ELECTROMAGNETISM - Workshop 2nd Set (Qns)
Superposition and symmetry

Professor D P Hampshire – 2nd Year Physics Lecture Course

The material for this examples class/workshop is split into three parts. Part I: Background material. Part II: contains worked examples. Please make sure you can answer worked examples in Part II from scratch without reference to the worked solutions. Part III gives some additional unseen questions.

Workshops - there are 3 functions for the examples classes:

i) Homework - Please make sure to bring a copy of your homework problem sheet and your marked homework scripts with you to the workshops/examples classes. We can go through the model solutions.

ii) Workshop/examples class problems – A dedicated sheet of problems will be provided for you to attempt during each workshop.

iii) Office hours: This lecture course is for 200-250 students. In first instance please use the workshop/examples class as my office hour – be brave, put your hand up and I can come over and chat with you about electromagnetism/homework/Judo/life/… Please ask the workshop support staff and your friends about difficult parts of the course. If all else fails – mention to me at a workshop that you have exhausted all reasonable resources/usual suspects and would like to (bring some friends who are also stuck and) chat in my office. (By appointment Friday 5.00 p.m. – Rm. 143).

1 Background material

Superposition plays a key role in Physics. Many problems can be very significantly simplified by using superposition – for example by considering a complex system as consisting of a collection of simple systems and then adding the contributions from the simple systems to find a total equal to that of the complex system.

1.1 Superposition

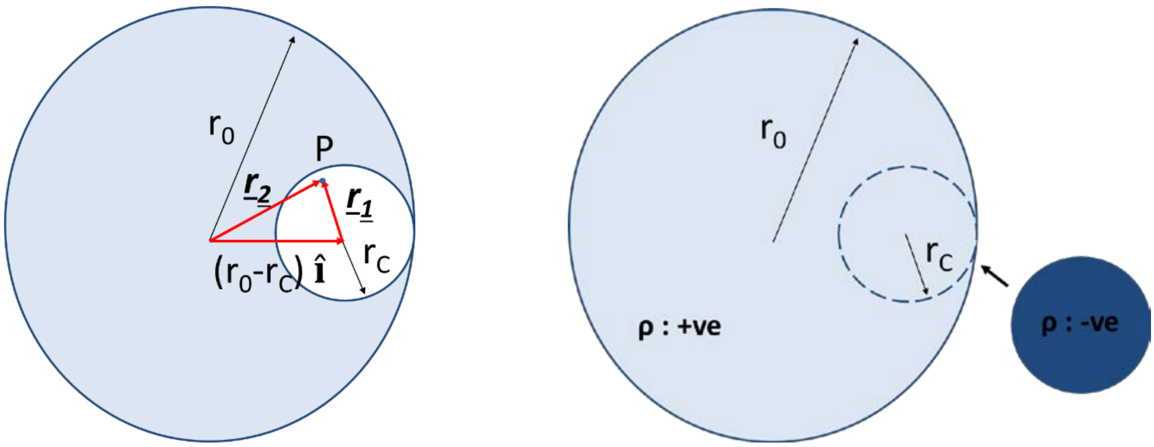


Figure 1 : A sphere of uniform charge density which has had a smaller cavity hollowed out of it.

Using superposition, we can find the field anywhere in a sphere of uniform charge density that includes a hollowed out spherical cavity. The hollowed out sphere on the LHS has the same charge distribution as the superposition of the charges from the two spheres on the RHS. Because both configurations have the same charge distribution, they must also produce same electric field everywhere. We can use simple analytic solution to find the vector additions of E-field produced by the two spheres (RHS) much more easily than that of the single hollowed out sphere (LHS).

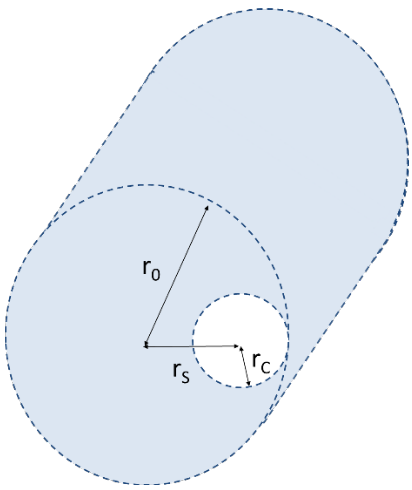


Figure 2 : A wire with uniform current density flowing along it which has had a cylinder drilled out of it. The separation between the axis of the hole and the axis of the original cylinder is r_s.

One can think about solving other simple analytic shapes with different current density and charge distributions.

1.2 High symmetry analytic shapes

There are many systems of interest where at least one of the objects of interest can be quite well approximated by something of high symmetry - a circle or a sphere, a cylinder or a square, a cube or a rectangle. By considering such high symmetry objects, one often finds analytic solutions that are useful to compare with say more precise computational solutions. Consideration of such objects can also provide material for examination questions so students must be au fait with using the calculus necessary to analyse such systems. The three examples below provide the type of mathematical framework that is often required in examinations:

Questions:

Given that the circumference of a circle is $2\pi r_0$, where r_0 is the radius, show that:

i) the area of a circle is πr_0^2.

ii) the surface area of a sphere $4\pi r_0^2$.

Given that the surface area of a sphere is $4\pi r_0^2$, where r_0 is the radius, show that:

iii) the volume of a sphere is $\frac{4}{3}\pi r_0^3$

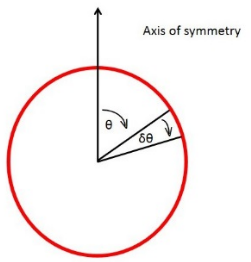
Given that the circumference of a circle is $2\pi r_0$, where r_0 is the radius, show that:

iv) the volume of a sphere is $\frac{4}{3}\pi r_0^3$

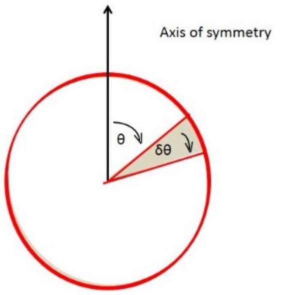
↓...

i) Given that the circumference of a circle is $2\pi r_0$, where r_0 is the radius, show that the area of a circle is πr_0^2.

Worked solution: Start by drawing a circle. Then add the axis that defines $\theta = 0$ and provides an axis of symmetry.



Shade in the differential element that will be integrated (or summed) to cover the entire surface of the circle.



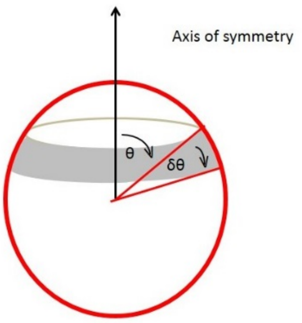
Then using the standard approach from calculus, the elemental area (δA) is given by:

$$\delta A = \frac{r_0 \mathrm{d}\theta}{2} r_0 = \mathrm{r}_0^2 \mathrm{d}\theta/2. \qquad \text{1-1}$$

and integrating we find as required:

$$A = \int_0^{2\pi} \mathrm{r}_0^2 \frac{\mathrm{d}\theta}{2} = \pi \mathrm{r}_0^2. \qquad \text{1-2}$$

ii) Given that the circumference of a circle is $2\pi r_0$, where r_0 is the radius, show that the surface area of a sphere $4\pi \mathrm{r}_0^2$.
Again, a clear drawing is essential:



In this case, the elemental area is of width $r_0 \mathrm{d}\theta$ and of length $2\pi r_0 \sin(\theta)$ so:

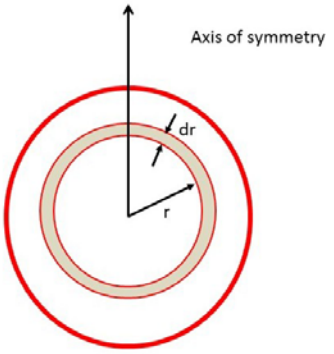
$$\delta \mathrm{A} = r_0 \mathrm{d}\theta \cdot 2\pi r_0 \sin(\theta). \qquad \text{1-3}$$

Hence integrating we find as required:

$$A = \int_0^{\pi} 2\pi \mathrm{r}_0^2 \sin(\theta) \mathrm{d}\theta = [-2\pi \mathrm{r}_0^2 \cos(\theta)]_0^{\pi} = 4\pi \mathrm{r}_0^2. \qquad \text{1-4}$$

the surface area of a sphere
↑...

iii) Given that the surface area of a sphere is $4\pi \mathrm{r}_0^2$, where r_0 is the radius, show that the volume of a sphere is $\frac{4}{3}\pi \mathrm{r}_0^3$



This gives the elemental volume (δV) to be:

$$\delta V = 4\pi r^2 \cdot \mathrm{dr}. \qquad 1\text{-}5$$

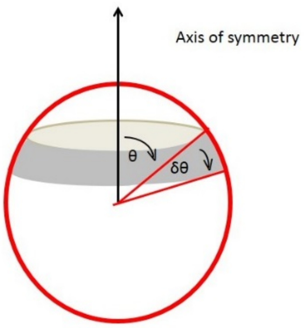
Integrating so the entire sphere is mapped out by summing all the elemental volumes gives:

$$V = \int_0^{r_0} 4\pi r^2 \cdot \mathrm{dr}. \qquad 1\text{-}6$$

So we find as required:

$$V = \frac{4}{3}\pi \mathrm{r}_0^3. \qquad 1\text{-}7$$

iv) Given that the that the area of a circle is πr_0^2 where r_0 is the radius, show that the volume of a sphere is $\frac{4}{3}\pi r^3$



We have already shown that the area of a circle is $\pi \mathrm{r}_0^2$. Producing a clear drawings is essential. In this case the elemental volume (shown above) is a disk of thickness[1] $r_0 \sin\theta \cdot \mathrm{d}\theta$ and radius $r_0 \sin\theta$. This gives the elemental volume (δV) to be:

$$\delta V = \pi (r_0 \sin\theta)^2 \cdot r_0 \sin\theta \cdot \mathrm{d}\theta. \qquad 1\text{-}8$$

Integrating so the entire sphere is mapped out by summing all the elemental volumes gives:

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[1] Note that the length of the curved surface of the elemental disk is longer than the thickness of the disk. This is particularly so near the poles. The length of the curved surface is $r_0 \cdot \mathrm{d}\theta$ and the (smaller) thickness of the disk is $r_0 \sin\theta \cdot \mathrm{d}\theta$.

$$V = \pi r_0^3 \int_0^{\pi} \sin^3\theta d\theta. \qquad 1\text{-}9$$

The integral is not straightforward but using the trigonometric identity:

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta). \qquad 1\text{-}10$$

leads to,

$$\int_0^{\pi} \sin^3\theta d\,\theta = \frac{1}{4}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right]_0^{\pi} = \frac{1}{4}((3 - -3) + (\frac{-1}{3} - \frac{1}{3})) = \frac{4}{3}. \qquad 1\text{-}11$$

So we find as required:

$$V = \frac{4}{3}\pi r_0^3. \qquad 1\text{-}12$$

2 Worked examples

2.1 Questions

1. Write down the Gauss' Law in integral form.

2. Consider a uniformly charged non-conducting sphere of radius r_0 with centre at the origin and volume charge density ρ.

a) Show that at a point within the sphere a distance r from the centre, $\underline{\boldsymbol{E}} = \frac{\rho}{3\varepsilon_o} r\hat{\mathbf{r}}$. [3 marks]
b) Material is removed from the sphere with radius r_0 to produce a spherical cavity of radius r_c with its centre at $(r_0 - r_c)\underline{\hat{\mathbf{i}}}$ along the x-axis as shown in Figure 1. Calculate the electric field,
(i) at the edge of the cavity furthest from the centre of the sphere.
(ii) at the edge of the cavity nearest the centre of the sphere.
[Hint: Using superposition theorem]
c) Show that the electric field throughout the cavity is uniform.

2.2 Answers

1. Gauss' Law:

$$\oint_S \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{S}} = \frac{Q_{Enclosed}}{\varepsilon_0}$$

2. a) $Q_{Enclosed} = \rho\, {}^4\!/_3\, \pi r^3$

Flux= $\oint_S \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{S}} = E 4\pi r^2$

$$E4\pi r^2 = \rho\, {}^4\!/_3\, \pi r^3 \frac{1}{\varepsilon_0}$$

$$=> \underline{\boldsymbol{E}} = \frac{\rho}{3\varepsilon_0} r\underline{\hat{\boldsymbol{r}}}$$

b) (i) at the edge of the cavity furthest from the centre of the sphere:

Use superposition theorem:
Let $\underline{\boldsymbol{E}}_B$ be the $\underline{\boldsymbol{E}}$-field due to the big sphere, and $\underline{\boldsymbol{E}}_S$ the field of the sphere that fills the cavity.

$$\underline{\boldsymbol{E}}_1 = \underline{\boldsymbol{E}}_B - \underline{\boldsymbol{E}}_S$$
$$= \frac{\rho}{3\varepsilon_0} r_0 \underline{\hat{\mathbf{i}}} - \frac{\rho}{3\varepsilon_0} r_C \underline{\hat{\mathbf{i}}}$$
$$= \frac{\rho}{3\varepsilon_0}(r_0 - r_C)\underline{\hat{\mathbf{i}}}$$

(ii) at the edge of the cavity nearest to the centre of the sphere:
Use superposition again:

$$\underline{\boldsymbol{E}}_2 = \underline{\boldsymbol{E}}_B - \underline{\boldsymbol{E}}_S = \frac{\rho}{3\varepsilon_0}(r_0 - 2r_C)\underline{\hat{\mathbf{i}}} + \frac{\rho}{3\varepsilon_0} r_C \underline{\hat{\mathbf{i}}} = \frac{\rho}{3\varepsilon_0}(r_0 - r_C)\underline{\hat{\mathbf{i}}}$$

c) Consider the point P inside the cavity in Figure 1. Using superposition,

$$\underline{\boldsymbol{E}}_P = \frac{\rho}{3\varepsilon_0}\underline{\boldsymbol{r}}_2 - \frac{\rho}{3\varepsilon_0}\underline{\boldsymbol{r}}_1$$

From the diagram, $\underline{\boldsymbol{r}}_2 - \underline{\boldsymbol{r}}_1 = (r_0 - r_C)\underline{\hat{\mathbf{i}}}$
=> $\underline{\boldsymbol{E}}_P = \frac{\rho}{3\varepsilon_0}(r_0 - r_C)\underline{\hat{\mathbf{i}}}$. Since this is true for any point P, the electric field in the hollow sphere is uniform

3 Unseen problems

1. (a) Twelve equal charges, q, are situated at the corners of a regular 12-sided polygon. The distance from each charge to the centre of the polygon is a_0. Find the net electric field at the centre? Use the convention that if the $\underline{\boldsymbol{E}}$-field points radially out from the centre it is positive and vice versa.
(b) If one of the 12's q is removed. What is the net electric field at the centre?
(c) What would the net electric field at the centre be if there were 13 charges, distributed evenly on a 13-sided polygon?
(d) If one of the 13 charges is removed. What is the net electric field at the centre?

2. The exponential decrease in the charge density of a spherical cloud of positive ions is of the form: $\rho(r) = \rho_0 \exp(-r^3/\lambda^3)$ out to its radius R. Show that the total charge in the cloud is given by $Q_{Total} = \frac{4\pi}{3}\rho_0 \lambda^3(1 - \exp(-R^3/\lambda^3)$. Find an expression for the spatial variation of the electric field inside the spherical cloud in terms of Q_{Total}?

3. A infinitely long conducting rod with circular cross-section has a conductivity that varies radially in the form $\sigma_n(r) = \sigma_0\left\{1 - \frac{r}{r_0}\right\}$ out to its radius r_0. Find an expression for the spatial dependence of the magnetic field inside the rod in terms of the total current I_{Total} that flows through the rod.

4.

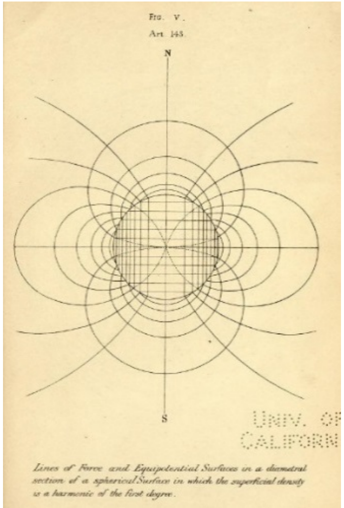

Figure 3 : Maxwell plotted the E-field lines produced by a uniformly polarised spherical dielectric in an applied field, $\underline{\boldsymbol{E}}_{\text{applied}}$. https://archive.org/stream/electricandmagne01maxwrich#page/n475/mode/2up

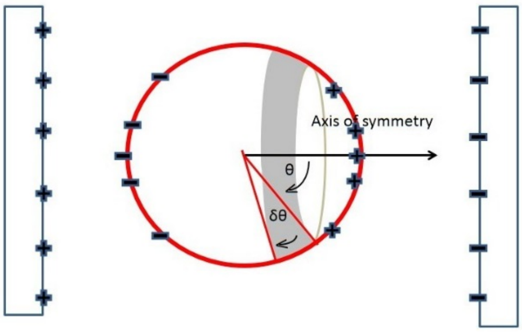

Figure 4 : A uniformly polarised spherical dielectric with a polarisation $\underline{\boldsymbol{P}}$ in an applied field $\underline{\boldsymbol{E}}_{\text{applied}}$. The surface charge density is $\sigma = \underline{\boldsymbol{P}} \cdot \hat{\mathbf{n}} = \mathrm{P}\cos\theta$

Consider a uniformly polarised spherical dielectric with a polarisation $\underline{\boldsymbol{P}}$ in an applied field $\underline{\boldsymbol{E}}_{\text{applied}}$. The surface charge density is $\sigma = \underline{\boldsymbol{P}} \cdot \hat{\mathbf{n}} = \mathrm{P}\cos\theta$. Show that the net $\underline{\boldsymbol{E}}$-field at the centre of the sphere is given by:

$$\underline{\boldsymbol{E}}_{\text{centre}} = \underline{\boldsymbol{E}}_{\text{applied}} - \frac{\underline{\boldsymbol{P}}}{3\varepsilon_0}$$

5. An infinitely long conducting rod with a circular cross–sectional area carries 10 A when an electric field of 10 $\mathrm{Vm^{-1}}$ is applied to it. A scientist drills a long hole close to the surface of the rod along the rod's entire length that removes 20% of the material in the rod as shown in Figure 2. The axis at the centre of the hole is r_S from the axis along the centre of the original rod. Show that the change in magnetic field at the centre of the rod when 10 $\mathrm{Vm^{-1}}$ is applied to it is given by $B = \mu_0/\pi r_S$.

