Quantum Theory - Worksheet 4

Problem 1

The ladder operators a_+ and a_- have been introduced in the Term 1 course to obtain the energy levels and energy eigenfunctions of a linear harmonic oscillator. In particular, you have seen that the Hamiltonian of this system, H, can be written both as $\hbar\omega(a_+a_- + 1/2)$ and as $\hbar\omega(a_-a_+ - 1/2)$, where ω is the angular frequency of the oscillator. You have also seen that its energy levels are $\hbar\omega(n + 1/2)$, $n = 0, 1, 2 \ldots$, and that

$$H\psi_n(x) = \hbar\omega(n + 1/2)\psi_n(x) \tag{1}$$

if $\psi_n(x) = A_n(a_+)^n\psi_0(x)$. In this last equation, $\psi_0(x)$ is the ground state wave function and A_n is a normalization constant such that the norm of $\psi_n(x)$ is 1. Moreover,

$$Ha_-\psi_n(x) = \hbar\omega(n - 1/2)a_-\psi_n(x)$$

and $a_-\psi_0(x) \equiv 0$. Recall also, from Problem 3 of Worksheet 3, that $a_+ = a_-^\dagger$ and $a_- = a_+^\dagger$.

(a) Deduce the following from the above information: (i) $a_+a_-\psi_n(x) = n\psi_n(x)$; (ii) $a_-a_+ = 1 + a_+a_-$; and (iii) $a_+a_- + 1 = a_+^\dagger a_+$. [Hint: compare the two forms of the Hamiltonian and use Eq. (1).]

(b) $a_+\psi_n(x) = c_{n+1}\psi_{n+1}(x)$, where c_{n+1} is a number, since $\psi_n(x) = A_n a_+^n\psi_0(x)$ and $\psi_{n+1}(x) = A_{n+1}a_+^{n+1}\psi_0(x)$. Show that $|c_{n+1}|^2 = n + 1$. [Hint: Do not attempt to calculate the normalization coefficients A_n, but use the equation $a_+a_- + 1 = a_+^\dagger a_+$ and the assumption that $\psi_n(x)$ and $\psi_{n+1}(x)$ are both normalized.]

(c) The result obtained in (b) fixes the modulus but not the argument of the numbers c_{n+1}. It is customary to chose these numbers to be all real and positive, so that $a_+\psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x)$. Use this last equation, and also the results obtained in (a), to show that $a_-\psi_{n+1}(x) = \sqrt{n+1}\,\psi_n(x)$. [Hint: What is $a_-a_+\psi_n(x)$?]

Problem 2

Suppose that $\hat{A}$ and $\hat{B}$ are two operators, not necessarily Hermitian, and that $\hat{B}$ has an eigenvector $|\psi\rangle$ with eigenvalue λ: $\hat{B}|\psi\rangle = \lambda|\psi\rangle$. (To avoid any worries you might have about the precise definition of the domains of these operators, you may assume that the domain of $\hat{A}$, $\hat{B}$, $\hat{A}^\dagger$, $\hat{B}^\dagger$ and any products of these operators is the entire Hilbert space in which these operators act. This is the case, for instance, if $\hat{A}$ and $\hat{B}$ can be represented by square matrices.)

Amongst the following equations, which ones are correct and which ones are in general incorrect?

$$\langle\psi|\hat{A}\hat{B}|\psi\rangle = \lambda\langle\psi|\hat{A}|\psi\rangle \tag{1}$$

$$\langle\psi|\hat{B}\hat{A}|\psi\rangle = \lambda\langle\psi|\hat{A}|\psi\rangle \tag{2}$$

$$\langle\psi|\hat{A}\hat{B}^\dagger|\psi\rangle = \lambda\langle\psi|\hat{A}|\psi\rangle \tag{3}$$

$$\langle\psi|\hat{B}^\dagger\hat{A}|\psi\rangle = \lambda\langle\psi|\hat{A}|\psi\rangle \tag{4}$$

$$\langle\psi|\hat{A}\hat{B}^\dagger|\psi\rangle = \lambda^*\langle\psi|\hat{A}|\psi\rangle \tag{5}$$

$$\langle\psi|\hat{B}^\dagger\hat{A}|\psi\rangle = \lambda^*\langle\psi|\hat{A}|\psi\rangle \tag{6}$$

Problem 3

Background information, relevant for part (b) of the problem: According to the definition used in the course, an operator $\hat{A}$ is said to be Hermitian when

$$\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}|\phi\rangle^* \tag{7}$$

for any vector $|\phi\rangle$ and $|\psi\rangle$ this operator acts on. For any operator acting in a finite dimensional space, Eq. (7) implies that $\hat{A} = \hat{A}^\dagger$ (i.e., in finite dimensional spaces any Hermitian operator is also self-adjoint, which is not true in infinite dimensional spaces).

Since operators acting in a finite dimensional space can always be represented by square matrices of a finite number of rows and columns, what's true for such matrices is also true for the operators they represent. However, things are rather more subtle in the case of operators acting in an infinite dimensional space. As this problem illustrates, treating operators acting on functions as if they were finite matrices may lead to wrong conclusions.

(a) Show that the function

$$\psi(x) = x^{-3/2}\exp[-\lambda/(4\hbar\, x^2)],$$

where λ is a real and positive constant, is a solution of the equation

$$-i\hbar\left[\frac{\mathrm{d}}{\mathrm{d}x}\,x^3\psi(x) + x^3\frac{\mathrm{d}\psi}{\mathrm{d}x}\right] = -i\lambda\psi(x).$$

Is this function square-integrable on $(-\infty, \infty)$?

(b) Suppose that P and Q are two finite Hermitian matrices — i.e., $\mathsf{P} = \mathsf{P}^\dagger$ and $\mathsf{Q} = \mathsf{Q}^\dagger$ (remember that for finite matrices, there is no difference between Hermiticity and self-adjointness, see the background information above). Show that the matrix $\mathsf{PQQQ} + \mathsf{QQQP}$ is also Hermitian.

(c) Now, consider the position and momentum operators Q and P, which, when acting on any function $\psi(x)$ transform this function in, respectively, $x\psi(x)$ and $-i\hbar\mathrm{d}\psi/\mathrm{d}x$. These two operators are Hermitian in the space of square-integrable functions on $(-\infty, \infty)$. However, is the operator $PQQQ + QQQP$ also Hermitian in that space? [Hint: Long calculations are not required here. Base your reasoning on the results established in (a).]