Mathematical Methods II
Weekly problem set 7

(a) Find the general solution of the first-order partial differential equation (PDE)

$$x\,\frac{\partial u}{\partial x} \;+\; \frac{1}{2}\,\frac{\partial u}{\partial y} \;=\; 0\,,$$

by searching for a solution in the form $u(x, y) = f(p)$ where $p = p(x, y)$ is a certain function of x and y which you need to determine, and $f(p)$ is an arbitrary function of its argument.

Solution

$$xu_x + \frac{1}{2}u_y = 0$$

Seek a solution $u(x, y) = f(p)$ where $p = p(x, y)$. Identify $A = x$, $B = 1/2$. Hence

$$\frac{dx}{A} = \frac{dy}{B}$$

$$\frac{dx}{x} = 2dy$$

Integrate

$$\ln x = 2y + c$$

$$x = Ae^{2y}$$

$$A = xe^{-2y}$$

Thus $p = A = xe^{-2y}$ i.e. the solution is a function of xe^{-2y}. As this is a 1^{st} order PDE only one function is required for the general solution. Hence

$$u(x, y) = f\left(xe^{-2y}\right)$$

(b) Now solve a more complicated PDE

$$x\,\frac{\partial u}{\partial x} \;+\; \frac{1}{2}\,\frac{\partial u}{\partial y} \;+\; 5\,u \;=\; 0\,,$$

by looking for a solution in the form $u(x, y) = h(x, y) \cdot f(p(x, y))$ where $h(x, y)$ is any particular solution of this equation, and $f(p(x, y))$ is the general solution of the equation in part (a).

You will need to seek a valid particular solution for $h(x, y)$. Check it is valid by substituting it into the PDE.

Solution

$$xu_x + \frac{1}{2}u_y + 5u = 0$$

We see that since we are looking for a solution $u = hf(p)$ to the same equation as above (except it has an additional term in u) $f(p)$ will in fact be the same, as the term in u does not contribute to it. We seek $f(p)$ just as we did above and find it is the same

$$\frac{dx}{A} = \frac{dy}{B}$$

$$\frac{dx}{x} = 2dy$$

$$\ln x = 2y + c$$

$$x = Ae^{2y}$$

$$A = xe^{-2y}$$

Thus $p = A = xe^{-2y}$. Now all that remains is to choose a particular solution $h(x, y)$ to multiply it by. One such solution is $h(x, y) = \exp[-10y]$, though any valid solution, however simple, will work. Hence our general solution is

$$u(x, y) = h(x, y) = e^{-10y} f\left(xe^{-2y}\right)$$

(c) Impose the boundary condition $u(x, y) = 3/x^2$ on the line $y = 0$ and hence derive the solution of the boundary value problem for the equation in part (b).

Solution

$$u(x, 0) = \frac{3}{x^2} \to e^0 f\left(xe^0\right) = f(x) = \frac{3}{x^2}$$

i.e. $f(z) = 3/z^2$. Remember that $z = p|_{BC}$, so the solution is

$$u(x, y) = \frac{3e^{-10y}}{\left(xe^{-2y}\right)^2} = \frac{3}{x^2 e^{6y}}$$