Quantum Theory - Worksheet 9

Throughout this worksheet we use the symbol $|j, m\rangle$ to represent a simultaneous eigenvector of the relevant $\hat{\mathbf{J}}^2$ and $\hat{J}_z$ operators.

Problem 1

Consider two systems, (1) and (2), in eigenstates of $\hat{\mathbf{J}}_1^2$ and $\hat{\mathbf{J}}_2^2$, but not necessarily in eigenstates of $\hat{J}_{1z}$ and $\hat{J}_{2z}$ (the z-components of, respectively, $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$). Let $\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$. As seen in the last lecture of the course, the bipartite system formed by (1) and (2) can be in a joint eigenstate $|j_1, j_2, J, M\rangle$ of $\hat{\mathbf{J}}_1^2$, $\hat{\mathbf{J}}_2^2$, $\hat{\mathbf{J}}_2$ and $\hat{J}_z$. This eigenstate can be written in the following way in terms of the eigenstates $|j_1, m_1\rangle_1$ of $\hat{\mathbf{J}}_1^2$ and $\hat{J}_{1z}$ and the eigenstates $|j_2, m_2\rangle_2$ of $\hat{\mathbf{J}}_2^2$ and $\hat{J}_{2z}$:

$$|j_1, j_2, J, M\rangle_{12} = \sum_{m_1, m_2} \langle j_1, j_2, m_1, m_2 | J, M\rangle |j_1, m_1\rangle_1 |j_2, m_2\rangle_2. \quad (1)$$

The coefficients $\langle j_1, j_2, m_1, m_2 | J, M\rangle$ are certain numbers called Clebsch-Gordan coefficients.

(a) Take two electrons, which can each be either in a state of spin up or a state of spin down. For electron 1, these two spin states correspond to the eigenstates $|1/2, 1/2\rangle_1$ and $|1/2, -1/2\rangle_1$, and similarly for electron 2 ($j_1 = j_2 = 1/2$ as an electron is a spin-1/2 particle, and $m_1, m_2 = 1/2$ for spin up or $-1/2$ for spin down). Use Eq. (1) and the information below to write the 2-electron states $|j_1, j_2, J, M\rangle$ in terms of the 1-electron states $|j_1, m_1\rangle_1$ and $|j_2, m_2\rangle_2$, (i) for $J = 0, M = 0$, (ii) for $J = 1, M = 0$, (iii) for $J = 1, M = 1$, and (iv) for $J = 1, M = -1$. I.e., calculate the relevant Clebsch-Gordan coefficients. (These four combinations of quantum numbers describe possible joint spin states of the two electrons of an atom of helium.)

Information: Bransden and Joachain — the textbook the syllabus of the course is based on — give these Clebsch-Gordan coefficients in the following form for $j_2 = 1/2$ (recall that $M = m_1 + m_2$):

J	m_2	$\langle j_1, j_2, m_1, m_2 \vert J, M\rangle$
$j_1 + 1/2$	$1/2$	$\sqrt{\frac{j_1 + M + 1/2}{2j_1 + 1}}$
$j_1 + 1/2$	$-1/2$	$\sqrt{\frac{j_1 - M + 1/2}{2j_1 + 1}}$
$j_1 - 1/2$	$1/2$	$-\sqrt{\frac{j_1 - M + 1/2}{2j_1 + 1}}$
$j_1 - 1/2$	$-1/2$	$\sqrt{\frac{j_1 + M + 1/2}{2j_1 + 1}}$

(b) Show that $\langle j_1, j_2, m_1, m_2 | J, M\rangle = 0$ when $M \neq m_1 + m_2$. [Hint: Note that $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$ and also that ${}_1\langle j_1, m_1 | j_1, m_1'\rangle_1 \, {}_2\langle j_2, m_2 | j_2, m_2'\rangle_2 = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$.]

(c) Suppose that $j_1 = 1$, $j_2 = 3/2$. For what values of J, m_1 and m_2 can $\langle j_1, j_2, m_1, m_2 | J, M\rangle$ be non-zero, (i) if $M = 5/2$, (ii) if $M = -1/2$?

Problem 2

Two systems in the states characterized by the state vectors $|j_1, m_1\rangle_1$ (for system 1) and $|j_2, m_2\rangle_2$ (for system 2) are combined to form a new system. What ranges of values are allowed for the quantum numbers J and M of the combined system if $m_1 = -m_2$?

Problem 3

(a) How is the commutator of the x- and z- components of the spin operator, $[\hat{S}_x, \hat{S}_z]$, related to its y-component, $\hat{S}_y$? [Hint: $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$.]

(b) The spherical harmonic $Y_{00}(\theta, \phi)$, which is the constant $1/\sqrt{4\pi}$, is an eigenfunction of *both* the x- and the z-component of the orbital angular momentum operator in the position representation, L_x and L_z. In fact, $L_x Y_{00}(\theta, \phi) = L_z Y_{00}(\theta, \phi) = 0$. (Two Hermitian operators may have joint eigenvectors even if they do not commute, but they cannot have a *complete* set of joint eigenvectors unless they commute.) Explain why, in contrast, $\hat{S}_x$ and $\hat{S}_z$ do not have any joint eigenstate amongst the possible spin states of a spin-1/2 particle such as an electron.

(c) Imagine an experiment in which the spin of an electron can be measured either in the x-direction or in the z-direction. Briefly explain why this electron cannot be in a spin state for which the probability of finding the value $\hbar/2$ would be 1 both for a measurement in the x-directions and for a measurement in the z-direction.

Note: This result implies that it is not possible to know the values of both the x- and z-component of the spin simultaneously. The electron may be in a state of well defined spin in the z-direction, or of well defined spin in the x-direction, but not in a state of well defined spin in both these directions.

(d) In the $\{|1/2, 1/2\rangle, |1/2, -1/2\rangle\}$ basis, the x- and z-components of the spin operator for a spin-1/2 particle are represented by the matrices S_x and S_z, with

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(continued on reverse)

These spin operators have only two eigenvalues, $\hbar/2$ and $-\hbar/2$. It is easy to see that the two column vectors

$$\chi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{and} \qquad \chi_{z+} = \begin{pmatrix}1\\0\end{pmatrix}$$

are normalized eigenvectors of, respectively, S_x and S_z corresponding to the eigenvalue $\hbar/2$, and similarly that

$$\chi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \qquad \text{and} \qquad \chi_{z-} = \begin{pmatrix}0\\1\end{pmatrix}$$

are normalized eigenvectors of S_x and S_z corresponding to the eigenvalue $-\hbar/2$.

Imagine that Alice and Bob each receive one electron and can measure its spin in either the x- or the z-direction, and that this pair of electron is in the entangled state

$$\psi_{AB}^- = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_A\begin{pmatrix}0\\1\end{pmatrix}_B - \begin{pmatrix}0\\1\end{pmatrix}_A\begin{pmatrix}1\\0\end{pmatrix}_B\right],$$

where the subscript attached to each column vector indicates whether this column vector represents a spin state of Alice's electron or one of Bob's electron. For simplicity, assume that Alice and Bob make their measurements exactly at the same time, so that there is no issue of how this state vector might evolve in time between these two measurements.

Suppose that both Alice and Bob measure the spin of their electron in the z-direction or that they both measure it in the x-direction. Show that for either choice of direction, either Alice finds $\hbar/2$ and Bob $-\hbar/2$ or Alice finds $-\hbar/2$ and Bob $\hbar/2$.

[Hint: Start with the case of measurements in the z-direction and calculate the probability, (i) that Alice finds $\hbar/2$ and Bob $-\hbar/2$, (ii) that Alice finds $-\hbar/2$ and Bob $\hbar/2$. Check that these two probabilities sum to 1 — which implies that the probability that both Alice and Bob find $\hbar/2$ or that they both find $-\hbar/2$ must be zero. Then repeat the calculation for measurements in the x-direction.

[Note: That Alice and Bob find perfectly anticorrelated results may seem strange in the light of part (c) of this question: Knowing the results found by Alice you can predict with certainty the results found by Bob, whether they both measure the spin in the x-direction or in the z-direction. This does not contradict what is written in part (c), though, because you cannot know with certainty what Bob found without knowing what Alice found. However, in a famous article published in 1935, Einstein and two of his collaborators, Podolsky and Rosen, argued that the fact that it is possible to predict Bob's result with certainty given Alice's means that the x- and z-components of Bob's electron must both have determinate values before his measurements; they proposed that this inconsistency with the predictions of Quantum Mechanics points to an incompleteness of the latter. (They reasoned in terms of position and momentum measurements, not spin measurements, but this does not make any difference in regards to their conclusions.) The Einstein-Podolsky-Rosen paper has been debated at great length. The issues it raises are fascinating but largely of a philosophical nature.

(e) Can you identify ψ_{AB}^- with one of the spin states you found in part (b) of Problem 2?

Problem 4

Show that the x-, y- and z-components of the spin operator for a spin-1/2 particle are represented by the following matrices in the $\{|1/2, 1/2\rangle, |1/2, -1/2\rangle\}$ basis:

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

[Hint: If $\hat{S}_+ = \hat{S}_x + i\hat{S}_y$ and $\hat{S}_- = \hat{S}_x - i\hat{S}_y$, then

$$\hat{S}_+|s,m\rangle = [s(s+1) - m(m+1)]^{1/2}\,\hbar|s,m+1\rangle,$$
$$\hat{S}_-|s,m\rangle = [s(s+1) - m(m-1)]^{1/2}\,\hbar|s,m-1\rangle,$$

with $\hat{S}_+|s,s\rangle = 0$ and $\hat{S}_-|s,-s\rangle = 0$. Here $s = 1/2$ and $m = \pm 1/2$. We use the letters S and s rather than J and j since we specifically consider the case of a spin angular momentum.]

Problem 5

At the end of Section 11.4 of the course notes, it is stated that L_z commutes with the Hamiltonian of an atom of hydrogen exposed to an external electric field $F_{\text{ext}}(t)\,\hat{\mathbf{z}}$, and therefore that the z-component of the orbital angular momentum of the electron is a constant of motion. This Hamiltonian takes on the following form:

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + eF_{\text{ext}}(t)z.$$

Show that it is indeed the case that $[H, L_z] = 0$.

Hint: In spherical polar coordinates (r, θ, ϕ) such that $z = r\cos\theta$,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{\hbar^2}\frac{\mathbf{L}^2}{r^2}$$

and

$$L_z = -i\hbar\frac{\partial}{\partial\phi}.$$