Workshop 7: Canonical Transformation to a Rotating Frame

Consider a point particle moving in 2 dimensions held within a rotating (with angular frequency ω) anisotropic harmonic potential $V(x,y) = (m/2)\{\omega_1^2[x\cos(\omega t) - y\sin(\omega t)]^2 + \omega_2^2[y\cos(\omega t) + x\sin(\omega t)]^2\}$ (such a potential is sometimes used to generate superfluid vortices in Bose-Einstein condensed atomic gases).

1. Determine the Hamiltonian H from the Lagrangian L for this system, using a Legendre transformation.

2. Consider a generating function of the form $F_2(x,y,P_X,P_Y) = P_X[x\cos(\omega t) - y\sin(\omega t)] + P_Y[y\cos(\omega t) + x\sin(\omega t)]$. Determine expressions for p_x, p_y, X, Y, from the implicit transformation equations (X and Y are the "new" coordinates, and P_X, P_Y their canonically conjugate momenta).

3. Determine the transformed Hamiltonian $H'(X,Y,P_X,P_Y)$ (which should be time-independent), in terms of the new coordinates and momenta, using the expressions you have calculated for p_x, p_y, X, Y.

4. Use Hamilton's equations to determine equations for $\dot{X}$, $\dot{P}_X$. In the case where $\omega = 0$, find the coupled equations of motion for X, P_X, in terms of their initial values.

5. What is the value of the Poisson bracket $\{X, P_X\}$? Justify your answer.