(c) $W = \int_{t_1}^{t_2} P\,dt = \frac{\mu_0 \alpha^2 l w}{4c} \int_{d/x}^{(d+h)/c} (ct-d)^2\,dt = \frac{\mu_0 \alpha^2 l w}{4c} \left[\frac{(ct-d)^3}{3c}\right]_{d/c}^{(d+h)/c} = \boxed{\frac{\mu_0 \alpha^2 l w h^3}{12c^2}}.$

Since $1/c^2 = \mu_0\epsilon_0$, this agrees with the answer to (a).

Problem 10.3

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}.} \quad \mathbf{B} = \nabla\times\mathbf{A} = \boxed{0.}$$

This is a funny set of potentials for a $\boxed{\text{stationary point charge}}$ q at the origin. ($V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$, $\mathbf{A} = 0$ would, of course, be the customary choice.) Evidently $\boxed{\rho = q\delta^3(\mathbf{r});\ \mathbf{J} = 0.}$

Problem 10.4

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -A_0\cos(kx-\omega t)\,\hat{\mathbf{y}}(-\omega) = \boxed{A_0\omega\cos(kx-\omega t)\,\hat{\mathbf{y}},}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \hat{\mathbf{z}}\frac{\partial}{\partial x}[A_0\sin(kx-\omega t)] = \boxed{A_0 k\cos(kx-\omega t)\,\hat{\mathbf{z}}.}$$

Hence $\nabla\cdot\mathbf{E} = 0$ ✓, $\nabla\cdot\mathbf{B} = 0$ ✓.

$$\nabla\times\mathbf{E} = \hat{\mathbf{z}}\frac{\partial}{\partial x}[A_0\omega\cos(kx-\omega t)] = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}}, \quad -\frac{\partial \mathbf{B}}{\partial t} = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}},$$

so $\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ ✓.

$$\nabla\times\mathbf{B} = -\hat{\mathbf{y}}\frac{\partial}{\partial x}[A_0 k\cos(kx-\omega t)] = A_0 k^2\sin(kx-\omega t)\,\hat{\mathbf{y}}, \quad \frac{\partial \mathbf{E}}{\partial t} = A_0\omega^2\sin(kx-\omega t)\,\hat{\mathbf{y}}.$$

So $\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$ *provided* $\boxed{k^2 = \mu_0\epsilon_0\omega^2,}$ or, since $c^2 = 1/\mu_0\epsilon_0$, $\boxed{\omega = ck.}$

Problem 10.5

$$V' = V - \frac{\partial\lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r};} \ \mathbf{A}' = \mathbf{A} + \nabla\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}} + \left(-\frac{1}{4\pi\epsilon_0}qt\right)\left(-\frac{1}{r^2}\hat{\mathbf{r}}\right) = \boxed{0.}$$

This gauge function transforms the "funny" potentials of Prob. 10.3 into the "ordinary" potentials of a stationary point charge.

Problem 10.6

Ex. 10.1: $\nabla\cdot\mathbf{A} = 0;\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Both Coulomb and Lorentz.}}$

Prob. 10.3: $\nabla\cdot\mathbf{A} = -\frac{qt}{4\pi\epsilon_0}\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = -\frac{qt}{\epsilon_0}\delta^3(\mathbf{r});\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Neither.}}$

Prob. 10.4: $\nabla\cdot\mathbf{A} = 0;\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Both.}}$

Problem 10.7

Suppose $\nabla\cdot\mathbf{A} \neq -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. (Let $\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = \Phi$—some known function.) We want to pick λ such that $\mathbf{A}'$ and V' (Eq. 10.7) *do* obey $\nabla\cdot\mathbf{A}' = -\mu_0\epsilon_0\frac{\partial V'}{\partial t}$.

$$\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = \nabla\cdot\mathbf{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = \Phi + \Box^2\lambda.$$

This will be zero provided we pick for λ the solution to $\Box^2\lambda = -\Phi$, which by hypothesis (and in fact) we know how to solve.

We *could* always find a gauge in which $V' = 0$, simply by picking $\lambda = \int_0^t V\,dt'$. We *cannot* in general pick $\mathbf{A} = 0$—this would make $\mathbf{B} = 0$. [Finding such a gauge function would amount to expressing $\mathbf{A}$ as $-\nabla\lambda$, and we know that vector functions *cannot* in general be written as gradients—only if they happen to have curl zero, which $\mathbf{A}$ (ordinarily) does *not*.]

Problem 10.8

¿From the product rule:

$$\nabla\cdot\left(\frac{\mathbf{J}}{\imath}\right) = \frac{1}{\imath}(\nabla\cdot\mathbf{J}) + \mathbf{J}\cdot\left(\nabla\frac{1}{\imath}\right), \quad \nabla'\cdot\left(\frac{\mathbf{J}}{\imath}\right) = \frac{1}{\imath}(\nabla'\cdot\mathbf{J}) + \mathbf{J}\cdot\left(\nabla'\frac{1}{\imath}\right).$$

But $\nabla\frac{1}{\imath} = -\nabla'\frac{1}{\imath}$, since $\boldsymbol{\imath} = \mathbf{r} - \mathbf{r}'$. So

$$\nabla\cdot\left(\frac{\mathbf{J}}{\imath}\right) = \frac{1}{\imath}(\nabla\cdot\mathbf{J}) - \mathbf{J}\cdot\left(\nabla'\frac{1}{\imath}\right) = \frac{1}{\imath}(\nabla\cdot\mathbf{J}) + \frac{1}{\imath}(\nabla'\cdot\mathbf{J}) - \nabla'\cdot\left(\frac{\mathbf{J}}{\imath}\right).$$

But

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \frac{\partial J_x}{\partial t_r}\frac{\partial t_r}{\partial x} + \frac{\partial J_y}{\partial t_r}\frac{\partial t_r}{\partial y} + \frac{\partial J_z}{\partial t_r}\frac{\partial t_r}{\partial z},$$

and

$$\frac{\partial t_r}{\partial x} = -\frac{1}{c}\frac{\partial \imath}{\partial x}, \quad \frac{\partial t_r}{\partial y} = -\frac{1}{c}\frac{\partial \imath}{\partial y}, \quad \frac{\partial t_r}{\partial z} = -\frac{1}{c}\frac{\partial \imath}{\partial z},$$

so

$$\nabla\cdot\mathbf{J} = -\frac{1}{c}\left[\frac{\partial J_x}{\partial t_r}\frac{\partial \imath}{\partial x} + \frac{\partial J_y}{\partial t_r}\frac{\partial \imath}{\partial y} + \frac{\partial J_z}{\partial t_r}\frac{\partial \imath}{\partial z}\right] = -\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla\imath).$$

Similarly,

$$\nabla'\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\imath).$$

[The first term arises when we differentiate with respect to the *explicit* $\mathbf{r}'$, and use the continuity equation.] thus

$$\nabla\cdot\left(\frac{\mathbf{J}}{\imath}\right) = \frac{1}{\imath}\left[-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\imath)\right] + \frac{1}{\imath}\left[-\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\imath)\right] - \nabla\cdot\left(\frac{\mathbf{J}}{\imath}\right) = -\frac{1}{\imath}\frac{\partial\rho}{\partial t} - \nabla'\cdot\left(\frac{\mathbf{J}}{\imath}\right)$$

(the other two terms cancel, since $\nabla\imath = -\nabla'\imath$). Therefore:

$$\nabla\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\left[-\frac{\partial}{\partial t}\int\frac{\rho}{\imath}\,d\tau - \int\nabla'\cdot\left(\frac{\mathbf{J}}{\imath}\right)d\tau\right] = -\mu_0\epsilon_0\frac{\partial}{\partial t}\left[\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\imath}\,d\tau\right] - \frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}}{\imath}\cdot d\mathbf{a}.$$

The last term is over the suface at "infinity", where $\mathbf{J} = 0$, so it's zero. Therefore $\nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. ✓