(a) $L_+S_- = (L_x+iL_y)(S_x-iS_y) = L_xS_x - iL_xS_y + iL_yS_x + L_yS_y$, and $L_-S_+ = (L_x-iL_y)(S_x+iS_y) = L_xS_x + iL_xS_y - iL_yS_x + L_yS_y$. [1 mark] Hence,

$$\frac{1}{2}(L_+S_- + L_-S_+) + L_zS_z = \frac{1}{2}(2L_xS_x + 2L_yS_y) + L_zS_z = L_xS_x + L_yS_y + L_zS_z$$

as required. [1 mark]

(b) $\underline{L}\cdot\underline{S}\psi^0_{2,1,-1,1/2} = [(L_+S_- + L_-S_+)/2 + L_zS_z]R_{21}Y_{1,-1}\chi_+$
$= R_{21}[(L_+S_- + L_-S_+)/2 + L_zS_z]Y_{1,-1}\chi_+$ as none of these operators affect n, l. [1 mark]

Now take each operator in turn — all S commute with all L so order doesn't matter. All S operators only affect χ while L operators only affect Y. Hence, $L_zS_zY_{1,-1}\chi_+ = (L_zY_{1,-1})(S_z\chi_+) = (L_zY_{1,-1})(\hbar/2)\chi_+ = (\hbar/2)(-\hbar)Y_{1,-1}\chi_+ = -(\hbar^2/2)Y_{1,-1}\chi_+$. [1 mark]

$L_-S_+Y_{1,-1}\chi_+ = (L_-Y_{1,-1})(S_+\chi_+) = 0$ as can't raise m_s above 1/2 (and can't lower m below -1 for $l=1$).

$L_+S_-Y_{1,-1}\chi_+ = (L_+Y_{1,-1})(S_-\chi_+) = (L_+Y_{1,-1})(\hbar/\sqrt{2})\chi_-$
$A_{1,-1} = \hbar\sqrt{2-1(-1+1)} = \hbar\sqrt{2}$, [2 marks]
and so

$$\begin{aligned}\frac{1}{2}(L_+S_- + L_-S_+)Y_{1-1}\chi_+ =& \frac{1}{2}(L_+Y_{1,-1})(\frac{\hbar}{\sqrt{2}}\chi_-)\\ =& \frac{1}{2}\hbar\sqrt{2}\frac{\hbar}{\sqrt{2}}Y_{1,0}\chi_- = \frac{\hbar^2}{2}Y_{1,0}\chi_-.\end{aligned}$$

In total

$$\underline{L}\cdot\underline{S}\psi^0_{2,1,-1,1/2} = R_{21}\frac{\hbar^2}{2}(Y_{1,0}\chi_- - Y_{1,-1}\chi_+) = \frac{\hbar^2}{2}(\psi^0_{2,1,0,-1/2} - \psi^0_{2,1,-1,1/2}).$$

The operator does not return the same function that we gave it, hence the unperturbed energy eigenfunctions are not eigenfunctions of $\underline{L}\cdot\underline{S}$. [1 mark]

(c) $[L_xS_x + L_yS_y + L_zS_z, L_z] = [L_xS_x, L_z] + [L_yS_y, L_z] + [L_zS_z, L_z]$
$= S_x[L_x, L_z] + S_y[L_y, L_z] + S_z[L_z, L_z]$ as the components of $\underline{S}$ and $\underline{L}$ commute. [1 mark]
This in turn $= S_x(-i\hbar L_y) + S_yi\hbar L_x = i\hbar(S_yL_x - S_xL_y) \neq 0$

$[L_xS_x + L_yS_y + L_zS_z, S_z] = [L_xS_x, S_z] + [L_yS_y, S_z] + [L_zS_z, S_z]$
$= L_x[S_x, S_z] + L_y[S_y, S_z] + L_z[S_z, S_z] = L_x(-i\hbar S_y) + L_yi\hbar S_x$
$= i\hbar(L_yS_x - L_xS_y) \neq 0$. [1 mark]

$[L_xS_x + L_yS_y + L_zS_z, J_z] = [L_xS_x + L_yS_y + L_zS_z, L_z]$
$+ [L_xS_x + L_yS_y + L_zS_z, S_z] = i\hbar(S_yL_x - S_xL_y + L_yS_x - L_xS_y) = 0$

So J_z commutes with the perturbation whereas L_z and S_z do not. We should therefore use n, l, j, m_j rather than n, l, m, m_s. [1 mark]