Answers to short question set #1: [Total: 10 seen; 2 unseen]

(a) (i) Prime since gives the shortest effective focal length and widest field of view ($s = f\,\theta$) [2 marks bookwork]

(ii) Naysmith or Coude since these are the most stable (Gravity vector stable and predictable). [2 marks bookwork]

(b) Dark current is measured by recording an image with the same exposure time (or more realistically an appropriately scaled longer exposure) as the actual observation, but without opening the camera shutter. [1 mark bookwork]

Sky background is measured by observing the sky adjacent to the science object. [1 mark bookwork]

The flat field is measured by observing a uniform field (e.g. severely out of focus or dithered image of sky). [1 mark bookwork]

Corrected data = $\frac{rawimage-dark}{flatfield-dark} - \frac{background-dark}{flatfield-dark} = \frac{rawimage-background}{flatfield-dark}$ [1 mark seen]

(c) The condition for constructive interference is: $d\sin\theta = \mathrm{n}\lambda$
Differentiating θ with respect to λ gives $d\theta\,/\,d\lambda = \mathrm{n}\,/\,\mathrm{d}\cos\theta$ [1 mark seen]
use $d\theta\,/\,\mathrm{d}\lambda = \mathrm{d}\theta\,/\,dx \times dx\,/\,\mathrm{d}\lambda$
and $d\theta\,/\,\mathrm{d}x = 1\,/\,f$ where f is the focal length of the telescope to derive final answer:
$\underline{d\lambda\,/\,\mathrm{d}x = d\cos\theta\,/\,\mathrm{n}f}$ [1 mark seen]

substitute numbers:
$n = 2$
$\theta = 30°$
$d\lambda\,/\,dx = 1\,\mathrm{nm}\,/\,\mathrm{mm} = 10^{-9}\,/\,10^{-3} = 10^{-6}$
$f = 0.15\,\mathrm{m}$
$d = nf\,(d\lambda\,/\,dx)\,/\,\cos\theta$
$d = 2 \times (150 \times 10^{-3}) \times 10^{-6}\,/\,\cos(30°)$
$\underline{d = 3.46 \times 10^{-7}\,\mathrm{m}}$ [1 mark partly unseen]
$\underline{\rho = 1\,/\,d = 2886\,\mathrm{lines/mm}}$ [1 mark partly unseen]

Answers to long question #1: [Total: 11 seen; 9 unseen]

(a i) Let the air column at the zenith be $X = 1$.

At angle z, the air column is $X = 1 / \cos(z)$ [1 mark seen]

The fractional loss of light is $-\mathrm{dI} / \mathrm{I} = k_{\mathrm{lambda}}\, \mathrm{dX}$ where k_λ is the extinction coefficient. [1 mark seen]

This implies $-\ln \mathrm{I} = k_\lambda \mathrm{X} + \mathrm{c}$

Evaluating at $\mathrm{X} = 0$ where $\mathrm{I} = \mathrm{I}_{\mathrm{obs}}$, and $\mathrm{X} = \mathrm{X}$ where $\mathrm{I} = \mathrm{I}_{\mathrm{corr}}$ gives:

$-\ln(\mathrm{I}_{\mathrm{obs}} / \mathrm{I}_{\mathrm{corr}}) = k_\lambda \mathrm{X}$ or $\mathrm{I}_{\mathrm{obs}} / \mathrm{I}_{\mathrm{corr}} = \exp(-k_\lambda \mathrm{X})$ [1 mark seen]

Using the equation $m_1 - m_2 = -2.5 \log(\mathrm{I}_1 / \mathrm{I}_2)$, or equivalently:
$m_{\mathrm{obs}} - m_{\mathrm{corr}} = -2.5 \log(\mathrm{I}_{\mathrm{obs}} / \mathrm{I}_{\mathrm{corr}})$ gives:
$m_{\mathrm{obs}} - m_{\mathrm{corr}} = -2.5 \log(e)\, k_\lambda \mathrm{X}$ [1 mark seen]

Let $A_\lambda = -2.5 \log(e)\, \mathrm{k}_\lambda$ and using $\mathrm{X} = \sec(z)$ from above gives: $\underline{m_{\mathrm{obs}} = m_{\mathrm{corr}} - \mathrm{A}(\lambda) \sec(z)}$ [1 mark seen]

(a ii) Using $m_{\mathrm{intrinsic}} = m_{\mathrm{observed}} - \mathrm{A}(\lambda) \sec(z)$

then: $15 = m_{\mathrm{observed}} - 0.15 \sec(30°)$ which gives $\underline{m_{\mathrm{observed}} = 15.17}$ [2 marks partly unseen]

(b i) At the diffraction limit: $\theta = 1.22\, \lambda / \mathrm{D}$
which gives: $\theta = 1.22 \times (550 \times 10^{-9}) / 2 \times 206265 \underline{= 0.07''}$. [1 mark partly seen]

(b ii) Flux of a 25^{th} magnitude star is: $m_* - m_0 = -2.5 \times \log(f_* / f_0)$

We know that a 0^{th} magnitude star has flux of $3.92 \times 10^{-8}\, \mathrm{W\, m^{-2}}\, \mu\mathrm{m}^{-1}$, so we can substitute numbers to derive $25 - 0 = -2.5 \log(f_* / 3.92 \times 10^{-8})$

which gives $\underline{f_* = 3.92 \times 10^{-18}\, \mathrm{W\, m^{-2}}\, \mu\mathrm{m}^{-1}}$. [1 mark partly unseen]

Total Power detected by telescope is:
$\mathrm{P} = \mathrm{Area} \times \mathrm{bandpass} \times \mathrm{efficiency} \times \mathrm{flux}$ [1 mark seen]

substitute numbers:
Area $= \pi\, (\mathrm{D} / 2)^2$ where D is the telescope diameter
total efficiency $= 0.6 \times 0.75$
bandpass $= 100\, \mathrm{nm} = 100 \times 10^{-9}\, \mathrm{m}$
flux $= 3.92 \times 10^{-18}\, \mathrm{W\, m^{-2}}\, \mu\mathrm{m}^{-1} \times 10^6$ to convert to SI units.

$\mathrm{P} = \pi\, (2 / 2)^2 \times (0.6 \times 0.75) \times (3.92 \times 10^{-18} \times 10^6) \times (100 \times 10^{-9}) \underline{= 5.5 \times 10^{-19}\, \mathrm{W}}$ [1 mark partly unseen]

(b iii) The energy of a photon in V-band, centered at 550 nm is:
$\mathrm{E} = hc / \lambda = hc / (550 \times 10^{-9}) \underline{= 3.6 \times 10^{-19}\, \mathrm{J.}}$ [1 mark partly unseen]

Total number of photons is the total power / photon energy:
$\mathrm{P} / \mathrm{E} = 5.54 \times 10^{-19} / 3.6 \times 10^{-19} \underline{= 1.5\, \mathrm{photons / second.}}$ [1 mark partly unseen]

But these photons fall on several pixels on the detector (due to diffraction). Number of pixels (N) is given by ratio of areas of airy disk to the pixel scale:
$\mathrm{N} = \pi\, (\theta / 2)^2 / \text{pixel-scale}^2$
$\mathrm{N} = \pi\, (0.07 / 2)^2 / 0.03^2 \underline{= 4.2\, \mathrm{pixels}}$ [1 mark partly unseen]

Hence photon arrival rate is $1.5 / 4.2 = \underline{0.35 \text{ photons / second / pixel}}$ [1 mark partly unseen]

(b iv) First we need to convert from magnitude per square-arcsecond to magnitude per pixel. If the light is spread across N-pixels, then the magnitude per pixel is $\mathrm{m_{per-pixel}} = \mathrm{m_{total}} + 2.5 \log(\mathrm{N})$
Hence 19^{th} mag / sq-arcsec corresponds to $19 + 2.5 \log(1^2 / 0.03^2) = \underline{26.6 \text{ mags / pixel}}$ [1 mark unseen]

In comparison, the planet has is 25^{th} magnitude (total), but light also spread across 4.2 pixels, so magnitude per pixel is $25 + 2.5 \log(4.2) = 26.6$ mags / pixel [1 mark unseen]

Hence sky background and planet have same magnitude per pixel, so photon arrival rate from sky is also 0.35 photons per second.

[Alternatively, students can go through same calculation as in (b iii) to derive same answer].

For background noise (B) to be greater than the read noise (σ_{read}), we require that $Bt > \sigma^2$ so $0.35\,t > 5^2$.
Hence $t > 71.4$ seconds. [1 mark unseen]

(b v) For **total** signal-to-noise, use the **total** photon arrival rate (1.5 photons / second) and $\sigma_{rd}^2 = 0$
$\mathrm{SNR} = St / \sqrt{St + Bt}$ [1 mark seen]
Hence $100 = 1.5\,t / \sqrt{1.5t + 1.5t}$ which gives $\underline{t = 13\,000\,\mathrm{s}\ (\sim 3.7\,\mathrm{hrs})}$ [1 mark unseen]

L2, Stars and Galaxies 2015 exam

David Alexander

June, DMA, Q3

7 short questions:

a) What simple physical model and what physical parameter of the model determines the observed colour of a star? Which two additional physical processes do we need to consider to understand the observed spectral-line properties of a star? [4 marks]

Solution

Stars radiate as (approximate) black bodies. The stellar colour is therefore largely determined by the star's effective temperature.
[1 mark for black body, 1 mark for temperature; Seen]

The additional physical processes that determine the spectral-line properties of stars are excitation and ionisation.
[1 mark for excitation, 1 mark for ionisation; Seen – will not receive a mark for stating either metallicity, temperature, or density since they are not physical processes].

b) The core of a star has the following properties: fully ionised hydrogen, a temperature of $T=10^7$ K, and an average proton density of $n_p=10^{32}$ m^{-3}. What is the dominant form of opacity (κ) in the core of the star? Calculate the mean free path of a photon in the core, assuming $\kappa=0.040$ m^2 kg^{-1}. [4 marks]

Solution

Given these stellar core conditions, electron scattering will dominate the opacity (the temperature is too low for Compton scattering). [1 mark; Seen]

The mean-free path of a photon is given by:

$$\ell = \frac{1}{n_p \sigma} = \frac{1}{\kappa\rho} \quad \textit{[1 mark; Seen]}$$

Since κ is given then just need to calculate the density, which is simply:

$$\ell = \frac{1}{0.040 \times \left(n_p m_p\right)} = \frac{1}{0.040 \times \left(10^{32} \times 1.67 \times 10^{-27}\right)} = 0.00015m \qquad \textit{[2 marks; Unseen]}$$

The same answer can also be obtained using the proton density and the Thomson cross section and the student should receive full marks if they get the correct answer with this approach.

c) The lifetime of the Sun on the main sequence is 10^{10} years. Assuming 26.2 MeV of energy is produced during each helium-fusing chain, what fraction of the mass of the Sun is converted to helium from hydrogen over the main-sequence lifetime of the Sun? [4 marks]

[1 eV = 1.60 x 10^{-19} J]

Solution

The energy release from the Sun in 10^{10} years, assuming a constant luminosity of 3.84 x 10^{26} W, is given by:

$$E = 10^{10} \times 3.16 \times 10^{7} \times 3.84 \times 10^{26} = 1.21 \times 10^{44}\,J$$

The total amount of radiative energy that could be released from the Sun, assuming that all of its mass is converted from hydrogen to helium (which is 4 times the mass of hydrogen) is:

$$E_{total} = \left(\frac{M}{4 \times m_H}\right) \times 26.2\,MeV = \left(\frac{1.99 \times 10^{30}}{4 \times 1.67 \times 10^{-27}}\right) \times 26.2 \times 1.60 \times 10^{-19} \times 10^{6} = 1.25 \times 10^{45}\,J$$

Therefore the fraction of the mass converted during the lifetime of the Sun is simply:

$$f = \frac{E}{E_{total}} = \frac{1.21 \times 10^{44}}{1.25 \times 10^{45}} = 0.097 \qquad \textit{(i.e., ~10\%)}$$

[2 marks for the overall approach in terms of calculating the energy release from the Sun (the student doesn't need to use exactly the same approach as that shown here) and the total amount of energy that could be released from the Sun; 2 marks for the correct answer; Unseen]

d) What determines the minimum possible mass of a main-sequence star? What limits the maximum possible mass of a main-sequence star? Give approximate values for the minimum and maximum masses of main-sequence stars. [4 marks]

Solution

The minimum mass is defined by the core temperature required for nuclear fusion to occur (T~4x10^6 K). [1 mark; Seen]

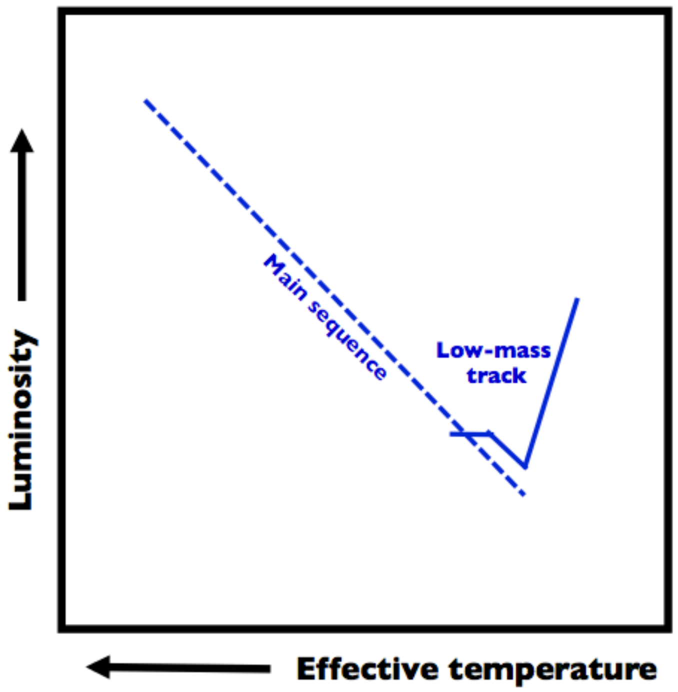
The maximum mass is defined as the point at which the star is unstable due to radiation pressure (which can be estimated from the Eddington limit). High-mass stars have strong radiation pressure which will drive away the outer regions of the star if it exceeds the gravitational pressure.
[1 mark – need to mention radiation pressure and/or Eddington limit; Seen]

The minimum mass of stars is ~0.08 solar masses; however, the mass limit depends on various assumptions and I will except any answer between ~0.05-0.15 solar masses.
[1 mark; Seen]

The maximum mass of stars is ~200 solar masses; however, the limit depends on various assumptions and I will except any answer between ~100-300 solar masses.
[1 mark; Seen]

e) Hayashi tracks trace the paths that protostars take on the Hertzsprung-Russell diagram before joining the main sequence for stars. Draw a Hertzsprung-Russell diagram highlighting the main sequence for stars and the Hayashi track of a low-mass star. [4 marks]

Solution



[1 mark for giving the correct axes and 1 mark for drawing on the correct main sequence. 2 marks for drawing the low-mass track, indicating the key features; Seen]

f) What are the dominant chemical elements of a white dwarf? Describe the physical process that generates the pressure that prevents white dwarfs from collapsing.
[4 marks]

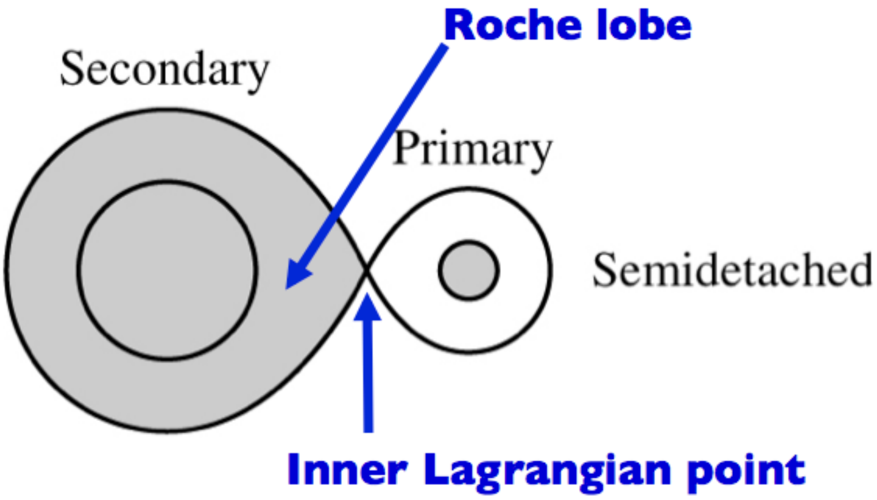
Solution

White dwarfs will be predominantly made of the elements heavier than Hydrogen since they are the degenerate core after nuclear fusion has ceased. Therefore the dominant chemical elements will be Helium, Carbon, and Oxygen. [2 marks – the student will receive 1 mark if they just say that a white dwarf is made of elements heavier than Hydrogen but they need to specify Helium, Carbon, and Oxygen to get 2 marks; Seen]

White dwarfs do not collapse because of "electron degeneracy pressure". Electron degeneracy pressure is the pressure exerted by electrons due to their tight confinement, which gives rise to high momentum (i.e., as expected given the Heisenberg Uncertainty principle and the Pauli Exclusion principle). [2 marks; Seen]

g) Sketch a semi-detached close-binary system, highlighting the principal components of the system. [4 marks]

Solution



[1 mark each for drawing the secondary and primary stars, 1 mark for highlighting the filled Roche lobe, and 1 mark for either drawing the equipotential contours or indicating the inner Lagrangian point; Seen]

L2, Stars and Galaxies 2015 exam

David Alexander

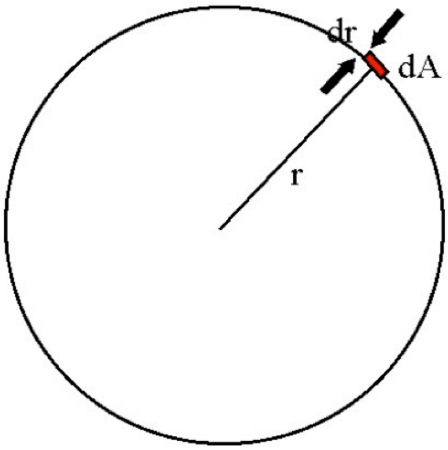
June, DMA, Q4

a) Show that the equation of hydrostatic equilibrium for a spherical mass distribution is given by

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \,,$$

where M_r is the enclosed mass within radius r and ρ is the density at radius r.
[6 marks]

Solution



Pressure force on element dA = [P(r+dr)-P(r)]dA
providing that dr is small, we may write the pressure force = dPdA [2 marks; Seen]

Gravitational pressure on element dA = ρ dr dA g, where for a spherical body:

$$g = \frac{GM_r}{r^2} \qquad \textit{[2 marks; Seen]}$$

For hydrostatic equilibrium, set the pressure force equal to the gravitational force:

$$dPdA = -\rho drdAg$$

and so when substituting for g:

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

[2 marks, Seen]

b) Using this, show that the pressure at the centre of a star with uniform density is given by

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4},$$

where M is the stellar mass and R is the stellar radius. [6 marks]

Solution

Starting from $$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

Assuming a constant density for M_r gives:

$$M_r = \frac{4}{3}\pi r^3 \rho$$

Substituting M_r in the equation of hydrostatic equilibrium, then:

$$\frac{dP}{dr} = -G\frac{4\pi}{3}\frac{\rho^2 r^3}{r^2}$$

$$= -\frac{4}{3}\pi G\rho^2 r$$ *[2 marks; Seen]*

Integrate from the centre to R, and let the pressure at the core be P_c. Then:

$$P = P_c - \frac{2\pi}{3}G\rho^2 r^2$$

Now, when r=R, P=0 (at least P~0 when compared to P_c), so:

$$P_c = \frac{2\pi}{3}G\rho^2 R^2$$ *[2 marks; Seen]*

Substituting for ρ:

$$= \frac{2\pi}{3}GR^2\left(\frac{3M}{4\pi R^3}\right)^2$$

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}$$ *[2 marks; Seen]*

c) Given the derivation in part b, do you expect the central pressure to provide an accurate measurement of the true pressure at the centre of a star? Briefly justify your answer. [3 marks]

Solution

No the central core pressure will not be accurate. It provides a lower limit on the true central pressure of the star because of the assumption of a uniform density throughout the star. The central regions of the star will have a much higher density than the average density of the star.

[3 marks; Seen]

d) Does particle or radiation pressure dominate at the centre of a star with temperature $T=15x10^6$ K, density $\rho=1.5x10^5$ kg m^{-3}, and mean-molecular mass $\mu=0.7$?
[5 marks]

[The radiation constant is a = 7.57 x 10^{-16} J m^{-3} K^{-4}]

Solution

The particle pressure is calculated from:

$$P = nkT = \frac{\rho kT}{\mu m_p} = \frac{1.5\times10^5\times1.38\times10^{-23}\times15\times10^6}{0.7\times1.67\times10^{-27}} = 2.66\times10^{16}\,Nm^{-2}$$

[2 marks; Unseen]

The radiation pressure is calculated from:

$$P = \frac{1}{3}aT^4 = \frac{1}{3}\times7.57\times10^{-16}\times\left(15\times10^6\right)^4 = 1.28\times10^{13}\,Nm^{-2}$$

[2 marks; Unseen]

Particle pressure dominates over radiation pressure and is therefore the dominant form of pressure in the star.

[1 mark; Unseen]

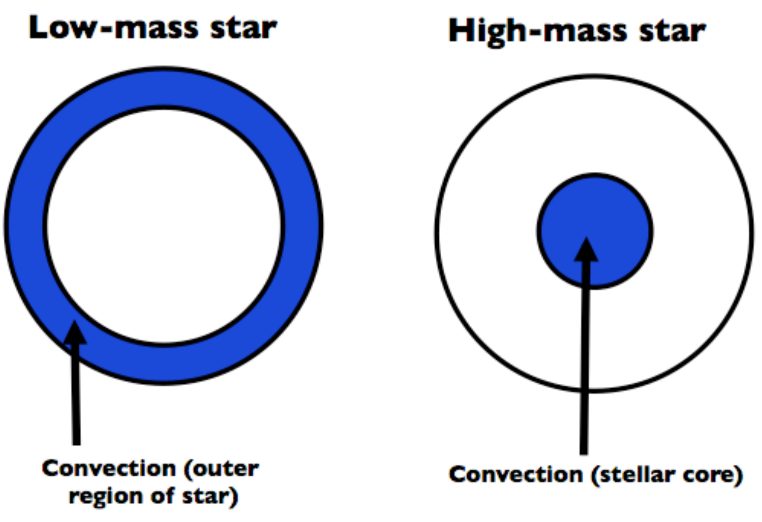
L2, Stars and Galaxies 2015 exam

David Alexander

June, DMA, Q5

(a) The Schwarzschild condition for convection is satisfied in different regions for different masses of stars. Draw cross sections through both a low mass (~1 solar mass) and high mass (~8 solar mass) main-sequence star to highlight the regions where convection is believed to occur in these stars. Why does convection occur in each of these regions? [4 marks]

Solution



Low mass stars - In the surface layers where the opacity increases with increasing temperature, and so radiation can be trapped in the lower layers, resulting in convection. [2 marks; Seen]

High mass stars - Central regions if lots of energy is produced in a small region – this requires a steep temperature gradient to carry the energy, driving convection. [2 marks; Seen]

(b) By considering the conditions required for an adiabatic mass element of material transported upwards in a star to be buoyant, show that the Schwarzschild criterion for convection to occur is $\frac{T}{P}\left(\frac{dP}{dT}\right)_{sur} < \frac{\gamma}{\gamma - 1}$, where P is the pressure, T is the temperature, and γ is the ratio of specific heats. Recall that the adiabatic gas law equation is $P = K\rho^{\gamma}$, where K is a constant and ρ is the density, and that the condition for convection to occur (unstable against convection) is $\left(\frac{dP}{d\rho}\right)_{sur} > \left(\frac{dP}{d\rho}\right)_{ad}$, where the subscript *ad* refers to the conditions of the adiabatic element and *sur* refers to the conditions of the surrounding gas. [8 marks]

Solution:

The mass element (that may convect) is represented by the adiabatic gas equation:

$P = K\rho^{\gamma}$ *rearrange and take the logarithmic differential then:*

$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P}$ *therefore* $\gamma = \frac{\rho}{P}\left(\frac{dP}{d\rho}\right) = \gamma_{ad}$ *[2 marks; Seen]*

As was mentioned the condition for convection to occur is $\left(\frac{dP}{d\rho}\right)_{sur} > \left(\frac{dP}{d\rho}\right)_{ad}$

Multiply by ρ/P on both sides and then re-express the adiabatic differential equation using the ratio of specific heats ($\gamma_{ad}=\gamma$):

$\frac{\rho}{P}\left(\frac{dP}{d\rho}\right)_{sur} > \gamma_{ad}$ *or* $\frac{P}{dP}\left(\frac{d\rho}{\rho}\right)_{sur} < \frac{1}{\gamma_{ad}}$ *i.e.,* $\frac{\rho}{P}\left(\frac{dP}{d\rho}\right)_{ad} = \gamma_{ad}$

[2 marks; Seen]

The pressure of the surrounding gas in the star is described by the ideal gas law:

$P \propto \rho T$

So if we take the logarithmic differential then we express it as

$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$ *or* $\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}$ *[2 marks; Seen]*

Now plugging this back into the original equation and rearranging gives us the condition when convection is prone to occur:

$$\frac{P}{dP}\left(\frac{dP}{P}-\frac{dT}{T}\right)_{sur} < \frac{1}{\gamma_{ad}}$$

$$\frac{P}{T}\left(\frac{dT}{dP}\right)_{sur} > \frac{\gamma_{ad}-1}{\gamma_{ad}}$$

$$\frac{T}{P}\left(\frac{dP}{dT}\right)_{sur} < \frac{\gamma_{ad}}{\gamma_{ad}-1}$$ *[2 marks; Seen]*

(c) Briefly describe the mixing-length theory of convection for stars. [2 marks]

Solution:

In the mixing-length theory of convection for stars, the adiabatic mass element rises and expands until it thermalizes with its environment. The point at which the mass element thermalizes determines the radial extent of the convection cell (the "mixing length").
[2 marks; Seen]

(d) Calculate the radial extent (ℓ) of the convection cells at the surface of the Sun using the following equation $\ell = \alpha H_P$, where the scale height (H_p) satisfies $\frac{1}{H_P} = -\frac{1}{P}\frac{dP}{dr}$ and α is a scaling factor. In your calculation assume that the pressure at the surface of the Sun is P=10^{13} N m^{-2} and that the density is equal to the average density of the Sun. Briefly justify the value of α. [6 marks]

[Hint: the radius of the Sun = 6.96 x 10^8 m]

Solution

$$\ell = \alpha H_P \qquad and \qquad \frac{1}{H_P} = -\frac{1}{P}\frac{dP}{dr}$$

For dP/dr assume the hydrostatic equilibrium equation, where

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

Therefore $\ell = \alpha \frac{r^2 P}{\rho G M_r}$ *[1 mark; Seen]*

The density of the star is simply $\rho = \frac{M}{\left(\frac{4}{3}\right)\pi r^3} = 1410 kgm^{-3}$ *[1 mark; Unseen]*

Therefore:

$$\ell = \alpha \frac{\left(6.96\times10^8\right)^2 \times 10^{13}}{1410\times6.67\times10^{-11}\times1.99\times10^{30}} = \alpha\times2.59\times10^7$$

[1 mark; Unseen]

α is likely to be ~1 – justification: this is the pressure scale height and over that distance the bubble would have expanded by a factor ~e and is therefore likely to have thermalized with the surrounding gas.

[3 marks; Unseen]

Level 2 Paper 4 Question 6

(a) Hubble's classified galaxies along a tuning fork, separating the roundish red galaxies (ellipticals), from the disky blue galaxies (spirals), dividing the latter into barred an non-barred. [2 marks, seen]
Students need to mention or describe **tuning fork**, and the division of spirals in **barred and non-barred** for full marks. This could also be done in the form of a sketch.

- star formation in spirals, not in ellipticals: young massive stars produce bluer light
- metallicty of stars in ellipticals tends to be higher: higher metallicities make stars redder.

[2 marks, seen]

(b) Proton and electron spin in neutral hydrogen may be aligned or anti-aligned, with the aligned state having higher energy. A spin flip from aligned to anti-aligned results in the emission of a 21 cm photon. [2 marks, seen]

Need to specify it is the relative spin of proton and electron that causes the energy difference. Stating it is a hyperfine transitions also gets full marks.

Molecular hydrogen has no hyperfine transition, so molecular gas does not emit 21 cm radiation: hence you cannot study star formation in molecular clouds using 21 cm. [2 marks, unseen]

(c) If spiral arms were material structures (i.e. always composed of the same material), then differential rotation would wind-up spiral arms very rapidly. Since many spirals are not tightly wound, this is not the case. That is the winding problem. [2 marks, seen]

This problem is resolved if the arms are *not* material structures, with stars and gas moving in and out of arms. [2 marks, seen]

(d) The kinetic energy K and potential energy of the cluster U are approximately

$$\begin{aligned} K &= \frac{1}{2}M\sigma^2 \\ U &= \frac{GM^2}{R}\,. \end{aligned}$$

[1 mark, seen]
Combining this with the virial theorem, ($K = U/2$) yields $\sigma^2 = GM/R$. [1 mark, seen]

Substituting numbers yields $M = 1.8 \times 10^{14} M_\odot$. [2 marks, unseen. Need correct rounding] which equals $M = 3.6 \times 10^{44}$ kg. (1 mark for not expressing answer in solar masses)

(e) Gravitational lensing is the deflection of light passing near a massive object[1 mark, seen]
due to the bending of space-time around it. [1 mark, seen]

If the Milky Way's halo consisted of massive compact halo object (MACHO), then if such an object were to pass in front of a star in the Large Magelanic Cloud (LMC) it might lens and hence temporarily amplify the flux received (micro-lensing)[1 mark, seen]

Although lensing was detected, the lens was a star (not a MACHO) - and so the Milky Way's halo does *not* consist of MACHO's. [1 mark, seen]

Level 2 Paper 4 Question 7

(a) Equating acceleration along a circular orbit, $a = V_c^2/R$, to the gravitational acceleration exerted by an enclosed mass M, GM/R^2 yields $V_c^2 = GM/R$. [2 marks, seen]

Substituting numbers yields $M = RV_c^2/G = 9.0 \times 10^{10} M_\odot$ [2 marks, unseen. Need correct rounding for full marks]
which equals 1.8×10^{41} kg (1 mark for providing answer in kilograms rather than solar masses, unseen)

(b) There are several correct ways of demonstrating this. Use the previous expression to find $V_c^2 R = GM$ and take the derivative with respect to R (for constant V_c)
yields $V_c^2 = G dM/dR$ [2 marks, seen]
hence $V_c^2 = 4\pi G R^2 \rho(R)$ or $\rho(R) \propto 1/R^2$ [2 marks, seen]

Showing that $M \propto R$ for a $1/R^2$ density distribution, yielding V_c^2 is constant also gets full marks.

Assuming $M = (4\pi/3)\rho R^3$ and substituting this in $V_c^2 = GM/R$ loses 3 marks.

(c) The light distribution does not follow $\rho \propto 1/R^2$, but there is no evidence of a strong gradient in the properties of stars - so for example discouraging an interpretation in terms of a strong (stellar) mass-to-light gradient. So the stellar mass density does not follow $\rho \propto 1/R^2$ so cannot be responsible for the observed flat rotation curve. [2 marks, seen]

(d) From (b) we find $\rho(R_\odot) = V_c^2/(4\pi G R_\odot^2)$ [1 mark, unseen]
hence $\rho(R_\odot) = 0.014 M_\odot\ \mathrm{pc}^{-3} = 9.5 \times 10^{-22}\ \mathrm{kg\ m}^{-3}$ [1 mark, unseen]

The density in stars is approximately $\rho_\star = 1 M_\odot\ \mathrm{pc}^{-3}$, (values that differ by order of a few are still OK) therefore the ratio $\rho_\star/\rho \sim 70$ [1 mark, unseen. Again, values that differ by order of a few are OK]

The reason that $\rho_\star \gg \rho$ yet the enclosed mass is dominated by dark matter and not stars, is because the stellar density is restricted to a plane, whereas the dark matter is (nearly) spherical. [1 mark, unseen]
Students need to realise that the finding $\rho_\star >> \rho$ is seemingly in contradiction with the rotation curve (enclosed mass) dominated by dark matter.

(e) The motions of stars near the galactic centre strongly suggest the presence of a very dense massive object. [2 marks, seen]

$\dot{M} = L/(0.1c^2) = 6.8 \times 10^{-11} M_\odot\ \mathrm{yr}^{-1}$ [2 marks, unseen. Need correct rounding]
$\dot{M} = 4.3 \times 10^{12}\ \mathrm{kg\ s}^{-1}$ (answer in kg s^{-1} loses 1 mark)

(f) The rate at which the BH emits ionising photons is $\dot{N} = L/E$ [1 mark, unseen]
so $\dot{N} = 1.7 \times 10^{46}\ \mathrm{s}^{-1}$ [1 mark, unseen]

Substituting this in the equation yields $R_S = 0.36$ pc [2 marks, unseen. Need correct rounding]
$R_S = 1.1 \times 10^{16}$ m (answer in meters loses 1 mark)