University of Durham

EXAMINATION PAPER

May/June 2014 Examination code: 042631/01

LEVEL 2 PHYSICS: THEORETICAL PHYSICS 2

SECTION A. Classical Mechanics
SECTION B. Quantum Theory 2

Time allowed: 3 hours

Examination material provided: None

Calculators: The following types **only** may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. CLASSICAL MECHANICS

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) What is a scleronomic constraint? How does it differ from a rheonomic constraint? Give an example of both of these types of constraint. [4 marks]

(b) Euler's equation gives the path $y(x)$ that produces an extreme value of

$$I(f) = \int_{x_1}^{x_2} f\left(y, \frac{dy}{dx}, x\right) dx.$$

State Hamilton's principle and how the calculus of variations problem solved by Euler's equation is relevant for the Lagrangian formulation of mechanics. [4 marks]

(c) Briefly describe the general motion of a lightly damped oscillator driven by a sinusoidal driving force. [4 marks]

(d) What are the normal modes of oscillation for a system of coupled oscillators, and why is it usually important that these oscillations are small? [4 marks]

(e) Using the implicit transformation equations $p = \partial F/\partial q$ and $P = -\partial F/\partial Q$, and the properties of the Poisson bracket of two arbitrary functions A and B, where

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q},$$

determine whether or not the generating function $F = qe^Q$ produces a canonical transformation. [4 marks]

(f) The Coriolis force on a mass m is $\underline{F} = -2m\underline{\omega} \times \dot{\underline{r}}$. What gives rise to this force and what are $\underline{\omega}$ and $\dot{\underline{r}}$? In which direction does the Coriolis force act on the Durham to London train as it heads due south? [4 marks]

(g) Draw views from 3 orthogonal directions of a prolate symmetric top. State in which directions the principal axes lie, and whether or not this choice is unique for your chosen prolate symmetric top. [4 marks]

(h) Euler's equations of motion for a rigid body are

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= N_1, \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= N_2, \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= N_3. \end{aligned}$$

Explain, briefly, which quantities the symbols in these equations represent, making clear what coordinate system is being used. [4 marks]

2. Two particles of mass m_1 and m_2 are connected by a light, inextensible string of length l. The particle with mass m_1 is constrained to move on top of a frictionless horizontal surface. The string passes through a hole at a point in the surface and the second particle is constrained to move vertically in a uniform gravitational field of strength g as the first particle changes its distance from the hole.

   (a) (i) Choose the zero of potential energy to be at distance l below the horizontal surface. In terms of the polar coordinates r and θ defining the position of the mass on the surface with respect to the hole through which the string passes, write down expressions for the kinetic and potential energies of the system. Hence determine the Lagrangian for the system. [4 marks]

   (ii) Explain why the coordinate θ is ignorable, and determine the corresponding constant of the motion, p_θ. What physical quantity does p_θ represent? [3 marks]

   (b) (i) Show that the total energy of the system can be written as

$$E = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + V_{\text{eff}},$$

   where the effective potential

$$V_{\text{eff}} = \frac{p_\theta^2}{2m_1 r^2} + m_2 g r.$$

   [2 marks]

   (ii) Solve for $t(r)$ by quadrature to give an expression for time as a function of radial coordinate. [2 marks]

   (c) Find an expression for the radius, r_{eq}, at which a stable circular orbit exists, and mark this radius on a sketch of the effective potential as a function of the radius. [4 marks]

   (d) Perform a Taylor series expansion of the effective potential about the point of stable equilibrium to second order in $x = r - r_{\text{eq}}$. Hence determine, in terms of m_1, m_2, g and p_θ, the angular frequency with which r oscillates when slightly perturbated away from the stable circular orbit. [5 marks]

3. A pendulum, of length l with a bob of mass m in a uniform gravitational field g, is driven by an external force such that its top has an x coordinate given by the function $x_0(t)$.

   (a) If the pendulum makes an angle θ with the downward vertical, show that the Lagrangian can be written as

   $$L = \frac{m}{2}(\dot{x}_0^2 + 2\dot{x}_0 l\dot{\theta}\cos\theta + l^2\dot{\theta}^2) + mgl\cos\theta.$$

   [4 marks]

   (b) Using the Legendre transformation $H(p_q, q) = p_q\dot{q} - L(q, \dot{q})$, determine the Hamiltonian of the system, $H(p_\theta, \theta)$, for $\theta \ll 1$. p_θ should be accurate to terms linear in θ and $\dot{\theta}$, and $H(p_\theta, \theta)$ accurate to quadratic order in p_θ and θ. Is $H(p_\theta, \theta) = E$, the total energy of the system? [5 marks]

   (c) Assume the small angle approximation from the start and repeat the steps above using the x coordinate of the pendulum bob, i.e. $x = x_0 + l\theta$, to find $L(x, \dot{x})$ and show that

   $$H'(p_x, x) = \frac{p_x^2}{2m} - mgl + \frac{mg}{2l}(x - x_0)^2.$$

   Is $H'(p_x, x) = E$? [5 marks]

   (d) Use Hamilton's equations of motion,

   $$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q},$$

   to find a second order differential equation for x.

   Describe how Green's functions could be used to solve this equation and the main features of the motion in the case when $x_0 = A\sin(\omega_0 t)$, where A is a constant. [6 marks]

SECTION B. QUANTUM THEORY 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Consider a spin-1/2 system in the eigenstate $|\psi\rangle = |1/2,\, z\rangle$ of the spin-operator acting in the z–direction, with eigenvalue $s = +\hbar/2$. Calculate the expectation values of the spin-operators in x– and z–directions of this state, given by $\langle S_x\rangle_{|\psi\rangle}$ and $\langle S_z\rangle_{|\psi\rangle}$ respectively. [4 marks]

   (b) What is the probability of obtaining a value of $\hbar/2$ in a measurement of S_x in (a)? [4 marks]

   (c) When is an operator Hermitian and when is it unitary? Consider a Hermitian operator $\hat{H}$ and show that the operator $\hat{U} = e^{i\hat{H}}$ is unitary. [4 marks]

   (d) State the commutation relation between position and momentum operators in one dimension. Use this to calculate the commutator $[\hat{p},\, \hat{x}^m]$ through recursion. [4 marks]

   (e) Show that

$$\left\langle jm \left| \left[\left(\Delta \hat{J}_x\right)^2 + \left(\Delta \hat{J}_y\right)^2 \right] \right| jm \right\rangle = \hbar^2 (j^2 + j - m)\,,$$

   where $\langle (\Delta \hat{O})^2 \rangle$ is the squared uncertainty of an operator $\hat{O}$. [4 marks]

   (f) Two systems (1) and (2) in the states characterised by $|j^{(1)},\, m^{(1)}\rangle$ and $|j^{(2)},\, m^{(2)}\rangle$ are combined to form a new system in the state $|J,\, M\rangle$. What ranges of values are allowed for the quantum numbers J and M of the combined system, if the quantum numbers $m^{(1)}$ and $m^{(2)}$ are fixed? [4 marks]

   (g) The Hamiltonian of a spin-1/2 particle in a magnetic field B in the z–direction is given by

$$\hat{H} = \omega \hat{S}_z\,,$$

   where ω is the Larmor frequency, and depends on the magnetic field. Determine the equations of motion for $\hat{S}_x$ and $\hat{S}_y$ in the Heisenberg picture. [4 marks]

5. A rigid rotator is subjected to a uniform magnetic field oriented along the z-axis, $\underline{B} = B_0\underline{\hat{e}_z}$. Its Hamiltonian is given by

$$\hat{H} = \frac{\hat{\underline{L}}^2}{2I} + \omega_0 \hat{L}_z \,,$$

where the Larmor frequency ω_0 and the moment of inertia I are constants. At time $t = 0$, the system wavefunction is given by

$$\langle \theta,\, \phi | \psi(0) \rangle = \sqrt{\frac{3}{4\pi}} \sin\theta \sin\phi \,.$$

(a) Express $\langle \theta,\, \phi | \psi(0) \rangle$ in spherical harmonics. [4 marks]

[Relevant spherical harmonics are given by

$$\begin{aligned}
Y_{00}(\theta,\, \phi) &= \sqrt{\frac{1}{4\pi}} \\
Y_{1\pm1}(\theta,\, \phi) &= \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} \\
Y_{10}(\theta,\, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta \\
Y_{2\pm2}(\theta,\, \phi) &= \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi} \\
Y_{2\pm1}(\theta,\, \phi) &= \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\phi} \\
Y_{20}(\theta,\, \phi) &= \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1).
\end{aligned}$$

Normalisation:

$$\int_{-1}^{1} \mathrm{d}\cos\theta \int_{0}^{2\pi} \mathrm{d}\phi Y_{lm}(\theta,\, \phi) Y^*_{l'm'}(\theta,\, \phi) = \delta_{ll'}\delta_{mm'}.$$
]

(b) Calculate the energy eigenvalues and corresponding eigenstates of $\hat{H}$. [2 marks]

(c) Compute the energy expectation value of the system in the state $|\psi(0)\rangle$. [3 marks]

(d) Calculate the time dependence of the state, i.e. $\langle \theta,\, \phi | \psi(t) \rangle$. [3 marks]

(e) Calculate the time evolution of the expectation value of $\hat{L}_x$ and of $\hat{L}_x^2$ with respect to this state. [8 marks]

[Hint: Use the ladder operators $\hat{L}_\pm$ and express the $\hat{L}_x$ through them. Also remember the matrix elements of the ladder operators, namely

$$\langle lm' | \hat{L}_\pm | lm \rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)} \,.$$
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6. Consider the Hamiltonian of the fermionic harmonic oscillator

$$\hat{H} = \epsilon \hat{N} = \epsilon \hat{b}^\dagger \hat{b}\,,$$

where ϵ is a positive parameter with units of energy and where

$$\hat{b}^\dagger \hat{b} + \hat{b}\hat{b}^\dagger \;=\; 1 \quad \text{and} \quad \hat{b}^2 = \left(\hat{b}^\dagger\right)^2 = 0\,.$$

(a) Show that $\hat{N}^2 = \hat{N}$ and that $\hat{N}$ is Hermitian. What does hermiticity imply for the eigenvalues of an operator? Determine the eigenvalues of $\hat{N}$. [10 marks]

(b) What are the eigenvalues and eigenstates of $\hat{H}$? [1 mark]

(c) Calculate the commutators $[\hat{N},\, \hat{b}]$ and $[\hat{N},\, \hat{b}^\dagger]$. [4 marks]

(d) Assume $|0\rangle$ to be the non-degenerate ground state, i.e. the eigenstate of $\hat{N}$ with the smallest eigenvalue. Using the commutation relations in (c), show that also $\hat{b}^\dagger|0\rangle$ is an eigenstate of $\hat{N}$ and show that $\hat{b}|0\rangle = 0$. [4 marks]

(e) What do the findings of (a)-(d) imply for the spectrum of this Hamiltonian? [1 mark]