University of Durham
EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2018	PHYS2611-WE01

Title:
Mathematical Methods in Physics

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper.
- Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** the answers that are not to be marked.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Mathematical Methods part 1
Section B: Mathematical Methods part 2

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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SECTION A: MATHEMATICAL METHODS PART 1

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) If

$$\begin{array}{lll} a_{11} = 1, & a_{12} = -1, & a_{13} = 0, \\ a_{21} = -2, & a_{22} = 3, & a_{23} = 1, \\ a_{31} = 2, & a_{32} = 0, & a_{33} = 4, \end{array}$$

use the Einstein summation convention to evaluate:

(i) $a_{1i}a_{2i}$.

(ii) $a_{i1}a_{2i}$.

(iii) $a_{1i}a_{2j}\delta_{ij}$.

(iv) $\delta_{ij}\,\epsilon_{ij\,k}$.

[4 marks]

(b) Determine whether the given sets defined below are vector spaces. If they are not, for each of them state an axiom that fails to hold.

(i) The set of (2×2) matrices A such that $A^2 = I$ where I is the identity matrix, with the usual addition of matrices and multiplication by a scalar.

(ii) The set of (2×2) matrices A with determinant equal to zero, with the usual addition of matrices and multiplication by a scalar.

[4 marks]

(c) (i) Find the value of a which makes the following determinant zero

$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & a & 2 \\ -1 & 1 & 2 \end{vmatrix}.$$

(ii) Without evaluating the following determinant, explain why it is zero

$$\begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & -1 \\ 4 & 8 & 2 \end{vmatrix}.$$

[4 marks]

(d) Calculate the following integrals

(i) $I_1 = \int\limits_{-\infty}^{\infty} \delta(2x)\left(e^{2(x-1)} + e^{-2(x-1)}\right)\,dx.$

(ii) $I_2 = \int\limits_{-2}^{2} \delta(x^2 - 3x - 4)x^4\,dx.$

[4 marks]

(e) The Laplace transform for a function $f(t)$ is defined as follows

$$\mathcal{L}[f(t)](s) \equiv \bar{f}(s) = \int_0^\infty f(t)\, e^{-ts} dt.$$

Calculate explicitly the Laplace transforms of the following functions

(i) $f(t) = e^{3t}$.

(ii) $f(t) = 2t + 1$.

[4 marks]

(f) Evaluate the line integral $I = \int_{\mathcal{C}} \underline{a} \cdot d\underline{r}$ along the curve $\mathcal{C}$ defined by

$$\underline{r}(u) = (1+u)\,\hat{\underline{i}} + 4u\,\hat{\underline{j}} + (1-3u)\,\hat{\underline{k}}, \quad 0 \le u \le 1,$$

where $\underline{a} = x^2\,\hat{\underline{i}} + yz\,\hat{\underline{j}} + y\,\hat{\underline{k}}$.

[4 marks]

(g) Establish whether the following vector fields are conservative

(i) $\underline{a}_1 = 2xz\,\hat{\underline{i}} + 2yz^2\,\hat{\underline{j}} + \,(x^2 + 2y^2 z - 1)\,\hat{\underline{k}}$.

(ii) $\underline{a}_2 = 2\underline{r}$, where $\underline{r}$ is the position vector.

[4 marks]

(h) Consider the surface $\mathcal{S}$ given by the following parametric equation

$$\underline{r}(\phi, z) = z^2 \cos\phi\,\hat{\underline{i}} + z^2 \sin\phi\,\hat{\underline{j}} + z\,\hat{\underline{k}}.$$

Calculate the scalar surface element dS.

[4 marks]

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2. (a) Diagonalise the following symmetric matrix,

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

i.e. find the matrices D, S and S^{-1} such that $D = S^{-1}AS$. Then use these matrices to find an expression for A^{-1} without computing it explicitly.
[14 marks]

(b) Given that $\underline{u} \times (\underline{v} \times \underline{w}) = (\underline{u} \cdot \underline{w})\, \underline{v} - (\underline{u} \cdot \underline{v})\, \underline{w}$, evaluate the identity

$$(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{d}) = -(\underline{c} \times \underline{d}) \times (\underline{a} \times \underline{b})$$

and prove that

$$\underline{a}\, [\underline{b}, \underline{c}, \underline{d}] - \underline{b}\, [\underline{c}, \underline{d}, \underline{a}] + \underline{c}\, [\underline{d}, \underline{a}, \underline{b}] - \underline{d}\, [\underline{a}, \underline{b}, \underline{c}] = 0,$$

where $[\underline{u}, \underline{v}, \underline{w}] = \underline{u} \cdot (\underline{v} \times \underline{w})$ is the triplet scalar product.
[6 marks]

[Hint: Use the cyclic permutation property of the triplet scalar product.]

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3. (a) Stokes' theorem for a vector field $\underline{a}$ on surface $\mathcal{S}$ enclosed by curve $\mathcal{C}$ states that

$$\int_{\mathcal{S}} (\nabla \times \underline{a}) \cdot d\underline{S} = \int_{\mathcal{C}} \underline{a} \cdot d\underline{r}.$$

Verify Stokes' theorem explicitly by computing both sides of the equation for

$$\underline{a} = (y^2 + x)\,\hat{\underline{i}} + y\,\hat{\underline{j}} + x^2 z\,\hat{\underline{k}}$$

over a rectangle $ABCD$ with vertices $A = (1,1,1)$, $B = (3,1,1)$, $C = (3,2,1)$ and $D = (1,2,1)$. Notice that the rectangle S lies in the plane $z = 1$.

[13 marks]

(b) Given that $\mathcal{L}[e^{\alpha t}](s) = 1/(s - \alpha)$, where α is a constant, find the inverse Laplace transform of

$$\bar{f}(s) = \frac{5s + 1}{s^2 - s - 12}.$$

[7 marks]

SECTION B: MATHEMATICAL METHODS PART 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Find the solution $y(x)$ of the ordinary differential equation

$$\frac{dy}{dx} + xy = 4x,$$

subject to the boundary condition $y(0) = 5$. [3 marks]

Verify your result by substitution into the equation above. [1 mark]

(b) Find a solution of the second-order homogeneous differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4x - 4,$$

by finding the roots of the auxiliary equation and using the method of trial functions with undetermined coefficients. [4 marks]

(c) If $\bar{f}(s)$ is the Laplace transform of $f(x)$, what is the Laplace transform of df/dx? [2 marks]

The Laplace transform of $\sin(x)$ is $\mathcal{L}[\sin(x)](s) = 1/(1+s^2)$. What is the Laplace transform of $\cos(x)$? [2 marks]

(d) Consider the differential equation

$$\frac{d^2f}{dx^2} + 5\frac{df}{dx} + 4f = 3e^{-5x}.$$

Knowing that e^{-x} and e^{-4x} are solutions to the associated homogeneous equation, use the Wronskian method to solve the inhomogeneous problem. [4 marks]

$$\left[\begin{array}{l}\text{Hint: If } f = k_1 f_1 + k_2 f_2, \text{ then } k_1' = -\frac{h(x)}{W(x)} f_2 \text{ and } k_2' = \frac{h(x)}{W(x)} f_1, \\ \text{with } W(x) \text{ the Wronskian and } h(x) \text{ the inhomogeneous term.}\end{array}\right]$$

(e) Consider the second order Euler differential equation. Show that with the change of variable $x = e^t$ it becomes a linear ordinary differential equation with constant coefficients. [4 marks]

(f) Consider the partial differential equation

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = 0.$$

Find the most general solution such that $u = 5$ on the parabola $y = x^2$. [4 marks]

(g) Write the generic one-dimensional Schrödinger equation with a time-independent potential $V(x)$. Use the method of separation of variables to write down the general time-dependence of the solution. [4 marks]

5. After separation of variables, the Schrödinger equation for the radial part of the wavefunction reads:

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{\hbar^2 l(l+1)}{2mr^2}\psi+V(r)\psi=E\psi.$$

(a) Consider the following differential equation:

$$x^2 f''(x)+2xf'(x)+(x^2-\alpha(1+\alpha))f(x)=0.$$

Show that $x=0$ is a regular singular point. [2 marks]

(b) Explain why it is natural to look for solutions of the form $f(x)=x^\sigma g(x)$, with $g(x)$ Taylor-expandable around $x=0$. [2 marks]

(c) From now on, assume $\alpha=0$. Find the possible values for σ by solving the indicial equation. Show that $\sigma=-1$ is a solution. [4 marks]

(d) Look for solutions of the form $f(x)=g(x)/x$. Substitute this expression for $f(x)$ into the differential equation to write the equation in terms of $g(x)$. Solve it, and use this result to show that if $\alpha=0$ the solution for $f(x)$ is a linear combination of the two functions

$$j_0(x)=\frac{\sin(x)}{x},\qquad y_0(x)=-\frac{\cos(x)}{x}.$$

[6 marks]

(e) Use this result to write the general solution for the radial part of the Schrödinger wavefunction $\psi(r)$, for the $s-$wave case ($l=0$) and in the absence of a potential. Impose that the solution is regular in $r=0$ and vanishes for $r=L$. Show that the energy is quantized. [6 marks]

6. Consider the differential equation

$$2x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{1}{2}y=0,$$

where $y=y(x)$.

(a) Does this equation contain singular points at finite values of x? If so, what type of singularities are these? [2 marks]

Use the Frobenius method to solve this equation as a series expansion around the point $x=0$. To achieve this follow the steps below:

(b) Represent $y(x)$ as the appropriate Frobenius series and plug it into the differential equation. [2 marks]

(c) Use this to find the indicial equation, determine its roots and characterize them. [6 marks]

(d) Derive the recursion relation for the coefficients a_n of the Frobenius series for each of the roots. [4 marks]

(e) Solve the recursion relations for the coefficients a_1, a_2, a_3 in terms of a_0. [4 marks]

(f) Finally write down the general solution of the differential equation as a series expansion with the first few terms fully determined. [2 marks]

