Workshop 1: Applications of the Lagrangian Approach

1. Consider a point mass (of mass m) falling vertically in the Earth's gravitational field (take the acceleration $= g$ to be constant, i.e., independent of the distance of the point mass from the Earth's surface, and ignore air resistance).

   (a) What is the kinetic energy T of the point mass? What is the potential energy V? Hence, determine the Lagrangian $L = T - V$.

   (b) Use the Euler-Lagrange equation to determine a second order differential equation for the vertical motion (z direction) of the point mass.

   (c) Find the equation of motion, and express the solution in terms of the point mass's initial position $z(0)$ and velocity $\dot{z}(0)$.

2. Consider a point mass (of mass m) attached to a spring (spring constant k), constrained somehow to move only in the x direction, where $x = 0$ is taken to be the equilibrium position of the point mass. This is just a one-dimensional simple harmonic oscillator (undamped), with characteristic oscillation frequency $\omega = \sqrt{k/m}$.

   (a) What is the kinetic energy T of the point mass? What is the potential energy V? Hence, determine the Lagrangian $L = T - V$.

   (b) Use the Euler-Lagrange equation to determine an equation for $\ddot{x}$.

   (c) Solve this linear, homogeneous, second-order differential equation using the trial solution $x \propto e^{\lambda t}$.

   (d) Find the general solution for the equation of motion, in terms of the point mass's initial position $x(0)$ and velocity $\dot{x}(0)$. Note that the position and velocity are both real, not complex.

   (e) The two independent real constant prefactors in the general oscillator solution, A and B, can equally well be represented in terms of two other real constants A' and ϕ', through $A = A' \sin\phi'$, and $B = A' \cos\phi'$. Use this, along with the trigonometric identity $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$, to show your solution can also be expressed in the form
   $$x(t) = A' \sin(\omega t + \phi'),$$
   and express A' and ϕ' in terms of the initial position and velocity.

3. Consider the situation of a small block (SB) of mass m sliding on a frictionless inclined plane (IP) of height h, which itself has a mass M and rests on a flat surface, without any friction between this surface and the inclined plane. Consider a set of cartesian axes such that the Y axis points up along the side of the initial location of the inclined plane, and the X axis points along the underside of the inclined plane (we ignore the Z direction). Take $(X_{\mathrm{SB}}, Y_{\mathrm{SB}})$ and $(X_{\mathrm{IP}}, Y_{\mathrm{IP}})$ to be the locations of the centre of mass of the SB, and the right-angled corner of the IP in this coordinate system, respectively.

   (a) Because the motion of the SB is constrained to remain on the upper surface of the IP, the value for the dynamical variable Y_{SB} can be determined exactly if the values for the dynamical variables X_{SB}, X_{IP} are known. Derive an equation (a constraint equation) describing this relationship.

   (b) Consider an alternative dynamical variable d, the distance of the SB centre of mass from the top of the IP. Determine expressions for X_{SB}, Y_{SB} in terms of X_{IP}, d, h and α. Taking the time-derivative, determine equivalent expressions for $\dot{X}_{\mathrm{SB}}$, $\dot{Y}_{\mathrm{SB}}$.

   (c) Trivially, the IP is constrained not to move in the Y direction, and so Y_{IP} and $\dot{Y}_{\mathrm{IP}}$ can be neglected. Determine an expression for the total kinetic energy T of the SB and IP system in terms of $\dot{X}_{\mathrm{IP}}$ and $\dot{d}$. Determine an expression for the total potential energy V in terms of X_{IP}, h and d.

   (d) Write down the Lagrangian L using your expressions for T and V. From the Lagrangian, derive equations for $\ddot{d}(t)$ and $\ddot{X}_{\mathrm{IP}}(t)$. Solve these differential equations, writing the solutions in terms of the initial positions and velocities $d(0)$, $X_{\mathrm{IP}}(0)$, $\dot{d}(0)$, $\dot{X}_{\mathrm{IP}}(0)$.

   (e) Assuming that the SB is initially at rest at the top of the IP, which is also at rest, derive an expression for the distance the IP has moved once the SB has reached the bottom of the IP, and hence (assuming $\alpha = \pi/4$ radians) determine how large m must be relative to M for this distance to be $= h/2$.