CM3: Oscillations about stable equilibrium

Two particles of mass m_1 and m_2 are connected by a light, inextensible rod of length l. The particles are constrained to slide on a smooth circular wire hoop of radius a, which is held in a vertical plane in a uniform gravitational field of strength g. θ and $\theta + \alpha$ are the angular coordinates, measured downwards from the horizontal, of m_1 and m_2 with respect to the centre of the hoop (α is fixed).

1. Find the Lagrangian of the system in terms of the generalised coordinate θ, and use the Euler-Lagrange equation,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$
to show that
$$\ddot{\theta} = \frac{g[m_1 \cos\theta + m_2 \cos(\theta + \alpha)]}{a(m_1 + m_2)}.$$

2. For the case with $m_1 = m_2 = m$, what is the equilibrium value of θ?

3. Assuming $m_1 = m_2 = m$, find the equation of motion for small oscillations about the equilibrium position, θ_{eq}, by rewriting the differential equation for $\ddot{\theta}$ in terms of $\phi = \theta - \theta_{\text{eq}}$ and solving for $\phi(t)$. Give your answer in terms of $\phi(0)$ and $\dot{\phi}(0)$.

4. What is the solution when $\alpha = \pi$ and what motion does this represent?

5. With what frequency does a single particle of mass $2m$, confined to the same circular hoop, undergo small oscillations about its equilibrium position?