RED: Updated Workshop 6 Solutions

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Meanings and definitions of the terms:

- $\vec{E}$ is the electric field, evaluated at observer spacetime coordinates $(t, \vec{r})$ with $\vec{r}$ the vector from the arbitrary origin
- q is the charge of the moving point
- ϵ_0 is the permittivity of free space
- c is the speed of light
- $v = |\vec{v}|$ with $\vec{v}(t_r) = \dot{\vec{\omega}}(t_r)$, magnitude of velocity of charge at time t_r
- $\vec{u} = c\hat{\vec{R}} - \vec{v}$, where $\hat{\vec{R}}$ is unit vector in direction from retarded position to observer: $\vec{R} = \vec{r} - \vec{\omega}(t_r)$
- $\vec{a}$ is the accelaration $\vec{a} = \partial\vec{v}/\partial t$

All terms $(\vec{R}, \vec{u}, \vec{v}, \vec{a})$ at RHS evaluated at retarted time $t_r = t - |\vec{r} - \vec{w}(t_r)|/c$, defined implicitly by the need for $|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$.

The terms are evaluated at retarded time because they are a result of applying the usual formulae to the potentials ϕ and $\vec{A}$. The terms in these potentials are evaluated at retarded time as explained in the notes, due to the finite time taken for information to be sent. Hence, when these terms trickle through to the resultant E and B fields, they must still be evaluated at retarded time. It is not the instantaneous potential configuration that matters, but the one at retarded time.

The Coulomb/velocity field goes as $\sim 1/R^2$, as expected in electrostatics. This can be seen by noting that $\vec{R} \cdot \vec{u} \sim R$, while c, v, u, a are independent. Therefore, the radiation field goes as $\sim 1/R$ in the same $R \to \infty$ limit.

The Poynting vector is given by,

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{1}{\mu_0 c}\vec{E} \times (\hat{\vec{R}} \times \vec{E}). \tag{1}$$

The total power radiated is the surface integral over the Poynting vector,

$$P = \oint \vec{S} \cdot d\vec{A} = \int (\vec{S} \cdot \hat{\vec{R}}) d\Omega = \int (\vec{S} \cdot \hat{\vec{R}}) R^2 \sin\theta d\theta d\phi, \tag{2}$$

with $d\Omega$ the usual solid angle measure $R^2 \sin\theta d\theta d\phi$. Here we use R not r because we are integrating over the suface of a sphere whose centre is not the coordinate origin but rather the retarded position of the charge. (Since we ultimately consider the $R \to \infty$ limit, the centre of the sphere is not all that relevant, but making $\hat{\vec{R}}$ the unit radial vector for the sphere is a fairly natural choice and simplifies much of the maths.)

We can now see that for a general case we will have terms of large R with three different behaviours: $1/R^2, 1/R^3, 1/R^4$. Given the R^2 from the vecor area measure, only the $1/R^2$ term will survive the $R \to \infty$ limit (the others will fall to zero) and contribute to the power radiated. This term is of course from the radiation field parts, hence the name. We also note that the radiation field is only non-zero if the charge is accelerating, which gives us a hint for the power radiated in the two examples that follow.

CASE 1

No acceleration $\vec{a} = 0$ but constant $\vec{v} \neq 0$. Only the velocity field is present, which is in the $\vec{u} = c\hat{\vec{R}} - \vec{v}$ direction:

$$\vec{E}(\vec{r}, t) = \xi \frac{R}{(\vec{R} \cdot \vec{u})^3}(c^2 - v^2)\vec{u}, \tag{3}$$

with $\xi = q/4\pi\epsilon_0$. Let's have a quick look at what this means geometrically. Recall $\vec{w}(t) = \vec{v}t$ in this constant velocity case, and $\vec{u}(t) = c\hat{\vec{R}} - \vec{v}(t)$. Hence

$$R\vec{u}(t_r) = Rc\hat{\vec{R}} - R\vec{v}(t_r) = c\vec{R} - R\vec{v}(t_r). \tag{4}$$

But $\vec{R} = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}t_r$ and $R = |\vec{R}| = c(t - t_r)$. So,

$$R\vec{u}(t_r) = c(\vec{r} - \vec{v}t_r) - c(t - t_r)\vec{v} = c(\vec{r} - \vec{v}t). \tag{5}$$

Strangely, the dependence on retarded time seems to have been removed in favour of current time. We can rewrite the E field as,

$$\vec{E}(\vec{r}, t) = \xi \frac{c(c^2 - v^2)}{(\vec{R} \cdot \vec{u})^3}\vec{\mathcal{R}}, \tag{6}$$

with $\vec{\mathcal{R}} = \vec{r} - \vec{v}t$ being the vector from the **current** position to the observer. This is unusual, and specific to this case of constant velocity, but the electric field that reaches the observer points in a direction as if it had come from the current position of the charge, not the retarded position one would expect!

For the magnetic field,

$$\vec{B}(\vec{r}, t) = \frac{\xi}{c}\frac{R}{(\vec{R} \cdot \vec{u})^3}(c^2 - v^2)\hat{\vec{R}} \times \vec{u} = \frac{\xi}{c}\frac{R}{(\vec{R} \cdot \vec{u})^3}(c^2 - v^2)\vec{v} \times \hat{\vec{R}}. \tag{7}$$

The final simplification can be made because the $\hat{\vec{R}}$ part of $\vec{u}$ vanishes in the cross product with itself. [NB: In the $v = 0$ limit, $\vec{u} = c\hat{\vec{R}}$ and so we recover the electrostatic result $\vec{E} = \xi\hat{\vec{R}}/R^2, \vec{B} = \vec{0}$. Now back to the general case.]

The Poynting vector is,

$$\begin{aligned}\vec{S} =& \frac{\xi^2}{\mu_0 c}\frac{R^2}{(\vec{R} \cdot \vec{u})^6}(c^2 - v^2)^2\vec{u} \times (\vec{v} \times \hat{\vec{R}}) \\ =& \frac{\xi^2}{\mu_0 c}\frac{R^2}{(\vec{R} \cdot \vec{u})^6}(c^2 - v^2)^2[(\vec{u} \cdot \hat{\vec{R}})\vec{v} - (\vec{u} \cdot \vec{v})\hat{\vec{R}}].\end{aligned} \tag{8}$$

As discussed earlier, only terms decreasing like $1/R^2$ (or slower) survive the $R \to \infty$ and integration. All terms above go as $1/R^4$ and thus the power radiated to infinity is zero! As we have seen in lectures, the velocity field does not radiate.

CASE 2

Now have acceleration $\vec{a} \neq 0$ but zero instantaneous velocity $\vec{v} = 0 \to \vec{u} = c\hat{\vec{R}}$. Now the radiation field is present:

$$\begin{aligned}\vec{E}(\vec{r}, t) =& \xi \frac{cR}{(\vec{R} \cdot \vec{u})^3}[c^2\hat{\vec{R}} + \vec{R} \times (\hat{\vec{R}} \times \vec{a})] \\ =& \frac{\xi}{c^2R^2}[(c^2 + R\hat{\vec{R}} \cdot \vec{a})\hat{\vec{R}} - R\vec{a}].\end{aligned} \tag{9}$$

For the magnetic field,

$$\vec{B}(\vec{r}, t) = \frac{\xi}{c^3}\frac{1}{R}\vec{a} \times \hat{\vec{R}}. \tag{10}$$

[NB: In the $a = 0$ limit, we once again recover the electrostatic result $\vec{E} = \xi\hat{\vec{R}}/R^2, \vec{B} = \vec{0}$. Now back to the general case.]

The Poynting vector is,

$$\begin{aligned}\vec{S} =& \frac{\xi^2}{\mu_0 c^5}\frac{1}{R^3}[(c^2 + R\hat{\vec{R}}\cdot\vec{a})\hat{\vec{R}} - R\vec{a}] \times (\vec{a}\times\hat{\vec{R}}) \\ =& \frac{\xi^2}{\mu_0 c^5}\frac{1}{R^3}[c^2\vec{a} + (Ra^2 - c^2\hat{\vec{R}}\cdot\vec{a} - R(\hat{\vec{R}}\cdot\vec{a})^2)\hat{\vec{R}}].\end{aligned} \tag{11}$$

As discussed earlier, only terms decreasing like $1/R^2$ (or slower) survive the $R \to \infty$ and integration. This simplifies the Poynting vector to,

$$\vec{S}_{rad} = \frac{\xi^2}{\mu_0 c^5}\frac{1}{R^2}(a^2 - (\hat{\vec{R}}\cdot\vec{a})^2)\hat{\vec{R}} = \frac{\xi^2}{\mu_0 c^5}\frac{a^2\sin^2\theta}{R^2}\hat{\vec{R}}, \tag{12}$$

if θ is the angle between $\vec{R}$ and $\vec{a}$ such that $(\hat{\vec{R}}\cdot\vec{a}) = a\cos\theta$. Hence, power radiated at infinity is

$$P = \frac{\xi^2 a^2}{\mu_0 c^5}\int \sin^3\theta d\theta d\phi = \frac{8\pi\xi^2 a^2}{3\mu_0 c^5} = \frac{8\pi q^2 a^2(\epsilon_0\mu_0)^2}{3(4\pi\epsilon_0)^2\mu_0 c} = \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{13}$$

This is Larmor's formula as promised.