MM2 Revision Lecture

Craig Testrow

craig.p.testrow@durham.ac.uk

Module Evaluation Questionnaires

- This year's Module Evaluation Questionnaires have been released.
- Please fill in your MEQs to help us improve the format and content of the modules.

Lecture Summary

Resources

- DUO
  - Lecture notes
  - Weekly problems/solutions
  - Workshop problems/solutions
  - Mid-term progress test
  - Past exams (5 years)
  - Extra reading
- Text book
  - Mathematical Methods for Physics and Engineering
    - Riley, Hobson and Benoe

Exam

- Format for MM2
  - 7 short questions: 7 x 4 marks = 28 marks
  - 1 long question: 1 x 30 marks
  - Answer **all** questions

- Topics that are in past exams but not covered this year (you don't need to learn these)
  - Frobenius method (like Taylor, but for singular points rather than ordinary points)
  - Reduction of order method (finding a 2^{nd} solution to a 2^{nd} order ODE if you already know 1 solution)
  - Spherical harmonics (related to Legendre polynomials; solutions in polar coordinates)

Course Summary

- 1 objective – solving ODEs and PDEs
  - i.e. finding an expression for dependent variables that does not include any derivatives.
- ODEs (one independent variable)
  - 1st order
  - 2nd order and higher
  - E.g. - 2nd order, 3rd degree ODE, independent variable x,

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = 2e^x$$

- PDEs (two or more independent variables)
  - E.g. - independent variables x and y,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = e^{xy}$$

1^{st} Order ODEs

- Techniques
  - Separation of variables L1P3
  - Exact equations L1P4
  - Inexact equations (integrating factors) L1P5
  - Homogeneous ODEs L2P1
  - Isobaric ODEs L2P2
  - Linear ODEs L2P4
  - Bernoulli ODEs L2P5

1^{st} Order ODEs

- Choosing a method
  - Can you separate the variables?
    - Yes – separate and integrate
    - No – check the form of the equation or check if exact

- Forms
  - Linear

$$\frac{dy}{dx} + p(x)y = q(x)$$

  - Bernoulli

$$\frac{dy}{dx} + p(x)y = q(x)y^n \text{ where } n \neq 0 \text{ and } n \neq 1$$

  - Homogeneous

    Form: $\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$ Condition: $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

  - Isobaric

    Form: $\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$ Condition: $f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x,y)$

1^{st} Order ODEs

- Solutions to the forms
  - Linear

    $$y = \frac{1}{\mu(x)} \int \mu(x) q(x) dx \text{ where } \mu(x) = e^{\int p(x) dx}$$

  - Bernoulli

    Change of variable $z = y^{1-n}$ to linearise the problem

  - Homogeneous – sub $y = vx$
  - Isobaric – sub $y = vx^m$

1^{st} Order ODEs

- Alternatively, check if the equation is exact
  - Exact:
    - Form: $A(x,y)dx + B(x,y)dy = 0$
    - Conditions: $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$, $\quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = Adx + Bdy = 0$
    - Solve for $u = constant$ since $du = 0$
  - Not exact:
    - Form: $\mu(x,y)A(x,y)dx + \mu(x,y)B(x,y)dy = 0$
    - Conditions: $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$, determine integrating factor so that $\frac{\partial \mu A}{\partial y} = \frac{\partial \mu B}{\partial x}$
    - Solve for $u = constant$ after applying integrating factor

1^{st} Order ODEs checklist

- Can you apply the method of separation of variables?
- Can you tell which method to apply based on the form of the ODE and any relevant conditions?
- Can you solve linear, Bernoulli, homogeneous and isobaric ODEs?
- Can you tell if an equation is exact and solve it?
- If an equation is not exact, can you find an integrating factor to make it exact and solve it?

2^{nd+} Order Linear ODEs

- General form of 2^{nd} order linear ODE L3P1

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0 = f(x)$$

- Classification L3P1
  - Constant or variable coefficients
  - Homogeneous $f(x) = 0$ or inhomogeneous $f(x) \neq 0$
- Complementary function L3P1
  - Solution to LHS (homogeneous problem)
- Particular integral L3P2
  - Solution to non-zero RHS (inhomogeneous problem)

2^{nd+} Order Linear ODEs

- Techniques
  - Solving ODE with constant coefficients
    - Auxiliary equation to find complementary function L3P2
    - Method of trial functions/undetermined coefficients to find particular integral L3P2
    - Laplace transform method L4P1
  - Solving ODE with variable coefficients
    - Legendre linear equations L4P4
    - Euler linear equations L4P5
    - Wronskian/variation of parameters method L5
    - Green's function method L6
  - Series solutions
    - Identifying ordinary/singular points L7
    - Taylor series solutions L8
  - Special functions
    - Legendre's differential equation/Legendre polynomials L9

2^{nd+} Order ODEs: Const. coeff. - General solution

- Complementary function
  - Auxiliary function – sub $y = Ae^{\lambda x}$
  - Obtain polynomial from ODE

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

  - Check roots for standard solutions
    - Real, distinct: $y_c = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
    - Real, repeat: $y_c = c_1 e^{\lambda_1 x} + x c_2 e^{\lambda_1 x}$
    - Complex: $y_c = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$
- Particular integral (when $RHS \neq 0$)
  - Trial functions – try a general function that matches the RHS, e.g.
    - $f(x) = 3e^x$ try solution $y_p = Ae^x$
    - $f(x) = 2x^3$ try solution $y_p = ax^3 + bx^2 + cx + d$
- General solution

$$y = y_c + y_p$$

2^{nd+} Order ODEs: Const. coeff. - Laplace transform method

- Change an ODE into an algebraic equation, solve it, reverse the transform
- Laplace transform definition
$$\bar{f}(s) \equiv \int_0^\infty e^{-sx} f(x)dx$$
- Laplace transform of n^{th} derivative
$$\overline{f^n}(s)$$
$$= s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0)$$
$$- f^{(n-1)}(0)$$
- May require you to solve partial fractions
- You do not need to memorise the table of standard transforms (for MM2 at least…)
  - E.g. $f(t) = e^{at}$ becomes$\bar{f}(s) = \frac{1}{s-a}$

2^{nd+} Order ODEs: Var. coeff. - Legendre and Euler eqns

- Legendre 2^{nd} order linear eqn

$$a_2(\alpha x+\beta)^2\frac{d^2y}{dx^2}+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0y=f(x)$$

  - Solved by subbing $(\alpha x+\beta)=e^t$
  - Find $\frac{dy}{dx}=\frac{dt}{dx}\frac{dy}{dt}=\frac{\alpha}{\alpha\text{x}+\beta}\frac{dy}{dt}$ and $\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{\alpha^2}{(\alpha\text{x}+\beta)^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$

- Euler 2^{nd} order linear eqn (Legendre special case)

$$a_2x^2\frac{d^2y}{dx^2}+a_1x\frac{dy}{dx}+a_0y=f(x)$$

  - Solved by subbing $x=e^t$
  - Find $\frac{dy}{dx}=\frac{dt}{dx}\frac{dy}{dt}=\frac{1}{e^t}\frac{dy}{dt}$ and $\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{1}{e^{2t}}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$

2^{nd+} Order ODEs: Var. coeff. - Wronskian

- Wronskian can be used to check solutions are linearly independent

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1$$

- Solutions are linearly independent if $W \neq 0$

- Wronskian/variation of parameters method finds a y_p constructed from same solutions as y_c

$$y_c = c_1 y_1 + c_2 y_2 \text{ and } y_p = k_1 y_1 + k_2 y_2$$

$$k_1 = -\int \frac{y_2 f(x)}{W} dx \text{ and } k_2 = \int \frac{y_1 f(x)}{W} dx$$

- $f(x) = RHS$ of ODE

- Note; No need to add y_c to y_p here, y_p IS the general solution

$2^{\text{nd+}}$ Order ODEs: Var. coeff. - Green's function method

- Green's function is useful because it allows us to find the solution to any RHS, given boundary conditions

- If L is a linear differential operator acting on $y(x)$ such that

$$Ly(x) = f(x)$$

then Green's function satisfies

$$LG(x,z) = \delta(x-z)$$

where $\delta(x-z)$ is the Dirac delta function, and

$$y(x) = \int_a^b G(x,z)f(z)dz$$

where z is the integration variable.

- Check the boundary conditions to form equations in the regions $x < z$ and $x > z$.

2^{nd+} Order ODEs: Var. coeff. - Identifying singular points

- Recall the 2^{nd} order linear equation

$$y'' + p(z)y' + q(z)y = 0$$

- For a complex point $z = z_0$, evaluate the nature of z_0 by testing p(z) and q(z).
  - Both converge (give finite answer) – ordinary point
  - One or both diverge (to infinity) – singular point

- If point is singular, test nature again
  - Find $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$
  - Both converge – regular singular point
  - One or both diverge – irregular singular point

- Singular points at infinity
  - Use a change of variable $z = 1/w$ where $w \to \infty$

2^{nd+} Order ODEs: Var. coeff. - Identifying singular points

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- Singular points at infinity
  - Use a change of variable $z = 1/w$ where $w \to \infty$

2^{nd+} Order ODEs: Var. coeff. - Taylor series

- Taylor series can be used to find solutions at ordinary points
- Any real function can be expressed as the sum of an infinite polynomial
- Taylor series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

- Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

2^{nd+} Order ODEs: Var. coeff. - Taylor series

- Taylor series solution

$$y(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n$$

- Maclaurin series solution

$$y(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} a_n\, z^n$$

- Derivatives

$$y'(z) = \sum_{n=0}^{\infty} n a_n\, z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1}\, z^n$$

$$y''(z) = \sum_{n=0}^{\infty} n(n-1) a_n\, z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, z^n$$

2^{nd+} Order ODEs: Var. coeff. - Taylor series

- The technique is to sub the series solutions into the equation and find a recurrence relation for a_n
- With luck you can express the solution as a sum of odd and even series or some similar combination
- Frobenius series is an extension to this idea that allows you to find solutions for singular points – not required this year

2^{nd+} Order ODEs: Var. coeff. - Legendre polynomials

- The Legendre differential equation is a special function, one that occurs frequently in physics

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

- and has a set of standard solutions – Legendre polynomials
  - E.g. $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- These can be calculated with Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

- Note: this is only the first of two solutions required for our 2nd order Legendre ODE. Finding the second is a more complex process and was not covered

2^{nd+} Order ODEs checklist

- Can you classify higher order ODEs?
- Can you use the auxiliary function and trial functions to solve homogeneous and inhomogeneous equations?
- Can you use Laplace transforms to solve 2^{nd} order ODEs with constant coefficients?
- Can you identify and solve Legendre and Euler equations?
- Can you use the Wronskian to demonstrate solutions are linearly independent?
- Can you use the Wronskian to solve equations?
- Can you identify the nature of singular/ordinary points?
- Can you use Taylor/Maclaurin series to find solutions at an ordinary point?
- Can you identify Legendre differential equations and find Legendre polynomials using the Rodrigues formula?

PDEs

- The general form of a 2nd order PDE is

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

- u is the variable we will solve for
- Most of the problems we look at do not have all 7 terms
- The technique you apply largely depends on which combination of terms you have

PDEs

- Techniques
  - Separation of variables L11,L12,L13
    - E.g. diffusion/heat equation
  - General solutions
    - 1st order PDEs
      - 1 derivative L14P1
      - 2 derivatives L14P3
      - With boundary conditions L14P3
      - With a non-derivative term L15P1
      - Homogeneous/inhomogeneous L15P3
    - 2nd order PDEs
      - $f(p)$ where $p = p(x, y)$ L16
      - With boundary conditions L17P1
      - E.g. wave equation, Laplace equation

PDEs
– Separation of variables

- Can you express the following?

$$u(x,t) = X(x)T(t)$$

- Partial derivatives you may need

$$u_x = X'T,\ u_t = XT',\ u_{xx} = X''T,\ u_{tt} = XT''$$

- Sub these into your PDE and divide to separate the variables (all X one side of equation, all T the other)
- Equate both sides to a constant, the separation constant
  - E.g. $\frac{X''}{X} = \frac{1}{k^2}\frac{T'}{T} = \mu$
- Separate this equation into two ODEs and solve them
- Multiply the solutions to obtain general solution

PDEs – Separation of variables

- To apply boundary conditions examine cases where

$$\mu < 0, \mu = 0, \mu > 0$$

- Add the solutions as a Fourier series

$$u(x,t) = u_{\mu=0} + \sum u_{\mu<0} + \sum u_{\mu>0}$$

- General Fourier

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{2n\pi x}{L} + \sum_{n=1}^{\infty} b_n sin\frac{2n\pi x}{L}$$

- Fourier coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)cos\frac{2n\pi x}{L}dx, \quad a_0 = \frac{1}{L}\int_0^L f(x)sin\frac{2n\pi x}{L}dx$$

PDE general solutions – 1^{st} order PDEs

- Seek a function made from a combination of independent variables
- You need to be able to reproduce the derivations of these solutions
- For u(x,y) seek u(p) where p=p(x,y)
- General equation

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y)$$

- There are slightly different methods for the cases
  - A=0 or B=0
  - C=0 and R=0
  - R=0
  - No terms are zero

PDE general solutions – 1st order PDEs

- A=0 or B=0

$$A(x,y)\frac{\partial u}{\partial x} + C(x,y)u = R(x,y)$$

  - Solve as a 1st order ODE

    $u = \frac{1}{\mu(x,y)}\int \mu(x,y)q(x,y)dx$ where $\mu(x,y) = e^{\int p(x,y)dx}$

- C=0 and R=0

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = 0$$

  - p here is equal to any multiple of the constant of integration of

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}$$

  - Solution: $u(x,y) = f(p)$

PDE general solutions – 1^{st} order PDEs

- R=0

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = 0$$

  - p here is equal to any multiple of the constant of integration of

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}$$

  - Solution: $u(x,y) = h(x,y)f(p)$
  - h is any non-trivial solution to the PDE

- The previous two techniques can be used to solve the homogeneous problem (i.e. when R=0)

- No terms are zero (inhomogeneous problem)
  - Find any solution that gives correct RHS, by inspection or integration (ignoring unrequired terms)

- Analogous to complementary function and particular integral in ODEs

PDE general solutions – 1^{st} order PDEs

- Applying boundary conditions
  - Sub in the conditions and note effect on the function
  - Try f(z) to find relation between function argument and the RHS, where $z = p|_{BC}$
  - If there is more than one way to write down the solution you will need an additional g(x,y) term in your solution
  - This typically happens for BCs at a point, rather than along a line, as there is usually a lot of choice when picking a solution that works

PDE general solutions – 2nd order PDEs

- Simplified general 2nd order equation
$$Au_{xx} + Bu_{xy} + Cu_{yy} = R(x, y)$$
- Again, you should be able to reproduce the derivation involving f(p) here
- Solutions $u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$
- where $\lambda = \frac{-B \pm (B^2 - 4AC)^{1/2}}{2C}$
- Limitation of method
  - Only works for derivatives of same order (e.g. wave equation)
  - Cannot mix 1st and 2nd order derivatives (e.g. diffusion equation)

PDEs checklist

- Can you apply the separation of variables method?
- Can you apply the 4 1^{st} order methods that depend on which terms are present?
- Can you apply boundary conditions to PDEs?
- Can you find the solution to 2^{nd} order PDEs?
- Can you show how the quadratic solution to 2^{nd} order PDEs is derived?
- Can you show how the form of the f(p) term is eliminated from the derivations for the 1^{st} and 2^{nd} order solutions?

General tips

- You don’t need to memorise most of the formulae in this course, with the exception of equation forms
  - i.e. You need to be able to identify types of equations
  - Can you tell an Euler from a Legendre?
  - Can you tell a linear from a Bernoulli?
  - Etc
- E-mail me or come and see me if you’re unsure about anything