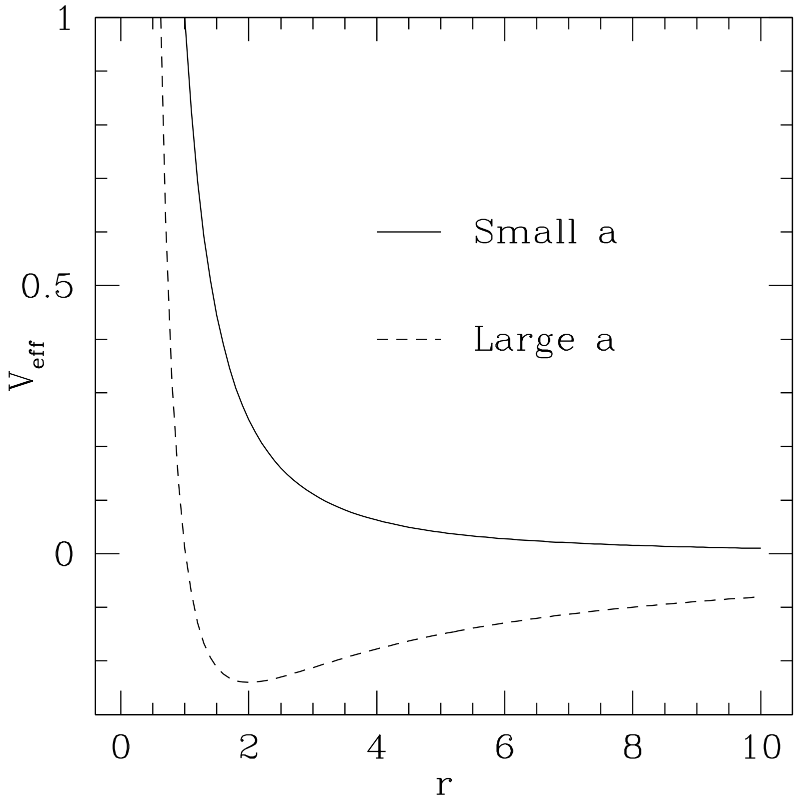
CM5 Solutions: Orbiting electrons

1. **(2 marks total)** The kinetic energy is $T = (m/2)(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2)$. If the motion takes place in a plane, then we can choose our coordinates such that $\theta = \pi/2$, in which case $\dot{\theta} = 0$ and $\sin\theta = 1$, from which the required Lagrangian ($= T - V$) follows. **[2 marks]**

2. **(2 marks total)** The associated constant of motion is the canonically conjugate momentum, defined as $J = \partial L/\partial\dot{\phi} = mr^2\dot{\phi}$. **[1 mark]**
   Eliminating $\dot{\phi}$ from $E = T + V$ leads to the required expression for $V_{\rm eff}$. **[1 mark]**

3. **(2 marks total)**



[2 marks]

4. **(4 marks total)** For a stable circular orbit to exist, $dV_{\rm eff}/dr = 0$ and $d^2V_{\rm eff}/dr^2 > 0$. $dV_{\rm eff}/dr = 0$ occurs at $r = r_c$, where

$$-\frac{J^2}{mr_c^3} + \frac{k}{r_c}e^{-r_c/a}\left(\frac{1}{r_c} + \frac{1}{a}\right) = 0,$$

i.e.

$$\frac{J^2}{mr_c^2} = ke^{-r_c/a}\left(\frac{1}{r_c} + \frac{1}{a}\right).$$

[2 marks]

$$\frac{d^2V_{\rm eff}}{dr^2} = \frac{3J^2}{mr^4} - \frac{k}{r}e^{-r/a}\left(\frac{2}{r^2} + \frac{2}{ar} + \frac{1}{a^2}\right).$$

evaluating this at $r = r_c$ and requiring that $d^2V_{\rm eff}/dr^2 > 0$ for a stable orbit leads to

$$\frac{3J^2}{mr_c^4} - \frac{k}{r_c}e^{-r_c/a}\left(\frac{2}{r_c^2} + \frac{2}{ar_c} + \frac{1}{a^2}\right) > 0.$$

Defining $x = r_c/a$, and using the expression for $J^2/(mr_c^2)$ to eliminate J leads to $x^2 - x - 1 < 0$, from which the physically sensible solution is $r_c/a < (1 + \sqrt{5})/2$. **[2 marks]**