(a) Normalization requires

$$\int_{-b}^{b} \psi^*(x)\psi(x)\,dx = 1$$

[1 mark, bookwork]

$$1 = A^2 \int_{-b}^{b} (b^2 - x^2)^2\,dx = A^2 \int_{-b}^{b} b^4 - 2b^2x^2 + x^4\,dx = A^2 \left[b^4 x - \frac{2b^2x^3}{3} + \frac{x^5}{5} \right]_{-b}^{b}$$

$$= A^2 \left(b^5 - \frac{2b^5}{3} + \frac{b^5}{5} \right) - A^2 \left(-b^5 + \frac{2b^5}{3} - \frac{b^5}{5} \right)$$

$$= A^2 2b^5 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15} A^2 b^5 = 1 \qquad \Rightarrow A = \frac{1}{4}\sqrt{\frac{15}{b^5}}$$

[1 mark, unseen]

Probability of being at $|x| < b/2$ is

$$P = \int_{-b}^{b} \psi^*(x)\psi(x)\,dx = A^2 \left[b^4 x - \frac{2b^2x^3}{3} + \frac{x^5}{5} \right]_{-b/2}^{b/2} = 2A^2 \left(\frac{b^5}{2} - \frac{2b^5}{2^3 \times 3} + \frac{b^5}{2^5 \times 5} \right)$$

$$= 2\frac{1}{16}\frac{15}{b^5} b^5 \left(\frac{1}{2} - \frac{1}{12} + \frac{1}{160} \right) = \frac{15}{8} \left(\frac{240 - 40 + 3}{480} \right) = \frac{15}{8} \left(\frac{203}{480} \right) = \frac{203}{256} = 0.793$$

[2 marks, unseen]

(b) The wavefunction is symmetric about $x = L/2$ and so by symmetry $c_2 = c_4 = 0$ [2 marks, unseen]
c_1 is the largest and positive as the wavefunction most resembles $\sin(\pi x/L)$. [1 mark, unseen]
The wavefunction is more centrally peaked than $\sin(\pi x/L)$ and so c_3 is negative [1 mark, unseen]

(c) Normalization requires

$$1 = \int (c\phi_1^*(x) + 2c\phi_2^*(x))(c\phi_1(x) + 2c\phi_2(x))\,dx = \int \left(c^2\phi_1^*\phi_1 + 2c^2\phi_2^*\phi_1 + 2c\phi_1^*\phi_2 + 4c^2\phi_2^*\phi_2 \right)\,dx$$

So using orthonormal properties we have $1 = c^2(1+4)$, $\quad c = 1/\sqrt{5}$ [2 marks, unseen]
Probability of measuring E_1 is $c_1^2 = 1/5$ i.e. 20% [1 mark, unseen]
Measurement collapses the wavefunction and so repeated measurements will always yield E_1. [1 mark, unseen]

(d) For the Coulomb potential $V(r) \propto 1/r = A/r$, hence

$$2\langle T \rangle = \langle \underline{r}.\hat{\underline{r}} \frac{d}{dr}\frac{A}{r} \rangle$$

$$= \langle -r\frac{A}{r^2} \rangle = -\langle V \rangle \quad \text{i.e.} \quad 2\langle T \rangle + \langle V \rangle = 0$$

[2 marks, unseen]

In the ground state $E = T + V = -13.6$ eV, but $V = -2T$ and so

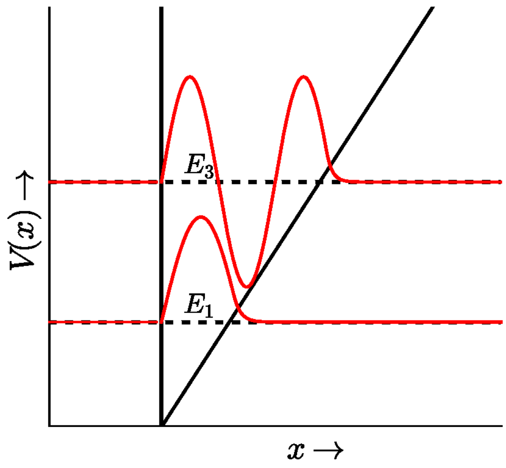
$$T - 2T = -13.6 \text{ eV} \qquad \Rightarrow T = 13.6 \text{ eV}$$

[2 marks, unseen]

(e) The sketch should show the following features:

1. wavefunction zero where potential is infinite [1 mark, unseen]

2. One extremum for the ground state and 3 for the 2nd excited state [2 marks, unseen]
3. Smooth transition from oscillatory to exponential decaying at the classical maximum radius [1 mark, unseen]



(f) That the energy levels are equally spaced with spacing $\hbar\omega$. [2 marks, bookwork]

Given $\phi_{n+1} = A_n a_+ \phi_n$ and $[H, a_+]\phi_n = \hbar\omega a_+ \phi_n$ gives

$$Ha_+\phi_n - a_+ H\phi_n = \hbar\omega a_+\phi_n$$

$$HA_n^{-1}\phi_{n+1} - a_+ E_n \phi_n = \hbar\omega A_n^{-1}\phi_{n+1}$$

$$A_n^{-1}E_{n+1}\phi_{n+1} - E_n A_n^{-1}\phi_{n+1} = \hbar\omega A_n^{-1}\phi_{n+1}$$

$$E_{n+1} - E_n = \hbar\omega \qquad \Rightarrow \qquad E_{n+1} = E_n + \hbar\omega$$

[2 marks, unseen]

(g) The radial probability distribution is given by

$$P(r)\,dr \propto r^2 R(r)^2\,dr$$

[1 mark, bookwork]

$$P(r)\,dr \propto r^4 \exp\left(-\frac{r}{a}\right)\,dr$$

At maximum

$$\frac{dP}{dr} = 0$$

[1 mark, unseen]

$$0 = \left(4r^3 - \frac{r^4}{a}\right)\exp\left(-\frac{r}{a}\right)$$

$$0 = r^3\left(4 - \frac{r}{a}\right)\exp\left(-\frac{r}{a}\right)$$

Hence turning point, which is the maximum is at $r = 4a$ [2 marks, unseen]

(h) The perturbation to the energy is given by

$$E_n^1 = \langle \psi_n | V'(x) | \psi \rangle$$

[1 mark, bookwork]

$$E_n^1 = \frac{2}{L}\epsilon \int_0^L \sin^2(n\pi x)\delta(x - L/2)\,dx$$

[1 marks, unseen]

$$E_n^1 = \frac{2}{L}\epsilon \sin^2(n\pi L/2)$$

[1 mark, unseen]

Hence for n even $E_n^1 = 0$ and for odd n $E_n^1 = 2\epsilon/L$ [1 mark, unseen]

(a) Using non-degenerate perturbation theory

$$\frac{E_n^1}{E_n^0} = \frac{1}{E_n^0}\langle\psi_n^0|H'|\psi_n^0\rangle.$$

For hydrogen

$$V(r) = \frac{-e^2}{4\pi\epsilon\, r}$$

[2 marks bookwork]

Hence

$$H' = \frac{1}{2mc^2}\left[\frac{\alpha^2 mc^2}{2n^2} + V(r)\right]^2 = \frac{1}{2mc^2}\left[\frac{\alpha^2 mc^2}{2n^2} - \frac{\hbar^2}{am}r^{-1}\right]^2$$

[2 marks unseen]

$$\frac{E_n^1}{E_n^0} = \frac{1}{2mc^2}\frac{2n^2}{\alpha^2 mc^2}\langle\psi_n^0|\left(\frac{(\alpha^2 mc^2)^2}{(2n^2)^2} - 2\frac{\alpha^2 mc^2}{2n^2}\frac{\hbar^2}{am}r^{-1} + \frac{\hbar^4}{a^2m^2}r^{-2}\right)|\psi_n^0\rangle$$

$$= \frac{1}{2mc^2}\frac{\alpha^2 mc^2}{2n^2}\langle\psi_n^0|\psi_n^0\rangle - 2\frac{1}{2mc^2}\frac{\hbar^2}{am}\langle\psi_n^0|r^{-1}|\psi_n^0\rangle + \frac{1}{2mc^2}\frac{2n^2}{\alpha^2 mc^2}\frac{\hbar^4}{a^2m^2}\langle\psi_n^0|r^{-2}|\psi_n^0\rangle$$

$$= \frac{\alpha^2}{4n^2} - \frac{\hbar^2}{am^2c^2}\langle r^{-1}\rangle + \frac{n^2\hbar^4}{\alpha^2a^2m^4c^4}\langle r^{-2}\rangle$$

[4 marks unseen]

(b) For $\psi_1^0 = (\pi a^3)^{-1/2}e^{-r/a}$

$$\langle r^{-1}\rangle = 4\pi\int_0^\infty (\pi a^3)^{-1/2}e^{-r/a}\; r^{-1}\; (\pi a^3)^{-1/2}e^{-r/a}\; r^2\, dr$$

[2 marks unseen]

$$= \frac{4\pi}{\pi a^3}\int_0^\infty r\; e^{-2r/a}\, dr$$

[1 mark unseen]

$$= \frac{4}{a^3}\frac{a^2}{4} = \frac{1}{a}$$

[2 marks unseen]

Similarly

$$\langle r^{-2}\rangle = \frac{4\pi}{\pi a^3}\int_0^\infty e^{-2r/a}\, dr = \frac{4}{a^3}\frac{a}{2} = \frac{2}{a^2}$$

[2 marks unseen]

Hence

$$\frac{E_1^1}{E_1^0} = \frac{\alpha^2}{4n^2} - \frac{\hbar^2}{am^2c^2}\frac{1}{a} + \frac{\hbar^4}{\alpha^2a^2m^4c^4}\frac{2}{a^2}$$

$$= \frac{\alpha^2}{4} - \frac{\hbar^2}{a^2m^2c^2} + \frac{2\hbar^4}{\alpha^2a^4m^4c^4}$$

[1 mark unseen]

Using $\alpha = 1/137$, $a = 5.29\times10^{-11}$ m and

$$\frac{\hbar}{amc} = \frac{1.05\times10^{-34}}{5.29\times10^{-11}\;\times\;9.11\times10^{-31}\;\times\;3\times10^{8}} = 7.29\times10^{-3} = \frac{1}{137}$$

[2 marks unseen]

we have

$$\frac{E_1^1}{E_1^0} = \frac{1}{4 \times 137^2} - \frac{1}{137.7^2} + \frac{2}{137.7^2}$$
$$= \frac{1}{137^2}\left(\frac{1}{4} - 1 + 2\right)$$
$$= \frac{1}{137^2}\,(0.25 - 1 + 2) = \frac{1}{137^2}\,(1.25) = 6.66 \times 10^{-5}$$

[2 marks unseen]

(a)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}\psi_2(x)e^{-iE_2t/\hbar}$$

[1 mark bookwork]

$$|\Psi(x,t)|^2 = \frac{1}{2}\left(\psi_1^*(x)e^{iE_1t/\hbar} + \psi_2^*(x)e^{iE_2t/\hbar}\right)\left(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}\right)$$

$$= \frac{1}{2}\left(\psi_1^*(x)\psi_1(x) + \psi_1^*(x)\psi_2(x)e^{i(E_1-E_2)t/\hbar} + \psi_2^*(x)\psi_1(x)e^{-i(E_1-E_2)t/\hbar} + \psi_2^*(x)\psi_2(x)\right)$$

[2 marks unseen]

This simplifies as $\psi_n(x)$ are real

$$= \frac{1}{2}\left(\psi_1(x)^2 + \psi_1(x)\psi_2(x)\left(e^{i(E_1-E_2)t/\hbar} + e^{i(E_1-E_2)t/\hbar}\right) + \psi_2(x)^2\right)$$

$$= \frac{1}{2}\left(\psi_1(x)^2 + 2\psi_1(x)\psi_2(x)\cos(E_1-E_2)t/\hbar) + \psi_2(x)^2\right)$$

[2 marks unseen]

Subbing in the expressions for $\psi_n(x)$ and E_n

$$|\Psi(x,t)|^2 = \frac{1}{L}\left(\sin^2(\pi x/L) + 2\sin(\pi x/L)\sin(2\pi x/L)\cos((1-4)\omega t/) + \sin^2(2\pi x/L)\right)$$

$$|\Psi(x,t)|^2 = \frac{1}{L}\left(\sin^2(\pi x/L) + \sin^2(2\pi x/L) + 2\sin(\pi x/L)\sin(2\pi x/L)\cos(3\omega t)\right)$$

[2 marks unseen]

(b)

$$\langle x\rangle = \int_0^L x|\Psi(x,t)|^2\,dx$$

[2 marks unseen]

$$= \frac{1}{L}\int_0^L x\sin^2(\pi x/L)\,dx + \frac{1}{L}\int_0^L x\sin^2(2\pi x/L)\,dx + \frac{1}{L}\int_0^L 2\sin(\pi x/L)\sin(2\pi x/L)\,dx\ \cos 3\omega t$$

[2 marks unseen]

$$= \frac{1}{L}\frac{L^2}{4} + \frac{1}{L}\frac{L^2}{4} - \frac{2}{L}\frac{8L^2}{9\pi^2}\cos 3\omega t$$

[2 marks unseen]

$$= \frac{L}{2} - \frac{16L}{9\pi^2}\cos 3\omega t$$

Frequency of oscilation $3\omega/(2\pi)$ (angular frequency 3ω also acceptable) and the amplitude is $16L/(9\pi^2) = 0.18L$

[2 marks unseen]

(c)

$$\langle E\rangle = \int \Psi^* H\Psi dx = \frac{1}{2}\int\left(\psi_1^*(x) + \psi_2^*(x)\right)H\left(\psi_1(x) + \psi_2(x)\right)\,dx$$

[1 mark bookwork]

Using $H\psi_n = E_n\psi_n$

$$= \frac{1}{2}\int\left(\psi_1^*(x) + \psi_2^*(x)\right)\left(E_1\psi_1(x) + E_2\psi_2(x)\right)\,dx$$

[1 mark unseen]

$$= \frac{1}{2}\int E_1\psi_1^*(x)\psi_1(x) + E_2\psi_2^*(x)\psi_1(x) + E_2\psi_1^*(x)\psi_2(x) + E_1\psi_2^*(x)\psi_1(x)\,dx$$

[1 mark unseen]

Using the orthonomal properties of the eigenfunctions we find

$$\langle E \rangle = \frac{1}{2}\left(E_1 + E_2 + E_2 \times 0 + E_1 \times 0\right) = \frac{1}{2}\left(E_1 + E_2\right)$$

[1 mark unseen]

Energy is conserved. [1 mark unseen]

Electromagnetism

Professor Hampshire June 18 Qn. 1

a) A plasma is a collection of positive and negative charges that are net neutral. Examples include, Ionosphere, fluorescent light bulbs, the atmosphere of stars, the plasma in a fusion tokamak,...

[4 marks – unseen]

b) In polar dielectrics, there are permanent dipole moments that rotate in response to an applied E-field. In non-polar dielectrics, there is the displacement of charge to produce dipoles can be in response to an applied E-field.

[4 marks – unseen].

c) Symmetry says the E-field points at right angles to the straight edge of the semi-circular sheet of charge. Using Coulomb's law: $\underline{\boldsymbol{E}} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$. Estimates are : $q \sim \rho_A \pi r_0^2/2$; $r \sim r_0/2$; and angular/symmetry considerations give a reduction of $\sim 1/\sqrt{2}$ (not required for order of magnitude estimate). $E = \rho_A/2\sqrt{2}\,\varepsilon_0$V.m^{-1} .

[4 marks – unseen].

d)

$$\underline{\boldsymbol{C}} = x^3 y\hat{\boldsymbol{k}}$$

$$LHS:\ \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{C}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^3 y \end{vmatrix} = x^3\,\hat{\boldsymbol{\imath}} - 3x^2 y\hat{\boldsymbol{\jmath}},\ \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{C}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & -3x^2 y & 0 \end{vmatrix} = (-6xy)\hat{\boldsymbol{k}}$$

RHS,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{C}} = 0$$

$$-\nabla^2 \underline{\boldsymbol{C}} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) x^3 y\hat{\boldsymbol{k}} = (-6xy)\hat{\boldsymbol{k}}$$

LHS = RHS as required

[4 marks – unseen].

e) Integrating and using Gauss' theorem gives: $\int \underline{\boldsymbol{J}}\, dS = -\int \frac{\partial \rho}{\partial t}\, dV$. The law holds for an arbitrary volume V, bounded by its surface, S. The LHS is the flux of the current density out through the surface S. This flux is equal to the charge leaving through the surface per second. The RHS is the decrease in the charge within the volume V per second. Hence if the equation requires that all changes in charge within any volume are accounted for by them leaving through the surface of the volume into its neighbour, it necessarily describes the conservation of charge.

[4 marks – unseen].

f)

$$\underline{\boldsymbol{k}}_0 = (3 + 2i)\hat{\boldsymbol{k}}\ \mathrm{m}^{-1}, \qquad \underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \exp i((k_{real} + ik_{imaginary}) \cdot x - \omega t)$$

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \exp i(3x - \omega t)\exp(-2x)$$

$$k_{real} = \frac{2\pi}{\lambda} \quad => \lambda = \frac{2\pi}{3} = 2.1\ \mathrm{m}$$

[4 marks – unseen]

g)

Using the definition of D, or otherwise:

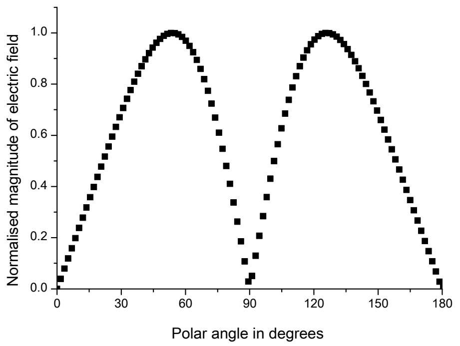
$$D = \varepsilon_0 E + P$$

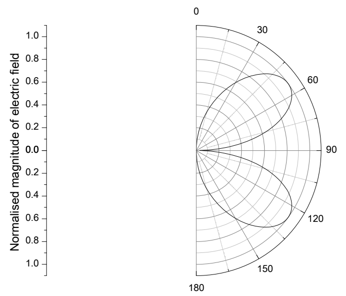
$$D = \varepsilon_0 E_{External} = \varepsilon_r \varepsilon_0 E_{Internal},\ \ \varepsilon_r = \frac{E_{External}}{E_{Internal}} = \frac{1}{0.2} = 5$$

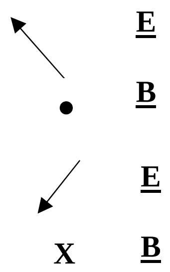
[4 marks – unseen]

Electromagnetism

Professor Hampshire June 18 Qn. 2







(a) The Polar diagram requested is on the RHS. Units for y-axis are: $1.189\frac{\mu_0 c I_0}{2\pi r}$ [4 Marks – unseen]

(b) Directions of **E**-fields and **B**-fields – **E** × **B** in the direction of propagation [4 Marks – unseen]

(c)

Taking amplitudes,

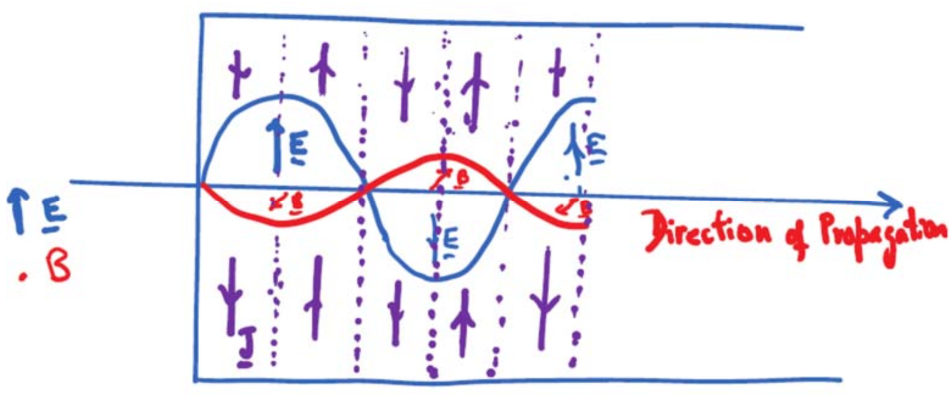
$$|\underline{\boldsymbol{E}}| = \frac{\mu_0 c I_0}{2\pi r}\frac{\sin(\pi\cos(\theta))}{\sin(\theta)}$$

By rearrangement,

$$I_0 = \frac{2\pi r E}{\mu_0 c}\frac{\sin(\theta)}{\sin(\pi\cos(\theta))}$$

Taking $\theta = 30°$

$$=> I_0 = \frac{2\pi \cdot 20\times10^3 \cdot 5}{(4\pi\times10^{-7})(3\times10^8)}\frac{1/2}{0.408} = 2042 \text{ A}$$ [4 marks – unseen]



d) Sketch:

[4 marks unseen]

e) The material is insulating so there are no currents produced by the E-field. The right-hand screw rule gives the direction of the B-field relative to the E-field. The B-field magnetises the reversible weakly paramagnetic material. Ampere's law gives the direction of the currents flowing associated with the field in the material. [4 marks - unseen]

Electromagnetism

Professor Hampshire June 18 Qn. 3

(a)

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2}$$

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \exp i(kx - \omega t) => k^2 = \mu_0 \epsilon_0 \omega^2$$

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = v_{phase}$$

[4 Marks – Seen]

(b)

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \sin(\frac{6\pi x}{a}) \sin(\frac{6\pi z}{a}) \exp i(ky - \omega t)$$

$$-\frac{\pi^2}{a^2} 36 - \frac{\pi^2}{a^2} 36 - k^2 = -\mu_0 \epsilon_0 \omega^2$$ [3 marks – unseen]

Condition for propagation: $k^2 > 0$ / k must be real.

$$\frac{\pi^2}{a^2} \cdot 72 - \mu_0 \epsilon_0 \omega_{min}^2 > 0$$

$$\omega_{min} > \frac{6\sqrt{2}\pi c}{a} = 7.99 \times 10^{10} \text{ Rad s}^{-1}$$

$$f > 1.27 \times 10^{10} \text{ Hz}$$

[3 Marks unseen]

(c) The condition for minimum angular frequency is,

$$\omega_{min} > \frac{6\sqrt{2}\pi c}{a}$$

In other solutions, the integer 6 can be replaced by any integer n:

$$\omega_{min} > \frac{\sqrt{2}\pi c}{a} \cdot n$$

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \sin(\frac{n_1 \pi x}{a}) \sin(\frac{n_2 \pi z}{a}) \exp i(ky - \omega t)$$

Any solution with n_1 and n_2 less than 6 is acceptable. Lower integer n implies a lower minimum frequency. n_1 does not have to equal n_2.

[4 Marks unseen]

(d) The minimum frequency is given by: $\omega_{min} > \frac{\pi c}{a\sqrt{\varepsilon_r}} \cdot ((n_1)^2 + (n_2)^2)$ where ε_r is the relative permittivity which is greater than unity and positive definite.
The minimum frequency decreases when the dielectric is added.

[3 Marks unseen]

The dispersion relation is:

$$\left(\frac{n_1 \pi}{a}\right)^2 + \left(\frac{n_2 \pi}{a}\right)^2 + k^2 = \mu_0 \varepsilon_r \epsilon_0 \omega^2$$

$$\frac{\omega}{k} = \left(\mu_0 \varepsilon_r \epsilon_0 - \left(\frac{n_1 \pi}{a\omega}\right)^2 - \left(\frac{n_2 \pi}{a\omega}\right)^2\right)^{-1/2}$$

The wavevector gets bigger so the wavelength gets smaller when the dielectric is added.

[3 Marks unseen]