QUANTUM MECHANICS 2 - WORKSHOP 4

The common eigenfunctions of the angular momentum operators L^2 and L_z are the spherical harmonics, Y_{lm}, which satisfy

$$L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$$

and

$$L_z Y_{lm} = m\hbar Y_{lm}.$$

The associated ladder operators are $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$ for which

$$L_\pm Y_{lm} = (L_x \pm iL_y) Y_{lm} = A_{lm\pm} Y_{lm\pm1},$$

where $A_{lm\pm}$ is a constant which depends on l, m and which of L_+ or L_- we are using.
So for example

$$L_- Y_{22} = A_{22-} Y_{21}.$$

a) Use the definitions above to explicitly show that $[L^2, L_z]Y_{lm} = 0$. If the observable L^2 is measured to be $30\hbar^2$, what are the possible values of a measurement of L_z?

b) Show that $L_x = \frac{1}{2}(L_+ + L_-)$. Use this, together with the orthornormal properties of wavefunctions, to calculate $\langle L_x \rangle$ for any spherical harmonic Y_{lm}.

c) Give a general argument showing that the eigenfunctions of L_+L_- are $\propto Y_{lm}$. Rewrite L_+L_- in terms of $L^2 = L_x^2 + L_y^2 + L_z^2$ and L_z remembering

$$[L_x, L_y] = i\hbar L_z.$$

Hence write down the eigenvalues of L_+L_-. Show explicitly that this is consistent with $A_{lm\pm} = \hbar\sqrt{(l(l+1) - m(m \pm 1)}$, i.e. $A_{lm-} = \hbar\sqrt{(l(l+1) - m(m-1)}$ and $A_{lm+} = \hbar\sqrt{(l(l+1) - m(m+1)}$.

d) Use the definition of L_x in b) above and the definition of $A_{lm\pm}$ in c) above to find the values of the coefficients a, b, c in order that $L_x\psi = \hbar q\psi$ for $\psi = aY_{11} + bY_{10} + cY_{1-1}$ Solve this explicitly to find the normalised eigenfunctions of L_x for $q = 1, 0, -1$.