University of Durham

EXAMINATION PAPER

May/June 2014 Examination code: 042591/01

LEVEL 2 PHYSICS: FOUNDATIONS OF PHYSICS 2B

SECTION A. Thermodynamics
SECTION B. Condensed Matter Physics
SECTION C. Modern Optics

Time allowed: 3 hours

Examination material provided: None

Calculators: The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. THERMODYNAMICS

Question 1 is compulsory. Question 2 is optional.

1. (a) One hundred 1.00 kg bricks are removed from a firing kiln, which operates at a temperature of 500 °C, and are allowed to cool in the atmosphere at 20.0 °C. If the process irreversibility is 18.8 MJ determine the entropy change of the Universe and the specific heat capacity of the material from which the bricks are constructed. [4 marks]

   (b) The Helmholtz function tells us the maximum possible work that a system can do or can have done on it. It is defined by $F = U - TS$, where the symbols have their usual meanings. Use this information to determine its total differential, dF, if the function has natural variables of temperature and volume, $F = F(T, V)$ and hence obtain the Maxwell relation,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$

   [4 marks]

   (c) Sketch pV and TS diagrams for a Carnot cycle, indicating clearly the directions of heat flow into and out of the engine and using T_H and T_L to denote the temperatures of the hot and cold reservoirs respectively. A Carnot cycle engine operates between reservoirs at 0.000 °C and 800.000 °C taking in heat energy of 500.000 kJ per cycle whilst rejecting 127.266 kJ per cycle. Using this information calculate the value of absolute zero on the Celsius scale. [4 marks]

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2. (a) What information is provided by the density of states, $g(\varepsilon)$, and distribution function, $f(\varepsilon)$, for a statistical mechanical distribution of indistinguishable particles? How are they related to the number of particles having energy ε? [3 marks]

(b) The Bose Einstein distribution function, for a system of particles of energy ε, is given by

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/(k_{\mathrm{B}}T)} - 1},$$

where μ is the chemical potential and T is the temperature. Show that as the temperature of such a system approaches absolute zero, the phenomenon of Bose Einstein condensation arises. [2 marks]

(c) The observable Universe can be modelled as a sphere of volume V consisting of photons at the temperature of the cosmic microwave background 2.7 K. Recall that photons obey Bose Einstein statistics and have density of states (in terms of angular frequency) of $g(\omega) = A\omega^2$, where $A = V/\left(\pi^2 c^3\right)$. Show that when written in terms of energy, the density of states becomes $g(\varepsilon) = V\varepsilon^2/\left(\pi^2\hbar^3 c^3\right)$. Hence determine the number of photons in the observable Universe, if it is 9.0 billion light years in diameter. [6 marks]

$$\left[\text{Hint: } \int_0^\infty \frac{x^2}{e^x - 1}dx = 2.4.\right]$$

(d) The partition function, Z, can be written in terms of the Helmholtz free energy, F, as $Z = e^{-F/(k_{\mathrm{B}}T)}$. Using this fact show that the Gibbs function, G, can be written as

$$G = k_{\mathrm{B}}T\left[-\ln Z + V\left(\frac{\partial \ln Z}{\partial V}\right)_T\right].$$

[4 marks]

(e) Determine the Gibbs function for a two level system whose state energies are $\varepsilon_0 = -\Delta/2$ and $\varepsilon_1 = \Delta/2$, in the two limits of $k_{\mathrm{B}}T \ll \Delta$ and $k_{\mathrm{B}}T \gg \Delta$. [5 marks]

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SECTION B. CONDENSED MATTER PHYSICS

Question 3 is compulsory. Questions 4 and 5 are optional.

3. (a) For a simple cubic lattice, sketch the set of planes with the Miller indices $(\bar{1}01)$ and (021). Include the x, y and z axes in your diagram. If the lattice constant a is 0.6 nm, determine the spacing for each of these two families of planes. [4 marks]

   (b) Describe van der Waals bonding. State how the bond energy is related to the bond length r. Give an example of an empirical function used to model the bond. [4 marks]

   (c) What assumptions form part of the Debye model of phonons in crystalline solids? The phonon dispersion relation for a simple cubic crystal is given by

$$\omega(\underline{K}) = \left(\frac{4C}{M}\right)^{1/2} \left|\sin\left(\frac{\underline{K} \cdot \underline{a}}{2}\right)\right|,$$

   where ω is the angular frequency, $\underline{K}$ is the phonon wavevector, $|\underline{a}|$ is the lattice constant, C is the interatomic force constant and M is the atomic mass. Obtain an expression for sound velocity in this system using the Debye model. [4 marks]

   (d) A metal with a single valence electron has a resistivity of 10^{-8} Ω m at $T = 300$ K and an electron density of 1×10^{28} m^{-3}. Use the Drude model to calculate the mean free path for the electrons. State any assumptions you make. [4 marks]

   (e) Define the term *Fermi energy*. Copper has a free electron density of 8.4×10^{28} m^{-3}. Determine the value of the Fermi energy for Cu. Comment on your result in comparison to the thermal energy of electrons at room temperature. [4 marks]

   (f) Define the *Hall coefficient*. Aluminium has a Hall coefficient of $+1.02 \times 10^{-10}$ m^3 C^{-1}. Determine the density of charge carriers in Al. Explain why the Hall coefficient is positive and the implications of this for the free electron theory. [4 marks]

4. (a) State the *Bragg Law* and use it to describe the process of *Bragg scattering*. A common method of describing X-ray diffraction patterns is to use the concept of the reciprocal lattice. Explain how peaks observed in an X-ray diffraction pattern relate to the reciprocal lattice. [6 marks]

   (b) Gold has a face centred cubic lattice structure and a lattice constant $a = 0.409$ nm. An X-ray powder diffraction spectrum is collected using the Debye-Scherrer method and the Cu Kα line with a wavelength of 0.15418 nm. Calculate the observed angles (2θ) of the first five peaks in the diffraction spectrum. Explain your method clearly. [10 marks]

   (c) A second X-ray diffraction pattern is collected from a sample of iron which has a body centred cubic crystal structure and a smaller lattice constant. Explain *qualitatively* the observed differences in the spectra between Au and Fe. [4 marks]

5. (a) Give an explanation for the physical origin of energy bands and band gaps in the nearly-free electron model. Explain the significance of the first Brillouin zone. [6 marks]

(b) A one-dimensional solid has an energy-wavevector $E(k)$ relation given by

$$E(k) = C\left(k^2 - \frac{a^2k^4}{2\pi^2}\right),$$

where k is the modulus of the wavevector, C is a constant and a is the lattice spacing. From this obtain expressions for the group velocity and effective mass of an electron in this energy band as a function of k. [6 marks]

(c) Determine expressions for the group velocity and effective mass at: (i) the centre and (ii) the boundary of the first Brillouin zone. Explain, with the aid of an appropriate diagram, why the expressions you have obtained are consistent with the bandstructure of the nearly-free electron model. [8 marks]

SECTION C. MODERN OPTICS

Question 6 is compulsory. Question 7 is optional.

6. (a) The Fourier transforms of $g(x)$ and $h(x)$ are

$$G(u) = \int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{i}2\pi ux}dx \quad \text{and} \quad H(u) = \int_{-\infty}^{\infty} h(x)\mathrm{e}^{-\mathrm{i}2\pi ux}dx \ ,$$

respectively. Give expressions for the Fourier transforms of: (i) $g(x)+h(x)$, (ii) $g(x) \otimes h(x)$, (iii) $\delta(x-d)$ and (iv) $\cos 2\pi x/\lambda_0$, where $\otimes$ and δ denote convolution and the Dirac δ-function, respectively, and λ_0 is a constant. [4 marks]

(b) The electric field of a cylindrically symmetric Gaussian beam can be written as

$$\mathcal{E}(\rho, z) \ = \ \frac{q_0}{q}\mathcal{E}_0\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}k\rho^2/2q} \ ,$$

where $\mathcal{E}_0$ is the amplitude, $k = 2\pi/\lambda$ is the spatial frequency, $q = z - \mathrm{i}z_{\mathrm{R}}$ is the complex beam parameter, q_0 its value at $z = 0$, and z_{R} is the Rayleigh range. Re-write $\mathrm{i}k\rho^2/2q$ in terms of real and imaginary parts and state the significance of each term. [4 marks]

(c) Consider the intensity distribution associated with Fraunhofer diffraction from an aperture consisting of 5 narrow slits. Sketch a phasor diagram corresponding to:

(i) a position with zero intensity, and

(ii) a position with intensity equal to 1/25 of the maximum.

[4 marks]

(d) State Babinet's principle. A vertical hair with diameter D is illuminated by a laser pointer with beam size $w_0 \gg D$. Sketch the far-field intensity distribution in the horizontal plane. [4 marks]

(e) Sketch the optical arrangement used in a $4f$ spatial filter. [4 marks]

(f) Two apertures with shapes corresponding to the form of football ($\sqcap$) and rugby goalposts (**H**) are placed successively in the input plane of a $4f$ spatial filter. Describe (or illustrate) the change in the intensity pattern in the Fourier plane when the football goalpost aperture is replaced by the rugby aperture. Comment on the use of filtering in the Fourier plane to convert a rugby goalpost input to a football goal post shape at the output. [4 marks]

7. (a) If monochromatic light with wavelength λ has an intensity distribution $\mathcal{I}_0|f(x', y')|^2$ in the input plane, then the intensity in an observation plane a distance z downstream can sometimes be written as

$$\mathcal{I}^{(z)} = \frac{\mathcal{I}_0}{\lambda^2 z^2} \left|\mathcal{F}[f(x', y')]\right|^2 \ ,$$

where $\mathcal{F}$ denotes a Fourier transform. Give two examples where this equation holds. [2 marks]

Re-write the Fourier transform as an integral and give two expressions that relate the Fourier variables to the wavelength and positions x and y in the observation plane. [3 marks]

(b) A screen containing two long vertical slits with height b and width a ($a \ll b$) and separation d ($d > a$) is illuminated normally by a laser beam with beam radius w_0, where $d \ll w_0 < b$, such that the variation of laser intensity in the horizontal direction can be neglected. Write an expression for the aperture function $f(x', y')$ corresponding to the screen. [3 marks]

(c) A student is setting up an experiment to confirm the validity of the result stated in (a) for the intensity distribution using the screen described above. Give a condition on z which must be satisfied. [2 marks]

Give another condition that allows the student to neglect diffraction of the laser beam in the vertical direction. [1 mark]

Write an expression for the intensity distribution in the observation plane, $\mathcal{I}^{(z)}$, assuming these conditions are fulfilled. [3 marks]

(d) The student finds that the bench is too short to satisfy all of the above conditions so decides to use a lens with focal length f. Explain (or illustrate) where the lens should be placed and where the observation plane should be. [2 marks]

If f = 100 mm, λ = 0.5 μm, a = 20 μm, and d = 500 μm, calculate the fringe spacing and position of the missing order in the diffraction pattern, and comment on whether it is possible to observe all the features in the diffraction pattern with a camera with pixel size 5×5 μm^2 and total area $1{\times}1$ mm^2. [4 marks]