ELECTROMAGNETISM – Workshop 5th Set (Qns)
Dispersion and attenuation of waves

Professor D P Hampshire – 2nd Year Physics Lecture Course

The material for this workshop is split into just two parts. Part I: contains Background material. Part II gives some worked examples. Part III gives some additional unseen questions.

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Introduction

Studying dispersion includes studying how waves spread out (or move through space as a function of time). We will need to solve the differential equations that follow from Maxwell's equations to describe the motion of the waves. Then we will need to characterise that motion by for example, identifying how fast the waves move and whether they spread out like ripples on a pond or don't spread out, like a particle moving through space. Such characterisation is known as studying the dispersion of the waves. This material considers the mathematical tools necessary to solve the underlying differential equations using complex numbers and the formulism for studying dispersion.

1 Dispersion and wave packets

1.1 Dispersive and ballistic motion

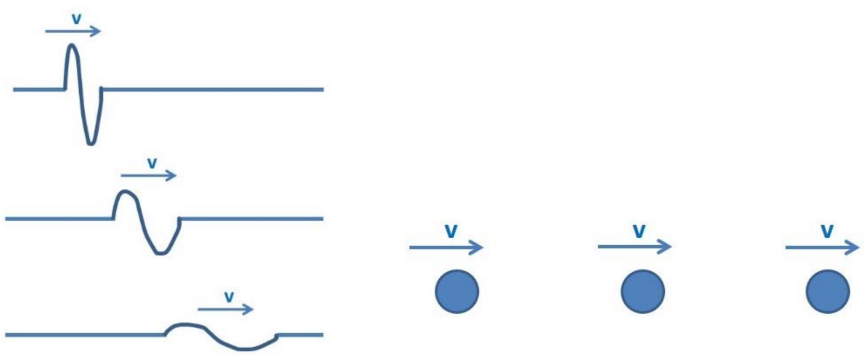

Figure 1 : LHS: A wave moving dispersively along a wire – it spreads out as it moves. RHS: A ball moving ballistically – its shape does not change as it moves.
Consider an infinite plane wave with a single wave-vector and frequency propagating in the x-direction. The wave-vector k points in the x-direction so

$$\phi(x,t) = \phi_o \exp \mathrm{i}(kx - \omega t) \qquad 1\text{-}1$$

If we build the full disturbance of a complex wave packet (or profile) using many component waves with different wavevectors, it can be described using

$$G(x,t) = \int_{-\infty}^{\infty} g(k) \exp\mathrm{i}(kx - \omega(k)t)\,\mathrm{d}k, \qquad 1\text{-}2$$

where $g(k)$ gives the amplitude of the component waves per unit range of k. For a wave packet that has a Gaussian distribution of k-vectors where k_0 is the characteristic (or average) wave-vector,

$$g(k) = \mathrm{A}\exp[-\sigma(k - \mathrm{k_o})^2]\,. \qquad 1\text{-}3$$

To calculate the propagation of the complex wave packet, we use a dispersion relation (which by definition gives the relationship between ω and k) where

$$\omega = \mathrm{f}(k) \qquad 1\text{-}4$$

Using Taylor's Theorem to simplify things we have

$$\omega = \omega_o + \alpha(k - \mathrm{k_o}) + \frac{\beta}{2}(k - \mathrm{k_o})^2 + \ldots \qquad 1\text{-}5$$

where

$$\alpha = \left(\frac{\partial \omega}{\partial k}\right)_{k=\mathrm{k_o}} \text{ and } \beta = \left(\frac{\partial^2 \omega}{\partial k^2}\right)_{k=\mathrm{k_o}}. \qquad 1\text{-}6$$

The general solution for the wave is

$$G(x,t) = \int_{-\infty}^{\infty} \mathrm{A}\exp[-\sigma(k - \mathrm{k_o})^2]\exp\mathrm{i}(kx - [\omega_o + \alpha(k - \mathrm{k_o}) + \frac{\beta}{2}(k - \mathrm{k_o})^2 + \ldots]t)\,\mathrm{d}k \qquad 1\text{-}7$$

Calculating the integral to second order gives:

$$G(x,t) = \mathrm{A}\left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}} \exp\mathrm{i}(\mathrm{k_o}x - \omega_o t)\exp[-\frac{\sigma\pi^2(x - \alpha t)^2}{\sigma^2 + (\pi\beta t)^2}] \qquad 1\text{-}8$$

where the half-width (which tells us whether the packet spreads out or not) is:

$$\left\{\frac{\sigma^2 + (\pi\beta t)^2}{\sigma\pi^2}\right\}^{\frac{1}{2}} \qquad 1\text{-}9$$

Studying dispersion is studying how waves spread out (or move through space as a function of time).

Important general results that you will need to remember are:

1 The phase velocity (of each component wave of a disturbance) is:

$$\mathrm{v_{phase}} = \frac{\omega}{k_{real}} = f\lambda \qquad \text{1-10}$$

2 The group velocity for the full disturbance is determined by how the position of the peak moves through space as a function of time. The condition $x - \alpha t = 0$ determines where the peak of the disturbance is so:

$$\mathrm{v_{group}} = \alpha = \left.\frac{\partial \omega}{\partial k_{real}}\right|_{k=\mathrm{k_o}} \qquad \text{1-11}$$

3 The full disturbance is ballistic (no spreading out) if the half-width of the full disturbance does not change as time proceeds. So the motion is ballistic if:

$$\frac{\partial^2 \omega}{\partial \mathrm{k}_{real}^2} = \beta = 0 \qquad \text{1-12}$$

The full disturbance is dispersive (spreads out) if:

$$\frac{\partial^2 \omega}{\partial \mathrm{k}_{real}^2} \neq 0 \Rightarrow \text{dispersive} \qquad \text{1-13}$$

1.2 Light in a vacuum

The properties of light follow directly from Maxwell's equations. In a vacuum: $\underline{\boldsymbol{E}}_{\boldsymbol{o}}$, $\underline{\boldsymbol{k}}$, $\underline{\boldsymbol{B}}_{\boldsymbol{o}}$ are perpendicular to each other; $\underline{\boldsymbol{B}}_{\boldsymbol{o}}$ is a factor c smaller than $\underline{\boldsymbol{E}}_{\boldsymbol{o}}$ and $\underline{\boldsymbol{E}}_{\mathrm{o}} \times \underline{\boldsymbol{B}}_{\mathrm{o}}$ is in the direction of $\underline{\boldsymbol{k}}$ and gives the direction of propagation.

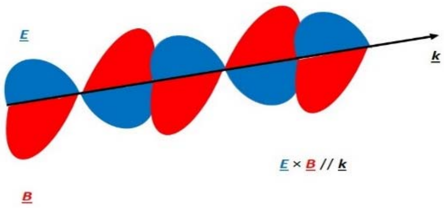

Figure 2 : An electromagnetic wave propagating through space showing the relative directions of $\underline{\boldsymbol{E}}$, $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{k}}$.

1.3 Worked Questions

1. Microwaves have a wavelength of 3 cm. Find their frequency.

2. If the $\omega - \mathrm{k}$ relation is given by $\omega = A\sin(4k)$, where A is a constant. Find a general expression for the phase velocity and group velocity.

3. Show that in a dielectric such as glass,

$$\mathrm{v_{group}} = \frac{\mathrm{c}}{\mathrm{n}+\omega\frac{\partial \mathrm{n}}{\partial \omega}},$$

where n is the refractive index of the dielectric and $\mathrm{n} = \frac{\mathrm{c}}{\mathrm{v}_{phase}}$, and it depends on ω.

4. Derive the wave equation for an electromagnetic wave travelling through a non-magnetic, non-dielectric, very highly conducting infinite linear isotropic medium (using Ohm's law and a similar approach to deriving the velocity of an electromagnetic wave in a vacuum or otherwise), which is given by:

$$\nabla^2 \underline{\mathbf{E}} = \mu_o \sigma_n \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}}$$

where μ_o is a fundamental constant and σ_n is electrical conductivity. When an electromagnetic penetrates a highly conducting media, the $\underline{\boldsymbol{E}}$-field is exponentially attenuated with a characteristic decay length δ. Show that $\delta = \left(\frac{2}{\mu_o \sigma_N \omega}\right)^{1/2}$.

1.4 Worked Answers

1. Microwaves have a wavelength of 3 cm. Find their frequency.

<u>Solution</u> Using $\mathrm{v_{phase}} = f\lambda,$
and the phase velocity of microwaves is c since they are electromagnetic waves.
Therefore, $f = \frac{\mathrm{v_{phase}}}{\lambda} = \frac{3\times10^8}{3\times10^{-2}},$
$f = 10^{10}$ Hz.

2. If the $\omega - \mathrm{k}$ relation is given by $\omega = \mathrm{A\sin4k}$, where A is a constant. Find a general expression for the phase velocity and group velocity.

<u>Solution</u> Using

$$\mathrm{v_{phase}} = \frac{\omega}{\mathrm{k}} = f\lambda,$$

$$\mathrm{v_{phase}} = \frac{\mathrm{A}}{\mathrm{k}}\sin(4k).$$

And the group velocity is given by $\mathrm{v_{group}} = \frac{\partial\omega}{\partial \mathrm{k}},$

$$\mathrm{v_{group}} = \frac{\partial}{\partial k} A\sin(4k) = 4A\cos(4k).$$

3. Show that in a dielectric such as glass,

$$v_{group} = \frac{c}{n+\omega\frac{\partial n}{\partial \omega}},$$

where n is the refractive index of the dielectric and $n = \frac{c}{v_{phase}}$, and it depends on ω.

Solution Using $v_{phase} = \frac{\omega}{k}$ and $n = \frac{c}{v_{phase}}$, we obtain;

$$k = \frac{\omega n}{c}.$$

Differentiating k with respect to ω;

$$\frac{dk}{d\omega} = \frac{1}{c}\left(\omega\frac{\partial n}{\partial \omega} + n\right).$$

Remembering that
$$v_{group} = \frac{\partial \omega}{\partial k} = \frac{1}{\frac{\partial k}{\partial \omega}},$$

therefore,
$$v_{group} = \frac{c}{n+\omega\frac{\partial n}{\partial \omega}}.$$

4. For the specific case of a conducting medium, Maxwell's 3rd and 4th equations are:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t},$$
$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0\varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} = \mu_0 \sigma_N \underline{\boldsymbol{E}} + \mu_0\varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}$$

where we have used Ohm's law: $\underline{\boldsymbol{J}} = \sigma_N \underline{\boldsymbol{E}}$ where σ_N is the electrical conductivity.

Taking the curl of Maxwell III

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial}{\partial t}\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}$$

.

Substituting Maxwell IV gives,

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} == -\mu_0 \sigma_N \frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \mu_0\varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2}$$

Using the vector identity

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) = \underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}) - \nabla^2 \underline{\boldsymbol{E}}$$

(Note: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$).

which gives:

$$\underbrace{\underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}})}_{\text{zero by Maxwell I}} - \nabla^2 \underline{\boldsymbol{E}} = -\mu_0 \sigma_N \frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \mu_0\varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2}$$

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \sigma_N \frac{\partial \underline{\boldsymbol{E}}}{\partial t} + \mu_0\varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2}$$

If we make the simplifying assumption that the medium is highly conducting, $\sigma_N \to \infty$ and the wave equation becomes:

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \sigma_N \frac{\partial \underline{\boldsymbol{E}}}{\partial t}$$

Consider a plane wave, oscillating $\underline{\boldsymbol{E}}$-field propagating in the x direction,

$$\underline{\tilde{\boldsymbol{E}}} = \underline{\boldsymbol{E}}_0 \exp i(kx - \omega t)$$

k: wave vector ($k = \frac{2\pi}{\lambda}$) – where λ is the wavelength, ω angular frequency ($\omega = 2\pi f$) – where f is the frequency.

$$\nabla^2 \underline{\widetilde{\boldsymbol{E}}} = -k^2 \underline{\widetilde{\boldsymbol{E}}}$$

$$\frac{\partial \underline{\widetilde{\boldsymbol{E}}}}{\partial t} = -i\omega \underline{\widetilde{\boldsymbol{E}}}$$

$$=> \nabla^2 \underline{\widetilde{\boldsymbol{E}}} - \mu_o \sigma_N \frac{\partial \underline{\widetilde{\boldsymbol{E}}}}{\partial t} = -k^2 \underline{\widetilde{\boldsymbol{E}}} + i\mu\sigma_N \omega \underline{\widetilde{\boldsymbol{E}}} = 0$$

So $k^2 = +i\mu_o \sigma_N \omega$, and $k = \left(\frac{\mu_o \sigma_N \omega}{2}\right)^{1/2} + i\left(\frac{\mu_o \sigma_N \omega}{2}\right)^{1/2}$. Substituting back into the travelling wave gives: $\underline{\widetilde{\boldsymbol{E}}} = \underline{\boldsymbol{E}}_o \text{exp}i\left(\frac{(\mu_o \sigma_N \omega)^{\frac{1}{2}}}{\sqrt{2}} \cdot x - \omega t\right) \cdot \text{exp} - (\frac{(\mu_o \sigma_N \omega)^{1/2}}{\sqrt{2}} x)$

[Note if $k = k_{real} + ik_{imaginary}$ then $\underline{\widetilde{\boldsymbol{E}}} = \underline{\boldsymbol{E}}_o \text{expi}((k_{real} + ik_{imaginary}) \cdot x - \omega t) = \underline{\boldsymbol{E}}_o \text{exp}i(k_{real} \cdot x - \omega t)\text{exp}(-k_{imaginary} \cdot x)$]

So the characteristic decay length for this material is $\left(\frac{2}{\mu_o \sigma_N \omega}\right)^{1/2}$. At optical frequencies the decay length is about 4 nm for copper.

2 Unseen problems

Maxwell's equations were derived from laws of nature that were found were broadly discovered by measuring lumps of magnetic and dielectric material. However they lead to equations that precisely describe electromagnetic waves:

1. Is there a more exquisite way to describe the propagation of light than using Maxwell's equations?

2. Write down Maxwell's 4 equations in differential form for the special case of a highly conducting medium.

3. By using appropriate Maxwell's equations for an electromagnetic wave travelling in a vacuum, show that,

$$\nabla^2 \underline{\boldsymbol{B}} = \varepsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} .$$

Then show that if $\underline{\widetilde{\boldsymbol{B}}} = \underline{\boldsymbol{B}}_o \text{exp}i(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t)$, where $\underline{\boldsymbol{B}}_o = B_{ox}\hat{\mathbf{i}} + B_{oy}\hat{\mathbf{j}} + B_{oz}\hat{\mathbf{k}}$, $\underline{\boldsymbol{k}} = k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}} + k_z\hat{\mathbf{k}}$ and $\underline{\boldsymbol{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, $\underline{\widetilde{\boldsymbol{B}}}$ is the solution to the above equation with the velocity $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$.

In general, $\underline{\boldsymbol{E}}$ field, $\underline{\boldsymbol{B}}$ field and $\underline{\boldsymbol{k}}$ are orthogonal to each other. Given that $\underline{\boldsymbol{B}}_o = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$ and $\underline{\boldsymbol{E}}_o = 10\hat{\mathbf{k}}$, show that the direction of propagation is parallel to $5\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

4. In an electromagnetic wave, given that the direction of electric field is $-3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and the direction of the propagation is $\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ at a given time, t = 0. Show that the magnetic field, $\underline{\boldsymbol{B}}$ at t = 0, if the magnitude of electric field is 0.15 Vm^{-1}, is given by $\underline{\boldsymbol{B}} = \frac{5\times10^{-10}}{\sqrt{195}}(11\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 7\hat{\boldsymbol{k}})$

5. A sinusoidal electromagnetic wave of frequency 6.10×10^{14}Hz travels in vacuum in the $+z$ direction. The $\underline{\boldsymbol{B}}$ field is parallel to the y-axis and has amplitude 5.80×10^{-4} T. Write down the vector equations for $\underline{\boldsymbol{E}}(\mathrm{z}, \mathrm{t})$ and $\underline{\boldsymbol{B}}(\mathrm{z}, \mathrm{t})$.

6. A 10 GHz plane wave travelling in free space has an amplitude $E_x = 1\ \mathrm{Vm}^{-1}$.

a) Find the phase velocity, the wavelength, and the wave vector.
b) Find the amplitude of the magnetic field intensity, *H,* given that in free space $B = \mu_0 H$.

7. An electromagnetic wave has a magnetic field given by:

$$\underline{\boldsymbol{B}}(\mathrm{x}, \mathrm{t}) = (8.25 \times 10^{-9}\mathrm{T})\sin[(1.38 \times 10^4\ \mathrm{rad\ m}^{-1})x - \omega t]\hat{\mathbf{j}}.$$

a) In what direction is the wave travelling?
b) What is the frequency *f* of the wave?
c) Write down the vector equation for $\underline{\boldsymbol{E}}(x,t)$.

8. Properly prepared distilled water can be considered insulating with $\varepsilon_r = 81$ and $\mu_r = 1$. Given that the phase velocity (v_{ph}) of such an insulating dielectric is given by $v_{ph} = \sqrt{\frac{1}{\varepsilon_r \varepsilon_0 \mu_r \mu_0}}$, find the refractive index and the phase velocity.

9. In a region of space, the magnitude of the electric field varies according to $E = 0.05\sin(2000t)\ \mathrm{NC}^{-1}$ where t is in seconds. Show that the maximum displacement current through a 1 m^2 area perpendicular to E is 8.85×10^{-10} A.

10. Show that the *B*-field of a 3 MHz EM wave will be attenuated by a factor 4.8×10^6 after travelling a distance of 2 m through a highly conducting medium where $\sigma_N = 5\ \Omega^{-1}\mathrm{m}^{-1}$?

11. Prove that the *E* field and *B* field in a very highly conducting material are out of phase with each other by $\frac{\pi}{4}$.

<u>Propagation and Attenuation</u> play a critical role when the EM wave interacts with materials that are conducting. The energy in the wave can be converted into heat in agreement with Ohm's law.

12. One of the worked examples above shows that the wave-equation for a conducting (i.e. not highly conducting) medium is given by:

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \sigma_N \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial \mathrm{t}^2}$$

Without doing any numerical calculations write down expressions (in terms of angular frequency, electrical conductivity and fundamental constants) that show you know in principle how to calculate the decay length, the wavelength, and the group velocity of electromagnetic waves in conducting media at any frequency.

13. Propagation of EM wave through plasma. A plasma is a net neutral collection of charged particles (ions and electrons) with an electron density N. We can usually ignore the response of the ions because they are much slower than the electrons to react.

The complex conductivity of a plasma where there is no scattering is given by:

$$\sigma_{\mathrm{N}} = \frac{Nqv}{E} = \frac{Nq^2}{m(\tau^{-1} - i\omega)} = \frac{iNq^2}{m\omega}$$

Using the general dispersion relation for an infinite, linear, isotropic homogeneous material (i.e. $k^2 = \omega^2\mu\varepsilon + i\mu\sigma_N\omega$) or otherwise, demonstrate that the dispersion relation for a plasma is given by :

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

where the characteristic (angular) plasma frequency is given by,

$$\omega_{\mathrm{p}} = \left(\frac{\mathrm{Ne}^2}{\mathrm{m_e}\varepsilon_{\mathrm{o}}}\right)^{\frac{1}{2}}$$

Prove that if there is no scattering and if $\omega \gg \omega_P$ for the plasma then:

$$v_g \approx c\left(1 - \frac{\omega_P^2}{2\omega^2}\right).$$

Further, show that

$$v_{ph} + v_g \approx 2c.$$

If a 110 MHz radio pulse from the Crab Nebula (distance ~ 6 x 10^{19} m) arrives 1.5 s later than a 115 MHz pulse (produced at the same time) estimate the plasma frequency in hertz and also the average electron density of the interstellar medium.
[**Hint:** Use $(1+x)^n \approx 1 + nx$, where x is small.]

14. Using the expression for the group and phase velocity of a EM wave in a plasma, show that in plasma, $v_{ph} \cdot v_g \approx c^2$.

Miscellaneous Questions.

15. Show that a magnetic field specified by $B_x = B_o x \sin\omega t, B_y = 0, B_z = 0$ cannot exist.

16. A small spherical volume of radius *r* within a conductor has an excess free charge density which decays according to $\rho_{\mathrm{free}} = \rho_{\mathrm{o}} e^{-\frac{t}{\tau}}$. Estimate the magnitude of the resultant current density at the surface of the sphere as this excess charge density decays.