Theoretical Physics 2019/20 — Problem QT2.2

Background reading: section 1.2 of Part 1 of the course notes (about the Stern-Gerlach experiment), section 2.10 of Part 2 (about orthonormal bases) and sections 3.1 and 3.2 of Part 3.

Recall that the two vectors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

form an orthonormal basis for the Hilbert space of 2-component complex column vectors. Any spin state of an electron can be represented by an element of that space, i.e., by a column vector of the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where α and β are two complex numbers.

Now, because electrons have a spin-dependent magnetic moment, their spin state may be modified by a magnetic field. Consider an electron at rest in a magnetic field $\mathbf{B}$ parallel to the $x - z$ plane (the y-component of $\mathbf{B}$ is zero). The Hamiltonian of this system can be represented by a 2×2 matrix. This matrix is given by the following equation in the $\{\chi_+, \chi_-\}$ basis:

$$H = -\frac{\gamma\hbar}{2} \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}, \tag{1}$$

where γ is a constant and B_x and B_z are the x- and z-components of $\mathbf{B}$. We assume that $B_x \neq 0$ (recall that we also assume that the y-component of $\mathbf{B}$ is zero).

(a) Show that $H = -\gamma\, \mathbf{B} \cdot \mathbf{S}$, where

$$\mathbf{S} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\mathbf{x}} + \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\mathbf{y}} + \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathbf{z}}.$$

(The vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the x-, y- and z-directions.)

(b) Consider the two column vectors χ_a and χ_b defined by the following equations:

$$\chi_a = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} -k \\ 1 \end{pmatrix}, \qquad \chi_b = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} 1 \\ k \end{pmatrix},$$

with $k = B_x/(B_z + |\mathbf{B}|)$. Show that both χ_a and χ_b are eigenvectors of H and write down the corresponding eigenvalues, E_a and E_b, in terms of the constant $\gamma\hbar$ and of $|\mathbf{B}|$. [Hint: You are not asked to find the eigenvectors and eigenvalues of this matrix from scratch; you may answer this question simply by showing that the column vector obtained by acting with H on χ_a is χ_a times a certain scalar E_a, and similarly for χ_b.]

(c) Why can you say that χ_a and χ_b form an orthonormal basis of this Hilbert space?