University of Durham

EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2018	PHYS3651-WE01

Title:
Planets and Cosmology 3

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper.
- Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** the answers that are not to be marked.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Cosmology
Section B: Planetary Systems

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A: COSMOLOGY

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) In Big Bang models based on the Friedmann equation, how does the age of the Universe scale with the value of the Hubble constant, H_0, if other parameters are unchanged? For a given H_0, explain qualitatively, with physical reasoning, the dependences of the age on the current density parameter of matter, $\Omega_{M,0}$, and of the cosmological constant, $\Omega_{\Lambda,0}$. [4 marks]

(b) A universe with Hubble constant H_0 and present-day density parameters in matter, radiation and cosmological constant of $\Omega_{M,0}$, $\Omega_{\gamma,0}$ and $\Omega_{\Lambda,0}$, respectively, is described by the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3},$$

where k and Λ are constants. Derive expressions for the cosmological constant Λ, and the Hubble parameter at redshift $z = 1$, in terms of some or all of H_0, $\Omega_{M,0}$, $\Omega_{\gamma,0}$ and $\Omega_{\Lambda,0}$. [4 marks]

(c) A population of identical disk galaxies is distributed uniformly in space. Their rotation curves are flat in their outer parts with amplitudes of ± 200 km s^{-1}, and the radial extent of their dark haloes is 10 per cent of the average separation between these galaxies. If they contribute $\Omega_g = 0.2$ to the local mass density parameter, estimate their space density in Mpc^{-3} for $H_0 = 70$ km s^{-1} Mpc^{-1}. [4 marks]

(d) In a flat Friedmann-Robertson-Walker universe, the flux f that an observer on the Earth receives from a source at redshift z with luminosity L is given by

$$f = \frac{L}{4\pi r^2(1+z)^2},$$

where r is the comoving distance between the observer and the source. Explain the origin of the $(1+z)^2$ factor. [4 marks]

(e) For a universe containing only matter and a cosmological constant, using the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + \frac{1}{3}\Lambda,$$

where ρ and P are respectively the density and pressure of matter, and Λ is the cosmological constant, find the value of the deceleration parameter $q = -(\ddot{a}/a)/(\dot{a}/a)^2$ at the time when matter and the cosmological constant had a density ratio of 2:1. Was the expansion decelerating or accelerating at that time? [4 marks]

(f) In a flat Friedmann-Robertson-Walker universe, an object at redshift $z = 0.2$ is observed to have an angular diameter of $\Theta = 0.5$ arcseconds. If the luminosity distance of the object is known to be $d_L = 1000.0$ Mpc, determine its comoving diameter. [4 marks]

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(g) Using the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2},$$

where $\rho(t)$ is the total density of all matter species at time t, show that the total density parameter of all matter species, $\Omega(t) = 8\pi G\rho(t)/3H^2$, satisfies

$$|\Omega(t) - 1| = \frac{|k|c^2}{\dot{a}^2}.$$

Explain what the flatness problem is and why a cosmological model with an inflationary period driven by a cosmological constant in the very early universe solves it. [4 marks]

(h) In the radiation dominated era, assuming that the curvature of the universe can be neglected, use the Friedmann equation to find the expression for the expansion rate $H(t)$ as function of the cosmic time t. Before Big Bang Nucleosynthesis, when the temperature T of the universe was $T = 10^{10}$ K, the density of the universe $\rho(t)$ satisfied

$$\rho(t)c^2 = \frac{1}{2}g_* \frac{4\sigma T^4}{c},$$

where $g_* = 10.75$ is the number of effective bosonic relativistic degrees of freedom. Find the age of the universe at this time in seconds. [4 marks]

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2. (a) Summarise briefly one piece of observational evidence which suggests that the expansion of the Universe is accelerating due to the presence of dark energy. [4 marks]

(b) Some forms of dark energy may be described by a contribution to the mass-energy density ρ with equations of state of the form $P = w\rho c^2$, where w is a constant and P is pressure, and which obey the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0.$$

Observations indicate that w is close to -1, the value for a cosmological constant.

(i) If $w = -1 + \delta$, where δ is a constant, show that $\rho \propto a^{-3\delta}$. [2 marks]

(ii) If $\delta < 0$ show that a universe containing only dark energy ($\Omega_{DE,0} = 1$) will undergo a 'big rip', reaching infinite size after a time $\Delta t = 2/(3|\delta|H_0)$ from now. [4 marks]

(c) For a universe with present-day $\Omega_{\Lambda,0} = 0.7$, $\Omega_{M,0} = 0.3$, $H_0 = 70$ km s^{-1} Mpc^{-1} and $\delta = 0$, we may define a Λ-era as commencing when $\Omega_\Lambda = 10\ \Omega_M$. By what factor will the universe have expanded by the time this epoch begins? [3 marks]

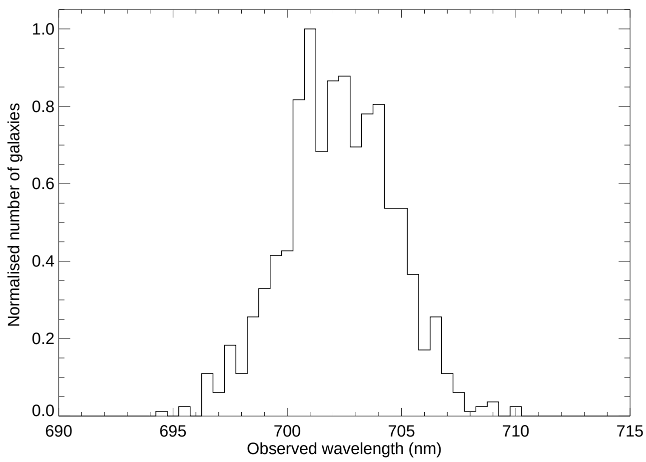
(d) Consider a pair of galaxies following the Hubble expansion in the universe of part (c). Following the start of this Λ-era (in which you may neglect the matter contribution to ρ), the galaxies communicate with each other via electromagnetic signalling. Show that there is a finite maximum value of the expansion factor, a_{last}, following which any emitted signal will no longer reach the other galaxy. Evaluate a_{last} if the current separation of the galaxy pair is 100 Mpc. [7 marks]

3. (a) A cluster of galaxies spans a region of sky approximately 1 degree in diameter. Its constituent galaxies emit the Hα emission line (rest-wavelength 656.3 nm) with the distribution of observed wavelengths shown below. Stating any assumptions and the main sources of uncertainty estimate:

(i) the redshift of the cluster; (ii) its internal 3-D velocity dispersion in km s^{-1}; (iii) its physical radius in Mpc; (iv) its total mass in solar masses. [10 marks]

Assume $H_0 = 70$ km s^{-1} Mpc^{-1} and a matter-only universe with density parameter $\Omega_{M,0} = 0.3$, and the following relationship between angular diameter distance, d_A and redshift:

$$d_A \simeq \frac{1}{1+z}\left(\frac{c}{H_0}\right)\left[z - \frac{z^2}{2}\left(1 + \frac{\Omega_{M,0}}{2}\right)\right].$$



(b) Cosmological theory suggests that the average density of matter within the boundaries of a gravitationally-collapsed structure such as a galaxy cluster should be $\alpha\Omega_M(z)^{-0.7}$ times the cosmic mean mass density at this redshift, where α is a constant and $\Omega_M(z)$ is the matter density parameter at redshift z. Starting from the Friedmann equation, show that:

$$\Omega_M(z) = \frac{\Omega_{M,0}(1+z)}{1+\Omega_{M,0}z}.$$

[4 marks]

(c) Use the results of (a) and (b) to estimate the constant α. Discuss briefly one other method for determining the cluster's mass, including the observations required and any necessary assumptions. [6 marks]

4. The mass of a proton is $m_p = 938.27$ MeV$/c^2$, and that of a neutron is $m_n = 939.57$ MeV$/c^2$. In the very early Universe, neutrons and protons were in thermal equilibrium maintained by weak interactions, and their number densities satisfied the following equilibrium relation

$$\frac{n_n}{n_p} = \exp\left(-\frac{\Delta m c^2}{k_{\mathrm{B}} T}\right),$$

where $\Delta m = m_n - m_p$ and T is the temperature of the Universe.

(a) Explain why this thermal equilibrium could not be maintained forever and the meaning of freeze-out. [4 marks]

(b) Freeze-out took place when the temperature T_f was given by $k_{\mathrm{B}} T_f = 0.7$ MeV. Neglecting free neutron decay so that all neutrons at freeze-out ended up in Helium-4 nuclei during the Big Bang nucleosynthesis (BBN), show that the mass fraction of Helium-4 nuclei immediately after BBN, Y_{He}, is 0.27. [4 marks]

(c) The relative difference between neutron and proton masses is 0.14%. A cosmologist, when solving part (b), incorrectly assumes that a neutron is 1.4% more massive than a proton. What is the calculated value for Y_{He}? [4 marks]

(d) Realising that the result of part (c) is too small compared with $Y_{\mathrm{He}} = 0.27$, which is supported by observations, the cosmologist sets off to find what has gone wrong but insists that the Δm value used in part (c) is correct. What is the freeze-out temperature, T_f', the cosmologist has to assume to get $Y_{\mathrm{He}} = 0.27$? [4 marks]

(e) The cosmologist understands that T_f is the temperature at which the expansion time scale τ_{exp} and the weak interaction time scale τ_{weak} were equal to each other, and that $\tau_{\mathrm{exp}}(T) = \alpha g_*(T)^{-1/2} T^{-2}$, $\tau_{\mathrm{weak}}(T) = \beta T^{-5}$, where $g_*(T)$ is the number of effective bosonic relativistic degrees of freedom at temperature T, and α, β are numerical constants. The textbook value is $g_*(T_f) = 10.75$ at freeze-out, but the cosmologist thinks that the correct value, g_*', should be different. Assuming that the values of α, β used in the calculation are correct, find the value of g_*' that the cosmologist has to assume to get the desired value for T_f'. Given that the standard model of particle physics predicts that the maximum value of g_* is 106.75, do you think the cosmologist's idea to solve the discrepancy in Y_{He} will work? [4 marks]

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SECTION B: PLANETARY SYSTEMS

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) State four of the main properties of our Solar System. [4 marks]

   (b) Show that the velocity of an extra-solar planet, relative to the star it orbits, at the pericentre of its orbit is given by

$$v = \left(\frac{2\pi\mu}{T}\right)^{\frac{1}{3}} \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}},$$

   for an orbital period T, eccentricity e and $\mu = G(M_* + M_p)$ where M_* and M_p are the stellar and planetary masses respectively. [4 marks]

   (c) Tidal forces are responsible for the gradual increase in the Moon's orbital distance from the Earth (and the reciprocal slowing of the Earth's rotation). Explain qualitatively, with the aid of a diagram, why Triton shows a gradual decrease in its orbital separation from Neptune, given that it has a retrograde orbit. [4 marks]

   (d) The mass-radius relation for a planet of mass M_p and radius R_p is given by

$$2\beta \left(\frac{M_p Z^5}{A^5}\right)^{\frac{1}{3}} = R_p \left(\frac{\alpha Z^2}{A^{\frac{4}{3}}} + \gamma M_p^{\frac{2}{3}}\right),$$

   for a planet composed of material of atomic number Z and atomic mass A. The constants are $\alpha = 6.50 \times 10^7$ J m kg$^{-\frac{4}{3}}$, $\beta = 5.70 \times 10^6$ J^2 s^2 kg$^{-\frac{8}{3}}$ and $\gamma = 4.00 \times 10^{-11}$ J m kg^{-2}. Calculate the average density of a planet composed primarily of carbon ($Z = 6, A = 12$), with a mass of 1.00×10^{24} kg. [4 marks]

   (e) Sketch a diagram showing the main features of the interior of the Earth, and explain how observations of P and S waves from earthquakes have helped constrain this model. [4 marks]

   (f) A K0 star with mass $M_* = 0.780 M_\odot$ at a distance of 15.0 pc is seen by the *Gaia* satellite to show regular sinusoidal positional shifts of amplitude $\pm 2.00 \times 10^{-5}$ arcseconds, with a period of 0.730 years. The angular shift θ in arcseconds is related to the mass of the planetary object inducing this reflex motion M_p as

$$\theta = \frac{M_p a}{M_* d},$$

   where M_* is the stellar mass, a is the radius of the circular planetary orbit in astronomical units, and d is the distance to the system in parsecs. Calculate the mass of the planet causing the observed motion of the star. [4 marks]

   (g) Outline briefly the reasons why gas giants are thought to have formed further from the Sun than terrestrial planets, and how this led to them being so much more massive than terrestrial planets. [4 marks]

6. (a) The rocket equation can be written as

$$\Delta v = v_{\rm ex} \ln\left(\frac{m_0}{m}\right).$$

Define each of the symbols in the equation. Explain how this equation can be used to demonstrate that multi-stage rockets are required to launch rockets that escape the Earth's gravity. You may take the Earth's mass to be 5.97×10^{24} kg and its radius to be 6.37×10^{6} m. [6 marks]

(b) The Apollo asteroids are a class of rocky bodies that orbit the Sun and cross the Earth's orbit. Their orbits have semi-major axes larger than the Earth's.

(i) A small Apollo asteroid lies in an orbit with a perihelion distance $r_{min} = 0.310$ AU and eccentricity $e = 0.823$. As it crosses the Earth's orbit its true anomaly is $\theta = 2.13$ rad. Calculate the time it takes to travel from perihelion to the point it crosses Earth's orbit. [10 marks]

[Hint: we relate the true anomaly to the eccentric anomaly E by

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right),$$

and the eccentric anomaly relates to the mean anomaly M via Kepler's equation $M = E - e\sin E$.]

(ii) The asteroid's orbit is coplanar with the Earth. Calculate the kinetic energy of the impact if the Earth and the asteroid should ever collide, given that the asteroid's mass is 1.00×10^{10} kg, that the angle between the velocity vector and the radial vector of the asteroid is $\phi = 0.680$ rad when it crosses the Earth's orbit, and that the Earth can be approximated to be moving in a circular orbit with a velocity magnitude of 29.7 km s^{-1}. You may neglect any effect of the Earth's gravity. [4 marks]

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7. (a) Describe qualitatively the two main factors affecting large-scale atmospheric motion. [4 marks]

(b) The effective temperature at the top of an atmosphere is given by

$$T_e = \left(\frac{(1-A)L_\odot}{16\pi r^2 \epsilon\sigma}\right)^{\frac{1}{4}},$$

for a planet with albedo A, distance from the Sun r and emissivity ϵ. Molecules may be retained in an atmosphere if the escape velocity of the planet $v_{\rm esc}$ exceeds the root mean square velocity of the molecules $v_{\rm rms}$ by a factor of 10, where

$$v_{\rm rms} = \sqrt{\frac{3k_{\rm B}T}{\mu_{\rm a} m_{\rm H}}}.$$

Here T is the temperature of the molecules, and $\mu_{\rm a}$ is their mass in atomic mass units.

(i) By assuming that the molecules in an atmosphere have a temperature equal to the effective temperature of the atmosphere, show that molecules are retained if

$$\mu_{\rm a} > C\left(\frac{R_p(1-A)^{\frac{1}{4}}}{M_p r^{\frac{1}{2}} \epsilon^{\frac{1}{4}}}\right),$$

for a planet of mass M_p and radius R_p, where the constant C is

$$C = \frac{75k_{\rm B}}{Gm_{\rm H}}\left(\frac{L_\odot}{\pi\sigma}\right)^{\frac{1}{4}}.$$

[5 marks]

(ii) Given the following data, demonstrate that Mars should lose both molecular hydrogen and water vapour from its atmosphere, but that it should retain carbon dioxide. You may assume that Mars radiates as a blackbody.

Mass of Mars	6.42×10^{23} kg
Radius of Mars	3.39×10^{6} m
Semi-major axis of Mars' orbit	1.52 AU
Eccentricity of Mars' orbit	0.093
Albedo of Mars	0.150
Atomic mass of hydrogen	1 atomic mass unit
Atomic mass of carbon	12 atomic mass units
Atomic mass of oxygen	16 atomic mass units

[7 marks]

(c) The atmosphere of Mars is composed primarily of carbon dioxide, however it is considerably thinner than would be expected based on comparisons with Earth and Venus. Briefly explain why this is the case. [4 marks]

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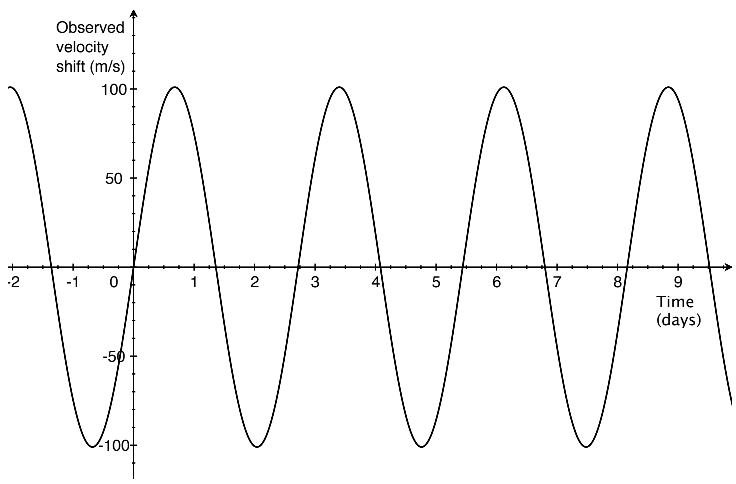
8. (a) Which three types of objects, other than planets, can cause a discernible periodic radial velocity shift in a star, without being directly imaged in optical light? How do we tell such objects apart from planets? [5 marks]

(b) A planet is seen to undergo regular equatorial transits across a G1 star (mass $M_* = 1.08 M_\odot$, radius $R_* = 1.28 R_\odot$), during which the luminosity of the star drops by 1.31%. A subsequent radial velocity study finds the data for the star to be well modelled by the sinusoidal function shown in the figure below.



In radial velocity studies we can relate the observed semi-velocity amplitude K to the period T, mass of the planet M_p and orbital inclination i as

$$K = \left(\frac{2\pi G}{T}\right)^{\frac{1}{3}} \frac{M_p \sin i}{M_*^{\frac{2}{3}}},$$

assuming circular orbits and that $M_* \gg M_p$. Calculate the density of the planet and comment on its nature. [8 marks]

[Hint: the radius of the Sun is $R_\odot = 6.96 \times 10^8$ m.]

(c) An extraterrestrial civilisation places an artificial beacon emitting an extremely regular pulse immediately above the north pole of the planet, such that it is visible to an external observer throughout the planet's whole rotational day. Sketch a figure showing the observed pulse period against time for an observer on Earth, indicating the point at which the planet is closest to the Earth. Briefly explain the main features of your sketch. [7 marks]