A hydrogen atom is subjected to a perturbation from an external electric field E_{ext} in the z direction, hence $H' = eE_{\text{ext}} r\cos\theta$.

(a) The unperturbed $n = 1$ level in Hydrogen is non-degenerate (ignoring spin) and has wavefunction $\psi^0_{100} = (\pi a^3)^{-1/2} e^{-r/a}$. Show that there is no first order correction to the ground state energy, i.e., $E^1_1 = \int \psi^{0*}_{100} H' \psi^0_{100} dV = 0$

(b) The unperturbed $n = 2$ level in Hydrogen is 4-fold degenerate, with energy eigenfunctions

$$
\begin{aligned}
\varphi_1 \equiv& \psi^0_{200} = \sqrt{\frac{1}{2\pi a}\,\frac{1}{2a}}\left(1 - \frac{r}{2a}\right) e^{-r/2a} \\
\varphi_2 \equiv& \psi^0_{211} = -\frac{1}{\sqrt{\pi a}}\,\frac{r}{8a^2} e^{-r/2a} \sin\theta e^{i\phi} \\
\varphi_3 \equiv& \psi^0_{210} = \frac{1}{\sqrt{2\pi a}}\,\frac{r}{4a^2} e^{-r/2a} \cos\theta \\
\varphi_4 \equiv& \psi^0_{21-1} = \frac{1}{\sqrt{\pi a}}\,\frac{r}{8a^2} e^{-r/2a} \sin\theta e^{-i\phi}
\end{aligned}
$$

Use degenerate perturbation theory to determine the first order correction, E^1_2, to the $n = 2$ level by writing down a 4×4 matrix equation with terms $W_{jk} = \langle\varphi_j|H'|\varphi_k\rangle$ for $j, k = 1, 2, 3, 4$. [Hint: do the angle integrals first, as many of these turn out to be zero!]

(c) Solve the matrix to determine all possible values of E^1_2. Into how many different energy levels does $n = 2$ split?

(d) Write down the wavefunctions $\chi = \alpha|\varphi_1\rangle + \beta|\varphi_2\rangle + \gamma|\varphi_3\rangle + \delta|\varphi_4\rangle$ for which we could have used non-degenerate perturbation theory.

Useful Integrals:

$$\int_0^\infty x^p e^{-qx} dx = \frac{p!}{q^{p+1}}$$