ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 12+13+14+15

10 Dielectrics

10.1 Microscopic properties of dielectrics

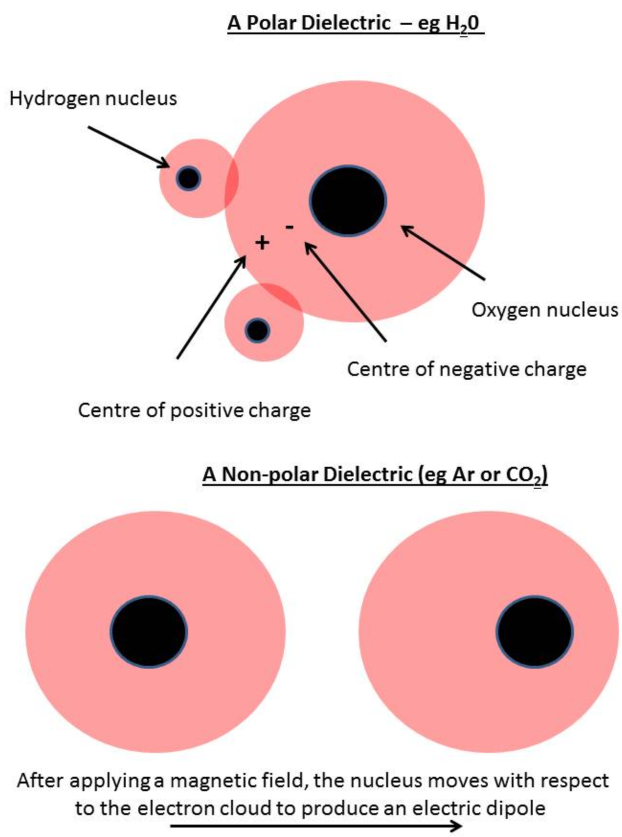


Figure 1 : A polar dielectric and a non-polar dielectric

Dielectrics produce their own electric field in response to an applied electric field – hence the name ‘dielectric’ : 2nd electric field.

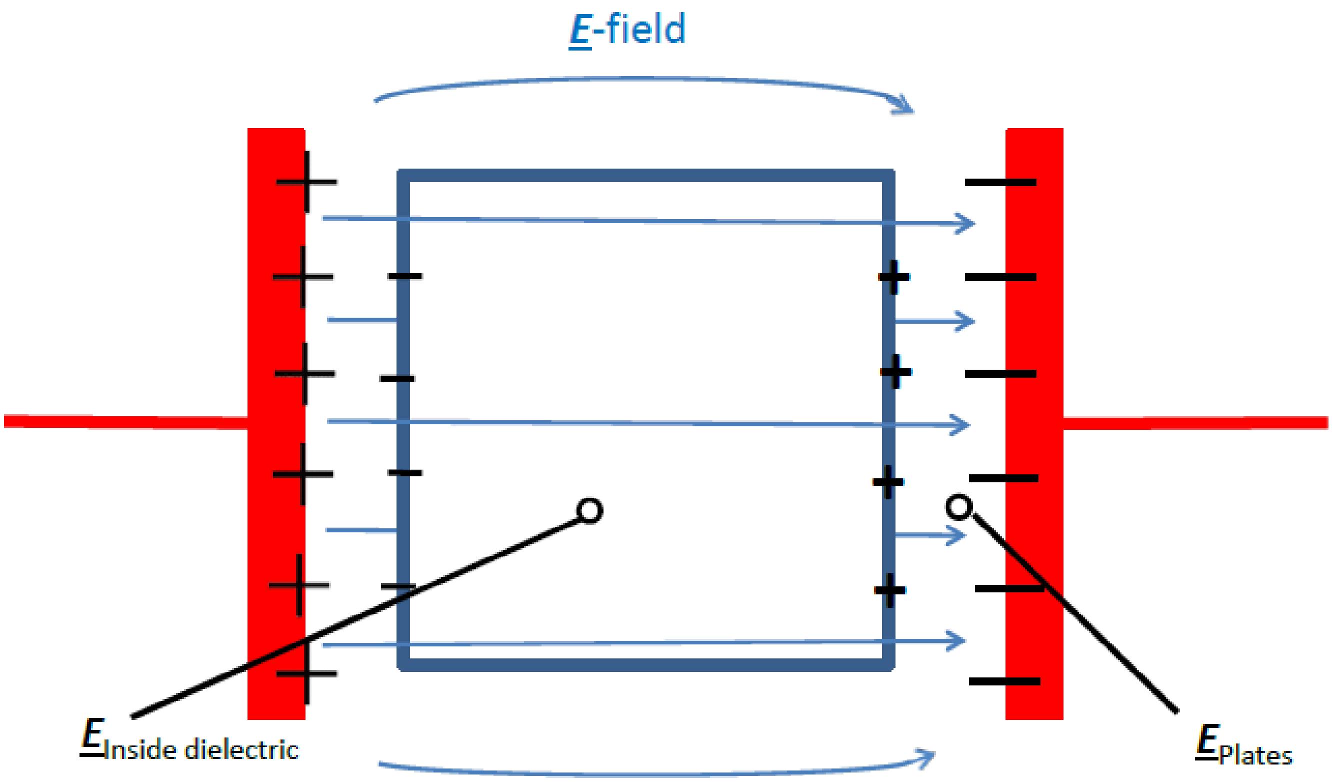


Figure 2 : A polarized cube of dielectric between two capacitor plates. The dielectric field opposes the applied electric field. The net $\underline{\boldsymbol{E}}$-field inside the dielectric is lower than the applied field.

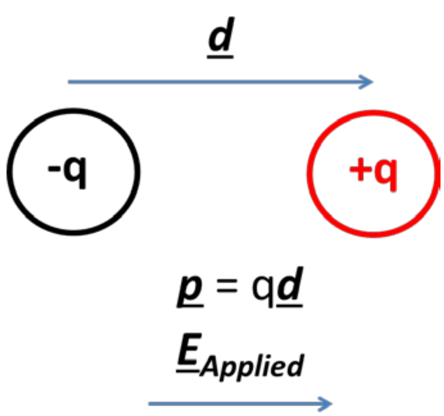


Figure 3 : An electric dipole moment.

The fundamental response of a dielectric:

$$\underline{\boldsymbol{p}} = q\underline{\boldsymbol{d}} \; - \; \text{Definition of the electric dipole moment } (\boldsymbol{p}) \qquad 10\text{-}1$$

$$\underline{\boldsymbol{P}} = \mathrm{N}\underline{\boldsymbol{p}} - \text{ Definition of the polarisation, } \boldsymbol{P} \ (\mathrm{Cm}^{-2}) \tag{10-2}$$

where N is the number of dipoles per unit volume(m^{-3}).

$$\underline{\boldsymbol{P}} = \varepsilon_o(\varepsilon_r - 1)\underline{\boldsymbol{E}} - \text{ Definition of the relative dielectric constant or permittivityy } (\varepsilon_r) \tag{10-3}$$

$$\underline{\boldsymbol{P}} = \varepsilon_o \chi_e \underline{\boldsymbol{E}} - \text{Definition of the electric susceptibility, } \chi_e \text{, of the medium} \tag{10-4}$$

Note that as either ε_r or χ_e increases, $\underline{P}$ increases for a given $\underline{E}$-field => the material is a stronger dielectric

10.2 Current density and charge density in dielectrics

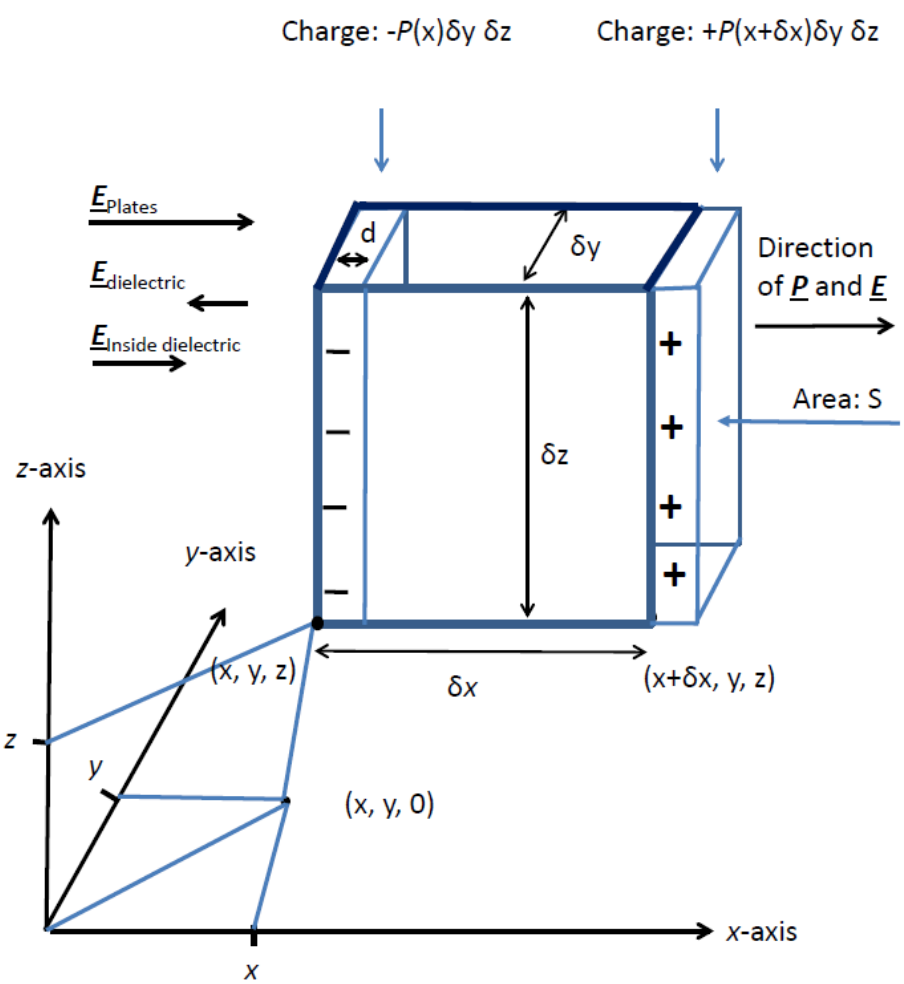


Figure 4 : A cube of dielectric in an $\underline{\boldsymbol{E}}$-field consisting of displaced +ve charge.

$$Q_{+ve} = \underbrace{Nq}_{\text{charge/volume}} \times \underbrace{Sd}_{\text{Volumeof+vecharge}} \tag{10-5}$$

$$\sigma = \frac{Q_{+ve}}{S} = Nqd = P \tag{10-6}$$

$$\sigma = \underline{\boldsymbol{P}} \cdot \hat{\mathbf{n}} \tag{10-7}$$

The positive charge that moves into the cube from the neighbouring cube is (c.f. $\sigma = P \cdot \hat{n}$) at x is given by :

$$Q_{In}(x) = P(x)\Delta y \Delta z \tag{10-8}$$

$$Q_{Out}(x + \Delta x) = P(x + \Delta x)\Delta y \Delta z \tag{10-9}$$

$$Q_{net} = -[P(x + \Delta x) - P(x)]\Delta y \Delta z = -\frac{\partial P}{\partial x}\Delta x \Delta y \Delta z \tag{10-10}$$

$$\rho_b = -\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z} = -\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{P}} \tag{10-11}$$

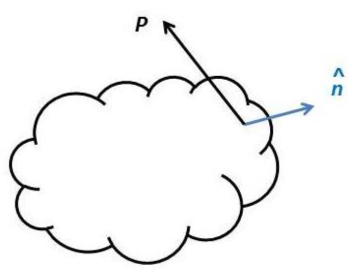


Figure 5 : An arbitrary volume of polarized material

If the polarisation changes with time, the moving charges are equivalent to a current. The current or charge flowing out per second ($\frac{\partial Q_{Out}}{\partial t}$) of an arbitrary volume is:

$$\begin{aligned}\frac{\partial Q_{Out}}{\partial t} &= -\int \frac{\partial \rho_{In}}{\partial t} dV = \frac{\partial}{\partial t}\int \underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{P}} dV = \frac{\partial}{\partial t}\int \underline{\boldsymbol{P}} \cdot d\underline{\boldsymbol{S}} \\ &= \int \underline{\boldsymbol{J}}_b \cdot d\underline{\boldsymbol{S}}\end{aligned} \tag{10-12}$$

Hence the current density is :

$$\underline{\boldsymbol{J}}_{\mathrm{b}} = \frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}} \qquad 10\text{-}13$$

10.3 Microscopic description of dielectrics

$$\mathrm{m}\frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{q}E - \frac{\mathrm{m}v}{\tau} - \mathrm{m}\omega_0^2 x \qquad 10\text{-}14$$

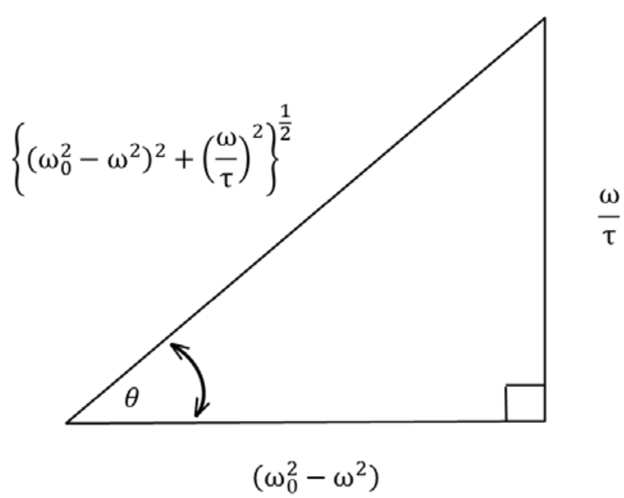
$$-\omega^2\tilde{x}\mathrm{m} = \mathrm{q}\tilde{E} + \frac{\mathrm{i}\omega\mathrm{m}\tilde{x}}{\tau} - \mathrm{m}\omega_0^2\tilde{x} \qquad 10\text{-}15$$

$$\mathrm{m}\left(\omega_0^2 - \omega^2 - \frac{\mathrm{i}\omega}{\tau}\right)\tilde{x} = \mathrm{q}\tilde{E} \qquad 10\text{-}16$$

Using Euler's equation

$$\frac{\tilde{x}}{\tilde{E}} = \frac{\mathrm{q}e^{\mathrm{i}\theta}}{\mathrm{m}\left((\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)^{\frac{1}{2}}} \qquad 10\text{-}17$$

where $\tan\theta = \frac{\omega}{\tau(\omega_0^2 - \omega^2)}$.



From the definitions of sine and cosine, $\cos\theta = \frac{(\omega^2 - \omega_0^2)}{\left((\omega^2 - \omega_0^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)^{\frac{1}{2}}}$ and $\sin\theta = \frac{\omega}{\tau\left((\omega^2 - \omega_0^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)^{\frac{1}{2}}}$

Writing the definition of polarization ($\widetilde{\mathrm{P}}$) and permittivity ($\tilde{\varepsilon}_r$) in complex form:

$$\tilde{P} = (\tilde{\varepsilon}_r - 1)\varepsilon_0\tilde{E} = \mathrm{N}\mathrm{q}\tilde{x} \qquad 10\text{-}18$$

Which gives:

$$\tilde{\varepsilon}_r = \varepsilon_{\text{real}} + i\varepsilon_{\text{imaginary}} = 1 + \frac{\text{Nq}}{\varepsilon_0}\frac{\tilde{x}}{\tilde{E}} \qquad \text{10-19}$$

Using $e^{i\theta} = \cos\theta + i\sin\theta$,

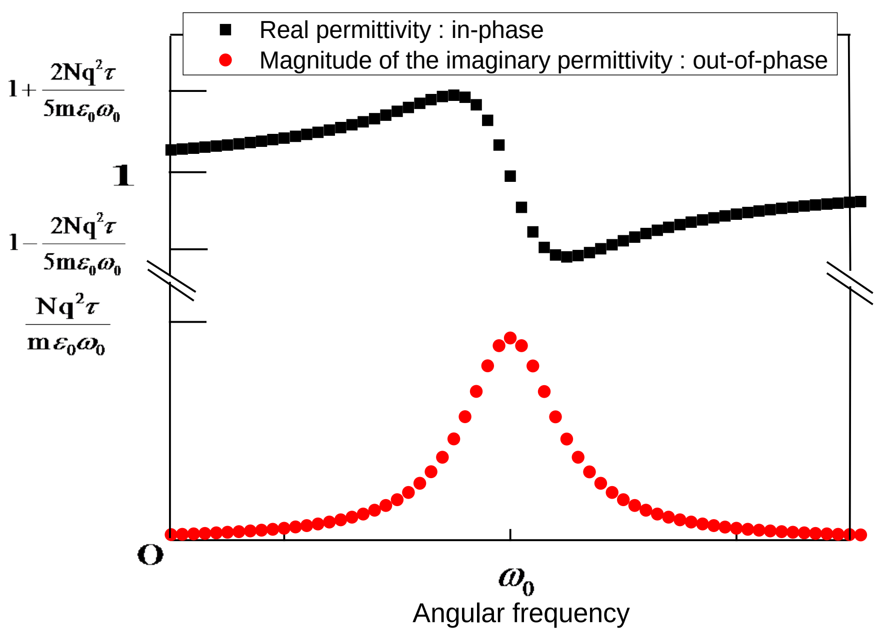
$$\tilde{\varepsilon}_r = 1 + \frac{\text{Nq}^2}{\text{m}\varepsilon_0}\frac{(\cos\theta + i\sin\theta)}{\left((\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)^{\frac{1}{2}}} \qquad \text{10-20}$$

Hence the real ($\varepsilon_{\text{real}}$) and imaginary parts ($\varepsilon_{\text{imaginary}}$) of the relative permittivity are:

$$\varepsilon_{\text{real}} = 1 + \frac{\text{Nq}^2}{\text{m}\varepsilon_0}\frac{(\omega_0^2 - \omega^2)}{\left((\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)} \qquad \text{10-21}$$

and

$$\varepsilon_{\text{imaginary}} = \frac{\text{Nq}^2}{\text{m}\varepsilon_0\tau}\frac{\omega}{\left((\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)} \qquad \text{10-22}$$



The variation of relative permittivity with angular frequency near a resonance.

10.4 The auxiliary field D.

We can write Maxwell I :

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_o} = \frac{\rho_{\mathrm{free}} + \rho_{\mathrm{dielectric}}}{\varepsilon_o} = \frac{\rho_{\mathrm{free}} - \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{P}}}{\varepsilon_o} \qquad \text{10-23}$$

This leads to a definition for the electric displacement field $\underline{\boldsymbol{D}}$ where:

$$\underline{\boldsymbol{D}} = \varepsilon_o \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}} \ - \text{definition of } \underline{\boldsymbol{D}} \qquad \text{10-24}$$

So Maxwell I becomes:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{D}} = \rho_{\mathrm{free}} \qquad \text{10-25}$$

$\underline{D}$ is useful shorthand commonly used in many calculations (no new Physics):

$$\underline{\boldsymbol{D}} = \varepsilon_o \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}} - \text{the defintion of } \underline{\boldsymbol{D}}, \text{the electric displacement field.} \qquad \text{10-26}$$

11 Magnetic Materials`

11.1 Microscopic properties for magnetic materials - Ampere’s model

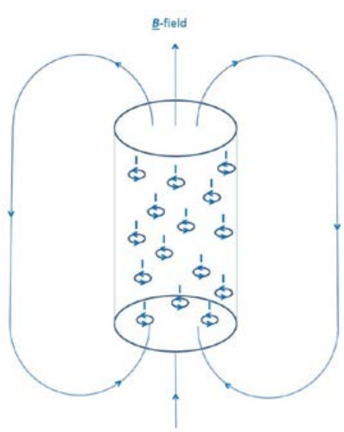


Figure 6 : The local and macroscopic fields produced by a magnetic material

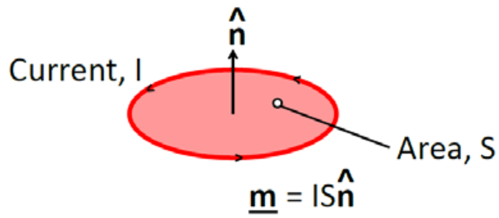


Figure 7 : A magnetic dipole moment.

The fundamental response of a magnetic material:

$$\underline{\boldsymbol{m}} = \mathrm{IS}\hat{\boldsymbol{n}} - \text{ Definition of the magnetic dipole moment } (\boldsymbol{m}) \qquad 11\text{-}1$$

where I is the current flowing around a loop of area S.

$$\underline{\boldsymbol{M}} = \mathrm{N}\underline{\boldsymbol{m}} - \text{ Definition of the magnetization } (\boldsymbol{M}) \qquad 11\text{-}2$$

where N is the number of magnetic dipoles per unit volume.

11.2 Currents densities in magnetic materials

a) Bulk current density

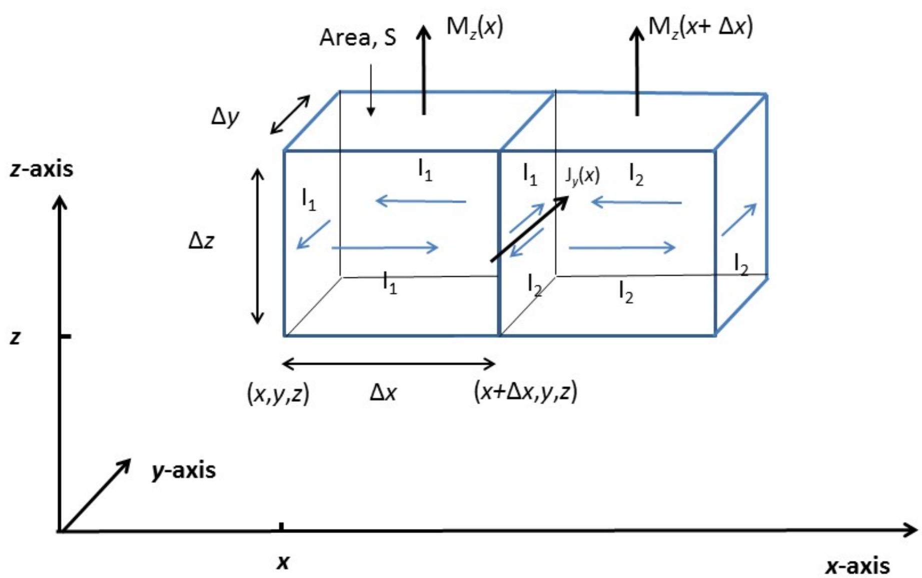


Figure 8 : Two magnetized cubes next to each other. The magnetization in each cube points in the z-direction because the current circulates in the x-y plane

$$m_{\mathrm{z}} = M_{\mathrm{z}}(x).\,\delta\mathrm{x}.\,\delta\mathrm{y}.\,\delta\mathrm{z} = I_1(x)A = I_1(x)\delta\mathrm{x}.\,\delta\mathrm{y} \qquad 11\text{-}3$$

$$I_1 = M_{\mathrm{z}}\Delta\mathrm{z} \qquad 11\text{-}4$$

The current along the interface between the cubes (I_{net})

$$I_{\mathrm{net}} = M_z.\Delta z - (M_z(x+\Delta x).\Delta z = -\Delta M_z.\Delta z \qquad 11\text{-}5$$

$$I_{\mathrm{net}} = -\frac{\Delta M_z}{\Delta x} \cdot \Delta x.\Delta z \qquad 11\text{-}6$$

$$J_y = \frac{I_{net}}{\Delta x.\Delta z} = -\frac{\Delta M_z}{\Delta x} \qquad 11\text{-}7$$

$$J_y = -\frac{\partial M_z}{\partial \mathrm{x}} \qquad 11\text{-}8$$

There is also a contribution to J_y if M_{x} varies, given by

$$J_y = \frac{\partial M_x}{\partial z} \qquad 11\text{-}9$$

.

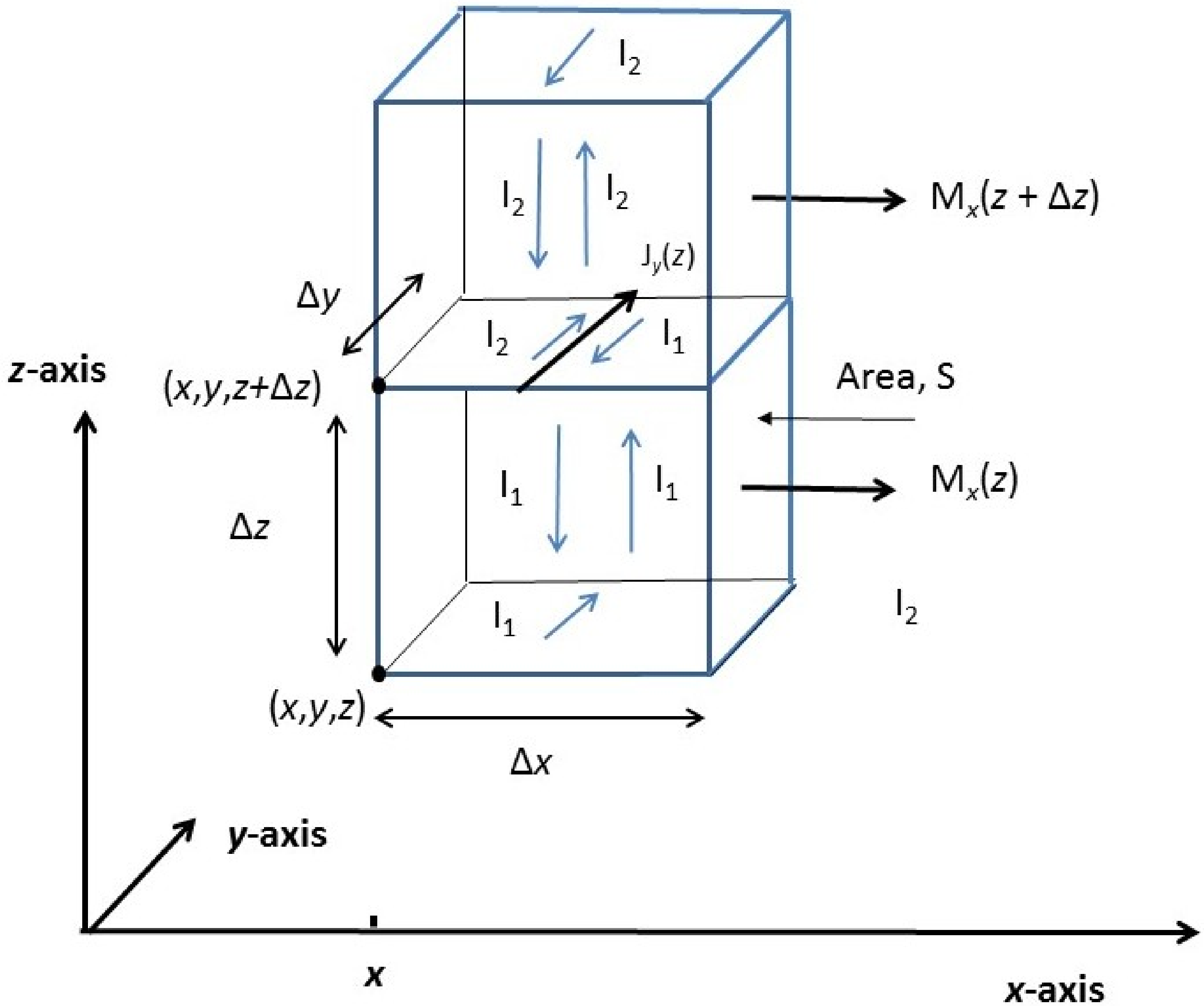


Figure 9 : Two magnetized cubes next to each other. The magnetization in each cube points in the x-direction because the current circulates in the *y-z* plane

Hence in 3D,

$$\underline{\boldsymbol{J}} = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \qquad 11\text{-}10$$

b) Surface Current Density

In a magnetized cylinder, the circulating currents in the bulk of the material cancel. The field from the material comes entirely from the circulating surface current.

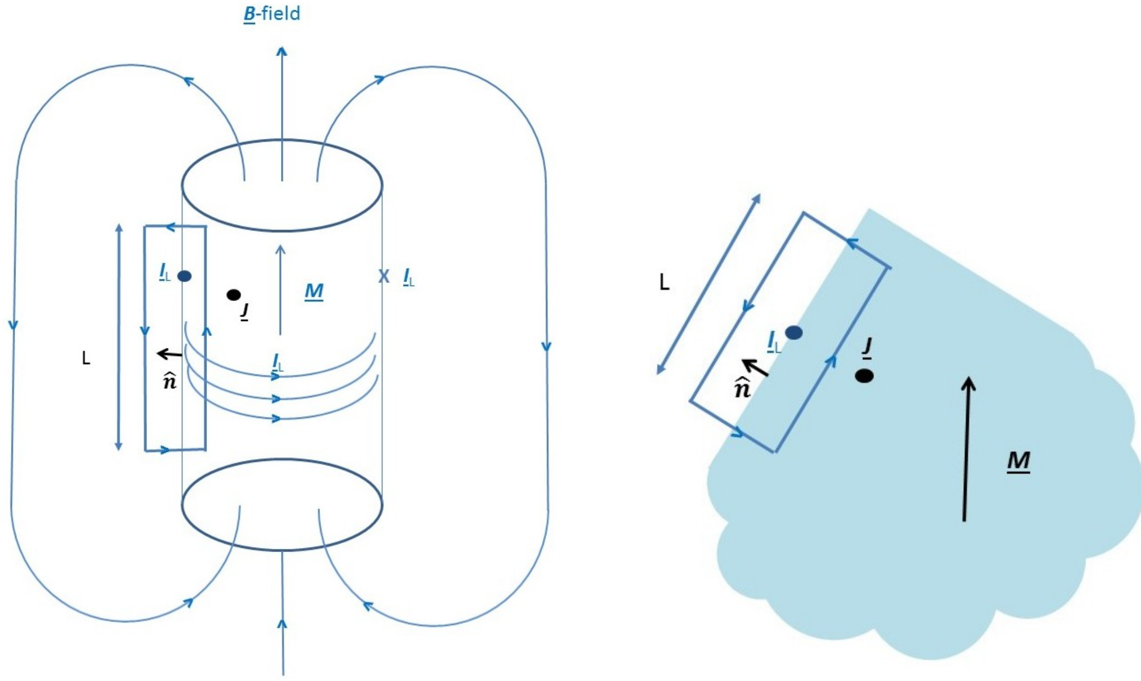


Figure 10 : The current flowing at the surface of a cylinder and an arbitrary shape. There is no bulk current density if the magnetisation is uniform.

For a uniformly magnetized material, there is a discontinuous change in $\underline{\boldsymbol{M}}$ at the surface. The current through the rectangle shown in the figure (normal to the board) can equally well be considered as a surface current per unit length or a bulk current ($\underline{\boldsymbol{J}}$) where

$$\int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = I_L L \qquad \text{11-11}$$

Using

$$\underline{\boldsymbol{J}} = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \qquad \text{11-12}$$

we have

$$\int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}}\right) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int \underline{\boldsymbol{M}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = ML = I_L L \qquad \text{11-13}$$

$$\Rightarrow I_L = M \qquad \text{11-14}$$

More generally

$$\Rightarrow \underline{\boldsymbol{I}}_{\mathrm{L}} = \underline{\boldsymbol{M}} \times \hat{\mathbf{n}} \qquad \text{11-15}$$

11.3 The auxiliary field, H

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \qquad 11\text{-}16$$

The total current density ($\underline{\boldsymbol{J}}$) through an ILIH material is

$$\underline{\boldsymbol{J}} = \underline{\boldsymbol{J}}_{\text{free}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial t} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \qquad 11\text{-}17$$

where:

$$\underline{\boldsymbol{J}}_{\text{free}} = \underline{\boldsymbol{J}}_{\text{Ohms law}} + \underline{\boldsymbol{J}}_{\text{Experimentalist}} \qquad 11\text{-}18$$

Maxwell IV can be rearranged as

$$\underline{\boldsymbol{\nabla}} \times \left(\underline{\boldsymbol{B}} - \mu_0 \underline{\boldsymbol{M}}\right) = \mu_0 \underline{\boldsymbol{J}}_{\text{free}} + \mu_0 \frac{\partial}{\partial t}\left(\underline{\boldsymbol{P}} + \varepsilon_0 \underline{\boldsymbol{E}}\right) \qquad 11\text{-}19$$

This leads to a definition for the magnetic field strength, $\underline{\boldsymbol{H}}$, defined by

$$\underline{\mathbf{B}} = \mu_o\left[\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}\right] - \text{ Definition of the magnetic field strength } (\boldsymbol{H}) \qquad 11\text{-}20$$

So we can rewrite MIV in terms of $\underline{\boldsymbol{D}}$, $\underline{\boldsymbol{H}}$ and ρ_{free}.

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}} = \underline{\mathbf{J}}_{\text{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial t} \qquad 11\text{-}21$$

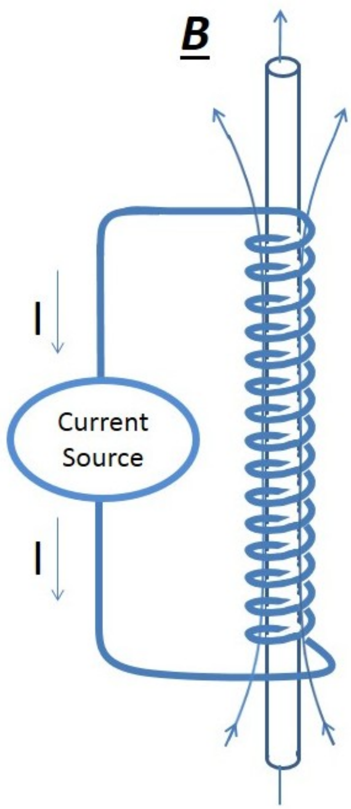


Figure 11 : A magnetized cylinder. The net field ($\underline{\boldsymbol{B}}$) is the sum of the magnetic field produced by the coil and the magnetic field produced by the material.

For a magnetized cylinder

$$\underline{\boldsymbol{B}}_{\mathrm{net}} = \underline{\boldsymbol{B}}_{\mathrm{applied}} + \mu_{\mathrm{o}}\underline{\boldsymbol{M}} \qquad 11\text{-}22$$

$$\underline{\boldsymbol{M}} = \chi_H \underline{\boldsymbol{H}} - \text{Definition of the magnetic susceptibility } (\chi_H) \qquad 11\text{-}23$$

$$\underline{\boldsymbol{B}} = \mu_{\mathrm{o}}\mu_{\mathrm{r}}\underline{\boldsymbol{H}} - \text{ Definition of relative permeability } (\mu_{\mathrm{r}}): \qquad 11\text{-}24$$

Note that μ_{r} can be 10^6.

Summary: Hence if we know $\underline{\boldsymbol{M}}$, we know the surface current per unit length $\underline{\boldsymbol{M}} \times \hat{\mathbf{n}}$ and the bulk current density $\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}}$.

12 The general dispersion relation

12.1 Aide-memoire

We can re-write Maxwell's equations using the definitions and derivations we have made (no new Physics):

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{D}} = \rho_{\text{free}} \qquad 12\text{-}1$$

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0 \qquad 12\text{-}2$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} \qquad 12\text{-}3$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}} = \underline{\mathbf{J}}_{\text{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial \mathrm{t}} \qquad 12\text{-}4$$

where $\underline{\boldsymbol{D}} = \varepsilon_o \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$, and $\underline{\boldsymbol{B}} = \mu_o\left(\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}\right)$

and $\rho = \rho_{\text{free}} - \underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{P}}$ and $\underline{\boldsymbol{J}} = \underline{\boldsymbol{J}}_{\text{free}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}}$

12.2 Propagation of transverse electromagnetic waves in materials

12.2.1 The general dispersion relation for an Infinite Linear-Isotropic-Homogeneous(ILIH) media

We can derive the general dispersion relation using Maxwell's fourth equation is:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad 12\text{-}5$$

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho_{\text{free}} + \rho_{\text{bound}}}{\varepsilon_0} = \frac{\rho_{\text{free}}}{\varepsilon_0} - (\varepsilon_r - 1)\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} \qquad 12\text{-}6$$

Rearranging, this gives:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho_{\text{free}}}{\varepsilon_r \varepsilon_0} = 0 \qquad 12\text{-}7$$

Maxwell IV can be rewritten:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \left(\sigma_n \underline{\boldsymbol{E}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}}\right) + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad 12\text{-}8$$

(Do not get confused: σ_n is the electrical conductivity from Ohm's law (Siemens or Ω^{-1}). σ is the surface charge density (C. m^{-2}).

This can be rearranged as

$$\underline{\boldsymbol{\nabla}} \times \left(\underline{\boldsymbol{B}} - \mu_0 \underline{\boldsymbol{M}}\right) = \mu_0 \sigma_n \underline{\boldsymbol{E}} + \mu_0 \frac{\partial}{\partial \mathrm{t}}\left(\underline{\boldsymbol{P}} + \varepsilon_0 \underline{\boldsymbol{E}}\right) \qquad 12\text{-}9$$

Using the materials properties definitions for ε_r , $\underline{\boldsymbol{H}}$, μ_r:

$$\underline{\boldsymbol{P}} = \varepsilon_o(\varepsilon_r - 1)\underline{\boldsymbol{E}} \qquad 12\text{-}10$$

$$\underline{\mathbf{B}} = \mu_o\left[\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}\right] = \mu_o\mu_r\underline{\boldsymbol{H}} \qquad 12\text{-}11$$

Maxwell IV can be written:

$$\underline{\boldsymbol{\nabla}} \times \frac{\underline{\mathbf{B}}}{\mu_r} = \mu_0\sigma_n\underline{\boldsymbol{E}} + \mu_0\frac{\partial}{\partial t}\left(\varepsilon_r\varepsilon_0\underline{\boldsymbol{E}}\right) \qquad 12\text{-}12$$

We substitute this into the curl of Maxwell's 3rd equation to find:

$$\underline{\boldsymbol{\nabla}} \times \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}\right) = -\frac{\partial}{\partial t}\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = -\frac{\partial}{\partial t}\left[\mu_o\mu_r\left(\sigma_n\underline{\boldsymbol{E}} + \varepsilon_o\varepsilon_r\frac{\partial \underline{\boldsymbol{E}}}{\partial t}\right)\right] \qquad 12\text{-}13$$

Hence using the vector identity $\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) = \underline{\boldsymbol{\nabla}}\left(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}\right) - \nabla^2\underline{\boldsymbol{E}}$ gives,

$$-\nabla^2\underline{\mathbf{E}} = -\underbrace{\mu_o\mu_r}_{\mu}\sigma_n\frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \underbrace{\mu_o\mu_r}_{\mu}\underbrace{\varepsilon_o\varepsilon_r}_{\varepsilon}\frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad 12\text{-}14$$

$$\Rightarrow \nabla^2\underline{\boldsymbol{E}} - \mu\sigma_n\frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \mu\varepsilon\frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} = 0 - \text{ Wave equation} \qquad 12\text{-}15$$

Equally taking the curl of MIV

$$\nabla^2\underline{\boldsymbol{B}} - \mu\sigma_n\frac{\partial \underline{\boldsymbol{B}}}{\partial t} - \mu\varepsilon\frac{\partial^2 \underline{\boldsymbol{B}}}{\partial t^2} = 0 \ - \text{ Wave equation} \qquad 12\text{-}16$$

We can substitute a plane wave solution into the wave equation.

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \exp i\left(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t\right) \ - \text{ plane wave equation} \qquad 12\text{-}17$$

Hence we find the dispersion relation (which is by definition the relationship between ω and k).

$$-k^2 + i\omega\mu\sigma_n + \mu\varepsilon\omega^2 = 0 \qquad 12\text{-}18$$

$$k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma_n$$

$$-\text{ General dispersion relation for an infinite, linear, isotropic, homogenous medium}$$

12.2.2 Electromagnetic waves in conducting/dielectric/magnetic materials

We now use the general dispersion relation to calculate the approximate decay length, wave-vector and wavelength of a wave of frequency ~ 10^{15} Hz propagating through tungsten - ρ_n = 1 mΩ.cm. Tungsten is neither magnetic nor a dielectric.

Solution

$$\nabla^2 \underline{\boldsymbol{E}} - \mu\sigma_n \frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} = 0$$

Assume that the plane wave has the form of $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \mathrm{expi}(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t)$, then substitute it into the wave equation,

$$\Rightarrow (ik)^2 \underline{\boldsymbol{E}} - \mu\sigma_n(-i\omega)\underline{\boldsymbol{E}} - \mu\varepsilon(-i\omega)^2 \underline{\boldsymbol{E}} = 0$$

Using

$$k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma_n.$$

Given $f \sim 10^{15}$Hz, $\rho_n = 1 \times 10^{-5}\Omega$m or $\sigma_n = \frac{1}{\rho_n} = 10^5 \ \Omega^{-1}m^{-1}$, $\mu = \mu_0$, and $\varepsilon = \varepsilon_0$

$$k^2 = 4\pi \times 10^{-7} \times 8.85 \times 10^{-12} \times (2\pi \times 10^{15})^2 + i(2\pi \times 10^{15}) \times 4\pi \times 10^{-7} \times 10^5$$

$$k^2 = 4.4 \times 10^{14} + i7.9 \times 10^{14}.$$

Note: Tungsten cannot be characterised as either a good insulator or a good conductor

Draw Argand diagram:

$$\tan\phi = \frac{7.9}{4.4} \Rightarrow \phi = 60.88° , \qquad \phi/2 = 30.44°$$

$$|k| = [(4.4 \times 10^{14})^2 + (7.9 \times 10^{14})^2]^{\frac{1}{4}} = 3.0 \times 10^7$$

hence, .

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \mathrm{expi}((2.59 \times 10^7 + i1.52 \times 10^7)x - \omega t)$$

$$= \underline{\boldsymbol{E}}_o \mathrm{expi}(2.59 \times 10^7 x - \omega t)\exp(-1.52 \times 10^7 x)$$

Taking either the real (or imaginary) part:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \cos(2.59 \times 10^7 x - \omega t) \exp(-1.52 \times 10^7 x)$$

The wave vector is the real part of k, $2.59 \times 10^7 m^{-1}$ and the wavelength is $\lambda = \frac{2\pi}{2.59 \times 10^7} \approx 2.4 \times 10^{-7}$m.
The decay length is the reciprocal of imaginary part of k, $\delta = \frac{1}{k_{imaginary}} = \frac{1}{1.52 \times 10^7} \approx 66$nm.

