ELECTROMAGNETISM

Level 2 Physics problems – Foundations of physics 2

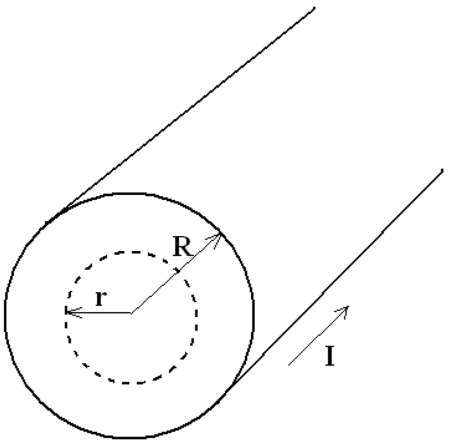
Solution 4 Cycle 2 Version 1

Professor D P Hampshire – 2nd Year Physics Lecture Course

Information underlined or indicated by red text is required for marks to awarded. The mark scheme is a guide and solutions should not be considered to be unique. Marks are awarded for correct relevant Physics.

1. Ampère's law;

$$\oint \underline{\boldsymbol{B}}.\,d\underline{\boldsymbol{l}} = \mu_0 I_{encl} \qquad 1\text{-}1$$



<u>Inside the wire:</u>
On the circle of radius r, B is tangential to the circle, and of constant magnitude. The circumference of the circle is 2πr, therefore,

$$\oint \underline{\boldsymbol{B}}.\,d\underline{\boldsymbol{l}} = B\oint dl = 2\pi r \cdot B$$

The current flowing inside the loop of radius r will vary depending on the radius of r. The area current density J, of the whole wire is given by,

$$J = \frac{I}{area} = \frac{I}{\pi R^2}$$

The area of the loop is given by πr^2, so the current passing through the loop is given by:

$$I_{encl} = area \cdot \mathrm{J} = \pi r^2 \frac{I}{\pi R^2} = \frac{r^2}{R^2} I \qquad 1\text{-}2$$

So, from Ampère's law,

$$2\pi r B = \mu_0 \frac{r^2}{R^2} I$$

$$=> B = \frac{\mu_0 r I}{2\pi R^2}$$ 1-3

Outside the wire:
The current inside the loop is simply I, so Ampère's law gives: 1-4

$$2\pi r B = \mu_0 I$$

$$=> B = \frac{\mu_0 I}{2\pi r}$$ 1-5

1 mark for correct answers, 1-3 and 1-5
[Qn 1: 1 mark total]

2.

a) $\boldsymbol{J}$ is the current density. Because J varies with r, to find the total current flowing through the cylinder, J must be integrated from r = 0 to r = a in 2D polar co-ordinates.

$$I = \int_0^{2\pi} \int_0^a J(r) r \, dr d\theta = \frac{2I_0}{\pi a^2} \int_0^{2\pi} \int_0^a \left[1 - \left(\frac{r}{a}\right)^2\right] r dr d\theta$$ 2-1

Where $rdrd\theta$ is the Jacobian for this coordinate system. Evaluating the r integrand first then the theta integrand.

$$= \frac{2I_0}{\pi a^2} \int_0^{2\pi} \frac{a^2}{2} - \frac{a^4}{4a^2} d\theta = \frac{I_0}{2\pi} \int_0^{2\pi} d\theta$$

$$= I_0$$ 2-2

1 mark for correct set up 2-1 and correct answer 2-2.

b) By the same reasoning as in 1), the current outside the wire will be given by,

$$B = \frac{\mu_o I_o}{2\pi r}$$ 2-3

1 mark if correct result 2-3.

c) Inside the wire, it is the current enclosed by a loop of radius r that is important. To find this current, repeat the integral in a), but change the limits from 0→a to 0→r.

$$I = \int_0^{2\pi} \int_0^{r} J(r') r' \, dr' d\theta$$ 2-4

$$I = \frac{2I_0}{\pi a^2} \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4a^2} d\theta = \frac{4\mathrm{I}_0}{2a^2}\left(r^2 - \frac{r^4}{2a^2}\right)$$

So, Ampère's law gives;

$$2\pi rB = \frac{4I_0\mu_0}{2a^2}\left(r^2 - \frac{r^4}{2a^2}\right)$$

$$=> B = \frac{I_0\mu_0}{2\pi a^2} r\left(2 - \frac{r^2}{a^2}\right) \qquad 2\text{-}5$$

1 mark for correct set up 2-4.
1 mark for correct answer 2-5.
[Qn 2: 4 marks total]

3. $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \rho/\epsilon_0$;

Measure spatial dependence of force between two charges. 3-1

$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0$;

Use test current (or current loop) to measure the spatial dependence of any B-field. 3-2

$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \boldsymbol{B}}{\partial t}$;

Measure the voltage induced in a solenoid. 3-3

$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \epsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$;

Measure spatial dependence of forces between two wires carrying a current. 3-4

1 mark for any two of 3-1, 3-2, 3-3 or 3-4 correct.
Or 2 marks if all of 3-1, 3-2, 3-3 or 3-4 correct.
[Qn 3: 2 marks total]

4.

a)

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \mathrm{i}\left(k_x E_{0x} + k_y E_{0y} + k_z E_{0z}\right)\exp[i\left(k_x x + k_y y + k_z z - \omega t\right)] \qquad 4\text{-}1$$

This can only equal zero if,

$$\left(k_x E_{0x} + k_y E_{0y} + k_z E_{0z}\right) = 0$$

This means,

$$\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{E_0}} = \left(k_x E_{0x} + k_y E_{0y} + k_z E_{0z}\right) = 0 \qquad 4\text{-}2$$

i.e. $\underline{\boldsymbol{k}}$ is perpendicular to $\underline{\boldsymbol{E}}_0$. The same goes for $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0$. 4-3

1 mark if equation 4-2 and statement 4-3 is written.

b) Use;

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \boldsymbol{B}}{\partial t} = i\omega \underline{\boldsymbol{B}} \qquad 4\text{-}4$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0x} & E_{0y} & E_{0z} \end{vmatrix} \exp i(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t) \qquad \text{4-5}$$

$$= \begin{pmatrix} ik_y E_{0z} - ik_z E_{oy} \\ ik_z E_{0x} - ik_x E_{0z} \\ ik_x E_{0y} - ik_y E_{0x} \end{pmatrix} \exp i(\underline{\boldsymbol{k}}.\underline{\boldsymbol{r}} - \omega t) = i\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = i\omega \underline{\boldsymbol{B}} \qquad \text{4-6}$$

$$=> \underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = \omega \underline{\boldsymbol{B}} \qquad \text{4-7}$$

Use;

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} = -i\omega \mu_0 \varepsilon_0 \underline{\boldsymbol{E}} = -\frac{i\omega}{c^2} \underline{\boldsymbol{E}} = -i\omega \frac{k^2}{\omega^2} \underline{\boldsymbol{E}} = -i\frac{k^2}{\omega} \underline{\boldsymbol{E}} \qquad \text{4-8}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \begin{pmatrix} ik_y B_{0z} - ik_z B_{0y} \\ ik_z B_{0x} - ik_x B_{0z} \\ ik_x B_{0y} - ik_y B_{0x} \end{pmatrix} \exp i(\underline{\boldsymbol{k}}.\underline{\boldsymbol{r}} - \omega t) = i\underline{\boldsymbol{k}} \times \underline{\boldsymbol{B}} = -i\underline{\boldsymbol{B}} \times \underline{\boldsymbol{k}} \qquad \text{4-9}$$

$$=> k^2 \underline{\boldsymbol{E}} = \omega \underline{\boldsymbol{B}} \times \underline{\boldsymbol{k}} \qquad \text{4-10}$$

1 mark if equation 4-7 is derived using steps 4-4 and 4-6.
1 mark if equation 4-10 is derived using steps 4-8 and 4-9.
[Qn 4: 3 marks total]

Total for all questions 10 marks