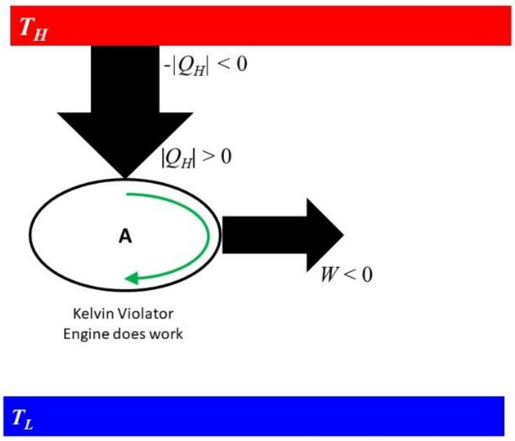
Foundations of Physics 2B/3C 2017/2018

Thermodynamics – Summer Examinations 2018

Short Solutions, version 3 (external) – all conceptual

(Parts of answer in **bold** what is expected from students, rest explanations)



a) **Heat $-|Q_H|$ is removed** by the engine **from the hot reservoir**, and goes into engine $|Q_H| > 0$

[1 mark – concept]

As the **engine operates in a cycle, $dU = 0$** around the cycle and $\oint dS = 0$ so from the first law $\delta Q = -\delta W$ and hence the **engine does work $W = -|Q_H|$**

[1 mark – concept]

Similarly the **engine has zero entropy change**, so only entropy change is from the reservoirs:

[1 mark – concept]

$$\Delta S_H = \int \frac{\delta Q_H}{T_H} = -\frac{|Q_H|}{T_H} < 0 \quad ; \quad \Delta S_L = 0.$$

The **total entropy change** of the universe is thus $\Delta S_H < 0$, which **violates the entropy statement of the second law.** **[1 mark – concept]**

b) The **distribution function** tells us the **probability a state of energy ε is occupied.**

[1 mark – concept]

The **density of states** tells us **how many energy levels are in the interval $\varepsilon \to \varepsilon + d\varepsilon$.**
[1 mark – concept]

As $T \to T_c$ the chemical potential approaches the energy of the lowest available state, $\mu \to \varepsilon_0$, and hence

$$\lim_{T\to 0}\left(\exp\left(\frac{\mu - \varepsilon}{k_B T}\right)\right) \to 1.$$

The **denominator** of the distribution function thus "**blows up" and all the particles migrate to the lowest available energy state.** **[2 marks – concept]**

c) If $\beta_p < 0$ **for** $\Delta T > 0$, we must have $\Delta V < 0$ – the material contracts on heating.

[1 mark – concept]

The **Nernst statement** of the third law tells us that **as we approach $T = 0$ K the entropy change approaches zero too.** **[1 mark – concept]**

Using the Maxwell Relation, we can write $\beta_p = -\frac{1}{V}\left(\frac{\partial S}{\partial p}\right)_T$, so if $\Delta S \to 0$ then

$$\lim_{T\to 0} \boldsymbol{\beta_p} = \mathbf{0}.$$

[2 marks – concept]

d) The total differential of the Helmholtz function is

$$\boldsymbol{dF = dU - TdS - SdT = -pdV - SdT},$$

if the first law is substituted, $(dU = TdS - pdV)$.

[1 mark – concept]

For $\boldsymbol{F}$ **to be conserved,** $\boldsymbol{dF} = \mathbf{0}$ and this is the case for $dT = 0$ and $dV = 0$ i.e. an **isothermal and isobaric** process.

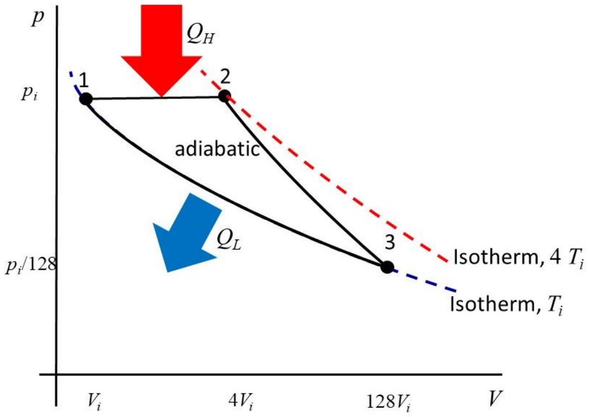
[1 mark – concept]

At constant temperature, $\boldsymbol{T_0}$, $\boldsymbol{\Delta F = \Delta U - T_0 \Delta S}$. If ΔS increased to maximum, $\boldsymbol{\Delta S = S_2 - S_1}$ **and hence** $\boldsymbol{\Delta F = F_2 - F_1}$ **must become as negative as possible.**

[1 mark – concept]

At constant temperature, $\boldsymbol{dF = -pdV = \delta W}$, so therefore the work tends to largest negative value possible **i.e. maximum work is done.** **[1 mark – concept]**

e)



[1 mark shape – concept]

[1 mark axis and heats– concept]

The thermodynamic states are:

- **1:** $(\boldsymbol{p_i, V_i, T_i})$
- **2:** $(\boldsymbol{p_i, 4V_i, 4T_i})$

as ideal gas and $pV = RT$.

- 3: $(\boldsymbol{p_i/128, 128V_i, T_i})$. We have adiabatic expansion between states 2 and 3 to the original temperature so $p_2V_2^\gamma = p_3V_3^\gamma \quad \Rightarrow \quad T_2V_2^{\gamma-1} = T_3V_3^{\gamma-1}$

$V_3 = \sqrt[\gamma-1]{\frac{T_2}{T_3}}V_2 = \sqrt[0.4]{4} \times 4V_i = 128V_i$. Hence $p_3 = p_i/128$ using the ideal gas law.

[2 marks – concept]

Foundations of Physics 2B/3C 2017/2018

Thermodynamics – Summer Examinations 2018

Long Solution 1, version 2 – all problems

(Parts of answer in **bold** what is expected from students, rest explanations)

a) The energy lost by the hotter block is gained by the cooler one, $\boldsymbol{\Delta Q_{Cu} = -\Delta Q_{Fe}}$ **as they come to an equilibrium temperature,** $\boldsymbol{T_E}$

$$\int_{T_{Cu}}^{T_E} C_p^{Cu} dT = -\int_{T_{Fe}}^{T_E} C_p^{Fe} dT \quad \Rightarrow \quad C_p^{Cu}(T_E - T_{Cu}) = -C_p^{Fe}(T_E - T_{Fe})$$

$$T_E = \frac{C_p^{Cu} T_{Cu} + C_p^{Fe} T_{Fe}}{C_p^{Cu} + C_p^{Fe}} = 469 \text{ K}.$$

[1 mark – problem]

The entropy changes are given by $\Delta S = \int \frac{\delta Q}{T}$, with $\delta Q = C_p dT$. Hence,

$$\Delta S_{Cu} = \int_{T_{Cu}}^{T_E} \frac{C_p^{Cu} dt}{T} = C_p^{Cu} \ln\left(\frac{T_E}{T_{Cu}}\right) = -2.81 \text{ kJ K}^{-1};$$

$$\Delta S_{Fe} = \int_{T_{Fe}}^{T_E} \frac{C_p^{Fe} dt}{T} = C_p^{FE} \ln\left(\frac{T_E}{T_{FE}}\right) = 4.24 \text{ kJ K}^{-1}.$$

[2 marks – problem]

Hence, $\boldsymbol{\Delta S_U = \Delta S_{Cu} + \Delta S_{Fe} = 1.43 \text{ kJ K}^{-1}}$. **[1 mark – problem]**

b) Statistically entropy is $\boldsymbol{S = k_B \ln \Omega}$, **with number of microstates** $\boldsymbol{\Omega = \frac{N!}{\prod_j n_j!}}$

[1 mark – problem]

The number of microstates for the above arrangement is

$$\Omega = \frac{N_A!}{\left(\frac{7N_A}{10}\right)!\left(\frac{2N_A}{10}\right)!\left(\frac{N_A}{10}\right)!}.$$

[1 mark – problem]

Hence

$$S = k_B \ln\left[\frac{N_A!}{\left(\frac{7}{10}\right)!\left(\frac{2N_A}{10}\right)!\left(\frac{N_A}{10}\right)!}\right] = k_B\left[\ln N_A! - \ln\left(\frac{7N_A}{10}\right)! - \ln\left(\frac{2N_A}{10}\right)! - \ln\left(\frac{N_A}{10}\right)!\right]$$

$$= k_B\left[N_A \ln N_A - N_A - \frac{7N_A}{10}\ln\left(\frac{7N_A}{10}\right) + \frac{7N_A}{10} - \frac{2N_A}{10}\ln\left(\frac{2N_A}{10}\right) + \frac{2N_A}{10} - \frac{N_A}{10}\ln\left(\frac{N_A}{10}\right) + \frac{N_A}{10}\right] = k_B\left[N_A \ln N_A - \frac{7N_A}{10}\ln\left(\frac{7N_A}{10}\right) - \frac{2N_A}{10}\ln\left(\frac{2N_A}{10}\right) - \frac{N_A}{10}\ln\left(\frac{N_A}{10}\right)\right].$$

[2 marks – problem]

$$S = k_B N_A \ln\left[\frac{N_A}{\left(\frac{7N_A}{10}\right)^{7/10}\left(\frac{2N_A}{10}\right)^{2/10}\left(\frac{N_A}{10}\right)^{1/10}}\right] = R\ln\left[\frac{10}{\left(\sqrt[10]{7}\right)^7\left(\sqrt[10]{2}\right)^2}\right]$$
$$= 6.66\ \mathrm{J\ mol^{-1}}$$

[2 marks – problem]

c) **The block will reach the temperature of the reservoir,** and will lose energy $\Delta Q_R = -\Delta Q_B$ to it

$$\Delta Q_B = \int_{T_1}^{T_R} CTdT = \frac{C}{2}\left(T_R^2 - T_1^2\right) < 0$$

[1 mark – problem]

The block has entropy change

$$\Delta S_B = \int_{T_1}^{T_R} \frac{CTdT}{T} = C(T_R - T_1) < 0.$$

[1 mark – problem]

The reservoir remains at T_R so has entropy change,

$$\Delta S_R = \int \frac{\delta Q_R}{T_R} = \frac{\Delta Q_R}{T_R} = -\frac{\Delta Q_B}{T_R} = -\frac{C}{2}\left(T_R - \frac{T_1^2}{T_R}\right).$$

[2 marks – problem]

The irreversibility is given by $I = T_E \Delta S_U$, so

$$I = T_E C(T_R - T_1) - \frac{T_E C}{2}\left(T_R - \frac{T_1^2}{T_R}\right) = T_E C\left(\frac{T_R}{2} - T_1 + \frac{T_1^2}{T_R}\right).$$

[1 mark – problem]

d) "Dividing the TdS equation by dT at constant entropy, S,

$$0 = C_V\frac{dT}{dT} + T\left(\frac{\partial p}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_S \quad\Rightarrow\quad C_V = -T\left(\frac{\partial p}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_S$$

[1 mark – problem]

The equation of state for an ideal gas is $pV = RT$, **so** $\left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V}$ whilst on an **adiabatic,** $pV^\gamma = \mathrm{const}_1$**, so** $TV^{\gamma-1} = \mathrm{Const}_2 \quad\Rightarrow\quad V^{\gamma-1} = \mathrm{Const}_2/T$

$$(\gamma-1)V^{\gamma-2}\left(\frac{\partial V}{\partial T}\right)_S = -\frac{\mathrm{Const}_2}{T^2} = -\frac{V^{\gamma-1}}{T^2} \quad\Rightarrow\quad \left(\frac{\partial V}{\partial T}\right)_S = -\frac{1}{\gamma-1}\frac{V}{T}$$

[2 marks – problem]

$$\therefore C_V = -\frac{TR}{V}\left(-\frac{1}{\gamma-1}\frac{V}{T}\right) = \frac{R}{\gamma-1}.$$

[1 mark – problem]

For a mono-atomic gas, $\gamma = \frac{5}{3}$. In this case**,** $C_V = \frac{R}{\frac{5}{3}-1} = \frac{3R}{2}$ **as expected.**

[1 mark – problem]

Foundations of Physics 2B/3C 2017/2018

Thermodynamics – Summer Examinations 2018

Long Solution 2, version 3 (external) – all problems

(Parts of answer in **bold** what is expected from students, rest explanations)

a) The total differential of F is $\boldsymbol{dF = dU - TdS - SdT = -mdB - SdT}$ using the first law ($dU = TdS - mdB$). **Hence $\boldsymbol{F = F(T, B)}$.**

[1 mark – problem]

$$\boldsymbol{dF = \left(\frac{\partial F}{\partial B}\right)_T dB + \left(\frac{\partial F}{\partial T}\right)_B dT.}$$

[1 mark – problem]

Comparing coefficients we find $\left(\frac{\partial F}{\partial B}\right)_T = \boldsymbol{-m}$ **and** $\left(\frac{\partial F}{\partial T}\right)_B = \boldsymbol{-S}$. As the **Helmholtz** is an **exact function,** we have $\left(\frac{\partial^2 F}{\partial T \partial B}\right) = \left(\frac{\partial^2 F}{\partial B \partial T}\right)$, **so**

$$\boldsymbol{\left(\frac{\partial}{\partial T}\left(\frac{\partial F}{\partial B}\right)_T\right)_B = -\left(\frac{\partial m}{\partial T}\right)_B = -\left(\frac{\partial S}{\partial B}\right)_T = \left(\frac{\partial}{\partial B}\left(\frac{\partial F}{\partial T}\right)_B\right)_T \Rightarrow \left(\frac{\partial m}{\partial T}\right)_B = \left(\frac{\partial S}{\partial B}\right)_T}$$

[2 marks – problem]

b) Using the Maxwell Relation,

$$\boldsymbol{\Delta S = \int \left(\frac{\partial m}{\partial T}\right)_B dB.}$$

[1 mark – problem]

From Curie's Law,

$$\boldsymbol{\left(\frac{\partial m}{\partial T}\right)_B = -\frac{CB}{\mu_0 T^2}.}$$

[1 mark – problem]

For a field going from $B' \rightarrow 2B'$ at T_0 we have

$$\boldsymbol{\Delta S = \int_{B'}^{2B'} -\frac{CB}{\mu_0 T^2} dB = \left[-\frac{CB}{2\mu_0 {T_0}^2}\right]_{B'}^{2B'} = -\frac{3C(B')^2}{2\mu_0 T_0^2}.}$$

[2 marks – problem]

c) Writing $S = S(T, B)$, we have

$$\boldsymbol{dS = \left(\frac{\partial S}{\partial T}\right)_B dT + \left(\frac{\partial S}{\partial B}\right)_T dB.}$$

Now, **multiply through by $\boldsymbol{T}$ and use the Maxwell Relation from (a)**

$$\boldsymbol{TdS = T\left(\frac{\partial S}{\partial T}\right)_B dT + T\left(\frac{\partial m}{\partial T}\right)_B dB.}$$

[2 marks – problem]

At constant temperature, $dT = 0$ and $\boldsymbol{\delta Q = TdS}$, so we get

$$\therefore \boldsymbol{\delta Q = T_0\left(\frac{\partial S}{\partial B}\right)_T dB}$$

[1 mark – problem]

If the field goes from $0 \to B_0$ the heat required for magnetisztion is, and using the Maxwell Relation

$$\boldsymbol{\Delta Q = \int_0^{B_0} T_0\left(-\frac{CB}{\mu_0 {T_0}^2}\right) dB = -\frac{CB_0^2}{2\mu_0 T_0}.}$$

[2 marks – problem]

d) The first law for a magnet is $dU = TdS - mdB$. **"Dividing" by $\boldsymbol{dB}$ at constant temperature**

$$\boldsymbol{\left(\frac{\partial U}{\partial B}\right)_T = T\left(\frac{\partial S}{\partial B}\right)_T - m = T\left(\frac{\partial m}{\partial T}\right)_B - m.}$$

[1 mark – problem]

Using the Curie law, $\boldsymbol{\left(\frac{\partial U}{\partial B}\right)_T = T\left(-\frac{CB}{\mu_0 T^2}\right) - \frac{CB}{\mu_0 T} = -\frac{2CB}{\mu_0 T}}$.

[1 mark – problem]

In general, we have $\boldsymbol{U = U(B,T)}$ as field and temperature are the two things we can change. Thus,

$$\boldsymbol{dU = \left(\frac{\partial U}{\partial B}\right)_T dB + \left(\frac{\partial U}{\partial T}\right)_B dT.}$$

[1 mark – problem]

At constant field, $dB = 0$ and $\boldsymbol{dU = TdS}$, **so** $\boldsymbol{\left(\frac{\partial U}{\partial T}\right)_B = T\left(\frac{\partial S}{\partial T}\right)_B = C_B}$.

As B and T are all functions of state (exact differentials), the above can be integrated separately to get the equation of state,

$$\boldsymbol{U = \int\left(\frac{\partial U}{\partial B}\right)_T dB + \int\left(\frac{\partial U}{\partial T}\right)_B dT = \int\left(-\frac{2CB}{\mu_0 T}\right)dB + \int C_B dT = -\frac{CB^2}{\mu_0 T} + f(T) + \mathrm{const},}$$

where $\boldsymbol{f(T)}$ is some function of temperature, since $\boldsymbol{C_B = C_B(T)}$ in general.

[Accept $\boldsymbol{U = -\frac{CB^2}{\mu_0 T} + C_B T + \mathrm{const}}$, if students assume constant heat capacity.]

[1 mark – problem]

e) Now $\boldsymbol{dF = -mdB - SdT}$, **so** $\boldsymbol{-m = \left(\frac{\partial F}{\partial B}\right)_T}$ **and** $\boldsymbol{-S = \left(\frac{\partial F}{\partial T}\right)_B}$.

[1 mark – problem]

Therefore,

$$\boldsymbol{-m = -\frac{kB}{T}}$$

In other words this describes **a paramagnetic material!**

[1 mark – problem]

$$\boldsymbol{S = -5aT^4 - \frac{kB^2}{T^2}}$$

$$\boldsymbol{U = F + TS = aT^5 - \frac{kB^2}{2T} + T\left(-5aT^4 + \frac{kB^2}{2T^2}\right) = -4aT^5 - \frac{kB^2}{T}}.$$

[1 mark – problem]

L2 Foundation of Physics 2B Optics 2017-18

February 7, 2018

(a) If waves travelling at 1.00 m s^{-1} break on the beach every 100 s, what is the spatial frequency of the wave? Specify the units in your answer. [4 marks]

[CONCEPT] Spatial frequency $u = 1/vT$, where v is the speed of the wave.[1]
$u = 1/(1.00 \times 100)^{[1]} = 0.01^{[1]}$ $\text{m}^{-1[1]}$.
This corresponds to 1 wave every 100 m.

(b) Write expressions for the following: (i) a plane wave in the xz plane with wave vector $\underline{k} = (k_x, 0, k_z)$ and (ii) a paraxial spherical wave propagating along the z axis with source point at the origin. [4 marks]

[BOOKWORK/CONCEPT]
(i)

$$\mathcal{E} = \mathcal{E}_0 \mathrm{e}^{\mathrm{i}(k_x x^{[1]} + k_z z^{[1]})} \, .$$

(ii)

$$\mathcal{E} = \mathcal{E}_0 \frac{\mathrm{e}^{\mathrm{i}kz}}{\mathrm{i}kz}^{[1]} \mathrm{e}^{\mathrm{i}kx^2/2z[1]} \, .$$

(c) A paraxial spherical wave propagating along the z axis with source point $(0, 0, -s_1)$ is incident on a thin lens lying in the $z = 0$ plane with focal length f. Write an expression for the field immediately downstream of the lens. What is the position of the image? [4 marks]

[BOOKWORK/CONCEPT]

$$\mathcal{E} = \mathcal{E}_0 \frac{\mathrm{e}^{\mathrm{i}ks_1}}{\mathrm{i}ks_1}^{[1]} \mathrm{e}^{-\mathrm{i}kx^2/2f[1]} \mathrm{e}^{\mathrm{i}kx^2/2s_1[1]} \, .$$

The position of the new focus is $z = s_2$ where

$$-\frac{1}{s_2} = -\frac{1}{f} + \frac{1}{s_1}^{[1]} \, ,$$

with the minus sign indicating that the curvature after the lens is negative forming an image downstream, i.e. $z = s_2 > 0$.

(d) Calculate the beam radius of a red laser pointer with wavelength $\lambda = 633$ nm on the Moon. The initial beam size, on Earth, is $w_0 = 1.00$ mm, and the distance to the Moon is 3.84×10^8 m. [4 marks]

[BOOKWORK/CONCEPT] The beam radius at a distance z is given by

$$w = \Delta\theta z^{[1]} ,$$

where the angular spread of a laser is

$$\Delta\theta = \frac{\lambda}{\pi w_0}^{[1]} .$$

Substituting in the numbers we get

$$w = \frac{633 \times 10^{-9}}{3.14 \cdot 1.00 \times 10^{-3}} 3.84 \times 10^8 = 7.74 \times 10^{4[1]} \mathrm{m}^{[1]} .$$

(e) A circular aperture with radius R_a placed in the $z = 0$ plane is illuminated by uniform monochromatic light with wavelength λ propagating in the positive z direction. Describe, or sketch, the light intensity in the xy plane at a distance: (i) $z = R_\mathrm{a}^2/\lambda$ and (ii) $z = R_\mathrm{a}^2/(2\lambda)$. Explain, briefly, the origin of the observed patterns in each case. [4 marks]

[CONCEPT] (i) For $z = R_\mathrm{a}^2/\lambda$ the aperture size is equal to the size of the first Fresnel zone,[1] all the light interferes constructively, and we end up with a maximum on axis. Sketch should show a circular light distribution with maximum intensity on-axis.[1] (ii) For $z = R_\mathrm{a}^2/(2\lambda)$ the aperture size is equal to the size of the first two Fresnel zones[1] the light from the second zone is out of phase and exactly cancels the contribution from the first giving zero intensity on axis. Sketch shows a ring with zero intensity on-axis.[1]

(f) Show that for a diffraction grating consisting of N-slits, the position of the first zero is given by $x = \lambda z/(Nd)$, where z is distance to the observation plane, d is the slit spacing, and λ is the wavelength. [4 marks]

[CONCEPT] The first zero occurs when the N phasors are equally distributed around the circle,[1] i.e., the angular spacing between adjacent phasors, $2\pi ud = 2\pi/N$.[1] This gives $u = x/(\lambda z) = 1/(Nd)$[1] and hence the position of the first zero is $x = (\lambda/Nd)z$.[1]

L2 Foundation of Physics 2B Optics 2017-18

February 6, 2018

A Michelson interferometer consists of a beamsplitter that divides an input with amplitude $\mathcal{E}_0$ into two equal amplitude 'arms' with lengths ℓ_1 and ℓ_2. Two perfect mirrors retro-reflect the light such that the two paths interfere at the beamsplitter.

(a) Write an expression for the output field after the two paths recombine at the beamsplitter. State any assumptions you make. [5 marks]

[CONCEPT]

$$\mathcal{E} = 0.5^{[1]}\mathcal{E}_0(\mathrm{e}^{\mathrm{i}2k\ell_1\,[1]} + \mathrm{e}^{\mathrm{i}2k\ell_2\,[1]}) \ .$$

We are assuming **plane waves**[1] and that we can neglect any **phase changes on reflection** at the beam splitter [1] .

(b) Write an expression for the intensity at the output. [3 marks]

[BOOKWORK/CONCEPT] Taking the modulus squared[1] we find

$$\mathcal{I} = 0.5\mathcal{I}_0^{[1]}\left[1 + \cos 2k(\ell_2 - \ell_1)\right]^{[1]} \ ,$$

where $\mathcal{I}_0$ is the maximum intensity.

(c) The path difference, $\ell_2 - \ell_1$, is chosen such that the intensity at the output is equal to one-half of its maximum possible value. Write an expression for $\ell_2 - \ell_1$ in terms of the wavelength, λ. [2 marks]

[BOOKWORK/CONCEPT]

$$\ell_2 - \ell_1 = \left(m + \frac{1}{2}\right)^{[1]}\frac{\lambda}{4} \ .^{[1]}$$

where m is an integer.

(d) A gravitational wave arriving at a Michelson interferometer increases the length of one arm by $\Delta\ell$, and decreases the length of the other arm by $\Delta\ell$. Write an expression for the output intensity as a function of $\Delta\ell$, assuming that $\Delta\ell$ is small. [4 marks]

[CONCEPT] Using the hint with $A = 2k(\ell_2 - \ell_1) = (2m+1)\pi/2$[1] (the unperturbed phase difference) and $B = 4k\Delta\ell$[1] (due to the gravitational wave) we have $\cos(A+B) = -B = -4k\Delta\ell$,[1] therefore

$$\mathcal{I} = 0.5\mathcal{I}_0 \left[1 - 4k\Delta\ell\right]^{[1]} .$$

(e) If the power circulating in each arm is 0.8 MW and the minimal detectable signal is 1 μW, the wavelength is 0.5 μm and the length of each arm is 4 km, estimate the minimum strain, $\Delta\ell/\ell$, that can be detected in principle. [4 marks]

[CONCEPT]

$$\frac{\Delta P}{P} = -8\pi \frac{\Delta\ell}{\lambda}^{[1]} .$$

which gives $\Delta\ell = (10^{-6}/0.8 \times 10^{6}) \cdot (5 \times 10^{-7}/24) = 3 \times 10^{-20}$ and the smallest strain $\Delta\ell/\ell \approx 10^{-23}$.

(f) Give two reasons why Young's double-slit interferometer is less well suited to measure gravitational waves than a Michelson interferometer. [2 marks]

[CONCEPT] Young's double slit is inefficient in the use of laser power as most of the wavefront is not used.[1] The optical paths in Young's double slit form a diamond and the path difference is not particularly sensitive to a deformation of the diamond.[1]

Hints:
$\cos(A+B) = \cos A \cos B - \sin A \sin B$.
For small B, $\sin B = B$, $\cos B = 1$, and $\cos(A+B) = \cos A - B \sin A$.

L2 Foundation of Physics 2B Optics 2017-18

February 6, 2018

An aperture containing 5 narrow slits with spacing, d, is placed in the $z = 0$ plane and illuminated normally using uniform monochromatic light with amplitude, $\mathcal{E}_0$, and wavelength, λ. The slits can be assumed to be effectively infinitely long in the y direction and centred around $x = 0$.

(a) Write an expression for the field at a point (x, z), where $z \gg d$. State any approximations you make. [6 marks]

[BOOKWORK/CONCEPT] Assuming the light from each slit can be approximated by a paraxial cylindrical wave, and then using the Fraunhofer approximation,[1] we can write

$$\mathcal{E} = \frac{\mathcal{E}_0 \mathrm{e}^{\mathrm{i}k\bar{r}}}{\sqrt{\mathrm{i}kz}} \left(\mathrm{e}^{\mathrm{i}4\pi dx/\lambda z[\mathbf{1}]} + \mathrm{e}^{\mathrm{i}2\pi dx/\lambda z[\mathbf{1}]} + 1^{[\mathbf{1}]} + \mathrm{e}^{-\mathrm{i}2\pi dx/\lambda z[\mathbf{1}]} + \mathrm{e}^{-\mathrm{i}4\pi dx/\lambda z[\mathbf{1}]} \right) .$$

(b) Draw, or describe, a phasor diagram to represent the field at $x = [\lambda/(2d)]z$. What is the ratio of the intensity at this position compared to the maximum intensity? [4 marks]

[CONCEPT] The phasor diagram consists of two phasors in one direction[1] and three (5 in total[1] in the opposite direction.[1] The intensity is $\mathcal{I}_0/(kz)$ compared to a peak intensity of $25\mathcal{I}_0/(kz)$.[1]

(c) How many positions of zero intensity are there between $x = 0$ and $x = [\lambda/(2d)]z$? Sketch the phasor diagrams, or specify the phasor angles, in each case. [3 marks]

[CONCEPT] There are two zeros.[1] One phasor is fixed say along the horizontal axis. At the first zero two phasors have moved to $\pm(1/5)2\pi$ and two have moved $\pm(2/5)2\pi$.[1] At the second zero two phasors have moved to $\pm(2/5)2\pi$ and two have moved $\pm(4/5)2\pi$ which looks identical.[1]

(d) Two glass plates that shift the phase by π are placed in front of the slits at input coordinates, $(x', 0) = (\pm d, 0)$. Draw, or describe, the modified phasor diagrams at $[\lambda/(2d)]z$ and at $x = 0$. [2 marks]

[CONCEPT] At $x = [\lambda/(2d)]z$ all five phasors are in the same direction.[1] At $x = 0$ three are at 0 and two are in the opposite direction at $\pm(1/2)2\pi$.[1]

(e) Describe, briefly, the similarities and differences between the diffraction patterns without and with the glass plates. Alternatively, sketch a graph of both patterns on the same axes. [2 marks]

[CONCEPT] The patterns are the same[1] except displaced by $[\lambda/(2d)]z$.[1]

(f) Instead of a simple phase shift, different glass plates are used to rotate the plane of polarization of light emerging from the slits at $x' = \pm d$ by 90°. Describe the expected pattern. What is the peak intensity? [3 marks]

[CONCEPT] For orthogonal polarisations we can add intensities so the pattern looks like the sum of a **double-slit**[1] intensity pattern (with spacing $2d$) plus a **triple-slit**[1] intensity pattern (with spacing $2d$). The peak intensity is $13\mathcal{I}_0/(kz)$.[1]

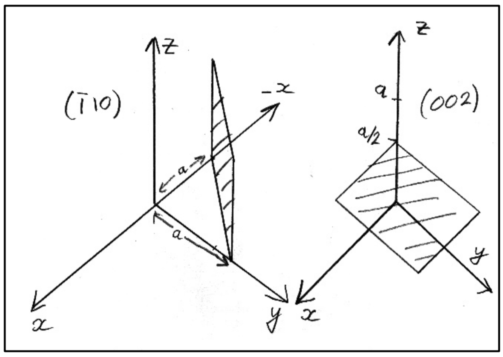
Examination Questions May/June 2018
Foundations of Physics 2B, Condensed Matter Physics, Q7 (5 parts):
SOLUTIONS

Synopsis Problem

a. Sketch shown in diagram. The spacing of each of the families of planes is given by:

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

where a is the lattice constant (0.4 nm) Substituting in the appropriate $(h\,k\,l)$ values gives: 0.28 nm and 0.20 nm
2 marks for diagram and 1 mark for each answer. **[4 marks]**

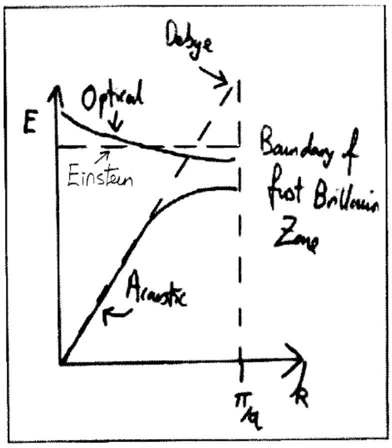


Synopsis

b. Carbon dioxide molecules are held together by covalent bonding. Covalent bonding occurs in systems where identical or similar atoms bond. It arises from the sharing of one electron from each atom. In a covalent bond the electrons have opposite spins. This produces a spatially symmetric wavefunction with a high density of electrons between the atoms. The individual CO_2 molecules in solid CO_2 are held together by Van-der-Waals bonding. This is a much weaker type of bonding resulting from the coulomb interaction between electric dipoles, it varies as $1/r^6$. In both cases the repulsive component of the bond comes from the Pauli exclusion principle at very close distances, increasing the energy as the spatial components of the wavefunctions overlap. **[4 marks]**

Synopsis

c. The sketch shows the phonon dispersion curves for the optical and acoustic branches in the first Brillouin zone – up to π/a. The acoustic branch is described by the Debye approximation having a constant velocity (the velocity of sound waves at long wavelengths). The optical branch is described by the Einstein approximation when all the phonon modes have the same energy (or frequency). **[4 marks]**



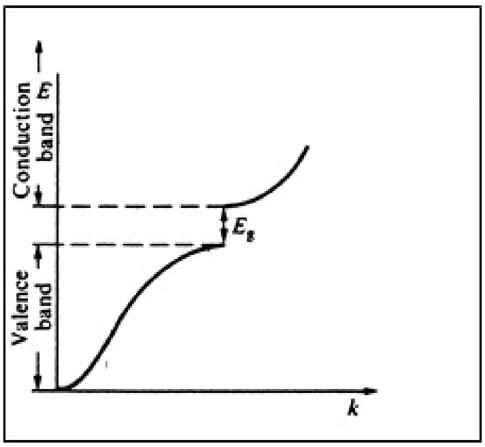
Conceptual

d. The Drude model treats electrons as classical particles with mass m_e, charge $-e$. Under a constant applied electric field $\underline{E}$ the electrons will undergo a constant acceleration. The Drude model assumes that the electron drift velocity increases linearly (with constant acceleration) until the electron undergoes a collision event. At the time of the collision event the electron drift velocity then drops to zero. The electron then begins to accelerate again under the influence of the electric field. The time between individual collision events is randomly varying. The Drude model uses the Drude scatting time τ which is the mean time between collision events. The mean time is related to the Drude conductivity. **[4 marks]**

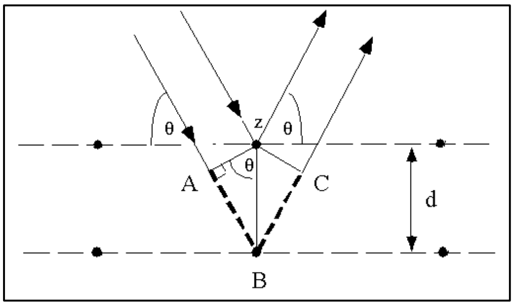
Conceptual

e. The energy bandgap arises from the Bragg scattering of electrons from planes of atoms leading to regions where there are no energy states. This can be considered to originate from destructive interference between the electron wavefunctions and the fourier components of the periodic potential in the crystal, where the periodic potential represents the presence of atoms.

The diagram shows the $E\,(k)$ behaviour for the first Brillouin zone. **[4 marks]**



Examination Questions May 2018
Foundations of Physics 2B, Condensed Matter Physics, Q8
SOLUTION



Bookwork

(a) The Bragg law arises from considering the path length difference in the reflections from adjacent planes. **[1]** In the diagram the path length difference is A-B-C, when this corresponds to constructive interference a peak appears. This leads to the Bragg law $2d\sin\theta = n\lambda$ where d is the plane spacing, λ is the X-ray wavelength and n is the order (usually only consider n = 1). **[1]**

Bragg scattering involves the partial reflection of incident X-ray waves from parallel planes of atoms in a crystalline solid. Each plane of atoms reflects a proportion or the intensity of the incident X-ray wave – related to the electron density. Bragg law models crystal as parallel planes. **[1]** **[1 mark for sketch]** **[4 marks]**

Bookwork

(b) The structure factor describes the intensity pattern of scattered X-rays from a crystal. **[1]**

The structure factor is given by:

$$S(h,k,l) = \sum_{cell} f \exp - 2\pi i (hx_j + ky_j + lz_j)$$ **[1]**

The $h\,k\,l$ values are from the particular Bragg peak being considered. The $x\,y\,z$ values are the fractional coordinates of each of the unique atoms within a single unit cell. **[1]** The Structure Factor gives either a non-zero or zero value. **[1]** (The atomic form factor is not considered in this course). The Structure Factor therefore describes interference within a unit cell which is not considered by the Bragg Law. **[1]**

[5 marks]

Problem

(c) From Bragg law we know $\lambda = 2d\sin\theta$

and: $d = \dfrac{1}{\sqrt{\dfrac{h^2}{a_1^2}\dfrac{k^2}{a_2^2}\dfrac{l^2}{a_3^2}}}$ **[1]** which for a cubic lattice becomes $d = \dfrac{a}{\sqrt{h^2+k^2+l^2}} = \dfrac{a}{\sqrt{N}}$

where *a* is the lattice constant. **[1]**

The structure factor describes the intensity pattern of scattered X-rays from a crystal, for a bcc lattice this is:

- $S(hkl) = 2f$ when $h + k + l$ is even
- $S(hkl) = 0$ when $h + k + l$ is odd .

(f is the atomic form factor which we don't worry about here.) **[1]**

The first four peaks will therefore occur for the planes (110), (200), (211) and (220) corresponding to *N* values of 2, 4, 6 and 8. **[1]**
We can use *N* to determine the observed values of theta.

Rearranging the above gives $\sin^2\theta = \frac{\lambda^2}{4a^2}N$ i.e. $N \propto \sin^2\theta$ **[1]**

Using the data from the question we can tabulate values from N and $\sin^2\theta$ to determine the values of θ and 2θ as required. **[1]**

peak	N	sin² θ	sin θ	θ	2 θ
1	2	0.1800	0.4243	25.10	50.21
2	4	0.3600	0.6000	36.87	73.74
3	6	0.5400	0.7348	47.29	94.59
4	8	0.7200	0.8485	58.05	116.10

The final column in the table gives the observed 2θ values as required. **[2]**

[8 marks]

Problem

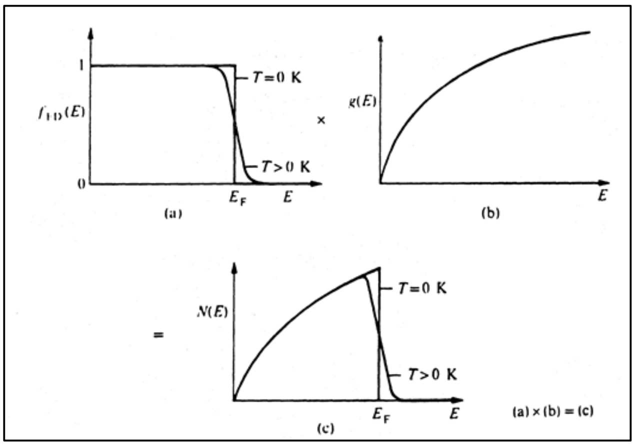
(d) The different cubic crystal structure will result in a different Structure Factor. We have been told that the lattice constant does not change, we note that the change in temperature is very small so we can assume that the Bragg peaks will not shift their angle. **[1]** Their intensity distribution will change. **[1]** The Structure Factor for the simple cubic lattice is non-zero for all values of N. Therefore twice as many peaks will appear corresponding to odd and event values of N. **[1]** (Note that N = 7 does not exist).

[3 marks]

Examination Questions May/June 2018
Foundations of Physics 2B, Condensed Matter Physics, Q9
SOLUTION

Synopsis

a) The energy density of states function gives the number of available electron energy states per unit energy range which can be occupied. **[1]** The Fermi-Dirac function gives the probability of these states being occupied. **[1]** The *Fermi Energy* is the highest occupied electron state in a system when it is in the ground state (equivalent to 0 K). **[1]** Below the Fermi Energy all energy states are filled, above the Fermi Energy all energy states are empty. **[1]** These function together give the total electron energy distribution as shown in the sketch. **[3 for sketch]**



[7 marks]

Problem

b) The Fermi energy is related to the electron density by $E_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{\frac{2}{3}}$. **[1]** Rearranging

we get $n = \frac{\left(2m_e E_F\right)^{\frac{3}{2}}}{3\pi^2\hbar^3}$ **[1]** giving $\frac{\left(2\times 9.11\times10^{-31}\times 7.0\times 1.60\times10^{-19}\right)^{\frac{3}{2}}}{3\times 3.14^2\times\left(1.054\times10^{-34}\right)^3} = 8.42\times10^{28}$ m^{-3}. **[1]**

The Fermi velocity is obtained from the Fermi wavevector $k_F = \left(3\pi^2 n\right)^{\frac{1}{3}} =$

$\left(3\times 3.14^2\times 8.42\times10^{28}\right)^{\frac{1}{3}}$ = 1.36×10^{10} m^{-1} **[1]** and $v_F = \frac{\hbar k_F}{m_e}$ = $\frac{1.054\times10^{-34}\times 1.36\times10^{10}}{9.11\times10^{-31}}$

=1.57×10^6 ms^{-1}. **[1]** This is about an order of mangnitude larger than the velocity predicted by classical theory and is a consequence of electrons being fermions. **[1]** **[6 marks]**

c) We **assume** that the free electron density is the same as the atomic density **[1]**. Density of copper in kg m^3 is then electron density times atomic mass. **[1]**

Density $= 8.42\times10^{28}\times 63.5\times 1.66\times10^{-27}$ = 8.87×10^3 kg m^{-3}. **[1]** **[3 marks]**

Conceptual

d) The confinement of electrons in a 2D layer alters the density of states function. **[1]** The density of states function is then constant with respect to energy instead of $E^{\frac{1}{2}}$ **[1]** This alters the available of electron energy states at the band edge **[1]** it gives favourable electronic properties for some devices **[1]**

[4 marks]