i) A rigid rotator has energy levels $E_l = l(l+1)\hbar^2/(2I)$ where I is the moment of inertia and l is the angular quantum number. For $l = 3, 4$ and 5, write down the energy, the degeneracy and the possible values for the magnetic quantum number m for each level. [1 mark]

ii) The energy eigenfunction for the rigid rotator above in the $l = 3$, $m = -2$ state is

$$Y_{3,-2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta \cos\theta \, e^{-2i\phi}$$

Write down the probability dP of finding the electron with this wavefunction in the solid angle $d\Omega = \sin\theta d\theta d\phi$ around position θ, ϕ. Integrate this over ϕ to get the probability density per unit θ. At what values(s) of θ does this have a maximum? Evaluate your answer in degrees to 3 significant figures. [3 marks]

Evaluate $\langle\theta\rangle$ (Note that $\langle\theta\rangle$ and the most probable value of θ are, in general, **not** the same.) and $\langle\cos\theta\rangle$ [2 marks]

What is the probability of finding the electron in the region $0 < \theta < \pi/3$. Evaluate your answer to 3 significant figures. [1 mark]

iii) Demonstrate explicitly that $Y_{3,-2}$ is an eigenfunction of the the angular momentum operators

$$\mathbf{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad \text{and } L_z = -i\hbar\frac{\partial}{\partial\phi}$$

but is *not* an eigenfunction of

$$L_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

[3 marks]

Useful integrals:

Use an online integrator such as `http://www.wolframalpha.com/`
where you can evaluate definite integrals by typing e.g.
`integrate cos x from 0 to pi`.