Chapter 11

Gravitational lensing

CO §28.4

Any object with mass causes the distortion ('warping') of space-time around it leading to the deflection of light rays - this phenomenon predicted by the theory of General Relativity (GR) is called gravitational lensing (GL). In addition to **light deflection**, GL may also create **image distortions, changes in brightness and multiple images of a lensed object** - all of which have been observed. Typically the effects are small meaning that our view of the Universe is not greatly distorted by GL. On the largest scales, GL of the Cosmic Microwave Background (CMB) has been used to constrain cosmology. It has also been used to measure the masses of galaxies and clusters of galaxies from the distortion introduced by GL of background galaxies, and to study faint distant galaxies (taking advantage of the increase in brightness). On smaller scales, GL can constrain the particle nature of the dark matter and has been used to image the event horizon of a black hole (Chapter 10). GL in the solar system provided the first verification of GR.

11.1 The lens equation

We first attempt to compute the deflection angle in Newtonian mechanics. The observed deflection of light rays grazing the Sun agrees with the GR prediction but disagrees with our 'Newtonian' estimate, providing the first confirmation of GR.

11.1.1 Bending of light

The shape of the orbit of an unbound test particle around a mass M is hyperbolic[1]: the incoming particle is 'deflected' by the encounter with the more massive object by an angle α which depends on M, the initial speed, v, and impact parameter, b. Provided the deflection angle is small,

$$\alpha = \frac{2GM}{b\,v^2}\,, \tag{11.1}$$

as derived in the Appendix (Eq. 11.10). Notice that this does not depend on the mass of the test particle. We now make the rather bold assumption that this applies to light as well and simply substitute v by c to obtain the 'Newtonian' prediction for GL.

For starlight grazing the Sun, we take $M = M_\odot$, $b = R_\odot \approx 7\times 10^5$km, the radius of the Sun, to find [2] $\alpha = 0.87$ arcsec. Our numerical value agrees with Einstein's 1911 estimate as well as the calculation by *Johann Soldner* almost a century earlier.

Curiously, the answer is wrong! After developing the theory of General Relativity (GR), Einstein recalculated the deflection angle for light as,

$$\alpha = \frac{4GM}{b\,c^2}\,, \tag{11.2}$$

exactly *twice* the Newtonian prediction[3] of Eq. (11.1). The difference in prediction between Newtonian physics and GR motivated Eddington to lead an expedition to measure α during 1919's total solar eclipse[4]. Eddington's measurements were revealed during a talk at the Royal Society, and confirmed GR's prediction[5].

[1] For example the motion of an unbound asteroid around the Sun.

[2] In radians, $\alpha = 4.2\times 10^{-6} \ll 1$, validating our assumption.

[3] The intuitive reason for the larger deflection takes account of *time-dilation*: it is as if the photon spends more time near the lens, so that the gravitational force has more time to act. Accounting for gravitational time-dilation - 'clocks appear to run slower when in a gravitational potential' - is essential to obtain the required precision in Global Positioning Systems (GPS).

[4] The expedition measured the apparent movement of stars on the sky for sight lines grazing the Sun compared to more distant sight lines. This is (only) possible during a solar eclipse, when the moon blocks most of the Sun light, allowing the observation of stars close in angle to the Sun.

[5] But subsequent measurements found less convincingly in favour of GR - probably due

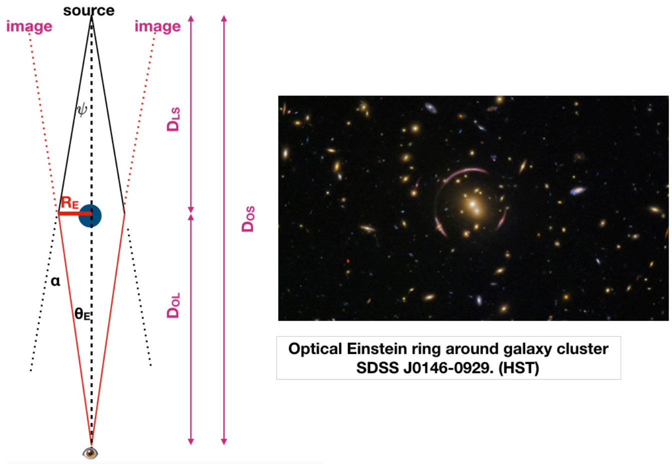

Figure 11.1: *Left panel:* Lensing configuration in a case in which observer, lens and source are co-linear. The source is imaged in a ring, called Einstein ring. *Right panel:* (Approximate) Einstein ring, when a distant galaxies is lensed by a foreground cluster of galaxies, SDSS J0146-0929..

11.2 Deflection through gravitational lensing

We can compute the image formed by GL for some simple configurations.

11.2.1 Point-like lens and source

Co-linear

Consider the special case when observer, lens and source are co-linear as illustrated in Fig. 11.1. Introducing the distances observer-lens, observer-source, and source lens, as in the figure, we find

$$\theta_E = \frac{R_E}{D_{OL}}; \quad \psi = \frac{R_E}{D_{LS}}; \quad \psi + \theta_E = \alpha; \quad \alpha = \frac{4GM}{R_E\, c^2}\,, \tag{11.3}$$

to measurement errors. Of the other two predictions that Einstein made - gravitational redshift was not detected and some astronomers were not convinced by the explanation for the perihelion precession of Mercury. As a consequence, the theory of GR was initially not as widely accepted as you may have thought, see this recent review.

where M is the mass of the lens. The first three relations are simple geometry[6], the last one is the lensing equation for the deflection angle from Eq. (11.2). Combining these yields

$$\theta_E = \left(\frac{4GM}{c^2}\right)^{1/2} \left(\frac{D_{LS}}{D_{OS}D_{OL}}\right)^{1/2} . \tag{11.4}$$

Notice that in this case the observer sees the source lensed in a **ring** called an *Einstein ring*[7]. *If* we can measure the distances observer-lens and observer-source, then the angular extent of the Einstein ring yields the mass of the lens, M.

This is a very powerful way of measuring masses of astrophysical objects, because we need not make any assumptions about what form this mass takes (baryonic or dark, for example), nor do we need to assume anything about its dynamical state.

The panel to the right shows an *approximate* optical Einstein ring, where a background galaxy is lensed by a foreground cluster. Neither the galaxy nor the cluster are point-masses: as far as lensing is concerned, the Einstein ring is a measure of the mass enclosed by R_E, the 'Einstein radius'.

Not co-linear

Figure 11.2 illustrates the configuration. Starting with simple geometry we find under the assumption of small angles, that

$$\theta_S = \theta_I - \beta = \theta_I - \frac{D_{SL}}{D_{OS}}\alpha; \quad R = \theta_I\, D_{OL}\, . \tag{11.5}$$

Combined with the lensing equation, $\alpha = 4GM/(Rc^2)$, yields a quadratic equation for the observable angle, θ_I, between the directions lens-image,

$$\theta_I^2 - \theta_I\theta_S = \frac{4GM}{c^2}\frac{D_{LS}}{D_{OS}D_{OL}} \equiv \theta_E^2\, , \tag{11.6}$$

[6]These relation are valid in the small-angle approximation. Notice that this approximation does not apply to the *cartoon* illustration, but works very well in real applications.

[7]In the cartoon, the observer sees *two images* in the plane of the paper. However, now rotate the plane along the axis observer-source: the observer sees the two images *in each of these planes* - i.e. they see a ring.

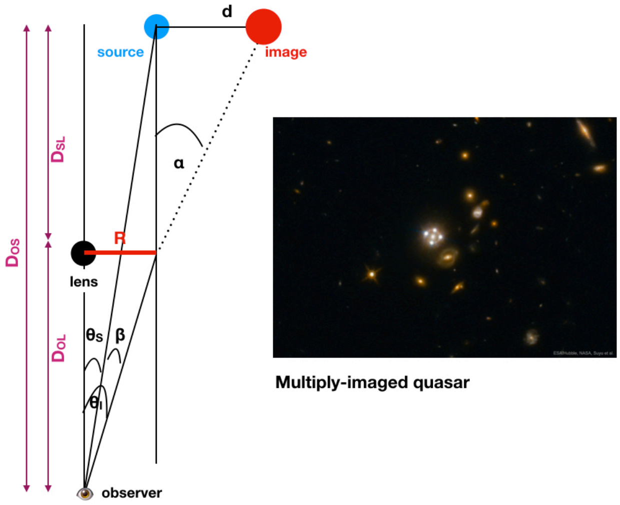

Figure 11.2: *Left panel:* Lensing configuration in a case in which observer, lens and source are not co-linear. *Right panel:* 4 images of a quasar lensed by an intervening galaxy.

using Eq. (11.4) for θ_E. This equation has two roots[8], $\theta_{I,\pm}$, which can be used to determine the two unknowns, θ_S and θ_E,

$$\theta_{I,+} + \theta_{I,-} = \theta_S; \quad (\theta_{I,+} - \theta_{I,-})^2 = (\theta_{I,+} + \theta_{I,-})^2 + 4\theta_E^2 \,. \tag{11.7}$$

Measuring the directions to the two images yields θ_E which yields the mass of the lens, M, provided all distances can be measured.

11.2.2 More complex lens-source configurations

More complex configurations occur when the source or the lens are spatially extended (for example the source is a galaxy, the lens a cluster of galaxies). The fact that the deflection angle may vary across a spatially extended source leads to image distortions. An extended *lens* may generate more than two images of a source, as illustrated by the 4 images[9] produced when a back-

[8]One root is positive, one is negative meaning the image is seen to the left of the source, under the convention that angles are positive when measured clockwise from the direction to the lens.

[9]The fifth central object is the lensing galaxy itself.

ground quasar is lensed by (the extended dark matter halo of) a foreground galaxy, shown in the right panel of Fig. 11.2.

11.3 Applications of gravitational lensing

Previously we only discussed how GL deflects light. In addition to displacing the image, GL may enlarge the image as compared to the source, make it look brighter, and create multiple images of the same source. Here we will show examples of these effects and discuss their applications.

11.3.1 GL in clusters of galaxies

GL is strong for sight lines passing close to the centre of a cluster of galaxies, causing the appearance of multiple images of the same galaxy, stretched in the form of tangential (and sometimes radial) arcs. The number of images, how strongly they are stretched and even their relative brightness, all depend on the projected mass distribution. These effects are referred to as **strong gravitational lensing**. Strong GL makes it possible to *infer* the mass distribution of the lens - i.e. the cluster - by modelling all of these effects. Fig. 11.3 shows a cluster (left) and its projected mass distribution (right) in the top panel. Gratifyingly, estimates of cluster masses using GL are in reasonable agreement with those based on the dynamics of galaxies or as inferred from the hot X-ray gas in clusters discussed in Chapter 8.

The effects of GL are weaker for background galaxies with lines-of-sight at larger impact parameters from the cluster. However, GL will still distort images slightly, imaging an intrinsically round source into a slightly elongated image. For any individual galaxy, it is not possible to blame the image elongation and its orientation on GL, since the galaxy that is lensed may be intrinsically elongated. However, GL will induce similar elongations and image orientations for galaxies that are close to each other on the sky - meaning distortions are correlated. This is called **weak gravitational lensing**: measuring the correlated distortions of galaxies on the sky, to infer the required lensing mass distribution.

Back to strong GL. Cluster mass distributions have characteristic lines where the effects of GL are very strong, with images of background galaxies factors of several brighter than they would be in the absence of lensing. These characteristic lines are called *caustics* and the reason they appear is similar

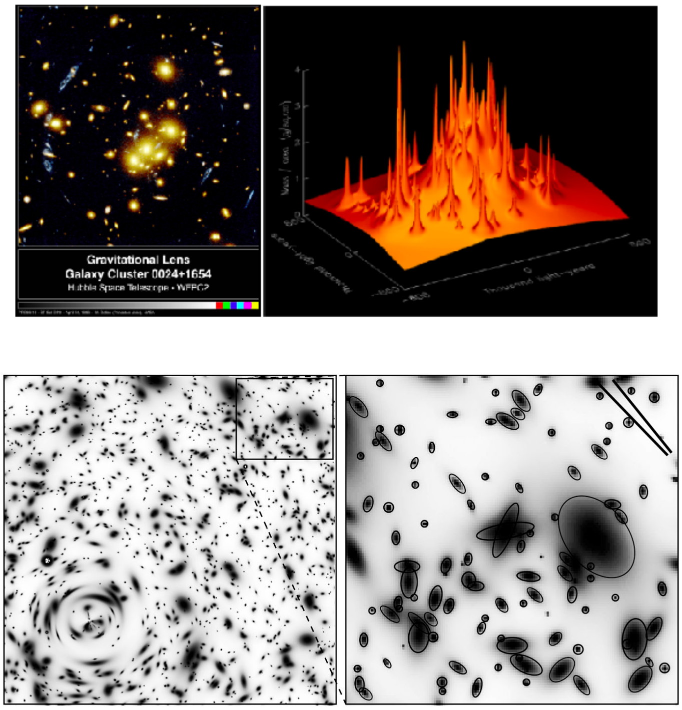

Figure 11.3: *Top left panel:* Optical image of a cluster of galaxies. Modelling of the multiple images of background galaxies due to GL makes it possible to infer the project mass distribution, which is shown in the *top right panel*, from Freese 2009. *Bottom panel:* At large impact parameter, GL does not produce multiply-imaged galaxies any more. However, it is still possible to infer statistically small stretchings in the shapes of background galaxies caused by GL, since these are correlated on the sky. This effect is exaggerated in this cartoon: close to the cluster's centre (bottom left), GL strongly deforms images of background galaxies into tangential arcs. Further away from the centre, distortions are much less but can be detected because they are *corrrelated* on the sky - galaxies close on the sky are stretched in similar directions.

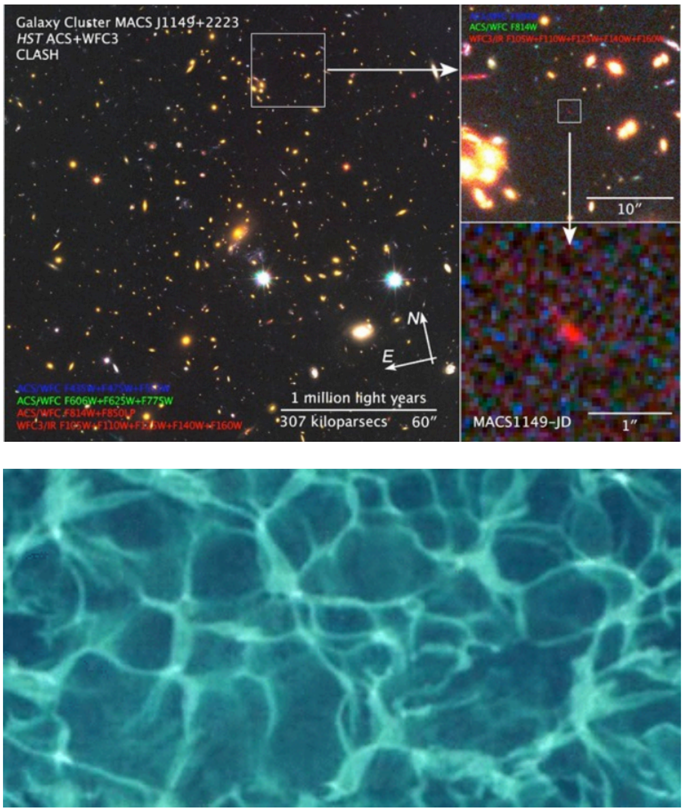

Figure 11.4: The physics of lensing caustics. *Bottom panel:* caustics - lines where sun light gets focussed on the bottom of a swimming pool. *Top panels:* Identifying very distant galaxies using GL near caustics in galaxy cluster MACS J1149-2223 (see text for details). From Heap 2015.

to the appearance of bright lines at the bottom of an outdoor swimming pool due to the focussing of sun light. This is illustrated in Fig. 11.4. Once a mass model for a cluster is constructed, it is possible to find where the caustics are - that is, where the lines are along which background galaxies will be strongly lensed with their image much brighter than it would be without GL. With flux enhancements by factors of 10-30, this enables us to study very distant galaxies that would be too faint to study without the help of GL. The top right panel identifies a faint red galaxy which may be one of the most distant galaxies ever identified. Without GL by the cluster, it would be impossible to find, let alone study, such infant galaxies.

11.3.2 GL and the nature of the dark matter

The nature of the dark matter (DM) is currently unknown with some type of elementary particle not part of the Standard Model of particle physics leading the pack of candidates. Because GL is agnostic about the nature the mass that causes lensing, it might be possible to constrain the DM's nature.

Dark matter in the Milky Way's halo

The MACHO collaboration proposed the following ingenious method to probe the MW's dark matter halo. Assume that the DM takes the form of Massive Astrophysical Compact Halo Objects - MACHO's - for example compact objects with masses of the order or the mass of a star or planet. Such MACHO's in the MW's halo would GL stars in other galaxies. The effect is generally weak, but occasionally, a MACHO might be passing *almost exactly* in front of a background star, and we may be able to detect the star becoming brighter by a factor of a few for a short time - due to GL. However, even if the whole dark matter halo of the MW was composed of such hypothetical objects, then the chance that any one background star is lensed is disappointingly small, 1 chance in 10^7. However, the collaboration realised that, by observing stars in the Large Magellanic Cloud (LMC) - a satellite of the MW (see chapter 5 on the local group) - it would be possible to observe of order 10^7 stars each night so we could expect of order 1 star to be lensed by a MACHO at anyone time. Notice that the brighting is time-dependent, because the lens moves - as does the observer and the source.

However, how can we conclude that a star gets brighter due to GL rather than simply because it is a variable star? An important aspect of GL is that

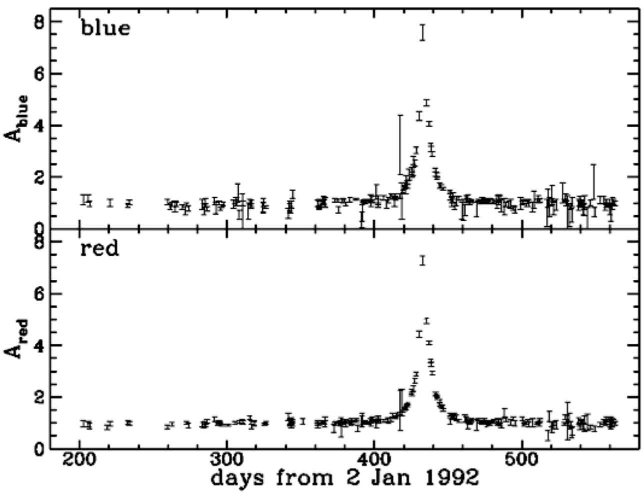

Figure 11.5: Flux of a star in the Large Magellanic Cloud (LMC, a satellite galaxy of the Milky Way) as a function of time, for a blue filter (top) and a red filter (bottom). The flux is normalised to its value at the start of the observations. Around day 420, the flux of the star increases by a factor of ~ 8 in both filters, due to gravitational lensing caused by an object moving across the sight line to the star.

the lensing properties are *independent* of the wavelength of light. Therefore if a star gets lensed, its brightness in a red and in a blue filter will increase by the exact same amount - whereas if it were an intrinsically variable star that would generally not be the case. Secondly, a variation due to lensing has a particular shape **and** a given star will vary *only once*. Combining all this should make it possible to recognise whether the flux of a star varies because it is intrinsically a variable star or because it is undergoing GL.

The collaboration detected several lensing events, one of which is shown in Fig. 11.5. However, in each case the lens was also detected in the image: it was not a dark matter MACHO but rather another star in the LMC. After years of observing, no lensing by MACHO's was detected. This negative result proved that the MW's dark matter halo*cannot* be dominated by MACHO's in the mass range $2.5 \times 10^{-7} \leq m/\mathrm{M}_{\odot} \leq 10^{-1}$.

Dark matter in other halos

The dark matter may clump on small scales producing DM halos with substructures. This is the case in the popular 'cold dark matter' (CDM) model, illustrated in the *left panel* of Fig. 11.6, which shows a numerical simulation of such a halo. Some of these substructures may host satellites of the main galaxy in such a DM halo. However, the CDM model predicts the existence of far more substructures than there are observed satellites around galaxies. Therefore many DM halos are dark: they do not host a visible galaxy.

How can we test whether real DM halos also contain so many dark substructures? Consider a case where a background galaxy is lensed into several tangential arcs due to GL by a massive intervening galaxy, as in the *right panel* of Fig. 11.6. The blue arcs are images of the *same* background galaxy, galaxies labelled G1-G3 are part of the same halo that causes the GL. In this particular image, there is another galaxy, labelled G4, that happens to fall on top of the right arc. As a consequence, the right arc is distorted by GL from G4 - but not the left arc.

This situation illustrates how we could detect dark DM substructures: examine in great detail the arcs produced by strong GL, and test whether we detect deformations of one image of the lensed galaxy but not the other(s). If such distortion is not caused by a detectable galaxy, then we may have found the first dark DM structure. No such cases have been found (yet). If you were to detect this, then drop everything you are doing to announce your discovery to the world!

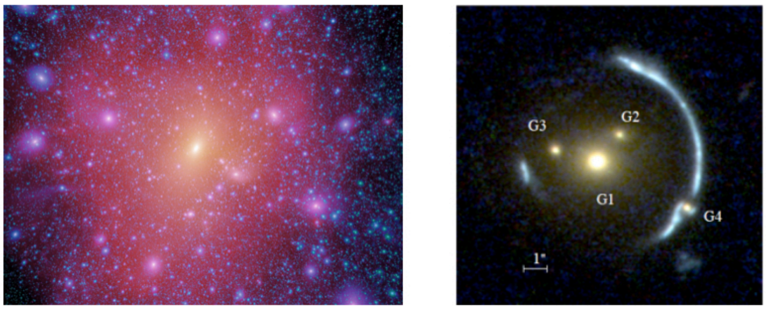

Figure 11.6: *Left panel:* theoretical density structure of a halo in case the DM is of the 'cold dark matter' type, from the Aquarius project. Regions of higher DM density are bright and yellow. This DM model predicts that the halo has a general smooth spherically symmetric density profile of the form $\rho(r) \propto 1/r^2$ around a centre that is close to the centre of the image, but on top of this contains many thousands of 'substructures' (the blobby density enhancements). Some of these may host satellite galaxies, but most must be completely dark since there are far fewer satellites observed compared to the number of substructures that are predicted. *Right panel:* Observed HST image of a background galaxy doubly imaged in the form of tangential arcs (the blue arcs) due to GL, lensing galaxies are labelled G1-G3. If one of the arcs happened to fall on top of a DM substructure, it would be additionally deformed. In this particular case, extra lensing is due to galaxy labelled G4, which distorts the right arc but not the left arc. Image from Lin 2009.

11.4 Summary

After having studied this lecture, you should be able to

- derive the Newtonian lensing equation (11.1)
- explain why alignment produces an Einstein ring, and derive its radius.
- explain how micro-lensing works and has been used to search for massive compact halo objects in the Milky Way halo
- explain how gravitational lensing has been used to estimate the mass of galaxy clusters, and hence infer that they contain dark matter.

Appendix: Gravitational deflection of a test mass

Consider the case were an unbound test mass (mass m) is gravitationally deflected by a more massive object (mass $M \gg m$). The orbit of m is of course hyperbolic. Here we examine the case where the deflection is small and we want to compute the *deflection angle* α - the angle between the initial and the final velocity. The geometry is illustrated in Fig. 11.7.

The deflection is caused by the component of the gravitational force perpendicular to the orbit, $F_\perp$. According to Newtonian mechanics

$$m\frac{dv_\perp}{dt} = F_\perp = \frac{GMm}{r^2}\cos(\theta) = \frac{GMmb}{r^3}\,. \tag{11.8}$$

Notice that the the solution $v_\perp(t)$ does not depend on m .

A first order-of-magnitude estimate follows by evaluating the force at closest approach, $F_\perp \approx GmM/b^2$, and estimating that the force acts over a time scale of order $\Delta t \approx b/v$. The induced velocity to the left is then $v_\perp \approx F_\perp \Delta t/m = GM/(bv)$, so that the deflection angle $\alpha = v_\perp/v \approx GM/(bv^2)$.

We can do better by simply integrating Eq. (11.8):

$$v_\perp = \int_{-\infty}^{\infty} \frac{GMb}{v(b^2+x^2)^{3/2}}\, vdt = \frac{GM}{bv}\int_{-\infty}^{\infty}\frac{dx'}{(1+x'^2)^{3/2}} = \frac{2GM}{bv}\,, \tag{11.9}$$

so that the deflection angle

$$\alpha = \frac{2GM}{bv^2}\,. \tag{11.10}$$

In this derivation I used the fact that $x = vt$ in the first step, and I set $x' = x/b$ in the second. Notice that the derivation assumes that v is constant, consistent with the assumption of a small deflection. To obtain the numerical value of the definite integral, you may want to change variables once more setting $x' = \tan(\phi)$.

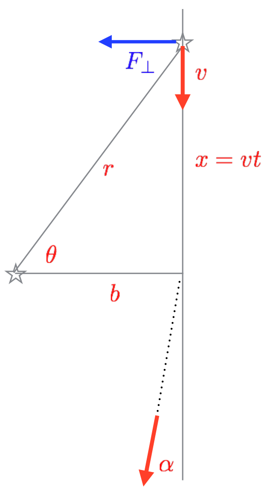

Figure 11.7: Geometry illustrating the deflection of a test mass by a more massive deflector. The impact parameter is b and the deflection angle α.