University of Durham

EXAMINATION PAPER

May/June 2017 Examination code: PHYS2581-WE01

FOUNDATIONS OF PHYSICS 2A

SECTION A. Quantum Mechanics 2
SECTION B. Electromagnetism

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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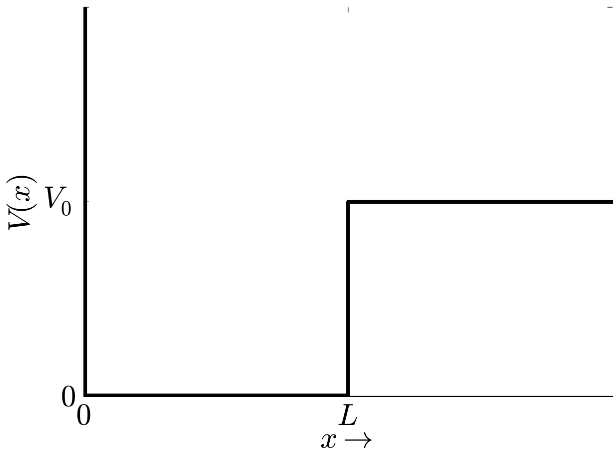
SECTION A. QUANTUM MECHANICS 2

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) The semi-infinite square well has a potential

$$V = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < L \\ V_0 & x > L \ , \end{cases}$$

as shown in the figure.



Assuming that V_0 is sufficiently large that there exists at least two bound solutions of Schrödinger's equation, sketch the wave functions corresponding to the ground and first excited states without solving any equations. Your sketches should show if the peaks and troughs of the wave functions are shifted relative to those of the corresponding infinite square well. [4 marks]

(b) Which of the following operators are necessarily Hermitian given that A and B are Hermitian operators and c is a real constant

i) $A + B$

ii) cA

iii) AB

iv) $AB + BA$?

[4 marks]

(c) In a certain system A has eigenvalues a_1 and a_2 corresponding to eigenfunctions $\psi_1 = (u_1 + u_2)/\sqrt{2}$ and $\psi_2 = (u_1 - u_2)/\sqrt{2}$ where u_1 and u_2 are normalized energy eigenfunctions with energies E_1 and E_2. At $t = 0$, A is measured to have the value a_1 which implies the wavefunction is

$$\Psi = \frac{u_1}{\sqrt{2}}\, e^{-iE_1 t/\hbar} + \frac{u_2}{\sqrt{2}}\, e^{-iE_2 t/\hbar}.$$

Find how $\langle A \rangle$ varies subsequently with time. [4 marks]

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(d) The normalized eigenfunctions of an electron in a hydrogen atom are ψ_{nlm} with quantum numbers n, l and m and energy $E_n = -13.6/n^2$ eV. An electron is described by the wavefunction

$$\Psi = \frac{4}{N}\psi_{100} + \frac{2}{N}\psi_{211} + \frac{\sqrt{5}}{N}\psi_{21-1}\,.$$

Find the value of N required to normalise Ψ. [2 marks]

Find the expectation value of the energy, $\langle E\rangle$. [2 marks]

(e) Consider a particle described by a wavefunction expressed in terms of spherical polar coordinates $\psi(r, \theta, \phi)$. Explain, in terms of this wavefunction, how to determine a radial probability distribution $\rho(r)$ for the particle's distance from the origin. [4 marks]

(f) Consider an eigenstate, ψ, of both the z-component of the angular momentum, L_z, operator and of the angular momentum squared, L^2, operator. Consider also that $L_z\psi = 2\hbar\psi$, and $L_+\psi = 0$, where L_+ is an angular momentum ladder operator. What is the eigenvalue of L^2 for this state? How many possible values of L_z are there for eigenstates which have this same value for L^2? Explain your answers. [4 marks]

(g) Suppose in the Stern–Gerlach experiment, the beam of silver atoms had been split into three by the inhomogeneous magnetic field. Knowing that the valence electron has zero orbital angular momentum, what would this tell us about the spin of the electron? Explain your answer. [4 marks]

(h) Consider a particle in a one-dimensional infinite square well potential, with ground state ψ_0 and excited states $\psi_1, \psi_2, \psi_3, \ldots$. Explain why adding a delta-function perturbation $H' = \alpha\delta(x - c)$ (where $x = c$ is the centre of the well) will not affect the odd excited states $\psi_1, \psi_3, \psi_5, \ldots$. [4 marks]

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2. The raising and lowering operators of a one-dimensional harmonic oscillator with potential $V = m\omega^2 x^2/2$, where m is the mass of the particle, are respectively

$$a_+ = \frac{1}{\sqrt{2}}\left(\alpha x - \frac{ip}{\alpha\hbar}\right)$$

and

$$a_- = \frac{1}{\sqrt{2}}\left(\alpha x + \frac{ip}{\alpha\hbar}\right).$$

Here p is the momentum operator and $\alpha = \sqrt{m\omega/\hbar}$.

(a) Given that $[x, p] = i\hbar$ evaluate the commutator $[a_-, a_+]$. [3 marks]

(b) Use one of the operators a_+ or a_- to show that the normalized ground state wavefunction is given by

$$\psi_0 = \left(\frac{\alpha^2}{\pi}\right)^{1/4} \exp(-\alpha^2 x^2/2). \quad \text{[8 marks]}$$

(c) From this result derive the wavefunction of the first exicited state, ψ_1. Normalizing ψ_1 is not required. [4 marks]

(d) Show that ψ_0 satisfies the time-independent Schrödinger equation and hence find the energy of the ground state. [5 marks]

$$\left[\quad \text{Hint : You may make use of the following standard integral,} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2}\, dx = \frac{\sqrt{\pi}}{\alpha}. \quad \right]$$

3. A simple model of an atom based gyroscope consists of an atom of mass m, free to move only on a circle of radius R. The time-independent Schrödinger equation for this system is then

$$E\psi(\phi) = -\frac{\hbar^2}{2mR^2}\frac{d^2\psi(\phi)}{d\phi^2},$$

where ϕ is the angle coordinate describing the position on the circle with $\psi(\phi + 2\pi) = \psi(\phi)$ (i.e., *periodic boundary conditions*), R is a fixed value and the ground state energy eigenfunction is $\psi_0 = 1/\sqrt{2\pi}$, with eigenenergy $E_0 = 0$.

(a) Show that, for $n = 1, 2, 3, \ldots$, the wavefunctions

$$\psi_{n-}(\phi) = \frac{1}{\sqrt{\pi}}\sin(n\phi), \qquad \psi_{n+}(\phi) = \frac{1}{\sqrt{\pi}}\cos(n\phi)$$

are excited state energy eigenfunctions, and that for each value of n they form degenerate pairs. [6 marks]

(b) Consider a perturbation $H' = g\delta(\phi)$, where $\delta(\phi)$ is a Dirac delta function centred on $\phi = 0$, g is a constant, and a "reflection operator", M, defined through $M\psi(\phi) = \psi(-\phi)$.

(i) Show that $[H', M]\psi(\phi) = 0$. [4 marks]

(ii) Show that $\psi_{n-}(\phi)$ and $\psi_{n+}(\phi)$ are also eigenfunctions of M and determine their eigenvalues. [4 marks]

(c) In non-degenerate perturbation theory, the first order correction to the excited state energy eigenvalues would be determined through $\langle\psi_{n-}|H'|\psi_{n-}\rangle$, $\langle\psi_{n+}|H'|\psi_{n+}\rangle$. Calculate these, and explain why this is sufficient to determine the appropriate first order corrections, even though $\psi_{n-}(\phi)$ and $\psi_{n+}(\phi)$ are actually energy degenerate. [6 marks]

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SECTION B. ELECTROMAGNETISM

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Write down the Maxwell equation that describes the electric field produced by a change in magnetic flux. Explain how this equation can be tested experimentally. [4 marks]

   (b) A point charge of 3 μC is located at the position $\underline{r}$, where $\underline{r} = 2\hat{\underline{i}} + \hat{\underline{j}}$, $\hat{\underline{i}}$ and $\hat{\underline{j}}$ are unit vectors in the x and y-directions respectively and $\underline{r}$ is in metres. A second charge of 1 μC is located at $\underline{r} = 5\hat{\underline{i}} + 3\hat{\underline{j}}$. Calculate the magnitude of the force on the 1 μC charge. [4 marks]

   (c) Briefly describe how a radio receiver works. [4 marks]

   (d) Describe what a magnetic dipole moment is. [4 marks]

   (e) Provide a pictorial representation of the spatial variation of the fields of an electromagnetic wave propagating through vacuum. Your diagram should make clear the directions of the $\underline{E}$ field and the $\underline{B}$ field for the wave and the direction of propagation. [4 marks]

   (f) Explain the difference between an intensive variable and an extensive variable. [4 marks]

   (g) A electron is travelling in a helical motion of a constant radius of 5 mm with a magnetic field along the axis of the helix of 5 μT. Calculate the speed of the electron in the direction orthogonal to the magnetic field. [4 marks]

5. (a) Discuss the role of the scattering time in the Drude model - the Drude model describes the electrical conductivity of a material that scatters charge carriers. [2 marks]

(b) The dispersion relation for a high frequency electromagnetic wave propagating through an infinite plasma is given by:

$$k^2 = \mu_0 \varepsilon_0 \omega^2 - (ne^2 \mu_0)/m_e,$$

where k is the magnitude of the wavevector, ω is the angular frequency and n is the density of electrons in the plasma. Show that the the frequency dependence of the group velocity (v_g), of the electromagnetic waves in the plasma at high frequencies is given by:

$$v_g \approx c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) ,$$

where ω_p is known as the plasma angular frequency and $\omega_p^2 = (ne^2)/(m_\mathrm{e}\varepsilon_0)$. [6 marks]

(c) Scientists in Durham have produced a spherical plasma of diameter 1 m. One of the scientists simultaneously produces a 10 GHz pulse and a 20 GHz pulse of electromagnetic waves and fires them through the centre of the plasma. A detector that receives the two pulses after they have passed through the plasma finds they arrive 10^{-6} seconds apart. Find an approximate value for the plasma angular frequency. [8 marks]

The scientists use polarised electromagnetic waves and find the time between the pulses increases. Discuss how that can occur. [4 marks]

6. (a) The wave equation for an electromagnetic wave propagating in vacuum is given by:

$$\nabla^2 \underline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}.$$

Calculate the group velocity of an electromagnetic wave obeying this wave equation. [2 marks]

(b) A plasma, produced inside a fusion energy tokamak, is being heated by microwaves travelling along a hollow square metallic tube (known as a waveguide) of width a where the electric field is given by:

$$\underline{E} = \underline{E}_0 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi z}{a}\right) \exp[i(ky - \omega t)],$$

where x, y and z are the conventional Cartesian coordinates, ω is the angular frequency, k is the wavevector, $\underline{E}_0$ gives the magnitude and polarisation of the wave and points in the z-direction. The axis of the tube is parallel to the y-axis. Given that $a = 0.1$ m, find the minimum frequency at which this particular wave can propagate along the waveguide. [6 marks]

(c) Above this minimum frequency, determine whether a pulsed wave propagates without changing shape. [2 marks]

(d) Calculate an expression for the associated magnetic field of the microwaves. [6 marks]

(e) A scientist fills the waveguide with a dielectric. Discuss what happens to the velocity of the microwaves and the energy of the photons associated with the microwaves.[4 marks]

