Mathematical Methods in Physics

Examination June 2017

Question 1

(a) (Unseen)

(i) It is a vector space. [2 marks]

(ii) It is not a vector space. For instance, scalar multiplication is not a close operation. Also, there are no inverse elements. [2 marks]

(b) (Unseen)

(i) They are linearly dependent. The third is equal to the first minus twice the second. [2 marks]

(ii) They are linearly independent. In fact the expression $c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = 0$ is only satisfied for $c_1 = c_2 = c_3 = 0$. [2 marks]

(c) (Unseen)

(i) $U_1^{-1} = U_1^\dagger, \qquad U_2^{-1} = U_2^\dagger$, then

$$(U_1U_2)^{-1} = U_2^{-1}U_1^{-1} = U_2^\dagger U_1^\dagger = (U_1U_2)^\dagger.$$

[2 marks]

(ii)

$$(i(A-A^\dagger))^\dagger = -i(A^\dagger - A) = i(A - A^\dagger).$$

[2 marks]

(d) (Unseen)

(i) For instance

$$\underline{r}(t) = x\,\underline{\hat{i}} + (3-x)\,\underline{\hat{j}} + (18-7x)\,\underline{\hat{k}}.$$

[2 marks]

(ii)

$$\underline{r}(t) = (\cos t + 1)\,\underline{\hat{i}} + (\sin t - 2)\,\underline{\hat{j}} + \underline{\hat{k}}.$$

[2 marks]

(e) (Unseen)

$$\frac{ds}{du} = \pm\sqrt{\frac{d\underline{r}}{du}\cdot\frac{d\underline{r}}{du}} = \pm\sqrt{2},$$

Then

$$\underline{\hat{t}} = \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{du}\,\frac{du}{ds} = \frac{1}{\sqrt{2}}(-\sin u\,\underline{\hat{i}} + \cos u\,\underline{\hat{j}} + \underline{\hat{k}}),$$

[2 marks]

and

$$\frac{d\underline{\hat{t}}}{ds} = \frac{d\underline{\hat{t}}}{du}\,\frac{du}{ds} = -\frac{1}{2}(\cos u\,\underline{\hat{i}} + \sin u\,\underline{\hat{j}}).$$

It follows that $\rho = 2$. [2 marks]

(f) (Unseen)

$$\frac{\partial\underline{r}}{\partial x} = \underline{\hat{i}} + 2x\,\underline{\hat{j}}, \qquad \frac{\partial\underline{r}}{\partial z} = \underline{\hat{k}}.$$

Then

$$d\underline{S} = \left(\frac{\partial\underline{r}}{\partial x}\times\frac{\partial\underline{r}}{\partial z}\right)\,dx\,dz = (2x\,\underline{\hat{i}} - \underline{\hat{j}})\,dx\,dz.$$

[2 marks]

Hence

$$I = \int_0^3 dz\int_0^2 dx\,(2x^3-2) = 12.$$

[2 marks]

(g) (Unseen)

$$c_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x-irx}dx = \frac{1}{2\pi}\left[\frac{e^{x(1-ir)}}{1-ir}\right]_{-\pi}^{\pi} = \frac{(-1)^r}{2\pi}\left(\frac{e^{\pi}-e^{-\pi}}{1-ir}\right) = \frac{\sinh\pi}{\pi}(-1)^r\frac{1+ir}{1+r^2}.$$

Then

$$e^x = \frac{\sinh\pi}{\pi}\sum_{-\infty}^{\infty}\frac{1+ir}{1+r^2}\,(-1)^r\,e^{irx} \qquad -\pi\le x\le\pi.$$

[4 marks]

(h) (Unseen)

$$\begin{aligned}\bar{f}(s) &= \int_0^{\infty}\cos t\,H(t-\pi)\,e^{-ts}dt = \int_{\pi}^{\infty}\cos t\,e^{-ts}dt\\ &= -\frac{1}{2}\left[\frac{e^{-t(s-i)}}{s-i}+\frac{e^{-t(s+i)}}{s+i}\right]_{\pi}^{\infty} = -e^{-\pi s}\frac{s}{s^2+1}.\end{aligned}$$

[4 marks]

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Question 2

(a) (Unseen)

The eigenvalues are: $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 3$. [4 marks]

The forms of the corresponding eigenvectors are: $v_1 = (0, 0, c)^T$, $v_2 = (a, -a, 0)^T$ and $v_3 = (b, b, 0)^T$, with a, b, c constants. The eigenvectors are clearly orthogonal. A possible choice for an orthonormal set of eigenvectors is: $v_1 = (0, 0, 1)^T$, $v_2 = (1/\sqrt{2}, -1/\sqrt{2}, 0)^T$ and $v_3 = (1/\sqrt{2}, 1/\sqrt{2}, 0)^T$. [4 marks]

(b) (Unseen)

For the integral on the left hand side, since the rectangle $ABCD$ lies in the plane $z = 0$, we only need the z-component of the curl

$$(\nabla \times \underline{F})_z = \left(\frac{\partial}{\partial x}((x+y)\, e^{xy}) - \frac{\partial}{\partial y}(y\, e^{xy}) \right) \underline{\hat{k}} = y^2\, e^{xy} \underline{\hat{k}}.$$

[2 marks]

Then

$$d\underline{S} = \underline{\hat{k}}\, dxdy,$$

[1 mark]

hence the surface integral is:

$$\int_{\mathcal{S}} (\nabla \times \underline{F}) \cdot dS = \int_1^3 dy \int_0^1 dx y^2\, e^{xy} = \int_1^3 dy (e^y - 1) y$$

$$= \quad y\, e^y|_1^3 - e^y|_1^3 - \left.\frac{y^2}{2}\right|_1^3 = 2\, e^3 - 4.$$

[3 marks]

For the integral on the right hand side there are four different $d\underline{r}$. Since $d\underline{S}$ has been chosen along the positive k-axis, the integration is anticlockwise along the perimeter of the rectangle $ABCD$:

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \int_0^1 dx \underline{F} \cdot \underline{\hat{\imath}}\,|_{y=1} - \int_0^1 dx \underline{F} \cdot \underline{\hat{\imath}}\,|_{y=3} + \int_1^3 dy \underline{F} \cdot \underline{\hat{\jmath}}\,|_{x=1} - \int_1^3 dy \underline{F} \cdot \underline{\hat{\jmath}}\,|_{x=0},$$

[3 marks]

which is

$$
\begin{aligned}
&= \int_0^1 e^x\,dx - \int_0^1 3\,e^{3x}\,dx + \int_1^3 (y+1)\,e^y\,dy - \int_1^3 y\,dy \\
&= e - 1 + (1 - e^3) + (4\,e^3 - 2\,e - e^3 + e) + \left(\frac{1}{2} - \frac{9}{2}\right) = 2\,e^3 - 4.
\end{aligned}
$$

[3 marks]

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Question 3

(a) (Unseen)

(i)

$$\text{div}\,\underline{v} = 2x + 4y, \qquad \text{grad}(\text{div}\,\underline{v}) = 2\,\hat{\underline{i}} + 4\,\hat{\underline{j}}, \qquad \text{grad}(\text{div}\,\underline{v}) \cdot \underline{u} = 2x^2 + 4(y - z).$$

[3 marks]

(ii)

$$\nabla \cdot (f\underline{r}) = \nabla f \cdot \underline{r} + f \nabla \cdot \underline{r} = f' \, \frac{\underline{r} \cdot \underline{r}}{r} + 3f = f'r + 3f.$$

[3 marks]

Set $f'(r)r + 3f(r) = 0$ then

$$f'(r) = -\frac{3f(r)}{r} \longrightarrow \frac{df}{f} = -3\frac{dr}{r} \longrightarrow \ln f = \ln r^{-3} + c' \longrightarrow f = c/r^3.$$

[4 marks]

(b) (Unseen)

(i) The function $f(t)$ is:

$$f(t) = \begin{cases} 1 + t/b & -b \le t \le 0 \\ 1 - t/b & 0 \le t \le b \end{cases}$$

[2 marks]

Its Fourier transform is:

$$\begin{aligned}
\mathcal{F}[f(t)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-b}^{0} \left(1 + \frac{t}{b}\right) e^{-i\omega t} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{b} \left(1 - \frac{t}{b}\right) e^{-i\omega t} dt \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{b} \left(1 - \frac{t}{b}\right) \cos \omega t \, dt \\
&= \sqrt{\frac{2}{\pi}} \left(\left[\frac{\sin \omega t}{\omega}\right]_0^b - \left[\frac{t \sin \omega t}{\omega b}\right]_0^b + \frac{1}{b} \left[\frac{-\cos \omega t}{\omega^2}\right]_0^b \right) \\
&= \sqrt{\frac{2}{\pi}} \, \frac{(1 - \cos \omega b)}{b\,\omega^2}.
\end{aligned}$$

[4 marks]

(ii) This function, $g(t)$, consists of two segments of the same shape as $f(t)$ in (i), but with each one shifted and scaled. The first segment has width one and it is centered at $t = 1/2$. Hence for this segment $b = 1/2$ and the argument t becomes $(t - 1/2)$. The second segment has width two and it is centered at $t = 2$. Hence it has $b = 1$ and the argument t becomes $(t - 2)$. Hence, the Fourier transform of this function is

$$\mathcal{F}[g(t)](\omega) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\omega^2}\left(e^{-iw/2}\,2(1 - \cos\omega/2) + e^{-i2w}\,(1 - \cos\omega)\right).$$

[4 marks]

(a) $\frac{dy}{dx} + \frac{2y}{x} = \frac{4}{x}$ (1)

First solve the homogeneous equation $\frac{dy}{dx} = -\frac{2y}{x}$

$\Rightarrow y_{hom} = C/x^2$ [1mark]

To solve the inhomogeneous eq-n use the varying const method: $C \rightsquigarrow C(x)$ $\Rightarrow y' = \frac{C'}{x^2} - \frac{2C}{x^3}$: plug in the eq-n (1) $\Rightarrow$

$\frac{C'}{x^2} - \cancel{\frac{2C}{x^3}} + \cancel{\frac{2C}{x^3}} = \frac{4}{x}$ $\Rightarrow C' = 4x$

$\Rightarrow C = 2x^2 + \text{const.}$ $\Rightarrow y_{inh} = 2 + \frac{\text{const}}{x^2}$ [2marks]

[if the answer is correct, will accept any method]

Substitute back into (1) [1mark]

$\Rightarrow \cancel{-\frac{2C}{x^3}} + \frac{4}{x} + \cancel{\frac{2C}{x^3}} = \frac{4}{x}$ ✓ works

unseen

(b) $y'' - 6y' + 5y = 0$ use $y = Ae^{\lambda t}$

$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$ $\Rightarrow \lambda = 1, 5$ roots [2marks]

$y_{homo} = Ae^{5t} + Be^{t}$

unseen

$y_{part.\,inhomo} = ax + bx^2 + d$ trial f-n.

RHS $= 4 + x + 10x^2$

plug in eq-n: $2b - 6a - 12bx + 5ax + 5bx^2 + 5d = \text{RHS}$

sol-n: $a = 5$; $b = 2$; $d = 6$

$y_{general\ inho} = Ae^{5t} + Be^{t} + 5x + 2x^2 + 6$ ✓

[2marks]

unseen

(c) n^{th} order lin. homo. ODE with const coef-s a_i.

$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0 y = 0$

One searches for the general sol-n by substituting

$y(x) = \sum_{i=1}^{n} c_i e^{\lambda_i x}$ [1mark]

where each λ_i is a root of the auxiliary eq-n:

$\sum_{j=1}^{n} a_j \lambda^j = 0$. [clearly there are n roots] [1mark]

When 1 root e.g. λ_1 has the k-fold degeneracy

the roots are $\underbrace{\lambda_1 \ldots \lambda_1}_{\text{K repeated roots } \lambda_1} \lambda_{k+1}\ \lambda_{k+2} \ldots \lambda_n$

Then the correct gen. sol-n is obtained by:

$y_{gen\ homo}(x) = (C_1 + C_2 x + \ldots + C_k x^{k-1})e^{\lambda_1 x} + C_{k+1}e^{\lambda_{k+1}x} + \ldots + C_n e^{\lambda_n x}$.

[still n independent coefficients $c_1 \ldots c_n$] [2marks]

bookwork

bookwork

(d) $1 \cdot y'' + p(x)\, y' + q(x)\, y = 0$ ⟸ 2^{nd} order linear ODE

Singularities are when $p(x = x_0) = \infty$
$q(x = x_0) = \infty$

regular – singular pt $x = x_0$ when ✓

$(x - x_0)\, p(x)$ is analytic at $x \to x_0$ (finite)

and

$(x - x_0)^2\, q(x)$ is analytic at $x \to x_0$ (finite) ✓

if non-analytic ⟹ essential singularity

[2 marks]

unseen

Legendre eq-n

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)\, y = 0$$

$\uparrow$ $x \to \pm 1$ $\qquad \uparrow$ $1/x \to 0$ sing

(essential sing at $x \to \infty$)

regular-sing points are at

$x = \pm 1$ [2 marks]

unseen

(e) Using the Rodriguez formula we get

$P_0 = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2 - 1)$ [1 mark]

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ [1 mark]

$$\int_{-1}^{1} dx\, P_\ell\, P_k = \begin{cases} \ell \text{ even},\ k \text{ odd} = 0 \\ \ell = 0,\ k = 2 \quad = \int_{-1}^{1}\left(\frac{3}{2}x^2 dx - \frac{1}{2}dx\right) \neq 0 \quad \underbrace{\frac{x^3}{2}\Big|_{-1}^{1}}_{1} - 1 \\ \ell = 1\ ;\ k = 3 \\ \Downarrow \\ \frac{1}{2}\int_{-1}^{1} dx\,(5x^4 - 3x^2) = \frac{1}{2}(x^5 - x^3)\Big|_{-1}^{1} = 0. \end{cases}$$

unseen

all $\int$'s $\equiv 0$.

As expected ✓

[2 marks]

End

unseen

(f) $F = -3y$ clearly satisfies $y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} = 3x$
by substitution [1 mark]

Need now gen. sol. n of $y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} = 0$

$F(x,y) = F(p(x,y))$ $\qquad dp = 0 = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$

$y\cancel{\frac{dF}{dp}}\frac{\partial p}{\partial x} - x\cancel{\frac{dF}{dp}}\frac{\partial p}{\partial y} = 0$ ⟸

⟹ $-y\cancel{\frac{\partial p}{\partial y}}\frac{dy}{dx} = x\cancel{\frac{\partial p}{\partial y}}$

⟹ $y\,dy + x\,dx = 0$ $\qquad p: x^2 + y^2 = \text{const.}$

[2 marks]

Gen. sol. n of the inh. eq. n : $F(x,y) = p(x^2 + y^2) - 3y$

$\uparrow$ arbitrary f. n.

[1 mark]

bookwork

(g) 1-dim wave eq-n : $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$ [1 mark]

$\psi(x,t) = \psi(p)$; $p = x + \lambda t$

$\frac{\partial^2 \psi}{\partial x^2} = \frac{d^2\psi}{dp^2}$; $\frac{\partial^2 \psi}{\partial t^2} = \frac{d^2\psi}{dp^2} \cdot \lambda^2$

⟹ eq-n ⟹ $1 = \lambda^2 / c^2$ ⟹ $\lambda = \pm c$

gen sol-n $\psi(x,t) = f(x + ct) + g(x - ct)$

[1 mark]

2dim Laplace : $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ [1 mark]

$p = x + \lambda t$

⟹ $1 + \lambda^2 = 0$; $\lambda = \pm i$

⟹ $\psi(x,t) = f(x + iy) + g(x - iy)$ ✓

[1 mark]

|End|

unseen

a) As the Legendre linear eq-n it takes the form:

$$(x-1)^2 \frac{d^2y}{dx^2} - 2y = 0 \qquad \text{i.e. } \alpha x + \beta = x - 1$$ [2 marks]

The substitution is $x - 1 = e^t$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x-1}\frac{dy}{dt}$; $\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\frac{1}{(x-1)^2}$ [1 mark]

$\Rightarrow$ the eq-n becomes: $\cancel{\frac{(x-1)^2}{(x-1)^2}}\left(\ddot{y} - \dot{y}\right) - 2y = 0$ [2 marks]

Eq-n w. const coeff-s ✓

b) Use : $y = e^{\lambda t}$ to solve it [1 mark]

$\Rightarrow$ auxiliary eq-n $\lambda^2 - \lambda - 2 = 0$ $\Rightarrow$ 2 sol-s $\lambda_\pm = 2; -1$

$y_1 = e^{2t} = e^{2\log(x-1)} = (x-1)^2$ [2 marks]

$y_2 = e^{-1} = \frac{1}{1-x}$ [3 marks]

2 particular sol-s

bookwork

c) Wronskian $W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ [1 mark]

$W = |y_1 y_2' - y_2 y_1'| = |-3| = 3 \neq 0$ [2 marks]

hence y_1 and y_2 are linearly independent.

$y(x)_{\text{general}} = c_1 (x-1)^2 + c_2 \frac{1}{1-x}$ c_1, c_2 ∀ constants. [2 marks]

unseen

d) $x = 0$ is a regular point of the equation.

Hence search for the power-series in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

compute $y'(x)$ and $y''(x)$ [2 marks]

$\Rightarrow$ substitute to the eq-n.

bookwork

Hence obtain a recursion relation for a_n's.

It can then be solved to determine 2 particular sol-s [2 marks]

End

book-work

a) $\ddot{x} + \omega_0^2 x = 0 \Rightarrow$ general sol-n : $x = A\cos\omega_0 t + B\sin\omega_0 t$

[1mark]

unseen

To remain in equilibrium at rest $\begin{cases} x(0) = 0 \\ \dot{x}(0) = 0 \end{cases} \Rightarrow x(t) = \emptyset$

[2marks]

b) The inhomogeneous ODE with the force :

$$\ddot{x} + \omega_0^2 x = f_0 \sin(at)$$

book-work

Laplace transform : $L[x]_{(s)} = \int_0^\infty e^{-st} x(t)\, dt$ [2marks]

Laplace tr. of $\ddot{x}$:

$$L[\ddot{x}]_{(s)} = s^2 L[x]_{(s)} - \underbrace{s x(0) - \dot{x}(0)}_{\equiv \emptyset \text{ from initial cond.-s}}$$

L[eq-n] $\Rightarrow$ $(s^2 + \omega_0^2) L[x](s) = f_0 \underbrace{L[\sin(at)]_{(s)}}_{as/(s^2+a^2) \text{ from the [hint]}}$ [2marks]

(A) $\Rightarrow L[x](s) = \dfrac{f_0\, a s}{(s^2+\omega_0^2)(s^2+a^2)}$

[2marks]

when $\omega_0^2 \neq a^2 \Rightarrow \dfrac{f_0 a}{\omega_0^2 - a^2}\left(\dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+\omega_0^2}\right)$ [2marks]

unseen

use [hint] to do inverse Laplace transform : $\Downarrow$ $\omega_0^2 \neq a^2$

$$x(t) = \frac{f_0 a}{\omega_0^2 - a^2}\left(\frac{\sin at}{a} - \frac{\sin\omega_0 t}{\omega_0}\right)$$

harmonic motion $\Leftarrow$

[2marks]

c) for $a^2 = \omega_0^2$ go back to (A) : $L[x](s) = \dfrac{f_0\, a s}{(s^2+a^2)^2}$

[3marks]

use the 2^{nd} hint to do the inverse Laplace transform :

$$x(t) = \frac{f_0}{2a^2}\left(\sin at - at\cos at\right)$$ [2marks]

this term grows linearly with time.

This trajectory is the resonance motion of the pendulum disturbed by the force w frequency $a^2 = \omega_0^2$

[2marks]

End.