University of Durham

EXAMINATION PAPER

May/June 2012 Examination code: 043522/02

LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3 PAPER 2

SECTION A. QUANTUM AND ATOMIC PHYSICS
SECTION B. QUANTUM AND NUCLEAR PHYSICS
SECTION C. QUANTUM AND PARTICLE PHYSICS

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. QUANTUM AND ATOMIC PHYSICS
Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) Show graphically that the expectation value of the dipole moment operator $\hat{d} = -ez$ for the 2p0 orbital in hydrogen is zero. The 2p0 orbital has the functional form

$$u_{210}(r, \theta, \phi) = \frac{1}{4\sqrt{2\pi}a_0^{3/2}} \frac{r\cos\theta}{a_0} e^{-|r|/2a_0},$$

where $r\cos(\theta) = z$. [4 marks]

(b) Explain the origin of the natural linewidth of an atomic transition. Calculate the clock uncertainty of an atomic clock based on an atomic transition of resonant wavelength $\lambda = 121.6$ nm and lifetime $\tau = 16.0$ ns. [4 marks]

(c) Explain, briefly, the origin of the expression for the maximum force $F_{\max} = \hbar k\Gamma/2$ due to a laser beam of wavevector k resonant with a transition of natural linewidth Γ. [4 marks]

(d) The general expression for the Zeeman shift (the change in energy caused by the Zeeman effect) of the m_F magnetic sub-levels of any alkali metal atom of nuclear spin I is given by

$$\Delta E = \pm\frac{\mathcal{A}}{2}\left(1 + \frac{8m_F\mu_B B}{\mathcal{A}(2I+1)} + \frac{4\mu_B^2 B^2}{\mathcal{A}^2}\right)^{1/2},$$

where $\mathcal{A}$ is the zero-field hyperfine splitting, μ_B is the Bohr magneton and B is the magnetic field. What is the relationship between I and m_F for m_F levels that show a linear Zeeman effect? [4 marks]

(e) The unperturbed wavefunctions for an infinite cubic well whose sides range from 0 to a in the x-, y- and z-directions are

$$\psi^0_{n_x,n_y,n_z} = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x\pi x}{a}\right) \sin\left(\frac{n_y\pi y}{a}\right) \sin\left(\frac{n_z\pi z}{a}\right).$$

Derive an expression for the first-order correction to the energy of the ground state when the potential is subject to the following perturbation

$$H' = \alpha\delta(x - a/2)\delta(y - a/2)\delta(z - 3a/4),$$

where α is a constant. [4 marks]

2. A particle of mass M is in a perturbed infinite square well potential of the form

$$V(x) = \begin{cases} \infty & , \ x \leq 0 \\ \beta(b-x) & , \ 0 < x < b \\ 0 & , \ b \leq x \leq a \\ \infty & , \ x \geq a \end{cases} \quad ,$$

where β and b are constants and a is the width of the well.

What are the eigenstates ψ_n^0 and eigenenergies E_n^0 of the unperturbed system and what is the perturbation hamiltonian $H'(x)$? [6 marks]

Calculate, to first order, the eigenenergies of the system.

$$[\text{Hint: } \sin^2 u = \tfrac{1}{2}(1 - \cos 2u)]$$

[10 marks]

If $b = a$, explain why the first order correction E_n^1 is independent of n. [4 marks]

3. In an electric dipole allowed transition, what is the relationship between the Einstein A-coefficient A_{Elec}, the lifetime τ_{Elec} of the excited state and the natural linewidth Γ, assuming that no other line broadening mechanisms are significant? [1 mark]

   With the aid of diagrams, and making reference to Γ, explain how the force a cloud of atoms experiences in two counter-propagating red-detuned laser beams is velocity dependent and, therefore, leads to Doppler cooling of the atom cloud. Limit your arguments to one dimension, that of the propagation direction of the laser beams. [4 marks]

   The Einstein A-coefficient for an electric dipole allowed transition is given by

$$A_{\mathrm{Elec}} = \frac{\omega_0^3 d^2}{3\pi\varepsilon_0 \hbar c^3},$$

   where d is the electric dipole matrix element and ω_0 is the angular transition frequency. Find an expression for the Einstein A-coefficient A_{Mag} for a magnetic dipole allowed transition. [3 marks]

   In the hydrogen atom, the lifetime τ_{Mag} of the magnetic dipole allowed ground state hyperfine transition is 8.7709×10^{24} times larger than the lifetime of the electric dipole allowed $1s \to 2p$ transition, which has a dipole moment matrix element of $d = 6.3158 \times 10^{-30}$ C m. The energies of the two transitions differ by a factor of 2.6823×10^{-4}. Calculate the magnetic dipole matrix element μ. [5 marks]

   The $2p0$ and $3p0$ orbital wavefunctions of the hydrogen atom are

$$u_{210} = \frac{1}{4\sqrt{2\pi}a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta \qquad \text{and}$$

$$u_{310} = \frac{\sqrt{2}}{81\sqrt{\pi}a_0^{3/2}} \left[6 - \frac{r}{a_0}\right] \frac{r}{a_0} e^{-r/3a_0} \cos\theta.$$

   Use a sketch of the wavefunctions along the z-axis to explain whether or not you would expect the dipole matrix element for the $1s \to 3p$ transition to be larger or smaller than the $1s \to 2p$ transition. [7 marks]

SECTION B. QUANTUM AND NUCLEAR PHYSICS
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) A particle has orbital angular momentum quantum number l and spin quantum number s. What is the range of possible values of its total angular momentum quantum number j? If a neutron has $l = 4$ what are its possible values of j and how many states are there in each multiplet? [4 marks]

   (b) In the independent particle shell model of the nucleus the first six levels in order of increasing energy are

$$1\mathrm{s}_{1/2},\ 1\mathrm{p}_{3/2},\ 1\mathrm{p}_{1/2},\ 1\mathrm{d}_{5/2},\ 2\mathrm{s}_{1/2},\ 1\mathrm{d}_{3/2}.$$

   Predict the spin and parity of the ground state of the nucleus of ${}_{15}\mathrm{P}^{29}$. [4 marks]

   (c) The nuclear magnetic dipole moment μ_j is defined as the maximum projection along a chosen axis of the vector $\underline{\mu}_j = g_j(\mu_\mathrm{N}/\hbar)\underline{J}$ where $\underline{J}$ is the total angular momentum vector and μ_N is the nuclear magneton. If g_j is given by

$$g_j = \frac{g_L + g_S}{2} + \frac{(g_L - g_S)}{2}\left[\frac{l(l+1) - s(s+1)}{j(j+1)}\right],$$

   find (μ_j/μ_N) as a function of j for a proton with $j = l + 1/2$. For a proton $g_L = 1$ and $g_S = 5.58$. [4 marks]

   (d) List four properties of the nuclear force. [4 marks]

   (e) The Yukawa potential $V(r) = -(g^2/r)\exp(-r/R) = -g\phi(r)$ can be interpreted as arising from the exchange of a particle described by the field ϕ. Show that ϕ satisfies the relativistic equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \frac{m^2c^2}{\hbar^2}\phi,$$

   for some R. State the form of R. If the range of the nuclear force is 1.2 fm make a prediction for the rest mass-energy of the exchange particle in MeV. [4 marks]

   [$\hbar c = 197$ MeV fm]

5. (a) The Liquid Drop Model binding energy used in the Semi-Empirical Mass Formula (SEMF) for a nuclide with mass number A and atomic number Z is

$$B(Z,A) = a_{\rm V}A - a_{\rm S}A^{2/3} - a_{\rm C}\frac{Z(Z-1)}{A^{1/3}} - a_{\rm sym}\frac{(A-2Z)^2}{A} + \delta$$

where δ equals $+a_{\rm P}A^{-3/4}$ for even-even nuclei, $-a_{\rm P}A^{-3/4}$ for odd-odd nuclei and zero for even-odd or odd-even nuclei. Briefly justify the signs of the Surface and Coulomb terms. [2 marks]
Briefly justify the Z and A dependence of the Coulomb and Symmetry terms. [4 marks]

(b) The SEMF gives the mass-energy of a nucleus as

$$M(Z,A)c^2 = Zm_{\rm p}c^2 + (A-Z)m_{\rm n}c^2 - B(Z,A)$$

which for isobars is quadratic in Z. Derive a formula for the most stable value of Z for isobars of mass number A when A is odd. [4 marks]
What is the most stable value of Z if $A = 101$? [2 marks]

(c) ${}_{20}\mathrm{Ca}^{48}$ is an isotope of Calcium that decays by the rare double beta decay to an isotope of Titanium,

$${}_{20}\mathrm{Ca}^{48} \rightarrow_{22} \mathrm{Ti}^{48} + 2e^- + 2\bar{\nu}.$$

Find the energy released Q (in MeV) in this decay using the SEMF. You may neglect the masses of the neutrinos. [6 marks]
How can you tell that this process is indeed allowed to occur spontaneously? [2 marks]

[$(m_{\rm n} - m_{\rm p})c^2 = 1.294$ MeV, $m_ec^2 = 0.511$ MeV,
the SEMF coefficients are $a_{\rm V} = 15.5$ MeV, $a_{\rm S} = 16.8$ MeV,
$a_{\rm C} = 0.72$ MeV, $a_{\rm sym} = 23$ MeV and $a_{\rm P} = 34$ MeV.]

6. The Schrödinger equation for neutron-proton scattering at low centre of mass energy E is

$$-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + (V(r) - E)u = 0,$$

where the spherically symmetric wavefunction is $\psi(r) = u(r)/r$. What is M called and how is it defined in this instance? [2 marks]

Assuming that the neutron-proton potential can be represented by the square well form $V(r) = -V_0$ for $0 \le r \le R$ and $V(r) = 0$ for $r > R$, write down the solutions for u inside and outside the well. Bear in mind the boundary condition on ψ that must be satisfied as $r \to 0$. [3 marks]

For $E \ll V_0$ show that u outside the well is approximated by a linear form $u_{\mathrm{out}} \propto (r - a)$ and show that the following expression holds for the scattering length a:

$$a = R - \frac{\tan\left(R\sqrt{2MV_0}/\hbar\right)}{\sqrt{2MV_0}/\hbar}.$$

[4 marks]

At very low energy the spin dependence of the neutron-proton force may be represented by such a square well potential with

$$V_0 = A + B\frac{(\underline{S}_1 \cdot \underline{S}_2)}{\hbar^2}.$$

Here $\underline{S}_1$ and $\underline{S}_2$ are the spin vectors of the two nucleons while $A = 23.2$ MeV and $B = 11.2$ MeV. The radius of the well $R = 2.5$ fm.

Calculate the potential well depths for triplet and singlet ($^3\mathrm{S}_1$ and $^1\mathrm{S}_0$) spin configurations and obtain values for the scattering lengths for spin triplet and singlet neutron-proton scattering. [11 marks]

[Use $\hbar c = 197$ MeV fm and $M = 469$ MeV$/c^2$.]

SECTION C. QUANTUM AND PARTICLE PHYSICS
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) Which fundamental interactions are described by the Standard Model? Which elementary particles (gauge bosons) are responsible for their transmission? [4 marks]

   (b) Consider the following reactions and state if they are allowed in the Standard Model or not. If they are forbidden, name the principle disallowing them; if they are allowed, draw at least one labelled Feynman diagram related to the reaction. [4 marks]

      (i) $e^+e^- \to t\bar{t}$

      (ii) $\mu^-\gamma \to e^-\gamma$

      (iii) $H \to \mu^-\bar{\nu}_\mu e^+\nu_e$

      (iv) $gg \to ZZ$

   (c) What is the difference between discrete and continuous symmetries? Give an example of each type. [4 marks]

   (d) What is the relation between luminosity, event rate and cross section? In which units are these quantities measured? How is the cross section related to the quantum mechanical transition amplitude? [4 marks]

   (e) Draw all lowest-order Feynman diagrams for the process $e^+e^- \to W^+W^-$, ignoring the mass of the electron. Which basic property of the electroweak interactions is tested through this process? [4 marks]

8. The search for the Higgs boson is one focus of current experimental efforts at the Large Hadron Collider (LHC), and first indications point at a Higgs boson with a mass of about 125 GeV.

   In parts (a)-(c), consider the two main decay routes of the Higgs boson, namely $H \to b\bar{b}$ and $H \to \tau^+\tau^-$. Including the mass of the fermions, calculate the ratio of the respective decay widths, where each of them can be taken from

$$\Gamma_{H\to f\bar{f}} = \frac{|\underline{p}_f|}{8\pi M_H^2} |\mathcal{M}|^2 \,.$$

   (a) Determine the decay kinematics in the rest frame of the Higgs boson, namely the energies and three-momenta of the fermions. [6 marks]

   (b) Estimate the ratio of the matrix elements squared for each decay from the respective coupling strengths and the internal degrees of freedom for each fermion. [4 marks]

   (c) What is the ratio $\Gamma_{H\to\tau^+\tau^-}/\Gamma_{H\to b\bar{b}}$, if $m_\tau \approx 1.8$ GeV and $m_b \approx 4.5$ GeV? [2 marks]

   (d) A further phenomenologically important decay of the Higgs boson is into a pair of photons. Comment on the size of its decay width and support your answer with a lowest-order Feynman diagram for this decay. [2 marks]

   (e) Of course, the Higgs boson needs also to be produced at the LHC, with the main production due to the parton-level process $gg \to H$. Name at least two additional production channels at the LHC of the Higgs boson and draw corresponding lowest-order Feynman diagrams for all three processes. [4 marks]

   (f) Why is the gluon-initiated process the dominant one? [2 marks]

9. At various experiments, $\Upsilon(4S)$ mesons (bound states of a b and a $\bar{b}$-quark) with a mass of $M_{\Upsilon(4S)} = 10.58$ GeV are being produced in the reaction $e^+e^- \to \Upsilon(4S)$. The width of the $b\bar{b}$-meson is $\Gamma_{\Upsilon(4S)} \approx 20$ MeV, and the $\Upsilon(4S)$ mesons nearly always decay into a pair of B-mesons, each with a mass of $m_B = 5.28$ GeV. These $B\bar{B}$ systems have been used to study in great detail the inner structure of the B-mesons.

In the rest of this question, you may ignore the mass of the electron and positron.

(a) What is the minimum energy of the electron and positron in the centre-of-mass frame of the reaction? [2 marks]

(b) Using the formula

$$\Gamma_{\Upsilon(4S)\to B\bar{B}} = \frac{|\underline{p}_B|}{8\pi M^2_{\Upsilon(4s)}} \left|\mathcal{M}_{\Upsilon(4S)\to B\bar{B}}\right|^2 ,$$

the masses of the involved particles and the decay width given above, estimate numerically the value of the unknown matrix element squared $|\mathcal{M}_{\Upsilon(4S)\to B\bar{B}}|^2$. [4 marks]

(c) Given the lifetime of the B mesons of $\tau_B = 1.5$ ps, calculate their average decay length in the centre-of-mass frame of the $\Upsilon(4S)$. [3 marks]

[Hint: Use the relation of energy and rest mass, $E = m_0/\sqrt{1-v^2}$ to deduce the velocity v. Do not forget to take into account time dilation!]

(d) To increase the decay length, the B mesons and therefore the $\Upsilon(4S)$ need to have a larger momentum. What value must the B momentum have for the B-meson to have an average decay length of 20 mm? [1 mark]

(e) What is the difference between baryons and mesons? What is the quark content of the mesons B^+, B^-, B^0 and $\bar{B}^0$? [4 marks]

(f) There is a phenomenon called $B^0\bar{B}^0$ mixing. Draw at least one lowest order Feynman diagram responsible for this effect. Comment on the significance of this phenomenon. [6 marks]