# Mathematical Methods II
# PDF 2

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## Key Points

- Solving 1st order linear ODEs.
- Solving Bernoulli type non-linear ODEs.
- Solving homogeneous 1st order ODEs.
- Solving isobaric 1st order ODEs.

## Solving 1st order ODEs (cont.)

- **Linear 1st order ODEs**: Linear 1st order ODEs are a special type of inexact ODE and have the form,
$$a(x)\frac{dy}{dx} + b(x)y = c(x).$$
If we divide by $a(x)$ we arrive at the canonical form,
$$\frac{dy}{dx} + p(x)y = q(x),$$
where $p(x) = b(x)/a(x)$ and $q(x) = c(x)/a(x)$. If an ODE can be rearranged into this form there is a standard method for solving it using an integrating factor. Applying the integrating factor $\mu = \mu(x)$ gives
$$\mu\frac{dy}{dx} + \mu py = \mu q,$$
$$\frac{d}{dx}(\mu y) = \mu q.$$
Integrating gives,
$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)dx,$$
which is our solution for $y$. We can use the equalities above to determine our required integrating factor. It must satisfy the following,
$$\frac{d}{dx}(\mu y) = \frac{d\mu}{dx}y + \mu\frac{dy}{dx} = \mu\frac{dy}{dx} + \mu p(x)y,$$

which gives,

$$\frac{d\mu}{dx} = \mu p(x).$$

Thus $\mu$ is given by,

$$\mu(x) = exp\left(\int p(x)dx\right).$$

**e.g. PDF2.1** Solve the following 1$^{\text{st}}$ order ODE

$$\frac{dy}{dx} + \left(\frac{1+x}{x}\right)y = \frac{e^x}{x}$$

Determine $p(x)$ and $q(x)$

$$p(x) = \frac{1+x}{x}$$
$$q(x) = \frac{e^x}{x}$$

Find the integrating factor, $\mu(x) = e^{\int p(x)dx}$

$$\mu(x) = e^{\int[(1+x)/x]dx}$$
$$= e^{\int[(1/x)+1]dx}$$
$$= e^{(\ln x + x)} = e^{\ln x}e^x$$
$$= xe^x$$

Write down the equation including the integrating factor and solve

$$\frac{d}{dx}(xe^x y) = (xe^x)\frac{e^x}{x}$$
$$xe^x y = \int e^{2x}dx$$
$$y = \frac{1}{xe^x}\left[\frac{e^{2x}}{2} + c\right]$$
$$y = \frac{e^x}{2x} + \frac{c}{xe^x}$$

- **Bernoulli ODEs**: Bernoulli equations are of the general form

$$\frac{dy}{dx} + b(x)y = c(x)y^n$$

where $n \neq 0$,$n \neq 1$. This is a non-linear equation, but we can reduce it to a linear equation by using the substitution

$$z = y^{1-n}.$$

Once this has been done, we may solve the equation with an integrating factor, just as with other 1$^{\text{st}}$ order linear ODEs.

**e.g. PDF2.2** Solve the following Bernoulli type ODE

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$$

Here, $n = 4$. Rearrange the equation for convenient -ve exponents by $\div y^4$

$$\frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3}y^{-3} = e^x$$

Use the substitution $z = y^{-3}$, work out $dz/dx$

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$\frac{1}{y^4}\frac{dy}{dx} = -\frac{1}{3}\frac{dz}{dx}$$

Substitute these back into the equation and rearrange into a linear form

$$-\frac{1}{3}\frac{dz}{dx} + \frac{1}{3}z = e^x$$

$$\frac{dz}{dx} - z = -3e^x$$

This can now be solved as a linear equation. Note that $p(x) = -1$ and $q(x) = -3e^x$. Find the integrating factor

$$\mu(x) = e^{\int -1 dx} = e^{-x}$$

Solve the equation

$$(e^{-x}z) = e^{-x}(-3e^x)$$

$$\begin{aligned} z &= -\frac{1}{e^{-x}}\int 3dx \\ &= -e^x(3x + c) \end{aligned}$$

Resubstitute $z = y^{-3}$

$$y = (ce^x - 3xe^x)^{-1/3}$$

- **Homogeneous ODEs**: Homogeneous ODEs have the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

where $A$ and $B$ are homogeneous functions of $x$ and $y$ and $F$ is a function of $y$ over $x$. Homogeneous functions obey the rule

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

That is, if you multiply each argument by a common factor $\lambda$, the value of the function will be multiplied by that factor raised to the $n^{\text{th}}$ power for all real $\lambda$. $n$ is called the degree of homogeneity.

If an ODE is homogeneous we can solve it by making the following substitutions

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

This allows us to separate the variables $x$ and $v$ and solve the equation.

**e.g. PDF2.3** Solve the homogeneous equation

$$(y^2 + xy)dx - x^2 dy = 0$$

First, rearrange it into the standard form

$$x^2 dy = (y^2 + xy)dx$$

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}$$

Test for homogeneity

$$f(\lambda x, \lambda y) = \frac{(\lambda y)^2 + (\lambda x)(\lambda y)}{(\lambda x)^2} = \frac{\lambda^2(y^2 + xy)}{\lambda^2(x^2)} = f(x, y)$$

therefore, this ODE is homogeneous. Now apply the substitutions $y = vx$ and $dy/dx = v + xdv/dx$ to the ODE.

$$v + x\frac{dv}{dx} = \frac{((vx)^2 + x(vx))}{x^2} = \frac{v^2x^2 + vx^2}{x^2} = v^2 + v$$

$$x\frac{dv}{dx} = v^2 + v - v = v^2$$

Separate the variables and integrate

$$\int \frac{1}{v^2} dv = \int \frac{1}{x} dx$$

Following the working and subsituting $v$ out we arrive at a solution

$$x = Ae^{-x/y}$$

Note: Our primary interest in homogeneity is related to $2^{\text{nd}}$ order linear ODEs of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

These equations are homogeneous when $f(x) = 0$ and non-homogeneous when $f(x) \neq 0$.

- **Isobaric ODEs**: Isobaric ODEs are a generalisation of homogeneous ODEs with the form

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)} = F\left(\frac{y}{x}\right)$$

The difference is that isobaric functions satisfy the following equality

$$f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x, y).$$

We can take advantage of dimensional consistency to solve isobaric ODEs. If we assign a relative weight of $m$ to the exponents of $y$ and $dy$, and a relative weight of 1 to the exponents of $x$ and $dx$ we are able to make the following substitutions

$$y = vx^m.$$

$$\frac{dy}{dx} = mvx^{m-1} + x^m \frac{dv}{dx}$$

This will make the ODE separable, and thus solvable.

**e.g. PDF2.4** Solve the following isobaric equation,

$$x(1 - 2x^2 y)\frac{dy}{dx} + y = 3x^2 y^2$$

First arrange into $dy$ and $dx$ terms.

$$xdy - 2x^3 ydy + ydx - 3x^2 y^2 dx = 0$$

Now assign the weights ($y, dy \to m$ and $x, dx \to 1$). Constants are assigned a weight of zero, essentially meaning they can be ignored. This gives the following sums for the 3 terms in the equation

$$xdy \to 1 + m$$

$$2x^3 ydy \to 0 + 3 + m + m = 3 + 2m$$

$$ydx \to m + 1$$

$$3x^2 y^2 dx \to 0 + 2 + 2m + 1 = 3 + 2m$$

We equate the 4 sums and look for a value of m that satifies the equalities.

$$m + 1 = m + 1 = 3 + 2m = 3 + 2m$$

This is satisfied if

$$m + 1 = 3 + 2m$$

Hence

$$m = -2$$

We can now substitute $y = vx^{-2}$ and $\frac{dy}{dx} = \frac{dv}{dx}x^{-2} - 2vx^{-3}$ into the equation, separate it, integrate and solve it.

$$x(1 - 2x^2 vx^{-2})\left(\frac{dv}{dx}x^{-2} - 2vx^{-3}\right) + vx^{-2} - 3x^2(vx^{-2})^2 = 0$$

$$x^{-1}\frac{dv}{dx} - 2vx^{-2} - 2vx^{-1}\frac{dv}{dx} + 4v^2 x^{-2} + vx^{-2} - 3v^2 x^{-2} = 0$$

$$\frac{dv}{dx} - vx^{-1} - 2v\frac{dv}{dx} + v^2 x^{-1} = 0$$

$$(1-2v)\frac{dv}{dx}+(v^2-v)x^{-1}=0$$
$$\frac{(1-2v)}{v(1-v)}dv = x^{-1}dx$$

If we integrate and subsitute $y$ back in we find the solution

$$4xy(1-x^2y)=1$$