## CM7: Poisson Brackets

When considering the motion of a particle in 3 spatial dimensions, one may describe the Poisson bracket for two functions $F$ and $G$ of the Cartesian coordinates $q_x$, $q_y$, $q_z$ and their canonically conjugate momenta $p_x$, $p_y$, $p_z$ as

$$\{F, G\} = \sum_{k=x,y,z} \left( \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right).$$

Consider a Lagrangian of the general form

$$L = \frac{m}{2}(\dot{q}_x^2 + \dot{q}_y^2 + \dot{q}_z^2) - V,$$

where $V$ is an unspecified potential energy function of the coordinates $q_x$, $q_y$, $q_z$.

1. Show that the canonically conjugate momenta $p_x$, $p_y$, $p_z$ are then equal to the mechanical momenta $m\dot{q}_x$, $m\dot{q}_y$, $m\dot{q}_z$. Hence determine the Poisson brackets $\{p_x, p_y\}$, $\{p_y, p_z\}$, $\{p_z, p_x\}$.

Still considering the same general form of Lagrangian, the angular momentum vector is defined as $\underline{J} = m\underline{q} \times \dot{\underline{q}}$, where $\underline{q} = (q_x, q_y, q_z)$.

2. Determine the components $J_x$, $J_y$, $J_z$ in terms of the coordinates and their canonically conjugate momenta.
3. Use these to determine the Poisson brackets $\{J_x, J_y\}$, $\{J_y, J_z\}$, $\{J_z, J_x\}$, and also the nested Poisson bracket $\{J_y, \{J_y, \{J_y, J_x\}\}\}$.

Consider the Lagrangian, expressed in spherical coordinates as

$$L = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta\right] - V.$$

4. Determine the canonically conjugate momenta $p_r$, $p_\theta$, $p_\phi$ to $r$, $\theta$, $\phi$, and the Poisson brackets $\{r, p_r\}$, $\{\theta, p_\theta\}$, $\{\phi, p_\phi\}$.