(a) Dark skies (away from cities etc.)
Clear skies (no clouds or above cloud line)
Good seeing (large $r_0$)
Low water vapour content (low background) [4 marks bookwork]

(b) Telescope collecting area is proportional to $D^2$
At the diffraction limit, the resolution is $\theta \propto \lambda \,/\, D$
Hence the energy per unit area $\propto$ collecting area / resolution$^2 \propto D^4$ [2 marks seen]

(c) Total power:
Power $= f \Delta\lambda \pi (D/2)^2 \epsilon_{ff}$ [1 mark seen]
Power $= 1.1 \times 10^{-17} \times 0.2 \times \pi (8/2)^2 \times 0.25$
Power $= 2.76 \times 10^{-17}$ W [1 mark unseen]

Photon energy at 2.2$\mu$m:
$\mathrm{E}_\gamma = hc/\lambda$
$\mathrm{E}_\gamma = 6.63 \times 10^{-34} \times 3 \times 10^8 \,/\, (2.2 \times 10^{-6})$
$\mathrm{E}_\gamma = 9.04 \times 10^{-20}$ J [1 mark unseen]

Photon arrival rate, $\dot{N}_\gamma$ is:
$\dot{N}_\gamma = \mathrm{Power} \,/\, \mathrm{E}_\gamma$
$\dot{N}_\gamma = 2.76 \times 10^{-17} \,/\, 9.04 \times 10^{-20}$
$\underline{\dot{N}_\gamma = 305\,\mathrm{s}^{-1}}$ [1 mark unseen]

(d) The apparent difference between the predictions of Newton and Einstein is 6 milli-arcseconds. To differentiate the two possibilities, we require a positional accuracy of 10% of the difference:
$\Delta\theta = 6\,\mathrm{mas} \,/\, 10$
$\Delta\theta = 0.6\,\mathrm{mas}$ [2 marks unseen]

Using the equation given in the question, we need to work out the S/N required for the observations, which is related to $\Delta\theta$ by:
$\Delta\theta = \theta_{\mathrm{dl}} \,/\, (\mathrm{S/N})$

The diffrection limit of an 8.0 meter telescope is:
$\theta_{\mathrm{dl}} = 1.22\,\lambda \,/\, D$ [1 mark seen]
$\theta_{\mathrm{dl}} = 1.22 \times 2.2 \times 10^{-6} \,/\, 8.0 \times 206265 \times 1000$ mas
$\theta_{\mathrm{dl}} = 69$ mas [1 mark unseen]

Therefore the S/N required is:
$\mathrm{S/N} = \theta_{\mathrm{dl}} \,/\, \Delta\theta$
$\mathrm{S/N} = 69 \,/\, 0.6$

S/N = 115 [1 mark unseen]

Since the sky background dominates, the S / N is given by:

S / N = $\frac{\dot{N}_\gamma t}{\sqrt{\dot{N}_{\rm bg} t}}$ [1 mark seen]

$115 = \frac{305t}{\sqrt{4\times 15000\times t}}$ [1 mark unseen]

(note the factor of 4 in denominator occurs because we are told the star light covers 4 pixels).

Rearranging for $t$ gives:

$\sqrt{t} = \frac{115\sqrt{4\times 15000}}{305}$

t = 8529 s [1 mark unseen]

(e) Images taken towards the galactic center suffer from strong dust obscuration. [1 mark seen]

The attenuation by dust is lower at longer wavelengths. [1 mark seen]

(a) Dark skies (away from cities etc.)
Clear skies (no clouds or above cloud line)
Good seeing (large $r_0$)
Low water vapour content (low background) [4 marks bookwork]

(b) Telescope collecting area is proportional to $D^2$
At the diffraction limit, the resolution is $\theta \propto \lambda \,/\, D$
Hence the energy per unit area $\propto$ collecting area / resolution$^2 \propto D^4$ [2 marks seen]

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Power $= f\Delta\lambda\pi(D/2)^2\epsilon_{ff}$ [1 mark seen]
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$\dot{N}_\gamma =$ Power / $E_\gamma$
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(note the factor of 4 in denominator occurs because we are told the star light covers 4 pixels).

Rearranging for $t$ gives:

$\sqrt{t} = \frac{115\sqrt{4 \times 15000}}{305}$

$\underline{t = 8529\,\mathrm{s}}$ [1 mark unseen]

(e) Images taken towards the galactic center suffer from strong dust obscuration. [1 mark seen]

In the optical/infrared, the attenuation by dust scales (approximately) as $1\,/\,\lambda$ hence longer wavelengths are less sensitive to dust attenuation. [1 mark seen]

# L2, Stars and Galaxies 2017 exam

David Alexander

## *June, DMA, Q3*

## *7 short questions:*

a) Calculate the minimum and maximum wavelength that corresponds to the Brackett series (n=4) of hydrogen. Recall that

$$E = -13.6eV\left(\frac{1}{m^2} - \frac{1}{n^2}\right).$$ [4 marks]

[Hint: 1 eV is $1.60\text{x}10^{-19}$ J]

**Solution:**

*The Brackett series is from n=4 to m=5 and m=∞ therefore the range in energies are*

*E=0.306 eV (for m=5)*
*E=0.850 eV (for m=∞)*

$$\lambda = \frac{hc}{E}$$

*λ=1.46—4.06 μm* *or* $\lambda=1.46\text{x}10^{-6}$–$4.06\text{x}10^{-6}$ *m*

*[2 marks for correctly calculating the energy range and 2 marks for correctly converting the energy range into wavelength (with 1 of the marks for obtaining the correct numerical result; Unseen)]*

b) Two stars have the same apparent magnitude and the same physical radius. However, star A has a temperature of 3,000 K while star B has a temperature of 50,000 K. On the basis of these properties, how much further away is star B than star A? [4 marks]

**Solution**

$L = 4\pi R^2 \sigma T_e^4$ *but also* $L = 4\pi d^2 f$

*Since both f (essentially the apparent magnitude) and R (the stellar radius) are the same for both star A and star B then:*

$$\frac{d_b}{d_a} = \sqrt{\frac{T_b^4}{T_a^4}} = 300$$

*[3 marks for taking the correct approach, 1 mark for the getting the correct numerical answer: Unseen]*

c) List four observational signatures that indicate that a binary star system. [4 marks]

**Solution:**

*Visual separation of stars.*

*Astrometric deviations in the position of a star.*

*Dips in the brightness of a star due to eclipsing stars.*

*Velocity shifts in the spectroscopic features of a star.*

*Other valid answers accepted.*

*[1 mark for each valid answer up to a maximum of 4 marks; Seen]*

d) The equation of hydrostatic equilibrium for a star is derived by balancing gravity and pressure. What are the contributors to the pressure in a main-sequence star? Which of these dominates in the Sun? [4 marks]

**Solution**

*Gas (or thermal) pressure and radiation pressure. [2 marks; Seen]*

*The dominant pressure force for a star like the Sun is gas pressure. [2 marks; Seen]*

e) Estimate the time it would take for a photon to escape from the core of a star on a "random walk". Assume the stellar core has a radius of $2.00\times10^{8}m$, a mean-free path of 0.005 m, and scattering with a $10^{-8}$ s delay each time a photon is scattered. What is likely to be the dominant form of opacity in core of the Sun? [4 marks]

**Solution:**

*Number of steps are:*

$$N=\left(\frac{d}{\ell}\right)^{2} \qquad \textit{where d is the radius of the stellar core}$$

$$N=1.60\times10^{21}\text{ steps}$$

*The equation for the total travel time of the photon is:*

$$t=\frac{N\ell}{c}+\left(N\times10^{-8}\right) \qquad \textit{therefore}$$

$$t=1.60\times10^{13}s$$

*The dominant form of opacity in the core is likely to be electron scattering.*

*[Essentially 1 mark for each stage; i.e., 2 marks for the number of steps calculation, 1 mark for the correct numerical answer, and 1 mark for the correct qualitative answer; Seen]*

f) What region and what physical process drives radial pulsations in stars? Why doesn't the Sun show strong radial pulsations? [4 marks]

**Solution**

*Radial pulsations originate in the partial ionisation zone. The partial ionisation of the hydrogen and helium gas in this region allows the stellar material to be strongly compressed – the energy produced through compression goes into ionising the gas rather than significantly raising the gas temperature and pressure. [2 marks; Seen]*

*The Sun doesn't pulsate radially because convection in the outer regions dampens the radial pulsations. [2 marks; Seen]*

g) The main-sequence lifetime of a star with a mass 10 times that of the Sun is $10^8$ years. Assuming 26.7 MeV of energy is produced during each helium-fusing chain and a luminosity that is 3,000 times that of the Sun, what fraction of the mass of the star is converted to helium from hydrogen over the main-sequence lifetime of the star?
[4 marks]

[Hint: 1 eV = 1.60 x $10^{-19}$ J]

**Solution:**

*The energy release from the star in $10^8$ years, assuming a constant luminosity of 3000 times that of the Sun, is given by:*

$$E = 10^8 \times 3.16 \times 10^7 \times 3000 \times 3.84 \times 10^{26} = 3.64 \times 10^{45} J$$

*The total amount of radiative energy that could be released from the star, assuming that all of its mass is converted from hydrogen to helium (which is 4 times the mass of hydrogen) is:*

$$E_{total} = \left(\frac{M}{4 \times m_H}\right) \times 26.7 MeV = \left(\frac{10 \times 1.99 \times 10^{30}}{4 \times 1.67 \times 10^{-27}}\right) \times 26.7 \times 1.60 \times 10^{-19} \times 10^6 = 1.27 \times 10^{46} J$$

*Therefore the fraction of the mass converted during the lifetime of the star is simply:*

$$f = \frac{E}{E_{total}} = \frac{3.64 \times 10^{45}}{1.27 \times 10^{46}} = 0.29 \;\; (i.e., \sim 29\%)$$

*[2 marks for the overall approach in terms of calculating the energy release from the star (the student doesn't need to use exactly the same approach as that shown here) and the total amount of energy that could be released from the star; 2 marks for the correct answer; Unseen]*

# L2, Stars and Galaxies 2017 exam

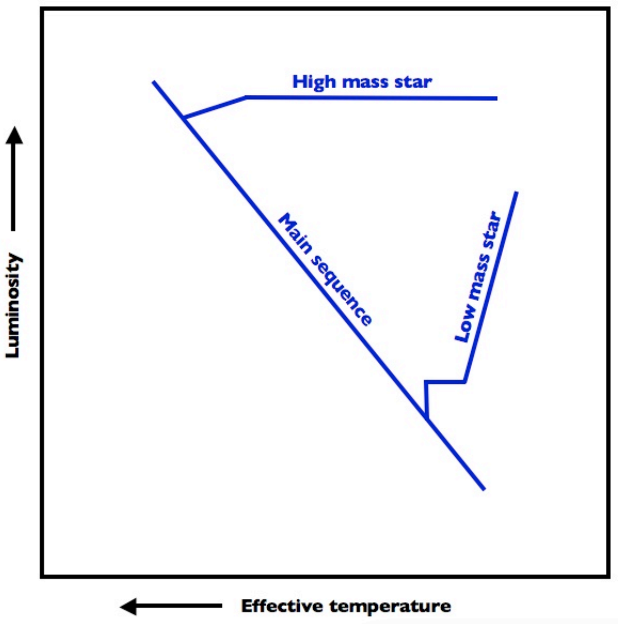
David Alexander

## *June, DMA, Q4*

## *Long question*

a) Draw a Hertzsprung-Russell diagram to illustrate the evolution of a low-mass (~1 solar mass) and high-mass (~25 solar masses) star. Clearly label the axes, mark the track of the main sequence, and sketch tracks to indicate the post-main sequence evolution of these stars up to (and including) the giant-branch phase. [4 marks]

**Solution**




*[Seen: 4 marks for diagram – 1 mark for correct axes, 1 mark for correct main sequence line, 1 mark for correct high-mass star line (need to demonstrate that it is effectively flat in luminosity as it gets redder/cooler), 1 mark for low-mass star line (need to demonstrate that becomes both more luminous and redder/cooler)]*

b) List four major factors that drive the evolution of stars. [4 marks]

**Solution**

*Abundance changes: increasing mean molecular mass, which increases the temperature and density at the core.*

*Fusion size: increase in the fusion region size due to an increase in the internal temperature of the star, leading to (for example) shell burning.*

*Slow contraction: slow gravitational collapse/contraction, which increases density, temperature and pressure.*

*Fast gravitational collapse: fast gravitational collapse when fusion ceases, which increases density, temperature, and pressure.*

*Mass loss: decreases the mass of the star (e.g., winds; planetary nebula; supernova).*

*Other valid answers accepted.*

*[4 marks; 1 mark for each answer up to a maximum of 4 marks; Seen]*

c) Show that the energy ($E$) available from the gravitational collapse of a spherically symmetrical star of constant density is $E \sim -\frac{3}{10}\frac{GM^2}{R}$, where $M$ is the mass of the star and $R$ is the post-collapse radius. Assume that the star is in virial equilibrium but state any other assumptions that you make in your derivation. [6 marks]

[Hint: the gravitational potential of mass $dm$ in a star of mass $M$ is $dU = -\frac{GMdm}{r}$, where $dm = 4\pi r^2 \rho dr$ is a shell at radius $r$ of density ρ and thickness $dr$]

**Solution**

*Taking* $dU_g = -GM4\pi r\rho dr$ *integrate over all shells assuming constant density for M*

$$U_g = -4\pi G\int_0^R M\rho r dr$$

*where* $M = \frac{4}{3}\pi r^3 \rho$ *which gives* *[Seen: 2 marks]*

$$U_g = -\frac{16\pi^2 G\rho^2}{3}\int_0^R r^4 dr \quad \textit{and} \quad U_g = -\frac{16\pi^2}{15}G\rho^2 R^5$$ *[Seen: 2 marks]*

*Converting back from density to mass where*

$$\rho^2 = \frac{M^2}{16/9\,\pi^2 R^6} \quad \textit{which gives} \quad U_g = -\frac{9}{15}\frac{GM^2}{R}$$

*Now applying the virial theorem (where only half of the potential energy is liberated in the gravitational collapse) finally gives*

$$E \sim -\frac{3}{10}\frac{GM^2}{R}$$ *[Seen: 2 marks]*

d) Neutron stars may result from the collapse of a massive star. What reaction causes the production of neutrinos in this physical process? [3 marks]

**Solution**

*Electron capture: neutrinos are produced during the stellar-core collapse because the protons and electrons are compressed strongly enough to form neutrons. To maintain the fermion number a neutrino is released.*

$$p^{+} + e^{-} \Rightarrow n + \nu_{e}$$

*[Seen: 3 marks]*

e) Assuming that the energy available from the gravitational collapse of a 2 solar mass core is released in the form of neutrinos, and that the energy of a typical neutrino is 6 MeV, how many neutrinos are produced from the collapse? Calculate the neutrino flux if the star was at a distance of 10 kpc. [3 marks]

[Hint: 1 eV = 1.60 x $10^{-19}$ J]

**Solution:**

*The energy available in the gravitational collapse is:*

$$E \sim -\frac{3}{10}\frac{GM^2}{R}$$

*Likely radii for the collapsed core (a neutron star) are 10-30 km. Note: since the post-collapse radius is so much smaller than the pre-collapse radius we only need to consider the final (post-collapse) radius of 10-30 km.*

*Therefore:*

$$E \approx (2.13 - 6.40)\times 10^{46}\,J$$ *for a 10-30 km radius [1 mark if answer in given range]*

*The number of neutrinos produced is:*

$$N_{neutrino} = \frac{E}{\left(6\times 1.60\times 10^{-13}\right)} = (1-3)\times 10^{58}\,!$$ *[1 mark if answer in given range]*

*The neutrino flux expected on Earth if the star was at 10 kpc is:*

$$n_{neutrino} = \frac{N_{neutrino}}{4\pi d^2} = \frac{N_{neutrino}}{4\times\pi\times\left(10\times 3.09\times 10^{19}\right)^2} = (1-3)\times 10^{16}\,m^{-2}$$

*[1 mark if the answer is in the given range and with the correct units]*

*[3 marks; Unseen]*

# L2, Stars and Galaxies 2017 exam

David Alexander

## *June, DMA, Q5*

## *Long question:*

a) What are degenerate stars and how are they fundamentally different from main-sequence stars? [2 marks]

**Solution:**

*A degenerate star is a star where nuclear fusion has ceased, unlike a main-sequence star. The star is held up through degeneracy pressure.*

*[2 marks; Seen]*

b) Derive an equation that expresses the degeneracy pressure from non-relativistic electrons, noting that $P \sim \frac{1}{3} n_e p v \;\; p_x = \hbar n_e^{1/3}$ and $n_e = \left(\frac{Z}{A}\right)\frac{\rho}{n_H}$. [5 marks]

**Solution:**

$p^2 = p_x^2 + p_y^2 + p_z^2 = 3p_x^2$ *therefore*

$p = \sqrt{3} p_x$ *[2 marks; Seen]*

*Since* $P \sim \frac{1}{3} n_e p v$ *then*

$p \sim \sqrt{3}\hbar \left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{1/3}$ *or alternatively* $p = \sqrt{3}\hbar n_e^{1/3}$ *[1 mark; Seen]*

*For non-relativistic electrons* $p = v m_e$ *and therefore*

$v \sim \frac{\sqrt{3}\hbar}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{1/3}$ *or alternatively* $v = \frac{\sqrt{3}\hbar n_e^{1/3}}{m_e}$ *[1 mark; Seen]*

*The exerted pressure will therefore be:*

$P = \frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}$ *i.e.,* $P = \frac{\hbar^2}{m_e} n_e^{1/3} n_e^{1/3} n_e = \frac{\hbar^2}{m_e} n_e^{5/3}$ *[1 mark; Seen]*

c) The pressure at the centre of a Carbon-dominated white dwarf with a mass of 1 solar-mass and radius of 5,000 km is ~$10^{22}$ N m$^{-2}$. Calculate whether electron degeneracy pressure is sufficient to hold up the white dwarf. What other element could dominate the composition of a white dwarf? [5 marks]

**Solution:**

*To determine the electron degeneracy pressure they need to insert values of Z, A, and ρ. For a Carbon-dominated white dwarf, Z/A=0.5, while ρ can be determined given the mass and radius of the star:*

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} \qquad \textit{therefore} \qquad \rho = \frac{1.99\times10^{30}}{\frac{4}{3}\times3.14\times(5000\times1000)^3} = 3.80\times10^{9}\,kgm^{-3}$$

*and so*

$$P = \frac{(1.055\times10^{-34})^2}{9.11\times10^{-31}}\left[\left(\frac{1}{2}\right)\frac{3.80\times10^{9}}{1.67\times10^{-27}}\right]^{5/3} = 1.52\times10^{22}\,Nm^{-2}$$

*White dwarfs can also be Helium dominated or Oxygen dominated – either answer is accepted. However, they cannot be dominated by heavier elements (as that would imply the progenitor was more massive than that expected to produce a white dwarf) and they cannot be dominated by Hydrogen. If a list of elements is provided that includes Hydrogen and/or heavier elements than Oxygen then a mark will not be awarded.*

*[5 marks:2 marks for calculating the density and Z/A value, 1 mark for the final numerical answer, and 2 marks for the correct qualitative answer; Unseen]*

d) White dwarfs cool through the release of kinetic energy. Estimate the cooling time for a Carbon-dominated white dwarf of 1 solar mass with a constant temperature of 10,000 K and a constant luminosity of 0.001 the luminosity of the Sun. Why is this estimate a lower limit on the true cooling time? [5 marks]

**Solution:**

$$t_{cool} = \frac{E_{WD}}{L_{WD}} = \left(\frac{3kT_{WD}}{2}\right)\left(\frac{M_{WD}}{Am_H}\right)\left(\frac{1}{L_{WD}}\right)$$

$$t_{cool} = \left(\frac{3\times 1.38\times 10^{-23}\times 10000}{2}\right)\left(\frac{2\times 10^{30}}{12\times 1.67\times 10^{-27}}\right)\left(\frac{1}{3.84\times 10^{23}}\right) = 5.38\times 10^{13}\,s$$

*The cooling time is a lower limit since it assumes a constant luminosity and temperature and so doesn't take into account of the increased cooling time as the temperature and luminosity decreases.*

*[5 marks: 2 marks for the correct equation/approach, 1 mark for the correct numerical answer, and 2 marks for the correct qualitative answer; Unseen]*

e) Briefly explain why the radius decreases with mass in white dwarfs. [3 marks]

**Solution:**

*The white-dwarf radius decreases with mass because to withstand the increased gravitational force the momentum of the electrons also needs to increase. This is achieved by the electrons being confined to a smaller volume which then leads to greater momentum and therefore larger electron degeneracy pressure. [3 marks; Seen]*

Level 2 Paper 4 Question 6 June 2017, answers

(a) The massive blue stars made in the burst will make the galaxies appear bluer in optical colours. [2 marks, unseen].
The red colour of an elliptical (compared to the blue colour of a star forming galaxy) is due to the absence of massive, short-lived stars, implying they have not undergone recent star formation. [2 marks, bookwork]

(b) Since light is absorbed, the optical flux will decrease [1 mark, bookwork]
Since blue light is absorbed more than red light, the stars will appear redder. [1 mark, bookwork]
The absorbed light heats the dust grains, which radiate in the IR. The absorbed energy is therefore re-emitted at longer wavelengths. [2 marks, bookwork]

(c) Massive stars are hotter than low mass stars, and hence produce more photons with high enough energy to photo-ionise the gas [2 marks, unseen. Need to mention hotter and energy of the photons. Being brighter does not help]
The H$\alpha$ line is produced when ionised gas recombines. [2 marks, bookwork]

(d) The accretion luminosity is of order $L = \eta \dot{M} c^2$ (where $\dot{M}$ is the accretion rate and $\eta \approx 0.1$ an efficiency parameter.[2 marks, seen in the lectures on stars, briefly repeated here]
Therefore $\dot{M} = 1.0 \times 10^{10} L_\odot/(0.1\, c^2) = 6.8 \times 10^{-3} M_\odot\ \mathrm{yr}^{-1}$. [2 marks, unseen. Need to be correct within an order of magnitude. Lose 1 mark if not in $M_\odot\ \mathrm{yr}^{-1}$.]

(e) Clusters of galaxies are gravitationally bound systems of 10s to 100s of galaxies. [1 mark, bookwork]
X-rays are emitted by hot gas in the cluster. Assuming this hot gas is in hydrostatic equilibrium [1 mark, bookwork]
with the cluster's gravitational potential, the required mass can be computed. [1 mark, bookwork]
Gravitational lensing of background galaxies can be used to model the mass profile and infer the mass of the cluster. [1 mark, bookwork]
No extra assumptions need to be made.

Level 2 Paper 4 Question 6 June 2017 - answers
Rotation curve, $V_c(R)$, is the circular velocity $V_c$ as a function of radius $R$. [1 mark, bookwork]
For a flat rotation curve, $V_c$ is a constant. [1 mark, bookwork]
The enclosed mass provides the gravitational acceleration to keep a particle on a circular orbit, hence

$$\frac{V_c^2}{R} = \frac{\mathrm{GM}_{\mathrm{enc}}}{R^2},$$

therefore $V_c^2\, R = \mathrm{G}\, M_{\mathrm{enc}}$. [1 marks, bookwork]
Taking the derivative wrt $R$, taking account that $V_c$ is a constant, and that $dM_{\mathrm{enc}}/dR = 4\pi\, R^2\, \rho(R)$ [1 mark, bookwork]
yield $V_c^2 = 4\pi\, \mathrm{G}\, R^2\, \rho(R)$, hence $\rho(R) \propto 1/R^2$. [1 mark, bookwork]

Several lines were discussed in the lectures, but there are other arguments that were not discussed that may be correct as well. These may also get full marks. Lines discussed in the lectures include:
Oort's constants describe the motion of stars near the Sun. Their measured values do not agree with expectations if light traces mass. [2 marks, bookwork],
the enclosed stellar mass profile, inferred from star light, does not result in the observed flat rotation curve, as measured from HI (21 cm) observations [2 marks, bookwork]
unless the stellar mass-to-light ratio were to vary significantly.

From above, we obtain $M_{\mathrm{enc}} = \frac{V_c^2 R}{\mathrm{G}}$ [1 mark, unseen]
therefore $M_{\mathrm{enc}} = 9.0 \times 10^{10} M_\odot$. [2 marks, unseen].
Not using solar masses loses 1 mark, not specifying units loses two marks]

The total luminosity $L = \int_0^\infty 2\pi R\Sigma(R) dR$ [1 mark, unseen]
$= 2\pi\Sigma_0 R_h^2 \int_0^\infty x \exp(-x) dx = 2\pi\Sigma_0\, R_h^2$ [1 mark unseen].
Therefore $\Sigma_0 = L/(2\pi R_h^2) = 180\ L_\odot\ \mathrm{pc}^{-2}$ [1 mark, unseen]
The enclosed luminosity $L_{\mathrm{enc}} = L\, |{-(x+1)}\exp(-x)|_0^{R_\odot/R_h} = L\,(1-(1+R_\odot/R_h)\, \exp(-R_\odot/R_h)) = 7.5 \times 10^9 L_\odot$ [1 mark, unseen]

From above, $M_{\mathrm{enc}}/L_{\mathrm{enc}} = 12 \mathrm{M}_\odot/\mathrm{L}_\odot$ [1 mark, unseen]
If all stars were like the Sun, then the stellar mass to light ratio would be $1\mathrm{M}_\odot/\mathrm{L}_\odot$, which is too small. Therefore, no: solar-type stars cannot provide the measured enclosed mass. [1 mark, unseen]
The relation between stellar mass and luminosity is approximately $L \propto M^3$ [1 mark, unseen in the galaxy lectures],
meaning that the mass-to-light ratio is *smaller* for more massive stars: these therefore cannot provide the required mass [1 mark, unseen]