Mathematical Methods II
Lecture 4

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Key Points

- Solving 2$^{\text{nd}}$ order ODEs using Laplace transforms
- Solving Legendre equations
- Solving Euler equations

Solving linear 2$^{\text{nd}}$ order ODEs with constant coefficients (cont.)

- **Laplace transform method:** Laplace transforms are useful for solving linear ODEs with constant coefficients. Taking the Laplace tranform of such an equation tranforms it into a purely algebraic equation. Once this equation has been solved an inverse Laplace transform can be applied to obtain the general solution of the ODE. Laplace can be used as an alternative to the trial function method.

  The Laplace transform of a function $f(x)$ is defined as

  $$\bar{f}(s) \equiv \int_0^\infty e^{-sx} f(x) dx.$$

  The Laplace transform of the n$^{\text{th}}$ derivative of $f(x)$ is given by

  $$\bar{f^n}(s) = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f^{'}(0) - ... - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

  $\bar{f}(s)$ is the Laplace transform of the required solution; we solve for this. The primes and bracketed superscripts denote differentiation w.r.t x. e.g. The transform for a 2$^{\text{nd}}$ order derivative is given by

  $$\bar{f^n}(s) = s^2 \bar{f}(s) - s f(0) - f^{'}(0).$$

  These equations, combined with the table of Laplace transforms allows us to solve linear ODEs with constant coefficients.

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\lvert a\rvert$
$\cosh at$	$s/(s^2-a^2)$	$\lvert a\rvert$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0)=\begin{cases}1 & \text{for } t\geq t_0\\ 0 & \text{for } t<t_0\end{cases}$	e^{-st_0}/s	0

Table 13.1 Standard Laplace transforms. The transforms are valid for $s > s_0$.

e.g. 4.1 Find the Laplace transform of $f(x) = c$.

Start with the definition

$$\bar{f}(s) = \int_0^\infty ce^{-sx}dx$$

$$= \left[-\frac{c}{s}\frac{1}{e^{sx}}\right]_0^\infty = -\frac{c}{s}\frac{1}{e^\infty} + \frac{c}{s}\frac{1}{e^0} = \frac{c}{s}$$

Note that the Laplace transform of a product of functions is not equal to the product of the transformed functions, i.e.

$$\mathcal{L}[fg] \neq \mathcal{L}[f]\mathcal{L}[g]$$

The relationship becomes complicated when both functions are variable. We will restrict ourselves for now to the case of a constant multiplied by a function, which for constant a has the simple result

$$\mathcal{L}[ag] = a\mathcal{L}[g]$$

e.g. 4.2 Solve the following ODE using the Laplace transform method

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{-x}$$

subject to the boundary conditions $y(0) = 2, y'(0) = 1$.

First, note the required substitutions

$$\mathcal{L}\left[\frac{d^2y}{dx^2}\right](s) = s^2\bar{y}(s) - sy(0) - y^{'}(0)$$

$$\mathcal{L}\left[\frac{dy}{dx}\right](s) = s\bar{y}(s) - y(0)$$

$$\mathcal{L}[y](s) = \bar{y}(s)$$

$$\mathcal{L}[e^{-x}](s) = \frac{1}{s+1}$$

Now, take the Laplace transform of the equation

$$s^2\bar{y}(s) - sy(0) - y^{'}(0) - 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{2}{s+1}$$

Collecting terms and subbing in our boundary conditions, this reduces to

$$(s^2 - 3s + 2)\bar{y}(s) - 2s + 5 = \frac{2}{s+1}$$

Now solve for $\bar{y}(s)$ using partial fractions

$$(s^2 - 3s + 2)\bar{y}(s) = \frac{2 + (2s-5)(s+1)}{s+1} = \frac{2s^2 - 3s - 3}{s+1}$$

$$\bar{y}(s) = \frac{2s^2 - 3s - 3}{(s+1)(s-1)(s-2)} = \frac{1}{3(s+1)} + \frac{2}{s-1} - \frac{1}{3(s-2)}$$

Finally, we can reverse the Laplace transform to find the general solution to the ODE

$$y(x) = \frac{1}{3}e^{-x} + 2e^{x} - \frac{1}{3}e^{2x}$$

Recall

Factor in denominator	**Term in partial fraction decomposition**
$ax + b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Source: http://tutorial.math.lamar.edu/Classes/Alg/PartialFractions.aspx

Solving linear ODEs with variable coefficients

- **Legendre linear equations:** Legendre's linear equation has the form

$$a_n(\alpha x+\beta)^n\frac{d^ny}{dx^n}+...+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0y=f(x)$$

where α, β and a_n are constants. The defining feature of this type of ODE is that the coefficient function of each derivative has a power of x equal to the order of the derivative (assuming $\alpha \neq 0$).

The 2^{nd} order Legendre is therefore

$$a_2(\alpha x+\beta)^2\frac{d^2y}{dx^2}+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0y=f(x)$$

and may be solved by making the substitution $\alpha x+\beta=e^t$. This gives

$$\frac{dy}{dx}=\frac{dt}{dx}\frac{dy}{dt}=\frac{\alpha}{\alpha x+\beta}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{\alpha^2}{(\alpha x+\beta)^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

e.g. 4.3 Solve the following equation using the substitute $x+2=e^t$

$$(x+2)^2y''+(x+2)y'+y=3x$$

Note that the equivalence between the following two operations

$$\frac{d}{dx}=\frac{dt}{dx}\frac{d}{dt}$$

Substitution: $x+2=e^t$. Find the derivatives. Note that $t=\ln|x+2|$ and $dt/dx=1/(x+2)$

$$y'=\frac{dt}{dx}\frac{dy}{dt}=\frac{1}{x+2}\dot{y}$$

$$y''=\frac{d}{dx}\frac{dy}{dx}=-\frac{1}{x+2}\dot{y}+\frac{1}{x+2}\frac{dt}{dx}\frac{d\dot{y}}{dt}$$

$$=\frac{1}{(x+2)^2}\left[\ddot{y}-\dot{y}\right]$$

Sub into ODE, $(x+2)$ terms cancel, and $RHS=3x=3(e^t-2)$

$$\ddot{y}-\dot{y}+\dot{y}+y=\ddot{y}+y=3(e^t-2)$$

Find auxiliary equation

$$\lambda^2+1=0\rightarrow\lambda=\pm i$$

Complementary equation

$$y_c(t) = c_1 e^{it} + c_2 e^{-it}$$

Particular equation $3e^t - 6$ so try $Ae^t + B$

$$\frac{d^2}{dt^2}(Ae^t + B) + Ae^t + B = 3e^t - 6$$

$$Ae^t + Ae^t + B = 2Ae^t + B = 3e^t - 6$$

$\Rightarrow A = \frac{3}{2}$ and $B = -6$

$$y_p(t) = \frac{3}{2}e^t - 6$$

$$y(t) = c_1 e^{it} + c_2 e^{-it} + \frac{3}{2}e^t - 6$$

Change variable t to x, recall $t = \ln|x+2|$

$$y(x) = c_1(x+2)^i - c_2(x+2)^i + \frac{3}{2}(x+2) - 6$$

$$= (c_1 - c_2)(x+2)^i + \frac{3}{2}x - 3$$

- **Euler linear equations:** Euler's equation is a special case of Legendre's equation, where $\alpha = 1$ and $\beta = 0$

$$a_n x^n \frac{d^n y}{dx^n} + ... + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

The 2^{nd} order Euler is therefore

$$a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

which may be solved by making the substitution $x = e^t$, though if $f(x) = 0$ substituting $y = x^\lambda$ leads to a simple algebraic equation in λ.

e.g. 4.4 Solve the following equation using the substitute $x = e^t$

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

Work out the derivatives and note $t = \ln|x|$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt} = \frac{1}{e^t}\dot{y}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{dt}{dx}\frac{d\dot{y}}{dt} = -\frac{1}{e^{2t}}\frac{dy}{dt} + \frac{1}{e^{2t}}\frac{d\dot{y}}{dt} = \frac{1}{e^{2t}}\left(\ddot{y} - \dot{y}\right)$$

Sub into the equation and cancel e^t terms

$$e^{2t}\frac{1}{e^{2t}}\left(\ddot{y} - \dot{y}\right) + e^t\frac{1}{e^t}\dot{y} - 4y = 0$$

$$(\ddot{y} - \dot{y}) + \dot{y} - 4y = 0$$
$$\ddot{y} - 4y = 0$$

Find the auxiliary equation

$$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

Thus

$$y = c_1 e^{2t} + c_2 e^{-2t} = c_1 x^2 + c_2 x^{-2}$$

e.g. 4.5 Solve the following equation using the substitute $y = x^\lambda$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

Make the substitution

$$\lambda(\lambda - 1)x^\lambda + \lambda x^\lambda - 4x^\lambda = 0$$
$$(\lambda^2 - 4)x^\lambda = 0$$

Since $x^\lambda = 0$ is only true in the trivial case where $x = 0$ we can conclude that

$$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

Thus

$$y = c_1 x^2 + c_2 x^{-2}$$