PHYS2581 Foundations 2A: QM2.1 solution

i) Normalisation requires

$$\int \psi^* \psi \, dx = 1 \qquad \Rightarrow \qquad \int_{-\infty}^{\infty} A^2 \exp(-2ax^2)\, dx = 1$$

Wolfram alpha gives $\int_{-\infty}^{\infty} \exp(-2ax^2)\, dx = \sqrt{\frac{\pi}{2a}}$

so $A^2 = \sqrt{2a/(\pi)} \quad \Rightarrow \quad A = \left(\frac{2a}{\pi}\right)^{1/4}$ [1 mark]

ii) The expectation value

$$\langle x \rangle = A^2 \int_{-\infty}^{\infty} x \exp(-2ax^2)\, dx$$

[1 mark]

Wolfram: $\int_{-\infty}^{\infty} x \exp(-2ax^2)\, dx = 0$ so $\langle x \rangle = 0$ (or zero by symmetry about $x = 0$) [1 mark]

iii) The expectation value of the momentum,

$$\langle p \rangle = A^2 \int_{-\infty}^{\infty} \exp(-ax^2) \times -i\hbar \frac{d}{dx}(\exp(-ax^2))\, dx = -i\hbar A^2 \int_{-\infty}^{\infty} (-2ax) \exp(-ax^2)\, dx$$

[2 marks]

Wolfram (or otherwise e.g. integrand is an odd function of x): $= 0$ [1 mark]

iv) The expectation value of the square of the momentum

$$\langle p^2 \rangle = A^2 \int_{-\infty}^{\infty} \exp(-ax^2) \times -\hbar^2 \frac{d^2}{dx^2} \left(\exp(-ax^2)\right)\, dx$$

[1 mark]

$$\langle p^2 \rangle = -\hbar^2 A^2 \int_{-\infty}^{\infty} \exp(-ax^2) \frac{d}{dx} \left(-2ax \exp(-ax^2)\right)\, dx$$

$$\langle p^2 \rangle = -\hbar^2 A^2 \int_{-\infty}^{\infty} \exp(-ax^2) \left((4a^2x^2 - 2a) \exp(-ax^2)\right)\, dx = \hbar^2 A^2 \sqrt{\frac{\pi a}{2}}$$

by Wolfram of otherwise [1 mark]

$$\langle p^2 \rangle = \hbar^2 \sqrt{\frac{2a}{\pi}} \sqrt{\frac{\pi a}{2}} = \hbar^2 a$$

[1 mark]

v) Hence

$$\langle T \rangle = \frac{\langle p^2 \rangle}{(2m)} = \frac{\hbar^2 a}{2m}$$

[1 mark]