University of Durham

EXAMINATION PAPER

May/June 2015 Examination code: PHYS3651WE01

PLANETS AND COSMOLOGY 3

SECTION A. Cosmology
SECTION B. Planetary Systems

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\,\mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\,\mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\,\mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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SECTION A. COSMOLOGY

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Explain what is meant by the cosmological principle. Relative to an observer, galaxies follow a Hubble expansion law, $\vec{v} = H\vec{r}$, where $\vec{v}$ and $\vec{r}$ denote the vector velocities and positions of galaxies respectively, and H is the Hubble parameter. Show that the same expansion law applies for an observer in any other galaxy. [4 marks]

(b) A supernova is observed in a galaxy at redshift $z = 2$. Calculate the redshift of a galaxy located mid-way in comoving distance between Earth and the supernova. Hence calculate the observed redshift of the supernova as seen by an observer in this intervening galaxy. Assume a matter-dominated flat cosmology for which the comoving distance-redshift relation is $r(z) = (2c/H_0)[1 - (1 + z)^{-0.5}]$. [4 marks]

(c) The Friedmann equation for a Universe in which the mass energy density, ρ, comprises only radiation and relativistic particles is

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

where a is the expansion factor, H is the Hubble parameter and k is a constant. Express k in terms of the density parameter, Ω_0, and the Hubble constant, H_0. For the case $\Omega_0 = 1$ show that the age of the Universe is $t_0 = 1/(2H_0)$. [4 marks]

(d) A Universe comprises identical galaxies with an average space density of 0.01 Mpc^{-3}. The rotation curves of these galaxies are flat in their outer parts with amplitudes of 200 km s^{-1} due to the presence of spherical dark matter haloes. Calculate the maximum radius out to which these haloes extend, in Mpc, if these galaxies contribute $\Omega_{\mathrm{gal}} = 0.1$ to the present-day density parameter. Assume $H_0 = 70$ km s^{-1} Mpc^{-1}. [4 marks]

(e) Consider a dark energy fluid with equation of state $p_{\mathrm{DE}} = -2\rho_{\mathrm{DE}}c^2$, which obeys the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left[\rho + \frac{p}{c^2}\right] = 0,$$

where p and ρ are the pressure and density of the fluid and a is the expansion factor. Use the fluid equation to infer the redshift evolution of the fluid density, $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{DE}}(z = 0)$. If the present-day density parameters in this dark energy and non-relativistic matter are $\Omega_{\mathrm{DE},0} = 0.7$ and $\Omega_{M,0} = 0.3$, respectively, calculate the redshift at which $\Omega_{\mathrm{DE}} = \Omega_M$. [4 marks]

(f) The dimensionless deceleration parameter is defined as

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2},$$

and is a function of time. State the physical meaning of q. [2 marks]

Assume the Universe is spatially flat, and neglect dark energy. Making use of the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_{\mathrm{tot}} + \frac{3p_{\mathrm{tot}}}{c^2}\right),$$

where $\rho_{\rm tot}$ and $p_{\rm tot}$ are respectively the energy density and pressure of matter plus radiation, evaluate q at the time of matter-radiation equality. [2 marks]

(g) In a flat universe, for an observer at redshift $z = 0$, the angular size Θ and *physical* length l of an object at redshift z are related by

$$l = \frac{r}{1+z}\Theta,$$

where r is the comoving distance between the observer and the object. Determine the angular diameter distance $d_{\rm A}$ from this expression. [2 marks]

If the universe is always matter-dominated, use the expression for $r(z)$ in 1(b) to evaluate the redshift at which $d_{\rm A}$ takes its maximum possible value. [2 marks]

(h) Assume dark matter particles to be massive neutrinos, which have a present-day density parameter given by

$$\Omega_{\nu,0} = \frac{\sum_{i=1}^{N_\nu} m_i c^2}{94 \text{ eV}} \left[\frac{H_0}{100 \text{ km s}^{-1}\text{Mpc}^{-1}}\right]^{-2},$$

where $N_\nu = 3$ is the number of neutrino species and m_i is the neutrino mass (assumed to be the same for all three species). Calculate m_i and express it in kg, if $\Omega_{\nu,0} = 0.30$ is required. Use $H_0 = 70$ km s^{-1} Mpc^{-1} in your calculation. [4 marks]

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2. (a) The Planck function for the energy density of black-body radiation per unit frequency, ν, and per unit volume is

$$u(T,\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1},$$

where T is the temperature. Use this to calculate the photon density, n_γ, in the present-day cosmic microwave background radiation (CMB) at temperature $T_0 = 2.73$ K. Hence calculate the redshift, $z_{\rm recomb}$, at which the mean photon energy in the background radiation equals the ionization potential of atomic hydrogen, 13.6 eV. [5 marks]

(b) If the present-day baryon density parameter $\Omega_b = 0.044$ and the Hubble parameter $H_0 = 70$ km s^{-1} Mpc^{-1}, calculate the present-day baryon:photon number density ratio. Hence explain why the $z_{\rm recomb}$ given by the method in (a) is a poor estimate of the redshift at which the CMB was produced. Assume a pure hydrogen composition. [5 marks]

(c) A more detailed calculation shows that the hydrogen ionization fraction decreases from 0.90 at $z_1 = 1488$ to 0.10 at $z_2 = 1318$. If this redshift interval defines the epoch of recombination, starting from the Friedmann equation show that its time duration is approximately $\Delta t_{\rm recomb} \simeq A/(H_0\sqrt{\Omega_0})$ in a matter-only Universe with density parameter Ω_0, where $\Omega_0 = \Omega_b + \Omega_{DM}$. Ω_{DM} is the present-day density parameter in dark matter, $\Omega_b = 0.044$ as in part (b), and A is a constant which should be evaluated. You may assume that $\Omega_0 z_2 \gg 1$. [5 marks]

(d) For what range of values of Ω_{DM} would the entire recombination epoch instead occur while the Universe is radiation-dominated, i.e. when the density parameter of the photons in the background radiation (ignoring the contribution of relativistic particles such as neutrinos) exceeds that in dark matter and baryons combined? Assume $\Omega_b = 0.044$ and $T_0 = 2.73$ K. [5 marks]

[Hint: The integral $\int_0^\infty \frac{x^2}{e^x - 1}dx = 2.408$.
The total energy density in blackbody radiation is $4\sigma T^4/c$.]

3. In the radiation-dominated era, the energy density of the Universe is dominated by relativistic particles. Let T be the temperature of the relativistic plasma. The number of relativistic bosonic degrees of freedom (d.o.f.), g_*, is

$$g_*(T) = \sum_{\text{bosons}} g_{\text{boson}} + \frac{7}{8} \sum_{\text{fermions}} g_{\text{fermion}},$$

where g_{boson} (g_{fermion}) is the number of d.o.f. of a given boson (fermion) particle.

(a) In the standard cosmological model, right before the onset of primordial nucleosynthesis, the relativistic particles in the Universe include photons, electrons, 3 species of neutrinos, and their corresponding antiparticles. Evaluate g_* at this time and explain your reasoning. [6 marks]

(b) The expansion time scale is defined as $\tau_{\text{exp}} \equiv H^{-1}$, where H is the Hubble parameter at a given time. Show that $\tau_{\text{exp}} \propto T^{-2}$ in the radiation-dominated era, assuming $g_* =$ constant (no need to evaluate or explicitly write down the proportionality coefficient). [2 marks]

The weak interaction time scale, τ_{weak}, which determines the rate of the reactions that convert neutrons to protons and vice versa, depends on temperature as $\tau_{\text{weak}} \propto T^{-5}$. The freeze-out temperature, T_{fo}, is approximately given by $\tau_{\text{exp}} = \tau_{\text{weak}}$, and in the standard cosmological model $kT_{fo} = 0.7$ MeV. Given that the mass difference between neutrons and protons, Δm, satisfies $\Delta mc^2 = 1.3$ MeV, calculate the ratio between the neutron number density (N_n) and the proton number density (N_p) at freeze-out. [2 marks]

(c) A cosmologist incorrectly assumes that there are 4 neutrino species instead of 3 as in the standard model. The calculation is otherwise correct, including using the correct value of τ_{weak}. Find the value of g_* the cosmologist gets in (a), and show that the resulting τ_{exp} is $\sqrt{43/50}$ times its correct value. [3 marks]

By using $\tau_{\text{exp}}(T'_{fo}) = \tau_{\text{weak}}(T'_{fo})$, show that the cosmologist will get an incorrect freeze-out temperature, T'_{fo}, which satisfies $T'_{fo} = \left(\frac{43}{50}\right)^{-1/6} T_{fo}$. [3 marks]

(d) After primordial nucleosynthesis, nearly all free neutrons present at freeze-out end up in ^{4}He nuclei (ignore free neutron decays), which have a mass fraction:

$$Y = \frac{2N_n}{N_n + N_p}.$$

What percentage error in Y would the cosmologist get in the cosmologist's calculation as a result of assuming 4 neutrino species? [4 marks]

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4. (a) The horizon scale, at a cosmic time t, is defined as

$$l_{\mathrm{H}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt',$$

in which $a(t)$ is the scale factor and c is the speed of light. Explain its physical meaning, and find expressions for it in terms of c and t for the radiation-dominated era (when $a(t) \propto t^{1/2}$) and the matter-dominated era (when $a(t) \propto t^{2/3}$), respectively. [6 marks]

(b) Assuming that the Universe is flat and dominated by matter from recombination until today (with no cosmological constant), the *physical* radius of the last scattering surface (LSS) is $3ct_0$ today (when $t = t_0$). Show that the ratio between the physical radius of the LSS at recombination ($t = t_{\mathrm{rec}}$) and $l_{\mathrm{H}}(t_{\mathrm{rec}})$ is $a_{\mathrm{rec}}^{-1/2}$ with $a_{\mathrm{rec}} \equiv a(t_{\mathrm{rec}})$. Given that $a_{\mathrm{rec}} = 0.001$, explain what is meant by the horizon problem. [4 marks]

(c) Inflation is assumed to be driven by some constant (vacuum) energy density ρ_{vac}, such that the rate of the Hubble expansion during inflation is a constant, $H = 10^{36}\ \mathrm{s}^{-1}$, and $a(t) \propto \exp(Ht)$. Find the value of l_{H} when inflation ends at $t = t_f = 10^{-34}$ s, in unit of Mpc, assuming that inflation starts at $t = 0$. [3 marks]

(d) t_{eq} is the time at matter-radiation equality, and $a_{\mathrm{eq}} = a(t_{\mathrm{eq}})$. Show that

$$a(t_f) = \left(\frac{t_f}{t_0}\right)^{1/2} a_{\mathrm{eq}}^{1/4}.$$

As in (b), assume in your calculation that the Universe is flat and now dominated by matter. Also, assume that the radiation-dominated era starts immediately at t_f. [3 marks]

By comparing the physical sizes of the LSS and l_{H} at t_f, and taking $a_{\mathrm{eq}} = 3 \times 10^{-4}$ and $t_0 = 3 \times 10^{17}$ s, explain how inflation solves the horizon problem. [4 marks]

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SECTION B. PLANETARY SYSTEMS

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) Summarise the main properties of the Solar System required to be reproduced in any successful model of its formation. [4 marks]

   (b) Show that the centre of mass of a system composed of two bodies interacting only through gravity moves with a constant velocity. [4 marks]

   (c) Comet 67P Churyumov-Gerasimenko has a period of 6.44 yr and a perihelion distance of 1.24 AU. Work out its distance from the Sun and the angle its velocity vector subtends with respect to the radial vector when it has a speed of 9.20 km s^{-1}. [4 marks]

   [Hint: the angle between the velocity vector and the radial vector in an elliptical orbit, ϕ, can be found from

   $$\sin\phi = \left(\frac{a^2(1-e^2)}{r(2a-r)}\right)^{\frac{1}{2}},$$

   for a position a distance r from the focus in an orbit with eccentricity e and semi-major axis a.]

   (d) Calculate the ratio between the forces due to radiation pressure and gravity for a black spheroidal dust grain with a radius of 10 μm and a density of 2.2×10^3 kg m^{-3} in the vicinity of a red giant star of mass 3.0 $M_\odot$ and luminosity 2.0×10^2 $L_\odot$. Comment on what this means for dust around this star. [4 marks]

   (e) Show that the surface temperature of a planet, $T_{\rm g}$, is related to the effective temperature of the outer layer of the atmosphere $T_{\rm e}$ by

   $$T_{\rm g}^4 = (n+1)T_{\rm e}^4,$$

   where n is the optical depth of the atmosphere. Given the effective temperature for Mars was 178 K a billion years after it was formed, what value of atmospheric optical depth was required for liquid water to be present on its surface at that point in its history? [4 marks]

   (f) Estimate the minimum radius of an asteroid with density $\rho = 4.5 \times 10^3$ kg m^{-3} for which the central pressure exceeds its material strength of $S = 4.0 \times 10^8$ N m^{-2}. Why does this explain the irregular appearance of many smaller asteroids? [4 marks]

   (g) Ground-based planetary transit surveys have a sensitivity limit of $\Delta m = 1.00 \times 10^{-3}$ (i.e. 1 milli-magnitude). Use this to place a limit on the radius of a star (in solar radii) for which you would be able to detect a transit from an Earth analogue from the ground. Comment on the suitability of such a system for hosting a habitable planet. [4 marks]

   [The radius of the Sun $R_\odot = 6.96 \times 10^8$ m, and the radius of the Earth is 6.37×10^6 m.]

6. (a) Show that the speed of a rocket initially of mass M changes by Δv after ejecting a mass m of fuel with velocity v_{e} where

$$\Delta v = v_{\mathrm{e}} \ln\left(\frac{M}{M-m}\right).$$

Assume no external forces act on the rocket. [6 marks]

(b) A future Europa sample recovery mission is composed of a lander and a sample return spacecraft, both of which are propelled by rockets. When it reaches the Jovian system, the sample return spacecraft is parked in a circular orbit of radius 2.17×10^6 km around Jupiter, and the lander sent down to Europa. It collects a sample of Europan water from beneath the icy crust of the moon, and starts its journey back to the sample return spacecraft. If this is via a Hohmann transfer orbit, show that the impulsive velocity changes are required to be:

(i) $\Delta v = 3.82\,\mathrm{km\,s^{-1}}$ to leave Europa and place the lander in the Hohmann transfer orbit;

(ii) $\Delta v = 2.39$ km s^{-1} to place the lander in the same orbit as the sample return spacecraft.

[10 marks]

[The mass of Europa is 4.80×10^{22} kg, and its radius is 1.57×10^3 km. You may assume Europa follows a circular orbit around Jupiter of radius 6.71×10^5 km. The mass of Jupiter is 1.90×10^{27} kg.]

(c) The lander has an unladen mass of 1.12×10^3 kg, and a rocket exhaust velocity of 3.10 km s^{-1}. If the total mass (including fuel) of the lander is 8.33×10^3 kg when it lifts off from Europa, calculate the mass of Europan water that the lander is able to return to the sample return spacecraft. [4 marks]

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7. (a) Sketch the P and S wave shadows cast by the Earth's core as the waves propagate away from an earthquake with an epicentre close to the Earth's surface. Explain why these results imply the presence of a liquid outer core in the Earth, whose density is higher than the surrounding mantle. [4 marks]

   (b) Eris is the largest known object in the new class of dwarf planets, with a mass of 1.67×10^{22} kg and a radius of 1.16×10^{3} km. Given that it is composed of a rocky core of uniform density $\rho_{\rm c} = 4.50 \times 10^{3}$ kg m^{-3}, enveloped by an icy mantle of uniform density $\rho_{\rm m} = 1.00 \times 10^{3}$ kg m^{-3}, show that the core has a mass of 1.31×10^{22} kg. [6 marks]

   (c) Calculate the angular distance around the surface of Eris, as measured from the centre of the dwarf planet, at which the P and S wave shadow from a seismic event would first be detected. [4 marks]

   (d) The surface temperature of Eris is 20.0 K at perihelion. Calculate the minimum mass of a molecule that Eris would be able to retain in its atmosphere, and comment on the likelihood of observing such an atmosphere around Eris. [6 marks]

8. (a) Show that the maximum observed radial velocity change of a star, K, induced by the barycentric motion of a star-planet system, is related to the orbital period P, the inclination of the star-planet orbit to our line-of-sight i, and the masses of the star and planet M_* and M_p respectively, as

$$K = \left(\frac{2\pi G}{P}\right)^{\frac{1}{3}} \frac{M_\mathrm{p} \sin i}{(M_* + M_\mathrm{p})^{\frac{2}{3}}}$$

if their orbits are circular. [6 marks]

(b) A nearby ($d = 20.0$ pc) K0V star is observed to show sinusoidal radial velocity variations on two different timescales, measured from wavelength variations in a Mg I line with rest wavelength $\lambda = 518$ nm. The first variation has a period $P_1 = 3.71$ days and induces a maximum wavelength shift of 5.01×10^{-5} nm; and the second has a period of $P_2 = 21.4$ days and a maximum wavelength shift of 1.27×10^{-4} nm. Sketch the observed radial velocity variations of the star, and show that the limits on the masses of the planets causing the barycentric motion are $M_1 > 0.187 M_\mathrm{J}$ and $M_2 > 0.848 M_\mathrm{J}$. [6 marks]

[The K0V star has a mass of 0.776 $M_\odot$, and Jupiter mass $M_\mathrm{J} = 1.90 \times 10^{27}$ kg.]

(c) The *Gaia* satellite is able to detect astrometric shifts in the position of the star of at least 2.00×10^{-5} arcseconds. Use this to determine a further mass constraint on the longer-period planet in order for its presence to be inferred in *Gaia* data. If it were to be detected, determine whether transit observations would be possible for this system. [4 marks]

[Hint: the observed astrometric shift of a star θ (in arcseconds) is given by

$$\theta = \frac{M_\mathrm{p} a}{M_* d},$$

for a star at distance d (in pc), orbited by a planet of orbital semi-major axis a (in AU).]

(d) The effective temperature at the top of a planet's atmosphere, T_e, is related to the parent stellar temperature T_* and radius R_* as

$$T_\mathrm{e} = T_* \sqrt{\frac{R_*}{2a}} (1 - A)^{\frac{1}{4}},$$

for a planet of albedo A. Calculate the maximum eccentricity of a planetary orbit in order for it to lie entirely within the Habitable Zone for this K0V star, assuming the planet has an albedo $A = 0.500$. Comment on the habitability of such a planet. [4 marks]

[The K0V star has a radius of 5.92×10^8 m and a surface temperature of 5.24×10^3 K.]

