Mathematical Methods II
PDF 7

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Key Points

- Taylor series solutions

Series solutions to linear ODEs (ctud)

- **Series solutions:** Series solutions are a relatively straightforward way to assess the solution of an ODE at a given point. They can be truncated to give approximate local solutions to the ODE. Or they can be taken to the n^{th} degree in an effort to seek a general solution.

  Last time we looked at determining the nature of points of an ODE. We found that a given point $z = z_0$ can be an ordinary, regular singular or irregular singular point. Once we know the nature of a point we can decide how to approach a series solution for the ODE at that point.

  For ordinary points we can find the Taylor series of the ODE at that point. Singular points require a more general approach, where we generate a Frobenius series. We will focus on solving ODEs around ordinary points.

- **Taylor series:** The Taylor series is a series expansion of a function about a point. The 1D Taylor expansion of a real function $f(x)$ about a point $x = a$ is given by

  $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

  This expression can be considered as a statement that any real function of x can be represented as the sum of an infinite polynomial, as long as in a given range of x $f(x)$ is a continuous, single-valued function with continuous derivatives up to the n^{th} order (where $n = \infty$).

  The reason we have $x - a$ terms rather than just x terms is that it generalises the function, allowing easy access to information about the behaviour of the function near some point a, a distance from x, by testing the limits of the function as $x \to a$. We often simplify things by setting $a = 0$, producing what is known as a Maclaurin series.

  $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n + ...$$

The main advantage of a Taylor series is that it allows you to easily calculate the values of even highly complex functions. Here is $\ln x$ represented as a Taylor series

$$\ln x = \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} + \mathcal{O}(x^4)$$

$\mathcal{O}(x^n)$ means 'terms with orders of x to the power n and higher'. Notice that we cannot express $\ln x$ as a Maclaurin series, since we would have to divide by $a = 0$. The series is often expressed as $\ln|1+x|$ or $\ln|1-x|$ instead.

Here is e^x represented as a Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we test $e^1 = 2.718$ to 3 dp, with 5 terms from the series we get

$$e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708$$

- **Series solutions at ordinary points:** Recall the general form of the 2$^{\text{nd}}$ order complex homogeneous linear ODE

$$y'' + p(z)y' + q(z)y = 0.$$

We can express a solution to this equation as a Taylor series

$$y(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

If we reframe our coordinates and take z_0 as the origin ($z_0 = 0$) then we can simplify this equation, producing a Maclaurin series

$$y(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} a_n z^n.$$

Remember that this series will converge for $|z| < r$, where r is the radius of convergence, which is now simply the distance from $z = 0$ to the nearest singular point.

Since every solution has a finite value at an ordinary point it is always possible to obtain two independent solutions from which we can construct a general solution to the complex homogeneous linear ODE. Since we are dealing primarily with 2$^{\text{nd}}$ order ODEs it would be useful to know what the derivatives of the series solution w.r.t z are

$$y' = \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n$$

To get the RHS terms we are just adding 1 or 2 to each n term that appears, for the first and second derivatives respectively. We are *shifting the index*. Note that it would seem appropriate to start the left hand sums from $n = 1$ and $n = 2$ respectively, but since the first terms are 0 when $n = 0$ we can start from there.

e.g. PDF7.1 Find the series solutions about $z = 0$ of

$$y''(z) + y(z) = 0$$

Here, we can tell by inspection that $z = 0$ is an ordinary point ($p = 0$, $q = 1$) we can go on to find two independent solutions by making the substitutions

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}z^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}z^n + \sum_{n=0}^{\infty} a_n z^n = 0$$

Which we can rewrite as

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] z^n = 0$$

For this equation to work we require that each coefficient of z (the square bracket) is equal to zero. If they were were not, as long as $z \neq 0$ (the trivial case) the result will not match the zero RHS.

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad \text{for } n \geq 0.$$

This is a two-term recurrance relation that allows us to readily calculate the even coefficients if we start from a_0, or odd coefficients if we start from a_1. This in turn allows us to find two independent solutions of the ODE. We can set either $a_0 = 0$ or $a_1 = 0$.

Let's set $a_0 = 1$ and let $a_1 = 0$. So

$$a_0 = 1$$

$$a_1 = 0$$

$$a_2 = -\frac{a_0}{(0+2)(0+1)} = -\frac{1}{2!}$$

$$a_3 = 0$$

$$a_4 = -\frac{a_2}{(2+2)(2+1)} = -\frac{(-1/2)}{12} = \frac{1}{24} = \frac{1}{4!}$$

$$a_5 = 0$$

Similarly, setting $a_0 = 0$ and letting $a_1 = 1$ gives

$$a_0 = 0$$

$$a_1 = 1$$

$$a_2 = 0$$

$$a_3 = -\frac{a_1}{(1+2)(1+1)} = -\frac{1}{3!}$$

$$a_4 = 0$$

$$a_5 = -\frac{a_3}{(3+2)(3+1)} = \frac{1}{5!}$$

Since $y = \sum_{n=0}^{\infty} a_n z^n$, this gives the solutions

$$y_1(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = cosz$$

$$y_2(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = sinz$$

We can now say that our general solution to the ODE is

$$y = c_1 y_1 + c_2 y_2 = c_1 \cos z + c_2 \sin z$$

We were able to express this solution in a *closed form* (i.e. in terms of elementary functions) - this is not usually the case!

e.g. PDF7.2 Find the series solutions about $z = 0$ of

$$y''(z) - \frac{2}{(1-z)^2} y(z) = 0$$

Again, by inspection we can tell that $z = 0$ is an ordinary point, so we can find two independent solutions by substituting

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}$$

If we sub these into the ODE and mulitply by an expanded $(1-z)^2$ to remove the fraction, we get

$$(1 - 2z + z^2) \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} a_n z^n = 0.$$

Since we have used the negative index term substitutions, when we multiply out the brackets we won't have any terms higher than z^n

$$\sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} n(n-1) a_n z^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n z^n - 2 \sum_{n=0}^{\infty} a_n z^n = 0$$

Now we need to shift the index of each term, so we have only terms in z^n

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n - 2\sum_{n=0}^{\infty}n(n+1)a_{n+1}z^n + \sum_{n=0}^{\infty}(n^2-n-2)a_n z^n = 0$$

Reducing this to a single sum we can write

$$\sum_{n=0}^{\infty}(n+1)[(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n]z^n = 0$$

Just like the previous example we require that the coefficient of z^n must be zero at each n

$$(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n = 0 \qquad \text{for } n \geq 0.$$

This means we can determine a_2 in terms of a_0 and a_1, and so on for $n \geq 2$. This is a three-term recurrence relation. Three-term recurrence relations and higher are generally a nuisance to solve, however this one has two simple solutions. First lets choose $a_n = a_0$ for all n. If we test this with $a_0 = 1$ we see that it satisfies the condition for the coefficient

$$(n+2)\times 1 - 2n \times 1 + (n-2)\times 1 = 2n - 2n + 2 - 2 = 0$$

So since $a_n = a_0 = 1$ for all n, recalling $y = \sum_{n=0}^{\infty} a_n z^n$, we can write the first solution as

$$y_1(z) = 1 + z + z^2 + z^3 + ... = \frac{1}{1-z}$$

i.e. the sum of an infinite geometric series (for $-1 \leq z \leq 1$, since it would otherwise diverge). The second solution can be found if $a_1 = -2a_0$, $a_2 = a_0$ and $a_n = 0$ for $n > 2$. If again we set $a_0 = 1$, we find that

$$y_2(z) = 1 - 2z + z^2 = (1-z)^2$$

which is a polynomial solution to the ODE. Thus our general solution is

$$y(z) = \frac{c_1}{1-z} + c_2(1-z)^2$$

Just as a check, let's test if our solutions are independent using the Wronskian.

$$W = y_1y_2' - y_1'y_2 = \frac{1}{1-z}[-2(1-z)] - \frac{1}{(1-z)^2}(10z)^2 = -3$$

$W \neq 0$, so y_1 and y_2 are linearly independent.