Lecture 8: The Expansion History of the Universe

To determine how the expansion factor a of the Universe evolves with cosmic time we must solve the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2}. \tag{8.0}$$

The solution of this equation depends both on the curvature (the value of the constant k) and how the energy density ρc^2 depends on a.

8.1 The Matter Dominated Era

[Liddle sec:11]

At the present day we have seen that the energy density of matter $\rho_{m,0}c^2$ greatly exceeds that of radiation. Consequently

$$\rho \approx \rho_{\rm m} \propto a^{-3}$$

at late times. As we will see, this is not the case in the early universe.

If we make the simplifying assumption that $k = 0$, which is equivalent to $\rho_0 = \rho_{\rm crit,0} \equiv 3{\rm H}_0^2/8\pi G$, i.e. $\Omega_{\rm m,0} = 1$ then equation (8.0) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = {\rm H}_0^2 \frac{\rho}{\rho_{\rm crit,0}} = {\rm H}_0^2\, a^{-3}.$$

Hence

$$a^{1/2}\,\dot{a} = {\rm H}_0$$

and by integrating

$$\int_0^a a'^{1/2}\,da' = {\rm H}_0 \int_0^t dt'$$

we find

$$\frac{2}{3}\,a^{3/2} = {\rm H}_0 t$$

or

$$a = \left(\frac{3{\rm H}_0\, t}{2}\right)^{2/3} = \left(\frac{t}{t_0}\right)^{2/3} \tag{8.1}$$

[One can solve the Friedmann equation for other values of Ω_m, but the solutions are more cumbersome.]

8.2 The Radiation Dominated Era

We saw in the last lecture that the energy density in photons and other relativistic particles changes more rapidly than the energy density in matter. Hence at sufficiently early times the Universe will be radiation dominated.

In the radiation dominated regime $\rho \propto T^4 \propto a^{-4}$ and so Friedmann's equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \mathrm{H}_{\mathrm{eq}}^2 \left(\frac{a}{a_{\mathrm{eq}}}\right)^{-4},$$

where H_{eq} is the value of the Hubble constant at the time when $a = a_{\mathrm{eq}}$. Note at these early times, the assumption of $\Omega = 1$ used here will always be an excellent approximation (see lecture 3). Solving this equation we find

$$a = a_{\mathrm{eq}} \left(\frac{t}{t_{\mathrm{eq}}}\right)^{1/2}. \tag{8.2}$$

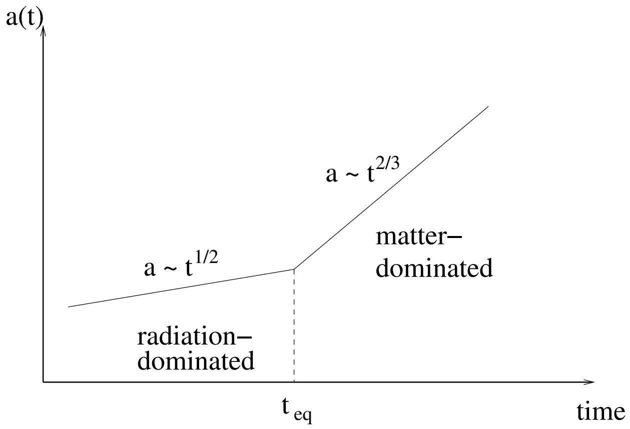

Figure 1: The expansion factor–time dependence in the radiation- and matter-dominated eras of an $\Omega = 1$ Universe.

8.3 Matter-Radiation Equality

This is the epoch when the energy density in radiation (relativistically moving particles) equalled the energy density in matter.

The energy density in Cosmic black-body radiation today is

$$\epsilon_{\mathrm{phot},0} = \rho_{\mathrm{phot},0} c^2 = \frac{4\sigma T_{\mathrm{CMB}}^4}{c}$$

where σ is Stefan's constant and $T_{\mathrm{CMB}} = 2.726$ K. However, one predicts that there is also a neutrino background (see Lecture 15) which contributes to the energy density in the relativistic component

$$\rho_{\mathrm{rel}} = \rho_{\mathrm{phot}}(1 + 3 \times 7/8 \times (4/11)^{4/3}) = 1.68\rho_{\mathrm{phot}}.$$

[This arises because the neutrino temperature T_ν is related to the photon-field temperature, T_γ as $T_\nu = T_\gamma (4/11)^{1/3}$ – see Liddle Advanced Topic A3.]

The energy density in matter is

$$\rho_{\mathrm{mass},0}\, c^2 = \Omega_{\mathrm{mass},0}\, \rho_{\mathrm{crit},0}\ c^2 = \Omega_{\mathrm{mass},0} \frac{3H_0^2 c^2}{8\pi G}$$

Thus present day fractional energy density in relativistic matter is

$$\left.\frac{\rho_{\mathrm{rel}}}{\rho_{\mathrm{mass}}}\right|_{\mathrm{today}} = 1.68 \frac{4\sigma\, T_{\mathrm{CMB}}^4}{c}\ \Omega_{\mathrm{mass},0}^{-1}\ \frac{8\pi G}{3H_0^2 c^2} \approx 7.4\times 10^{-5}\ \Omega_{\mathrm{mass},0}^{-1} \left(\frac{H_0}{75\,\mathrm{km\,s^{-1}Mpc^{-1}}}\right)^{-2}$$

which is very small.

However since $T \propto (1+z)$ we have $\rho_{\mathrm{rel}} \propto (1+z)^4$, while $\rho_{\mathrm{mass}} \propto (1+z)^3$. Thus

$$\frac{\rho_{\mathrm{rel}}}{\rho_{\mathrm{mass}}} = \left.\frac{\rho_{\mathrm{rel}}}{\rho_{\mathrm{mass}}}\right|_{\mathrm{today}} (1+z)$$

an so the two energy densities should be equal ($\rho_{\mathrm{rel}}/\rho_{\mathrm{mass}} = 1$) at

$$(1+z_{\mathrm{eq}}) \approx 13,500\ \Omega_{\mathrm{mass},0} \left(\frac{H_0}{75\,\mathrm{km\,s^{-1}Mpc^{-1}}}\right)^2 .$$

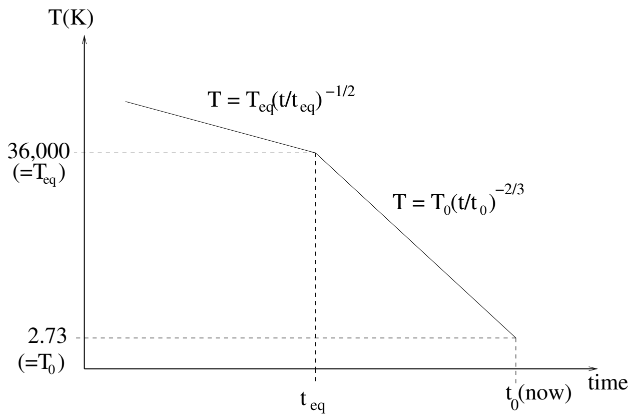

Figure 2: Radiation temperature vs. time for an $\Omega = 1$ Universe.

8.4 The Time-Temperature Relation

The hot early Universe is dominated by radiation (relativistically moving particles) and so will evolve according to (8.2), $a \propto t^{1/2}$. By the present, mass dominates and solution (8.1) [assuming $\Omega = 1$] becomes appropriate, $a \propto t^{2/3}$.

The transition between these two regimes will be smooth, but we can make a fairly accurate approximation by assuming an abrupt transition between the two solutions at $z_{\rm eq}$, the redshift at which the energy densities in matter and radiation are equal, where

$$\frac{1}{a_{\rm eq}} = (1 + z_{\rm eq}) \approx 13,500\ \Omega_{\rm mass,0} \left(\frac{H_0}{75\,{\rm km\,s^{-1}Mpc^{-1}}} \right)^2 .$$

Using this and the relation

$$T = \frac{T_{\rm CMB}}{a} = \frac{2.726\ {\rm K}}{a}$$

between temperature and expansion factor, we can calculate the relation between temperature and time back to very early times.

From above $(1 + z_{eq}) = 13,500$, giving $T_{eq} = 2.73/a_{eq} = 36,000$ K.

From eqn. (8.2), $t_{eq} = a_{eq}^{3/2} t_0$ with $t_0 = 2/3H_0^{-1} = 8.7$ Gyr for $\Omega_0 = 1$, giving $t_{eq} \simeq 5500$ yr. In fact $\Omega_m \simeq 0.25$, giving $t_{eq} \sim 30000$ yr.

Examples

1 Assuming $\Omega = 1$, $T_{\rm CMB} = 2.7\,$K and $\mathrm{H}_0 = 75\,$km s^{-1} Mpc^{-1} estimate the age, $t_{\rm eq}$, and temperature, $T_{\rm eq}$, of the Universe when the energy densities of matter and radiation were equal?

Repeat your calculation for a hypothetical universe in which $\mathrm{H}_0 = 7.5\,$km s^{-1} Mpc^{-1}. Compare this with the redshift of reionisation in this universe.

2 How are $t_{\rm eq}$ and $T_{\rm eq}$ altered if instead of there being three species of light neutrino there are four?

3 Estimate the age of the universe when it had the following temperatures: i) $T = 5.4$ K ii) $T = 3000$ K iii) $T = 10^{10}$ K .

4 Sketch the Time-Temperature relation for an $\Omega = 1$ Universe, marking on the point where the energy density in matter and radiation are equal. Indicate how this relation is changed if $\Omega_m < 1$. (Assume the present day Hubble Constant and CMB temperature are to be kept constant.)