University of Durham

EXAMINATION PAPER

May/June 2017 Examination code: PHYS3631-WE01

FOUNDATIONS OF PHYSICS 3B

SECTION A. Statistical Physics
SECTION B. Condensed Matter Physics part 1
SECTION C. Condensed Matter Physics part 2

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklets for Sections B and C, in order, inside your booklet for Section A, before they are collected by the invigilator.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A. STATISTICAL PHYSICS

Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) We toss N similar coins and we find that n coins show heads and $N - n$ tails. The pair $(n, N - n)$ gives the distribution of heads and tails for the N coins. How many different distributions of N coins are possible and what is the number of microstates $\Omega(n, N - n)$ for each distribution? Give a distribution with the smallest possible $\Omega(n, N - n)$. [4 marks]

   (b) What is the Boltzmann entropy for the distribution $(n, N-n)$ of N coins? Determine the value of n for which the distribution $(n, N - n)$ has the largest entropy. State an assumption in statistical physics that allows us to anticipate this result.

   Assume that N and n are large enough that Stirling's approximation holds: $\ln N! \simeq N \ln N - N$. [4 marks]

   (c) A system of N particles is in thermal equilibrium with a heat bath at a temperature T. The single-particle energy levels ϵ_i are non-degenerate. Give the single-particle partition function Z_1 and show that the total energy per particle, $U = \sum_i n_i\epsilon_i$, is equal to

   $$U = -N\frac{\partial \ln Z_1}{\partial \beta},$$

   where $\beta = +1/k_{\mathrm{B}}T$ and n_i is the number of particles with energy ϵ_i.

   [4 marks]

   (d) (i) Write the Fermi-Dirac distribution, defining any symbols that appear in the expression, and explain what the distribution describes.

   (ii) For fermions in three dimensions, plot the Fermi-Dirac distribution for two temperatures, $T = 0$ and $T > 0$. Indicate the position of the Fermi energy in the two plots and explain if the Fermi energy changes with temperature.

   [4 marks]

   (e) (i) Using the first law of thermodynamics $dU = TdS - PdV$, give the definition of temperature in thermodynamics, in terms of the change in entropy as we increase the energy of the system.

   (ii) Based on this definition, or otherwise, briefly explain how it is possible to bring certain systems to negative temperatures. Give an example.

   (iii) When the temperature of a system is negative, is the energy of the system greater or smaller than what it would be if the temperature were infinite?

   (iv) Is it possible to bring a monatomic gas in a volume V to thermal equilibrium at a negative temperature?

   [4 marks]

2. N spin-one ions, with magnetic dipole moment μ, are localised on a lattice. The lattice of ions is in thermal equilibrium at temperature T and is in a magnetic field B directed along the z direction. The single-particle states for the spins are characterised by the quantum number m (with $m = 1, 0, -1$), which gives the z component of their spin, $S_z = m\,\hbar$. The corresponding single particle energies are $\epsilon_m = -m\,\mu B$.

   (a) Derive the single-particle partition function Z_1 and give the probabilities p_1, p_0, p_{-1} that an ion will be in a state with either $m = 1$, $m = 0$, or $m = -1$. [4 marks]

   (b) Derive the total energy, $U = N \sum_m p_m\, \epsilon_m$, of the ions. Give the limits of U for low T and for high T. Explain briefly why the result is expected. [4 marks]

   (c) Show (i) that the entropy of the ions must be zero at low T and (ii) that it must tend to $Nk_\mathrm{B} \ln 3$ at high T. (iii) Show that the entropy depends on the magnetic field B and the temperature T through their ratio B/T. The detailed derivation of the entropy S for all temperatures T is not required to answer this question. [4 marks]

   $$\left[\begin{array}{c}\text{Hint: Use: } S = -Nk_\mathrm{B} \sum_m p_m \ln p_m. \text{ Alternatively, you may use:}\\ F = -Nk_\mathrm{B}T \ln Z_1 \text{ and } F = U - T\,S\,.\end{array}\right]$$

   (d) Derive the heat capacity under constant volume $C_V = dU/dT$. Show that C_V vanishes for high T and provide a physical argument for this result. [4 marks]

   (e) Based on (c) above, draw a qualitative sketch of the entropy versus temperature for two values, B_1, and $B_2 > B_1$, of the applied magnetic field. Make an approximate estimate of the lowest temperature at which the slope of S vs T will start increasing from zero, for both curves. Use the plot to explain how the system of ions can be cooled. [4 marks]

3. Consider a gas of bosons in a fixed volume V, whose single-particle energy levels are ϵ_i with degeneracy g_i. The distribution $\{n_0, n_1, \ldots,\}$ gives the number of bosons n_k with energy ϵ_k, for $k = 0, 1, \ldots$.

   (a) Show that for large $g_k \gg 1$, the number of microstates for the distribution $\{n_i\}$ is given by:
   $$\Omega(\{n_i\}) = \prod_i \frac{(n_i + g_i)!}{n_i! g_i!}\,.$$
   Obtain the number of microstates, $\Omega^{\text{dilute}}(\{n_i\})$, in the limit of a dilute system i.e. when $g_k \gg n_k$, for $k = 0, 1, \ldots$. [4 marks]

   (b) Explain whether $\Omega^{\text{dilute}}(\{n_i\})$ is greater, smaller, or equal to the number of microstates, $\Omega^{\text{class}}(\{n_i\})$, for a distribution of N distinguishable particles with the same n_i and g_i. Express $\Omega^{\text{class}}(\{n_i\})$ in terms of $\Omega^{\text{dilute}}(\{n_i\})$. How does the entropy, S^{dilute}, of the bosons in the dilute limit compare with the entropy, S^{class}, of the distinguishable particles? [4 marks]

   (c) Show that the Boltzmann entropy of the boson gas is given by:
   $$S = k_{\mathrm{B}} \sum_i g_i \Big[(1 + f_i)\ln(1 + f_i) - f_i \ln f_i\Big]$$
   where $f_k = n_k/g_k$. Assume that n_k and g_k are sufficiently large that Stirling's approximation holds: $\ln n_k! = n_k \ln n_k - n_k$.
   [4 marks]

   (d) For the same gas of bosons obtain the fractional occupations f_k that minimize the grand potential energy $\Phi_G = U - TS - \mu N$.
   $U = \sum_i g_i f_i\, \epsilon_i$ is the total energy, S is the entropy in (c), T, μ are the temperature and chemical potential of the gas and $N = \sum_i g_i f_i$ is the number of bosons.
   [4 marks]

   (e) The lowest energy level is $\epsilon_0 = 0$. Show that for any T, the chemical potential of the boson gas is $\mu(T) \leq 0$. Show that $\mu(T) = 0$ for any T below the Bose-Einstein transition temperature T_B.
   Show that the number, $N_{th}(T)$, of bosons of mass M which are excited away from the ground state ϵ_0, for $T < T_B$, is given by:
   $$N_{th}(T) = 2.315\,\frac{2\pi V}{h^3}\,(2Mk_{\mathrm{B}}T)^{\frac{3}{2}}\,, \quad \text{where} \quad \int_0^\infty \frac{\sqrt{x}\,dx}{e^x - 1} = 2.315\,.$$
   Give the equation that determines the Bose-Einstein transition temperature T_B.
   [4 marks]

[Hint: You may use without proof that: (a) $\ln(1 + x) \simeq x$, when $|x| \ll 1$ and (b) the density of states for a gas of particles with mass M in volume V is:
$$g(\epsilon)\,\delta\epsilon = \frac{2\pi V}{h^3}(2M)^{\frac{3}{2}}\,\epsilon^{\frac{1}{2}}\,\delta\epsilon\,.$$]

SECTION B. CONDENSED MATTER PHYSICS part 1
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) Show that the group velocity of an electron at the bottom of an energy band at $k = 0$ in the nearly-free electron model is

$$v_{group} = \frac{\hbar k}{m^*},$$

where $\hbar k$ is the crystal momentum of the Bloch electrons and m^* is the effective mass of the electron. Use this result to show that a completely filled band makes no contribution to the current carried by a crystal. [4 marks]

(b) In the presence of an applied magnetic field B the periodicity $\Delta\left(\frac{1}{B}\right) = \frac{2\pi e}{\hbar S}$ of the de Haas-van Alphen oscillations measures the extremal cross-sectional area S in k-space of the Fermi surface normal to the direction of $\underline{B}$. Calculate the period expected for a single crystal of silver metal within the free electron model given that the magnitude of the Fermi wavevector is 1.20 $\times 10^{10}$ m^{-1}. [4 marks]

(c) Give a brief explanation of the origin of diamagnetism in Langévin's model and state the Langévin equation for the susceptibility of a diamagnetic solid. Identify all the terms in the equation. [4 marks]

(d) What are ferromagnetic spin waves and ferromagnetic magnons? [4 marks]

(e) Provide a brief explanation of what is meant by direct exchange and superexchange. [4 marks]

5. (a) Consider a one-dimensional chain of atoms of lattice constant a. Starting from the energy−wavevector relationship, $E(k)$ for free electrons, show that a weak periodic potential of period, a, causes the wavefunctions at $k = \pm\pi/a$ to be standing waves instead of travelling waves of the form $\exp(\pm ix/a)$. What are the functional forms of the standing waves? [6 marks]

   (b) By considering the distribution of probability densities of these standing waves show that the weak periodic potential causes band gaps to open up at the wavevectors $k = \pm\pi/a$ and that the energy gap, U, is equal to the coefficient of the Fourier component of the crystal potential associated with $G = 2\pi/a$. [4 marks]

   (c) Squarium is a two-dimensional metal that has a crystal structure composed of monovalent atoms in a primitive square lattice of cell dimensions $a =$ 0.300 nm. Draw the lattice in reciprocal space. Show on your diagram the first three Brillouin zones and calculate the area of the first Brillouin zone in m^{-2}. Calculate the radius of the free electron Fermi sphere in m^{-1}. [6 marks]

   (d) The higher-order Brillouin zones in squarium appear to be fragmented into detached segments. Show how these fragments of the Fermi surface can be recombined into a connected area in the reduced or periodic zone scheme. [4 marks]

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6. (a) Sketch the general form of the temperature dependence of the spontaneous magnetisation of a ferromagnet. Indicate both the Curie temperature, T_{C}, and the saturation magnetisation, M_{sat}, on your sketch. [4 marks]

(b) The classical Weiss mean field theory of ferromagnetism interprets the magnetisation, M, of a ferromagnetic solid as being composed of both the applied magnetic field B_0 and an 'internal' magnetic field, or exchange field, B_{E}, arising from the interacting magnetic moments in the solid. In the paramagnetic phase this magnetisation is described by the expression

$$\mu_0 M = \chi_P(B_0 + B_E) = \chi_P(B_0 + \mu_0 N_{\mathrm{W}} M)$$

where χ_P is the Curie form of the paramagnetic susceptibility and N_{W} is the mean-field constant. Use this expression to obtain an expression for the Curie-Weiss susceptibility with $T > T_{\mathrm{C}}$. [3 marks]

(c) A ferromagnetic solid contains Gd^{3+} ions arranged in a primitive cubic unit cell with a unit cell parameter of $a = 0.750$ nm and with a mean-field constant of $N_{\mathrm{W}} = 100$. The Curie temperature of the material is 250 K and each gadolinium ion has 7 electrons in the unfilled $4f$ shell. Use Hund's rules to calculate the principal quantum numbers S, L and J as well as the Landé g-factor. [2 marks]

Use these results to calculate both:

(i) the saturation magnetisation of the ferromagnet at absolute zero, [2 marks]

(ii) the Curie-Weiss susceptibility of the solid above T_{C} at 290 K given that

$$C = \frac{\mu_0 N g^2 \mu_{\mathrm{B}}^2 J(J+1)}{3k_{\mathrm{B}}} = \frac{\mu_0 N m_{eff}^2}{3k_{\mathrm{B}}}.$$

[3 marks]

(d) Sketch the typical hysteresis curves of both a magnetically soft material and one that is magnetically hard. Give one example of a technologically useful application for each material. [6 marks]

SECTION C. CONDENSED MATTER PHYSICS part 2
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) When a reverse bias of 0.20 V is applied to a p-n junction diode at room temperature, a current of $10\,\mu$A flows. Calculate the current flow when it is forward biased with the same voltage. [4 marks]

(b) An electron in a 2-D semiconductor has energy, E, as a function of wavevector, $\underline{k}$, given by

$$E(\underline{k}) = Ak^2 + Bk_x^4 k_y^4,$$

where A and B are positive constants, and $k^2 = k_x^2 + k_y^2$. Determine the components of the inverse effective mass tensor, $(1/m^*)_{ij}$, for the above 2-D semiconductor as a function of the wavevector $\underline{k}$. [4 marks]

(c) Sketch the typical phase diagrams as a function of temperature T and applied magnetic field B_{a} for both type I and type II superconductors. Indicate the Meissner, vortex, and normal states on your sketches. [4 marks]

(d) A dielectric material has a complex relative permittivity which depends on the angular frequency, ω, according to

$$\epsilon_r(\omega) = 1 + \frac{\chi(0)}{1 + i\omega\tau},$$

where $\chi(0)$ is the static susceptibility and τ is the timescale for molecular rearrangement. The loss tangent is defined as $\tan\delta = \epsilon_r''/\epsilon_r'$, where ϵ_r' and ϵ_r'' are respectively the real and imaginary part of the relative permittivity. If the material has a relative permittivity of 3 at zero frequency, and $\tau = 1 \times 10^{-4}$ s, at what frequency does the loss tangent peak occur? [4 marks]

(e) A ferroelectric material can be described by the following Landau free energy

$$G(P,T) = g_0 + \frac{1}{2}g_2 P^2 + \frac{1}{4}g_4 P^4,$$

where P is the polarization, $g_2 = \gamma(T - T_0)$, T is the temperature of the system, and T_0 is the so-called Curie temperature. γ, g_0 and g_4 are positive constants in the model. Sketch the free energy as a function of the polarization for $T > T_0$ and $T < T_0$. Which temperature regime favours the ferroelectric phase, and which regime favours the dielectric phase? [4 marks]

8. (a) The carrier concentration of a p-type semiconductor is measured using the Hall effect. Assuming holes are the dominant charge carriers, show that the induced electric field, $\underline{E}_\mathrm{H}$, is given by

$$\underline{E}_\mathrm{H} = R_\mathrm{H} \underline{B}_\mathrm{a} \times \underline{j}\,,$$

where $R_\mathrm{H} = 1/(pe)$ is the Hall coefficient, $\underline{B}_\mathrm{a}$ is the applied magnetic field, $\underline{j}$ is the current density of the electrons, p is the concentration of holes, and e is the electronic charge. [4 marks]

(b) The semiconducting material is purified until it only contains acceptor impurities with concentration $N_\mathrm{A} = 1.00 \times 10^{18}$ acceptors/m^3. Compute the temperature above which the material shows an intrinsic behaviour, given that the energy band gap is $E_\mathrm{g} = 1.00$ eV, and the conduction and valence band effective densities of states are $N_\mathrm{C} = 4.00 \times 10^{25}$ m^{-3} and $N_\mathrm{V} = 1.00 \times 10^{25}$ m^{-3}. For simplicity, assume N_C and N_V are independent of temperature. [5 marks]

(c) Compute the magnitude of the generated electric field E_H when a current of 100 mA is passed along a sample of width 4 mm and thickness 1 mm in the presence of a magnetic field of 1 T at:
(i) $T = 300$ K, [4 marks]
(ii) $T = 500$ K. [4 marks]
Assume holes are always the dominant charge carriers.

(d) Briefly explain why holes can remain as the dominant charge carriers in the intrinsic regime where the densities of electrons and holes are the same. [3 marks]

9. (a) A superconducting film of uniform thickness d in the z-direction is exposed to an applied magnetic field B_{a} in the x-direction. The centre of the film is at $z = 0$. Starting from the London equation

$$\underline{\nabla} \times \underline{j} = -\frac{nq^2}{m}\underline{B},$$

show that the total magnetic field in the x-direction across the film is described by

$$B(z, B_{\mathrm{a}}) = B_{\mathrm{a}} \frac{\cosh\left(\frac{z}{\lambda}\right)}{\cosh\left(\frac{d}{2\lambda}\right)}.$$

$\lambda = \sqrt{m/(\mu_0 n q^2)}$ is the London penetration depth, $\underline{j}$ is the current density, n, q, and m are the density, charge, and mass of the electrons. [6 marks]

(b) If the thickness of the superconducting film is much smaller than the London penetration depth, $d \ll \lambda$, show that the magnetization is approximately given by

$$M(z, B_{\mathrm{a}}) \approx \frac{B_{\mathrm{a}}}{8\mu_0\lambda^2}(4z^2 - d^2).$$

[5 marks]

(c) The contribution of the magnetic energy to the Gibbs free energy density is given by

$$G_{\mathrm{s}}(z, B_{\mathrm{a}}) - G_{\mathrm{s}}(z, 0) = -\int_0^{B_{\mathrm{a}}} M(z, B'_{\mathrm{a}})\, dB'_{\mathrm{a}}.$$

Show that the average magnetic energy density within the thin film is

$$\langle G_{\mathrm{s}}(B_{\mathrm{a}})\rangle - G_{\mathrm{s}}(0) = \frac{B_{\mathrm{a}}^2 d^2}{24\mu_0\lambda^2}.$$

[6 marks]

(d) The film is no longer a superconductor when $\langle G_{\mathrm{s}}(B_{\mathrm{a}})\rangle$ equals to the Gibbs free energy density of a normal phase, G_{n}. For simplicity, we may assume G_{n} is independent of the applied magnetic field. If this occurs at $B_{\mathrm{c}} = 10^{-3}\,\mathrm{T}$ for a film of thickness 10 nm, what is the expected critical magnetic field for a film of the same material when its thickness is 5 nm? [3 marks]

[Hint: You may use the following relations without any proof:
$B(z) = \mu_0 M(z) + B_{\mathrm{a}}$ and $\cosh(x) \simeq 1 + x^2/2$ for $x \ll 1$.]

