Mathematical Methods in Physics

Workshop 5

5.1

a) Show that the Parseval relation

$$\int_{-\infty}^{\infty} \hat{f}(\omega)(\hat{g}(\omega))^* d\omega = \int_{-\infty}^{\infty} f(t) g^*(t) dt$$

holds. In order to do so, start with the left hand side of this relation. Use the definition of Fourier transform and integrate first with respect to ω using the integral representation of the Dirac-delta function.

b) Find the Fourier transform of the function $f(t) = e^{-|t|}$.

c) Using the Parseval relation and the result in part b) evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(\omega^2+1)^2} d\omega.$$

5.2

Consider a signal $f(t)$ obtained by sampling a function $x(t)$ at regular intervals T, i.e.

$$f(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT).$$

This signal is passed through an electronic filter whose response $g(t)$ is represented by straight lines joining $(0,0)$ to $(T, 1/T)$ to $(2T, 0)$ and is zero for all other values of t. The output $h(t)$ of the filter is the convolution of the input $f(t)$ with $g(t)$.

a) Calculate the Fourier transform of the functions $f(t)$ and $g(t)$.

b) Using the convolution theorem and the inverse Fourier transform, show that the output of the filter can be written as follows

$$h(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \left(\frac{\sin(\omega T/2)}{\omega T/2} \right)^2 e^{-i\omega[(n+1)T-t]} d\omega.$$

5.3

Consider the following differential equation

$$\frac{d^2 f(t)}{dt^2} - f(t) = \frac{1}{1+t^2}.$$

a) Take the Fourier transform of the differential equation using the differentiation rule.

b) Find a particular solution of the differential equation in the form of a convolution integral. Use the result in (5.1).