University of Durham

EXAMINATION PAPER

May/June 2014 Examination code: 043661/01

LEVEL 3 PHYSICS: THEORETICAL PHYSICS 3

SECTION A. Relativistic Electrodynamics
SECTION B. Quantum Theory 3

Time allowed: 3 hours

Examination material provided: None

Calculators: The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. RELATIVISTIC ELECTRODYNAMICS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Show that the Minkowski scalar product $a^\mu b_\mu$ of two 4-vectors a and b is invariant under Lorentz transformations. [4 marks]

   (b) The polarization tensor for a vector boson with mass m and 4-momentum p^μ is
   $$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2c^2}\right).$$
   Here, $g^{\mu\nu}$ is the metric tensor. Calculate $p_\mu T^{\mu\nu}$ and $\epsilon_{\alpha\beta\mu\nu}T^{\mu\nu}$. [4 marks]

   (c) Show that an isolated photon cannot spontaneously create an electron-positron pair. [4 marks]

   (d) State the definition of the 4-force, f^μ. Use the definition to express the components of f^μ in terms of the energy E and three momentum $\underline{p}$ and their derivatives with respect to time. [4 marks]

   (e) An observer moves with constant velocity v with respect to a point charge q, and measures both the electric, $\underline{E}$, and magnetic, $\underline{B}$, fields a distance d from the point charge to be non-zero. What value does the observer find for $\underline{E}\cdot\underline{B}$? [4 marks]

   (f) The Lienard-Wiechert potential of a point charge q with 4-velocity u^μ is
   $$A^\mu = \frac{q}{4\pi\epsilon_0}\frac{u^\mu}{u^\nu R_\nu},$$
   where R_ν is the 4-distance between the observer and the point charge. The right-hand side of the expression must be evaluated at the retarded time $t_{\rm ret}$. Evaluate this expression in the instantaneous rest frame of the point charge at $t_{\rm ret}$ and show that you obtain the expected result. [4 marks]

   (g) Use the covariant form of the inhomogeneous Maxwell equations to derive the wave equation for the 4-potential A^μ in the Lorenz gauge. [4 marks]

2. a) Give the definition of the electric, $\underline{E}$, and magnetic, $\underline{B}$, fields in terms of the scalar, Φ, and vector, $\underline{A}$, potentials. [4 marks]

b) State the definition of the field-strength tensor in terms of the 4-potential $A^\mu = (\Phi, c\underline{A})$. [2 marks]

c) Show that the field-strength tensor can be written in terms of the electric and magnetic fields as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

[6 marks]

Let S' be the rest frame of a medium with a charge-density of ρ_0 and a current density $\underline{J}'$. In this frame, the electric field, $\underline{E}'$, is related to the current density, $\underline{J}'$, by Ohm's law $\underline{J}' = \sigma \underline{E}'$, where σ is the conductivity.

The frame S' moves with velocity $\underline{v}$ with respect to the frame S. In S the 4-current density is

$$j^\mu = a v^\mu + \frac{\sigma}{c} F^{\mu\nu} v_\nu,$$

where a is a constant and $v^\mu = \gamma(c, \underline{v})$ is the 4-velocity of the medium with $\gamma = 1/\sqrt{1 - \frac{|\underline{v}|^2}{c^2}}$.

d) Compute a by calculating $j^\mu v_\mu$, using the fact that in the frame S', $j'^\mu = (\rho_0 c, \sigma \underline{E}')$. [2 marks]

e) Calculate the 3-current density $\underline{J}$ in S in terms of $\underline{v}$ and the electric, $\underline{E}$, and magnetic fields, $\underline{B}$ (as measured in S). Interpret your result. [6 marks]

3. An observer at rest in an inertial frame S notices that at time $t = 0$ a point charge q of mass m is at the origin of the coordinate system with a momentum $\underline{p}_0$ with Cartesian coordinates $\underline{p}_0 = (0, 0, p_0)$. In S there is also a constant electric field $\underline{E} = (0, E, 0)$.

   a) Show that the momentum of the point charge at any later time t is given by $\underline{p}(t) = (0, qtE, p_0)$. [4 marks]

   b) Show that the velocity, $\underline{v}$, of the point charge can be expressed in terms of its momentum, $\underline{p}$, and energy, E, as

   $$\underline{v} = \frac{c^2}{E}\,\underline{p}$$

   and use this to compute $\underline{v}$ as a function of time. What happens in the limit $t \to \infty$? [8 marks]

   c) Compute the position (as measured in S) of the point charge as a function of time. [5 marks]

   d) Is the point charge uniformly accelerated? Justify your answer. [3 marks]

$$\left[\text{Hint}: \int \frac{\mathrm{d}t}{\sqrt{1 + a^2t^2}} = \frac{\operatorname{arcsinh}(at)}{a}\right]$$

4. In an inertial frame S, an event has coordinates x^μ. The inertial frame S' moves with velocity v with respect to S along the axis $\underline{n}$ and at $t = t' = 0$ the origin of the two frames coincide. The axis is given by Cartesian coordinates in S as

$$\underline{n} = \frac{1}{\sqrt{2}}(1, 1, 0).$$

a) Find the matrix representation for the Lorentz transformation Λ relating the coordinates x^μ of an event in S to the coordinates for the same event x'^ν, as measured in S'. [12 marks]

b) Starting from the invariance of the Minkowski scalar product under Lorentz transformations show that every matrix Λ representing a Lorentz transformation satisfies

$$\Lambda^T \cdot g \cdot \Lambda = g$$

where g is the metric and Λ^T is the transpose of Λ. What values are allowed for the determinant of Λ? [8 marks]

SECTION B. QUANTUM THEORY 3

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) State the three-dimensional, time-dependent Schrödinger equation for a particle in a potential given by $V(\underline{r})$. Write down a general ansatz for the solution in the case of a spherical potential, by separating time-dependence and radial and angular coordinates. What functions describe the angular dependence? [4 marks]

   (b) State the Klein-Gordon equation for a free particle. For what kind of particles is this equation applicable? Give an example of a particle whose behaviour is described by the Klein-Gordon equation. [4 marks]

   (c) State the Dirac equation for a free particle. For what kind of particles is this equation applicable? Give an example of a particle whose behaviour is described by the Dirac equation. [4 marks]

   (d) Write down an expression for the density operator. Express the ensemble average of an operator A in terms of the density matrix. Write down the Liouville equation. [4 marks]

   (e) Assuming in all cases the systems are at a temperature, T, write down the distribution function of a system of non-interacting bosons and the distribution function of a system of non-interacting fermions. Finally, write down the distribution function for the Maxwell-Boltzmann case. Draw the energy dependence of these three distribution functions. [4 marks]

   (f) The spherical Bessel functions $j_l(\rho)$ are defined by

$$j_l(\rho) = (-1)^l \rho^l \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^l \frac{\sin\rho}{\rho} \,.$$

   Give the explicit expression for $j_0(\rho)$ and $j_1(\rho)$. Expand $j_0(\rho)$ and $j_1(\rho)$ for small values of ρ and write down the leading term. [4 marks]

   (g) The Dirac equation can be expressed in terms of γ-matrices. Express the γ-matrices in terms of the α- and β-matrices. Write down the algebra of the γ-matrices. How many γ-matrices are there? Which of these matrices are hermitian and which are antihermitian? [4 marks]

   (h) How does one couple the electromagnetic field, expressed in terms of the vector potential $\underline{A}$ and the scalar potential ϕ, to particles described by quantum mechanical wave functions? Give as an example the Klein-Gordon equation with a coupling to the electromagnetic field. [4 marks]

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6. A massive particle with spin $\underline{s}$ has a magnetic moment $\underline{\mu}$, where

$$\underline{\mu} = 2\mu_B \underline{s} \ ,$$

with $\mu_B = e\hbar/(2mc)$. In a magnetic field $\underline{B}$ such a particle has energy $\epsilon = -\underline{\mu}\cdot\underline{B}$. Consider a system of N identical particles with spin $s = 1/2$ that only interact via their spin with an external magnetic field $\underline{B}$. Assume no interaction of the particles with each other and no other interactions.

   (a) What are the possible energy values of each of these particles, if there is a magnetic field in the z-direction? [2 marks]

   (b) Write down the energy of the whole system of N particles. [2 marks]

   (c) Write down the partition function of this system. [4 marks]

   (d) What is the probability that the spin of one of the particles is parallel or anti-parallel to the magnetic field? [2 marks]

   (e) Calculate the average magnetic moment of one of the particles. [6 marks]

   (f) Give an expression for the energy of the whole system using the average magnetic moment of one of the particles. [4 marks]

7. (a) Derive the Klein-Gordon equation for a free particle using the correspondence principle. [4 marks]

   (b) Derive the radial equation of a Klein-Gordon field coupled to a spherical potential $V(r)$. [4 marks]

   (c) Derive the Dirac equation for a free particle starting from a linear ansatz and using the correspondence principle. Derive the three independent relations between the matrices α_i and β which appear. [9 marks]

   (d) What kind of functions solve the free Dirac equation? Insert these solutions in the Dirac equation to get an equation for eigenvalues and eigenfunctions. Assume now that the Dirac particle is at rest and derive the explicit expressions for the spinors which arise and give their eigenvalues. [3 marks]

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8. Calculate the total cross section for s-wave scattering of a particle of wavenumber k, on a hard shell with radius R, as follows:

    (a) Write down the potential of a hard shell with radius R. [2 marks]

    (b) Write down the corresponding Schrödinger equation. [4 marks]

    (c) What are the solutions of the above Schrödinger equation? [3 marks]

    (d) Determine all parameters of the solution. [6 marks]

    (e) Calculate the total cross section. [4 marks]

    (f) What cross section is obtained in the limit $k \to 0$? [1 mark]