Foundations of Physics 2B/3C 2016/2017

Thermodynamics – Summer Examinations 2017

Short Questions Complete – all conceptual (Vetted)

a) Heat is taken in - hot temperature. Work is done, Heat rejected at cold temperature engine is returned to its initial state. [3 marks, 1 per point]

$$\oint dU = \oint \delta Q + \oint \delta W.$$

Internal energy is a function of state, so $\oint dU = 0$, but heat and work are inexact so$\oint \delta Q \neq 0, \oint \delta W \neq 0$ [1 mark]

b) $\Delta S_U = \Delta S_1 + \Delta S_2$. with $dS = \frac{\delta Q_{rev}}{T}$, with $\delta Q_{rev} = CdT$ [1 mark]

$$\Delta S_U = \int_{T_1}^{T_E} \frac{C}{T} dT + \int_{T_2}^{T_E} \frac{C}{T} dT = C \ln\left(\frac{T_E}{T_1}\right) + C \ln\left(\frac{T_E}{T_2}\right) = C \ln\left(\frac{T_E^2}{T_1 T_2}\right).$$

[1 mark]

Entropy statement entropy must always increase

$$\Delta S_U \geq 0 \Rightarrow \ln\left(\frac{T_E^2}{T_L T_H}\right) \geq 0 \Rightarrow T_E^2 \geq T_1 T_2.$$

[1 mark]

$$(T_1^2 + 2T_1T_2 + T_2^2) \geq 4T_1T_2$$
$$T_1^2 - 2T_1T_2 + T_2^2 = (T_1 - T_2)^2 \geq 0$$

This holds for all T_1 and T_2. [1 mark]

c) Approximate by

$$\left(\frac{\partial p}{\partial T}\right)_{i \to f} = \frac{L}{T\Delta V}$$

$$\int dp = \int \frac{L}{T\Delta V} dT \quad \Rightarrow \quad p = \frac{L}{\Delta V} \ln(T) + \text{const.}$$

[2 marks]

$(T, p) = (T_0, p_0)$ at some point

$$p = p_0 + \frac{L}{\Delta V} \ln\left(\frac{T}{T_0}\right).$$

[1 mark]

ΔV is small, so the phase boundary is steep. [1 mark]

d)

$$C_\alpha = T\left(\frac{\partial S}{\partial T}\right)_\alpha \quad \Rightarrow \quad \Delta S = \int_{T_1}^{T_2} \frac{C_\alpha}{T} dT = C_\alpha[\ln(T_2) - \ln(T_1)].$$

[1 mark]

Nernst heat theorem - temperature approaches absolute zero, the entropy change also tends to zero.

$\lim_{T\to 0}(-\ln(T)) = \infty$, so C_α must go to zero more quickly [2 marks]

C_p and C_V must both go to zero - Mayer's equation becomes invalid,

[1 mark]

Foundations of Physics 2B/3C 2016/2017

Thermodynamics – Summer Examinations 2017

Long Question 1 (Vetted)

a) Total derivative $dG = dU - TdS - SdT + mdB + Bdm$, so $dG = -SdT + Bdm$.

[1 mark]

$$G = G(T, m) \quad \Rightarrow \quad dG = \left(\frac{\partial G}{\partial T}\right)_m dT + \left(\frac{\partial G}{\partial m}\right)_T dm.$$

Comparing terms shows that $\left(\frac{\partial G}{\partial T}\right)_m = -S$ and $\left(\frac{\partial G}{\partial m}\right)_T = B$. Gibbs is a function of state,

$$\left(\frac{\partial^2 G}{\partial m \partial T}\right) = \left(\frac{\partial}{\partial m}\left(\frac{\partial G}{\partial T}\right)_m\right)_T = -\left(\frac{\partial S}{\partial m}\right)_T = \left(\frac{\partial B}{\partial T}\right)_m = \left(\frac{\partial}{\partial T}\left(\frac{\partial G}{\partial m}\right)_T\right)_m = \left(\frac{\partial^2 G}{\partial T \partial m}\right).$$

[2 marks]

b) $S = S(m, T)$

$$TdS = T\left(\frac{\partial S}{\partial T}\right)_m dT + T\left(\frac{\partial S}{\partial m}\right)_T dm.$$

[2 marks]

Definition $C_m = T\left(\frac{\partial S}{\partial T}\right)_m$ for the first term, use Maxwell relation on the second

$$TdS = C_m dT - T\left(\frac{\partial B}{\partial T}\right)_m dm.$$

[2 marks]

c) Subtracting the two TdS

$$(C_B - C_m)dT = -\left[T\left(\frac{\partial B}{\partial T}\right)_m dm + T\left(\frac{\partial m}{\partial T}\right)_B dB\right].$$

[1 mark]

$$T = T(B, m) \quad \Rightarrow \quad dT = \left(\frac{\partial T}{\partial B}\right)_m dB + \left(\frac{\partial T}{\partial m}\right)_B dm.$$

[1 mark]

$$(C_m - C_B)\left[\left(\frac{\partial T}{\partial B}\right)_m dB + \left(\frac{\partial T}{\partial m}\right)_B dm\right] = \left[T\left(\frac{\partial B}{\partial T}\right)_m dm + T\left(\frac{\partial m}{\partial T}\right)_B dB\right].$$

[1 mark]

Comparing coefficients of dB (or dm)

$$(C_m - C_B)\left(\frac{\partial T}{\partial B}\right)_m = T\left(\frac{\partial m}{\partial T}\right)_B \quad \Rightarrow \quad C_m - C_{mB} = T\left(\frac{\partial m}{\partial T}\right)_B \left(\frac{\partial B}{\partial T}\right)_m.$$

[2 marks]

$\left(\frac{\partial m}{\partial B}\right)_T = -\frac{B}{T^2}$ and $\left(\frac{\partial B}{\partial T}\right)_m = m.$

$$C_m - C_B = -T\left(\frac{B}{T^2}\right)m = -\frac{mB}{T}.$$

[2 marks]

d)

i) $dG = -SdT + Bdm$ and so $B = \left(\frac{\partial G}{\partial m}\right)_T$

[1 mark]

$$B = 2amT \quad \Rightarrow \quad m \sim \frac{B}{T}.$$

A paramagnet. [2 marks]

ii) $U = G + TS - mB.$ [1 mark]

$$S = -\left(\frac{\partial G}{\partial T}\right)_m = -[am^2 + 2bT + 3cT^2].$$

$$U = am^2T + bT^2 + cT^3 - T(am^2 + 2bT + 3cT^2) - mB = -bT^2 - 2cT^3 - mB.$$

[2 marks]

Foundations of Physics 2B/3C 2016/2017

Thermodynamics – Summer Examinations 2017

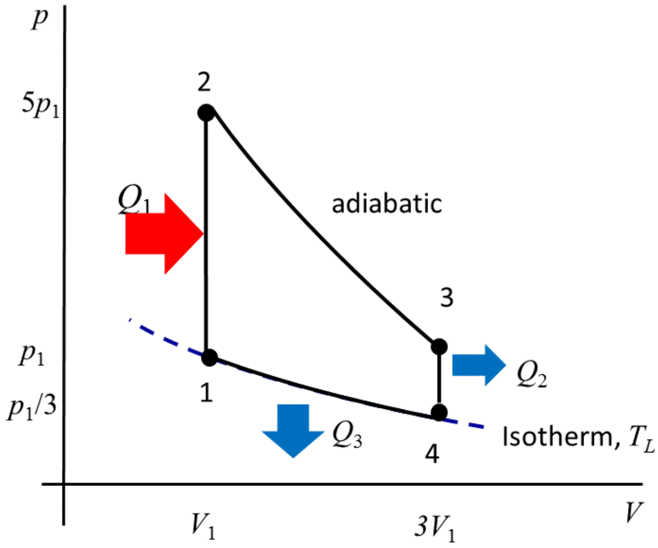
Long Question 2 (Vetted)

a) 1: $(p_1, V_1, T_1) = (p_1, V_1, T_L)$
2: $(p_2, V_2, T_2) = (5p_1, V_1, 5T_L)$, volume constant and $p_1V_1 = RT_L$
3:$(p_3, V_3, T_3) = (p_3, 3V_1, T_3)$.
On the adiabatic, $pV^\gamma =$ const. so $p_3 = \frac{5}{3^\gamma} p_1 = 1.07p_1$.

$$p_3V_3 = RT_L \quad \Rightarrow \quad T_3 = 3V_1 \times \frac{1.07p_1}{R} = 3.22T_L$$

4: $(p_4, V_4, T_{L4}) = (\frac{p_1}{3}, 3V_1, T_L)$

[4 marks]



No heat interaction along the adiabatic. Isochoric heating sees heat added, heat rejected in isochoric cooling. Heat is also removed at the low temperature isotherm.

[1 mark]

[1 mark for shape, 1 for heat directions]

b) $Q_1 = C_V(T_2 - T_1) = 4C_VT_L$; $Q_2 = C_V(T_4 - T_3) = -2.22C_VT_L$. [2 marks]

$$Q_3 = \int_{V_4}^{V_1} pdV = \int_{3V_1}^{V_1} \frac{RT_LdV}{V} = RT_L \ln\left(\frac{1}{3}\right) = -RT_L \ln 3\,.$$

[1 mark]

$$\eta = \frac{|Work|}{Heat\ In} = 1 - \frac{|Q_L|}{Q_H}$$

[1 mark]

$$Q_L = Q_2 + Q_3 = -2.22C_VT_L - RT_L \ln 3.$$

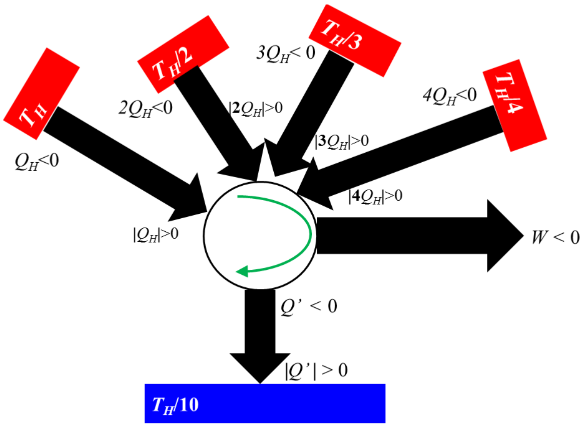
$$\eta = 1 - \frac{T_L(2.22C_V + R\ln 3)}{4C_VT_L} = 1 - \frac{2.22}{4} - \frac{R}{4C_V}\ln 3$$

[1 mark]

$\gamma = C_p/C_V$ and $C_p - C_V = R$, so $\frac{R}{C_V} = \frac{C_p}{C_V} - 1 = 0.4.$ [1 mark]

$\eta = 33.5\ \%$ [1 mark]

Adiabatic and isothermal steps were swopped would increase the value of Q_H and decrease Q_L [1 mark]



c) Maximum efficiency so equality in the Clausius. [1 mark]

$$\oint \frac{\delta Q_{rev}}{T} = 0$$

$$\Rightarrow \frac{Q}{T_H} + \frac{2Q}{T_H/2} + \frac{3Q}{T_H/3} + \frac{4Q}{T_H/4} + \frac{Q'}{T_H/10} = 0$$

$$\frac{30Q}{T_H} + \frac{10Q'}{T_H} = 0$$

$$Q' = -3Q$$

[2 marks]

$dU = 0$ and the first law says $\delta Q = -\delta W$. T

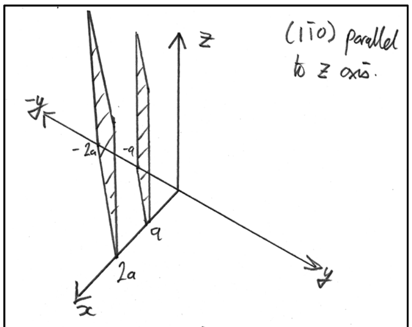
$$W = -(Q_H + 2Q_H + 3Q_H + 4Q_H + Q') = -7Q_H.$$

[1 mark]

$$\eta = \frac{|Work|}{Heat\ In} = \frac{7Q_H}{10Q_H} = 70\%$$

[1 mark]

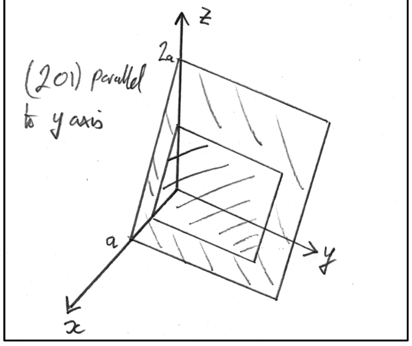
Examination Questions May/June 2017
Foundations of Physics 2B, Condensed Matter Physics, Q4 (5 parts):
SOLUTIONS



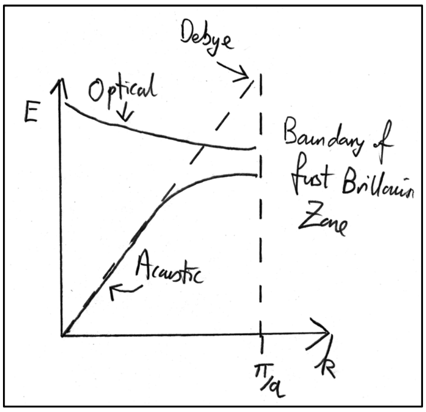
Synopsis **a.** Sketch shown in diagram. The spacing of each of the families of planes is given by:

$$d_{hkl} = \frac{a}{\sqrt{h^2+k^2+l^2}}$$

Problem where a is the lattice constant (0.5 nm) Substituting in the (h k l) values gives: 0.35 nm and 0.22 nm
2 marks for diagram and 1 mark for each answer.
[4 marks]

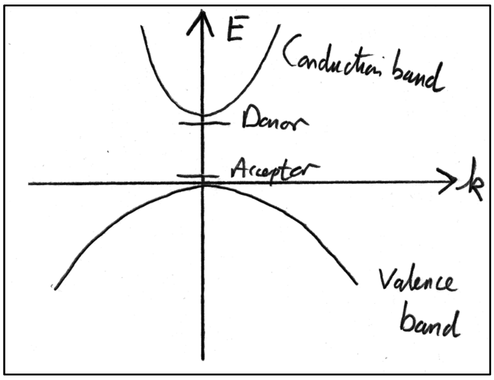


Synopsis **b.** Covalent bonding arises in systems with similar or identical atoms. The repulsive force in the covalent bond arises from electrons being subject to the Pauli Exclusion Principle at very close distances, increasing the energy as the spatial component of the wavefunction overlaps. The attractive force arises from the sharing of one electron from each atom. In a covalent bond the wavefunction must be spatially symmetric giving a high density of electrons between the atoms $\left|\psi\left(\underline{r}\right)\right|^2$. This arises only when the electrons have opposite or antisymmetric spins **[4 marks]**



Synopsis **c.** The sketch shows both the optical and acoustic branches. The acoustic branch results from the two atoms in the basis moving in phase, the optical branch results from the two atoms moving out of phase. The Debye approximation is for the acoustic branch and assumes that energy (or frequency) is proportional to wavevector. This applies for sound waves propagating through a crystal. 2 marks for correctly labelled sketch and 2 marks for description as given. **[4 marks]**

Conceptual **d.** The Wiedemann-Franz law states that the ratio of the electrical to thermal conductivities in metals is proportional to temperature with a common constant for all metals. The Drude model treats electrons as classical particles with mass m_e, charge $-e$. Using the classical equipartition theorem to determine the thermal conductivity and the Drude electrical conductivity it predicts that the ratio of electrical to thermal conductivities is $\frac{\kappa}{\sigma} = \frac{3}{2}\left(\frac{k_\mathrm{B}}{e}\right)T$. The reason this works reasonably well is that this expression does not include the Drude scatting time τ which the Drude model significantly overestimates (as a consequence of treating electrons as classical particles). **[4 marks]**



Conceptual **e.** Sketch shows the conduction and valence bands. Bands must be parabolic because $E = \frac{\hbar^2 k^2}{2m_e^*}$. The effective mass is inversely proportional to the curvature so the conduction band must show a much larger curvature in the sketch – because it has a smaller associated effective mass. Donor and acceptor levels are just below/above the conduction/valence bands. They have relatively small binding energies because of the small effective mass. The binding energies can also be influenced by the dielectric response (and other factors not discussed in the lecture course). Generally donor binding energies are smaller than acceptor binding energies. Free electrons released from donor levels will occupy the conduction band. 2 marks for correctly labelled diagram and 2 for explanation. **[4 marks]**

Examination Questions May/June 2017
Foundations of Physics 2B, Condensed Matter Physics, Q5
SOLUTION

Synoptic **a)** Bragg Law is $\lambda = 2d\sin\theta$. **[1]** Physical meaning is that incident X-ray waves are partially reflected from parallel planes of atoms. **[1]**
When the path length difference is equal to integer multiples of the wavelength you get constructive interference **[1]** and diffracted waves occur. This happens when the Bragg Law is satisfied. **[1]** The scattering wavevector Δk is the change in wavevector between the incident and outgoing X-ray wave. For there to be a non-zero scattering intensity the scattering wavevector must be the same as any reciprocal lattice vector **[1]** This is generally written as $\Delta k = G$. **[1]** The Structure Factor describes the intensity of the scattered X-ray wave and allows for interference within the unit cell which can sometimes lead to zero intensity where a Bragg peak is expected. **[1]**

[7 marks]

Problem **b)** To show the data is consistent with the FCC structure we must use the structure factor rules. **[1]** For the fcc lattice the structure factor rules are S = 0 when h k and l are mixed parity, and S = 4 when h k l are all even or all odd. **[1]**

Applying the structure factor rules means that the first five peaks occur when $N = h^2 + k^2 + l^2$ = 3, 4, 8, 11 and 12 **[1]** corresponding to the planes (111), (200), (220), (311) and (222). **[1]**

Starting with the Bragg Law $\lambda = 2d\sin\theta$ and using $d = \frac{a}{\sqrt{h^2+k^2+l^2}} = \frac{a}{\sqrt{N}}$ **[1]** we get

$\sin\theta = \frac{\lambda\sqrt{N}}{2a}$ in other words $\sin\theta \propto \sqrt{N}$. **[1]** Using the data in the table we get:

Peak	N	2θ	sinθ	sinθ/√N
1	3	37.13	0.3184	0.1838
2	4	43.14	0.3676	0.1838
3	8	62.66	0.5200	0.1838
4	11	75.13	0.6097	0.1838
5	12	79.10	0.6368	0.1838

[2 marks for data]

This confirms that the data is consistent with the fcc lattice as final column is same value. **[1]**

Using the value in the table 0.1838 to determine the lattice constant we get 0.1838 = $\frac{\lambda}{2a}$ giving a = 0.408 nm or the lattice constant of gold. **[1]**

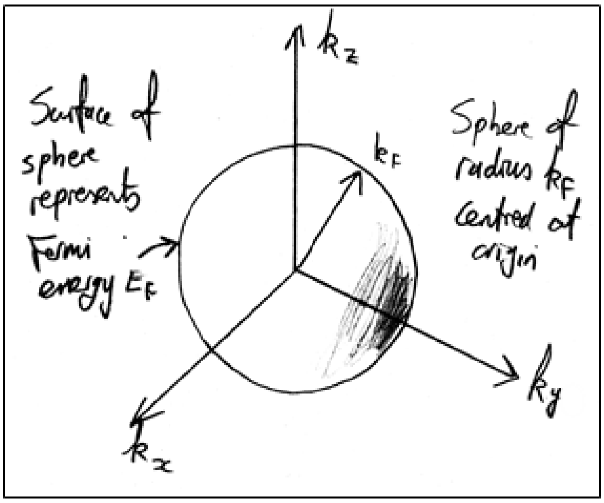
[10 marks]

Problem **c)** Heating the metal first to 500 K will result in thermal expansion of the metal increasing the lattice constant *a*. **[1]** This will result in the X-ray peaks shifting to smaller angles in proportion to the shift in lattice constant. **[1]** At 1500 K the gold will have melted resulting in the loss of the crystal structure so no X-ray peaks will be present. **[1]**

[3 marks]

Examination Questions May/June 2017
Foundations of Physics 2B, Condensed Matter Physics, Q6
SOLUTION

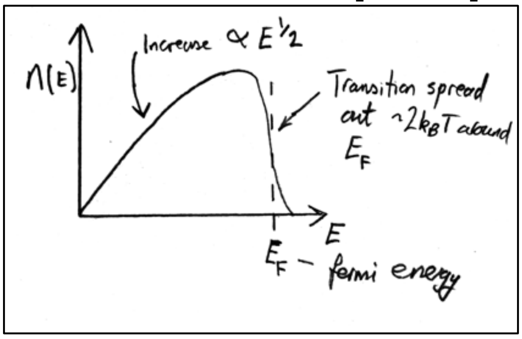
Synopsis **a)** The *Fermi Energy* is the highest occupied electron state in a system when it is in the ground state. **[1]** From the solution of the free electron Schrödinger equation it can be shown that energy is related to wavevector by $E = \frac{\hbar^2 k^2}{2m_e}$. **[1]** In 3D we have $k^2 = k_x^2 + k_y^2 + k_z^2$ **[1]**. The surface of constant energy in *k*-space or reciprocal space is therefore a surface of constant k^2 or a sphere. The radius of the sphere is then the Fermi wavevector k_F . **[1]** This is shown in the diagram. **[3]** marks for diagram with correct labels.



[7 marks]

Problem **b)** The Fermi energy is related to the electron density by $k_{\mathrm{F}} = \left(3\pi^2 n\right)^{\frac{1}{3}}$ (from the Fermi sphere). This gives $E_{\mathrm{F}} = \frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{\frac{2}{3}}$. **[1]** The electron density can be calculated from the atomic density assuming there is one free electron per atom **[1]**. Atomic density is then $\rho = \frac{\text{Density}}{\text{Atomic Mass}} = \frac{10500}{108\times1.66\times10^{-27}}$ $5.857 \quad 10^{28}$ m^{-3}. **[1]** Finally this gives E_{F} as $\frac{\hbar^2}{2m_e}\times\left(3\pi^2\times5.857\times10^{28}\right)^{\frac{2}{3}} = 8.811\times10^{-19}$ J (5.50 eV) **[1]** This is many orders of magnitude higher than the thermal energy at 300K (~25 meV). This is due to the quantum nature of electrons in the free electron model where energy increases as electron density increases **[1]**. **[5 marks]**

Conceptual **c)** Sketch shows free electron density as function of temperature, increasing as $E^{1/2}$ up to the Fermi energy E_{F} this results from the density of states in 3D **[1]**. At the Fermi energy the density of states drops to zero as electron energy states above E_F are empty **[1]**, the transition is broadened out by approximately ~$2k_{\mathrm{B}}T$. **[1]** Plus **[2]** for correctly labelled sketch.



[5 marks]

Problem **d)** Heating the metal by 200 degrees will lead to thermal expansion **[1]**. This will give a slightly lower atomic density and therefore electron density **[1]**. A lower electron density will lower the Fermi energy. **[1]**

[3 marks]

MO 2016-17 Short questions

1. (i) $G(u) + H(u)$ [1] [BOOKWORK]
(ii) $G(u) * H(u)$ [1]
(iii) $G(u)\,H(v)$ [1]
(iv) $G(u) * \mathcal{F}[1/h(x)]$ [1]

2. $\mathcal{E} = \mathcal{E}_0\, e^{i(k_x x + k_z z)}$ [1] [BOOKWORK]

$\mathcal{E} = \mathcal{E}_0\, e^{ik_z z}\left(e^{ik_x x} + e^{-ik_x x}\right)$ [1]

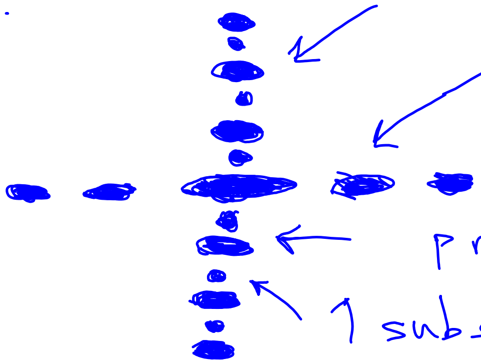
$I = I_0 \cos^2 k_x x$ [2] [CONCEPT]

3. $\mathcal{E} = \mathcal{E}_0\, e^{ikz}\, e^{ik(x^2+y^2)/2z}$ [1] [BOOKWORK]

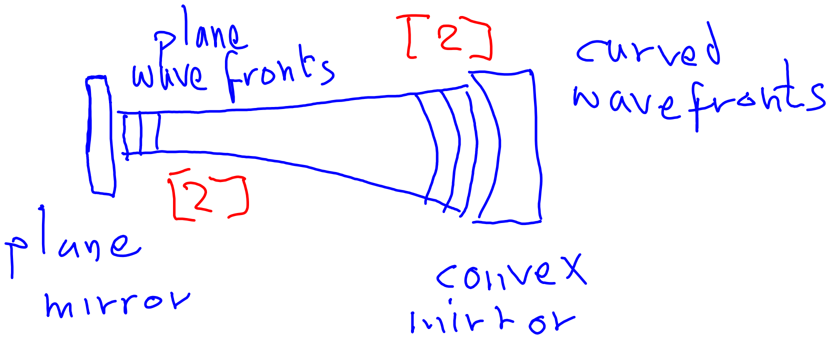
Place lens at $z = f$ [1] phase imprinted $e^{-ik(x^2+y^2)/2f}$ [2] [CONCEPT]

4. $w = \dfrac{\lambda}{\pi w_0} z$ [1] $= \dfrac{5.00 \times 10^{-7}}{3.14 \times 1.00 \times 10^{-3}}\, 3.84 \times 10^{8}$ [1] $= 61.1\ \mathrm{km}$ [2] [CONCEPT]

5. 3 slit pattern from ☰ [1]
sinc^2 pattern from ▯ [1]



principal maxima [1]
1 subsidiary maxima [1] [CONCEPT]

6. 

[CONCEPT]

L2 Foundation of Physics 2B Modern Optics 2016-17

MO EXAM Q.8

February 13, 2017

(a) [BOOKWORK]

$$\mathcal{I}^{(z)} = \frac{\mathcal{I}_0}{\lambda^2 z^2} |\mathcal{F}[\mathrm{f}(x', y')]|^2 \ . \text{ [1]}$$

[CONCEPT] The function $\mathrm{f}(x', y')$ is dimensionless[1], the Fourier transform is an integral over x' and y' so have units of $(\text{Length})^2$,[1] which cancels with the factor λz also have units of $(\text{Length})^2$.[1]

(b) [UNSEEN]

$$\mathcal{F}[\cos(2\pi u_0 x)] = \tfrac{1}{2}[\delta(u + u_0) + \delta(u - u_0)] \ . \text{ [1]}$$

where $u = x/(f\lambda)$.[1]. Substituting in the above

$$\mathcal{I}^{(z)} = \frac{\mathcal{I}_0}{4\lambda^2 z^2} \left[\delta\left(\frac{x}{\lambda f} + u_0\right) + \delta\left(\frac{x}{\lambda f} - u_0\right) \right]^2 \ . \text{ [2]}$$

(c) [CONCEPT] (i) Sketch should include two δ-functions[1] at $x = \pm f\lambda u_0$[1]. (ii) For large u_0 the sketch is the same [1] except that now the δ-functions are farther apart.[1]

(d) [CONCEPT] The spacing is $\Delta x = 2f\lambda u_0$.[1] It is larger[1] for red light.

(e) [CONCEPT] We are assuming that the lens has infinite spatial extent.[1] For a lens with finite diameter, the δ-functions in the focal plane are replaced by Airy patterns.[1]

(f) [CONCEPT] For a lens with diameter D the first zero of the Airy pattern is at $1.22f\lambda/D$.[1] and two Airy patterns are just resolved when their spacing is $\Delta x = 1.22f\lambda/D$.[1]. So the minimum value of u_0 is when

$$2f\lambda u_0 = 1.22\frac{f\lambda}{D} \ . \text{ [1]}$$

and hence

$$u_0 = \frac{0.61}{D} \ . \text{ [1]}$$

L2 Foundation of Physics 2B Modern Optics 2016-17

MO EXAM Q.8

February 13, 2017

(a) [BOOKWORK] Sketch should include a labelled input plane at $z = -2f$,[1] a lens at $z = -f$,[1] a lens at $z = f$,[1] and the output plane at $z = 2f$.[1]

(b) [BOOKWORK] Spatial filtering is performed by placing a spatial filter (where the transmission is a function of position)[1] in the Fourier plane at $z = 0$.[1]

(c) [CONCEPT] f$(x') = 1$. [1]

(d) [CONCEPT] The sketch should be an Airy pattern[2] with the first zero at $1.22f\lambda/D$.[2]

(e) [CONCEPT] The input field is a square wave[1] with a period d.[1] For a lens with finite diameter, the δ-functions in the focal plane are replaced by Airy patterns.[1]

(f) [CONCEPT] The transmission is 50%.[1]

(g) [CONCEPT] It is identical to part (d).[2]

(h) [CONCEPT] The field is the same square wave as in part (e) but now with the dc offset removed.[2]. The intensity pattern is uniform except for dark lines every $d/2$.[2]