University of Durham
EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2018	PHYS3661-WE01

Title:
Theoretical Physics 3

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper.
- Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** the answers that are not to be marked.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Relativistic Electrodynamics
Section B: Quantum Theory 3

A list of physical constants is provided on the next page.

Revision:	1

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

Continued Next Page

SECTION A: RELATIVISTIC ELECTRODYNAMICS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Express the electric and magnetic fields $\underline{E}, \underline{B}$ in terms of the 4-potential A^μ. [4 marks]

   [Hint: See (e) for $F^{\mu\nu}$ written in terms of the physical fields $\underline{E}, \underline{B}$.]

   (b) A particle of rest mass m moves with velocity $c/2$ along the x-axis. A second particle of rest mass $2m$ has the same spatial part of the 4-momentum. What is its velocity? [4 marks]

   (c) Give the definition and physical consequences of two events with 4-vectors x_1^μ and x_2^μ having a timelike and spacelike separation. [4 marks]

   (d) Compute $a^\mu v_\mu$, where a^μ is the 4-acceleration and v^μ is the 4-velocity of a given particle. [4 marks]

   (e) Express the spatial components of the Maxwell equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$ in terms of the electric and magnetic fields. [4 marks]

   [Hint:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}.$$

   ]

   (f) The 4-potential for a parallel-plate capacitor oriented normal to the y-axis is $A^\mu = (Ey, 0, 0, 0)$, where E is the electric field between the plates and y the distance from the negatively charged plate. Show that in a frame moving with velocity v in the x-direction, there is a magnetic field in the z-direction, and calculate its magnitude. [4 marks]

   (g) If $(\underline{E}_1, \underline{B}_1)$ and $(\underline{E}_2, \underline{B}_2)$ are the electromagnetic fields from two different sources, show that $\underline{E}_1 \cdot \underline{E}_2 - c^2 \underline{B}_1 \cdot \underline{B}_2$ and $\underline{E}_1 \cdot \underline{B}_2 + \underline{B}_1 \cdot \underline{E}_2$ are Lorentz invariants. [4 marks]

   (h) Write the gauge-transformation of the 4-potential A^μ in contra-variant form. Use this to show that the elements of the field strength tensor are unchanged under a gauge transformation. [4 marks]

2. (a) A photon with frequency ν moves along the x-axis before it scatters off an electron, with mass m, which is initially at rest. Compute the frequency of the scattered photon, ν', as a function of the scattering angle of the photon, i.e. the angle the outgoing photon makes with the x-axis. Show that for a non-vanishing scattering angle, the photon always loses energy in the collision. [6 marks]

   (b) Now consider the process where the electron also moves, with velocity $\underline{v} = (-v, 0, 0)$ head-on towards the incoming photon which is travelling in the opposite direction. Compute the frequency ν' of the scattered photon as a function of the scattering angle. [8 marks]

   (c) Find the minimal velocity of the electron such that the process results in a gain of energy for the photon. [6 marks]

3. A particle of charge q and rest mass m travels with a relativistic velocity $\underline{v}_0$ as it enters a medium, where it is slowed down by a force proportional to the velocity $\underline{v}$ of the particle, $\underline{F} = -\alpha \underline{v}$, where $\alpha > 0$ is a constant. In the following you should ignore the effects of radiation back-reaction.

   (a) Show that for motion along one direction the following holds

$$\frac{\mathrm{d}(\gamma m v)}{\mathrm{d}t} = m\gamma^3 \frac{\mathrm{d}v}{\mathrm{d}t} = m\gamma^3 a.$$

   Use this to show that after the particle has entered the medium the acceleration $\underline{a} = \frac{\mathrm{d}\underline{v}}{\mathrm{d}t}$ is related to the velocity of the particle. [4 marks]

The electromagnetic fields generated by a point charge q in vacuum and in arbitrary motion are given by

$$\underline{E}(\underline{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{R}{(\underline{R} \cdot \underline{u})^3} [(c^2 - v^2)\underline{u} + \underline{R} \times (\underline{u} \times \underline{a})],$$

$$\underline{B}(\underline{r}, t) = \frac{1}{c} \hat{\underline{R}} \times \underline{E}(\underline{r}, t),$$

where $\underline{R}$ is the vector between the point charge and the observer, $\underline{v}$ is the velocity of the point charge, $\underline{u} = c\hat{\underline{R}} - \underline{v}$, and $\underline{a}$ is the acceleration of the point charge. $\underline{R}, \underline{u}, \underline{v}$, and $\underline{a}$ are all evaluated at the retarded time. $\hat{\underline{R}}$ is a unit vector in the direction of $\underline{R}$: $\hat{\underline{R}} = \underline{R}/|\underline{R}|$. In the following you can assume that within the medium the radiation propagates as in vacuum.

   (b) Identify the electric radiation field from the equations above, and show that for the trajectory in question it is given by

$$\underline{E}_{\mathrm{rad}}(\underline{r}, t) = \frac{c}{4\pi\epsilon_0} \frac{q}{R} \frac{1}{(\hat{\underline{R}} \cdot \underline{u})^3} \left[\left(\hat{\underline{R}} \cdot \underline{a} \right) \hat{\underline{R}} - \underline{a} \right].$$

   [4 marks]

   (c) Show that the Poynting vector for the radiation fields is given by

$$\underline{S}_{\mathrm{rad}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{1}{R^2} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^6} \hat{\underline{R}},$$

   where θ is the angle between $\underline{R}$ and $\underline{v}$, and $\beta = v/c$. [4 marks]

The total power radiated to infinity by the point charge at the time t is given by Liénard's result

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left(\underline{a}^2 - \left| \frac{\underline{v} \times \underline{a}}{c} \right|^2 \right).$$

   (d) Use Liénard's result and the result of (a) to calculate the total energy emitted as electromagnetic radiation by the particle as it slows down from the initial velocity v_0 to complete rest. [8 marks]

4. Consider a point charge q of rest mass m in an electromagnetic field. The field strength tensor is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

The 4-force f^μ acting on the point charge is defined by

$$f^\mu \equiv \frac{dp^\mu}{d\tau},$$

where τ is the proper time and p^μ the 4-momentum of the point charge.

(a) Show that f^μ is a 4-vector, and demonstrate how it is related to the usual force $\underline{F} = d\underline{p}/dt$. [4 marks]

(b) The 4-force acting on the point charge due to the electromagnetic field is given by

$$f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu,$$

where u_ν is the 4-velocity of the point charge. Show that the spatial components of this equation correspond to the Lorentz force law. Also, interpret the equation obtained from the 0-component of the above equation. [8 marks]

(c) The point charge moves under the influence of a uniform magnetic field $\underline{B}$. Starting with the assumption that the velocity $\underline{v}$ of the point charge is perpendicular to $\underline{B}$, show that the point charge moves with constant speed in a circle. Ignoring radiation from the charge, compute the radius of this circle in terms of the magnitude of the magnetic field, the mass, m, and speed, v, of the point charge. [6 marks]

What happens if the initial velocity is not perpendicular to $\underline{B}$? [2 marks]

SECTION B: QUANTUM THEORY 3

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) [This sub-question has been deleted as a result of strike action.]

(b) The differential cross-section is defined by the well-known formula

$$\frac{d\sigma}{d\Omega} = \frac{1}{F} \frac{dn(\theta, \phi)}{d\Omega} \, .$$

Define all the symbols in this expression. Verify that the cross-section has the appropriate units. [4 marks]

(c) What are the common eigenfunctions of the orbital momentum operators $\underline{L}^2$ and L_z and what are their eignevalues? Write down the partial wave expansion of a wave-function $\psi(r, \theta)$ in a centrally symmetric potential. [4 marks]

(d) Imagine that a physical system or a process is described by a wave-function which is a real-valued function of the three-dimensional space. Can such a wave-function describe a scattering process and if so, what would be the values for the total cross-section, the phase shifts, and the scattering amplitudes? [4 marks]

(e) [This sub-question has been deleted as a result of strike action.]

(f) [This sub-question has been deleted as a result of strike action.]

(g) The density matrix operator ρ is defined as follows,

$$\rho = \sum_{\alpha=1}^{N} |\alpha\rangle W_\alpha \langle\alpha| \, .$$

Explain what we are summing over in the expression above and what is the meaning of W_α. What is the value of $\mathrm{Tr}(\rho)$ when pure states $|\alpha\rangle$ are normalised to 1? [4 marks]

6. A particle of mass m with momentum $\underline{p} = \hbar \underline{k}$ is scattered by the central potential $V(r)$ given by,

$$V(r) = \begin{cases} -V_0 \dfrac{1}{r} & : \text{ for } 0 \leq r < r_0 \\ \\ -V_1 \dfrac{e^{-\mu r}}{r} & : \text{ for } r_0 \leq r < \infty \end{cases}$$

where V_0, r_0 and μ are positive constants, and $V_1 = V_0\, e^{\mu r_0}$, so that the potential is continuous at $r = r_0$.

(a) The expression for the scattering amplitude in the first Born approximation, f_B, is given by

$$f_B = -\frac{1}{\Delta} \int_0^\infty dr\, r\, \sin(\Delta r)\, U(r)\,, \quad \text{where} \quad U(r) = \frac{2m}{\hbar^2} V(r)\,.$$

Define all the quantities in this formula. What is the physical meaning of Δ? [3 marks]

(b) Compute the scattering amplitude f_B by evaluating the integral in (a) for the potential $V(r)$ given above. [8 marks]

(c) Consider the limit $\Delta\, r_0 \ll 1$ and simplify your expression for the scattering amplitude working to the order $(\Delta\, r_0)^1$. Hence evaluate the differential cross-section in this approximation. [2 marks]

(d) Compute the total cross-section for this scattering process in the same limit as in (c). [4 marks]

(e) Show that at high energies, the cross-section in (d) can be written as

$$\sigma_B(E) = \frac{a}{E} + \frac{b}{E^2} + \mathcal{O}\left(\frac{1}{E^3}\right)$$

where E is the energy of the incoming particle. Give expressions for the constants a and b in terms of V_0, μ, m and r_0. [3 marks]

7. Particles of mass m are scattered by the central potential

$$V(r) = \frac{\lambda}{r^2},$$

where λ is a positive constant.

(a) Write down the time-independent Schrödinger equation for this problem. Explain briefly how the equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} - \frac{2m\lambda}{\hbar^2}\frac{1}{r^2}\right) R_{kl} = 0$$

for the radial component $R_{kl}(r)$ of the wave function arises and all the symbols used in the expression above. [2 marks]

(b) Specify the regular solutions for $R_{kl}(r)$ far away from the origin in terms of free spherical waves given by spherical Bessel functions. Show that this leads to the asymptotic expression for $R_{kl}(r)$ at $r \to \infty$

$$R_{kl}(r) \to A_l(k)\,\frac{1}{kr}\,\sin\left(kr - \frac{\pi l}{2} + \delta_l(k)\right)$$

where $\delta_l(k)$ are the phase-shifts and $A_l(k)$ are some r-independent coefficients. [4 marks]

(c) By comparing the free case, $\lambda = 0$, with the interacting case, $\lambda \neq 0$, prove that the phase-shifts are given by,

$$\delta_l = \frac{\pi}{2}\left(l + \frac{1}{2} - \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2m\lambda}{\hbar^2}}\right)$$

and simplify this expression in the weak-coupling limit $2m\lambda/\hbar^2 \ll 1$. [8 marks]

(d) Compute the differential cross section using the expression that is valid for $\delta_l \ll 1$,

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\sum_{l=0}^{\infty}(2l+1)\,\delta_l\,P_l(\cos\theta)\right|^2.$$

[6 marks]

[Hint: You can use the summation formula: $\displaystyle\sum_{l=0}^{\infty} P_l(\cos\theta) = \frac{1}{2\sin(\theta/2)}$.]

8. [This question has been deleted as a result of strike action.]

