University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 043651/01 or 044231/01

LEVEL 3 PHYSICS: PLANETS AND COSMOLOGY 3
LEVEL 4 PHYSICS: PLANETS AND COSMOLOGY 4

SECTION A. COSMOLOGY
SECTION B. PLANETARY SYSTEMS

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. COSMOLOGY

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Define the Hubble Parameter, $H(t)$, at arbitrary cosmic time t in terms of the expansion factor a and its time derivative. [2 marks]

   A quasar at redshift $z = 2.5$ emits a hydrogen Hα emission line at a rest-frame wavelength of 656.3 nm. What is the observed wavelength of the line? [2 marks]

   (b) Starting from the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$

   where a is the expansion factor, k is a constant and ρ is density, define the present-day critical density in terms of the Hubble Constant, H_0, and fundamental constants and show that for a matter-dominated universe with the critical density, the age of the universe is $(2/3)H_0^{-1}$. [4 marks]

   (c) A spiral galaxy is observed in the 21 cm emission line of neutral hydrogen and found to have a flat rotation curve in its outer parts with a value of ± 200 km s^{-1} (relative to the galaxy's systemic velocity) at radii $r = 10 - 60$ kpc. Assuming spherical symmetry, calculate the mass density in the galaxy's halo at $r = 40$ kpc in units of M$_\odot$ pc^{-3}. [4 marks]

   (d) The present-day ratio of the mass-energy density in the cosmic microwave background (CMB) to the critical density, i.e. the density parameter $\Omega_{\gamma,0}$, is 4.4×10^{-5}. If the density parameter for matter is currently $\Omega_{M,0} = 0.25$ and the CMB temperature is 2.73 K, what was the temperature of the background radiation at the epoch when $\Omega_\gamma = 0.1\Omega_M$? [4 marks]

   (e) Consider a dark energy model with equation of state $p = w\rho c^2$, which obeys the cosmological fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left[\rho + \frac{p}{c^2}\right] = 0,$$

   where p and ρ are the pressure and density of the fluid, respectively. By solving the fluid equation, show that the ratio of the density parameters of dark energy to matter evolves from its present-day value $\Omega_{DE,0}/\Omega_{M,0}$ as $\Omega_{DE}/\Omega_M = (\Omega_{DE,0}/\Omega_{M,0})a^{-3w}$. [4 marks]

   (f) State the definition of the deceleration parameter, q. [2 marks]

   Starting from the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_M + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}$$

   (where a is the expansion factor, ρ_M is the matter density, P is the pressure and $\Lambda \equiv 8\pi G\rho_{\rm vac}$ is the cosmological constant) show that

$$q = \frac{\Omega_M}{2} - \Omega_\Lambda$$

where Ω_M and Ω_Λ are the matter and vacuum energy density parameters. You may assume that the matter is non-relativistic. [2 marks]

(g) Give the definition of angular diameter distance, being careful to state whether lengths are physical or comoving. [2 marks]

A galaxy formation theory predicts that the redshift 2 universe will contain many compact galaxies with a physical diameter of 1 kpc. Assuming that the universe is flat, dominated by non-relativistic matter and that $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, determine the expected angular size of such galaxies in degrees. [2 marks]

[Hint: the co-moving distance to an object at redshift z is

$$r = \frac{c}{H_0}\int_0^{z_1} \frac{dz}{(1+\Omega_{M,0}z)^{1/2}(1+z)}$$

where H_0 is the present-day value of the Hubble parameter and $\Omega_{M,0}$ is the present-day matter density parameter.]

(h) The neutron to proton ratio freezes out when the universe has a temperature of $T = 8 \times 10^9$ K. At this time the neutron-proton ratio is 0.16 and the age of the universe is 1.5 s. Neutrons then decay with a half life of 614 s until the temperature of the universe drops to $T = 1 \times 10^9$ K.

A cosmologist makes the (erroneous) assumption that the energy density is dominated by *non-relativistic matter* at these temperatures. What neutron-proton ratio should they expect when $T = 1 \times 10^9$ K [4 marks]

[Hint: you may assume that the universe has the critical density and that g_*, the effective number of bosonic degrees of freedom, remains constant.]

2. (a) One method of measuring the masses of clusters of galaxies is to use X-ray observations of the temperature and density profiles of the hot gas in the intracluster medium (ICM). Describe briefly two other methods for measuring the masses of galaxy clusters. [6 marks]

(b) A cluster contains an ICM whose gas density, ρ_{gas}, depends on radius, r, as $\rho_{gas} \propto r^{-2}$. Within 1 Mpc the total enclosed mass (gas and dark matter) is 10^{15} M$_\odot$, of which the gas mass fraction is 15 per cent. Calculate the gas temperature (in K) and the electron density (in cm^{-3}) at this radius. Assume spherical symmetry and an isothermal ICM of pure ionized hydrogen in hydrostatic equilibrium, i.e.:

$$\frac{1}{\rho_{gas}}\frac{dP}{dr} = -\frac{GM(<r)}{r^2},$$

where P is the gas pressure and $M(<r)$ is the total mass interior to radius r. [6 marks]

(c) State briefly how measurements of the cluster gas mass fraction may be used to estimate the present-day value of the matter density parameter Ω_M, assuming that the gas mass fraction is equal to the universal baryon fraction. [2 marks]

(d) Measurements of the gas mass fraction (f_{gas}) in clusters can also constrain Ω_Λ, since the inferred f_{gas} for any given cluster depends on the assumed cosmological parameters as $f_{gas} \propto d_A^{1.5}$, where d_A is the angular diameter distance to the cluster, whilst the true gas fraction is not expected to vary between clusters. For a cluster at $z = 0.5$ an observer measures $f_{gas} = 0.14$, assuming a cosmology with $\Omega_M = 0.3$ and $\Omega_\Lambda = 0$. Assuming that the true, intrinsic gas fraction of this cluster is actually the universal value, $f_{gas} = 0.19$, use this information to estimate Ω_Λ. You may assume $\Omega_M = 0.3$ and the following approximation for d_A:

$$d_A = \frac{1}{1+z}\left(\frac{c}{H_0}\right)\left[z - \frac{(1+q_0)}{2}z^2\right],$$

where $q_0 = 0.5\Omega_M - \Omega_\Lambda$. [6 marks]

3. (a) Recent measurements have found that the present-day matter density parameter, $\Omega_{M,0} \sim 0.3$, that microwave background temperature fluctuations are correlated on large angular scales and that the abundance of magnetic monopoles is so low that it is undetectable. Briefly describe why these are seen as problems for the standard big bang model, and explain how they are solved if the universe undergoes a brief period of inflation. [9 marks]

(b) In a universe containing only non-relativistic matter and a constant vacuum energy density the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

where a is the expansion factor, ρ the matter density, $\Lambda \equiv 8\pi G\rho_\Lambda$ is the cosmological constant and k is a constant.

Show that the matter density parameter at some time t can be related to the value at time t_0 by

$$\frac{\Omega_M}{\Omega_{M,0}} = \frac{1}{a}\frac{(1-\Omega_M)}{(1-\Omega_{M,0})}\frac{(\Lambda - 3kc^2)}{(\Lambda a^2 - 3kc^2)}.$$

[4 marks]

(c) Assume that the universe is flat, and that $\Omega_{M,0} = 0.25$. Show that

$$\Omega_M = \left(1 + a^3\frac{1-\Omega_{M,0}}{\Omega_{M,0}}\right)^{-1}$$

and hence determine the expansion factor at which the energy density in non-relativistic matter is nine times the vacuum contribution. [3 marks]

In the future, galaxy formation will cease because matter will become too diffuse. Determine the expansion factor at which this will occur, assuming that galaxy formation requires $\Omega_M > 0.01$. [2 marks]

Using these results, briefly discuss whether we should be surprised to measure similar values for the matter and vacuum energy densities at the present day. [2 marks]

4. (a) Briefly outline the thermal history of the early Universe. Include in your answer the Planck era, the Grand Unification Theory era, the Electro-Weak phase transition, Nucleosynthesis, Quark-Hadron phase transition, and Recombination. Your answer should briefly state the significance of these eras/events. [8 marks]

   (b) The present-day temperature of the cosmic neutrino background is 2 K. Estimate the number density of neutrinos per cubic metre. Assume that there are three species of neutrinos. [4 marks]

   (c) One explanation for the matter-antimatter asymmetry of our Universe is that the decay of a new particle, X, is asymmetric such that,on average, each X decays into one quark and $(1-\epsilon)$ anti-quarks. Assume the mass of the X particle is 10^{14} GeV, that the quark mass is negligible, and that the reactions which create X particles freeze out when the age of the universe is 10^{-36} s and the effective number of bosonic degrees of freedom, $g_* \sim 100$.

   What is the temperature (in K) of the Universe when freeze-out occurs? [3 marks]

   Estimate the number density of X particles at this temperature. You may assume that the effective number of degrees of freedom associated with X particles, $g_X = 4$. [2 marks]

   By the present day, each particle present at the time of freeze out, except for the excess of quarks produced by the decay of X particles, has annihilated to produce one photon. Estimate the ratio of the number density of baryons to photons at the present day if $\epsilon \sim 10^{-6}$. [3 marks]

SECTION B. PLANETARY SYSTEMS

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) Sketch the locations of the constituents of the Solar System, highlighting the main types of objects found at different radii from the Sun. Where are the minor planets predominantly found? What definition do we use to distinguish between planets and minor planets? [4 marks]

   (b) Derive the rocket equation

$$v - v_0 = v_e \ln\left(\frac{m_0}{m}\right)$$

   defining the meaning of all the symbols. [4 marks]

   (c) Using Kepler's equation, $M = E - e\sin E$, where M is the mean anomaly, E is the eccentric anomaly and e is the orbital eccentricity, calculate the eccentric anomaly of Saturn to the nearest hundredth of a degree five Earth years after its perihelion passage. [4 marks]

   [1 Saturnian year = 29.4 Earth years, and the eccentricity of Saturn's orbit is $e = 0.056$.]

   (d) How do P and S seismic waves produced by earthquakes differ in their properties? Explain with the aid of a diagram why the Earth exhibits a P-wave shadow zone even though P-waves can travel through all parts of the Earth. [4 marks]

   (e) For a plane parallel isothermal atmosphere the pressure, p, as a function of altitude h is given by: $p = p_0 \exp(-h/H)$, where H is the pressure scale height. Assuming a constant gravitational acceleration, and that the atmosphere behaves like an ideal gas, calculate the pressure scale height at Venus' surface assuming a pure carbon dioxide (CO_2) atmosphere.

   [Venus has a gravitational acceleration of 8.87 m s^{-2} and temperature of 737 K at its surface. Carbon and Oxygen atoms have masses of 12 and 16 atomic mass units respectively.]

   (f) The *Gaia* mission will potentially detect many extrasolar planets by the astrometric method over the course of its planned 5-year mission. It will have the capability to detect stellar motions with angular semi-major axes as small as 20 micro-arcseconds, although at least two full orbits will be required to be observed for the mission to flag the star as a possible astrometric planet detection. Calculate a lower limit on the mass of a planet detectable by *Gaia* orbiting a $1M_\odot$ star at 100 pc, in Jupiter masses. [4 marks]

   [The mass of Jupiter is $M_{\rm Jupiter} = 9.55 \times 10^{-4} M_\odot$.]

   (g) Describe the main features of the solar nebular theory and list four of the main observed properties of our Solar System that it must explain. [4 marks]

6. (a) Briefly summarise the distribution and characteristics of objects in the Kuiper Belt and Oort Cloud. [3 marks].

   (b) A new comet is detected as it enters the inner Solar System, on its approach to the Sun. Astronomers measure its trajectory and determine it had an aphelion distance of 1.0×10^4 AU, and will reach a perihelion distance of 0.10 AU. Calculate the eccentricity of the orbit to 5 significant figures. How long after this perihelion passage would we have to wait until the next time we see this comet? [6 marks]

   (c) The European Space Agency decides to launch a probe to intercept the comet as it crosses the plane of the ecliptic, at a distance 0.70 AU from the Sun. Fortunately a Hohmann transfer orbit is possible for this manoeuvre, with the probe at perihelion when it intercepts the comet. Sketch this orbit, and calculate the launch date given the comet crosses the plane of the ecliptic on January 22nd 2017. Assuming the probe's rockets are fired when it is far from the Earth, but still orbiting the Sun in the same orbit as the Earth, what is the impulsive velocity change required to place the probe in this transfer orbit? [6 marks]

   (d) The probe passes through the cometary tail as part of its encounter, and measures the mean radius of the spherical dust particles in the tail to be 0.25 μm, and their mean density to be 1000 kg m^{-3}. Calculate a limit on the radiation pressure coefficient, $Q_{\rm pr}$, such that a particle with the mean properties is not ejected from the Solar System by radiation pressure. [5 marks]

7. (a) Give a brief description of how observations of the exterior properties of planetary bodies can be used to constrain models of their interior structure. [6 marks]

(b) A planet of mass M_P and radius R_P consisting of atoms of atomic mass A and atomic number Z has:

$$V_\mathrm{e} = -\alpha \left(\frac{M_\mathrm{P}}{A}\right)^{\frac{4}{3}} \left(\frac{Z^2}{R_\mathrm{P}}\right),$$

$$V_\mathrm{G} = -\gamma \frac{M_\mathrm{P}^2}{R_\mathrm{P}},$$

$$\text{and } T_\mathrm{d} = \beta \left(\frac{Z M_\mathrm{P}}{A}\right)^{\frac{5}{3}} \frac{1}{R_\mathrm{P}^2},$$

where α, β and γ are constants, and $V_\mathrm{e}, V_\mathrm{G}$ and T_d are the total electrostatic energy, gravitational energy and degeneracy energy of the planetary body. Use the Virial Theorem to derive a mass-radius relation for the planet, and hence show the planet has a maximum radius of

$$R_\mathrm{max} = \frac{\beta}{\sqrt{\alpha\gamma}} \frac{Z^{\frac{2}{3}}}{A},$$

and mass at maximum radius of

$$M_\mathrm{max} = \left(\frac{\alpha}{\gamma}\right)^{\frac{3}{2}} \frac{Z^3}{A^2}.$$

[10 marks]

(c) By applying the formula for the Roche radius, r_R, for a small moon of density ρ_m, orbiting a planet of density ρ_P and radius R_P:

$$r_\mathrm{R} = 2.44 \left(\frac{\rho_\mathrm{P}}{\rho_\mathrm{m}}\right)^{\frac{1}{3}} R_\mathrm{P},$$

calculate the closest distance a small moon of mainly water ice composition, and so density $\rho_m = 950$ kg m^{-3}, can approach a gas giant planet of mass M_max, composed mainly of hydrogen ($A = 1, Z = 1$). Give the answer in units of the planet's radius. [4 marks]

[Use the following values: $\beta/\sqrt{\alpha\gamma} = 1.1 \times 10^8$ m, and $(\alpha/\gamma)^{3/2} = 2.0 \times 10^{27}$ kg.]

8. (a) Describe four characteristic physical and/or orbital properties of an extrasolar planet that can be derived from transit studies, and how they are derived. Assume the properties of the planet's star are known perfectly. [4 marks]

(b) The transit time for a planet in a circular orbit is given by

$$t_{\rm tr} = \frac{P}{\pi}\left(\frac{(R_* + R_{\rm P})cos(\delta)}{a}\right),$$

where P is the period, R_* and $R_{\rm P}$ are the stellar and planetary radii respectively, a is the semi-major axis and δ is the latitude of the planet's track across the stellar disc. With the aid of a diagram, derive this equation in the special case of an equatorial transit (i.e. $\delta = 0$). [3 marks]

(c) The *Kepler* observatory has picked up a new transit across the face of an F7V star. It measures a drop in flux of 0.160% (assuming limb darkening is negligible), a transit time of 16.0 hours, and an orbital period of 142 days. Show that the planet's orbit has a semi-major axis of $a = 8.91 \times 10^{10}$ m, and hence calculate the inclination, i, of the orbit of the planet to our line-of-sight, assuming the orbit is circular. Given that follow-up Doppler observations measure a velocity semi-amplitude of 9.34 m s^{-1} from a perfectly sinusoidal radial velocity curve, show that the planet is most likely a gas giant. [8 marks]

(d) Assuming the planet has an albedo $A = 0.5$, demonstrate that the planet is outside the habitable zone for this star. If the semi-major axis of its orbit were a factor of 2 larger, would it be possible for this planet to host lifeforms similar to those we see on Earth? [5 marks]

[The F7V star has a mass $M_* = 1.40 M_\odot$, radius $R_* = 1.49 \times 10^9$ m and effective surface temperature $T_{\rm e} = 6200$ K.]