University of Durham

EXAMINATION PAPER

May/June 2014 Examination code: 043641/01

LEVEL 3 PHYSICS: ADVANCED PHYSICS 3

SECTION A. Soft Condensed Matter Physics
SECTION B. Optical Properties of Solids
SECTION C. Modern Atomic and Optical Physics

Time allowed: 3 hours

Examination material provided: None

Calculators: The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. SOFT CONDENSED MATTER PHYSICS
Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) A mixture of two fluids has a free energy per unit volume $f_{\rm m}(\phi, \chi)$, for a uniform mixture of composition ϕ. Here χ is an externally imposed control parameter, which depends on temperature. Sketch a typical curve of $f_{\rm m}(\phi, \chi)$ as a function of ϕ for a fixed value of χ for which demixing occurs. Mark the regions in which an intially homogenous mixture is (i) linearly unstable to the growth of small perturbations that lead to demixing, (ii) metastable to the onset of demixing and (iii) stable to the onset of demixing. [4 marks]

(b) A simple lattice model of a mixture of two polymeric fluids, each comprising polymer chains with a degree of polymerization N, has a free energy per unit volume given by

$$f_{\rm m} = \frac{1}{N}\phi \ln(\phi) + \frac{1}{N}(1-\phi)\ln(1-\phi) + \chi\phi(1-\phi).$$

(i) Which contributions to this free energy are of entropic origin, and which are of energetic origin? [2 marks]

(ii) Show that the coexistence line in the phase diagram is given by

$$\frac{1}{1-2\phi}\ln\left(\frac{\phi}{1-\phi}\right) = -N\chi.$$

[2 marks]

(c) The freely jointed chain model of a polymer molecule has N links, each of length a, with the orientations of successive links being uncorrelated with each other. By considering the statistical properties of the chain's end-to-end vector, calculate the root-mean-squared end-to-end distance of the chain, which gives a measure of the "size" of the molecule. [4 marks]

(d) A block of rubber of initial dimensions L_x, L_y, L_z undergoes an extensional deformation along the x axis, leading to new sample dimensions of $\lambda_x L_x, \lambda_y L_y, \lambda_z L_z$ with $\lambda_x > 1$, $\lambda_y < 1$ and $\lambda_z < 1$. This results in an increase in the material's free energy per unit volume given by

$$\Delta f = \frac{nk_{\rm B}T}{2}(\lambda_x^2 + \lambda_y^2 + \lambda_z^2 - 3).$$

Here n is the number of cross-linked chain-like molecular strands per unit volume, and T is temperature. Considering the case of a uniaxial ($\lambda_y = \lambda_z$) and incompressible ($\lambda_x\lambda_y\lambda_z = 1$) deformation, calculate the elongational modulus of the material by taking the derivative of Δf with respect to a strain variable e, defined such that $\lambda_x = 1 + e$. [4 marks]

(e) (i) Sketch the phase diagram for an A_mB_n block copolymer showing the different phases as a function of percentage of the polymer that comprises monomers of type A and type B and their level of mutual incompatibility. [3 marks]

(ii) Explain why interfaces between areas with a high concentration of A and of B are not expected to be flat. [1 mark]

2. (a) Sketch the steady state flow curve of shear stress σ as a function of shear rate $\dot{e}$ for a Newtonian fluid. What feature of this curve gives the fluid's viscosity? [2 marks]

(b) Sketch a typical steady state flow curve for a shear thinning fluid. Discuss how the apparent viscosity of this fluid varies with the imposed shear rate $\dot{e}$, and how this can be seen from the shape of the curve you have sketched. [2 marks]

(c) Following the imposition to a soft material of a step shear strain of small amplitude e_0 at time $t = 0$, such that the shear strain $e(t) = e_0\Theta(t)$ where $\Theta(t)$ is the Heaviside step function, the material's shear stress response is given by $\sigma(t) = e_0 G(t)$ where $G(t)$ is known as the stress relaxation function.

A particular model for the rheology of soft materials has a stress relaxation function given by

$$G(t) = G_0 \int_{\tau_0}^{\infty} d\tau P(\tau) \exp(-t/\tau),$$

with a spectrum of relaxation timescales

$$P(\tau) = (x-1)\tau_0^{x-1}\tau^{-x},$$

for $\tau \geq \tau_0$. Here τ_0 is a constant microscopic timescale, G_0 is a constant modulus and x is a dimensionless control parameter, which you should assume adopts only values $x > 1$.

(i) For $x > 2$ and small values of the flow rate $\dot{e}$, the model's steady state flow curve is given by $\sigma = \eta\dot{e}$ where $\eta = \eta(x) = \int_0^\infty dt G(t)$. Calculate $\eta(x)$ and comment on its behaviour as x approaches the value 2 from above. [4 marks]

(ii) For $1 < x < 2$ the steady state flow curve is given by $\sigma = A\dot{e}^{x-1}$ where A is a constant. Sketch this curve and comment on how its behaviour in the limit of small $\dot{e}$ is consistent with your comment in (i) above. [4 marks]

(iii) Calculate the time-dependence of the model's stress relaxation function $G(t)$ for times $t/\tau_0 \gg 1$, recognising that the integral

$$\int_0^{\Delta} ds\, s^{x-2} \exp(-s)$$

tends to a constant as $\Delta \to \infty$. [8 marks]

3. (a) The chemical potential μ, of aggregates size N of an amphiphile is given by,

$$\mu_N = \mu = \epsilon_N + \frac{k_{\rm B}T}{N} ln \frac{X_N}{N}$$

where X_N is the volume fraction of aggregate of size N.

(i) Define ϵ_N and describe the origin of both terms of the chemical potential. [3 marks]

(ii) How must ϵ_N vary with N in order for stable aggregation to occur and why? [1 mark]

(b) For a bilayer,

$$\epsilon_N = \epsilon_\infty + \frac{\alpha k_{\rm B}T}{\sqrt{N}}$$

were α is a constant.

(i) With reference to the geometry of a disk, derive the dependence of ϵ_N on $\frac{1}{\sqrt{N}}$. [2 marks]

(ii) By writing down the equation in part (a) for $N = 1$ and $N = m$, find the volume fraction of aggregates size m as a function of the volume fraction of aggregates of size one for a bilayer and hence show that smaller bilayers are unstable with respect to larger ones. [4 marks]

(c) A bilayer sheet lies in the $x - y$ plane with length 1 in the y direction and length L in the x direction. It is fixed along the lines $x = 0$ and $x = L$.

(i) Write down a function $H(x)$ that describes height of the bilayer as a function of position in the x direction for those sinusoidal fluctuations which obey $H(x) = 0$ at $x = 0, L$ (assume the curvature is zero in the y direction). [2 marks]

(ii) For a bending modulus, $k = 20k_{\rm B}T$, derive an equation for the energy associated with each of these fluctuations. [6 marks]

[Hint: The local energy contribution due to bending $E_{bend} = \frac{k}{2}(c_1 + c_2)^2$ and you may assume that local curvature is well approximated by the local second derivative of $H(x)$]

(iii) Using the equipartition theorem estimate the amplitude of the fluctuations and hence describe the overall shape of the bilayer as a function of the length scale at which it is observed. [2 marks]

SECTION B. OPTICAL PROPERTIES OF SOLIDS
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) Titanium dioxide may be considered to exhibit zero absorption and has a (power) reflection coefficient $R = 0.121$ at a wavelength of 1550 nm for light at normal incidence from a vacuum. What is its relative permittivity at this wavelength? [4 marks]

(b) Graphene is essentially a 2-dimensional array of carbon atoms in which the bonds between carbon atoms may be considered as built up from sp^2 hybrid orbitals. If one of these orbitals is of the form

$$|sp^2\rangle = \frac{1}{\sqrt{3}}|s\rangle + \frac{1}{\sqrt{6}}|p_x\rangle + \frac{1}{\sqrt{2}}|p_y\rangle$$

where the individual s and p orbitals may be assumed to be orthonormal and have energies $|\epsilon_s\rangle = -17.52$ eV, $|\epsilon_\mathrm{p}\rangle = -8.97$ eV respectively, what is the energy of this sp^2 hybrid orbital? [4 marks]

(c) The standard expression for the free electron density of states is

$$D(E) = \frac{1}{2\pi^2}\left(\frac{2m_\mathrm{e}}{\hbar^2}\right)^{3/2} E^{1/2}.$$

Given that a direct gap semiconductor has a band gap of 1 eV and may be assumed to have parabolic conduction and valence bands characterised by effective masses 0.1 m_e and 0.2 m_e respectively, what is the numerical value of the joint density of states at photon energy 1.5 eV? [4 marks]

(d) Define the term luminescent efficiency (η_R), explaining the significance of any symbols employed in doing so in relation to the associated rate equations. [4 marks]

(e) In response to an applied time-dependent electric field $E = E_\mathrm{o}\cos(\omega \mathrm{t})$ a particular material exhibits an induced polarisation which is well approximated by

$$P = \epsilon_\mathrm{o}\left(\chi^{(1)}E + \chi^{(2)}E^2\right),$$

where $\chi^{(1)}$ and $\chi^{(2)}$ are first and second order susceptibilities respectively. Demonstrate why this leads to the phenomenon of second harmonic generation (SHG). [4 marks]

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5. Unpolarised light (i.e. light of general polarisation) is incident from a vacuum normal to the planar surface of a birefringent material which has its principal optic axis (POA) at an angle θ to the normal. Describe the general nature of the resultant light propagation within the birefringent material when (a) $\theta = 0°$, (b) $\theta \neq 0°$. [6 marks]

   A particular birefringent material is characterised by ordinary and extraordinary refractive indices $n_{\mathrm{o}}, n_{\mathrm{e}}$ respectively and has a POA at angle $\theta = 15°$ to the normal to its surface. An electromagnetic wave propagating within the material may be taken to be of the form $\underline{E}\,(\underline{r}, t) = \underline{E}_{\mathrm{o}} e^{i(\underline{k}\cdot\underline{r}-\omega t)}$ where $\underline{E}_{\mathrm{o}}, \underline{k}$ and ω give the polarisation direction, wavevector and angular frequency respectively. Make a sketch of the scenario and determine the direction of $\underline{k}$ if $\underline{E}_{\mathrm{o}}$ within the material is polarised parallel to the surface and $\underline{k}$ and $\underline{E}_{\mathrm{o}}$ both lie in the plane defined by the normal to the surface and the direction of the POA. [14 marks]

   [Hint: consider the implications of $\nabla \cdot (\epsilon \underline{E}) = 0$, with ϵ in the form of an appropriate permittivity tensor.]

6. An ionic material has a frequency dependent relative permittivity of the form

$$\tilde{\epsilon}_{\text{rel}} = \epsilon_1(\nu) + i\epsilon_2(\nu) = \epsilon_\infty + \frac{(\epsilon_{\text{st}} - \epsilon_\infty)\nu_{\text{TO}}^2}{(\nu_{\text{TO}}^2 - \nu^2 - i\gamma\nu)}$$

in the infrared region of the spectrum.

(a) Explain the significance of the various terms in the above expression. [4 marks]

(b) By employing an appropriate assumption demonstrate how the well known Lyddane-Sachs-Teller (LST) relationship can be obtained from the above expression. [6 marks]

(c) Obtain separate expressions for $\epsilon_1(\nu)$ and $\epsilon_2(\nu)$. Given the specific values $\epsilon_{\text{st}} = 15, \epsilon_\infty = 11, \nu_{\text{TO}} = 7.0 \times 10^{12}$ Hz and a small but finite value for γ, sketch and explain the form of $\epsilon_1(\nu)$ indicating key values on your sketch. In particular, identify the Reststrahlen band frequency region. [10 marks]

SECTION C. MODERN ATOMIC AND OPTICAL PHYSICS
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) The ion in the $^{27}\mathrm{Al}^{+}$ ion quantum logic clock is in the Lamb-Dicke regime. Discuss, briefly, how the first-order Doppler shift can be eliminated under this regime. [4 marks]

(b) Calculate the maximum lifetime that the upper state in the $^{1}S_{0} \rightarrow {}^{3}P_{1}$ transition at 657 nm in $^{40}\mathrm{Ca}$ must have in order to levitate the atoms against gravity using a resonant laser. [4 marks]

(c) The $^{2}S_{1/2} \rightarrow {}^{2}P_{3/2}$ cooling transition used in a magneto-optical trap (MOT) of $^{7}\mathrm{Li}$ at 671 nm has a linewidth of $\Gamma = 2\pi(5.9\ \mathrm{MHz})$. The Doppler velocity of the MOT is given by

$$v_{\mathrm{D}} = \left(\frac{\hbar\Gamma}{M}\right)^{\frac{1}{2}}.$$

Calculate the recoil velocity v_{r}. [4 marks]

(d) Explain, briefly, why the fine structure splitting in atoms is larger than the hyperfine splitting by a factor approximately equal to the proton-to-electron mass ratio. [4 marks]

(e) Discuss, briefly, the relative merits of ion quantum logic clocks versus neutral atom optical lattice clocks. [4 marks]

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8. The ^{87}Sr optical lattice clock involves two stages of laser cooling to reach temperatures of $\sim 1\ \mu$K. The $5s\ ^1S_0 \rightarrow 5s5p\ ^1P_1$ "blue" transition at 461 nm is used in the first cooling stage and the $5s\ ^1S_0 \rightarrow 5s5p\ ^3P_1$ "red" transition at 689 nm is used in the second cooling stage. The natural linewidths of the two cooling stages are $\Gamma_{\rm B} = 32$ MHz and $\Gamma_{\rm R} = 7.1$ kHz, respectively.

   (a) Explain why two transitions are required to reach $\sim 1\ \mu$K. [4 marks]

   (b) Once the ^{87}Sr atoms are cooled to $\sim 1\ \mu$K, they are loaded into a "magic wavelength" ($\lambda_{\rm m} = 813$ nm) optical lattice, where the $\Gamma_{\rm C} = 1$ mHz linewidth $5s\ ^1S_0 \rightarrow 5s5p\ ^3P_0$ "clock" transition at 698 nm is interrogated. Explain why the clock transition has such a narrow linewidth, what the term "magic wavelength" means and why the optical lattice is required to be at this wavelength. [6 marks]

   (c) Assume that the magic wavelength optical lattice is "one-dimensional", formed from the interference of a retro-reflected focused laser beam of power $P = 150$ mW propagating along the z-axis. The intensity profile is given by

$$I(r,z) = \frac{3P}{2\pi w(z)^2}\cos^2\left(\frac{2\pi z}{\lambda_{\rm m}}\right)\exp\left(-\frac{2r^2}{w(z)^2}\right) + \frac{P}{\pi w(z)^2}\exp\left(-\frac{2r^2}{w(z)^2}\right).$$

   The beam waist is $w(z) = w_0(1 + z^2/b^2)^{1/2}$, where $w_0 = 50\ \mu$m is the beam waist at the radially smallest part of the lattice, $b = \pi w_0^2/\lambda_{\rm m}^2$ and $r^2 = x^2 + y^2$. For an atom interacting with the lattice beam, the ratio of $I(r,z)$ to the saturation intensity $I_{\rm sat}$ is given by $I(r,z)/I_{\rm sat} = 2\Omega^2/\Gamma^2$, where Ω is the Rabi frequency and Γ is the linewidth of the strongest atomic transition of wavelength λ in the vicinity of the lattice laser wavelength and $I_{\rm sat} = \pi hc/(3\lambda^3\tau)$. The maximum potential experienced by the atom is $U_{\rm max} = \hbar\Omega^2/(4\delta)$, where δ is the detuning of λ from $\lambda_{\rm m}$ and τ is the lifetime of the upper state of the strong transition. Derive an expression for $U_{\rm max}$ in terms of the lattice laser power P and beam waist w_0, and evaluate $U_{\rm max}$. [10 marks]

9. The wavefunctions of the 1s0 and 2p0 states of the hydrogen atom are

$$\psi_{1\text{s}0}(r,\theta,\phi) = Y_{0,0}R_{1,0} = \frac{1}{\sqrt{\pi}a_0^{3/2}}e^{-r/a_0} \qquad \text{and}$$

$$\psi_{2\text{p}0}(r,\theta,\phi) = Y_{1,0}R_{2,1} = \frac{1}{4\sqrt{2\pi}a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}\cos\theta,$$

where the Bohr radius $a_0 = 5.29 \times 10^{-11}$ m.

(a) Show graphically that the electric dipole moment d associated with the 1s0→2p0 transition is non-zero. [4 marks]

(b) Calculate the electric dipole moment associated with the 1s0→2p0 transition. [12 marks]

[Hint: The dipole moment operator is $\mathbf{d} = -e\hat{\mathbf{z}}$. Use spherical polar coordinates where the Jacobian is $r^2 \sin\theta$. The following integral may be useful

$$\int r^4 e^{-br} dr = -\frac{1}{b^5}\left[e^{-br}\left(b^4r^4 + 4b^3r^3 + 12b^2r^2 + 24br + 24\right)\right],$$

where $b = 3/(2a_0)$.]

(c) The Einstein A-coefficient for an electric dipole transition is

$$A = \frac{\omega_0^3 d^2}{3\pi\epsilon_0\hbar c^3}.$$

The 1s0→2p0 transition occurs at a wavelength $\lambda = 121.6$ nm. Calculate the lifetime of the 2p0 state as it decays to the 1s0 state by spontaneous emission. [4 marks]

