4 marks]

5. Consider the harmonic oscillator in one dimension with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x) = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 ,$$

where m is the mass of the oscillator, ω is the frequency of the oscillator, $\hat{p}$ is the momentum operator, $\hat{x}$ is the displacement, and $\hat{V}(x)$ is the potential.

(a) If we introduce a creation operator $\hat{a}^\dagger$ and an annihilation operator $\hat{a}$ show that we can write the Hamiltonian in the form

$$\hat{H} = (\hat{a}^\dagger\hat{a} + \frac{1}{2})\hbar\omega ,$$

and give expressions for $\hat{a}^\dagger$ and $\hat{a}$ in terms of $\hat{x}$ and $\hat{p}$. [2 marks]

(b) Given that $[\hat{x}, \hat{p}] = i\hbar$ show that $\hat{a}^\dagger$and $\hat{a}$ satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. [2 marks]

(c) Define the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$ with eigenvectors $|n\rangle$, $\hat{N}|n\rangle = n|n\rangle$, $(n = 0, 1, 2, 3, \ldots)$. Show that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, and $a|n\rangle = \sqrt{n}|n-1\rangle$. [4 marks]

(d) Expressing $\hat{a}$ in terms of $\hat{x}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ show that $\hat{a}|0\rangle = 0$ yields a differential equation for the ground state wave function $\psi_0(x) = \langle x|0\rangle$ with solution

$$\psi_0(x) = N \exp\left(-\alpha\frac{x^2}{2}\right) ,$$

where $\alpha \equiv \frac{m\omega}{\hbar}$ and N is a normalization constant. [6 marks]

(e) By rewriting the kinetic energy $\hat{T}$, and potential energy $\hat{V}$, in terms of creation and annihilation operators using the results obtained in (a), show that their expectation values in energy eigenstates of the harmonic oscillator satisfy

$$\langle n|\hat{T}|n\rangle = \langle n|\hat{V}|n\rangle .$$

[6 marks]

6. Consider a rigid rotator (rotor) with Hamiltonian

$$\hat{H} = \frac{1}{2J}\hat{L}^2$$

where J is the moment of inertia.

(a) Give the commutation relations between the angular momentum operators $\hat{L}_{x,y,z}$ themselves and with $\hat{L}^2$. [4 marks]

(b) What are the eigenvalues and joint eigenfunctions of the operators $\hat{L}^2$ and $\hat{L}_z$? [2 marks]

(c) Specify the eigenvalues and eigenfunctions of the Hamiltonian above. Discuss potential degeneracies. [2 marks]

(d) The rotational levels of a diatomic molecule may be obtained by treating it as a rigid rotator. If the molecule is placed in an external magnetic field $\underline{B}$ the interaction energy is $\mu\underline{B}\cdot\hat{\underline{L}}$, where μ is the magnetic moment. Explain how the energy eigenvalues in (c) are modified in this case. [2 marks]

(e) At a given time $t = 0$ the rotor is in the state

$$|\psi\rangle = N\left[\cos^2\theta + \sin^2\theta\cos(2\phi)\right] ,$$

where N is a normalisation constant.

Express the state $|\psi\rangle$ in terms of spherical harmonics and fix N through the normalisation requirement on $|\psi\rangle$. [6 marks]

(f) In the state $|\psi\rangle$ in (e) above, what are the probabilities for energy measurements to yield the values $3\hbar^2/J$, $\hbar^2/J$, and 0? [4 marks]

Hint: At various stages in this question, you will need some of the relations and definitions listed below.
Explicit representations:

$$\begin{aligned}
Y_{00}(\theta,\,\phi) &= \sqrt{\frac{1}{4\pi}} \\
Y_{1\pm1}(\theta,\,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{\pm i\phi} \\
Y_{10}(\theta,\,\phi) &= \sqrt{\frac{3}{4\pi}}\,\cos\theta \\
Y_{2\pm2}(\theta,\,\phi) &= \sqrt{\frac{15}{32\pi}}\,\sin^2\theta e^{\pm 2i\phi} \\
Y_{2\pm1}(\theta,\,\phi) &= \mp\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta e^{\pm i\phi} \\
Y_{20}(\theta,\,\phi) &= \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)
\end{aligned}$$

Normalisation:

$$\int_{-1}^{1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi Y_{lm}(\theta,\,\phi)Y^*_{l'm'}(\theta,\,\phi) \;=\; \delta_{ll'}\delta_{mm'}$$

