Relativistic Electrodynamics

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Maxwell said:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \qquad \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \varepsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$$

$$\underline{F} = \frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B})$$

In other words:

$$\partial_\mu F^{\mu\nu} = \frac{j^\nu}{c\epsilon_0}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$f^\mu = \frac{dp^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} u_\nu$$

Preamble

These lecture notes were LaTeXed from my hand-written copies by Mr. Alex J. Dent during the quieter moments of a summer studentship at the Institute of Particle Physics Phenomenology, Department of Physics. The notes are based on those from earlier incarnations of the course by Prof. Peter Richardson and Dr. Adrian Signer, but contain a significant amount of additional material. I take responsibility for any remaining mis-prints - please report any you might find to `jeppe.andersen@durham.ac.uk`.

This course in Relativistic Electrodynamics introduces the mathematical language of Special Relativity expressed in tensor notation to Maxwell's Equations. Along the way we will solve interesting physics-problems involving the electromagnetic field generated from particles in relativistic motion. My job is to guide you through this extremely satisfying course in theoretical physics, covering the mathematical language of Lorentz-covariant formalism, detailed theoretical treatments of the electromagnetic field generated by moving objects, and touching upon symmetries in Lagrangian formulations, which ultimately led to the postulations of the existence of the Higgs boson, several decades before its discovery.

There are several aspects of the teaching:

- **Lectures** will be based on the content in these lecture notes - they identify the most important content (for this course) of the material in "Modern Electrodynamics" by A. Zangwill.
- **Workshops** will give examples of applications of the theory developed during the lectures; and the workshops will be an extra chance for you to ask me or the PhD students helping me any questions you might have to the material.
- Working with the **Homework** and the workshop material will teach you how to apply your new-found understanding and develop the skills you need to finish with a successful exam.

Your job is to help me by engaging with the material – we each learn and assimilate knowledge in different ways. Please find one that suits you. Please ask questions during the lectures[1] – you will (probably) not be the only one sitting there with the question. Anyone can buy the book - the lectures are your unique chance, offered only by a university course, to ask direct questions to help in *your* understanding. Please make use of it.

[1] In these times of change and "asynchronous" lectures, the chance to ask questions will be during office hours and workshops.

Defined learning outcome:
Subject-specific Knowledge:

Having studied this module, students will have developed a working knowledge of tensor calculus, and be able to apply their understanding to relativistic electromagnetism.

Subject-specific Skills:

In addition to the acquisition of subject knowledge, students will be able to apply the principles of physics to the solution of complex problems. They will know how to produce a well-structured solution, with clearly-explained reasoning and appropriate presentation.

The "weekly" problems and the example classes are vital for you to achieve these goals.

Books

1. "Modern Electrodynamics", A. Zangwill
2. "Modern Problems in Classical Electrodynamics", C. A. Brau
3. "Introduction to Electrodynamics", David J Griffiths.
   sec. 10.1, 10.2, 11.1, 11.2, 12.1, 12.2, 12.3
4. SR "Introduction to special Relativity", W Rindler
5. SR "Relativity made Relatively Easy", Steane
6. ED "Classical Electrodynamics", J. D. Jackson

Structure

The content of the course can be categorised into the following logical chapters

1. Basics of relativity, space-time, tensor calculus
2. Relativistic kinematics and dynamics
3. Scalar and vector potential, gauge transformations
4. Radiation from special sources (point charge, dipole)
5. Relativistic fields,...

RED builds on the 1st and 2nd year courses on Special Relativity and Electromagnetism. It prepares you for final year particle theory and General Relativity. It is an introduction to the theoretical concepts that led to the postulation of the Higgs Boson! It introduces notation used by the experts to effectively work with theories for relativistic particles and fields.

Modern physics exposes the underlying symmetries of nature. One must make sure to use a notation and language that does not hide the symmetries of your theory!

Contents

Chapter 1

Special Relativity

1.1 Introduction - Electromagnetism and Special Relativity

The rich structure of classical (i.e. non-quantum) electromagnetism is based on the four Maxwell's equations (detailing the connections between the electromagnetic fields and sources) and the Lorentz force (detailing the relation between the force experienced by a particle of charge q and velocity v in the presence of electromagnetic fields $\underline{E}$ and $\underline{B}$). The Maxwell Equations, connecting the electromagnetic fields $\underline{E}, \underline{B}$ with the sources of the charge density ρ, and current density $\underline{j}$, are given by[1]

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \tag{i}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{ii}$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{iii}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \varepsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}. \tag{iv}$$

The Lorentz force, experienced by a particle with charge q travelling through an $\underline{E}$ and $\underline{B}$ field with velocity $\underline{v}$, is given by

$$\underline{F} = q\ (\underline{E} + \underline{v} \times \underline{B}), \tag{1.1}$$

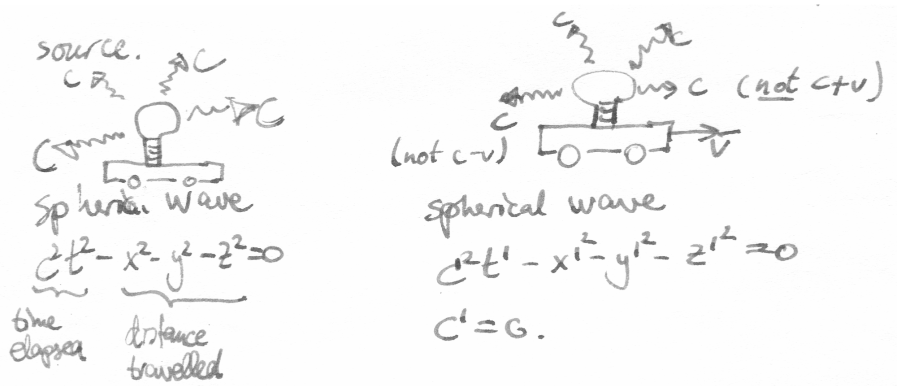
The Maxwell equations predict the existence of electromagnetic waves, which propagate at a fixed speed (in a vacuum),

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}},$$

where c is the predicted speed of light, μ_0 is the permeability and ϵ_0 is the permittivity. But this speed is measured relative to what? This was a big question at the end of the 19th century. Maxwell's Electromagnetism predicts the speed of light in vacuum to be independent of the

[1] Appendix A lists the form of the differential operators in various coordinate systems, and contain further useful results from vector analysis.

frame of the observer or the speed of the emitter. This may seem odd in both a particle and in a wave interpretation of light. If light is a particle, surely its speed depends on the speed of the emitter? If it is a wave, surely the measured speed depends on the speed of the medium? The Michelson–Morley experiment in 1887 settled that indeed, the prediction from Maxwell's electromagnetism was right, and the speed of light is the same in all frames, and independent on the speed of the emitter. This though means that electromagnetism is **not invariant** under Galilean transformations (simple addition of velocities).



Several other simple setups may make you question the applicability of Maxwell's equations and the Lorentz force. Consider for example a simple setup of an infinite and uniform magnetic field and no electric field. Let a charged particle travel through this field with velocity $\underline{v}$ (which is not parallel to the magnetic field $\underline{B}$). According to the Lorentz force this particle will experience a force

$$\underline{F} = q(\underline{v} \times \underline{B}) \qquad \text{(there is no electric field)}$$

and therefore undergo some acceleration (perpendicular to the current velocity), and not travel in a straight line. However, if the situation is analysed from the rest frame of the particle, then the velocity of the particle is obviously $\underline{v} = 0$, and there is no force generated by the magnetic field! One might therefore conclude that the particle would experience no acceleration, and therefore any other inertial frame would have the particle travelling in a straight line. This would be in contradiction with the previous conclusion, and the conclusion as to whether a particle undergoes acceleration or not would seem to depend on which inertial frame is used for the analysis (remember, this would violate the first law of Newton!). Luckily such conclusion is premature, and is the result of an incomplete understanding of the frame-dependence of the electric and magnetic fields. This is the focus of chapter 3.

As it turns out Maxwell's electromagnetism gives remarkably consistent answers during changes between frames, when the sources for the fields are included. To illustrate this, consider a wire-loop and a horse-shoe magnet. Consider the situation where the magnet is moving towards a stationary wire-loop, as illustrated on figure 1.1. In this frame there can be **no magnetic force on the electrons in the wire-loop**, since the *loop* is at rest (and therefore macroscopically $\underline{v} = 0$ for each conducting electron). But since the *magnet* is moving, all points in space will have $\frac{\partial B}{\partial t} \neq 0$, and an electric field is generated by the third Maxwell equation

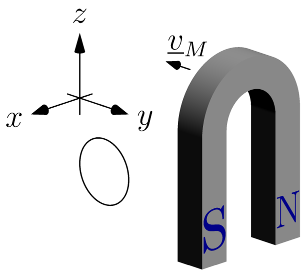


Figure 1.1: A horse-shoe magnet travelling over a stationary wire-loop. It is no coincidence that the field-lines are not drawn.

$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$. The electrons in the wire-loop will therefore feel a Lorentz force due to the generated electric field

$$\underline{F} = q\,(\underline{E} + \underline{v} \times \underline{B}) \qquad \text{(in this setup } \underline{v} = 0\text{).} \tag{1.2}$$

A calculation of the electromotive force in the wire-loop therefore proceeds as

$$\mathcal{E} = \oint_{\mathcal{C}} \underline{F}(\underline{r})/q \cdot \mathrm{d}\underline{l} = \oint_{\mathcal{S}} (\underline{\nabla} \times \underline{E}(\underline{r})) \cdot \mathrm{d}\underline{a} = -\oint_{\mathcal{S}} \frac{\partial \underline{B}(\underline{r})}{\partial t} \cdot d\underline{a} = -\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\mathcal{S}} \underline{B}(\underline{r}) \cdot \mathrm{d}\underline{a} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{1.3}$$

where we used the curl theorem $\oint_{\mathcal{C}} \underline{E} \cdot d\underline{l} = \oint_{\mathcal{S}} (\underline{\nabla} \times \underline{E}) \cdot d\underline{a}$ (Eq. (A.36)), the third Maxwell equation, and finally interchanged the order of integration and differentiation with time. Also, since the wire-loop is fixed, the only change in flux through the wire-loop is due to the changing magnetic field. The result in equation (1.3) is the Faraday's flux rule.

Let us now instead analyse the situation in the frame where the magnet is fixed at rest in space, and the loop rides with velocity $\underline{v}$ through the magnetic field. Any effect here can be due only to the magnetic field, as there is no electric field present. If we consider the wire-loop to be the curve $\mathcal{C}$, and have the wire-loop move with velocity $\underline{v}_{\mathcal{C}}$ (the same velocity for every differential element in $\mathcal{C}$ and $\mathcal{S}$), then there will be a contribution to the change in the integral not just because of the local change in $\underline{B}$, but also because the wire-loop ($\mathcal{C}$) and therefore surface $\mathcal{S}$ change with time. We start by calculating the change in the flux Φ by considering the situation at time t and $t + \delta t$:

$$\delta\Phi = \Phi(t+\delta t) - \Phi(t) = \oint_{\mathcal{S}(t+\delta t)} \underline{B}(t+\delta t) \cdot \mathrm{d}\underline{a} - \oint_{\mathcal{S}(t)} \underline{B}(t) \cdot \mathrm{d}\underline{a} \tag{1.4}$$

$$= \oint_{\mathcal{S}(t)} \delta t\, \frac{\partial}{\partial t}\underline{B}(t) \cdot \mathrm{d}\underline{a} + \left\{ \oint_{\mathcal{S}(t+\delta t)} - \oint_{\mathcal{S}(t)} \right\} \underline{B}(t) \cdot \mathrm{d}\underline{a} + \mathcal{O}(\delta t^2), \tag{1.5}$$

The contribution from the last two surface integrals can be simplified due to the *flux theorem* (equation (A.35)) and the fact that $\underline{\nabla} \cdot \underline{B} = 0$ by Maxwell's second law. Taken together, these imply that the surface integral over any closed surface of the magnetic field $\underline{B}$ is zero. The two surfaces $\mathcal{S}(t+\delta t)$ and $\mathcal{S}(t)$ form the top and bottom lid of a cylinder, where the sides of the cylinder are spanned by the wire-loop during the movement from t to $t + \delta t$. We can call this area for $\mathcal{S}_{\text{cylinder}}$. We note that the minus sign in the brackets of equation (1.4) ensures that the surface vector on the second integral is counted outwards from the volume enclosed by the cylinder, and therefore

$$\left\{\oint_{\mathcal{S}(t+\delta t)} - \oint_{\mathcal{S}(t)}\right\} \underline{B}(t)\cdot \mathrm{d}\underline{a} + \oint_{\mathcal{S}_{\text{cylinder}}} \underline{B}(t)\cdot \mathrm{d}\underline{a} = 0 \tag{1.6}$$

such that

$$\left\{\oint_{\mathcal{S}(t+\delta t)} - \oint_{\mathcal{S}(t)}\right\} \underline{B}(t)\cdot \mathrm{d}\underline{a} = -\oint_{\mathcal{S}_{\text{cylinder}}} \underline{B}(t)\cdot \mathrm{d}\underline{a}. \tag{1.7}$$

The surface element $\mathrm{d}\underline{a}$ for the cylinder can be parametrised as

$$\mathrm{d}\underline{a} = \mathrm{d}\underline{l} \times \underline{v}\ \delta t, \tag{1.8}$$

where $\mathrm{d}\underline{l}$ is the differential line-element along the curve $\mathcal{C}(t)$. The change in the flux can then be written

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \lim_{\delta t\to 0}\frac{\delta\Phi}{\delta t} = \oint_{\mathcal{S}(t)} \frac{\partial \underline{B}}{\partial t}\cdot \mathrm{d}\underline{a} \ - \ \oint_{\mathcal{C}} (\underline{v}\times\underline{B}(t))\cdot \mathrm{d}\underline{l}, \tag{1.9}$$

where we used the standard properties of the vector products $\underline{B}(t)\cdot(\mathrm{d}\underline{l}\times\underline{v}) = (\underline{v}\times\underline{B}(t))\cdot \mathrm{d}\underline{l}$ to rewrite the last integral. Now we use the third of Maxwell's equations to write $\partial\underline{B}/\partial t = -\underline{\nabla}\times\underline{E}$, and use Stoke's theorem to relate the surface integral of this curl to the line integral of $\underline{E}$. We finally arrive at

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\oint_{\mathcal{C}} (\underline{E} + \underline{v}\times\underline{B})\cdot \mathrm{d}\underline{l}. \tag{1.10}$$

Now, we set out to calculate the electromotive force generated by a Lorentz force on the electrons in this setup of a fixed magnet and moving coil. The electromotive force is

$$\mathcal{E} = \oint_{C} (\underline{E} + \underline{v}\times\underline{B})\cdot \mathrm{d}\underline{l}, \tag{1.11}$$

but a quick glance at equation (1.10) shows that also in this scenario do we find that electromotive force in agreement with **Faraday's flux rule**.

$$\mathcal{E} = \oint_{C} (\underline{E} + \underline{v}\times\underline{B})\cdot \mathrm{d}\underline{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{1.12}$$

This is exactly same result as before, even if the physical explanation for the effect is different![2]

Maxwell's electromagnetism therefore appears fully consistent under this reversal of reference frames, but they violate the Galilean idea for transformations between moving frames. The

[2]This discussion ignored the "complications" added by the charged electrons moving round the coil - however, the contribution from this complication is zero, since the velocity of the electrons can be split as $\underline{v} = \underline{v}_{\mathcal{C}} + \underline{v}_d$, with $\underline{v}_d$ the drift velocity. Since (for a thin coil) $\underline{v}_d || \mathrm{d}\underline{l}$, the contribution from the drift to the electromotive force is zero.

Dutch physicist Hendrik Anton Lorentz observed that Maxwell's equations were symmetric (as in unchanged) under special transformations of space and time, which became known as *Lorentz transformations*. This led Einstein to reject the idea that Galilean transformations should describe the transformation of coordinates between frames, and elevate the Lorentz transformation to describing how coordinates transform between moving frames. The rules for Lorentz transformations between frames of references are then the result of just **two postulates:**

1. **The principle of relativity:** The laws of physics are the same (have the same form) in all inertial reference systems (Inertial Frames).

2. **Universal speed of light:** The speed of light in a vacuum is the same for all inertial observers, regardless of the motion of the source.

The transformation between non-accelerated frames (**Inertial Frames**) are explained by **special** relativity. Transformation between frames which are themselves accelerated requires **general** relativity. In inertial frames objects under the influence of **zero total force** move at constant velocity. The principle of relativity guarantees that the results of experiments are the same in all inertial frames.

These postulates give rise to seemingly surprising phenomena that you have already studied:

1. Relativity of simultaneity - two events that are simultaneous in one inertial frame are not in general simultaneous in another inertial frame

2. Time dilation - moving clocks run slowly

3. Lorentz contractions - moving objects are shortened

We will in the following develop the formalism of four-vectors and tensors. This will prove to be a very efficient language for the study of physics obeying the laws of Special Relativity (SR).

1.2 Lorentz Transformations

First, we define the object that is being transformed. These are the coordinates of an "**event**": An occurrence which can be described by a single set of coordinates $\underline{x}$ and time t (space-time coordinates) in a given inertial frame (IF). Eg. a flash of light or a clap of hands. An event has no extension - neither temporal nor spatial.

The Lorentz transformation relates the coordinates for an event (t, x, y, z) measured in one inertial frame to those for the same event, but measured in a different inertial frame. Consider two IF, S and S' in the **standard configuration** (S' boosted along x-axis in S): The frames are set up such that they coincide at $t = t' = 0$. An event has coordinates $(t, \underline{x}) = (t, x, y, z)$ measured in S and $(t', \underline{x'}) = (t', x', y', z')$ in S'. We want to know the coordinates in S' as a function of the coordinates in S, i.e - $t'(t, x, y, z)$, $x'(t, x, y, z)$, $y'(t, x, y, z)$ and $z'(t, x, y, z)$. This is illustrated in figure 1.2. We are used to projecting 3D objects onto paper and interpret the three axes of $\underline{x}, \underline{y}, \underline{z}$ to be those of a three-dimensional coordinate system. That suffices for Gallilean transformations, where the time is the same for all frames. However, all four coordinates of space-time are involved in the Lorentz transformations, so the coordinate systems

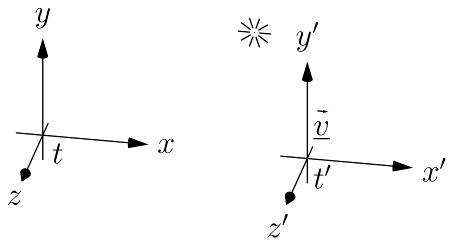


Figure 1.2: Two inertial frames in the standard configuration, and an event.

need a fourth axes. Instead of drawing a fourth axes on the coordinate systems in figure 1.2, the three-dimensional frames are labelled with the fourth coordinates t and t'.

Assuming the transformation relating the coordinates for the same event as measured in the two frames is linear in the coordinates, there is one unique transformation, which fulfils Einstein's postulates for special relativity. The derivation of the form of the Lorentz transformations is given in appendix B. The result (when the frames are arranged in the standard configuration) is a Lorentz transformation from S to S' given by

$$\begin{aligned} ct' &= \gamma ct - \gamma\beta x, \\ x' &= -\gamma\beta ct + \gamma x, \\ y' &= y, \\ z' &= z, \end{aligned}$$

with the notation: $\underline{\beta} = \frac{\underline{v}}{c}$, $\beta = |\beta| = \frac{v}{c} = \frac{|\underline{v}|}{c}$, $\gamma = \gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$ The **inverse** of the transformation can be found by letting $\underline{v} \to -\underline{v}$, i.e $\underline{\beta} \to -\underline{\beta}$.

We see that

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2, \tag{1.13}$$

since

$$\begin{aligned} c^2t'^2 - x'^2 &= \gamma^2c^2t^2 + \gamma^2\beta^2x^2 - 2\gamma^2\beta ctx - \left(\gamma^2\beta^2c^2t^2 + \gamma^2x^2 - 2\gamma^2\beta ctx\right) \\ &= c^2t^2\gamma^2\left(1-\beta^2\right) - x^2\gamma^2(1-\beta^2), \\ &= c^2t^2 - x^2. \end{aligned}$$

Here we used that $\gamma^2(1-\beta^2) = 1$. We therefore see that the quantity $c^2t^2 - x^2 - y^2 - z^2$ is invariant under the special LT considered.

We now **define Lorentz Transformations** as the set of linear transformation which leave $c^2t^2 - x^2 - y^2 - z^2$ invariant. We note that the boost along a single axis that we have considered is a special case of such Lorentz transformations. We also not that standard Galilean rotations (mixing x, y, z) are also a special case of Lorentz transformations, since they leave the length of the vector $\underline{x} = (x, y, z)$ invariant and therefore $-x^2 - y^2 - z^2$ is invariant, and t does not change under a Galilean rotation.

1.3 Vectors and 4-Vectors

Let us consider a "normal" Euclidean space (3 dimensions, no time coordinate necessary). For a fixed coordinate system (CS) K, a point P can be ascribed coordinates $\underline{x} = (x, y, z)$. A transformation (rotation, translation,...) is described by picking another coordinate system: K'. The same point P has a different set of coordinates in K': $\underline{x'} = (x, y, z)$. A coordinate transformation is found by giving $\underline{x}'$ as a function of $\underline{x}$ (or vice versa).

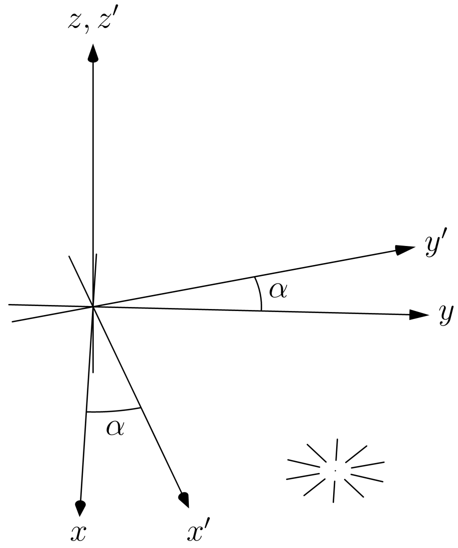


Figure 1.3: The frame S' is obtained by a rotation of S around the z-axis by and angle α (counter clockwise).

Example: Consider the coordinates for a fixed point in the two frames S, S', where the frame S' is obtained by a rotation of S around the z-axis by and angle α (counter clockwise). This situation is illustrated on figure 1.3. The point (indicated by the dot in figure 1.3) has coordinates $\underline{x} = (x, y, z)$ in the frame S. The coordinates in the frame S' is then given by

$$\underline{x'} = \begin{cases} \cos(\alpha)x + \sin(\alpha)y + 0z \\ -\sin(\alpha)x + \cos(\alpha)y + 0z \\ 0x + 0y + z \end{cases},$$

or in its vector and matrix form:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

An arbitrary rotation can be described by a rotation matrix R, such that $\underline{x'} = R\underline{x}$, or

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1.14}$$

where $R = \frac{\partial \underline{x'}}{\partial \underline{x}}$. We note that rotations mix $\{x, y, z\}$-coordinates, and that the effect of the rotation can be written in *index notation* as

$$x'_i = R_{ij}\ x_j, \quad \text{with} \quad R_{ij} = \frac{\partial x'_i}{\partial x_j}. \tag{1.15}$$

The Lorentz transformations mix all the coordinates of $\{t, x, y, z\}$. It is therefore convenient to define a 4-Vector as (ct, x, y, z) where t is multiplied by c so all the components in the vector have the same units. The Lorentz boost considered earlier can the be written

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{\partial ct'}{\partial ct} & \frac{\partial ct'}{\partial x} & \frac{\partial ct'}{\partial y} & \frac{\partial ct'}{\partial z} \\ \frac{\partial x'}{\partial ct} & \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial ct} & \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial ct} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{1.16}$$

For a boost along the x-axis with the boost parameter (rapidity) ψ $\psi = \text{arctanh}(\beta)$, such that $\cosh(\psi) = \gamma$ and $\sinh(\psi) = \gamma \cdot \beta$, the Lorentz transformation can be represented by a matrix Λ with

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\psi) & -\sinh(\psi) & 0 & 0 \\ -\sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.17}$$

This seems somewhat similar to the form of a standard rotations, but obviously operates in a different vector space.

Note that using the definition of the group of Lorentz Transformations, a standard rotation (around eg. z-axis) is also a Lorentz Transformation, and can be represented by the Lorentz Transformation Λ:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{1.18}$$

An arbitrary (general) Lorentz Transformation needs 6 parameters. 3 rotation angles and 3 boost parameters. A single boost along an arbitrary direction $\underline{n}$ is described by 4 parameters, namely the 3 of the rotation and the velocity of the boost. The Lorentz Transformation matrix is found as given by $R^{-1} \cdot \Lambda_{\text{boost}} \cdot R$, where R described the rotation, and Λ_{boost} is the Λ of the standard configuration.

1.4 Index and Tensor Notation

We will now continue the development of the efficient formalism for expressing the laws of physics, exposing the properties under Lorentz transformations. In the following, Greek indices $(\mu, \nu, \lambda, \sigma, \ldots)$ run from $0 \ldots 3$ (just like you are used to Roman indices for writing matrix and vector multiplications in component formalism running from $1 \ldots 3$).

Using the *component* or *Tensor*-notation, the 4-Vector is expressed as

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{1.19}$$

A Lorentz Transformation is then represented by:

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}. \tag{1.20}$$

Note that there is an ordering to the indices: The first index (reading from left to right) is in this case μ, and is an upper (*contra-variant*) index, and the second (ν) is in this case a lower (*co-variant*) index.

The effect of the Lorentz transformation *can* be represented by a 4x4 matrix. For example a boost along the x-axis is given:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh(\psi) & -\sinh(\psi) & 0 & 0 \\ -\sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.21}$$

In this case, the first index μ signifies the row, the second index ν signifies the column. Great care must be exercised in general when attempts are made at interpreting tensor expressions as matrices, since the result of applying several tensor multiplications is not necessarily expressed as a simple product of the matrix representations of each individual tensor.

The Lorentz Transformation of a 4-vector is then given by:

$$x'^{\mu} = \sum \Lambda^{\mu}{}_{\nu}\, x^{\nu} \equiv \Lambda^{\mu}{}_{\nu}\, x^{\nu}, \tag{1.22}$$

where we have used *Einstein's Summation Convention*: It is implicitly understood that repeated indices (in this case ν) are summed over.

Definition of Contravariant 4-vector : A contravariant 4-vector is an object that transforms under a Lorentz Transformation as

$$q'^{\mu} = \Lambda^{\mu}{}_{\nu} q^{\nu}$$

is called a Contravariant 4-Vector. (Please check that such vectors do indeed form a vector space. This is called *Minkowski space-time*).

Covarient 4-Vectors For $q^{\mu} = (q^0, q^1, q^2, q^3)$ define $q_{\mu} = (q^0, -q^1, -q^2, -q^3)$.

The Metric The *metric* $g^{\mu\nu}$ is defined as

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.23}$$

Hence $q_{\mu} = g_{\mu\nu} q^{\nu}$, and also, $q^{\mu} = g^{\mu\nu} q_{\nu}$.

We can now raise and lower an index, i.e. transform between the contravariant and covariant vectors, by applying the metric,

$$g^{\mu}{}_{\nu} = g^{\mu\rho} g_{\rho\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\mu}{}_{\nu}. \tag{1.24}$$

A covariant 4-vector transforms as

$$x'{}_{\mu} = g_{\mu\nu}\, x'^{\nu} = g_{\mu\nu}\, \Lambda^{\nu}{}_{\rho}\, x^{\rho} = g_{\mu\nu}\, \Lambda^{\nu}{}_{\rho}\, g^{\rho\sigma}\, x_{\sigma} = \Lambda_{\mu}{}^{\sigma}\, x_{\sigma}.$$

Recall: $\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$, but $\Lambda_{\mu}{}^{\sigma} = \frac{\partial x'{}_{\mu}}{\partial x_{\sigma}}$, since it transforms covariant vectors.

Why would we introduce covariant 4-vector, you might ask, since they do not contain any more information than their contra-variant counterparts. However, consider the **Minkowski**

scalar product of a single 4-vector x^μ. This can now be written as a product of the contravariant and covariant form of the vector:

$$x^\mu x_\mu = x^0x_0 + x^1x_1 + x^2x_2 + x^3x_3 = x^0x^0 - x^1x^1 - x^2x^2 - x^3x^3 = ((ct)^2 - x^2 - y^2 - z^2).$$

We already demonstrated this is Lorentz Invariant (a Lorentz Scalar). We can use this Lorentz Invariance to derive a property of a special contraction of the Lorentz Transformation tensor, ${\Lambda^\mu}_\nu {\Lambda_\mu}^\rho$:

$$x^\mu x_\mu = x'^\mu x'_\mu = ({\Lambda^\mu}_\nu x^\nu)({\Lambda_\mu}^\rho x_\rho) = ({\Lambda^\mu}_\nu {\Lambda_\mu}^\rho) x^\nu x_\rho.$$

Since this is true for all x^μ, we find that

$${\Lambda^\mu}_\nu {\Lambda_\mu}^\rho = {\delta_\nu}^\rho (= {g_\nu}^\rho) \tag{1.25}$$

(note that the combination of Λ's is not the matrix product of the two Λ's!). Equation (1.25) immediately says that the inverse of the transformation ${\Lambda^\mu}_\nu$ is given by ${\Lambda_\mu}^\rho$. But since ${\Lambda^\mu}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$, its inverse (which we know is ${\Lambda_\mu}^\sigma$) can be expressed as ${\Lambda_\mu}^\sigma = \frac{\partial x^\sigma}{\partial x'^\mu}$.

This statement looks deceptively simple when expressed in tensor notation. However, it must be remembered that the left hand side of Eq. (1.25) is not expressed as a matrix multiplication - the index μ refers to the rows of the matrices representing both ${\Lambda^\mu}_\nu$ and ${\Lambda_\mu}^\nu$. To make it into a matrix multiplication, the row and column index for one of the matrices would need to be interchanged - this is called the transpose matrix. Let M_1 denote the matrix representing the tensor ${\Lambda^\mu}_\nu$: $M_1 \equiv {\Lambda^\mu}_\nu$ and M_2 denote the matrix representing the tensor ${\Lambda_\mu}^\nu$. We then have

$$M_1^{-1} = M_2^T. \tag{1.26}$$

One finds using Eq. (1.25) or Eq. (1.26) that the determinant of the matrix representing the general Lorentz Transformation is given by $\det(\Lambda) = \pm 1$.

Now, any contravariant 4-vector y^μ transforms like x^μ. From this one finds that not only is $x^\mu x_\mu$ Lorentz Invariant, but also $y^\mu x_\mu$ is Lorentz invariant. Proof:

$$y'^\mu x'_\mu = ({\Lambda^\mu}_\nu y^\nu)({\Lambda_\mu}^\sigma x_\sigma) = ({\Lambda^\mu}_\nu {\Lambda_\mu}^\sigma) y^\nu x_\sigma = y^\nu x_\nu,$$

or:

$$\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\mu} y^\mu x_\sigma = \frac{\partial x^\sigma}{\partial x^\nu} y^\nu x_\sigma = {\delta^\sigma}_\nu y^\nu x_\sigma = y^\nu x_\nu.$$

It does not matter which of the 4-vectors x, y enter in contravariant form, and which is covariant, since

$$y^\mu x_\mu = y^\mu (g_{\mu\nu} x^\nu) = (y^\mu g_{\mu\nu}) x^\nu = y_\nu x^\nu = y_\mu x^\mu.$$

One therefore writes the product as just $y.x$ or $x.y$. This is called the Minkowski scalar product.

1.5 Tensors and the Relativity Principle

The laws of Physics should now be expressed in terms of quantities with the right properties under LT:
Scalar: A single quantity (number) that does not change under a LT. Tensor of rank 0 with 0 indicies. 4^0 entries.
4-vector: An entity of 4 numbers that transform as: contravariant: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$. covariant: $x_\mu \to x'_\mu = \Lambda_\mu{}^\nu x_\nu$. Both transforms imply $\Lambda^\mu{}_\nu \Lambda_\mu{}^\rho = \delta_\nu{}^\rho$. A 4-vector is a Tensor of rank 1, it has 1 index and $4^1 = 4$ entries.
Tensor of rank 2: A quantity with 2 indicies and $4^2 = 16$ entries, which transforms as: contravariant: $F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$, covariant: $F_{\mu\nu} \to F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$, mixed $F^\mu{}_\nu \to F'^\mu{}_\nu = \Lambda^\mu{}_\rho \Lambda_\nu{}^\sigma F^\rho{}_\sigma$.
Note: count only <u>free</u> indicies(i.e not repeated indicies). eg $g^{\mu\nu}$ is a tensor of rank 2 (The Metric tensor). but $g^\mu{}_\mu = \delta^\mu_\mu = 4$ is a tensor of rank 0 (scalar).
Reason for this notation: By writing the law of physics in terms of (Minkowski) tensors, it is automatically ensured they fulfil relativity principle. The two Maxwell equations

$$\underline{\nabla} \cdot \underline{E} = -\frac{\rho}{\epsilon_0} \tag{1.27}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t} \tag{1.28}$$

are contained in the single simple statement that (where $F^{\mu\nu}$ and j^ν are defined later)

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\varepsilon_0} j^\nu. \tag{1.29}$$

It is clear that this statement manifestly respects the principle of relativity, since it is a statement between 4-vectors.

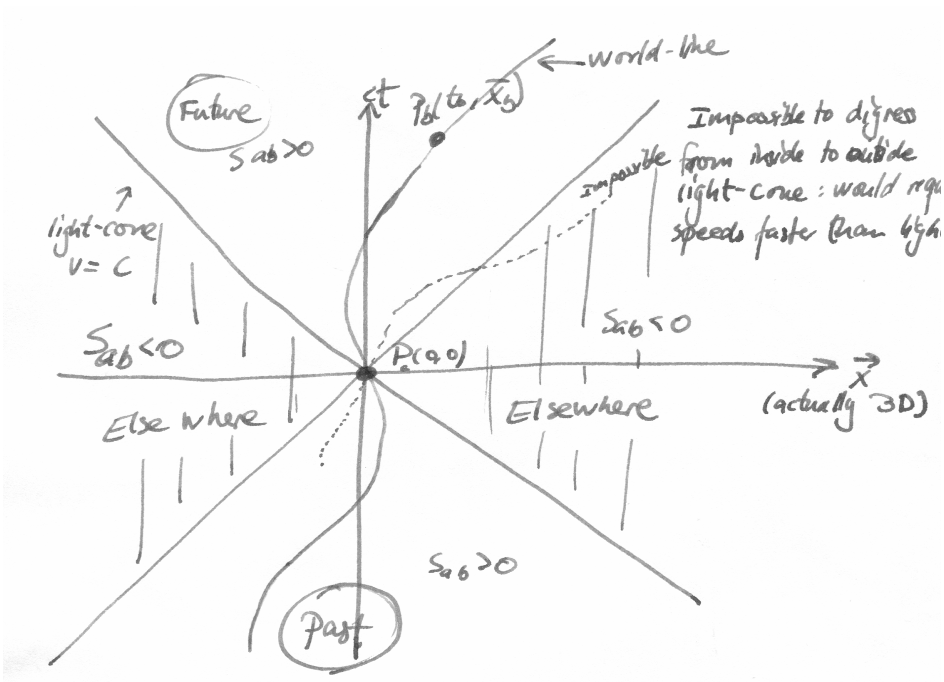
1.6 Basics of Relativity

Consider 2 events x^μ,y^μ (= 0 w.o.l.g). The (LT-) invariant distance between the two events,

$$s = (x-y)^2 = (x-y)(x-y) = (x-y)^\mu (x-y)_\mu.$$

Lorentz- invariant classification of the distance:
$s = 0$: light-like.
$s < 0$: Space-like (The contribution to the minkowski product from the spatial components are biggest). There exists a frame s' such that the events happen at the same time but at different positions.
$s > 0$: Time-like (The contribution from the 0th component is largest). There exists a frame s'' such that the events happen at the same point but at different times.
Consider a particle in a certain IF, where it has $\underline{x} = 0$ at $t = 0$.

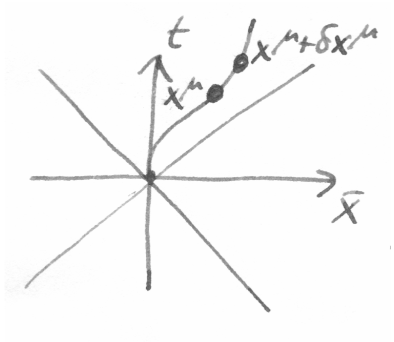
Future: These events occur after $P(0,0)$ for all observers. Only these points in phase-space can be influenced by P_a.



Past: These events occur before $P(0,0)$ for all observers. Only these points can influence/have a causal effect on P_a.
Elsewhere: For some observers, these happen before $P(0,0)$, and for some after. There can be no causal connection with P_a.

1.7 Proper time τ and time dilation

Consider a particle with a certain world line. The Proper Time (τ) is defined as the time as measured by a clock moving with the particle. Let the particle have an instantaneous velocity $\underline{v}(t)$ with respect to some inertial frame. In the time interval dt, its position changes $d\underline{x} = \underline{v}(t)dt$.



$$ds^2 = \delta x^\mu \delta x_\mu$$

ds^2 in $S = ds^2$ in particle frame.

$$\begin{aligned}
c^2 dt^2 - d\underline{x}^2 &= c^2 d\tau^2 \\
\rightarrow c^2 dt^2 - v^2 dt^2 &= c^2 d\tau^2 \\
\rightarrow d\tau &= \sqrt{1 - \frac{v^2}{c^2}}\, dt \\
&= \frac{dt}{\gamma(v(t))}
\end{aligned}$$

After a time difference $t_2 - t_1$ in S, the proper time difference is,

$$\triangle t = \int_{t=t_1}^{t=t_2} d\tau = \int_{t=t_1}^{t=t_2} \frac{dt}{\gamma(v(t))}.$$

Note: $\gamma(v) > 1 \rightarrow \triangle\tau < \triangle t \rightarrow$ A moving clock runs more slowly than a stationary clock. Time dilation.
"Train Paradox" - strictly cannot treat the setup within SR since there is no way of bringing two trains back together without acceleration. Life time of decaying particles.

1.8 Length Contraction

Consider a rod with length $l_0 = \triangle x = |\underline{x_e} - \underline{x_b}|$ as rest in an inertial frame. l_0 is called the proper length of the rod.

What is the length of the rod as measured by an observer moving with velocity v along the direction of the rod (Eg. x-axis).

To measure the length in S' we measure the position of the two endpoints of the rod at some time in S' (which does not correspond to simultaneous measurements in S),

$$l_0 = x_e - x_b = \gamma(x'_e + vt'_e) - \gamma(x'_b + vt'_b) = \gamma(x'_e - x'_b) = \gamma l',$$

as $[t'_e = t'_b]$, where l' is the length as measured by the moving observer.

$$l' = \frac{l_0}{\gamma(v)}.$$

$l' < l_0$, thus for a moving observer, the rod is contracted $\rightarrow$ length contraction(moving objects are shortened). No effect if the observer moves perpendicularly to the direction of the rod. Volume transforms as,

$$V' = \frac{V_0}{\gamma(v)}.$$

Why is the Pole & Barn paradox not a paradox?

1.9 4-velocity

So far we know only two types of four-vectors: $x^\mu, \partial_\mu = \frac{\partial}{\partial x^\mu}$ [Note $\frac{dx^\mu}{dt}$ <u>not</u> a 4-vector].

Now define $u^\mu = \frac{dx^\mu}{d\tau}$ [dx^μ: 4-vector, $d\tau$ scalar, dt transforms as oth component of 4-vector]

The 4-velocity and the 3-velocity are related as

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma \frac{dx^\mu}{dt} = \gamma\ (c, \underline{v}),$$

where of course $\gamma = \gamma(v) = 1/\sqrt{1 - v^2/c^2}$.

We have defined a new 4-vector that depends on the 3-velocity of a particle and the speed of light. Let is check the square of this 4- velocity. Since the contra-variant 4-velocity is $u^\mu = \gamma(c, \underline{v})$, we find for the co-variant version $u_\mu = g_{\mu\nu}u^\nu = \gamma(c, -\underline{v})$, and therefore

$$u^2 = u^\mu u_\mu = \gamma^2(c^2 - v^2) = c^2\gamma^2 \left(1 - \frac{v^2}{c^2}\right) = c^2.$$

So the Minkowski norm of the 4-velocity c^2 (independent of v), even if u^μ explicitly depends on the 3-velocity.

We could have evaluated this different: The Minkowski norm is independent of the inertial frame used for the evaluation. So we could pick the rest frame of the particle. In this frame $v = 0$, so $\gamma = 1$. Then $u^\mu = (c, 0)$ and therefore we immediately get $u^\mu u_\mu = c^2$. This is a general trick when evaluating Lorentz invariant quantities: Pick the frame which makes the calculation easy!

[Note that if we had defined the 4-velocity as $\frac{dx^\mu}{dt}$, then $\frac{dx^\mu}{dt} \cdot \frac{dx^\mu}{dt} = c^2 - v^2$ i.e <u>not</u> L.I]

1.10 4-Momentum

Probably the most important 4-vector of this course! Define $p^\mu = mu^\mu$ where m is the <u>rest mass</u> of the particle [sometimes denoted by m_0, but we will only ever refer to the rest masses, so no need to keep the subscript].

$$p^\mu = m\gamma(c, \underline{v}) = (m\gamma c, m\gamma\underline{v}) = (m_\gamma c, m_\gamma\underline{v}), \tag{1.30}$$

where m_γ relativistic inertial mass (hardly ever used, since frame dependent). The spatial components of the 4-momentum are very similar to the Newtonian 3-momentum (but $m \to m_\gamma = m \cdot \gamma$). What is the interpretation of the 0th component? We will see: $\frac{1}{c}\times$ Relativistic Energy.

Consider a particle with rest mass m. Let an unbalanced force $\underline{F}$ act on it. Classically, the kinetic energy of the particle is,

$$E_{kin} = \int \underline{F}.d\underline{x} = \int \frac{d\underline{P}}{dt}d\underline{x} = \int \frac{dp}{dt}dx.$$

We want to check if we can make a similar assignment in special relativity using the relativistic momentum
$$E_{\text{kin}} = \int \frac{d}{dt}(m\gamma v)dx,$$
using $dx = vdt$,
$$E_{\text{kin}} = \int \frac{d}{dt}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right) vdt.$$
Now $\frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = -\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-2v}{c^2}\right)\frac{dv}{dt}$, so:
$$\begin{aligned}
E_{\text{kin}} &= \int \left(m\gamma\frac{dv}{dt} + mv\frac{d\gamma}{dt}\right) vdt && (1.31)\\
&= \int \frac{m}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\frac{dv}{dt}\left(\left(1-\frac{v^2}{c^2}\right)+\frac{v^2}{c^2}\right) vdt && (1.32)\\
&= \int mc^2\frac{1}{c^2}v\frac{dv}{dt}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}dt && (1.33)\\
&= \int \frac{d}{dt}\left(mc^2\gamma\right)dt. && (1.34)
\end{aligned}$$
If the particle was initially at rest and the final velocity is v: $E_{\text{kin}} = mc^2(\gamma-1)$. This definition of the relativistic kinetic energy satisfies the non-relativistic limit, and the integral assignment using with the relativistic momentum. Useful check: consider Newtonian limit $\frac{v^2}{c^2} \ll 1$: Taylor expand $\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2} + \theta\left(\frac{v^4}{c^4}\right)$ so for $\frac{v^2}{c^2} \ll 1$,
$$E_{\text{kin}} = mc^2\left(1+\frac{v^2}{2c^2}-1\right) = \frac{1}{2}mv^2, \quad \text{(non-relativistic limit)}$$
as expected!

Define: Rest energy of particle: $E_0 = mc^2$. Total relativistic energy of particle: $E = E_0 + E_{\text{kin}} = m\gamma c^2$. So the 4-momentum is given by: $p^\mu = \left(\frac{E}{c}, \underline{p}\right)$ where E is the relativistic energy and $\underline{p}$ is the relativistic 3-momentum $\underline{p} = m\gamma\underline{v}$. The Minkowski scalar product of the 4-momentum of a particle is: $p^2 = p^\mu p_\mu = \frac{E^2}{c^2} - \underline{p}^2$. In the rest frame: $p^\mu p_\mu = m^2c^2 \to E^2 - (pc)^2 = (mc^2)^2$.

1.11 4-acceleration

Define $a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau}$ Recall $u^\mu = (\gamma c, \gamma\underline{v})$ & $\frac{d}{d\tau} = \gamma\frac{d}{dt} = \gamma$,
$$a^\mu = \frac{du^\mu}{d\tau} = \gamma\cdot\frac{d}{dt}(\gamma c, \gamma\underline{v}) = \gamma\cdot(\dot{\gamma}c, \dot{\gamma}\underline{v} + \gamma\underline{\dot{v}}) = \gamma\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\underline{v} + \gamma\underline{a}\right).$$

Define now the **proper acceleration** as the acceleration of a particle measured in its instantaneous rest frame. In this frame $v = 0 \to \gamma = 1, \dot{\gamma} = 0 \to a^\mu = (0, \underline{a})$. **Uniform acceleration** is used to describe the situation with constant proper acceleration. In the instantaneous rest-frame of the particle one has $a^\mu = (0, \underline{\alpha}) \therefore a^\mu a_\mu = -\alpha^2$ (with the latter valid in any frame).

We now want t compute $u^\mu a_\mu$:

a) Take general form and do lots of algebra (tedious)

b) Choose "best possible" frame $v = 0, u^\mu = (c, 0), a^\mu = (0, \underline{a}) \therefore u^\mu a_\mu = 0$

c) use $u^\mu u_\mu = c^2$, so $\frac{d}{d\tau}(u^\mu u_\mu) = 2u^\mu a_\mu \therefore u^\mu a_\mu = 0$

Conclusion: 4-acceleration is Minkowsky-perpendicular to 4 velocity!

1.12 4-force

Define the total four-force (or *Minkowski force*) acting on a particle as $f^\mu = \frac{dp^\mu}{d\tau}$ (it is a 4 vector),

$$\frac{d}{d\tau}p^\mu = \frac{d}{d\tau}\left(\frac{E}{c}, \underline{p}\right) = \gamma\frac{d}{dt}\left(\frac{E}{c}, \underline{p}\right) = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\underline{F}\right),$$

where $\underline{F} = \frac{dp}{dt} = \frac{d}{dt}(m\gamma\underline{v})$.

In the three-force $\underline{F}$, the derivative is with respect to time, not the proper-time ($\underline{F} = \frac{dp}{dt}$), and therefore the three-force $\underline{F}$ is not a 4-vector. Since $p^\mu = (E/c, \underline{p})$, the 3-momentum translate between frames as the spatial components of the 4-momentum. Since the Lorentz transformation is a linear transformation, the differentials $\mathrm{d}p_x, \mathrm{d}p_y, \mathrm{d}p_z$ etc. will also translate as the spatial components of a 4-vector. In S' in the standard configuration we have for the components of the three-force $\underline{F}'$,

$$F'_y = \frac{\mathrm{d}p'_y}{\mathrm{d}t'} = \frac{\mathrm{d}p_y}{\gamma\mathrm{d}t - \frac{\gamma\beta}{c}\mathrm{d}x} = \frac{\frac{\mathrm{d}p_y}{\mathrm{d}t}}{\gamma\left(1 - \frac{\beta}{c}\frac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{F_y}{\gamma\left(1 - \frac{\beta u_x}{c}\right)}.$$

A similar result is obtained for the z-component. The x-component is more involved, and we find

$$F'_x = \frac{\mathrm{d}p'_x}{\mathrm{d}t'} = \frac{\gamma p_x - \gamma\beta\mathrm{d}p^0}{\gamma\mathrm{d}t - \frac{\gamma\beta}{c}\mathrm{d}x} = \frac{F_x - \frac{\beta}{c}\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)}{1 - \frac{\beta u_x}{c}} = \frac{F_x - \frac{\beta(\underline{u}\cdot\underline{F})}{c}}{1 - \frac{\beta u_x}{c}},$$

where in the last step we used the relationship between the force, the velocity and the change in energy $\mathrm{d}E/\mathrm{d}t = \underline{u} \cdot \underline{F}$. If the particle is at rest in S (such that $\underline{u} = 0$) then $F'_\perp = \frac{1}{\gamma}F_\perp$ and $F'_\parallel = F_\parallel$ (with respect to the boost between S' and S. F parallel to boost is unchanged! F orthogonal to the boost is divided by γ.

While the Minkowski 4-Force $f^\mu = \frac{dp^\mu}{d\tau}$ has neat transformation properties, the ordinary force is useful, since we are often interested in calculating the trajectory as a function of the observer time t (not the rest-time of the particle itself, τ). This obviously requires us to calculate $\frac{dp^\mu}{dt}$ to find the change in velocity and position as measured by the observer (=us).

1.13 Newton's Laws And Relativity

1st law is built into relativistic principle.
3rd law (action- reaction) is obviously problematic: If two objects are separated in space, $s_{ab} < 0$, so causal connection forbidden! Suppose in some frame s the force of A on B at instant "t" is $F(t)$. The force of B on A is $-F(t)$ in this frame. But a moving observer will not see these two events of force measurement at equal times, so in general the 3rd law is violated.
2nd law: $\underline{F} = \frac{d\underline{p}}{dt}$ [Newton] holds in special relativity provided we use the relativistic momentum.(note: t, not τ!).

1.14 Motion Under A Constant Force

Question: Particle of mass m, under a constant force $\underline{F}$ starts from the origin. Find $x(t)$?

Solution: $\frac{dp}{dt} = F \to p = Ft +$ constant. $p(t=0) = 0 \therefore$ constant $= 0$, so

$$p(t) = \frac{mu(t)}{\sqrt{1 - \frac{u(t)^2}{c^2}}} = Ft.$$

Solving the quadratic equation for u leads to,

$$u(t) = \frac{\left(\frac{F}{m}\right) t}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}}.$$

Numerator: classical solution. Relativistic denominator ensures $u < c$.

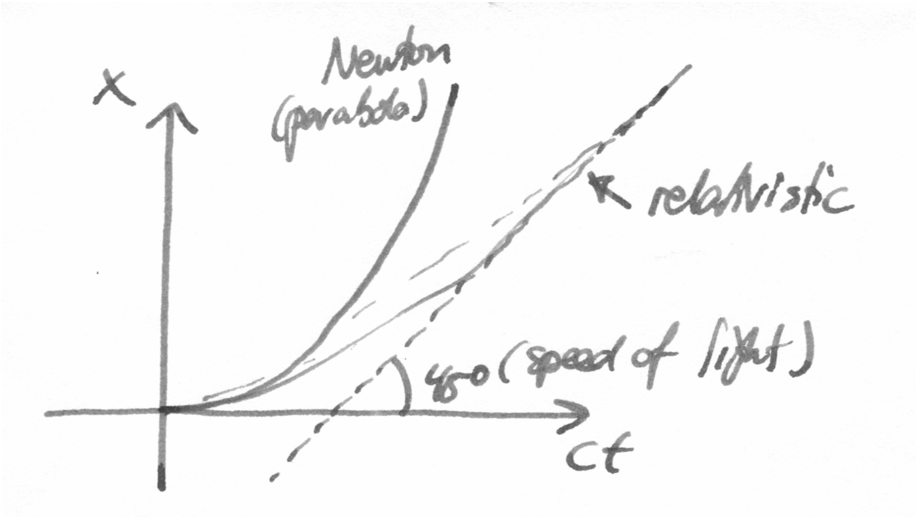
We can now find the position as a function of time

$$\begin{aligned} x(t) &= \frac{F}{m}\int_0^t \frac{t'}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}} dt', \\ &= \frac{mc^2}{F}\sqrt{1 + \left(\frac{Ft'}{mc}\right)^2}\Bigg|_{t'=0}^{t'=t}, \\ &= \frac{mc^2}{F}\left(\sqrt{1 + \left(\frac{Ft}{mc}\right)^2} - 1\right), \end{aligned}$$

In Newtonian mechanics the solution to movement underconstant force is a parabolic motion: $x(t) = \left(\frac{F}{2m}t^2\right)$. In relativity a constant force gives hyperbolic motion.

The **work** W applied to the particle is defined as the integral of the force along the path, $W = \int \underline{F} \cdot d\underline{l}$. With the appropriate definitions of the relativistic energy, the **Work-Energy Theorem** holds (that the net work equals increase in kinetic energy),

$$W = \int \underline{F} \cdot d\underline{l} = \int \frac{d\underline{p}}{dt} \cdot d\underline{l} = \int \frac{d\underline{p}}{dt}\frac{d\underline{l}}{dt}\, dt = \int \frac{d\underline{p}}{dt} \cdot \underline{u}\, dt,$$



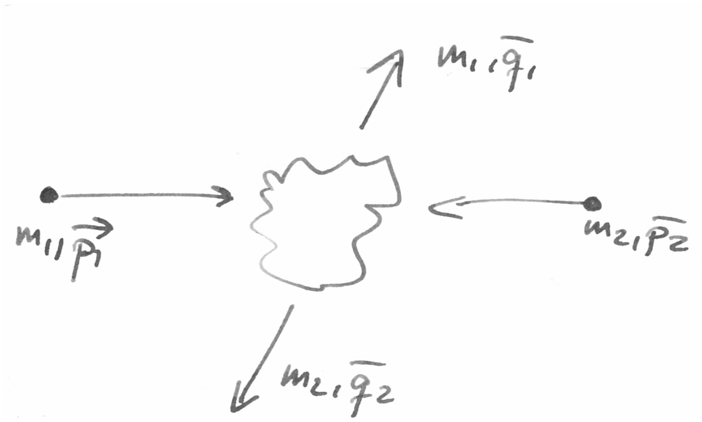
but $\frac{dp}{dt} \cdot \underline{u} = \frac{d}{dt}\left(\frac{m\underline{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right) \cdot \underline{u}$, and by using $-\frac{\dot{g}}{g^2} = \frac{d}{dt}\left(\frac{1}{g}\right)$ with $g = \sqrt{1 - \frac{\underline{u}\cdot\underline{u}}{c^2}}$ we find

$$= \frac{m\underline{u}}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}} \cdot \frac{d\underline{u}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\frac{\underline{u}\cdot\underline{u}}{c^2}}}\right) = \frac{dE_{rel}}{dt}.$$

So $W = \int \frac{dE_{rel}}{dt} dt = E_{\text{final}} - E_{\text{initial}}$ [the rest energy cancels out]. (This was used in paragraph 1.10 to define the relativistic kinetic energy).

1.15 Applications of Relativity

Elastic Scattering

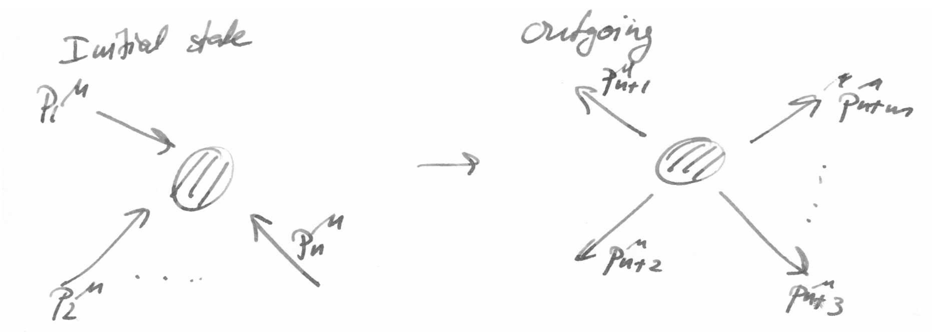


Incoming and outgoing particles are the same. Even when we know nothing about the details of the interaction we know the kinematic constraint of **4-momentum conservation**. This encodes conservation of both Energy (0th component) and momentum (spatial component). Eg. 2 particles with equal masses $m_1 = m_2 = m$. In the centre of mass frame we have $\underline{p_1} = -\underline{p_2} = \underline{p}$. $P_1^\mu = \left(\frac{E_1}{c}, \underline{p}\right)$, and $P_2^\mu = \left(\frac{E_2}{c}, -\underline{p}\right)$ as $m_1 = m_2$ we have $E_1 = E_2 = m\gamma(v)c^2 = E$ in centre of mass frame.

After the collision $q_1^\mu = \left(\frac{E_1'}{c}, \underline{q_1}\right)$ and $q_2^\mu = \left(\frac{E_2'}{c}, \underline{q_2}\right)$ and 4-momentum conservation gives, $P_1^\mu + P_2^\mu = q_1^\mu + q_2^\mu \to \left(\frac{2E}{c}, \underline{0}\right) = \left(\frac{E_1' + E_2'}{c}, \underline{q_1} + \underline{q_2}\right)$. But since $m_1 = m_2$, $|\underline{q_1}| = |\underline{q_2}|$, $E_1' = E_2' = E$ and $|\underline{q_1}| = |\underline{q_2}| = |\underline{p}|$. The scattering angle θ (and azimuthal angle ϕ) are the only degrees of freedom.

Check on the number of degrees of freedom: $(2 \times 4) - 2 - 4 = 2$. (2×4): Entries in 4-vectors of outgoing particles; "-2": on-shell conditions; "-4" momentum conservation.

Inelastic Scattering and Threshold Energy



Consider now then a process where the numbers of particles and their masses can change. 4-momentum conservation $p_1^\mu + p_2^\mu + ... + p_n^\mu = p_{n+1}^\mu + p_{n+m}^\mu$. Often useful to calculate the Minkowski Norm of the sum of 4-momenta on the LHS and the RHS.

Threshold Energy: Assume just 2 incoming particles of rest mass's m_1, m_2 and M out-agoing particles of rest masses $m_3, ..., m_{m+2}$. Is the process kinematically allowed? Depends on relative velocity of incoming particles and the masses,

$$
\begin{aligned}
p_1^\mu + p_2^\mu &= p_3^\mu + ... + p_{m+2}^\mu = Q_f^\mu, \\
(p_1 + p_2)^2 &= p_1^2 + p_2^2 + 2p_1p_2 = Q_f^2.
\end{aligned}
$$

Compute in rest frame of particle 1: $p_1^\mu = (m_1 c, \underline{0})$, $p_2^\mu = (m_2 c\gamma(v), m_2\gamma(v)\underline{v})$

$$\therefore p_1 \cdot p_2 = m_1 m_2 c^2 \gamma(v) \tag{1.35}$$

$$\therefore Q_f^2 = m_1^2 c^2 + m_2^2 c^2 + 2 m_1 m_2 c^2 \gamma(v) \tag{1.36}$$

$$\therefore \gamma(v) = \frac{Q_f^2 - m_1^2 c^2 - m_2^2 c^2}{2 m_1 m_2 c^2} \tag{1.37}$$

v: relative velocity of the incoming particles. The minimum Q_f^2 allowed by the final state particles is obtained if there is a inertial frame where <u>all</u> the final state particles are at rest. If such a frame exists, one would have $\left(Q_f^2\right)_{\text{min}} = c^2 \left(\sum_{i=3}^{m+2} m_i\right)^2$. This allows us to find the minimal value for v (i.e. the minimal energy in the collision), which allows for the production of the particles.

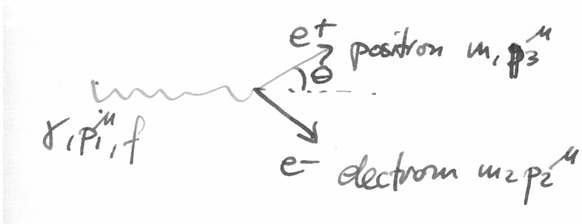
Example: $pp \to pp\pi^0$ As $m_p \simeq 1000 Mev/c^2 = 1 Gev/c^2$ and $m_\pi \simeq 140 Mev/c^2 = 0.14 Gev/c^2$.

$$\gamma(v)_{min} = \frac{c^2(2m_p + m_\pi)^2 - 2m_p^2 c^2}{2c^2 m_p^2} = \frac{2m_p^2 + 4m_p m_\pi + m_\pi^2}{2m_p^2} = 1 + 2\frac{m_\pi}{m_p} + \frac{m_\pi^2}{2m_p^2} \simeq 1.3 \to v \geq 0.63c$$

Example: Pair creation and photons Photons are the quanta of electromagnetic radiation. They are particles with zero rest mass (a relativistic concept that is impossible in Newtonian mechanics), travelling at the speed of light.

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) = \left(\frac{hf}{c}, \frac{hf}{c}\underline{n}\right) = \frac{hf}{c}(1, \underline{n}),$$

where h is Planck's constant, f is the frequency and $\underline{n}$ is the direction of travel ($|\underline{n}| = 1|$). As $p^2 = p^\mu p_\mu = \frac{h^2 f^2}{c^2} - \frac{h^2 f^2}{c^2} = 0$! Some points:



a) Photons never stop.
b) There is no rest frame for the photon.
c) Often we use the notation γ for photons (NOT the gamma-factor).

We now find that the spontaneous "decay" or conservation of γ is impossible (production of e^+e^- pair), since we have for $p_1 \to p_2 + p_3$:

$$p_1^\mu = (hf_1/c, hf_1/c\underline{n}) \qquad p_2^\mu = (E_2/c, \underline{p}_2) \qquad p_3^\mu = (E_3/c, \underline{p}_3). \tag{1.38}$$

But 4-momentum conservation gives us

$$p_1^\mu = p_2^\mu + p_3^\mu \quad \therefore \quad p_1^\mu - p_2^\mu = p_3^\mu \quad \therefore \quad p_1^2 - 2p_1 \cdot p_2 + p_2^2 = p_3^2. \tag{1.39}$$

We can evaluate each of these Lorentz invariants in their separate most convenient frames. So we find $p_1^2 = 0, p_2^2 = p_3^2 = m_e^2 c^2$. 4-momentum conservation therefore requires $2p_1 \cdot p_2 = 0$. However, evaluating directly $p_1 \cdot p_2$ we find

$$p_1 \cdot p_2 = \frac{hf_1}{c}(E_2/c - \underline{n} \cdot \underline{p}_2) = \frac{hf_1}{c}(E_2/c - |p_2| \cos\theta), \tag{1.40}$$

with θ the angle between the photon and the electron momentum. Since $E_2 > |p_2|c$ we therefore find $p_1 \cdot p_2 > 0$ in contradiction with the result from 4-momentum conservation. We therefore find that we cannot both have 4-momentum conservation and physical momenta for the particles, i.e. the process is kinematically forbidden.

In the presence of another particle X, the spontaneous production however *is* possible. X is usually a proton or other nucleus. In the rest frame of X: $P_1^\mu = \frac{hf}{c}(1, \underline{n})$ and $P_2^\mu = (Mc, \underline{0})$. We now want to compute the threshold frequency i.e minimum frequency required. Conservation of 4-momentum gives

$$(p_1 + p_2)^\mu = Q_f^\mu = (q_1 + q_2 + q_3)^\mu,$$

squaring and recalling $p_1^2 = 0$ and $p_2^2 = M^2c^2$,

$$p_1^2 + p_2^2 + 2p_1 \cdot p_2 = Q_f^2, \tag{1.41}$$

$$\therefore \quad 2p_1 \cdot p_2 = 2Mhf_1 = (Q_f^2 - M^2c^2). \tag{1.42}$$

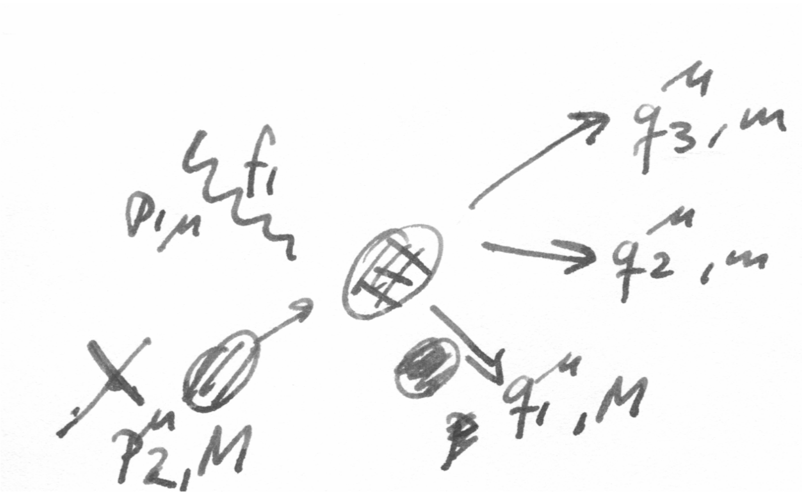
A lower bound for f_1 is found by using the lower bound on Q_f^2 obtained by considering the hypothetical frame where all final state particles are at rest. In such a frame we would find

$$(Q_f^2)_{min} = (2m_e + M)^2 c^2 = 4m_e^2 c^2 + 4m_e M c^2 + M^2 c^2 \tag{1.43}$$

so the threshold frequency is

$$f_{\text{thr}} = 2\frac{m^2c^2 + Mmc^2}{Mh}. \tag{1.44}$$

We find that $f_{\text{thr}} \to 2mc^2/h$ for $M \gg m$ and $f_{\text{thr}} \to \infty$ for $M \to 0$)



q_3^μ, m
f_1
p_1, μ
q_2^μ, m
p_2^μ, M
q_1^μ, M

Chapter 2

Electromagnetism

2.1 Scalar and Vector potentials

Maxwell's Equations:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \tag{i}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{ii}$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{iii}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \varepsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t} \tag{iv}$$

We will set ourselves an interesting problem: solve these beautiful equations for a general, time-dependent distribution of sources $\rho(\underline{r}, t), \underline{j}(\underline{r}, t)$. It will pay off to use the language of potentials rather than the physical fields themselves (two vector fields: 2x3=6 Degrees of Freedom minus the two constraints of the ME, a scalar and a vector potential: 4 DoF minus the constraints from the ME).

In Electrostatics we have : $\underline{\nabla} \times \underline{E} = 0$, so there exists a scalar potential ϕ with $\underline{E} = -\nabla\phi$. This is not so in Electrodynamics! However, we still have $\underline{\nabla} \cdot \underline{B} = 0$, so we can write $\underline{B}$ in terms of a vector potential $\underline{A}$:

$$\boxed{\underline{B} = \underline{\nabla} \times \underline{A}(\underline{r}, t).} \tag{a}$$

$\underline{A}(\underline{r}, t)$: Vector potential, 3 components.
If we put this into Faraday's Law (iii) we find ($\underline{E}, \underline{B}, \underline{A}$ and ϕ are from now on always understood to be dependent on $(\underline{r}, t)$):

$$\underline{\nabla} \times \underline{E} = -\frac{\partial}{\partial t} \left(\underline{\nabla} \times \underline{A} \right),$$

thus (by interchanging the differential operation in space and in time, i.e. interchanging $\nabla\times$ and $\partial/\partial t$):

$$\underline{\nabla} \times (\underline{E} + \frac{\partial \underline{A}}{\partial t}) = 0.$$

I.e. even if $\underline{E}$ itself has a non-zero curl because of the time-varying $\underline{B}$-field, $\underline{E} + \frac{\partial \underline{A}}{\partial t}$ is curl-less and can therefore be written in terms of a scalar potential (ϕ):

$$\underline{E} + \frac{\partial \underline{A}}{\partial t} = -\underline{\nabla}\phi,$$

such that:

$$\boxed{\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}.} \tag{b}$$

We have therefore expressed $\underline{E}$ in terms of potentials $(\phi, \underline{A})$.

This representation (a) and (b) obviously automatically fulfils the two homogeneous Maxwell equations (ii) and (iii) (which were used in connecting the potentials to the physical fields). The constraint on the potentials $(\phi, \underline{A})$ arising from (i) is found by putting (b) into (i):

$$-\underline{\nabla} \cdot \underline{E} = \nabla^2\phi + \frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{A}) = -\frac{\rho}{\epsilon_0}. \tag{c}$$

Next, we find the constraint arising from (iv) by putting (a) and (b) into (iv):

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \mu_0 \underline{j} - \mu_0\epsilon_0\underline{\nabla}\left(\frac{\partial\phi}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2 \underline{A}}{\partial t^2},$$

and using the $(BAC - CAB)$–rule $\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2\underline{A}$ gives:

$$\left(\nabla^2\underline{A} - \mu_0\epsilon_0\frac{\partial^2 \underline{A}}{\partial t^2}\right) - \underline{\nabla}\left(\underline{\nabla} \cdot \underline{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right) = -\mu_0\underline{j}. \tag{d}$$

Thus we see that two of the Maxwell's equations ((ii) and (iii)) are necessary for the physical fields to be definable in terms of a scalar and a vector potential. The remaining two of Maxwell's equations define the potentials in terms of the sources: (c) , (d). Is this satisfying? Well, it is not such nice equations in general. The next section discusses what can be done to make them prettier.

2.2 Gauge Transformations

Notice that the potentials $(\phi, \underline{A})$ are not uniquely defined by (a) or (b). Just like the gravitational potential (where a constant can be added without changing the dynamics) the potentials in electromagnetism are not unique.

We can, with some constraints, change the potentials $(\phi, \underline{A})$ without changing the physical fields $\underline{E}$ and $\underline{B}$: This is called the **Gauge Freedom**.

Let $(\phi, \underline{A})$ be one set of potentials for the physical fields $\underline{E}(\underline{r}, t), \underline{B}(\underline{r}, t)$. Now, let $\underline{A}'(\underline{r}, t) = \underline{A}(\underline{r}, t) + \underline{\alpha}(\underline{r}, t)$, $\phi'(\underline{r}, t) = \phi(\underline{r}, t) + \beta(\underline{r}, t)$ and require that $\underline{E}(\underline{r}, t)$ and $\underline{B}(\underline{r}, t)$ be unchanged. For $\underline{B}(\underline{r}, t)$ we find that:

$$\underline{\nabla} \times \underline{A} = \underline{B} = \underline{\nabla} \times \underline{A}' = \underline{\nabla} \times (\underline{A} + \underline{\alpha}) \therefore \underline{\nabla} \times \underline{\alpha} = 0$$

This requirement is automatically fulfilled if we let

$$\underline{\alpha} = \underline{\nabla}\lambda, \tag{2.1}$$

where λ is any **arbitrary** scalar field (since $\underline{\nabla} \times (\underline{\nabla}\lambda) = 0$ for any scalar field λ). At the same time, for any $\underline{\alpha}$ with $\underline{\nabla} \times \underline{\alpha} = 0$ we can find a λ with $\underline{\nabla}\lambda = \underline{\alpha}$.

Requiring that $\underline{E}$ is unchanged by the transformation we find:-

$$\begin{aligned}
\underline{E} &= -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \\
&= -\underline{\nabla}\phi' - \frac{\partial \underline{A}'}{\partial t} \\
&= -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} - \left(\underline{\nabla}\beta + \frac{\partial \underline{\alpha}}{\partial t}\right) \\
\therefore \underline{\nabla}\beta &+ \frac{\partial \underline{\alpha}}{\partial t} = 0,
\end{aligned}$$

and by using $\underline{\alpha} = \underline{\nabla}\lambda$ and interchanging the order of $\underline{\nabla}$ and $\frac{\partial}{\partial t}$:

$$\underline{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0.$$

I.e. the combination $\left(\beta(\underline{r}, t) + \frac{\partial \lambda(\underline{r}, t)}{\partial t}\right)$ is independent of the position $\underline{r}$. We can call it $k(t) = (\beta + \frac{\partial \lambda}{\partial t})$. Since λ was arbitrary , we can define a new λ', by subtracting $\int_0^t k(t')dt'$ from λ. Obviously, the new λ' still fulfils the requirement $\underline{\nabla}\lambda' = \underline{\nabla}\lambda = \underline{\alpha}$ (since $k(t)$ is independent of the position $\underline{r}$). With this new λ we have

$$\frac{\partial \lambda'}{\partial t} = \frac{\partial \lambda}{\partial t} - \beta - \frac{\partial \lambda}{\partial t} = -\beta \tag{2.2}$$

and therefore we find that if the fields $(\phi, \underline{A})$ are potentials for the physical fields $(\underline{E}, \underline{B})$ then so are the potentials $(\phi', \underline{A}')$, if they are related as

$$\underline{A}' = \underline{A} + \underline{\nabla}\lambda' \tag{2.3}$$

$$\phi' = \phi - \frac{\partial \lambda'}{\partial t}. \tag{2.4}$$

for any smooth field $\lambda'(\underline{r}, t)$. We could of course have chosen the original (and arbitrary) λ to already fulfil the requirement that $\partial\lambda/\partial t = -\beta$, and so we conclude that for an arbitrary scalar field $\lambda(\underline{r}, t)$, we can add $\underline{\nabla}\lambda$ to $\underline{A}$ and subtract $\frac{\partial \lambda}{\partial t}$ from $\underline{\phi}$ without changing the resulting physical fields $\underline{B}$, $\underline{E}$. Such changes are called **Gauge Transformations**.

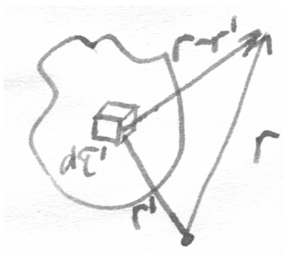
Such transformations are often exploited to simplify equations. In magnetostatics, one often chooses $\underline{\nabla} \cdot \underline{A} = 0$. In electrodynamics the most convenient gauge depends on the problem at hand.

2.3 Coulomb Gauge

The divergence of $\underline{A}'$ is $\underline{\nabla} \cdot \underline{A}' = \underline{\nabla} \cdot \underline{A} + \underline{\nabla} \cdot \underline{\nabla}\lambda$. Therefore, starting from one solution $\underline{A}'$ we can find another solution $\underline{A}$ with the additional requirement $\underline{\nabla} \cdot \underline{A} = 0$ by a gauge transformation, where λ solves the differential equation $\nabla^2\lambda = \underline{\nabla} \cdot \underline{A}'$. With this new $\underline{A}$, equation (c) becomes $\nabla^2\phi = -\frac{1}{\epsilon_0}\rho$. This is Poisson's equation for ϕ, and by setting $\phi = 0$ at ∞, we have the solution:

$$\phi(\underline{r}, t) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho(r', t)}{\mid r - r' \mid} dr'.$$

However, we still need to find the vector potential $\underline{A}$ in order to get the physical fields $\underline{E}$ (and of course $\underline{B}$)! Note: in these equations there is only one time t: ϕ everywhere depends on



the charge distribution everywhere at the same time. If the charge distributions changes, then ϕ changes everywhere *instantaneously*. This is NOT in contradiction with special relativity. Only $\underline{E}$ and $\underline{B}$ are physical (i.e. has measurable consequences through the Lorentz force), and somehow it is built into $\underline{A}$ to correct for the instant change in ϕ used when deriving the physical $\underline{E}$ and $\underline{B}$. The signal of change only moves at speed c.
Advantage of this gauge: Easy ϕ
Disadvantage: Hard A

Equation (d) in the Coulomb Gauge reads:

$$\nabla^2\underline{A} - \mu_0\epsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{j} + \mu_0\epsilon_0 \underline{\nabla}\left(\frac{\partial\phi}{\partial t}\right).$$

2.4 Lorenz Gauge and the Four-Potential

It is possible instead to require $\underline{\nabla} \cdot \underline{A} = -\mu_0\epsilon_0\frac{\partial\phi}{\partial t}$, in order to eliminate the middle term in (d). This gives for (d):

$$\nabla^2\underline{A} - \mu_0\epsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = -\mu_0 \underline{j}.$$

Equation (c) becomes:

$$\nabla^2\phi - \mu_0\epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\epsilon_0}\rho.$$

These two last equations look very similar! Define a new differential operator $\Box$ as:

$$\mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box \qquad \text{(d'Alembertian)},$$

then:

$$\Box\phi = \frac{1}{\epsilon_0}\rho, \quad \Box\underline{A} = \mu_0\underline{j}.$$

Since $\Box$ acts on each component independently, these 4 equations can also be written $\Box(\phi, \underline{A}) = (\frac{1}{\epsilon_0}\rho, \mu_0\underline{j})$. All entries in the 4-tuples can have the same units by a judicial multiplication by c, such that (using $c^2 = \frac{1}{\varepsilon_0\mu_0}$)

$$\Box(\phi, c\underline{A}) = \frac{1}{c\ \varepsilon_0}(\rho c, \underline{j}). \tag{2.5}$$

This notation is of course suggestive of the 4-tuples being 4-vectors. Indeed, since the derivative is linear, we see that $(\phi, c\underline{A})$ is a four-vector if $(\rho c, \underline{j})$ transforms as a four-vector. To show that this is indeed the case, we refer to Einstein's first postulate for special relativity, that the laws of physics be invariant in all frames of inertia (which are related by Lorentz transformations). This applies also to the conservation of charges expressed through the continuity equation

$$\underline{\nabla} \cdot \underline{j} + \frac{\partial\rho}{\partial t} = 0. \tag{2.6}$$

This equation is easily recast in covariant form, since

$$0 = \underline{\nabla} \cdot \underline{j} + \frac{\partial\rho}{\partial t} = \underline{\nabla} \cdot \underline{j} + \frac{\partial c\rho}{\partial ct} \tag{2.7}$$

$$\therefore \partial_\mu(c\rho, \underline{j}) = 0. \tag{2.8}$$

If we therefore put $j^\mu = (c\rho, \underline{j})$ (without assuming at first it actually is a four-vector), we therefore conclude from Eq. (2.7) that $\partial_\mu j^\mu$ must transform as a Lorentz invariant. We already know that $\partial_\mu = \partial/\partial x^\mu$ transforms under Lorentz transformations as a covariant four-vector

$$\frac{\partial}{\partial x'^\mu} = \partial'_\mu = \Lambda_\mu{}^\nu\ \partial_\nu. \tag{2.9}$$

We now assume that in the frame S', j'^μ is related to j^μ by a linear transformation $j'^\mu = L^\mu{}_\nu j^\nu$, and want to show that $L^\mu{}_\nu = \Lambda^\mu{}_\nu$. This follows from the observation that

$$d'_\mu j'^\mu = \Lambda_\mu{}^\nu\ L^\mu{}_\rho\ d_\nu j^\rho \tag{2.10}$$

is invariant if and only if $\Lambda_\mu{}^\nu\ L^\mu{}_\rho = \delta^\nu{}_\rho$, i.e. we get from the discussion around Eq. (1.25) that

$$L^\mu{}_\rho = \Lambda^\mu{}_\rho \tag{2.11}$$

and therefore that j^μ indeed transforms according to the Lorentz transformations, and therefore is a four-vector. We have therefore also shown that in the Lorenz gauge,

$$A^\mu = (\phi, c\underline{A}) \tag{2.12}$$

is a four-vector, and we will call this the four-potential.

2.5 Continuous Distribution

We will be looking at solving Maxwell's equations for the potentials for several cases. First for continuous distributions, using the Lorenz Gauge.

The differential operator $\Box$ is defined as

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

With that, the constraints on the potentials from the Maxwell equations are

$$\Box\phi = \frac{1}{\epsilon_0}\rho, \tag{i}$$

$$\Box\underline{A} = \mu_0\underline{j}, \tag{ii}$$

In the *static* case where all time-derivatives are zero, we find 4 copies of Possion's equation:

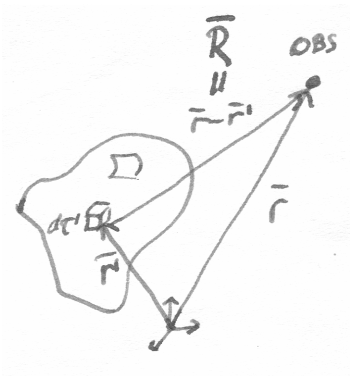
$$\nabla^2\phi = -\frac{1}{\epsilon_0}\rho \tag{2.13}$$

$$\nabla^2\underline{A} = -\mu_0\underline{j}, \tag{2.14}$$

with the well-known solutions

$$\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{r'})}{|\underline{r}-\underline{r'}|}d\tau'. \tag{2.15}$$

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}(\underline{r'})}{|\underline{r}-\underline{r'}|}d\tau' \tag{2.16}$$



Now, it would be *tempting* to suggest that since information travels at the speed of light, it is not the state of the source at time t that matters, but at an earlier time t_r (the retarded time) when the imaginary message about the state of the source left. Since the message travels a distance $|\underline{R}| = |\underline{r} - \underline{r}'|$ we have $t_r = t - \frac{|r-r'|}{c}$.

The *retarded time* t_r depends on the position r of the observer even for fixed t! Similar to distant star observations. The natural generalisation from the static case would then be given by the volume integrals with differential $d\tau'$

$$\phi(\underline{r},t) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{r}',t_r)}{R}d\tau', \tag{2.17}$$

$$\underline{A}(\underline{r},t) = \frac{\mu_0}{4\pi}\int_V \frac{\underline{j}(\underline{r}',t_r)}{R}d\tau'. \tag{2.18}$$

Where $\underline{R} = \underline{r} - \underline{r}'$, $\rho(\underline{r}',t_r)$ is the charge distribution at source point $\underline{r}'$ at the time t_r (which itself depends on both $\underline{r}'$ and the position of the observer $\underline{r}$), and $R = |\underline{R}|$. These expression are called the *retarded potentials*, since it is the source distributions at the retarded time (for a given observer position), which is integrated over.

That all sounds reasonable - but is it correct? We can test if this ansatz fulfils the equation for the potentials (these differential equations combined with Dirichlet boundary conditions $\phi(\infty) = 0, \underline{A}(\infty) = \underline{0}$ have one unique solution according to the existence and uniqueness theorem for differential equations (the Cauchy–Lipschitz theorem)).

The procedure is: For a given $\rho(\underline{r}',t)$, pick ϕ according to Eq. (2.17) (or A_x, A_y, A_z according to Eq. (2.18)). Show that with this ϕ we have $\mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = \frac{1}{\epsilon_0}\rho$, i.e. the Maxwell equation for the potential in the Lorenz gauge. The differential operator $\underline{\nabla}$ is with respect to r. And differentiation is easier than integration. The only difficulty is: $\phi(r',t_r)/R$ depends on r through R, <u>and</u> through t_r! So we start:

$$\underline{\nabla}\phi = \frac{1}{4\pi\epsilon_0}\int\left[(\underline{\nabla}\rho)\frac{1}{R} + \rho\left(\underline{\nabla}\frac{1}{R}\right)\right]d\tau'.$$

By the chain rule we find (with respect to r):

$$\underline{\nabla}\rho = \frac{\partial\rho}{\partial t}\underline{\nabla}t_r = -\frac{1}{c}\frac{\partial\rho}{\partial t}\underline{\nabla}R, \tag{2.19}$$

since $\frac{\partial}{\partial t} = \frac{\partial}{\partial t_r}$ (since $\underline{R}$, $\underline{r}'$, $\underline{r}$ are all independent of t). We now use some standard results from vector analysis:

$$\underline{\nabla}R = \hat{\underline{R}}, \qquad \underline{\nabla}\left(\frac{1}{R}\right) = -\frac{\hat{\underline{R}}}{R^2}, \tag{2.20}$$

so,

$$\underline{\nabla}\phi = \frac{1}{4\pi\epsilon_0}\int_V\left[\frac{-\dot{\rho}}{c}\frac{\hat{\underline{R}}}{R} - \rho\frac{\hat{\underline{R}}}{R^2}\right]d\tau'. \tag{a}$$

To find $\nabla^2\phi$ we need to calculate the divergence of this ($\underline{\nabla}\cdot\underline{\nabla}\phi = \nabla^2\phi$),

$$\nabla^2\phi = \frac{1}{4\pi\epsilon_0}\int_V\left\{-\frac{1}{c}\left[\frac{\hat{\underline{R}}}{R}\cdot(\underline{\nabla}\dot{\rho}) + \dot{\rho}\underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R}\right)\right] - \left[\frac{\hat{\underline{R}}}{R^2}\cdot(\underline{\nabla}\rho) + \rho\underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right)\right]\right\}d\tau',$$

but (using the result of the chain rule in Eq. (2.19)) $\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\underline{\nabla}R = -\frac{1}{c}\ddot{\rho}\hat{\underline{R}}$, $\underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R}\right) = \frac{1}{R^2}$, and $\underline{\nabla}\left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\delta^3(\underline{R})$. So the second and third term in the integral cancel each other, and we find

$$\nabla^2\phi = \frac{1}{4\pi\epsilon_0}\int_V \left[\frac{1}{c^2}\frac{\ddot{\rho}}{R} - 4\pi\rho\ \delta^3(\underline{R})\right] d\tau' = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\underline{r}, t)$$

This is what we needed to show.

Problem 10.8 in Griffith's will be performed in an example class and shows that the *retarded potentials* fulfil the Lorenz Gauge condition.

We can now derive the physical fields $\underline{E}$ and $\underline{B}$. First we note that

$$\frac{\partial\underline{A}}{\partial t} = \frac{\mu_0}{4\pi}\int_V \frac{\partial\underline{j}/\partial t}{R} d\tau', \tag{2.21}$$

and that therefore with the definitions of the fields in terms of the potentials:

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial\underline{A}}{\partial t}, \tag{2.22}$$

$$\underline{B} = \underline{\nabla}\times\underline{A}, \tag{2.23}$$

we find (using $c^2 = \frac{1}{\mu_0\epsilon_0}$):

$$\underline{E} = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\underline{r}', t_r)}{R^2}\hat{\underline{R}} + \frac{\dot{\rho}(\underline{r}', t_r)}{cR}\hat{\underline{R}} - \frac{\partial\underline{j}(\underline{r}', t_r)/\partial t}{c^2 R}\right] d\tau'. \tag{2.24}$$

Similarly, we find for the magnetic field $\underline{B}$:

$$\underline{B}(\underline{r}, t) = \frac{\mu_0}{4\pi}\int \left[\frac{\underline{j}(\underline{r}', t_r)}{R^2} + \frac{\partial\underline{j}(\underline{r}', t_r)/\partial t}{cR}\right] \times \hat{\underline{R}}\ d\tau'. \tag{2.25}$$

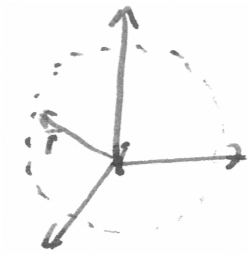
These are known as *Jefimenko's equations.* <u>Note:</u>

1. (a) and (b) are <u>not</u> just the solution from the static case, with the retarded time inserted! This worked for potentials, but would not give the right solution if performed on the fields themselves (Why is that?)

2. Actually, a perfectly valid solution to Maxwell's equations for potential would also have been obtained by using <u>advanced</u> time $t_a = t + \frac{R}{c}$. But this violates causality. Disregard for physics. Both solutions arise basically because $\Box$ involves $\frac{\partial^2}{\partial t^2}$ and therefore does not care about the direction of time.

2.6 Radiation from accelerated charges and changing currents

The generation of electromagnetic waves. They propagate to infinity and carry energy away from the source. Consider "only" localised sources. The total energy passing through an imaginary spherical shell surrounding the source is given by the surface integral of the Poynting vector,



$$P(\underline{r}) = \oint \underline{S} \cdot d\underline{a} = \frac{1}{\mu_0} \oint (\underline{E} \times \underline{B})\, d\underline{a}.$$

The power radiated,

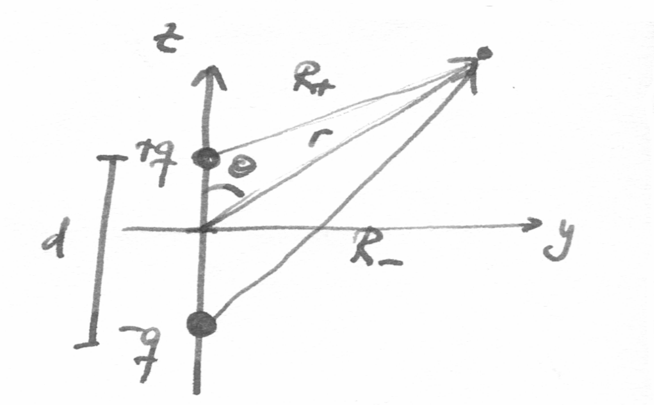
$$P_{rad} = \lim_{r \to \infty} P(\underline{r}).$$

Energy (per unit time) transported out to infinity. Area of a sphere grows like r^2, so for the limit to be non-zero, must have the Poynting vector deceasing no faster than $\frac{1}{r^2}$. Inspecting Eqs. (2.24)- (2.25), all the terms in the fields $\underline{E}$ and $\underline{B}$ from *static* charges decrease like $\frac{1}{r^2}$ so $S \sim \frac{1}{r^4}$. Therefore static sources do not radiate!

To calculate the power radiated to infinity we have to: Find the parts of $\underline{E}, \underline{B}$ that decrease at most as $\frac{1}{r}$, and construct the terms of the Poynting vector that decreases (at most) $\frac{1}{r^2}$, integrate over the surface, take limit $r \to \infty$.

Will defer the discussion of the field from moving point charges until after we have done some advanced topics of relativity. Much more satisfying discussion that the one in the book by Griffiths.

2.6.1 Electric Dipole Radiation



Consider two tiny metal spheres separated by a distance d, connected by wire. At time t the charge on the upper sphere is $q(t)$. On the lower sphere the charge is $-q(t)$. The charge is driven back and forth at an angular frequency ω: $q(t) = q_0\ \cos(\omega t)$. This is an oscillating electric dipole with dipole moment, $\underline{p}(t) = p_0\ \cos(\omega t)\hat{\underline{z}}$, $p_0 = q_0\ d$.

The retarded potential is the sum of the contribution from two point sources, evaluated at their respective retarded times

$$\phi(\underline{r},t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos\left(\omega\left(t-\frac{R_+}{c}\right)\right)}{R_+} - \frac{q_0\cos\left(\omega\left(t-\frac{R_-}{c}\right)\right)}{R_-}\right\}.$$

The law of cosines: $R_\pm = \sqrt{r^2 \mp rd\cos(\theta) + \left(\frac{d}{2}\right)^2}$. A series of approximations will allow us to simplify this expression and calculate the power radiated, by finding the Poynting vector. To do so, we need the expressions for the electric $\underline{E}$ and magnetic $\underline{B}$ fields. It is easier to find these from derivatives of the potentials, rather than directly from the Jefimemko's equations – this is usually true, since fewer integrals are involved, and usually it is simpler to differentiate than to integrate.

1. Approximation: $d \ll r$ (dipole is small compared to the distance to the observer). We expand and want to keep terms to first order in $\frac{d}{r}$. [there is no radiation at 0th order in r],

$$R_\pm \simeq r(1 \mp \frac{d}{2r}\cos(\theta)),$$

so:

$$\frac{1}{R_\pm} \simeq \frac{1}{r}(1 \pm \frac{d}{2r}\cos(\theta)).$$

Also

$$\begin{aligned}\cos\left[\omega\left(t-\frac{R_\pm}{c}\right)\right] &\simeq \cos\left[\omega\left(t-\frac{r}{c}\right) \pm \frac{\omega d}{2c}\cos(\theta)\right] \\ = \cos\left[\omega\left(t-\frac{r}{c}\right)\right]&\cos\left[\frac{\omega d}{2c}\cos(\theta)\right] \mp \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\frac{\omega d}{2c}\cos(\theta)\right].\end{aligned}$$

2. Approximation: "<u>perfect</u>" dipole limit: $d \ll \frac{c}{\omega}$ (or $d \ll \lambda$). To first order the expansions of the cos and sin then gives $\cos\left[\frac{\omega d}{2c}\cos(\theta)\right] \simeq 1$, $\sin\left[\frac{\omega d}{2c}\cos(\theta)\right] \simeq \frac{\omega d}{2c}\cos(\theta)$, and therefore

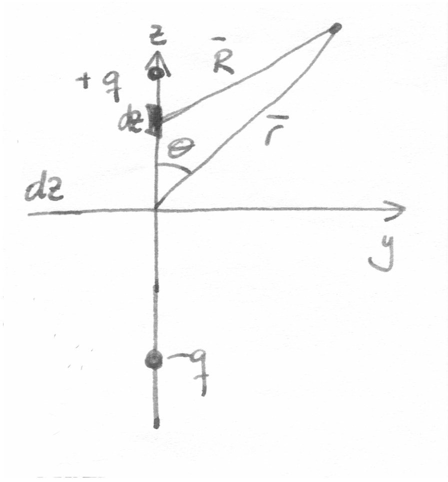
$$\cos\left[\omega\left(t-\frac{R_\pm}{c}\right)\right] \simeq \cos\left[\omega\left(t-\frac{r}{c}\right)\right] \mp \frac{\omega d}{2c}\cos(\theta)\sin\left[\omega\left(t-\frac{r}{c}\right)\right].$$

The scalar potential of the oscillating perfect dipole is then

$$\phi(r,\theta,t) = \frac{p_0}{4\pi\epsilon_0}\left(\frac{\cos(\theta)}{r}\right)\left\{-\frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}. \tag{2.26}$$

3. Approximation: <u>Radiation Zone</u> $r \gg \lambda$ (ie. $r \gg \frac{c}{\omega}$, such that the last term in Eq. (2.26) is suppressed relative to the first term),

$$\phi(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos(\theta)}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right].$$



The vector potential can be determined from the current in the wire and Eq. (2.18). The current is

$$\underline{I} = \frac{dq}{dt}\hat{\underline{z}} = -q_0\omega\sin(\omega t)\hat{\underline{z}}.$$

The vector potential is then

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi}\int_{\frac{-d}{2}}^{\frac{d}{2}} \frac{-q_0\omega\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\underline{z}}}{R}dz.$$

We now want to apply the approximations 1-3 used for the scalar potential also to the vector potential. We want to keep only the first order in d/r, using $d \ll r$. Given the approximations, $\frac{1}{R} \sim \frac{1}{r} + \mathcal{O}(d/r)$, and furthermore since $d \ll \lambda$ (i.e. $d \ll \frac{c}{\omega}$), the sin will vary slowly over the integral region.

We want to keep just the first order in $\frac{d}{r}$. The integration interval itself introduces a dependence on d. Given the accuracy required, we can then approximate the integral by integrand at $z = 0$ times integration length,

$$\underline{A}(r, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\underline{z}}.$$

We can now calculate the most significant parts of the fields

$$\underline{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\underline{\theta}}, \tag{2.27}$$

$$= \frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos(\theta)\left(-\frac{1}{r^2}\sin\left[\omega\left(t - \frac{r}{c}\right)\right] - \frac{\omega}{rc}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right)\hat{\underline{r}} - \frac{\sin(\theta)}{r^2}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\underline{\theta}}\right\}, \tag{2.28}$$

$$\simeq -\frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos(\theta)}{r}\right)\cos\left(\omega\left(t - \frac{r}{c}\right)\right)\hat{\underline{r}} \tag{2.29}$$

We also have (by writing $\hat{\underline{z}}$ in terms of spherical coordinates $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$, $\hat{\underline{z}} = \cos(\theta)\hat{\underline{r}} - \sin(\theta)\hat{\underline{\theta}}$)

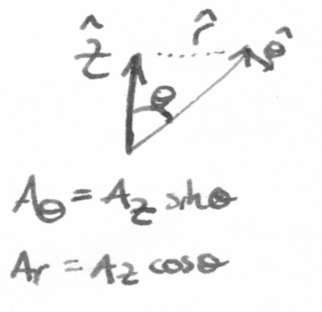
$$\frac{\partial\underline{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\cos(\theta)\hat{\underline{r}} - \sin(\theta)\hat{\underline{\theta}}\right),$$

so (using $\epsilon_0 c^2 = \mu_0$)

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin(\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\underline{\hat{\theta}}.$$

[Notice how the contribution to $\underline{E}$ from $\nabla\phi$ is cancelled by terms from $\frac{\partial A}{\partial t}$ in this gauge.]

We now calculate $\underline{B}$ from $\underline{\nabla}\times\underline{A}$. We will use spherical co-ordinates despite $\underline{A}$ lying along the $\underline{\hat{z}}$ direction, in order to be able to form Poynting vector. There is no azimuthal (ϕ) dependence so the curl is simple in spherical coordinates.



$$\underline{\nabla}\times\underline{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\underline{\hat{\phi}} \tag{2.30}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin(\theta)\cos\left[\omega\left(t - \frac{r}{c}\right)\right] + \frac{\sin(\theta)}{r}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}\underline{\hat{\phi}}, \tag{2.31}$$

Now use approximation 3 ($r \gg c/\omega$)

$$\underline{B} = \underline{\nabla}\times\underline{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin(\theta)}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\underline{\hat{\phi}}.$$

The result is: monochromatic (one ω) waves, radial direction, speed of light. $\underline{E}$ and $\underline{B}$ are in phase, mutually perpendicular. Transverse to direction of propagation. $\frac{E_0}{B_0} = c$.

The energy radiated is found by first calculating the Poyinting vector

$$\underline{S} = \frac{1}{\mu_0}(\underline{E}\times\underline{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin(\theta)}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}^2\underline{\hat{r}}.$$

We want average the Poynting vector over one time cycle: $\langle\cos^2\left[\omega\left(t - \frac{r}{c}\right)\right]\rangle = \frac{1}{2}$,

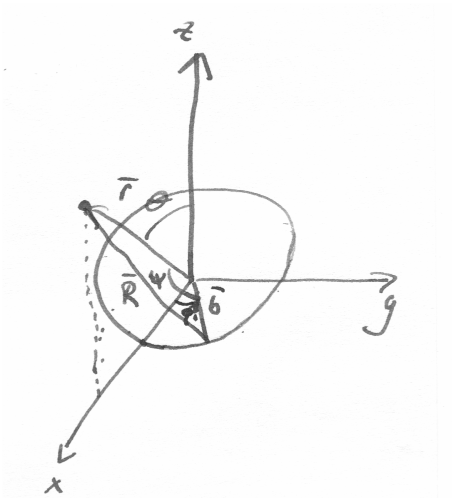
$$\langle\underline{S}\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2(\theta)}{r^2}\underline{\hat{r}}.$$

Conclusion: There is no radiation along the axis of the dipole, $(\theta = 0)$ or $(\theta = \pi)$. The intensity profile is shaped like a donut. The total power is found by integrating over sphere,

$$\langle P \rangle = \int \langle \underline{S} \rangle \cdot d\underline{a} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \int \frac{\sin^2(\theta)}{r^2} r^2 \sin(\theta) d\theta d\phi = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}.$$

[Independent of r as expected (once $r \to \infty$ used)].

2.6.2 Magnetic Dipole Radiation



A wire loop of radius b with alternating current,

$$I(t) = I_0 \cos(\omega t), \tag{2.32}$$

produces an oscillating <u>magnetic</u> dipole.

$$\underline{m}(t) = \pi b^2 I(t) \hat{\underline{z}} = m_0 \cos(\omega t) \hat{\underline{z}},$$
$$m_0 = \pi b^2 I_0,$$

where m_0 is the magnetic dipole moment.

The loop is uncharged, so the scalar potential ϕ is zero! The retarded vector potential is,

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\underline{j}(\underline{r}', t_r)}{R} d\tau',$$
$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos\left[\omega\left(t - \frac{R}{c}\right)\right]}{R} d\underline{l}'.$$

In cylindrical coordinates/plane polar for a point (r, θ, ϕ) $\underline{A}$ must point in the $\hat{\underline{\phi}}$ direction, with no contribution in the $\hat{\underline{\theta}}$ direction. For any contribution in the $+\hat{\underline{r}}$ direction, there is an equal and opposite contribution in the $-\underline{\hat{r}}$ direction. There are no infinitesimal contributions in the $\hat{\theta}$-direction. We therefore have

$$\underline{A}(\underline{r},t) = \frac{\mu_0 I_0 b}{4\pi}\underline{\hat{\phi}}\int_0^{2\pi} \frac{\cos\left[\omega\left(t-\frac{R}{c}\right)\right]}{R}\cos(\underline{\phi}')d\underline{\phi}', \tag{a}$$

the cosine arises from the projection of $\underline{\hat{\phi}'}$ on to $\underline{\hat{\phi}}$. (As usual, the primed co-ordinates are used for the source, un-primed relate to the coordinates of the observer). The law of cosines gives $R = \sqrt{r^2 + b^2 - 2rb\cos(\psi)}$ hence

$$\underline{r} = r\sin(\theta)\underline{\hat{x}} + r\cos(\theta)\underline{\hat{z}},$$
$$\underline{b} = b\cos(\phi')\underline{\hat{x}} + b\sin(\phi')\underline{\hat{y}},$$

so $rb\cos(\psi) = \underline{r}\cdot\underline{b} = rb\sin(\theta)\cos(\phi')$ and hence,

$$R = \sqrt{r^2 + b^2 - 2rb\sin(\theta)\cos(\phi')}.$$

We now study a set of approximations to this formula, relevant for specific situations.

Approximation 1: $b \ll r$ (i.e. observer far from dipole),

$$R \simeq r\left(1 - \frac{b}{R}\sin(\theta)\cos(\phi')\right),$$

so:

$$\frac{1}{R} \simeq \frac{1}{r}\left(1 + \frac{b}{R}\sin(\theta)\cos(\phi')\right), \tag{b}$$

and

$$\cos\left[\omega\left(t-\frac{R}{c}\right)\right] \simeq \cos\left[\omega\left(t-\frac{r}{c}\right) + \frac{\omega b}{c}\sin(\theta)\cos(\phi')\right]$$
$$\simeq \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\left(\frac{\omega b}{c}\sin(\theta)\cos(\phi')\right) - \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left(\frac{\omega b}{c}\sin(\theta)\cos(\phi')\right). \tag{c}$$

Again, this doesn't really appear to help much on its own.

Approximation 2: $b \ll \frac{c}{\omega}$ (or $b < \lambda$). In this case, $\cos\left(\frac{\omega b}{c}\sin(\theta)\cos(\phi')\right) \simeq 1$, and (c) can be approximated by

$$\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega b}{c}\sin(\theta)\cos(\phi')\sin\left[\omega\left(t-\frac{r}{c}\right)\right]. \tag{d}$$

Inserting the result of equation (d) into (a) one finds

$$\underline{A}(\underline{r},t) = \frac{\mu_0 I_0 b}{4\pi}\underline{\hat{\phi}}\int_0^{2\pi}\left\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right] + b\sin\theta\cos\phi'\left(\frac{b}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\right.$$
$$\left. - \frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\cos(\underline{\phi}')d\underline{\phi}'. \tag{2.33}$$

Dropping the 2nd order terms in b^2. The first term integrates to zero: $\int_0^{2\pi} \cos(\phi')d\phi' = 0$. The second term: $\int_0^{2\pi} \cos^2(\phi')d\phi' = \pi$. We then get:

$$\underline{A}(r,\theta,t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin(\theta)}{r}\right)\left\{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\underline{\hat{\phi}}.$$

[Static limit $\omega = 0$, reduces to familiar result of the vector potential from a current loop .]

Approximation 3: "Radiation zone" $r \gg \frac{c}{\omega}$ ($r \gg \lambda$). The first term is suppressed and we get,

$$\underline{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin(\theta)}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\underline{\hat{\phi}}.$$

We can now derive the physical fields by applying derivatives to the potentials. In particular, we will use the spherical form of the curl, after noticing that $\underline{A}$ has only A_ϕ component. The curl for this special form is therefore

$$\underline{\nabla} \times \underline{A} = \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)A_\phi\right)\underline{\hat{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\underline{\hat{\theta}},$$

and we find

$$\underline{E}(\underline{r},t) = -\underline{\nabla}\phi - \frac{\partial\underline{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin(\theta)}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\underline{\hat{\phi}},$$
$$\underline{B}(\underline{r},t) = \underline{\nabla} \times \underline{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin(\theta)}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\underline{\hat{\theta}}.$$

The fields generated from the magnetic dipole are perpendicular to each other, and transverse to the direction of propagation. The ratio of the amplitudes are $\frac{E_0}{B_0} = c$. This is very similar to the fields found from the electric dipole, but the magnetic dipole generates fields with directions of $\underline{B} \sim \underline{\hat{\theta}}$, $\underline{E} \sim \underline{\hat{\phi}}$.

The Poynting vector is found as

$$\underline{S} = \frac{1}{\mu_0}\left(\underline{E}\times\underline{B}\right) = \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin(\theta)}{r}\right)\cos\left[\omega\left(t-\frac{R}{c}\right)\right]\right\}^2\underline{\hat{r}}.$$

The time-averaged intensity is

$$\langle\underline{S}\rangle = \left(\frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\right)\frac{\sin^2(\theta)}{r^2}\underline{\hat{r}}.$$

The radiated power is found by integrating this over a sphere

$$\langle P\rangle = \int\langle\underline{S}\rangle\cdot d\underline{a} = \frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\int\frac{\sin^2(\theta)}{r^2}r^2\sin(\theta)d\theta d\phi = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3}.$$

The power radiated by electric and magnetic dipoles driven with same ω is

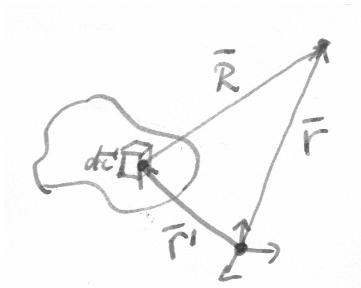
$$\frac{P_{\text{mag}}}{P_{\text{el}}} = \left(\frac{m_0}{p_0 c}\right)^2.$$

As $m_0 = \pi b^2 I_0$, $p_0 = q_0 d$, $I_0 = q_0\omega$ (current in electric case) and setting $d = \pi b$ for sake of comparison we get,

$$\frac{P_{\text{mag}}}{P_{\text{el}}} = \left(\frac{\omega b}{c}\right)^2 .$$

The power radiated from a magnetic dipole is small by approximation 2, and then squared!

2.7 Fields of Moving Point Charge



The retarded potentials for an arbitrary charge and current density distribution are given by (with the retarded time $t_r = t - \frac{R}{c}$)

$$\phi(\underline{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\underline{r}', t_r)}{R} d\tau', \tag{2.34}$$

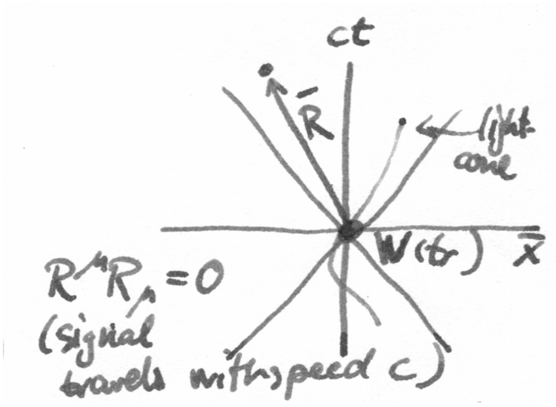
$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}(\underline{r}', t_r)}{R} d\tau'. \tag{2.35}$$

Now we want to calculate the retarded potentials of a point charge moving on the trajectory $w(t)$ (position of q at time t). The retarded time is determined implicitly by the requirement $|\underline{r} - \underline{w}(t_r)| = c(t - t_r)$ where $t_r = t - \frac{|\underline{r} - \underline{W}(t_r)|}{c}$. It is important to establish that as long as the charge and the observer are in vacuum, at most one point is at the distance $|\underline{r} - \underline{w}(t_r)| = c(t - t_r)$ at any particular time.

Proof: Suppose two point with retarded times, $t_1 \neq t_2$. $R_1 = c(t - t_1)$, $R_2 = c(t - t_2)$. $R_1 - R_2 = c(t_2 - t_1)$ so average speed of the particle towards $\underline{r}$ is the speed of light. This is impossible according to the laws of relativity, and therefore the assumptions must be wrong. Therefore there can be at most one (and therefore unique) retarded point for any given time t.

2.7.1 Lienard-Wiechert potentials

Compute the scalar and vector potentials or the 4 potential, $A^\mu = (\phi, c\underline{A})$. First work in a special inertial frame (S'): The instantaneous rest frame of q at time t_{ret}. In this frame we have



with $e = \frac{q}{4\pi\epsilon_0}$,

$$\phi' = \frac{q}{4\pi\epsilon_0|\underline{R'}|} = \frac{e}{R'}, \tag{2.36}$$

$$\underline{A} = 0. \tag{2.37}$$

We now want to translate this to an arbitrary inertial frame S. We know already that the 4-potential $A^\mu = (\phi, c\underline{A})$ is a 4-vector, so we could just Lorentz-transform the fields to S. However, we will use a little trick: We want to find an expression of 4-vectors, which take the above form in S'. We do not have too many 4-vectors at our disposal, and we can guess that it will have to involve the four-velocity, since the vector-potential from a charged particle depends on its velocity. Secondly, it should depend on the (four-)distance between the observer and the (retarded position of the) particle. We try with $A^\mu = \frac{e}{u_\nu r^\nu} u^\mu$. This is a 4-vector where u^μ is the 4-velocity of the charge at the time t_{ret}. We start by checking that in S' this is the correct 4-vector. We have

$$u_\nu R^\nu = (c\gamma - \gamma\underline{v}) \begin{pmatrix} c(t - t_{ret}) = |\underline{R}| \\ \underline{R} \end{pmatrix} = \gamma c \left(|\underline{R}| - \frac{\underline{v} \cdot \underline{R}}{c} \right).$$

In S': $u'^\mu = (c, 0)$; $u'_\nu R'^\nu = c|R'|$,

$$\rightarrow A^\mu = \frac{e}{u'_\nu R'^\nu} u'^\mu = \frac{e}{c|R'|}(c, \underline{0}) = \left(\frac{e}{R'}, \underline{0} \right).$$

So this 4-vector form gives the right answer in $\underline{S'}$. The 4-potential is a 4-vector so transforms according to the Lorentz-transformations like any other 4-vector. Therefore the expression is true in any S! With the usual relativistic notation of $\underline{\beta} = \underline{v}/c$ we have

$$A^\mu = \left. \frac{e u^\mu}{u_\nu R^\nu} \right|_{t=t_{\rm ret}} \tag{2.38}$$

$$\phi(\underline{r}, t) = \left. \frac{e}{R - \underline{\beta} \cdot \underline{R}} \right|_{t=t_{\rm ret}} \tag{2.39}$$

$$\underline{A}(\underline{r}, t) = \left. \frac{e\underline{\beta}}{c\left(R - \underline{\beta} \cdot \underline{R}\right)} \right|_{t=t_{\rm ret}} \tag{2.40}$$

2.7.2 The Physical Fields of a Moving Point Charge

We now want to find the physical fields $\underline{E}, \underline{B}$ from the potentials.

$$\underline{E}(\underline{r},t) = -\underline{\nabla}\phi(\underline{r},t) - \frac{\partial \underline{A}(\underline{r},t)}{\partial t} \tag{2.41}$$

$$\underline{B}(\underline{r},t) = \underline{\nabla} \times \underline{A}(\underline{r},t) \tag{2.42}$$

Difficulty: the differentiation is tricky, since $\underline{R} = \underline{r} - \underline{w}(t_r)$, $\underline{v} = \dot{\underline{w}}(t_r)$ are both evaluated at the retarded time t_r, which is defined implicitly through $|\underline{r} - \underline{w}(t_r)| = c(t - t_r)$, and so t_r itself is a function of $\underline{r}$ and t. After 2 pages of algebra (436-437 in Griffith's) and using $\underline{u} = c\hat{\underline{R}} - \underline{v}$, and the acceleration $\underline{a} = \frac{\partial v}{\partial t}$ (evaluated at the retarded time t_r) one finds

$$\underline{E}(\underline{r},t) = \frac{q}{4\pi\epsilon_0}\frac{R}{(\underline{R}\cdot\underline{u})^3}\left[(c^2 - v^2)\underline{u} + \underline{R}\times(\underline{u}\times\underline{a})\right], \tag{2.43}$$

$$\underline{B}(\underline{r},t) = \frac{1}{c}\hat{\underline{R}} \times \underline{E}(\underline{r},t). \tag{2.44}$$

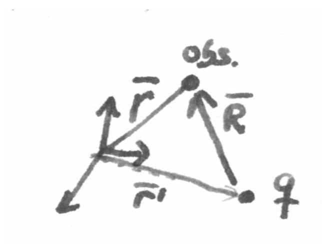
The magnetic field from a point charge is always perpendicular to the electric field from the point charge, and to the vector from the retarded point to the observer in $\underline{r}$. The first term in $\underline{E}$ falls as of $\frac{1}{R^2}$: This is called the *generalised coulomb field* (if $v = 0, a = 0$ it reproduces the Coulomb field of a stationary point particle). The second term falls off as $\frac{1}{R}$, and dominates at large distances. This is called the *radiation field* or *acceleration field*.

The Lorentz force on particle of charge Q and velocity $\underline{v}$ from the moving charge q is then given by:

$$\begin{aligned}\underline{F} &= Q(\underline{E} + \underline{v}\times\underline{B}) \\ &= \frac{qQ}{4\pi\epsilon_0}\frac{R}{(\underline{R}\cdot\underline{u})^3}\left\{\left[(c^2-v^2)\underline{u} + \underline{R}\times(\underline{u}\times\underline{a})\right] + \frac{\underline{v}}{c}\times\left[\hat{\underline{R}}\times\left[(c^2-v^2)\underline{u} + \underline{R}\times(\underline{u}\times\underline{a})\right]\right]\right\},\end{aligned} \tag{2.45}$$

where $\underline{R}$, $\underline{u}$, $\underline{v}$, and $\underline{a}$ are all evaluated at the retarded time (and $\underline{u} = c\hat{R} - \underline{v}$ is a three-vector, not part of the 4-vector u^μ).

2.8 Power Radiated by Accelerated Point Charge



Let us now evaluate the power radiated by the moving point charge. If we choose a coordinate system S' such that $\underline{r'} = 0$ at $t_{ret} = 0$, then $t_{ret} = t - \frac{R}{c}$. The Poynting vector is,

$$\underline{S}(\underline{r}, t) = \frac{1}{\mu_0}(\underline{E} \times \underline{B}) = \frac{1}{\mu_0 c}\left[\underline{E} \times (\underline{\hat{R}} \times \underline{E})\right].$$

By use of the standard "$BAC - CAB$" vector identity $\underline{A} \times (\underline{B} \times \underline{C}) = \underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B})$, we find that this equals

$$\underline{S}(\underline{r}, t) = \frac{1}{\mu_0 c}\left[E^2 \underline{\hat{R}} - (\underline{\hat{R}} \cdot \underline{E})\underline{E}\right]. \tag{2.46}$$

Not all of this radiation will propagate to infinity. To calculate the total power radiated to infinity at time t_{ret} we should at time t integrate the Poynting vector over a sphere of (large) radius R, centred at the position of the particle at time t_{ret} (All points of the sphere will then be reached simultaneously at time t by radiation emitted at time t_{ret}).

Like in the dipole-discussion, we will concentrate on terms in $\underline{S}$ which fall off like $1/R^2$. Those that fall off faster will be suppressed for significantly large R (and there are none that fall off slower). The acceleration fields represent the true radiation, which falls off as $1/R$ for $R \to \infty$

$$\underline{E_{\text{rad}}} = \frac{q}{4\pi\epsilon_0}\frac{R}{(\underline{R} \cdot \underline{u})^3}\left[\underline{R} \times (\underline{u} \times \underline{a})\right],$$

[The velocity field carries energy too - dragged along with the charge, but it does not contribute at large R].

As $\underline{E_{\text{rad}}}$ is perpendicular to $\underline{\hat{R}}$, the second term in Eq.2.46 for $\underline{S}$ vanishes, and we find

$$\underline{S_{\text{rad}}} = \frac{1}{\mu_0 c}E_{\text{rad}}^2 \underline{\hat{R}}.$$

We will use this to calculate the total power radiated to infinity in a few examples of increasing difficulty:

Charge at rest, but undergoing acceleration

At time t_r (where $\underline{v}(t_r) = 0$ we find $\underline{u} = c\underline{\hat{R}}$ we have (using $c^2 = \frac{1}{\epsilon_0\mu_0}$)

$$\underline{E_{\text{rad}}} = \frac{q}{4\pi\epsilon_0 c^2 R}\left[\underline{\hat{R}} \times (\underline{\hat{R}} \times \underline{a})\right] = \frac{\mu_0 q}{4\pi R}\left[(\underline{\hat{R}} \cdot \underline{a})\underline{\hat{R}} - \underline{a}\right],$$

Then, since

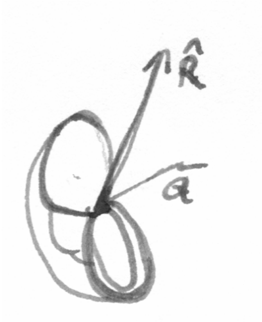
$$\left|(\underline{\hat{R}} \cdot \underline{a})\underline{R} - \underline{a}\right|^2 = \left(\left(\underline{\hat{R}} \cdot \underline{a}\right)\underline{\hat{R}} - \underline{a}\right)\left(\left(\underline{\hat{R}} \cdot \underline{a}\right)\underline{\hat{R}} - \underline{a}\right) = (\underline{\hat{R}} \cdot \underline{a})^2 - 2(\underline{\hat{R}} \cdot \underline{a})(\underline{\hat{R}} \cdot \underline{a}) - \underline{a}^2 = (a^2 - (\underline{\hat{R}} \cdot \underline{a})^2), \tag{2.47}$$

we find

$$\underline{S} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi R}\right)^2\left[a^2 - (\underline{\hat{R}} \cdot \underline{a})^2\right]\underline{\hat{R}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2(\theta)}{R^2}\right)\underline{\hat{R}},$$

where θ is the angle between $\underline{a}$ and $\hat{\underline{R}}$. We notice that no power is radiated in direction of acceleration. We find a donut-shaped power-distribution about the direction of instantaneous acceleration.

To find the total power radiated we notice that in the surface integral the surface normal $\hat{\underline{n}}$ is outward, and $\hat{\underline{R}} \cdot d\underline{a} = R^2 \sin(\theta) d\theta d\phi$. Therefore we can find the total power radiated (the



Poynting-vector integrated over a sphere at large R)

$$P = \oint \underline{S_{\text{rad}}} \cdot d\underline{a} \tag{2.48}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c^2} \int \frac{\sin^2(\theta)}{R^2} R^2 \sin(\theta) d\theta d\phi \tag{2.49}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c^2} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin^3(\theta) \tag{2.50}$$

As there is no ϕ dependence, the ϕ-integral just gives a factor $\int_0^{2\pi} d\phi = 2\pi$. By using a substitution of $x = \cos(\theta)$, we find

$$\int_0^{\pi} d\theta \sin^3(\theta) = -\int_1^{-1} \sin^2(\theta) d(\cos(\theta)) = \int_{-1}^{1} (1 - x^2) dx = x - \frac{1}{3}x^3 \Big|_{-1}^{1} = \frac{4}{3}. \tag{2.51}$$

Thus,

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}. \qquad [\underline{v} = 0] \tag{2.52}$$

This is known as the Larmor's formula and valid when $\underline{v} = 0$.

Charge with $\underline{v} \neq 0$ and undergoing acceleration

Obviously $\underline{E_{\text{rad}}}$ will be more complicate in this case. We will derive the form of the power radiated using 4-vectors and the requirement of invariance under Lorentz-transformation. And we will illustrate how it can be derived through the "straight-forward"-approach.

"Straight-forward" (but tedious) approach First we need to take into account a subtle effect: the power radiated at t does not equal the power observed passing through the sphere of radius R at a time $\frac{R}{c}$ later. This is related to the Doppler effect.

Let $\frac{dW}{dt}$ be the rate at which energy passes through an infinitesimal area of the sphere at radius R, then the rate at which the energy left the particle at time t_r was $\frac{dW}{dt_r} = \frac{dW}{dt} \times \frac{dt}{dt_r}$. So we need to find $\frac{dt}{dt_r}$. Now

$$c(t - t_r) = R \therefore c^2(t - t_r)^2 = R^2 = \underline{R} \cdot \underline{R}.$$

Differentiating this with respect to t,

$$2c^2(t - t_r)(1 - \frac{\partial t_r}{\partial t}) = 2\underline{R}\frac{\partial \underline{R}}{\partial t} \tag{2.53}$$

$$\therefore cR\left(1 - \frac{\partial t_r}{\partial t}\right) = \underline{R} \cdot \frac{\partial \underline{R}}{\partial t}. \tag{2.54}$$

As $\underline{R} = \underline{r} - \underline{w}(t_r)$, and knowing that $\underline{r}$ is fixed, we find

$$\frac{\partial \underline{R}}{\partial t} = -\frac{\partial \underline{w}(t_r)}{dt} = -\frac{\partial \underline{w}(t_r)}{dt_r}\frac{\partial t_r}{dt} = -\underline{v}\frac{\partial t_r}{dt} \tag{2.55}$$

This we substitute into Eq. (2.54) to find

$$cR(1 - \frac{\partial t_r}{\partial t}) = -\underline{R} \cdot \underline{v}\frac{\partial t_r}{\partial t}, \tag{2.56}$$

$$\therefore cR = \frac{\partial t_r}{dt}(cR - \underline{R} \cdot \underline{v}) = \frac{\partial t_r}{dt}(\underline{R} \cdot \underline{u}), \tag{2.57}$$

$$\therefore \frac{\partial t_r}{dt} = \frac{cR}{\underline{R} \cdot \underline{u}}. \tag{2.58}$$

Therefore, finally,

$$\frac{dW}{dt_r} = \left(\frac{\underline{R} \cdot \underline{u}}{Rc}\right)\frac{dW}{dt}. \tag{2.59}$$

As $\frac{\underline{R} \cdot \underline{u}}{Rc} = \frac{\hat{\underline{R}} \cdot \underline{v}}{c}$ and $\underline{u} = c\hat{\underline{R}} - \underline{v} \to \frac{\underline{u}}{c} = \hat{\underline{R}} - \underline{\beta}$,

$$\frac{dP}{d\Omega} = \left(\frac{\underline{R} \cdot \underline{u}}{Rc}\right)\frac{1}{\mu_0 c}E_{\mathrm{rad}}^2 R^2 = \frac{q^2}{16\pi^2\varepsilon_0^2}\frac{R^5c^5}{(\underline{R} \cdot \underline{u})^5}\frac{1}{R^2c^6}\frac{1}{\mu_0 c}|\underline{R} \times (\underline{u} \times \underline{a})|^2 \tag{2.60}$$

$$= \frac{q^2}{16\pi^2\varepsilon_0}\frac{1}{c}\frac{\left|\hat{\underline{R}} \times \left(\hat{\underline{R}} - \underline{\beta}\right) \times \dot{\hat{\beta}}\right|}{(1 - \hat{\underline{R}} \cdot \underline{\beta})^5}, \tag{2.61}$$

where we used $\underline{u} = c\hat{\underline{R}} - \underline{v}$ and therefore $\underline{u}/c = \hat{\underline{R}} - \underline{\beta}$ to get to the last result. This general expression then "just" needs to be integrated over the sphere.

This is difficult to integrate in full generality - you will get to do a few examples. The general result is given by Lienard's generalisation of Larmor's formula:

$$P = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \frac{|\underline{v} \times \underline{a}|^2}{c^2}\right), \tag{2.62}$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. The radiated intensity increases enormously for relativistic particles.

Derivation using 4-vectors and Relativistic Covariance We will now illustrate how the same result can be obtained by use of arguments from special relativity. We start by observing that the power radiated $P(t)$ is a Lorentz scalar. This can be seen by considering the 4-vectors $(ct, \underline{r})$ of the source and $(U_{\rm rad}/c, \underline{P}_{\rm rad})$, the total energy and the momentum radiated. If we consider first the instantaneous rest frame of the particle, S', then by symmetry, there is no preferred direction of the radiation (e.g. the Poynting vector is symmetric in $\theta \to -\theta$, and in $\phi \to -\phi$) and therefore the total momentum radiated in a time dt' is $d\underline{P}'_{\rm rad} = 0$. Similarly, $d\underline{r}' = 0$ because the particle is at rest, $\underline{v}' = d\underline{r}'/dt = 0$. Using now the standard Lorentz transformation properties for 4-vectors, we find

$$P = \frac{dU_{\rm rad}}{dt} = \frac{\gamma\,(dU'_{\rm rad} + \underline{v}\cdot d\underline{P}'_{\rm rad})}{\gamma\,(dt' + \underline{v}\cdot d\underline{r}'/c^2)} = \frac{dU'_{\rm rad}}{dt'}, \tag{2.63}$$

and therefore P is a Lorentz scalar.

For a frame, in which the particle is moving, one finds that the rate at which it radiates momentum is related to its velocity and the rate at which it radiates energy - and that the net radiation of momentum is in the direction of the particle's velocity $\underline{v}$

$$\frac{d\underline{P}_{\rm rad}}{dt} = \frac{\gamma\left(d\underline{P}'_{\rm rad} + \underline{v}dU'_{\rm rad}/c^2\right)}{\gamma\,(dt' + \underline{v}\cdot d\underline{r}'/c^2)} = \frac{\underline{v}}{c^2}\frac{dU'_{\rm rad}}{dt'}. \tag{2.64}$$

Larmor's formula holds in the instantaneous rest-frame of the particle. We will now apply the same procedure that we used in the study of the Lienard-Wiechert potentials, and write the Larmor's formula using 4-vectors in order to obtain an expression that holds in an arbitrary frame of inertia.

We have just derived that the rate of energy loss is a Lorentz scalar. In covariant notation, this must then be a contraction of 4-vectors. The natural (and indeed unique, based on the quadratic behaviour in the acceleration) generalisation to 4-vectors is simply to replace $a^2 \to -a^\mu a_\mu$ in Larmor's formula Eq. (2.52). It turns out to be smarter to use the energy-momentum tensor of the particle instead of its 4-acceleration directly, and make use of the relation $p^\mu = mu^\mu$, and therefore $a^\mu = 1/m\ dp^\mu/d\tau$. We then have

$$P(t) = -\frac{\mu_0 q^2}{6\pi c}\frac{1}{(mc)^2}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}. \tag{2.65}$$

To evaluate the Minkowski product we note that

$$\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 - \frac{d\underline{p}}{d\tau}\cdot\frac{d\underline{p}}{d\tau} \tag{2.66}$$

$$= (mc\gamma)^2\left[\left(\frac{d\gamma}{dt}\right)^2 - \frac{d(\gamma\underline{\beta})}{dt}\cdot\frac{d(\gamma\underline{\beta})}{dt}\right]. \tag{2.67}$$

We now calculate each of the pieces in the square bracket of Eq. (2.67) and find

$$\frac{d\gamma}{dt} = \gamma^3\underline{\beta}\cdot\dot{\underline{\beta}} \quad \frac{d\gamma\underline{\beta}}{dt} = \gamma\dot{\underline{\beta}} + \gamma^3(\underline{\beta}\cdot\dot{\underline{\beta}})\underline{\beta}. \tag{2.68}$$

After some algebra and use of the definition of $\gamma = 1/\sqrt{1-\beta^2}$ we find

$$P(t) = \frac{\mu_0\ q^2 c}{6\pi}\gamma^6 \left(\underline{\dot{\beta}}^2 - \left(\underline{\beta} \times \underline{\dot{\beta}}\right)^2\right)_{\mathrm{ret}}, \tag{2.69}$$

where it obviously is understood that the bracket is evaluated at the retarded time corresponding to t. Eq. (2.69) is equivalent to Eq. (2.62).

We can write the result differently by use of the form in Eq. (2.66) instead of Eq. (2.67). We find

$$P(t) = \frac{\mu_0\ q^2\gamma^2}{6\pi m^2 c}\left[\left|\frac{d\underline{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right], \tag{2.70}$$

which can be easier to evaluate than Eq. (2.69), if the force applied to the point particle and the momentum change is known.

On top of the observation that for a given acceleration, the radiation from relativistic particles radiate is significantly more intense than that from non-relativistic counterparts, it is worth noting that for a fixed magnitude of $\underline{v}$ and $\underline{a}$, Eq. (2.69) predicts that an acceleration parallel to the velocity $\underline{a}\|\underline{v}$ produces radiation with a rate $P \propto \gamma^6 a^2$, whereas an acceleration orthogonal to the velocity $\underline{a} \perp \underline{v}$ produces radiation at a rate $P \propto \gamma^4 a^2$.

2.9 Radiation Reaction

Accelerated charges radiate $\rightarrow$ For given force charged particles accelerate less that a neutral particles of same mass. This can be described as a “recoil force” (radiation reaction force), acting on accelerated charges

$$\underline{F_{\mathrm{rad}}} = \frac{\mu_0 q^2}{6\pi}\underline{\dot{a}}. \qquad \text{[non-relativistic]}$$

[Encouraged to read section 11.2.2]

Chapter 3

The Transformation of the Fields

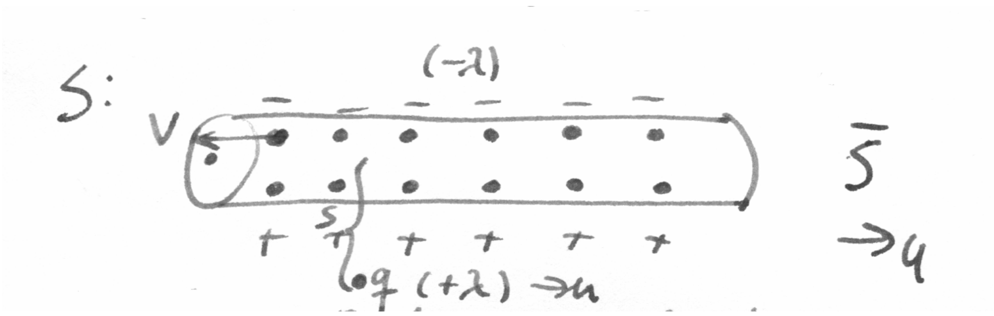
3.1 Magnetism as a Relativistic Phenomenon

We will here discuss how magnetism arises as a relativistic phenomenon. We will see that a postulation of the existence of Electric fields coupled with relativity leads to magnetism. The Maxwell's equations and the Lorentz force ($\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$) are fully consistent with special relativity, despite its apparent reference of a specific velocity relative to an observer. In the following we will re-express EM to expose this covariant behaviour.

First we will remind ourselves of the rules for "adding" velocities under relativity: Einstein's velocity addition rule. A particle moves distance dx in time dt as measured in S, and therefore the velocity of the particle in S is given by $u = \frac{dx}{dt}$. The frame S' is moving with velocity v in the x direction. In S', the particle therefore moved a distance $dx' = \gamma(dx - vdt)$ in an interval $dt' = \gamma(dt - \frac{v}{c^2}dx)$. The velocity in S' is therefore,

$$u' = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u - v}{1 - \frac{uv}{c^2}},$$

where we used L'Hôpital's rule in replacing the numerator and denominator with their derivatives with respect to t.



Consider now the following model of an uncharged, current-carrying wire at rest: It has a continuous charge distribution (λ) of positive and negative charges. The string of positive (negative) charges moving to the right (left) with speed v. The net current is to the right

$I = 2\lambda v$. A distance s from the wire, a point charge q is travelling to the right at speed $u < v$. In this frame there is no electric force (since there is no electric field from the wire - the positive and negative charges in the wire cancel), but there is a magnetic field.

Let us examine the situation from S', which we arrange to be moving to the right with speed u. In this frame, q is at rest, so there can be no magnetic contribution to the Lorentz force! The velocities of the positive and negative charges in this frame are,

$$v'_{\pm} = \frac{v \mp u}{1 \mp \frac{uv}{c^2}}.$$

We find that $|v_-| > |v_+|$ and therefore the Lorentz contractions of the spacing between negative charges is more severe than that between the positive charges. There are more negative charges that positive charges for a given length of wire (in this frame - the situation would of course be reversed, if the charge q had been moving in the opposite direction, and the frame S' was still chosen as the rest-frame of q). In S', the charge densities of the positive and negative charges differ, and we have

$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0,$$

where $\gamma_{\pm} = \frac{1}{\sqrt{1-\frac{v_{\pm}^2}{c^2}}}$, λ_0 is the charge density in the rest frame of the charges in the wire (which is different from the charge densities $\pm\lambda$ in the frame S, since in S the charges move with speed v: $\lambda = \gamma(v)\lambda_0$).

Concentrating on γ_+,

$$\begin{aligned}\gamma_+ = \frac{1}{\sqrt{1 - \frac{1}{c^2}\frac{(v-u)^2}{\left(1-\frac{uv}{c^2}\right)^2}}} &= \frac{c^2 - uv}{\sqrt{(c^2 - uv)^2 - c^2(v-u)^2}}, \\ \frac{c^2 - uv}{\sqrt{(c^2 - v^2)(c^2 - u^2)}} &= \gamma(v)\frac{1 - \frac{uv}{c^2}}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}.\end{aligned}$$

The result for γ_- is similar (but with the sign of v changed), and we find for the total charge density:

$$\lambda_{tot} = \lambda_+ - \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1 - \frac{u^2}{c^2}}}$$

The conclusion is that because of the unequal Lorentz contractions between positive and negative charges (because of direction of current and the charge), a current-carrying wire that is neutral in one frame will be charged in another!

The charge sets up an electric field at the position of the particle $E = \frac{\lambda_{tot}}{2\pi\epsilon_0 s}$. In the frame S' there is therefore an electric force on q!

$$\underline{F} = q\underline{E} = \frac{-\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1 - \frac{u^2}{c^2}}}$$

But if there is a force on q in S', then there must also be a force on q in S (both are Inertial frames, so there is an acceleration of the charge only if a force is applied). Remembering now how forces transform: if the the particle is at rest in S: $F'_\perp = \frac{1}{\gamma}F_\perp$, $F'_\parallel = F_\parallel$.

Since q is at rest in S', and F' is perpendicular to u, the force in S is:

$$F = \frac{1}{\gamma(u)}F' = \sqrt{1-\frac{u^2}{c^2}}F' = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}$$

The charge is attracted towards the wire by a force that is purely electrical in S', but non-electrical in S (where the wire is neutral). Electrostatics + relativity imply another force: the magnetic force. Using $c^2 = \frac{1}{\epsilon_0\mu_0}$, and $v' = 2\gamma v$ we find indeed,

$$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right).$$

Equivalent to $\underline{F} = q\underline{v} \times \underline{B}$ used in system S!

3.2 Lorentz-transformation of the E-M fields

We now want to derive how the fields $\underline{E}$, $\underline{B}$ transform. Let us be given $\underline{E}$, $\underline{B}$ in S. We want to find $\underline{E'}$, $\underline{B'}$ in S'. In deriving the laws of transformation, we will take as input from physics that charge is Lorentz-invariant.

Assumption: We will assume that the transformation laws for the fields are independent of how the fields themselves are produced (i.e. if two different configuration of sources produce the same field, then the transformed fields are the same too). Will use this to choose a specific setup of sources which we understand the transform-laws of. Also, we will assume that the transformation is linear: This means we can investigate each component independently.

Consider an arbitrary general $\underline{E}_0$ at a point in space-time. We want to know $\underline{E}$ in a different inertial frame (S). Consider an inertial frame S_0 where each component of $\underline{E}$ is generated by a simple parallel place capacitor. Start by considering the situation where the capacitor is at rest in S_0 and the plates are parallel to the $x-z$ plane. This means only the y-component of the electric field is non-zero and the field is given by

$$\underline{E}_0 = \frac{\sigma_0}{\epsilon_0}\underline{\hat{y}}$$

The surface charge in S_0 is σ_0. Consider another frame S moving along the x_0-axis in the standard configuration. In S, the plates are moving to the left, but the field still takes the form $\underline{E} = \frac{\sigma}{\epsilon_0}\underline{\hat{y}}$ (the symmetry of the problem ensures there can be no field-component along the $\underline{\hat{x}}$ or $\underline{\hat{z}}$ directions). The surface charge σ in S is increased compared to the surface charge σ_0 in S_0 duo to the Lorentz contraction of the moving plates and the charge being constant.

$$\sigma = \gamma(v_0)\sigma_0$$

so in this case

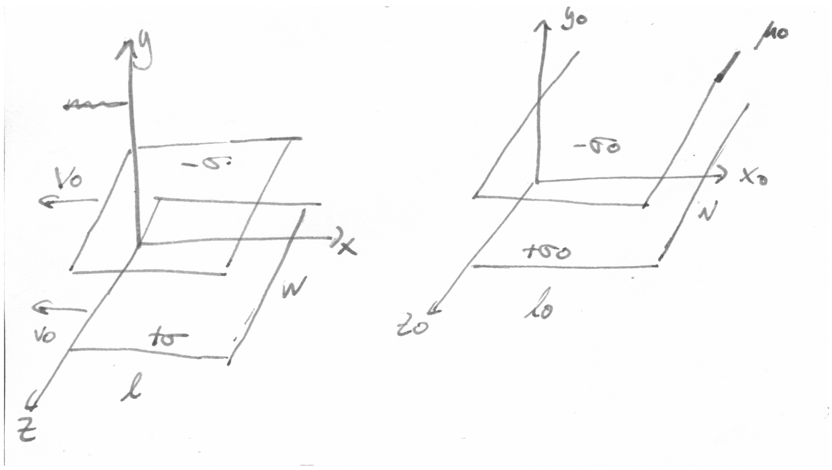
$$\underline{E}^\perp = \gamma(v_0)\underline{E_0}^\perp.$$

[One would have obtained the same results by having the plates parallel to the $x - y$ plane and considering field in z-direction].

The setup we just considered will not, however, give the most general transformation rules, since we considered a setup with no magnetic field in S_0. Will now generalise the result. We note that in S, there will be both a $\underline{E}$ and a $\underline{B}$. The magnetic field is generated by the surface currents from the moving capacitors. The currents are

$$\underline{k}_{\pm} = \mp \sigma v_0 \hat{\underline{x}}$$

The magnetic field generated by these surface currents will lie in the plane of the capacitor, and one finds by using the right-hand rule that $B_z = -\mu_0 \sigma v_0$.



Consider now a third inertial frame S', travelling to the right with speed v relative to S. In S' we would have

$$\begin{aligned} E'_y &= \frac{\sigma'}{\epsilon_0} \\ B'_z &= -\mu_0 \sigma' v' \\ v' &= \frac{v + v_0}{1 + \frac{v v_0}{c^2}} \qquad \text{(the speed relative to } S_0\text{)} \\ \gamma' &= \gamma(v') = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}} \\ \sigma' &= \gamma' \sigma_0 \qquad \text{(the surface charge density in } S'\text{)} \end{aligned}$$

We will now seek to express $\underline{E'}$,$\underline{B'}$ in terms of $\underline{E}$, $\underline{B}$. We start by writing E'_y and B'_z in terms of the charge density in S

$$\begin{aligned} E'_y &= \left(\frac{\gamma'}{\gamma_0}\right) \frac{\sigma}{\epsilon_0}, \\ B'_z &= -\left(\frac{\gamma'}{\gamma_0}\right) \mu_0 \sigma v'. \end{aligned}$$

One finds

$$\frac{\gamma'}{\gamma_0} = \frac{\sqrt{1-\frac{v_0^2}{c^2}}}{\sqrt{1-\frac{v'^2}{c^2}}} = \gamma(v)\left(1+\frac{vv_0}{c^2}\right).$$

Therefore, by using $\sigma v_0 = \frac{-B_z}{\mu_0}$,

$$E_y{}' = \gamma\left(1+\frac{vv_0}{c^2}\right)\frac{\sigma}{\epsilon_0} = \gamma\left(E_y - \frac{v}{c^2\epsilon_0\mu_0}B_z\right), \tag{3.1}$$

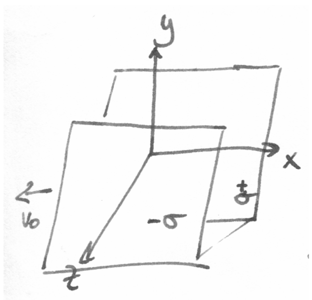
$$B_z{}' = -\gamma\left(1+\frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v+v_0}{1+\frac{vv_0}{c^2}}\right) = \gamma\left(B_z - \mu_0\epsilon_0 v E_y\right). \tag{3.2}$$

Using $\mu_0\epsilon_0 = \frac{1}{c^2}$ we find

$$E'_y = \gamma\left(E_y - vB_z\right),$$
$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

We now derived the transformation laws for the components E_y,B_z.

We can get the laws for the transformations for E_z and B_y similarly by considering them generally by a capacitor parallel to the $x-y$ plane. With a similar method as that just applied



one starts by finding

$$E'_z = \frac{\sigma'}{\epsilon_0}$$
$$B'_y = \mu_0\sigma' v' \tag{3.3}$$

where the sign on the magnetic field component is changed due to the right hand rule. One then derives

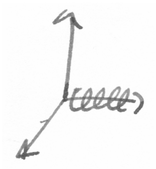
$$E'_z = \gamma\left(E_z + vB_y\right)$$
$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

We have thus found the transformation laws for two further components.

In order to derive the transformation laws for the x-components, we consider first the capacitors arranged parallel to the $y-z$-plane. With this setup there is no change in the surface charge when boosting in the x-direction. Therefore

$$E'_x = E_x \tag{3.4}$$

But this setup alone would not reveal how to transform the B_x- component, since there is no B_x component in S. Consider instead the B_x component generated by an additional solenoid pointing in the x-direction. In the frame S we have



$$B_x = \mu_0 n I \tag{3.5}$$

where n is the winding per unit length and I the current. In S' one finds $B_x = \mu_0 n' I'$. Because of length contraction, $n' = \gamma(v)n$. But since also time dilates $I' = \frac{1}{\gamma(v)}I$, since the current is the charge passing through a section of the wire per time interval, and the charge is considered constant. Time in S' runs slower by a factor $\gamma(v)$, $\frac{d}{dt'} = \frac{1}{\gamma}\frac{d}{dt}$. In the equation for the x-components of the magnetic fields, the two factors of γ cancel, so

$$B'_x = B_x. \tag{3.6}$$

In other words both E'_{para}, B'_{para} are unchanged.

In summary, the transformation laws for the physical fields are

$$\begin{array}{ll} E'_x = E_x & B'_x = B_x \\ E'_y = \gamma(E_y - vB_z) & B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \\ E'_z = \gamma(E_z + vB_y) & B'_z = \gamma(B_z - \frac{v}{c^2}E_y) \end{array} \tag{3.7}$$

We can consider the following simplifying special cases:
1)If $\underline{B} = 0$ in S: $\underline{B'} = \gamma \frac{v}{c^2}\left(E_z \underline{\hat{y}} - E_y \underline{\hat{z}}\right) = \frac{v}{c^2}\left(E'_z \underline{\hat{y}} - E'_y \underline{\hat{z}}\right) \rightarrow \underline{B'} = -\frac{1}{c^2}\left(\underline{v} \times \underline{E'}\right)$.
2) If $\underline{E} = 0$ in S: $\underline{E'} = \underline{v} \times \underline{B'}$.

3.3 The Field Tensor $F^{\mu\nu}$

We learned in the previous section that $\underline{E}$ and $\underline{B}$ do <u>not</u> transform like the spatial components of a 4-vector - they even mix under Lorentz-transformations. Can we find a Lorentz-covarient object with 6 components, which transform like $\underline{E}$,$\underline{B}$?
Answer: A antisymmetric, rank-2 tensor! Remembering 4-vectors transform like:

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu.$$

For a boost along the x-axis,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(y) & -\sinh(y) & 0 & 0 \\ -\sinh(y) & \cosh(y) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A rank-2 tensor is an object with two free indicies,where the elements transform as,

$$t^{\mu\nu} \to t'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\lambda}t^{\sigma\lambda}.$$

The rank-2 tensor can be represented as a matrix,

$$t^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix}.$$

A symmetric tensor: $t^{\mu\nu} = t^{\nu\mu}$ so in total 10 distinct components.
An anti-symmetric tensor: $t^{\mu\nu} = -t^{\nu\mu}$ leading to 6 distinct components,

$$\text{Anti-symmetric } t^{\mu\nu} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}.$$

Let us check the transformation rule for a few components. $t'^{01} = \Lambda^{0}{}_{\lambda}\Lambda^{1}{}_{\sigma}t^{\lambda\sigma}$. Consider boost along x-axis (as usual). Then $\Lambda^{0}{}_{\lambda} = 0$ unless λ is 0 or 1. $\Lambda^{1}{}_{\sigma} = 0$ unless σ is 0 or 1. So we need to check four terms only,

$$t'^{01} = \Lambda^{0}{}_{0}\Lambda^{1}{}_{0}t^{00} + \Lambda^{0}{}_{0}\Lambda^{1}{}_{1}t^{01} + \Lambda^{0}{}_{0}t^{10} + \Lambda^{0}{}_{1}\Lambda^{1}{}_{1}t^{11}.$$

But for the anti-symmetric tensor: $t^{00} = t^{11} = 0$ and $t^{01} = -t^{10}$ so,

$$t'^{01} = (\Lambda^{0}{}_{0}\Lambda^{1}{}_{1} - \Lambda^{0}{}_{1}\Lambda^{1}{}_{0})t^{01} = \left(\cosh^2(y) - \sinh^2(y)\right)t^{01} = t^{01}.$$

Thus:

$$t'^{01} = t^{01},\ t'^{02} = \gamma\left(t^{02} - \beta t^{12}\right),\ t'^{13} = \gamma\left(t^{03} + t^{31}\right),$$

$$t'^{23} = t^{23},\ t'^{31} = \gamma\left(t^{31} + \beta t^{03}\right), t'^{12} = \gamma\left(t^{12} - \beta t^{02}\right).$$

By comparing these transformation rules with those of Eq. (3.7), one sees that the rules for Lorentz-transforming the $\underline{E}$ and $\underline{B}$-fields are encoded in the rank-2 tensor with the following assignments:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

This combines $\underline{E}$ and $\underline{B}$ into a single object, the *field strength tensor*. Clearly $\underline{E}$ and $\underline{B}$ are interrelated - two faces of the same force.

Other assignments of the elements in $F^{\mu\nu}$ are possible, and would still have the right transformation properties. In fact, we could have also defined

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix} \tag{3.8}$$

This is called the *dual field strength tensor*.

There still seems like a lot of signs to remember. We will introduce E/M in tensor notation; this will take care of signs etc.

Even though $\underline{E}$,$\underline{B}$ change under LT, some combinations are invariant. $F^{\mu\nu}F_{\mu\nu}$ is Lorentz scalar. Note: This is not on the form of a matrix multiplication. To get it on that form $F^{\mu\nu}F_{\mu\nu} = -F^{\mu\nu}F_{\nu\mu}$. Now it is the trace of a matrix multiplication.
One finds: $-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \underline{E}^2 - c^2\underline{B}^2$ invariant!
Similarly: $F_{\mu\nu}\tilde{F}^{\mu\nu} = -4c\underline{E}\cdot\underline{B}$ invariant!

No further Lorentz invariant quantities can be built from field strength tensors. All other combinations reduce to 0 or the quantities we have already built: $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = \underline{E}^2 - c^2\underline{B}^2$. $g_{\mu\nu}F^{\mu\nu} = 0$ etc. Only two non-zero invariants arise from contractions with $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$: $\underline{E}^2 - c^2\underline{B}^2$ and $\underline{E}\cdot\underline{B}$.

The consequences: if $E \leq$ or $\geq B$ in one IF, then this is true in all IF. If $\underline{E}$ is perpendicular to $\underline{B}$ in one frame, then this true in all IF. If in one IF: $\underline{E} \neq 0, \underline{B} = 0$ then there can be no IF with $E = 0, B \neq 0$ and vice versa.

Chapter 4

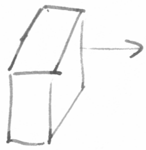
Electromagnetism in Relativistic Notation

This chapter will see the rewriting of the beautiful Maxwell Equations in relativistic notation. We will start by considering the sources (charge and current density) written in terms of the 4-current density.

4.1 4-Current Density and the Continuity Equation

Consider a volume element with many point charges. At rest, the charge density is given by

$\rho_0 = \frac{Q_{tot}}{V} = \frac{\sum_i q_i}{V}$, where V is the volume. Under a boost, the charged volume will be Lorentz contracted whereas the total charge is Lorentz-invariant. We therefore find for the charge density for a moving body $\rho = \frac{Q_{tot}}{V'} = \frac{\sum_i q_i}{V/\gamma(v)} = \gamma(v)\ \rho_0$. So the 4-current density in this case is



$$j^\mu = \rho_0\ \gamma(v)\ (c, \underline{v}) = \rho(c, \underline{v}) = (\rho c, \underline{j}). \tag{4.1}$$

With this, the continuity equation:

$$\frac{\partial}{\partial t}\left(\underline{\nabla} \cdot \underline{E}\right) = \frac{1}{\epsilon_0}\frac{\partial}{\partial t}\rho$$

can be written as

$$\partial_\mu j^\mu = 0, \tag{4.2}$$

as already explored in Section 2.4.

4.2 The 4-Potential

The 4-potential is defined as $A^\mu = (\phi, c\underline{A})$, and we showed in Section 2.4 that in the Lorenz gauge it does in deed transform as a four-vector.

Consider now a gauge-transformation of the potentials. The gauge-transformation can be written in 4-vector notation as

$$A^\mu \to A'^\mu = A^\mu - c\,\partial^\mu \lambda, \tag{4.3}$$

where λ is a suitable continuous and differentiable scalar function. In section 2.2 we discussed how the gauge freedom allowed us to fix $\underline{\nabla}\cdot\underline{A}$. In the Lorenz gauge, this is fixed as: $\underline{\nabla}\cdot\underline{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$. In this gauge, the solution to the Maxwell equations for the potentials is give by the Retarded Potentials. In 4-vector notation, the Lorenz Gauge condition $\underline{\nabla}\cdot\underline{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$ is simply written as

$$\partial_\mu A^\mu = 0. \qquad \text{Lorenz Gauge Condition} \tag{4.4}$$

Let us now rewrite the Maxwell equations for the potentials (i.e. for the four-potential) in the Lorentz gauge in co-variant notation to explicitly demonstrate the invariance of the laws of Electromagnetism under Lorentz transformations. We wrote the Maxwell equations for the potentials in the Lorenz gauge already in Section 2.4, but let is here derive them again: The first Maxwell equation says $\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0}$. Plug in $\underline{E} = -\underline{\nabla}\phi - \frac{\partial\underline{A}}{\partial t}$ to get

$$-\underline{\nabla}\cdot\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{\nabla}\cdot\underline{A} = \frac{\rho}{\epsilon_0} \quad \therefore \quad -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\epsilon_0}. \quad \text{[Lorenz Gauge]} \tag{4.5}$$

The fourth Maxwell equation says

$$\underline{\nabla}\times\underline{B} = \frac{1}{c^2}\frac{\partial\underline{E}}{\partial t} + \mu_0\underline{j} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\underline{\nabla}\Phi - \frac{\partial\underline{A}}{\partial t}\right) + \mu_0\underline{j} \tag{4.6}$$

We now use that

$$\underline{\nabla}\times\underline{B} = \underline{\nabla}\times(\underline{\nabla}\times\underline{A}) = \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \nabla^2\underline{A} = \frac{1}{c^2}\underline{\nabla}\frac{\partial\Phi}{\partial t} - \nabla^2\underline{A} \tag{4.7}$$

to rewrite Eq. (4.6) as

$$-\nabla^2\underline{A} + \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} = \mu_0\underline{j} = \frac{1}{c^2\epsilon_0}\underline{j}. \tag{4.8}$$

But the operator: $-\underline{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \partial_\mu\partial^\mu$. The first and the fourth Maxwell equations for the potentials then take the combined form

$$\partial_\mu\partial^\mu A^\nu = \frac{1}{c\epsilon_0}j^\nu. \qquad \text{[Lorenz Gauge]} \tag{4.9}$$

This is called the *Wave Equation*, since the solution for the homogeneous equation (i.e. $j^\nu = 0$ as in vacuum) is plane waves, and the inhomogeneous term j^ν is the source for these waves.

4.3 The Field-Strength Tensor

We showed that the construct,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix},$$

when transformed as a rank-2 tensor encodes the correct transformation properties of the $\underline{E}$ and $\underline{B}$-fields. We now note that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ - this makes it clear that it is actually a rank-2 tensor! And there are now no more arbitrary signs to remember. Only the definition of $\underline{E}$, and $\underline{B}$ in terms of potentials, and the metric tensor. The new definition of $F^{\mu\nu}$ is obviously anti-symmetric: $F^{\mu\nu} = -F^{\nu\mu}$.

We see also that $F^{\mu\nu}$ is gauge invariant, by using the form of the gauge transformation given in Eq. (4.3):

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \rightarrow \tag{4.10}$$

$$F'^{\mu\nu} = \partial^\mu (A^\nu - c\,\partial^\nu \lambda) - \partial^\nu (A^\mu - c\,\partial^\mu \lambda) = \partial^\mu A^\nu - \partial^\nu A^\mu - c(\partial^\mu \partial^\nu - \partial^\nu \partial^\mu)\lambda \tag{4.11}$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}. \tag{4.12}$$

$F^{\mu\nu}$ is gauge-invariant, but when written in terms of the 4-potential A^μ we can consider choosing the Lorentz gauge, and get the following simplification:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \frac{1}{c\epsilon_0} j^\nu - 0,$$

where the 0 comes from interchanging the order of differentiations and the gauge choice. Therefore, the inhomogeneous Maxwell equations can be written as:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\,\epsilon_0} j^\nu,$$

where each element is gauge-invariant. This form of the Maxwell equation therefore holds in any gauge (since technically it is an equation relating the physical fields $\underline{E}, \underline{B}$ with the sources).

We can calculate each element in the field-strength tensor explicitly:

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial}{\partial t}(cA_x) - \left(\frac{-\partial}{\partial x}\right)\Phi = \left(\nabla\Phi + \frac{\partial \underline{A}}{\partial t}\right)_x = -E_x,$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \frac{-\partial}{\partial x} cA_y - \left(-\frac{\partial}{\partial y} cA_x\right) = (\underline{A} \times \underline{A})_z = -cB_z.$$

We can also define the dual field strength tensor of Eq. (3.8) as,

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu p\sigma} F_{p\sigma},$$

or $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu p\sigma}F^{p\sigma}$, where $\epsilon^{\mu\nu p\sigma}$ is the 4-dimensional Levi-Civita tensor [Totally anti-symmetric ($\epsilon_{\mu\nu p\sigma} = -\epsilon_{\nu\mu p\sigma} = +\epsilon_{\nu p\mu\sigma}$). If two labels identical the anti-symmetric tensor evaluates to 0. Finally, we define $\epsilon_{0123} = -\epsilon^{0123} = +1$ by convention.] Calculating explicit elements, we find

$$\tilde{F}^{01} = \frac{1}{2}\left(\epsilon^{0123}F_{23} + \epsilon^{0132}F_{32}\right) = \frac{1}{2}\left(-F_{23} + F_{32}\right) = -F_{23} = -F^{23} = cB_x,$$

and so on, to get:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}.$$

Consider now $\partial_\mu \tilde{F}^{\mu\nu}$. Using the definition in terms for the field strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu p\sigma}F_{p\sigma}$ we find that $\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) = 0$, since each term is a contraction between a tensor symmetric in (μ,ρ) $(\partial_\mu\partial_\rho)$ or (μ,σ) $(\partial_\mu\partial_\sigma)$ and the anti-symmetric Levi-Civita tensor. Such a contraction is obviously zero, and turns out by inspection to give exactly the 2 remaining Maxwell equations.

All the 4 Maxwell equations can therefore be written elegantly as:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0}j^\nu \qquad \text{inhomogeneous Maxwell equation} \tag{4.13}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \text{homogeneous Maxwell equation} \tag{4.14}$$

These equations are written using manifestly gauge-invariant quantities, rather than using a specific gauge as in Eq. (4.9), and are therefore considered slightly more elegant and fundamental. The homogeneous Maxwell equations were of course necessary for the introduction of the potentials in the first place, which in turn were necessary for writing the field strength tensor as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

$$\partial_\mu F^{\mu\nu} = \frac{j^\nu}{c\epsilon_0}\left\{ \begin{array}{ccc} \nu = 0 & : & \underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0} \\ \nu = 1,2,3 & : & \underline{\nabla}\times\underline{B} = \mu_0\nu_0\frac{\partial\underline{E}}{\partial t} + \mu_0\underline{j} \end{array} \right. ,$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0\left\{ \begin{array}{ccc} \nu = 0 & : & \underline{\nabla}\cdot\underline{B} = 0 \\ \nu = 1,2,3 & : & \underline{\nabla}\times\underline{E} = -\frac{\partial\underline{B}}{\partial t} \end{array} \right\}.$$

The last set of equations are necessary for the use of potentials.

4.4 Lorentz Force in co-variant notation

We suggest that the well-known Lorentz force on a charged particle in an electro-magnetic field is contained in the relativistic notation

$$f^\mu = \frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu,$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, and u_ν is the covariant 4-velocity of the particle. To see that this is indeed the case, we list the relevant results:

$$p^\mu = (p_0, \underline{p}) = \left(\frac{E}{c}, \underline{p}\right), \tag{4.15}$$

$$f^\mu = \frac{dp^\mu}{d\tau} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \underline{F}\right), \tag{4.16}$$

$$u^\mu = \gamma\left(c, \underline{v}\right), \tag{4.17}$$

and calculate:

$$F^{\mu\nu}u_\nu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} \begin{pmatrix} \gamma c \\ -\gamma v_x \\ -\gamma v_y \\ -\gamma v_z \end{pmatrix} = \left(\gamma \underline{E}\cdot\underline{v}, \gamma c\left[\underline{E} - \underline{B}\times\underline{v}\right]\right). \tag{4.18}$$

So $\left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\underline{F}\right) = \left(q\frac{\gamma}{c}\underline{E}\cdot\underline{v}, q\gamma\left[\underline{E} + \underline{v}\times\underline{B}\right]\right)$. The 0- component gives us: $\frac{dE}{dt} = q\underline{E}\cdot\underline{v}$ where $\underline{E}$ is the electric field and $\frac{dE}{dt}$ is the real energy of point particle, and the 1,2,3-component is the Lorentz force.

4.5 Charges in an Electro-magnetic Field

Consider a point charge q in an EM-field. How does it move under the influence of external forces (ignoring the effects of radiation from the charge)? Using co-variant notation,

$$f^\mu = \frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu. \tag{4.19}$$

$$\underline{F} = \frac{d\underline{p}}{dt} = q\left(\underline{E} + \underline{v}\times\underline{B}\right), \quad p^\mu = (p_0, \underline{p}) = \left(\frac{E}{c}, \underline{p}\right), \quad \frac{dp^\mu}{d\tau} = \left(\frac{dE}{dt}, \gamma\underline{F}\right), \quad u^\mu = \gamma(c, \underline{v}) \tag{4.20}$$

This equation reduces to the Lorentz-force law

$$\underline{F} = \frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v}\times\underline{B}), \quad \frac{dE}{dt} = q\underline{E}\cdot\underline{v} \tag{4.21}$$

(remembering that $\underline{E}$ is the electric field, while E is the energy of the point charge). Simple special cases (all ignores the radiation from the accelerated particles): 1) $\underline{E} = 0$, uniform $\underline{\mathrm{B}}$-field. Energy of particle conserved, no work from external field on particle! $\rightarrow v = |\underline{v}|$ constant. If $\underline{\mathrm{v}}$ is perpendicular to $\underline{\mathrm{B}}$: circular motion occurs.

$$\underline{F} = \frac{d\underline{p}}{dt} = q\underline{v}\times\underline{B} = qvB = \frac{d}{dt}\left(m\gamma\underline{v}\right) = m\gamma\dot{\underline{v}}, \tag{4.22}$$

since

$$\frac{d\gamma}{dt} = 0,$$

so $|\underline{v}|$ is constant. We then find for the circular orbit

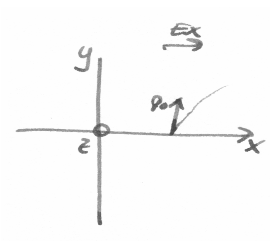
$$\frac{d\underline{p}}{dt} = p\frac{d\theta}{dt} = p\frac{v}{R}$$

and therefore

$$R = \frac{p}{qB} = \frac{\gamma(v)mv}{qB}, \tag{4.23}$$

where R is the radius of the circular orbit.

2) $\underline{B} = 0$, uniform $\underline{E}$-field in x-direction. For $t = 0$, point charge of mass m is on x-axis, with $\underline{p}(t = 0) = (0, p_0, 0)$, given $\underline{E} = (E_x, 0, 0)$, want $\underline{x}(t)$.



$$\frac{d\underline{p}}{dt} = q\underline{E} \to \dot{p_x} = qE_x,\ \dot{p_y} = 0, \dot{p_t} = 0,\ p_x(t) = qE_xt, p_y(t) = p_0, p_z(t) = 0$$

Next determine $\underline{v}$ from $\underline{p}$:

$$\underline{p} = m\gamma\underline{v}, E = mc^2\gamma \to v = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{m^2c^4 + c^2p_0^2 + (cqE_xt)^2}},$$

$$v_x = \frac{dx}{dt} = \frac{p_xc^2}{\sqrt{m^2c^4 + \underline{p}^2c^2}} = \frac{qE_xtc^2}{\sqrt{m^2c^4 + c^2p_0^2 + (cqE_xt)^2}} \to c \text{ for } t \to \infty,$$

$$v_y = \frac{dy}{dt} = \frac{p_yc^2}{\sqrt{m^2c^4 + p^2c^2}} = \frac{p_0c^2}{\sqrt{m^2c^4 + c^2p_0^2 + (cqE_xt)^2}} \to 0 \text{ for } t \to \infty.$$

Determine $\underline{x}$ from $\underline{v}$,

$$x(t) = \int v_x(t)dt = \frac{1}{qE_x}\sqrt{m^2c^4 + c^2p_0^2 + (cqE_xt)} + \text{constant},$$

$$y(t) = \int v_y(t)dt = \frac{p_0c}{qE_x}\arcsin\left(\frac{cqE_xt}{\sqrt{m^2c^4 + c^2p_0^2}}\right) + 0,$$

combining these together gives the x-component as a function of the y-component:

$$x = \frac{1}{qE_x}\sqrt{m^2c^4 + c^2p_0^2}\cosh\left(\frac{qE_x y}{p_0 c}\right).$$

This is a hyperbola - as expected, since in relativity a constant force gives a hyperbolic motion (compared to a parabolic motion in Newtonian mechanics).

4.6 A Little Taster Of Things To Come

Lagrangian and Hamiltonian for a relativistic particle

Recall: Lagrangian $L(x, \dot{x})$ is a function of a generalised co-ordinate.

Action: $S = \int_{t_1}^{t_2} L(x, \dot{x})dt$. The true path (in conformation space) is the one which minimises the action.

Generalised momentum: $p = \frac{\partial L}{\partial \dot{x}}$.

Lagrangian equation: $\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{d\dot{x}} = 0$.

Energy Function (Hamiltonian): $H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L$.

Modifications for relativistic systems: The action for a point mass must be Lorentz invariant: $S = k\int_a^b ds$ where $ds = \sqrt{c^2dt^2 - d\underline{x}^2} = \sqrt{c^2d\tau^2} = cd\tau$ where k is the transformation to be fixed by NR limit. Recalling $d\tau = \frac{dl}{\gamma}$ then: $S = k\int_{t1}^{t2}\frac{c}{\gamma}dt$ we can evaluate this in any IF, where the time of the events are seen in one IF.

$\rightarrow L = kc\sqrt{1 - \frac{v^2}{c^2}} = kc - k\frac{v^2}{2c} + \theta\left(\frac{v^4}{c^4}\right)$ In the NR limit, construct term does not affect the equation of motion. so drop it. So for a free point mass,

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} = -m\frac{c^2}{\gamma},$$

$$S = -mc\int_a^b ds.$$

Momentum: $\underline{p} = \frac{\partial L}{\partial \dot{\underline{x}}} = \frac{\partial L}{\partial \underline{v}} = \frac{\partial}{\partial \underline{v}}\left(-mc^2\sqrt{1 - \frac{v^2}{c^2}}\right) = m\gamma\underline{v}$.

Lagrange equation: $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \rightarrow \dot{\underline{p}} = 0$ [Free Particle].

Now must add the Electromagnetic potential and it must be Lorentz invariant.

$$S = \int_a^b \left(-mcds - \frac{q}{C}A_\mu dx^\mu\right) = \int_a^b \left(-mcds - \frac{q}{C}\Phi cdt + \frac{q}{C}c\underline{A}\cdot d\underline{x}\right),$$

where $d\underline{x} = \frac{dx}{dt}dt = \underline{v}dt$ and C is a constant fixed by NR limit.

$$= \int_{t_1}^{t_2}\left(-mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{C}\underline{A}\cdot\underline{v} - q\Phi\right)dt$$

This is the L for a charged point mass in a E/M potential. Generalised momentum is: $\underline{p}_{gen} = \frac{\partial L}{\partial \dot{\underline{x}}} = \frac{\partial L}{\partial \underline{v}} = m\underline{v}\gamma + q\underline{A} = \underline{p} + q\underline{A}$ [follow through and end up with the Lorentz force]. Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial \underline{x}}$. You can even define a Lagrangian for the EM field itself!

$$S = \int d^4x \mathcal{L}\left(A^\mu, \delta^\mu A^\nu\right) = -\frac{\epsilon_0}{4} F^{\mu\nu} F_{\mu\nu} \ !$$

[In a vacuum] with sources,

$$\mathcal{L} = -\frac{\epsilon_0}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} j^\nu A_\nu .$$

The Euler-Lagrange equation then gives the Maxwell equation!

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0} j^\nu$$

This is how modern Theoretical physics operates: Symmetry $\rightarrow$ Get Lagrangian $\rightarrow$ E-L equation $\rightarrow$ Solve!

Appendix A

Results from Vector Calculus

In this appendix we will briefly list useful results from vector analysis, which will be used repeatedly during the course. The results are discussed mostly in order to establish the notation used.

Consider a three-dimensional space. A **scalar field** $f(\underline{r})$ returns a number (scalar) for every point $\underline{r}$ in this space. A **vector field** $\underline{F}(\underline{r}) = (F_x(\underline{r}), F_y(\underline{r}), F_z(\underline{r}))$ returns a vector for every point $\underline{r}$.

A.1 Coordinate systems in three dimensions

A.1.1 Cartesian coordinates

A point, or rather its **position vector** $\underline{r}$, can be defined in terms of Cartesian coordinates (x, y, z) as

$$\underline{r} = x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}, \tag{A.1}$$

where $\hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}}$ are the basis vectors.

The **gradient** of a scalar field $f(\underline{r})$ is

$$\underline{\nabla} f(\underline{r}) = \frac{\partial f}{\partial x}\hat{\underline{x}} + \frac{\partial f}{\partial y}\hat{\underline{y}} + \frac{\partial f}{\partial z}\hat{\underline{z}}. \tag{A.2}$$

The **divergence**, $\underline{\nabla}\cdot\underline{F}(\underline{r})$, of a vector field $\underline{F}(\underline{r}) = (F_x(\underline{r}), F_y(\underline{r}), F_z(\underline{r}))$ represents the volume density of the outward flux of the vector field from an infinitesimal volume around a given point. $\underline{\nabla}\cdot\underline{F}(\underline{r})$ can be found in terms of Cartesian coordinates as

$$\underline{\nabla}\cdot\underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{A.3}$$

The **curl**, $\underline{\nabla}\times\underline{F}(\underline{r})$, of a vector field is defined as the line integral of the field along a closed curve, divided by the area enclosed, all in the limit of zero area. The local curl can be found as

$$\underline{\nabla}\times\underline{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\underline{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\underline{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\underline{z}}. \tag{A.4}$$

The **Laplacian of a scalar field** $f(\underline{r})$, the divergence of the gradient $\nabla^2 f = \underline{\nabla} \cdot \underline{\nabla} f$, can be calculated as

$$\nabla^2 \underline{f} = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f. \tag{A.5}$$

The **Laplacian of a vector field** $\underline{F}(\underline{r})$ in Cartesian coordinates is given by

$$\nabla^2 \underline{F} = \nabla^2 F_x \underline{\hat{x}} + \nabla^2 F_y \underline{\hat{y}} + \nabla^2 F_z \underline{\hat{z}}. \tag{A.6}$$

The results for differential elements in 1,2 and three dimensions (the line $\mathrm{d}\underline{l}$, surface $\mathrm{d}\underline{a}$ and volume $\mathrm{d}V$ elements) are

$$\mathrm{d}\underline{l} = \mathrm{d}x\ \underline{\hat{x}} + \mathrm{d}y\ \underline{\hat{y}} + \mathrm{d}z\ \underline{\hat{z}} \tag{A.7}$$

$$\mathrm{d}\underline{a} = \mathrm{d}y\ \mathrm{d}z\ \underline{\hat{x}} + \mathrm{d}x\ \mathrm{d}z\ \underline{\hat{y}} + \mathrm{d}x\ \mathrm{d}y\ \underline{\hat{z}} \tag{A.8}$$

$$\mathrm{d}V = \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \tag{A.9}$$

A.1.2 Spherical Polar Coordinates

Spherical polar coordinates describe the position vector $\underline{r}$ in terms of the radial distance $r \in [0, \infty)$, polar angle $\theta \in [0, \pi)$ and azimuthal angle $\phi \in [0, 2\pi)$, written in terms of the Cartesian coordinates as

$$r = \sqrt{x^2 + y^2 + z^2} \tag{A.10}$$

$$\theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \arccos \frac{z}{r} \tag{A.11}$$

$$\phi = \arctan \frac{y}{x}. \tag{A.12}$$

Contrary to the case for the Cartesian coordinate system, the orthogonal basis vectors in spherical polars depend on the position $\underline{r}$. In terms of the Cartesian vectors, they are expressed as

$$\underline{\hat{r}} = \sin\theta\ \cos\phi\ \underline{\hat{x}} + \sin\theta\ \sin\phi\ \underline{\hat{y}} + \cos\theta\ \underline{\hat{z}} \tag{A.13}$$

$$\underline{\hat{\theta}} = \cos\theta\ \cos\phi\ \underline{\hat{x}} + \cos\theta\ \sin\phi\ \underline{\hat{y}} - \sin\theta\ \underline{\hat{z}} \tag{A.14}$$

$$\underline{\hat{\phi}} = -\sin\phi\ \underline{\hat{x}} + \cos\phi\ \underline{\hat{y}} \tag{A.15}$$

The Cartesian coordinates are given in terms of the spherical polars as

$$x = r \sin\theta \cos\phi \tag{A.16}$$

$$y = r \sin\theta \sin\phi \tag{A.17}$$

$$z = r \cos\theta. \tag{A.18}$$

The differential operators expressed in terms of spherical polars are given by

$$\underline{\nabla} f = \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} \tag{A.19}$$

$$\underline{\nabla}\cdot\underline{F}(\underline{r}) = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(F_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \tag{A.20}$$

$$\underline{\nabla}\times\underline{F} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(F_\phi\sin\theta) - \frac{\partial F_\theta}{\partial\phi}\right)\underline{\hat{r}} \tag{A.21}$$

$$+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right)\underline{\hat{\theta}} \tag{A.22}$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\underline{\hat{\phi}} \tag{A.23}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{A.24}$$

$$\nabla^2\underline{F} = \left(\nabla^2F_r - \frac{2F_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\right)\underline{\hat{r}} \tag{A.25}$$

$$+\left(\nabla^2F_\theta - \frac{F_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\right)\underline{\hat{\theta}} \tag{A.26}$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\underline{\hat{\phi}} \tag{A.27}$$

The differential elements are

$$\mathrm{d}\underline{l} = \mathrm{d}r\ \underline{\hat{r}} + r\ \mathrm{d}\theta\ \underline{\hat{\theta}} + r\sin\theta\ \mathrm{d}\phi\ \underline{\hat{\phi}} \tag{A.28}$$

$$\mathrm{d}\underline{a} = r^2\sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi\ \underline{\hat{r}} + r\sin\theta\ \mathrm{d}r\ \mathrm{d}\phi\ \underline{\hat{\theta}} + r\ \mathrm{d}r\ \mathrm{d}\theta\ \underline{\hat{\phi}} + r\ \mathrm{d}r\ \mathrm{d}\theta\ \underline{\hat{\phi}} \tag{A.29}$$

$$\mathrm{d}V = r^2\sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi. \tag{A.30}$$

A.2 Results for Differential Operators

The curl of a curl of a vector field can be found as

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{F}) = \nabla(\underline{\nabla}\cdot\underline{F}) - \nabla^2\underline{F}. \tag{A.31}$$

It might help remembering this by referring to the BAC-CAB rule for standard vector products of vectors $\underline{A}, \underline{B}, \underline{C}$:

$$\underline{A}\times(\underline{B}\times\underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B}). \tag{A.32}$$

It is straightforward, and a good test of the understanding of how to apply the formulas, to verify that the curl of any gradient of a scalar field is zero

$$\underline{\nabla}\times(\underline{\nabla}f) = 0. \tag{A.33}$$

Also, the divergence of any curl of a vector field is zero

$$\underline{\nabla}\cdot(\underline{\nabla}\times\underline{F}) = 0. \tag{A.34}$$

A.3 Flux Theorem or Gauss' Law

The flux theorem in three dimensions is also known as Gauss' law, and relates the integral over a volume $\mathcal{V}$ of the divergence, $\underline{\nabla} \cdot \underline{F}$, of a vector field $\underline{F}$ to the integral of $\underline{F}$ over any surface $\mathcal{S}$ bounded by the volume:

$$\int_{\mathcal{V}} (\underline{\nabla} \cdot \underline{F}) \, \mathrm{d}V = \oint_{\mathcal{S}} \underline{F} \cdot \mathrm{d}\underline{a}. \tag{A.35}$$

A.4 Stokes' theorem (the curl theorem)

Stokes' theorem relates the closed line-integral of a vector field to the integral of the curl of the same field over any surface bounded by the closed line

$$\oint_{\mathcal{C}} \underline{F} \cdot \mathrm{d}\underline{l} = \int_{\mathcal{S}} (\underline{\nabla} \times \underline{F}) \cdot \mathrm{d}\underline{a}. \tag{A.36}$$

The integrals are oriented - meaning that the curve integral is taken in counter clockwise direction, and the surface vector is outwards pointing when using the right-hand rule in the direction of the curve integral (or both are changed).

Appendix B

The Form of the Lorentz Transformations

The Lorentz Transformations (LT) were derived in the level-1 Special Relativity course. This handout is only intended to remind you of the derivation, perhaps in a slightly more formal way. This derivation is based on the one in Rindler (1.6). More detailed study of the properties of the Lorentz transformation requires knowledge of group theory and the properties of the Lorentz group which we will not cover.

Consider a clock reading a time τ moving uniformly according to Newton's first law in an Inertial Frame (IF) S. The homogeneity of the space through which the clock moves requires that for equal changes in τ the coordinates $x^\mu = (ct, x, y, z)$, $(\mu \in \{0, 1, 2, 3\})$ also change by equal amounts, *i.e.*

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \text{constant} \qquad \Rightarrow \qquad \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0.$$

This must be true in any other internal frame with coordinates x'^μ. However

$$\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} = \sum \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \qquad \Rightarrow \qquad \frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}\tau^2} = \sum \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \sum \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$

Hence for any free motion of a clock the last term must vanish, *i.e.*

$$\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\sigma} = 0.$$

This means that the transformation is linear.

We can therefore assume a linear form of the transformations of the coordinates of an event. Without loss of generality, we can consider a boost between frames in the standard configuration (boost along x-axis, and frames overlapping at $t = t' = 0$. Since the boost is along the x-axis, perpendicular to y and z, we can arrange for $y = y' = z = z' = 0$. For this boost along the x-axis from S to S', the linear form of the transformation then reduces to finding coefficients

d, e, a, b in

$$\begin{aligned} t' &= dx + et \\ x' &= ax + bt \\ y' &= y \\ z' &= z \end{aligned}$$

If we consider the origin of S', *i.e.* $x' = 0$ then it must have position $x = vt$ in S which requires that x' does not depend on y or z. Therefore for the origin of S'

$$x' = 0 = avt + bt,$$

as this must be true for all t, $v = -\frac{b}{a}$.

Similarly if we consider the origin of S $x = 0$ and $x' = -vt'$ giving

$$-vt' = -evt = bt,$$

using $t' = et$ for $x = 0$. Hence $v = -\frac{b}{a}$, which means $a = e$.

We can then use the second postulate

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2.$$

Substituting the forms of t' and x' gives

$$c^2t^2 - x^2 - y^2 - z^2 = t^2\left(e^2c^2 - b^2\right) - x^2\left(a^2 - c^2d^2\right) + 2xt\left(c^2de - ab\right) - y^2 - z^2.$$

Equating the coefficients of t^2

$$e^2c^2 - b^2 = c^2 \quad \Rightarrow \quad c^2a^2 - v^2a^2 = c^2 \quad \Rightarrow \quad a^2 = \frac{1}{1 - \frac{v^2}{c^2}}.$$

Therefore

$$a = e = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Equating the coefficients of xt

$$c^2de = ab \quad \Rightarrow \quad c^2d = b = -va.$$

Therefore

$$b = -v\gamma.$$

This gives the Lorentz Transformations

$$\begin{aligned} ct' &= -\frac{v\gamma}{c}x + \gamma ct \\ x' &= \gamma x - v\gamma t \\ y' &= y \\ z' &= z \end{aligned}$$