University of Durham

EXAMINATION PAPER

May/June 2016 Examination code: PHYS3661-WE01

THEORETICAL PHYSICS 3

SECTION A. Relativistic Electrodynamics
SECTION B. Quantum Theory 3

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. RELATIVISTIC ELECTRODYNAMICS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Give the definition and physical interpretation of lightlike, timelike and spacelike separations between two events with coordinates x_1^μ and x_2^μ. [4 marks]

(b) State the definition of a covariant and contravariant 4-vector and the relation between them. Show that the sum of two covariant 4-vectors is also a covariant 4-vector. [4 marks]

(c) A polarization tensor for a photon is

$$T^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n}\right),$$

where p^μ and n^μ are lightlike 4-vectors. Calculate $T^\mu{}_\mu$ and $T^{\mu\nu}T_{\mu\nu}$. [4 marks]

(d) What is the speed of a particle (relative to c) if its kinetic energy is equal to three times its rest mass energy? [4 marks]

(e) An observer moves with constant velocity $\underline{v}$ relative to a point charge q. She measures the electric $\underline{E}$ and magnetic $\underline{B}$ fields at a distance $\underline{r}$ from the point charge. What value does she find for $\underline{E} \cdot \underline{B}$? [4 marks]

(f) The Lienard-Wiechert potential of a point charge q with 4-velocity u^μ is

$$A^\mu = \frac{q}{4\pi\epsilon_0} \frac{u^\mu}{u^\nu R_\nu},$$

where R_ν is the 4-distance between the observer and the point charge. The right-hand side of the expression must be evaluated at the retarded time t_{ret}. Evaluate this expression in the instantaneous rest frame of the point charge and show that you obtain the expected result. [4 marks]

(g) Use the covariant form of the inhomogeneous Maxwell equation to derive the wave equation in vacuum for the 4-potential A^μ in the Lorenz gauge. [4 marks]

(h) Express the 0-component of the Maxwell equation $\partial_\mu F^{\mu\nu} = j^\nu/(c\epsilon_0)$ in terms of the electric and magnetic fields. [4 marks]

[Hint: See Question 2 for $F^{\mu\nu}$ written in terms of the physical fields $\underline{E}, \underline{B}$.]

2. Consider a point charge q of rest mass m in an electromagnetic field. The field strength tensor is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

The 4-force f^μ acting on the point charge is defined by

$$f^\mu \equiv \frac{dp^\mu}{d\tau},$$

where τ is the proper time and p^μ the 4-momentum of the point charge.

(a) Show that f^μ is a 4-vector, and demonstrate how it is related to the usual force $\underline{F} = d\underline{p}/dt$. [4 marks]

(b) The 4-force acting on the point charge due to the electromagnetic field is given by

$$f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu,$$

where u_ν is the 4-velocity of the point charge. Show that the spatial components of this equation correspond to the Lorentz force law. Also, interpret the equation obtained from the 0-component of the above equation. [8 marks]

(c) In the following, you can ignore the effects of radiation from an accelerating charge. The point charge moves under the influence of a uniform magnetic field $\underline{B}$. Starting with the assumption that the velocity $\underline{v}$ of the point charge is perpendicular to $\underline{B}$, show that the point charge moves with constant speed in a circle. Compute the radius of this circle in terms of the magnitude of the magnetic field, the mass, m, and speed, v, of the point charge. What happens if the initial velocity is not perpendicular to $\underline{B}$? [8 marks]

3. An observer at rest in an inertial frame S notices that at time $t = 0$ a point charge q of mass m is at the origin of the coordinate system with a momentum $\underline{p}_0$, written in terms of its Cartesian coordinates as $\underline{p}_0 = (0, 0, p_0)$. In S there is also a constant electric field $\underline{E} = (0, E, 0)$.

   a) Show that the momentum of the point charge at any later time t is given by $\underline{p}(t) = (0, qtE, p_0)$. [4 marks]

   b) Show that the velocity $\underline{v}$ of the point charge can be expressed in terms of its momentum and energy E as

   $$\underline{v} = \frac{c^2}{E}\,\underline{p}$$

   and use this to compute $\underline{v}$ as a function of time. What happens in the limit $t \to \infty$? [8 marks]

   c) Compute the position (as measured in S) of the point charge as a function of time. [5 marks]

   $$\left[\text{Hint}: \int \frac{\mathrm{d}t}{\sqrt{1 + a^2 t^2}} = \frac{\operatorname{arcsinh}(at)}{a}\right]$$

   d) Is the point charge uniformly accelerated? Justify your answer. [3 marks]

4. The electromagnetic fields generated by a point charge q in arbitrary motion are given by

$$\underline{E}(\underline{r},t) = \frac{q}{4\pi\epsilon_0}\frac{R}{(\underline{R}\cdot\underline{u})^3}[(c^2-v^2)\underline{u}+\underline{R}\times(\underline{u}\times\underline{a})],$$

$$\underline{B}(\underline{r},t) = \frac{1}{c}\hat{\underline{R}}\times\underline{E}(\underline{r},t),$$

where $\underline{R}$ is the vector between the point charge and the observer, $\underline{v}$ is the velocity of the point charge, $\underline{u} = c\hat{\underline{R}} - \underline{v}$, and $\underline{a}$ is the acceleration of the point charge. $\underline{R}, \underline{u}, \underline{v}$, and $\underline{a}$ are all evaluated at the retarded time.

Consider a point charge q in the frame where, at time t, it is instantaneously at rest but undergoing an acceleration $\underline{a}$.

(a) Identify the electric radiation field from the equations above, and show that it is given by

$$\underline{E}_{\rm rad}(\underline{r},t) = \frac{\mu_0\, q}{4\pi R}\left[\left(\hat{\underline{R}}\cdot\underline{a}\right)\hat{\underline{R}} - \underline{a}\right].$$

[4 marks]

(b) Show that the Poynting vector for the radiation field is given by

$$\underline{S}_{\rm rad} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin\theta}{R}\right)^2\hat{\underline{R}},$$

where θ is the angle between $\underline{R}$ and $\underline{a}$. [4 marks]

(c) Calculate the total power radiated to infinity by the point charge at time t, by integrating the Poynting vector as follows:

$$P = \oint_S \left(\frac{\underline{R}\cdot\underline{u}}{Rc}\right)\underline{S}_{\rm rad}\cdot {\rm d}\underline{a}$$

and check that your answer is in agreement with the general result by Liénard for a point charge

$$P = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \left|\frac{\underline{v}\times\underline{a}}{c}\right|^2\right).$$

[6 marks]

(d) Consider now a new system, consisting of *two* point charges instantaneously at rest and located at the origin of the coordinate system. The first point charge $+q$ undergoes an acceleration $\underline{a}$ as in the example above. The second point charge $-q$ undergoes an acceleration $-\underline{a}$.

Find the total power radiated to infinity by this new system at the instant when the acceleration starts. [6 marks]

[Hint: $\underline{A}\times(\underline{B}\times\underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B})$ $\qquad \underline{S} = (\underline{E}\times\underline{B})/\mu_0$]

  Continued Next Page

SECTION B. QUANTUM THEORY 3

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) For $|\underline{r}| \gg |\underline{r}'|$ and $k|\underline{r}'| \approx 1$, show that

$$\frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} = \frac{e^{ik|\underline{r}|}}{|\underline{r}|} e^{-ik\hat{\underline{r}}\cdot\underline{r}'} + \mathcal{O}\left(\frac{|\underline{r}'|}{|\underline{r}|^2}\right),$$

where $\hat{\underline{r}} = \underline{r}/|\underline{r}|$. [4 marks]

(b) Find the algebraic relations which the coefficients α_k and β must satisfy for a solution Ψ of the Dirac equation

$$\left(-i\alpha_k\partial_k + \beta m\right)\Psi = i\frac{\partial\Psi}{\partial t},$$

to also satisfy the Klein-Gordon equation $\left(\partial_\mu\partial^\mu + m^2\right)\Psi = 0$. [4 marks]

(c) We can express the azimuthally symmetric scattering amplitude in terms of Legendre polynomials

$$f(k,\theta,\phi) = \sum_{\lambda=0}^{\infty} f_\lambda(k)\mathrm{P}_\lambda(\cos\theta) \quad \text{with} \quad \int_{-1}^{+1} dx\, \mathrm{P}_\lambda^2(x) = \frac{2}{2\lambda+1}.$$

Derive an expression for the total cross section as a sum of partial-wave contributions. [4 marks]

(d) The distribution function for an ensemble of non-interacting identical fermions is given by

$$f(E) = \frac{1}{e^{\beta(E-\mu)}+1}.$$

What are the quantities represented by β and μ? For $\mu > 0$, sketch the distribution at high and low temperatures and how the distribution interpolates between these limits. [4 marks]

(e) Explain how the scattering of two identical particles in quantum mechanics leads to terms in the cross section which are absent in classical mechanics. [4 marks]

(f) Compute the commutator $[H, \underline{L}]$ where H is the Dirac Hamiltonian in a central potential, $H = c\underline{\alpha}\cdot\underline{p} + \beta mc^2 + V(r)$, and $\underline{L} = \underline{r}\times\underline{p}$ is the orbital angular momentum operator. [4 marks]

(g) A scattering problem predicts an s-wave scattering phase shift of the form $\tan\delta_0(k) = -ka + \mathcal{O}(k^2)$. Find an expression for the cross section in terms of the real parameter a in the small k limit. What would a represent if we were to replace the potential with an approximation in the form of a hard sphere? [4 marks]

[Hint: the s-wave cross-section can be written $\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0(k)$.]

6. Consider the central potential for a particle of mass m given by

$$V(r) = -\frac{\hbar^2}{ma^2}\frac{1}{\cosh^2(r/a)},$$

where a is a real positive constant.

(a) Write down the time independent Schrödinger equation for a particle of initial momentum $\hbar\underline{k}$ scattering in this potential. Show that it can be written as

$$\left[\nabla^2 + \underline{k}^2 - U(r)\right]\Psi_k(\underline{r}) = 0,$$

giving an explicit expression for $U(r)$. [4 marks]

(b) The Laplacian in spherical coordinates can be written

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\underline{L}^2(\theta,\phi)}{\hbar^2 r^2}.$$

Separate the Schrödinger equation into radial and angular equations. Which functions solve the angular part? [4 marks]

(c) Using the fact that

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \eta^2 + \frac{2}{\cosh^2(x)}\right]\phi(x) = 0,$$

where η is a real constant, has the pair of solutions

$$\phi_\pm(x) = \frac{e^{\pm i\eta x}}{x}\left(\tanh(x) \mp i\eta\right),$$

find the most general solution of the s-wave radial equation. What is the equivalent solution to the s-wave angular equation? [4 marks]

(d) What boundary condition applies to the solution of the radial equation at the origin? Use this to find the wavefunction up to a total normalisation factor. [4 marks]

(e) Show that the solution can be written for large r in the form,

$$\lim_{r\to\infty}\Psi_k(r) = \lim_{r\to\infty}\frac{A}{r}\sin\left[kr + \delta_0(k)\right],$$

and compute the scattering phase $\delta_0(k)$. Use your result to express the s-wave cross section σ_0 as a function of the energy of the initial particle, given that

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0(k),$$

[4 marks]

7. In the chiral basis, the Dirac matrices are given in 2×2 block form as

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

where $k \in \{1, 2, 3\}$ and the 2×2 matrices are given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Write out the components of the matrices

$$\Sigma_\pm(p) = p_0 I \pm \sum_{k=1}^{3} p_k \sigma^k,$$

for a 4-momentum $p_\mu = (p_0, p_1, p_2, p_3)^{\mathrm{T}}$. Show that these matrices are Hermitian. [3 marks]

(b) Show how the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\Psi = 0,$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^{\mathrm{T}}$, can be decomposed into two coupled linear equations for the two 2-component vectors $\chi = (\Psi_1, \Psi_2)^{\mathrm{T}}$ and $\zeta = (\Psi_3, \Psi_4)^{\mathrm{T}}$. Then take the limit $m \to 0$ and comment on how this changes the relationship between χ and ζ. [4 marks]

(c) Consider plane wave Ansätze for χ and ζ of the form

$$\chi(x) = u(p)e^{-ip_\mu x^\mu} \qquad \text{and} \qquad \zeta(x) = v(p)e^{-ip_\mu x^\mu}.$$

Show that for these to be solutions of the Dirac equation when $m = 0$, we require

$$\Sigma_-(p)u(p) = 0 \qquad \text{and} \qquad \Sigma_+(p)v(p) = 0. \qquad \text{[4 marks]}$$

(d) Show that for $m = 0$ and non-zero $u(p)$ and $v(p)$, the matrix equations in part (c) require that $p^2 = p_\mu p^\mu = 0$. [3 marks]

(e) In a frame where the 3-momentum is aligned with the z-axis, find solutions when $m = 0$ for $u(p)$ and $v(p)$. Show that they are orthogonal. [3 marks]

(f) Show that the identity matrix can be written as

$$1 = \frac{\Sigma_+(p) + \Sigma_-(p)}{2p_0},$$

and use this to show that for $m = 0$, the vectors $u(p)$ and $v(p)$ are orthogonal regardless of the direction of their 3-momentum. [3 marks]

8. For a spin-1 system, the spin operators in the basis of S_z eigenstates are given by the matrices

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Explain the two types of randomness that can exist in a quantum ensemble. [2 marks]

(b) Compute the density matrix for a beam of spin-1 particles, 40% of which have S_x values of $+\hbar$ and 60% of which have S_z values of $-\hbar$. [4 marks]

[Hint: the eigenvector of S_x with eigenvalue $+\hbar$ is $v = \frac{1}{2}\left(1, \sqrt{2}, 1\right)^{\mathrm{T}}$.]

(c) Compute the expectation value of $(S_y)^2$ for the system described by the density matrix found in part (b). [3 marks]

(d) What is meant by a *pure* and a *mixed* state? Give *two* ways of determining whether a density matrix represents a mixed state, and use them to show that the spin-1 system under consideration is mixed. [4 marks]

(e) Show that if the system evolves unitarily according to the Schrödinger equation then the density matrix satisfies the von Neumann equation

$$\hbar \frac{\partial \rho}{\partial t} = -i[H, \rho].$$

[3 marks]

(f) Prove that under unitary evolution the trace of the density matrix squared is a constant, so that

$$\frac{\partial}{\partial t} \mathrm{Tr}\left(\rho^2\right) = 0.$$

Provide a physical interpretation of this result. [4 marks]

