University of Durham

EXAMINATION PAPER

May/June 2017 Examination code: PHYS3621-WE01

FOUNDATIONS OF PHYSICS 3A

SECTION A. Quantum Mechanics 3
SECTION B. Nuclear and Particle Physics

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A. QUANTUM MECHANICS 3

Answer Question 1 and **at least one** of Questions 2 and 3.

1. (a) (i) A hydrogen atom is placed in a time-dependent homogeneous electric field $\underline{\mathcal{E}}(t)$ oriented in the positive z-direction. State the potential energy imparted to the atomic electron by this electric field. [3 marks]

   (ii) What is the sudden approximation? [1 mark]

   (b) A time-independent Hamiltonian H_0 is perturbed by a time-dependent Hamiltonian $H'(t)$ such that $H'(t) = 0$ for $t < 0$ and $H'(t) = \hat{H}' \sin \omega t$ for $t \geq 0$, where neither $\hat{H}'$ nor ω depend on time. Suppose that $H_0 + H'(t)$ is the total Hamiltonian of a 1D system and that this system is in the eigenstate a of H_0 for $t < 0$. The rate of transition to continuum eigenstates of H_0 is given by Fermi's Golden Rule as $(2\pi/\hbar)\, \rho_{1\mathrm{D}}(E_\mathrm{f})\, |A^\dagger_{k_\mathrm{f}a}|^2$, where $\rho_{1\mathrm{D}}(E_\mathrm{f})$ is the density of states at the energy E_f.

   (i) How is the operator $A^\dagger$ related to $H'(t)$? [1 mark]

   (ii) How is the wave number k_f related to the energy of state a and to $\hbar\omega$ if $E_\mathrm{f} = \hbar^2 k_\mathrm{f}^2/(2m)$, where m is a mass? [2 marks]

   (iii) Define the matrix element $A^\dagger_{k_\mathrm{f}a}$ in terms of eigenfunctions of H_0. [1 mark]

   (c) Which of the following states of atomic hydrogen, if any, are allowed to decay spontaneously to the $(n = 1, l = 0, m = 0)$ state by emission of a single photon, according to the electric dipole selection rules?

(i)	$(n = 3, l = 0, m = 0)$	(ii)	$(n = 3, l = 1, m = 0)$
(iii)	$(n = 3, l = 1, m = 1)$	(iv)	$(n = 3, l = 2, m = 0)$
(v)	$(n = 3, l = 2, m = 1)$	(vi)	$(n = 3, l = 2, m = 2)$

   For each forbidden transition, specify a rule which forbids the transition. [4 marks]

   (d) Outline the principles of the Rayleigh-Ritz variational method for the calculation of the ground state energy of a quantum system. [4 marks]

   (e) (i) Neglecting spin-orbit coupling and other relativistic effects, and assuming that the nucleus has infinite mass, write down the Hamiltonian of helium. [2 marks]

   (ii) Explain why the product $u_{1\mathrm{s}\uparrow}(q_1)u_{1\mathrm{s}\downarrow}(q_2)$ could *not* represent the ground state wave function of helium, even approximately. (The functions $u_{1\mathrm{s}\uparrow}(q_1)$ and $u_{1\mathrm{s}\downarrow}(q_2)$ represent one-electron spin-orbitals of, respectively, spin up and spin down, and q_1 and q_2 denote the spatial and spin variables of, respectively, electron 1 and electron 2.) [2 marks]

   (f) Write down a linear combination of two 1s atomic orbitals which can be taken as an approximate, unnormalized molecular orbital describing a gerade state of the molecular ion H_2^+. Define, in a few words, all the symbols appearing in your equation, and explain (briefly) why this molecular orbital is bonding rather than antibonding. [4 marks]

2. A cloud of ^{202}Hg atoms, in thermal equilibrium at temperature T, is confined inside a cavity. These atoms may be in the ground state, in the $6^3\mathrm{P}_1$ or $6^3\mathrm{P}_2$ excited states, or in excited states irrelevant for this problem.

   (a) Given that the $6^3\mathrm{P}_1$ state can decay only to the ground state, that the wavelength of the photon emitted in this transition is 254 nm, and that the radiative lifetime of this excited state is 120 ns (irrespective of the value of the magnetic quantum number), what is the value of the Einstein B-coefficient B_{ab}? (Express your result in units of m^3 J^{-1} s^{-2}.) [5 marks]

   [Hint: $A_{ab} = \hbar\omega_{ba}^3 B_{ab}/(\pi^2 c^3)$, where ω_{ba} is the Bohr angular frequency and A_{ab} is the A-coefficient for the transition.]

   (b) An observer recording a spectrum of the photons emitted in that transition would find that the wavelength is distributed over a narrow range centered at 254 nm. What are the line broadening mechanisms you would expect to contribute to this line width, besides experimental errors? [3 marks]

   (c) Determine the temperature of the cloud if the rate of stimulated emission for the $6^3\mathrm{P}_1$ state is smaller than the rate of spontaneous emission by a factor of 3×10^{-4}. [7 marks]

   [Hint: The energy density distribution per unit angular frequency range is, here,
   $$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1}.$$]

   (d) The energies of the $6^3\mathrm{P}_1$ and $6^3\mathrm{P}_2$ states are, respectively, -5.55 eV and -4.98 eV. The cloud is at the temperature you calculated in part (c). Determine the ratio of the populations in energy levels $6^3\mathrm{P}_1$ and $6^3\mathrm{P}_2$. [5 marks]

   [Hint: ^{202}Hg atoms have no hyperfine structure. The degeneracy of the energy levels is $2J + 1$, where J is the corresponding total angular momentum quantum number.]

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3. In this question, S_z denotes the z-component of $\underline{S}$, the total spin operator: $\underline{S} = \underline{S}_1 + \underline{S}_2$, where $\underline{S}_1$ is the spin operator of electron 1 and $\underline{S}_2$ the spin operator of electron 2. The x-, y- and z-components of $\underline{S}_1$ are S_{1x}, S_{1y} and S_{1z}, and similarly for $\underline{S}_2$. Moreover, $\alpha(1)$ and $\beta(1)$ are, respectively, the spin-up and spin-down eigenstates of S_{1z}, and $\alpha(2)$ and $\beta(2)$ are the corresponding eigenstates of S_{2z}.

   (a) Show that $S_z\alpha(1)\beta(2) = 0$. [5 marks]

   (b) Briefly explain why $\alpha(1)\beta(2)$ is orthogonal to $\beta(1)\alpha(2)$. [3 marks]

   (c) Find an eigenstate of S_x of the form of a linear combination of $\alpha(1)\alpha(2)$ and $\beta(1)\beta(2)$ (the norm of this eigenstate does not need to be 1). [6 marks]

   [Hints:
   (i) Look for an eigenstate with a zero eigenvalue.
   (ii) $S_{1x}\alpha(1) = (\hbar/2)\beta(1)$, $S_{1x}\beta(1) = (\hbar/2)\alpha(1)$, and similarly for electron 2.]

   (d) An atom of helium, initially in the $\alpha(1)\alpha(2)$ spin state, is exposed to a magnetic field oriented in the x-direction which is turned on at $t = 0$. If one neglects spin-orbit coupling and other relativistic effects,

      (i) could this perturbation induce a transition from this initial state to a singlet state?

      (ii) could it induce a transition to a singlet state if the atom was initially in some triplet spin state other than $\alpha(1)\alpha(2)$?

   (Justify your answer. Do not go beyond the first order of time-dependent perturbation theory.) [6 marks]

SECTION B. NUCLEAR AND PARTICLE PHYSICS

Answer Question 4 and **at least one** of Questions 5, 6, 7 and 8.

4. (a) Sketch the area of the $N-Z$ plane where naturally occurring isotopes lie. N is the number of neutrons and Z the number of protons. Identify on the sketch the following regions: [4 marks]

   – stable nuclei
   – nuclei undergoing β^+ decays
   – nuclei undergoing β^- decays
   – nuclei undergoing α decays.

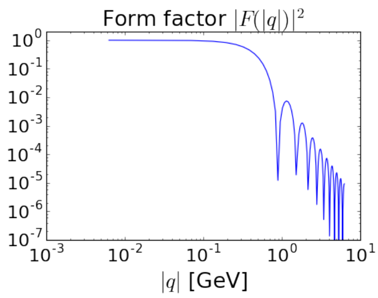
   (b) A nucleus of mass M decays into two nuclei of equal mass m. Express the momentum of the daughter nuclei in the rest frame of the decaying nucleus as a function of the ratio $\nu = m/M$. [4 marks]

   (c) What are the spin and parity predicted by the shell model for $^{37}_{20}$Ca? (The order of the lowest shells in the shell model for protons and neutrons is $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, ...) [4 marks]

   (d) Explain the origin of the first two terms of the semi-empirical mass formula

$$B(A,Z) = a_V A - a_s A^{2/3} + \dots \qquad [4 \text{ marks}]$$

   (e) Explain the significance of the form factor and estimate the spatial size of the charge distribution whose form factor is given below. [4 marks]



[Hint: $\hbar c = 0.197\,\text{GeV fm}$]

   (f) Do parity and angular momentum conservation allow the following decays? If yes give all possible values of the orbital angular momentum of the final state wave-function.

   (i) $\rho^0 \to e^+e^-$
   (ii) $\rho^+ \to \rho^0\pi^+$
   (iii) $\pi^0 \to \rho^0 f_2$
   (iv) $f_2 \to \rho^+\pi^-$

   where $\rho^{+,0,-}$ and ω are vector mesons with $J^P = 1^-$, the pions $\pi^{+,0,-}$ have $J^P = 0^-$. The f_2 meson has $J^P = 2^+$. [4 marks]

   (g) Explain why there are two different $J = 1/2$ baryons with quark content uds (the Ω^0 and Σ^0) but only one $J = 3/2$ baryon (the $\Sigma^{*,0}$) with the same quark content. [4 marks]

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(h) Explain why the charged pions are more likely to decay into a muon and muon-antineutrino pair rather than an electron and electron-antineutrino pair. [4 marks]

(i) For each of the following reactions give all single particles X that are allowed in the reaction in the Standard Model and and if there is more than one possibility explain which is most likely: [4 marks]

- $\mu^- \to e^- \nu_\mu X$
- $W^- \to X d$
- $t \to b \nu_e X$.

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5. $^{51}_{22}$Ti nuclei decay through β^- decay to excited states of $^{51}_{23}$V. These in turn relax to lower states through the emission of photons.

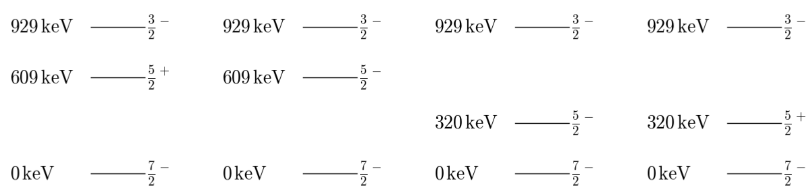
   a) The spin and parities of the nuclei are $3/2^-$ for $^{51}_{22}$Ti and $7/2^-$ for $^{51}_{23}$V. Calculate the shell model prediction for the spin and parity of these nuclei and compare with the experimental values. (The order of the lowest shells in the shell model for protons and neutrons is $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, $2p_{3/2}$, ...) [6 marks]

   b) Calculate the ratio
   $$r = \frac{E_\gamma}{M_1 - M_2}$$
   between the energy of the photon emitted in the transition $N_1 \to N_2 + \gamma$ and the difference between the masses M_1 and M_2 of two nuclei states N_1 and N_2. Comment on the accuracy of using the photon energy to determine the mass difference between states. [5 marks]

   c) The photon emissions observed with the highest rates are listed in the following table

Energy [keV]	multipolarity
320.0	M1+E2
928.6	??
608.6	M1+E2

   No photons with lower multipolarity are detected. Many more photons with energy 320.0 keV are seen than with energy 608.6 keV. The emissions come from two excited states above the ground state. Which of the following levels diagram is the right one? Explain your reasoning. [5 marks]



i) ii) iii) iv)

   d) Determine the multipolarity of the 928.6 keV emission? [2 marks]

   e) Why is $^{51}_{22}$Ti more likely to decay into the excited states rather than the ground state? [2 marks]

6. The differential cross section for the scattering of an electron from an extended charge distribution is

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{exp}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{*}_{\mathrm{Mott}} \left|F(\underline{q}^2)\right|^2,$$

where $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{*}_{\mathrm{Mott}}$ is the Mott scattering cross section and the form factor is

$$F(\underline{q}^2) = \int \exp\left(i\underline{q}\cdot\underline{x}\right) f(\underline{x})\mathrm{d}^3x,$$

where $\underline{q}$ is the change in the momentum of the electron and $f(\underline{x})$ is the charge distribution normalised such that

$$\int \mathrm{d}^3x f(\underline{x}) = 1\ .$$

a) Sketch the dependence of the form factor on $\underline{q}^2$ for a point particle and a homogeneous charged sphere. [2 marks]

b) Show that for a spherically symmetric charge distribution ($f(\underline{x}) = f(r)$, $r = |\underline{x}|$)

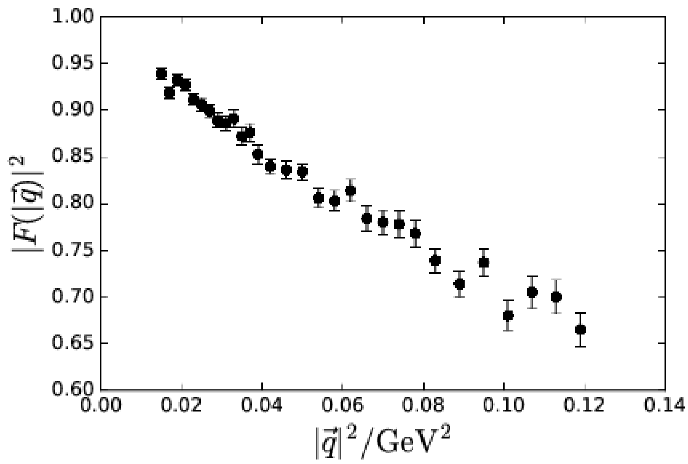
$$F(\underline{q}^2) = \frac{4\pi}{|\underline{q}|}\int f(r) r \sin\left(|\underline{q}|r\right)\mathrm{d}r. \qquad [4 \text{ marks}]$$

c) Consider the spherically symmetric charge distribution

$$f(r) = \frac{b\exp(-ar)}{r},$$

where a and b are constants. Determine b in terms of a and show that the form factor is

$$F(\underline{q}^2) = \left(1 + \frac{\underline{q}^2}{a^2}\right)^{-1}. \qquad [6 \text{ marks}]$$



d) The figure above shows $|F(\underline{q}^2)|$ for the charged pions. By expanding the form factor for small $|\underline{q}|$ and estimating the slope of the curve in the figure determine the constant a for the charge distribution of the pion. [4 marks]

e) Calculate the mean radius, $\langle r\rangle$, of the pion in fm. [4 marks]

[Hint: $\hbar c = 0.197\mathrm{GeV\,fm}$.]

7. The baryons which contain a single charm quark and two light quarks are:

$$\Lambda_c^+(cud, I=0); \qquad \Sigma_c^0, \Sigma_c^+, \Sigma_c^{++}(cdd, cud, cuu, I=1);$$

$$\Xi_c^0, \Xi_c^+(cds, cus); \qquad \Omega_c^0(css).$$

a) Assuming all baryons have the same colour wave-function, give an argument to justify why it is totally anti-symmetric. [3 marks]

b) What are the possible spins, S_{qq}, of the pair of light quarks in each of these baryons? Use the spins of the light quark pairs to obtain the possible spins of the baryons and give the number of $J=\frac{1}{2}$ and $J=\frac{3}{2}$ baryons containing a single charm quark. [7 marks]

c) Plot the positions of these baryons on diagrams with the third component of the isospin on the horizontal axis and strangeness on the vertical axis. Draw separate diagrams for the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ baryons. [2 marks]

[Hint: If there are baryons, X, with the same quark content it is conventional to call the $J=\frac{3}{2}$ state X^* and the $J=\frac{1}{2}$ state X. If there are two $J=\frac{1}{2}$ states the one with a $S=1$ light quark pair is called X' and the one with a $S=0$ quark pair X.]

The hyperfine mass splitting for baryons is

$$\Delta M_{ss} = \frac{16}{9}\pi\alpha_S\left|\psi(0)\right|^2 \sum_{\substack{i,j=1\\ i<j}}^{3} \frac{\underline{s}_i\cdot\underline{s}_j}{m_i m_j},$$

where the sum is over the three quarks in the baryon, and $\underline{s}_i$ and m_i are the spin and constituent masses of the quark respectively. Here $|\psi(0)|^2$ is proportional to the probability that the two quarks are at the same point in space.

d) Show that the mass of the $J=\frac{3}{2}$ baryons is

$$M\left(J=\frac{3}{2}\right) = m_1+m_2+m_3+\frac{4}{9}\pi\alpha_S\left|\psi(0)\right|^2\left(\frac{1}{m_1m_2}+\frac{1}{m_1m_3}+\frac{1}{m_2m_3}\right).$$

[2 marks]

e) Show that for the spin-1/2 baryons

$$\sum_{\substack{i,j=1\\ i<j}}^{3} \underline{s}_i\cdot\underline{s}_j = -\frac{3}{4}$$

and calculate the mass of the spin-1/2 Λ_c^+ and Ω_c^0 baryons. [6 marks]

8. a) Draw a Feynman diagram for the production of a quark-antiquark pair and a muon-antimuon pair in an electron-positron collider. Sketch the behaviour of the cross section ratio

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

for energies up to below the Z boson mass. Explain the values taken by the ratio. [5 marks]

b) The J/ψ is a $c\bar{c}$ vector meson with mass 3097 MeV and width 92.9keV. It can decay into a muon-antimuon pair. What are the possible spin and orbital angular momentum of the muon pair? Calculate the momentum of the muon after decay in the rest frame of the J/ψ. [3 marks]

c) The J/ψ can also decay into three pions. Calculate the angle between the first and second pions as a function of the magnitudes of their momenta. [5 marks]

d) Given the partial widths

$$\Gamma(J/\psi \to \mu^+\mu^-) = 5.5\,\text{keV}\;, \quad \Gamma(J/\psi \to ggg) = 69.4\,\text{keV}\;,$$

and the formula for the cross section for $J/\psi \to f$ as a function of the centre of mass energy E

$$\sigma_{e^+e^-\to f}(E) = \frac{3\pi\lambda^2}{16\pi}\frac{\Gamma_{J/\psi\to e^+e^-}\Gamma_{J/\psi\to f}}{(E - M_{J/\psi})^2 + \Gamma_{tot}^2/4},$$

sketch on a single plot the energy dependence of the cross sections for $f = ggg$ and $f = \mu^+\mu^-$ around the J/ψ mass. [2 marks]

e) Derive an estimate of the number of quark colours from the partial widths above along with the following approximations:

- The three-gluon and electromagnetic decay through a virtual photon are the only decay channels.
- Up to the factor due to the electric charge, the cross sections for the decay through a virtual photon for all allowed fermions are the same.

[5 marks]

[Hint: $m_e = 0.51\,\text{MeV/c}^2$, $m_\mu = 105.7\,\text{MeV/c}^2$, $m_\tau = 1777\,\text{MeV/c}^2$]

