Mathematical Methods in Physics

Workshop 2

2.1

Consider two dimensional vectors of the form $\mathbf{v}^T = (v_1, v_2)$. Take the inner product of two vectors to be defined as follows

$$\langle \mathbf{v}|\, \mathbf{w}\rangle \equiv \mathbf{v}^\dagger \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{w} = (v_1^*, v_2^*) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 2v_1^* w_1 + v_1^* w_2 + v_2^* w_1 + 2v_2^* w_2.$$

Verify that this inner product does satisfy the requirements for an inner product, i.e.

a) $\langle \mathbf{v}|\mathbf{w}\rangle = \langle \mathbf{w}|\mathbf{v}\rangle^*$,

b) $\langle \mathbf{v}|(\alpha\mathbf{u} + \beta\mathbf{w})\rangle = \alpha\langle \mathbf{v}|\mathbf{u}\rangle + \beta\langle \mathbf{v}|\mathbf{w}\rangle$, note that this computation is a little bit lengthy,

c) $\langle \mathbf{v}|\mathbf{v}\rangle > 0$ if $\mathbf{v} \neq 0$.

2.2

The matrices σ_1, σ_2, σ_3 are defined as follows

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices are called the Pauli matrices and are often used in the context of quantum mechanics.

a) Show that:

i) $(\sigma_i)^2 = I$,

ii) $\sigma_i\sigma_j = i\sigma_k, \quad (i, j, k) = (1, 2, 3) = (2, 3, 1) = (3, 1, 2)$,

iii) $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} I$, where the matrix I is the identity matrix, that is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) Show that

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}\, I + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}),$$

where

$$\boldsymbol{\sigma} = \sigma_1\, \mathbf{i} + \sigma_2\, \mathbf{j} + \sigma_3\, \mathbf{k}$$

and $\mathbf{a}$ and $\mathbf{b}$ are general non-zero vectors. For instance $\mathbf{a} = a_1\, \mathbf{i} + a_2\, \mathbf{j} + a_3\, \mathbf{k}$.

c) Deduce that $(\boldsymbol{\sigma} \cdot \mathbf{a})$ and $(\boldsymbol{\sigma} \cdot \mathbf{b})$ commute if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

2.3

You want to use the following identity

$$|A|\,\epsilon_{lmn} = A_{li}A_{mj}A_{nk}\epsilon_{ijk}, \tag{1}$$

to prove that $|A| = |A^T|$. In order to do so follow these steps:

a) Write the expression (1) for the matrix $|A^T|$.

b) Multiply both sides of your result in part a) by ϵ_{lmn} and pick out the expression (1) for $|A|$ on the right hand side. Notice that the expression $\epsilon_{lmn}\epsilon_{lmn}$ is a scalar (see Workshop 1).

2.4

Two matrices U and H are related by

$$U = e^{iaH},$$

where a is a real scalar.

a) If H is hermitian, show that U is unitary.

b) If U is unitary, show that H is hermitian. (Note that H is independent of a.)

c) Find the time independent differential operator H such that the evolution operator $U = e^{-itH/\hbar}$, for which

$$\psi(x,t) = e^{-itH/\hbar}\psi(x,0),$$

satisfies the time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\,\psi(x,t) + V(x)\psi(x,t).$$