Relativistic Electrodynamics

RED.2

Consider two inertial frames S and S' in the standard configuration and let $\Lambda^{\mu}{}_{\nu}$ be the Lorentz Transformation (a Rank-2 tensor) represented by a 4×4- matrix that relates the contravariant coordinates of an event as measured in S to the contravariant coordinates of the event as measured in S', *i.e.* $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$.
Let $\Lambda^{\mu}_{x\,\nu}(\Psi_1)$ and $\Lambda^{\mu}_{y\,\nu}(\Psi_2)$ denote the rank-2 tensors for the boost of rapidity $\Psi_1 = \ln(3)$ and $\Psi_2 = \ln(3)$ along the x-axis and y-axis respectively. The inverse transformations are given by boosts of $-\Psi_1$ and $-\Psi_2$ in the respective directions.

(a) Calculate the matrix representation for the Lorentz Transformation-tensor describing the combined effect of a boost along x, then along y, followed by a boost by $-\Psi_1$ in the x-direction, and then $-\Psi_2$ along the y-direction, i.e.

$$\Lambda^{\mu}{}_{\nu} = \Lambda^{\mu}_{y\,\rho}(-\Psi_2)\ \Lambda^{\rho}_{x\sigma}(-\Psi_1)\ \Lambda^{\sigma}_{y\,\lambda}(\Psi_2)\ \Lambda^{\lambda}_{x\nu}(\Psi_1). \tag{1}$$

[6 marks]

For normal Gallilean transformations, this series of transformations would result in no change at all (i.e. the identity).
Let x^{μ} denote the position 4-vector of a light-pulse travelling along the x-direction at time $t = 1m/c$ after it left the origin: $x^{\mu} = (1, 1, 0, 0)$m.

(b) Calculate the length travelled by the light-pulse as measured by an observer who has gone through the series of boost outlined in Eq. (1), i.e. $d = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Also, give the value of $x'^{\mu} x'_{\mu}$, where $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$. How does that compare to $x^{\mu} x_{\mu}$?

[4 marks]