(a) see attached figure. [S:2 marks, U:1 mark]

ii and iii will have finite numbers of bound states as the potential is finite [U:1 mark]

(b) $\int_0^1 \psi^* \psi dx = 1$ [S:1 mark]

$A^2 \int_0^1 x^2(1-x)^2 dx = A^2 \int x^2(1-2x+x^2)dx = A^2 \int x^2 - 2x^3 + x^4 dx$ [U:1 mark]

$A^2[x^3/3 - 2x^4/4 + x^5/5]_0^1 = A^2(1/3 - 1/2 + 1/5) = A^2(10-15+6)/30 = A^2/30$ and $A = \sqrt{30}$ [U:2 marks]

(c) prob $E_5 = |c_5^2| = 25/35$ [U:1 mark]

$< E >= 1/35E_1 + 9/35E_3 + 25/35E_5$ [U:1 mark]

$= 1/35E_1 + 9.9/35E_1 + 25.25/35E_1 = 707E_1/35 = 20.2E_1$ [U:1 mark]

no single measurement will give this number as its not one of the E_n's [S:1 mark]

(d) $p = -i\hbar d/dx$ [1 mark]

$[H,p]\psi = [p^2/2m, p]\psi + [V,p]\psi = [V,p]\psi$ [U:1 mark]

$[H,p]\psi = [V,p]\psi = Vp\psi - p(V\psi) = V.-i\hbar\frac{d\psi}{dx} - -i\hbar d/dx(V\psi)$ [U:1 mark]

$= -i\hbar(Vd\psi/dx - Vd\psi/dx - \frac{dV}{dx}\psi) = i\hbar\frac{dV}{dx}\psi$ [U:1 mark]

(e)

$$\frac{d<p>}{dt} = \frac{i}{\hbar} < [H,p] > + \left\langle \frac{\partial p}{\partial t} \right\rangle = \frac{i}{\hbar} < [H,p] >$$

as $\partial p/\partial t = 0$ from hint [U:1 mark]

$$\frac{d<p>}{dt} = \frac{i}{\hbar} i\hbar \frac{dV}{dx} = -\frac{dV}{dx} \qquad [U:1\ mark]$$

which is exactly what you expect in classical physics [S:1 mark]

as dV/dx is (conservative) force which is rate of change in momentum [U:1 mark]

(f) $< r >= \int_0^r \int_0^\pi \int_0^{2\pi} \psi^* r \psi r^2 \sin\theta dr d\theta dphi$ [S:1 mark]

$= 4\pi/(\pi a^3) \int_0^r e^{-2r/a} r^3 dr = 4\pi/(\pi a^3) 3!/(2/a)^4$

$= 4/a^3 \times 3.2a^4/(2.2.2.2) = 3a/2$ [S:1 mark]

probability $\psi^* \psi r^2 \sin\theta dr d\theta d\phi \propto r^2 e^{-2r/a}$ [S:1 mark]

maximum at $dP/dr = 0$ so $2re^{-2r/a} + r^2 . -2/ae^{-2r/a} = 0$ so $2r - 2r^2/a = 0$ ie $r = a$ [S:1 mark]

(g) 1st level $E_{111} = 3A$ where $A = \frac{\hbar^2\pi^2}{2mL^2}$ degeneracy 1 [S:1 mark]

2nd level $E_{112} = E_{121} = E_{211} = 6A$ degeneracy 3 [S:1 mark]

3rd energy level $E_{122} = E_{221} = E_{212} = 9A$ degeneracy 3

4th $E_{311} = E_{131} = E_{113} = 11A$ degeneracy 3 [S:1 mark]

5th energy level $E_{222} = 12A$ degeneracy 1

6th $E_{321} = E312 = E_{231} = E_{213} = E_{123} = E_{132} = 14A$ degeneracy 6 [S:1 mark]

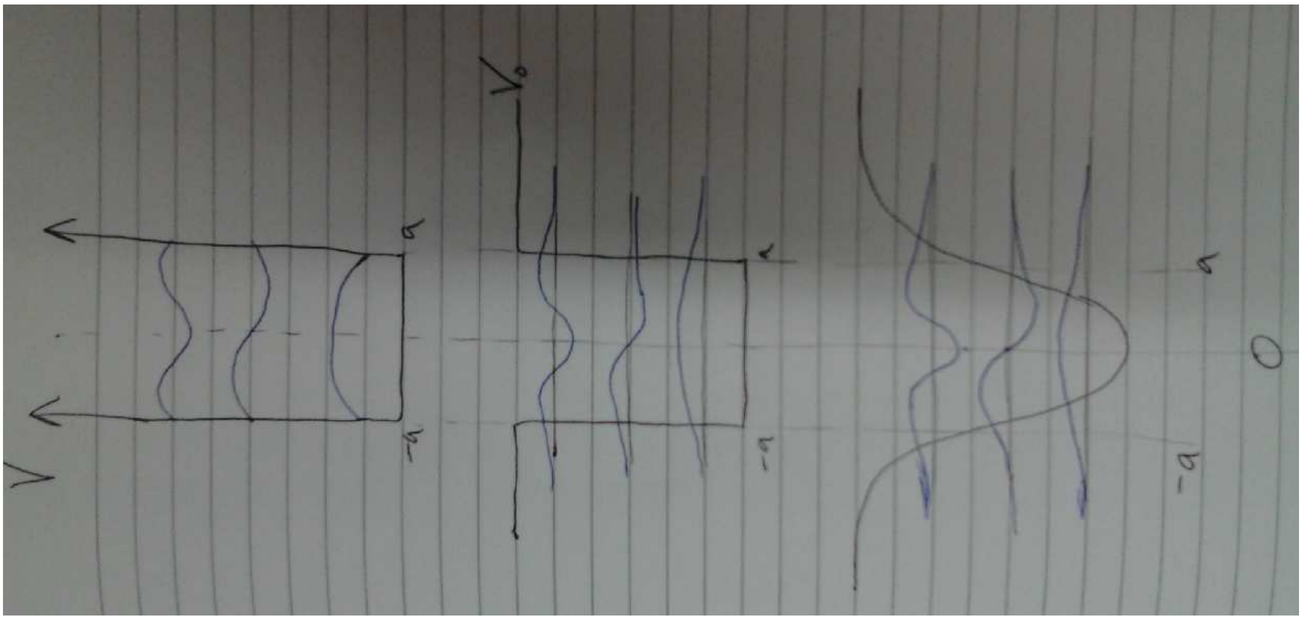
(h) $E_0^1 = \int \psi^* H' \psi dx$

$\int_{-\infty}^{+\infty} (\alpha/\pi)^{1/2} e^{-\alpha x^2} V_0 \delta(x-0) dx$ [U:1 mark]

$= V_0 (\alpha/\pi)^{1/2}$ [U:1 mark]

no, can't have degeneracy in 1-d. [S:1 mark]

in 2-d and higher systems then there can be rotational symmetry so we can get the same energies for different quantum numbers [S:1 mark]


V
a
-a
V_0
a
-a
a
0
-a

(a) $[L^2, L_z]Y_{lm} = L^2(L_zY_{lm}) - L_z(L^2Y_{lm})$ [S:1 mark]
$= L^2m\hbar Y_{lm} - L_zl(l+1)\hbar^2Y_{lm} = l(l+1)m\hbar^3Y_{lm} - m\hbar l(l+1)\hbar^2Y_{lm} = 0$ [S:1 mark]

$30 = l(l+1)$ so $l = 5$ which means m takes any value $= -5, -4..0..4, 5$ [U:1 mark]

(b) add to get $L_+ + L_- = 2L_x$ so $L_x = (L_+ + L_-)/2$ [S:1 mark]
$< L_x >= \frac{1}{2}\int\int Y^*_{lm}L_+Y_{lm}\sin\theta d\theta d\phi + \frac{1}{2}\int\int Y^*_{lm}L_-Y_{lm}\sin\theta d\theta d\phi$ [U:1 mark]
$= \frac{1}{2}A_{lm}\int\int Y^*_{lm}Y_{lm+1}\sin\theta d\theta d\phi + \frac{1}{2}A_{lm}\int\int Y^*_{lm}Y_{lm-1}\sin\theta d\theta d\phi = 0$
as normalisation is $\int\int Y_{lm}Y_{l'm'}\sin\theta d\theta d\phi = \delta(m-m')\delta(l-l')$ [S:1 mark]

(c) $L_x\psi = q\hbar\psi = q\hbar(aY_{11} + bY_{10} + cY_{1-1})$
$L_-\psi = L_-(aY_{11} + bY_{10} + cY_{1-1}) = aA_{11}Y_{10} + bA_{10}Y_{1-1}$
$= a\hbar\sqrt{2}Y_{10} + b\hbar\sqrt{2}Y_{1-1}$ [U:1 mark]
$L_+\psi = L_+(aY_{11} + bY_{10} + cY_{1-1}) = bA_{10}Y_{11} + cA_{10}Y_{10}$
$= \hbar(b\sqrt{2}Y_{11} + c\sqrt{2}Y_{10})$ [U:1 mark]
Hence $L_x\psi = \frac{\hbar}{2}(b\sqrt{2}Y_{11} + (c+a)\sqrt{2}Y_{10} + b\sqrt{2}Y_{1-1}) = \frac{\hbar}{\sqrt{2}}(bY_{11} + (c+a)Y_{10} + bY_{1-1})$ [U:2 marks]
equate coefficients $\frac{\hbar}{\sqrt{2}}b = \hbar aq$, $\frac{\hbar}{\sqrt{2}}(c+a) = \hbar bq$ and $\frac{\hbar}{\sqrt{2}}b = c\hbar q$
so for $q = 1$ we have $\frac{1}{\sqrt{2}}b = a$, $\frac{1}{\sqrt{2}}(c+a) = b$ and $\frac{1}{\sqrt{2}}b = c$
so $\psi = b(\frac{1}{\sqrt{2}}Y_{11} + Y_{10} + \frac{1}{\sqrt{2}}Y_{1-1})$ [U:2 marks]
normalise $\psi = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}Y_{11} + Y_{10} + \frac{1}{\sqrt{2}}Y_{1-1})$
$= \frac{1}{2}Y_{11} + \frac{1}{\sqrt{2}}Y_{10} + \frac{1}{2}Y_{1-1}$ [U:1 mark]
so for $q = -1$ we have $\frac{1}{\sqrt{2}}b = -a$, $\frac{1}{\sqrt{2}}(c+a) = -b$ and $\frac{1}{\sqrt{2}}b = -c$
so $\psi = b(-\frac{1}{\sqrt{2}}Y_{11} + Y_{10} - \frac{1}{\sqrt{2}}Y_{1-1})$
normalise $\psi = \frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}}Y_{11} + Y_{10} - \frac{1}{\sqrt{2}}Y_{1-1}) = -\frac{1}{2}Y_{11} + \frac{1}{\sqrt{2}}Y_{10} - \frac{1}{2}Y_{1-1}$ [U:2 marks]
and for $q = 0$ we have $\frac{1}{\sqrt{2}}b = 0$, $\frac{1}{\sqrt{2}}(c+a) = 0$ and $\frac{1}{\sqrt{2}}b = 0$ i.e. $b = 0$ and $c = -a$
so $\psi = a(Y_{11} - Y_{1-1})$
normalise $\psi = \frac{1}{\sqrt{2}}(Y_{11} - Y_{1-1})$ [U:2 marks]

(d) $L_z = 0$ [S:1 mark]

$L_x = 0$ and this is deterministic (unlike the general case) as $L^2 = 0$ so all components must be zero! [U:2 marks]

(a)

$$\begin{pmatrix} -E^1 & -\epsilon \\ -\epsilon & -E^1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$ [S : 1 mark]

solution is determinant=0 $(E^1)^2 - \epsilon^2 = 0$ [S:1 mark]

$E^1 = \pm\epsilon$ so $E = E^0 \pm \epsilon$ [S:1 mark]

$E^1 = \epsilon$ gives

$$\begin{pmatrix} -\epsilon & -\epsilon \\ -\epsilon & -\epsilon \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

so $-\alpha - \beta = 0$

normalise so $\psi = (\psi_1 - \psi_2)/\sqrt{2}$ [S:1 mark]

$E^1 = -\epsilon$ and get $\alpha - \beta = 0$

normalise so $\psi = (\psi_1 + \psi_2)/\sqrt{2}$ [S:1 mark]

(b) $< \psi_j^0|(H^0 + H')\psi_1 >= E_1 < \psi_j^0|\psi_1 >$ [U:1 mark]

$< \psi_j^0|(H^0 + H')(\alpha\psi_1^0 + \beta\psi_2^0) >= E_1 < \psi_j^0|(\alpha\psi_1^0 + \beta\psi_2^0 >$ [U:1 mark]

$< \psi_j^0|H^0(\alpha\psi_1^0 + \beta\psi_2^0) > + < \psi_j^0|H'(\alpha\psi_1^0 + \beta\psi_2^0) >= E_1\alpha\delta_{j1} + E_1\beta\delta_{j2}$
[U:2 mark]

$\alpha E_1^0\delta_{j1} + \beta E_2^0\delta_{j2} + \alpha W_{j1} + \beta W_{j2} = E_1\alpha\delta_{j1} + E_1\beta\delta_{j2}$ [U:2 marks]

(c) j=1: $\alpha E_1^0 + \alpha W_{11} + \beta W_{12} = E_1\alpha$ [U:1 mark]

$\alpha(E^0 + \Delta) - \beta\epsilon = E_1\alpha$ [U:1 mark]

j=2: $\beta E_1^0 + \alpha W_{21} + \beta W_{22} = E_1\beta$ [U:1 mark]

$\beta(E^0 - \Delta) - \alpha\epsilon = E_1\beta$

from $j = 1$: $\beta = (E^0 + \Delta - E_1)\alpha/\epsilon$ [U:1 mark]

substitute into $j = 2$: $\beta(E^0 - \Delta - E_1) - \alpha\epsilon = 0$

$(E^0 + \Delta - E_1)(E^0 - \Delta - E_1) - \epsilon^2 = 0$ [U:1 mark]

$(E^0)^2 - \Delta^2 - (E^0 + \Delta)E_1 - (E^0 - \Delta)E_1 + (E_1)^2 - \epsilon^2 = 0$

$(E_1)^2 + (E^0)^2 - \Delta^2 - 2E^0E_1 - \epsilon^2 = 0$ [U:2 mark]

$E_1 = [2E^0 \pm \sqrt{4(E^0)^2 - 4((E^0)^2 - \Delta^2 - \epsilon^2)}]/2$

$= E^0 \pm \sqrt{\Delta^2 + \epsilon^2)}$ [U:1 mark]

as the question tells us that the i=2 case has exactly the same solution as the i=1 case then one of E_1, E_2 has the + sign, the other has the -ve sign [U:1 mark]

Electromagnetism

Prof. Hampshire
June 2015 – Qn1 Short Qs.

a) For a good conductor, $\sigma_N \mu_0 \omega \gg \mu_0 \varepsilon \omega^2$
$$\sigma_N \gg \varepsilon \omega = 20 \cdot 8.85\times10^{-12} \cdot 2\pi \cdot 10^{11}$$
$$= 111.2\ \Omega^{-1}m^{-1}$$
Given $\sigma_N = 2\times10^8\ \Omega^{-1}m^{-1}$, the material is a good conductor — 4 Marks Seen.

b). A radio aerial is an arrangement of conductors that detect an electromagnetic wave and produces an ac. voltage. — 4 Marks Seen.

c). $5\mu C$: charge 1. $7\mu C$: charge 2.

$$F_1 = q_1 E_2 = \frac{q_1 q_2 r_{12}}{4\pi\varepsilon_0 (r_{12})^3} = q_1 q_2 \frac{[(2\hat{i}+3\hat{j}+4\hat{k})-(2\hat{i}+3\hat{j}+8\hat{k})]}{4\pi\varepsilon_0 (4)^3}$$
$$= -\frac{q_1 q_2 4\hat{k}}{4\pi\varepsilon_0 (4)^3} = \frac{-35\times10^{-12}\hat{k}}{4\pi\varepsilon_0 \cdot 16} = 1.967\times10^{-2}\hat{k}\ \text{Newtons}$$
— 4 Marks Seen.

d) A waveguide confines an electromagnetic wave to propagate in one direction, ideally dissipating no power while propagating. Microwave heating, optical fibre, ... — 4 Marks Seen.

e) $B = \mu_0(H+M)$ – Definition of H, $M = \chi H$ – Definition of χ
$\Rightarrow 6 = 1(1+\chi) \Rightarrow \chi = 5.$ — 4 Marks Seen.

f). Fresnel's equations are derived by requiring that Maxwell's equations are met (i.e. valid at all points in space and time) across the interface between two media. More specifically that the continuity of $\underline{E}$ and $\underline{H}$ are met across the interface. — 4 Marks Seen.

g). $\nabla\times\underline{B} = \mu_0 \underline{J} + \mu_0\varepsilon_0 \partial\underline{E}/\partial t$.
Measure the spatial dependence of the $\underline{B}$-field $(\nabla\times B)$ produced by the current flowing through a wire (J) charging a capacitor $\left(\frac{\partial E}{\partial t}\right)$ — 4 Marks Seen.

Electromagnetism

Prof. Hampshire.
June 2015 Qn 2.

a) The Poynting vector $\underline{S} = \underline{E} \times \underline{H}$ where $\underline{E}$: electric field and $\mu_0\underline{H} = \underline{B} - \mu_0 \underline{M}$, $\underline{H}$: field strength.] 2 Marks Seen.

b). Using $\underline{\nabla} \times \underline{E} = -\partial \underline{B}/\partial t \Rightarrow \underline{k} \times \underline{E}_0 = \omega \underline{B}_0$

$\underline{\nabla} \times \underline{B} = \mu_0 \varepsilon_0 \partial \underline{E}/\partial t \Rightarrow \underline{k} \times \underline{B}_0 = -\mu_0 \varepsilon_0 \omega \underline{E}_0$

$\Rightarrow \quad \underline{k} \perp \underline{B}_0 \perp \underline{E}_0$] 4 Marks Unseen.

c) $v = c/\sqrt{\varepsilon_r} = 3\times10^7$, $k = \omega/v_{phase} = \frac{3\times10^{15}}{3\times10^{7}} = 10^8\,m^{-1}$.

$\hat{k} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix}$, $\Rightarrow \underline{k} = \frac{10^8}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix} m^{-1}$.

$\hat{B}_0 = \frac{1}{\sqrt{9+36}}\begin{pmatrix}-3\\0\\6\end{pmatrix} \Rightarrow \underline{B} = \frac{3\times10^{-6}}{\sqrt{45}}\begin{pmatrix}-3\\0\\6\end{pmatrix} T$.] 4 Marks

MIV $\underline{k} \times \underline{B}_0 = -\mu_0 \varepsilon_0 \omega \underline{E}_0$

$\underline{k} \times \underline{B}_0 = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 0 & 1\\ -3 & 0 & 6\end{vmatrix} \frac{10^8}{\sqrt{5}} \cdot \frac{3\times10^{-6}}{\sqrt{45}} = \frac{3\times10^{2}}{15}(-15\hat{\jmath}) = -3\times10^{2}\hat{\jmath}$

$\underline{E}_0 = \frac{3\times10^{2} \cdot (3\times10^{8})^2}{3\times10^{15} \cdot 100}\hat{\jmath} = 90\hat{\jmath}\ V.m^{-1}$] 4 Marks

] 8 Marks Unseen.

d). $\underline{S} = \underline{E} \times \underline{H} = \frac{3\times10^{-6}}{4\pi\times10^{-7} \cdot \sqrt{45}}\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 90 & 0\\ -3 & 0 & 6\end{vmatrix} = 0.356(540\hat{\imath} + 270\hat{k})$

$= (192\hat{\imath} + 96\hat{k})\ Wm^{-2}$] 3 Marks Unseen

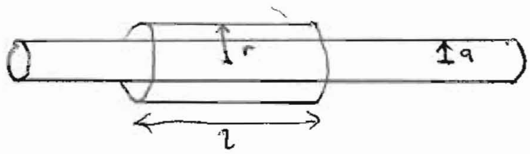
e). Energy $= P/_{Area} \cdot$ Area $\cdot$ time

$= \sqrt{(192)^2 + (96)^2} \cdot 5 \cdot 1 = 1073$ Joules.] 3 Marks Unseen.

Electromagnetism

Prof. Hampshire,
June 2015. Qn. 3

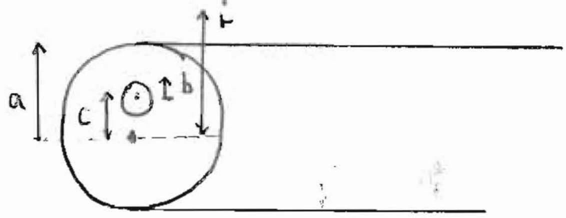
a). Gauss' law $\int \underline{E} . d\underline{S} = \Sigma Q/\varepsilon_0$.



$$E . 2\pi r l = \rho . \pi a^2 l/\varepsilon_0 \Rightarrow \underline{E} = \rho a^2/2\varepsilon_0 r \; \hat{r}.$$

4 Marks Seen

b)



Above the spherical cavity $E . 4\pi (r-c)^2 = \rho . \frac{4}{3}\pi b^3/\varepsilon_0$

From Superposition (a solid rod + a sphere of negative charge):

$$\underline{E} = \left(\rho a^2/2\varepsilon_0 r - \rho b^3/3(r-c)^2 \varepsilon_0 \right) \hat{r}.$$

6 Marks Unseen

c) Maximum change, ΔE_{MAX}, along the length

$$\Delta E = \rho b^3/3(r-c)^2 \varepsilon_0$$

Since cavities can be anywhere, for maximum ΔE, $r = c + b$

$$\Delta E_{MAX} = \rho b^3/3(b)^2 \varepsilon_0 = \rho b/3\varepsilon_0.$$

5 Marks Unseen

d) Percentage variation $\Delta E\%$

$$\Delta E\% = \frac{2 \rho b/3\varepsilon_0 . 100}{\rho a^2/2\varepsilon_0 (c+b)} = \frac{400 b (c+b)}{3a^2}$$

$$\approx \frac{400 bc}{3a^2} \text{ in the limit of } b \to 0.$$

5 Marks Unseen