Theoretical Physics 2019/20 — Problem QT2.1

Preparation and background reading: Read through sections 2.1 and 2.3 to 2.10 of Part 2 of the course notes (on DUO), in particular sections 2.1, 2.8 and 2.10. If you hesitate about how to tackle some of these questions or are not sure how to make sense of the model solution, do not hesitate to ask for hints or clarification in a workshop.

Consider the set of all functions of the form

$$c_0 + c_1\,x + c_2\,x^2 + c_3\,x^3,$$

where c_0, c_1, c_2 and c_3 are complex numbers and the variable x can take any real value between -1 and 1. It is not difficult to see that these functions form a complex vector space if vector addition and multiplication by a scalar are defined as follows (which is the obvious definition of these two operations for these functions):

$$(c_0 + c_1\,x + c_2\,x^2 + c_3\,x^3) + (d_0 + d_1\,x + d_2\,x^2 + d_3\,x^3) \equiv \\ (c_0 + d_0) + (c_1 + d_1)\,x + (c_2 + d_2)\,x^2 + (c_3 + d_3)\,x^3 \tag{1}$$

for any complex numbers c_0, c_1, c_2, c_3, d_0, d_1, d_2 and d_3, and

$$\lambda(c_0 + c_1\,x + c_2\,x^2 + c_3\,x^3) \equiv (\lambda c_0) + (\lambda c_1)\,x + (\lambda c_2)\,x^2 + (\lambda c_3)\,x^3 \tag{2}$$

for any complex numbers λ, c_0, c_1, c_2 and c_3. [You are encouraged to check that these two operations are indeed consistent with the axioms of a vector space listed in Section 2.1 of the notes; however, this is optional.]

(a) Suppose that $f(x)$ and $g(x)$ are two elements of this vector space. Show that defining their inner product by the following equation is consistent with the axioms of the inner product listed in Section 2.8 of the notes:

$$(f, g) = \int_{-1}^{1} f^*(x)\,g(x)\,\mathrm{d}x. \tag{3}$$

(b) Amongst the elements of this vector space are the four functions $v_0(x)$, $v_1(x)$, $v_2(x)$ and $v_3(x)$, equal, respectively, to the constant 1 for all x, to x, to x^2 and to x^3 (thus $v_n(x) \equiv x^n$). Defining the inner product as in part (b), show that $v_0(x)$ and $v_2(x)$ are orthogonal to $v_1(x)$ and $v_3(x)$ but that $v_0(x)$ is not orthogonal to $v_2(x)$ and that $v_1(x)$ is not orthogonal to $v_3(x)$. [See Section 2.10 of the notes.]

(c) Using the Gram-Schmidt orthogonalisation method explained in Section 2.10, or (if you prefer) the Gram-Schmidt orthonormalisation method you may have studied in a previous course, find two elements of this vector space, $w_2(x)$ and $w_3(x)$, that are orthogonal to each other as well as to $v_0(x)$ and $v_1(x)$.

(d) Go online, or open your favourite maths textbook, check out what the Legendre polynomials $P_2(x)$ and $P_3(x)$ are, and compare them to what you have found for $w_2(x)$ and $w_3(x)$. [Note that $P_0(x) \equiv v_0(x)$ and $P_1(x) \equiv v_1(x)$. Also, note that the functions $w_2(x)$ and $w_3(x)$ obtained by the Gram-Schmidt process are unnormalized, whereas the Legendre polynomials $P_n(x)$ are normalized in such a way that $P_n(1) = 1$ for all n.]