University of Durham

EXAMINATION PAPER

May/June 2012 Examination code: 043551/01 or 044191/01

LEVEL 3 PHYSICS: THEORETICAL PHYSICS
LEVEL 4 PHYSICS: THEORETICAL PHYSICS 4

SECTION A. QUANTUM MECHANICS
SECTION B. SPECIAL RELATIVITY AND ELECTROMAGNETISM

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. QUANTUM MECHANICS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Consider the operators described by the matrices

$$A = \begin{pmatrix} 3 & 1+i \\ 1-i & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 1+i \\ 1+i & 2 \end{pmatrix}.$$

Which one cannot correspond to an observable and why? [2 marks]

For the other operator, what are the possible outcomes of a measurement? [2 marks]

(b) State the generalised uncertainty relation for two operators $\hat{A}$ and $\hat{B}$. [2 marks]

Consider two Hermitian operators with the following matrix representations

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Can they be simultaneously measured with arbitrary precision? [2 marks]

(c) Consider the Hamiltonian $\hat{H} = \lambda \hat{L}_z$, where λ is a constant. Its eigenstates are the eigenstates of $\hat{L}^2$ and $\hat{L}_z$ indicated as $|l, m\rangle$. The system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{26}}(|3,2\rangle + 3|3,1\rangle + 4|2,0\rangle).$$

What are the possible outcomes of a measurement of the energy and with what probability? [4 marks]

(d) Consider a particle with angular momentum $l = 1$. Use the eigenbasis of $\hat{L}^2$ and $\hat{L}_z$, given by the states $|1,1\rangle$, $|1,0\rangle$ and $|1,-1\rangle$. The action of the angular momentum raising operator is given by

$$\hat{L}_+|l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}|l, m+1\rangle.$$

Find the matrix representation of $\hat{L}_+$. [4 marks]

(e) Consider a linear harmonic oscillator in 1-dimension, with creation operator $\hat{a}^\dagger$ and its Hermitian conjugate, the annihilation operator $\hat{a}$. The action of $\hat{a}$ is given by

$$\hat{a}|n\rangle = C_n|n-1\rangle,$$

where n are the eigenvalues of the operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$, $|n\rangle$ are the corresponding eigenstates and C_n is a constant. Find C_n such that all the eigenstates are normalized. [4 marks]

(f) How many lines are observed on the screen of a Stern-Gerlach experiment if a beam of spin 3/2 atoms passes through its magnets? [2 marks]

How many lines were observed in the original Stern-Gerlach experiment? Discuss why this result cannot be explained by a quantized integer angular momentum and why it requires half-integer spin? [2 marks]

(g) Consider a spin 1/2 particle. If we measure the operator

$$\hat{S}_d = \frac{1}{\sqrt{2}}\left(\hat{S}_x + \hat{S}_y\right),$$

what are the possible results? [4 marks].

[Hint: $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.]

(h) Consider a system of two identical particles and the following wave functions

$$\begin{aligned} \psi_S(q_1, q_2) &= \tfrac{1}{\sqrt{2}}\left[\psi(q_1, q_2) + \psi(q_2, q_1)\right], \\ \psi_A(q_1, q_2) &= \tfrac{1}{\sqrt{2}}\left[\psi(q_1, q_2) - \psi(q_2, q_1)\right], \end{aligned}$$

where q_i denotes a complete set of commuting observables for particle i.

Which of these wave functions must be used if the identical particles are bosons and which one for fermions? [2 marks]

If the two particles are identical fermions, explain how the previous answer implies the Pauli exclusion principle. [2 marks]

2. Consider a spin 1/2 particle in a magnetic field $\underline{B}$. The Hamiltonian is given by

$$\hat{H} = \gamma \underline{\hat{S}}.\,\underline{B} = \gamma(\hat{S}_x B_x + \hat{S}_y B_y + \hat{S}_z B_z),$$

where γ is a constant. Its eigenstates are eigenstates of $\hat{S}^2$ and $\hat{S}_z$, indicated as $|s, m_s\rangle$. In the basis given by the states $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$, we have

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The magnetic field $\underline{B}$ has components

$$B_x = \frac{1}{\sqrt{2}} B_0, \qquad B_y = 0, \qquad B_z = \frac{1}{\sqrt{2}} B_0.$$

(a) Find the matrix representation for the Hamiltonian $\hat{H}$ in the basis given by the states $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$. [4 marks]

(b) Calculate the eigenvalues and the normalized eigenvectors of $\hat{H}$. [6 marks]

(c) At time $t = 0$, the system is in the state $|\psi(t = 0)\rangle = |1/2, -1/2\rangle$. If the energy is measured at time $t = 0$, what values can be measured and with what probability? [6 marks]

(d) Calculate the state $|\psi(t)\rangle$ at time t. [4 marks]

3. Consider a 3d harmonic oscillator. The Hamiltonian is the sum of 1d harmonic oscillators for each Cartesian direction and takes the form

$$\hat{H} = \left(\frac{\hat{P}_x^2}{2m} + \frac{1}{2}m\omega_x^2\hat{Q}_x^2\right) + \left(\frac{\hat{P}_y^2}{2m} + \frac{1}{2}m\omega_y^2\hat{Q}_y^2\right) + \left(\frac{\hat{P}_z^2}{2m} + \frac{1}{2}m\omega_z^2\hat{Q}_z^2\right),$$

where $\hat{P}_x$, $\hat{P}_y$, $\hat{P}_z$ are the momentum operators in the x, y and z directions, respectively. Similarly, $\hat{Q}_x$, $\hat{Q}_y$, $\hat{Q}_z$ are the relevant position operators.

We define the creation operator in the x direction as $\hat{a}_x^\dagger = \frac{1}{\sqrt{2}}\left(\beta_x\hat{Q}_x - i\frac{1}{\hbar\,\beta_x}\hat{P}_x\right)$ where $\beta_x = \sqrt{\frac{m\omega_x}{\hbar}}$. The creation operators for the other directions are defined analogously.

The eigenstates of the Hamiltonian are fully determined by the eigenstates of the number operators, which we call n_x, n_y and n_z.

(a) The ground state has $n_x = n_y = n_z = 0$. Its wave function is

$$\psi_0 = \left(\frac{\beta_x^2}{\pi}\right)^{1/4}\left(\frac{\beta_y^2}{\pi}\right)^{1/4}\left(\frac{\beta_z^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}(\beta_x^2x^2+\beta_y^2y^2+\beta_z^2z^2)}.$$

Calculate the wave functions, without normalization, for the states with $n_x = 1$, $n_y = 0$, $n_z = 0$ and $n_x = 1$, $n_y = 1$, $n_z = 0$. [6 marks]

(b) The scalar product between two states is

$$\langle\psi|\phi\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\ \psi(x,y,z)^*\phi(x,y,z).$$

Show by explicit evaluation of the scalar product that the two states derived in (a) are orthogonal. Why is this result expected? [4 marks]

Now consider an isotropic 3d harmonic oscillator, which corresponds to setting $\omega_x = \omega_y = \omega_z \equiv \omega$. The Hamiltonian simplifies to

$$\hat{H} = \frac{\underline{\hat{P}}^2}{2m} + V(r), \qquad V(r) = \frac{1}{2}m\omega^2r^2$$

where $\underline{\hat{P}}^2 = \hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2$ and $\hat{Q}_r^2 = \hat{Q}_x^2 + \hat{Q}_y^2 + \hat{Q}_z^2 = r^2$.

(c) Do the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$ commute with the Hamiltonian? Explain briefly why this is the case and discuss the implications this has on the eigenstates of the Hamiltonian. [4 marks]

(d) It is useful to work in spherical coordinates and separate variables, $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$, where $Y_{lm}(\theta,\phi)$ are the spherical harmonics. The reduced radial equation for $u_{nl}(r) \equiv rR_{nl}(r)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(u_{nl}(r)) + \frac{\hbar^2 l(l+1)}{2mr^2}u_{nl}(r) + V(r)u_{nl}(r) = E_{nl}u_{nl}(r).$$

Use this equation to discuss the behavior of $R_{nl}(r)$ for $r \to 0$. [3 marks]

(e) Use the reduced radial equation to show that, for $r \to \infty$

$$R_{nl}(r) \sim p(r)e^{-\frac{1}{2}\beta^2r^2},$$

where $\beta = \sqrt{\frac{m\omega}{\hbar}}$ and $p(r)$ is a polynomial. [3 marks]

4. Consider an electron in a linear triatomic molecule formed by three equidistant atoms. We denote $|1\rangle$, $|2\rangle$ and $|3\rangle$ the three orthonormal states of this electron. The state $|1\rangle$ corresponds to the wave function of the electron localized at the first atom, $|2\rangle$ corresponds to the electron localized at the second atom and $|3\rangle$ corresponds to the electron localized at the third atom. In this basis, the Hamiltonian of the system is

$$\hat{H} = E_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where E_0 is a real constant and a is a positive real constant. The physical interpretation of this Hamiltonian is the following. The first term describes the energy of the electron when we neglect the possibility of it jumping from one atom to the other. The second term takes into account the possibility of the electron jumping between different atoms.

(a) Calculate the eigenvalues and normalized eigenstates of the Hamiltonian. [4 marks]

(b) At time $t = 0$ the electron is in the state $|\psi(t = 0)\rangle = |1\rangle$. Find the state of the electron at time t and discuss the localization of the electron as a function of time. [6 marks]

(c) For the state in part (b), are there any times for which the electron is perfectly localized at any of the atoms and if so calculate at which times. [6 marks]

(d) Let $\hat{D}$ be an observable described by the matrix

$$\hat{D} = \begin{pmatrix} -d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{pmatrix}.$$

If we measure $\hat{D}$ at time t, what values can be obtained and with what probabilities for the state defined in part (b)? [4 marks]

SECTION B. SPECIAL RELATIVITY AND ELECTROMAGNETISM
Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) Give the definitions and physical interpretations of lightlike, timelike and spacelike distances. [4 marks]

(b) The polarization tensor for the photon is

$$T^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n} \right),$$

where p^μ and n^μ are lightlike four vectors. Calculate $T^\mu{}_\mu$ and $T^{\mu\nu}T_{\mu\nu}$. [4 marks]

(c) An observer is moving with velocity $\underline{v}$ perpendicular to one of the faces of a cube with sides of length l. Describe the shape of the cube she sees and calculate its volume in the rest frame of the observer. [4 marks]

(d) State the relativistic relationship between the energy E, mass m and velocity v of a particle. Assuming that $v \ll c$, calculate the first two non-zero terms in the Taylor expansion of E in $\frac{v}{c}$ and interpret your result. [4 marks]

(e) Give the definition of the angular-momentum tensor $L^{\mu\nu}$ for a particle with 4-momentum p^μ and position 4-vector x^μ. Show that for a particle not acted upon by an external force the angular momentum tensor is conserved. [4 marks]

(f) An observer moves with constant velocity with respect to a point charge q. He measures the electric $\underline{E}$ and magnetic $\underline{B}$ fields at a distance d from the point charge. What value does he find for $\underline{E} \cdot \underline{B}$? [4 marks]

(g) The Lienard-Wiechert potential of a point charge q with 4-velocity u^μ is

$$A^\mu = \frac{q}{4\pi\epsilon_0} \frac{u^\mu}{u^\nu R_\nu},$$

where R_ν is the 4-distance between the observer and the point charge. The right-hand side of the expression must be evaluated at the retarded time t_{ret}. Evaluate this expression in the instantaneous rest frame of the point charge at t_{ret} and show that you obtain the expected result. [4 marks]

6. Consider a particle of mass m with 4-momentum $p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$, where E is the energy of the particle and $p_{x,y,z}$ the x-, y- and $z-$components of its three-momentum, respectively. An alternative representation of the 4-momentum of the particle is

$$p^\mu = (m_T c \cosh y, p_T \cos\phi, p_T \sin\phi, m_T c \sinh y),$$

where y is the rapidity of the particle, p_T the magnitude of the three-momentum transverse to the z-direction, ϕ is the azimuthal angle and m_T the transverse mass.

Determine m_T in terms of the mass and transverse momentum p_T of the particle. [4 marks]

The Large Hadron Collider collides two protons with 4-momenta

$$p_1^\mu = \frac{E_{\text{beam}}}{c}(1,0,0,1) \quad \text{and} \quad p_2^\mu = \frac{E_{\text{beam}}}{c}(1,0,0,-1),$$

where E_{beam} is the energy of the protons. As $E_{\text{beam}} \gg m_{\text{p}}c^2$ the mass of the proton can be neglected. A quark carrying a fraction x_1 $(0 \le x_1 \le 1)$ of the four momentum of the first proton collides with a quark carrying a fraction x_2 $(0 \le x_2 \le 1)$ of the four momentum of the second proton.

Calculate the 4-momentum of the resultant system and its invariant mass squared. [4 marks]

A single particle with mass m and rapidity y is produced in the collision of the two quarks. Using the conservation of 4-momentum or otherwise show that

$$x_1 = \frac{mc^2}{2E_{\text{beam}}}\exp(y),$$

and hence obtain the maximum possible rapidity of the particle. [6 marks]

A W^- boson is produced together with a number of additional particles. The W^- boson subsequently decays to an electron and an anti-electron neutrino. The mass of the decay products can be neglected.

Show that $m_W^2 c^2 \ge 2p_T^e p_T^\nu (1 - \cos(\Delta\Phi))$, where m_W is the mass of the W^- boson, $\Delta\Phi$ is the difference in the azimuthal angles of the electron and anti-electron neutrino, and p_T^e and p_T^ν are the magnitudes of the transverse momentum of the electron and anti-electron neutrino, respectively. [6 marks]

[Hint: $\cosh(A-B) = \cosh A \cosh B - \sinh A \sinh B$ and
$\cos(A-B) = \cos A \cos B + \sin A \sin B$.]

7. A particle is initially at rest with respect to an observer. At $t = 0$ it accelerates with a constant proper acceleration α in the x-direction. At time t the x-component of the particle's position is $x(t)$, velocity is $v(t)$ and acceleration is $a(t)$, as seen by the observer.

   (a) Show that the acceleration as measured by the observer is given by

   $$a = \frac{\alpha}{\gamma^3(v)}.$$

   [10 marks]

   $$[\text{Hint: } \frac{\mathrm{d}\gamma}{\mathrm{d}t} \equiv \frac{\gamma^3 va}{c^2}].$$

   (b) Show that the result of part (a) can be rewritten as

   $$\alpha = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma v).$$

   Use this result to find $x(t)$ for the initial conditions $x(0) = 0$ and $v(0) = 0$. [6 marks]

   (c) Find the proper time τ of the particle as a function of t. [4 marks]

   $$[\text{Hint: } \int \frac{\mathrm{d}t}{\sqrt{1 + k^2t^2}} = \frac{1}{k}\sinh^{-1}(kt)].$$

8. State the transformations of x^μ, y_μ, $F^{\mu\nu}$ and ${F_\mu}^\nu$ under an arbitrary Lorentz transformation Λ. Show that the Minkowski scalar product of two 4-vectors is invariant under Lorentz transformations. [6 marks]

   The explicit form of the dual field-strength tensor in terms of the electric $\underline{E}$ and magnetic $\underline{B}$ fields is

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}.$$

   Give the homogeneous Maxwell's equations in covariant form. [2 marks]

   Using the explicit form of the dual field-strength tensor rewrite these equations in terms of the electric and magnetic fields. [4 marks]

   In the inertial frame S, $\underline{E} = 0$ and $B_x = 0$. Using the transformation property of the dual field-strength tensor and the explicit form of Λ for a boost along the z-axis, derive the electric and magnetic fields as seen by an observer at rest in the inertial frame S'. The frame S' is moving along the z-axis with velocity v with respect to S and at $t = t' = 0$ the two frames coincide. [8 marks]