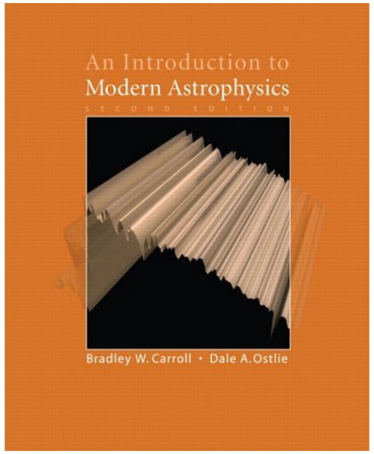
Lecture 10:
Stellar structure –
Getting the energy out: convection



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Chapter 10 of Carroll and Ostlie

(general approach for convection from Dina Prialnik "Theory of stellar structure and evolution")

Aims of lecture

Key concept: convection

<u>Aims:</u>

- Understand the Schwarzschild criterion for convection and have a basic understanding of the mixing length theory of convection in stars

- Know where convection occurs in stars and be able to assess whether convection occurs in the core of the Sun

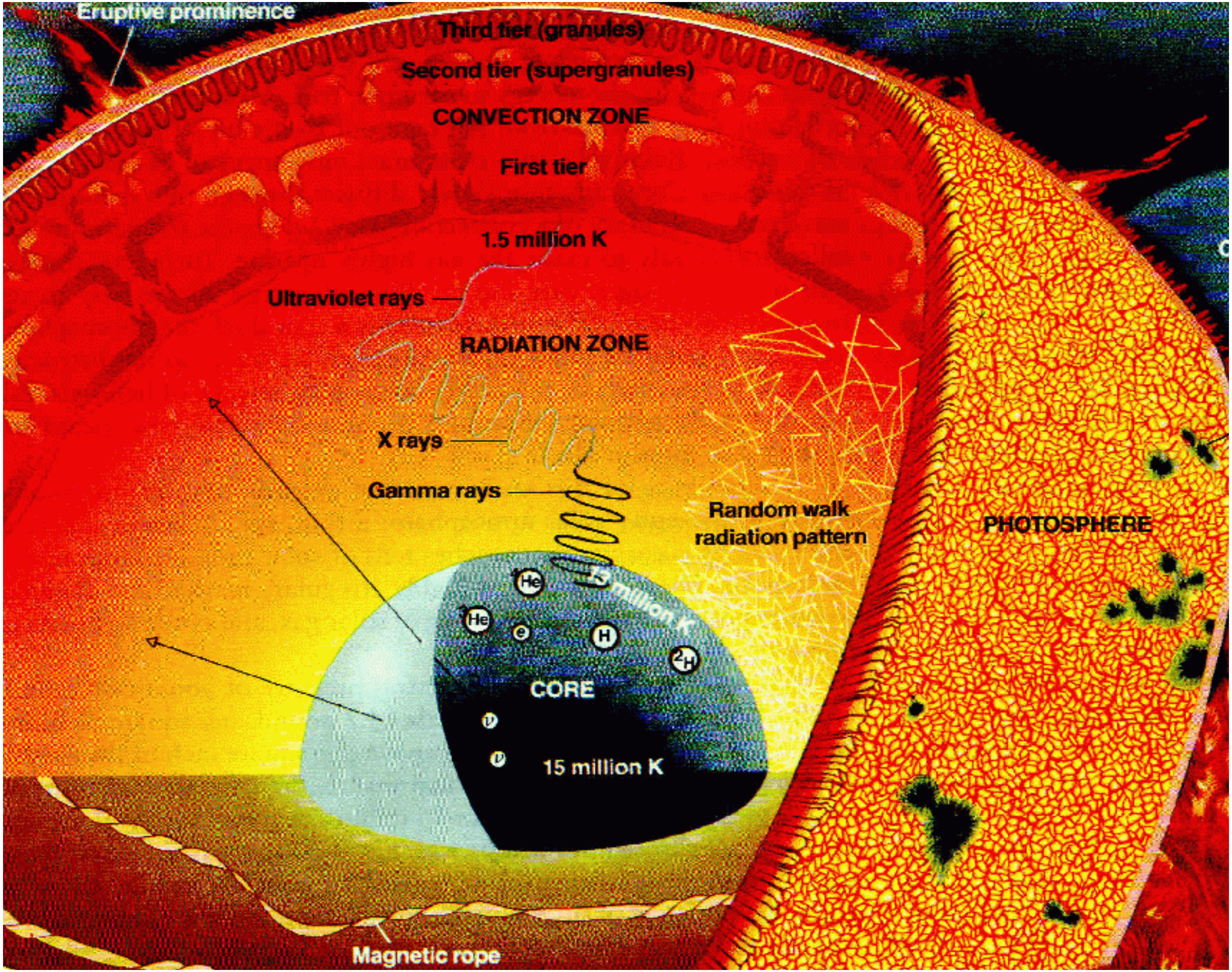
- Know and be able to show:

$$\left|\frac{dT}{dr}\right|_{sur} > \left(\frac{\gamma_{ad}-1}{\gamma_{ad}}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|_{sur}$$

Condition for convection to occur

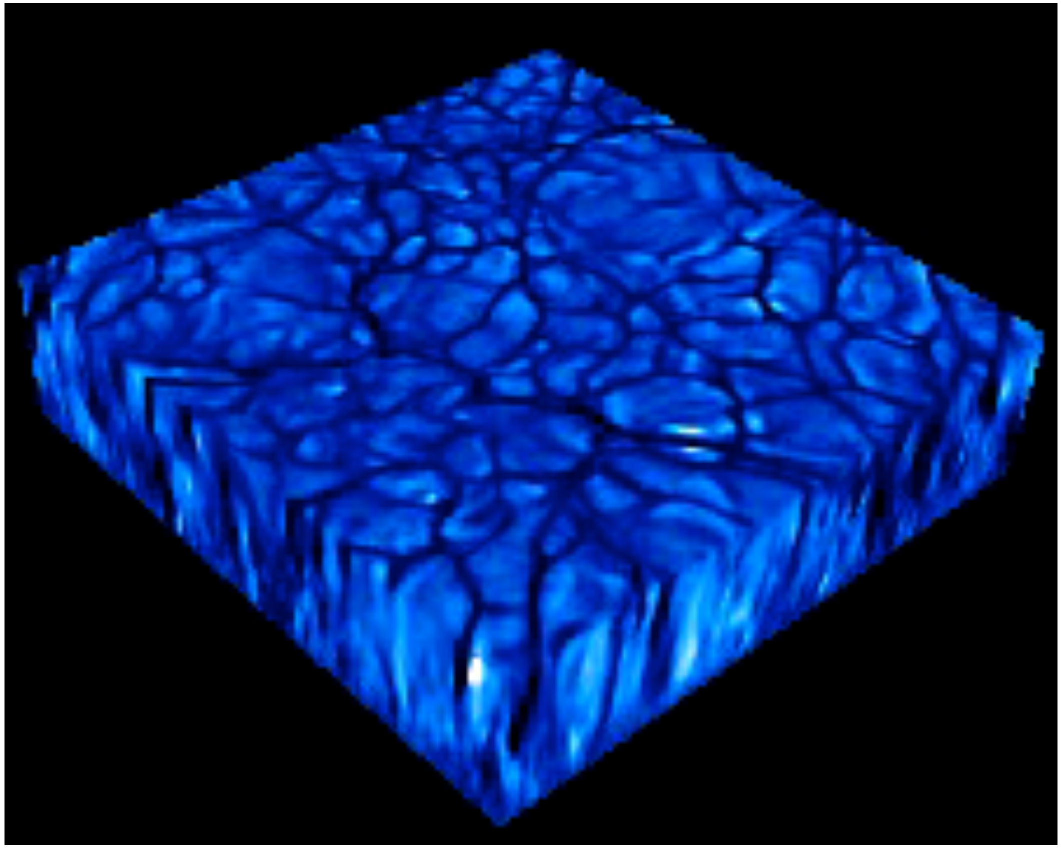
$$\ell = \alpha H_P$$

Convection mixing length

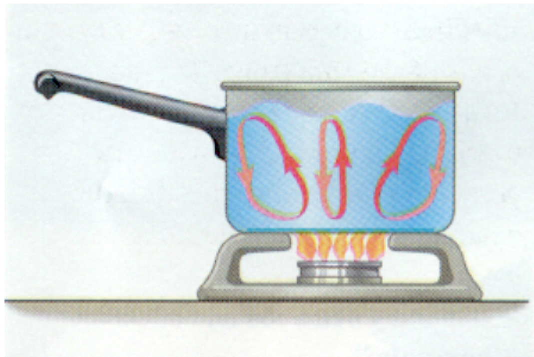

Eruptive prominence
Third tier (granules)
Second tier (supergranules)
CONVECTION ZONE
First tier
1.5 million K
Ultraviolet rays
RADIATION ZONE
X rays
Gamma rays
Random walk
radiation pattern
PHOTOSPHERE
15 million K
CORE
15 million K
Magnetic rope

Energy transport through convection

Radiation and conduction occurs when there is a temperature gradient (e.g., dT/dr). Convection only occurs in liquids and gases when the temperature gradient exceeds a critical value.



Convection: the efficient transport of energy through the mass motions of gas



Convection is a type of dynamical instability but is not destructive. It is thought to be a very efficient method of energy transport and chemical mixing in stars

Schwarzschild criterion for convection

First shown by Karl Schwarzschild in 1906

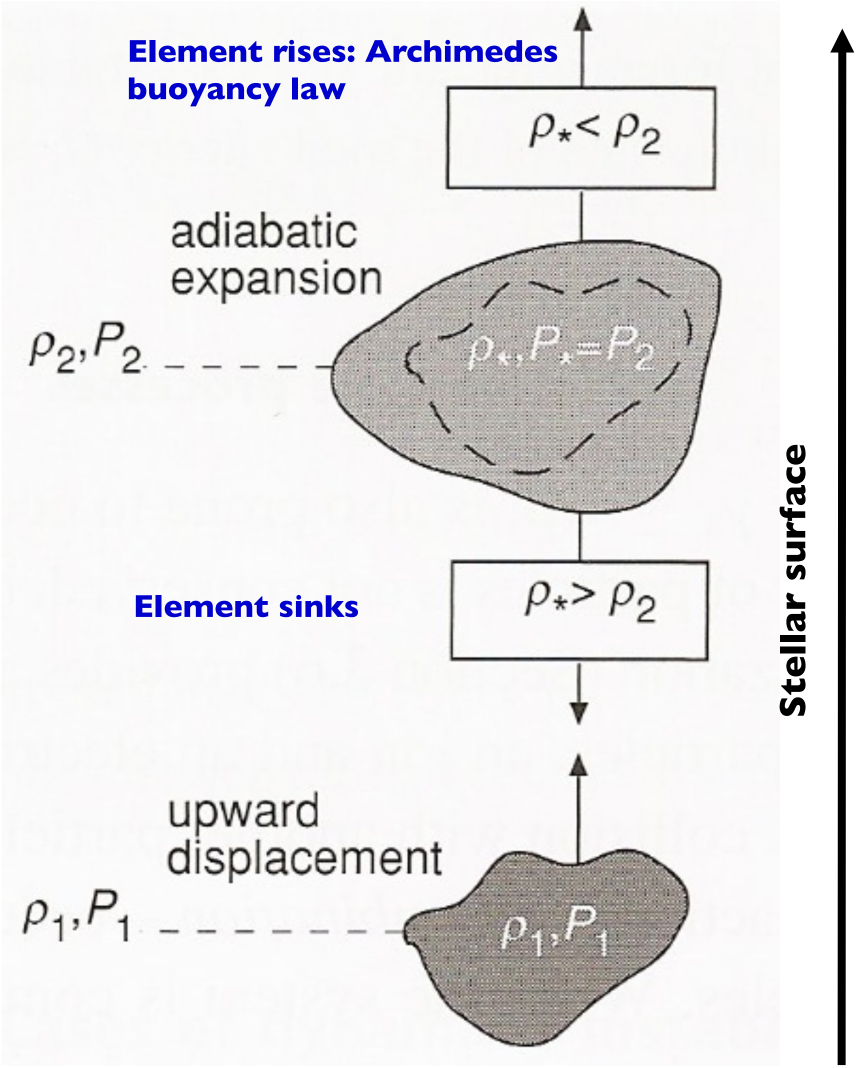
A mass element (Δm) rises out by a small amount. Since the pressure in a star decreases with increasing radius ($P_2<P_1$), the surrounding gas pressure will be lower than the pressure of the mass element.

The mass element will therefore expand until internal and external pressures are in balance. Given great difference in dynamical and thermal timescales no heat exchange occurs during this process – it is an adiabatic expansion. The gas pressure in the mass element is therefore:

$$P = K_a \rho^{\gamma}$$

K_a is a constant, γ is ratio of specific heats.

Mass element will **rise** if $\rho_* <$ density of the surrounding medium (i.e., $\rho_* < \rho_2$): unstable against convection. Otherwise it will **sink** back: stable against convection.



Adiabatic process: no exchange of heat with environment

Schwarzschild criterion for convection

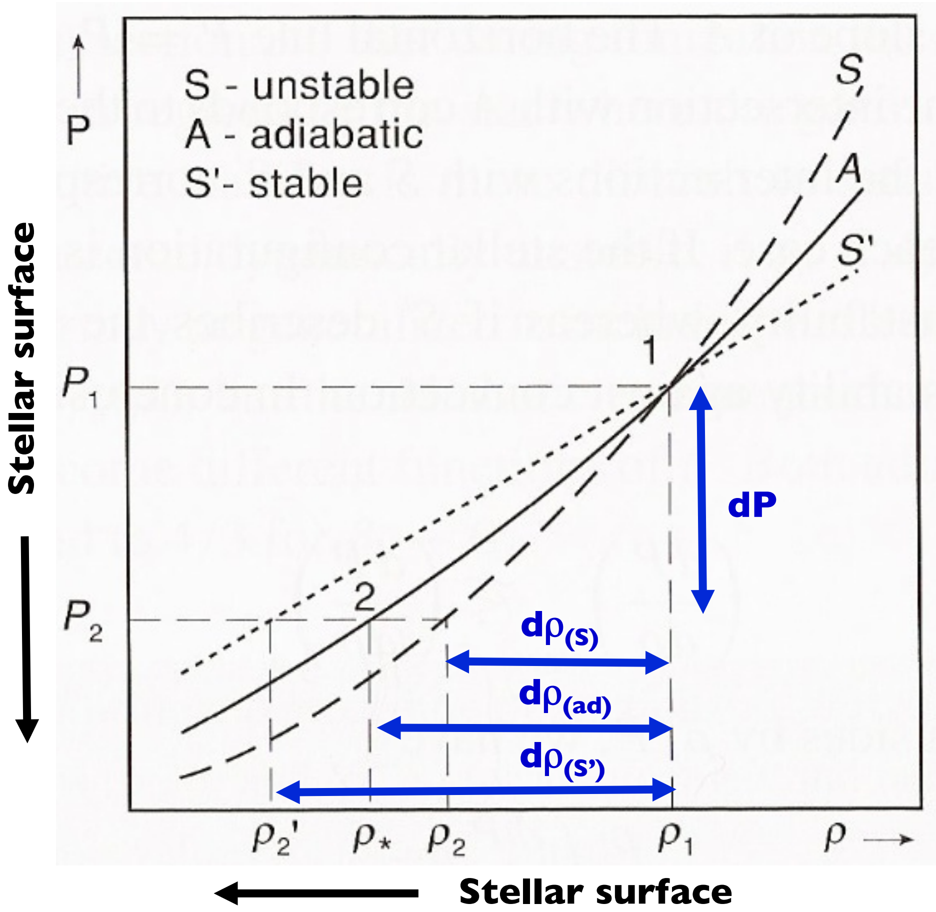
The mass element is represented by curve A (adiabatic relation):

$$P = K_a \rho^{\gamma}$$

S and S' are two potential configurations of the surrounding gas in the star.

Assess ρ for the mass element with different surrounding gas properties (S and S') at pressure P_2:

$\rho_* < \rho_2$ for surroundings S and so the element is prone to rise (<u>unstable against convection</u>) while $\rho_* > \rho_2$ for surroundings S' and so it will not rise (<u>stable against convection</u>).



Unstable against convection (i.e., that of curve S (sur) vs A (ad: mass element)) requires:

$$\left(\frac{dP}{d\rho}\right)_{sur} > \left(\frac{dP}{d\rho}\right)_{ad}$$

Convection is prone to occur when the density of the element is less than the surroundings (at a given pressure): $d\rho_{(ad)}$ is larger

Deriving the Schwarzchild criterion: notes

The ratio of specific heats is defined as

$$\gamma = \frac{C_P}{C_V} = \frac{s+2}{s}$$ where s are the number of degrees of freedom (dof)

For monatomic gas s=3 and therefore $\gamma = 5/3$ (partially ionised gas has more dof)

The ideal gas law (surrounding gas) and the logarithmic derivative are defined as

$$P = \frac{\rho k T}{\mu m_H}$$ so $$P \propto \rho T$$ and therefore

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$ and so $$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}$$

The adiabatic gas law (mass element) and the logarithmic derivative are defined as

$$P = K_a \rho^\gamma$$ so $$P \propto \rho^\gamma$$ and therefore

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$ and so $$\gamma = \frac{\rho}{P}\left(\frac{dP}{d\rho}\right)$$

Deriving the Schwarzschild criterion

We want to derive the temperature gradient for convection prone to occur

Unstable against convection (convection prone to occur) requires:

$$\left(\frac{dP}{d\rho}\right)_{sur} > \left(\frac{dP}{d\rho}\right)_{ad}$$

sur=surrounding gas in the star
ad=mass element

Multiply by ρ/P on both sides:

$$\frac{\rho}{P}\left(\frac{dP}{d\rho}\right)_{sur} > \frac{\rho}{P}\left(\frac{dP}{d\rho}\right)_{ad}$$

Equate to specific heat ratio for the adiabatic component:

$$\frac{\rho}{P}\left(\frac{dP}{d\rho}\right)_{sur} > \gamma_{ad} \qquad \text{or} \qquad \frac{P}{dP}\left(\frac{d\rho}{\rho}\right)_{sur} < \frac{1}{\gamma_{ad}}$$

Deriving the Schwarzschild criterion

Replace dρ/ρ with the logarithmic derivative of the ideal gas equation:

$$\frac{P}{dP}\left(\frac{dP}{P}-\frac{dT}{T}\right)_{sur} < \frac{1}{\gamma_{ad}}$$

Expand and show the condition where convection is prone to occur:

$$\frac{T}{P}\left(\frac{dP}{dT}\right)_{sur} < \frac{\gamma_{ad}}{\gamma_{ad}-1}$$

For a monatomic gas (neutral or fully ionised: $\gamma \sim 5/3$) this threshold ratio is <2.5

We can also divide by dr and rearrange for the temperature gradient to give:

$$\left|\frac{dT}{dr}\right|_{sur} > \left(\frac{\gamma_{ad}-1}{\gamma_{ad}}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|_{sur}$$

Equation 21

The temperature and pressure gradients are negative (i.e., they drop off with radius) and we therefore take the absolute values. When the temperature gradient exceeds this threshold then energy transport via convection can dominate over radiation transport.

Where does convection occur in stars?

Convection occurs when there is a strong temperature gradient. Recall that the temperature gradient for radiation transport is:

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\kappa\rho}{T^3}\frac{L_r}{r^2}$$

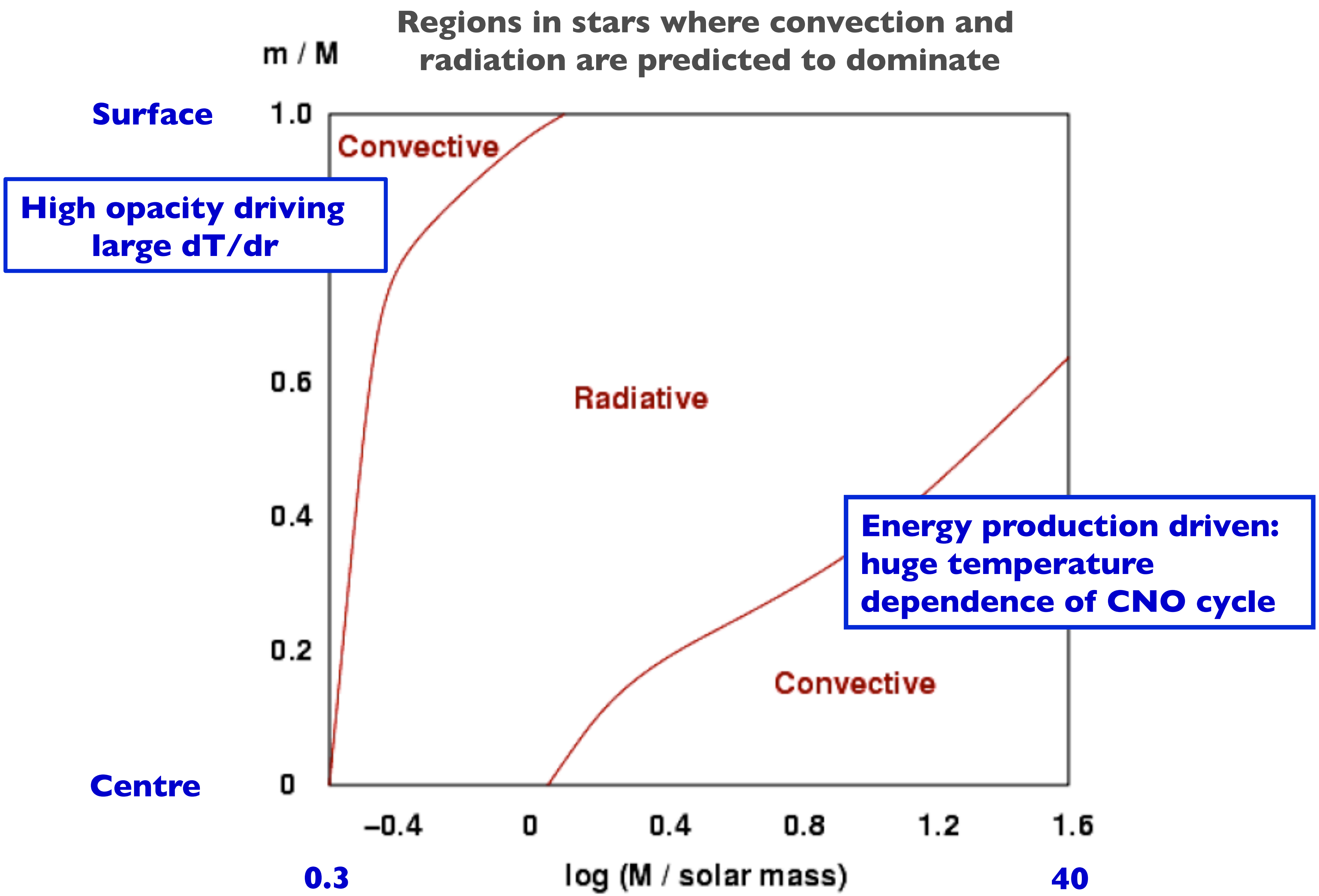
What factors drive a strong temperature gradient and hence could cause convection to develop in stars?

We may therefore expect to see convection in the:

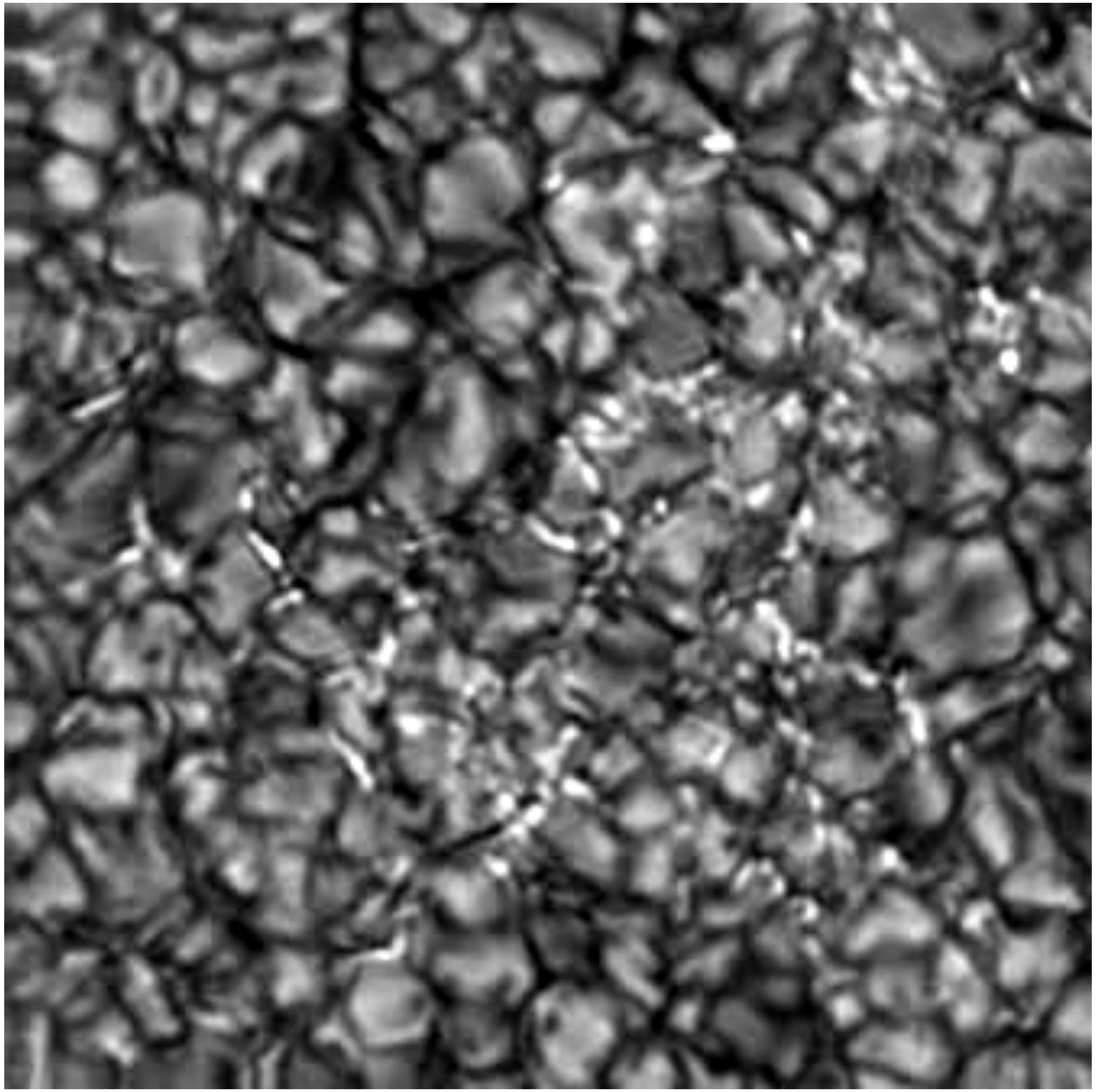
(1) Regions when opacity can be high. For example, in the regions close to the stellar surface in less-massive stars (i.e., cool stars where the temperature of the outer regions are relatively low – bound-bound and bound-free opacity).

(2) Regions where sufficient energy is produced to cause strong increase in luminosity and flux. For example, in the cores of luminous and massive stars where CNO cycle dominates (recall the T^{17} dependence on energy output).

Why does convection occur in these regions?



Convection in the outer layers of the Sun



Does convection occur in the Suns core?

We can assess whether we would expect a convective core in the Sun by equating the temperature gradient for radiation to the criterion for convection:

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\kappa\rho}{T^3}\frac{L_r}{r^2} = \left(\frac{\gamma_{ad}-1}{\gamma_{ad}}\right)\frac{T}{P}\frac{dP}{dr}$$

Now we know that the star must be in hydrostatic equilibrium so:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$ and therefore:

$$\frac{L_r}{M_r} > \frac{16\pi cGP_{rad}}{\kappa P}\left(\frac{\gamma_{ad}-1}{\gamma_{ad}}\right)$$ where $P_{rad} = \frac{1}{3}aT^4$

For the Sun: $\gamma_{ad} = 5/3; P = 2.34\times10^{16}\,Nm^{-2}; T = 15.7\times10^{6}\,K; \kappa = 0.138\,m^2kg^{-1}$

Putting these numbers into the above equation gives:

$\frac{L_r}{M_r} > 1.9\times10^{-3}\,Wkg^{-1}$ while for the Sun $\frac{L_r}{M_r} \approx 10^{-3}\,Wkg^{-1}$

The core energy generation is too low to cause convection in the core

Mixing length theory of convection

We understand the conditions required for convection to occur in stars. What more can we say about convection in stars; for example, how large are the convection cells (i.e., radial extent)? Understanding the details of convection in stars is one of the worst defects in current studies of stellar structure and evolution. One of the challenges is that the gas is turbulent and chaotic (non linear) and therefore only first-order statements can be made.

The current leading model is called the "mixing length theory of convection".

A hot rising bubble is expected to travel a distance before it will thermalise with the surroundings, giving up its excess heat at a constant pressure (since $P_* = P_a$). This distance is called the **mixing length** and is defined as:

$\ell = \alpha H_P$ where H_P is the scale height: **Equation 22**

$$\frac{1}{H_P} = -\frac{1}{P}\frac{dP}{dr}$$

and therefore

$$P = P_0 e^{-r/H_P}$$

Scale height: distance over which pressure declines by a factor of e

Mixing length theory of convection

From numerical models α is typically found to be α~0.5-3 and is often assumed to be α~1. Since the bubble will expand by a factor ~e as it rises over the scale height, it seems unlikely that α>>1.

Recall that when in hydrostatic equilibrium, dP/dr for a star will be:

$$\frac{dP}{dr} = -g\rho = -\frac{GM_r\rho}{r^2} \quad \text{and therefore} \quad H_p = \frac{P}{g\rho} = \frac{r^2 P}{\rho G M_r}$$

We would therefore expect the radial extent of the convection cells to be:

$$\ell = \alpha \frac{r^2 P}{\rho G M_r}$$

Estimate the mixing length for convection cells towards the surface of the Sun:

$$@ r \sim 0.8 R_\Theta, T \sim 10^6 K, \rho \sim 6000 kgm^{-3}, P \sim 10^{14} Nm^{-2}, M_r \sim 1.9 \times 10^{30} kg, r \sim 5.6 \times 10^8 m$$

Therefore $H_P \sim 4 \times 10^7 m$ and so for $\alpha = 0.5 - 3$

$$\ell \sim (0.2 - 1.2) \times 10^8 m \sim (0.03 - 0.17) R_\Theta$$

The convection cells could be a sizeable fraction of the radius of the Sun!