Lecture 3: Big Bang Models

3.1 Friedmann Equation

[Liddle sec:3.1]

Consider a spherical region of radius r within a homogeneous isotropic universe with density $\rho(t)$. Assume that pressure is negligible, i.e.that the universe is matter dominated (sometimes referred to as dust dominated).

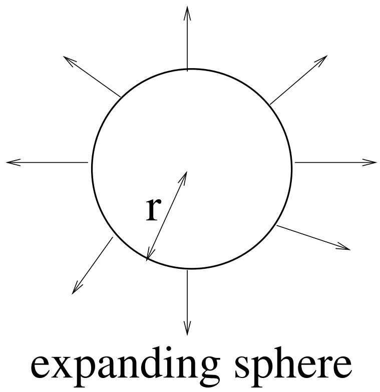

Figure 1: Evolution of a spherical region of the Universe.

If we consider only the material inside the sphere and apply Newton's law of gravity to determine the behaviour of the radius of the bounding sphere we find:

$$\ddot{r} = -\frac{GM}{r^2} = -\frac{4\pi G\rho(t)r^3}{3r^2} \tag{3.1}$$

and using the conservation of mass $\rho(t)r(t)^3 = \rho_0 r_0^3$ (a constant) and we have

$$\ddot{r} = -\frac{4\pi G\rho_0 r_0^3}{3}\,\frac{1}{r^2}. \tag{3.2}$$

Pre-multiplying by $\dot{r}$ and integrating with respect to t we obtain

$$\frac{1}{2}\dot{r}^2 = \frac{4\pi G\rho_0 r_0^3}{3}\,\frac{1}{r} + \text{constant}. \tag{3.3}$$

It is helpful to re-write this equation in term of the dimensionless scale factor a (eq. 2.1)

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2}.} \tag{3.4}$$

This is known as the **Friedmann equation**. Here k is a constant with the dimensions of $[L]^{-2}$. It can be positive, negative or zero. [Beware! In many books the unit of length is chosen so that k takes on the conventional values of $k = \pm 1$ or 0.]

We have derived this equation using Newtonian mechanics assuming a matter (or dust) dominated universe, but in fact it can be derived using General Relativity and can be shown to be valid even when the universe is dominated by radiation or other forms of energy.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\rho_{\rm mass}\, c^2 + \rho_\gamma\, c^2 + \rho_{\rm DE}\, c^2\right) - \frac{kc^2}{a^2}.$$

where $\rho_\gamma c^2$ and $\rho_{\rm DE} c^2$ are the energy densities in radiation and **dark energy**. We will return to this in lecture 9.

3.2 The Evolution of the Expansion Parameter

[Liddle sec:4.5/5.5,6.2]

Recall that $H(t) = \dot{a}/a$ (eq. 2.2), so that $H_0 = \dot{a}/a$ evaluated at the present time. Also, $a_0 = 1$ by definition. Hence from (3.4)

$$H_0^2 = \frac{8\pi G\rho_0}{3} - kc^2. \tag{3.6}$$

It is also often useful to write the Friedmann equation (3.4) as

$$H(t)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2} \tag{3.7}$$

Since we know the present-day density of the universe, we can use eq. 2.2 to express H or $\dot{a}$ as a function of the expansion factor

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a} - kc^2 \tag{3.8}$$

The present-day universe is expanding, so the first term dominates the equation at $a = 1$. However, as the universe gets bigger and bigger, the second term will eventually become dominant. If $k < 0$, $\dot{a}$ decreases with time, and eventually becomes constant. The universe expands forever. If $k = 0$, $\dot{a} \to 0$ as a gets larger and larger. The expansion never quite stops, but it becomes less and less as time goes on. For $k > 0$, the two terms cancel for at a finite value of a, and the universe has a finite maximum size.

In order to examine the time dependence of $H(t)$ in more detail, it is useful to define a **critical density**

$$\boxed{\rho_{\rm crit}(t) = \frac{3H^2(t)}{8\pi G}.} \tag{3.9}$$

Rearranging (3.7)

$$H(t)^2 = \frac{\rho(t)H(t)^2}{3H(t)^2/8\pi G} - \frac{kc^2}{a^2}.$$

Then using (3.9) we can write this as

$$H(t)^2\left(\frac{\rho(t)}{\rho_{\rm crit}(t)} - 1\right) = \frac{kc^2}{a^2}$$

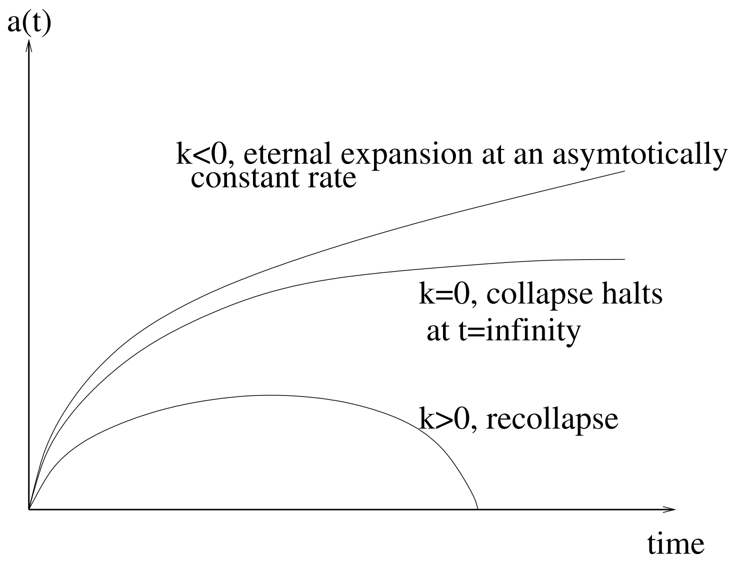

Figure 2: Evolution of expansion factor and the fate of the Universe for different values of k.

Applying this equation to the present-day, we see that the constant k must have the same sign as $\left(\frac{\rho_0}{\rho_{\rm crit,0}} - 1\right)$.

• If $\rho_0 > \rho_{\rm crit,0} \to k > 0$ the universe expands to a maximum size and then recollapses.

• If $\rho_0 < \rho_{\rm crit,0} \to k < 0$ the universe expands forever with $\dot{a}$ tending to a constant at late times.

• If $\rho_0 = \rho_{\rm crit,0} \to k = 0$ This is a special case for which $\rho = \rho_{\rm crit}$ at all times. In this type of universe the expressions for $a(t), H(t)$ etc are particularly simple.

At the present time this critical density has the value of

$$\rho_{\rm crit,0} = \frac{3H_0^2}{8\pi G} \approx 1.0 \times 10^{-26}\ {\rm kg\,m^{-3}} \approx 1.5 \times 10^{11}\ {\rm M_\odot\,Mpc^{-3}} \approx 6 m_{\rm p}\,{\rm m^{-3}} \tag{3.10}$$

Often it is convenient to express densities in units of the critical density. We do this by defining the **density parameter**

$$\boxed{\Omega(t) \equiv \rho(t)/\rho_{\rm crit}(t)} \tag{3.11}$$

Using this definition of $\Omega(t)$ we arrive at

$$|\Omega(t) - 1| = \frac{|k|c^2}{a(t)^2 H(t)^2} = \frac{|k|c^2}{\dot{a}(t)^2}$$

We can use this equation to determine the constant k by measuring the values of Ω and H at the present time.

$$k = \frac{\mathrm{H}_0^2}{c^2}\,(\Omega_0 - 1). \tag{3.12}$$

Remember that k is a genuine constant that does not depend on time. Consequently if Ω_0 is exactly equal to one, $k = 0$ exactly, and Ω is exactly equal to one forever. But if Ω_0 differs slightly from one this differnce will grow (remember that as the universe expands, $\dot{a}$ gets smaller). We can in general show that

$$\Omega(a) = \frac{\Omega_0}{a(1 - \Omega_0) + \Omega_0}. \tag{3.13}$$

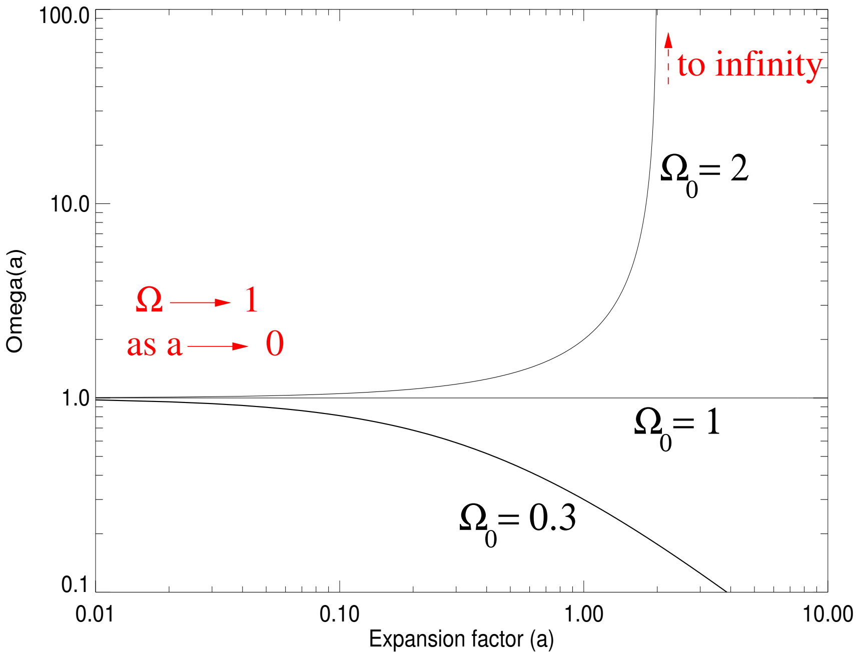

Figure 3: Evolution of Ω with expansion factor for different values of Ω_0.

3.3 The Age of the Universe

[Liddle sec:7.1/8]

We can estimate the present-day age of the universe by measuring the present value of the Hubble parameter. The **Hubble time** is defined as

$$t_{\mathrm{H}} \equiv 1/\mathrm{H}_0.$$

For a critical density universe it is straightforward to determine an explicit equation for the relation between a and t. **The indented section assumes $\Omega = \rho/\rho_{\mathrm{crit}} = 1$ and hence $k = 0$.**

For this special case, eq. 3.8 becomes

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a}$$

Since $\rho_0 = \rho^0_{\mathrm{crit}}$, we can use 3.9 to express this as

$$\dot{a}^2 = H_0^2 \frac{1}{a}$$

Rearranging and integration (noting $a = 0$ at $t = 0$) we find

$$a = \left(\frac{3\mathrm{H}_0}{2}\right)^{2/3} t^{2/3}, \tag{3.14}$$

or

$$a = \left(\frac{t}{t_0}\right)^{2/3}, \quad \text{where} \quad t_0 \equiv \frac{2}{3}\,\mathrm{H}_0^{-1} \equiv \frac{2}{3}\,t_{\mathrm{H}}. \tag{3.15}$$

This is the relation (for an $\Omega = 1$ universe) between the expansion factor of the universe, a and the time, t since the Big Bang (when $a = 0$). The present age of the universe is t_0 (ie the value when $a = 1$).

If $\Omega > 1$ then $t_0 < 2/3\,\mathrm{H}_0^{-1}$, since the universe decelerates more quickly. If $\Omega < 1$ then $t_0 > 2/3\,\mathrm{H}_0^{-1}$, and as $\Omega \to 0$ we find that $t_0 \to \mathrm{H}_0^{-1}$. Numerical integration of the Friedmann equation for various values of Ω_0 is shown in Fig.4, illustrating this dependence on Ω_0.

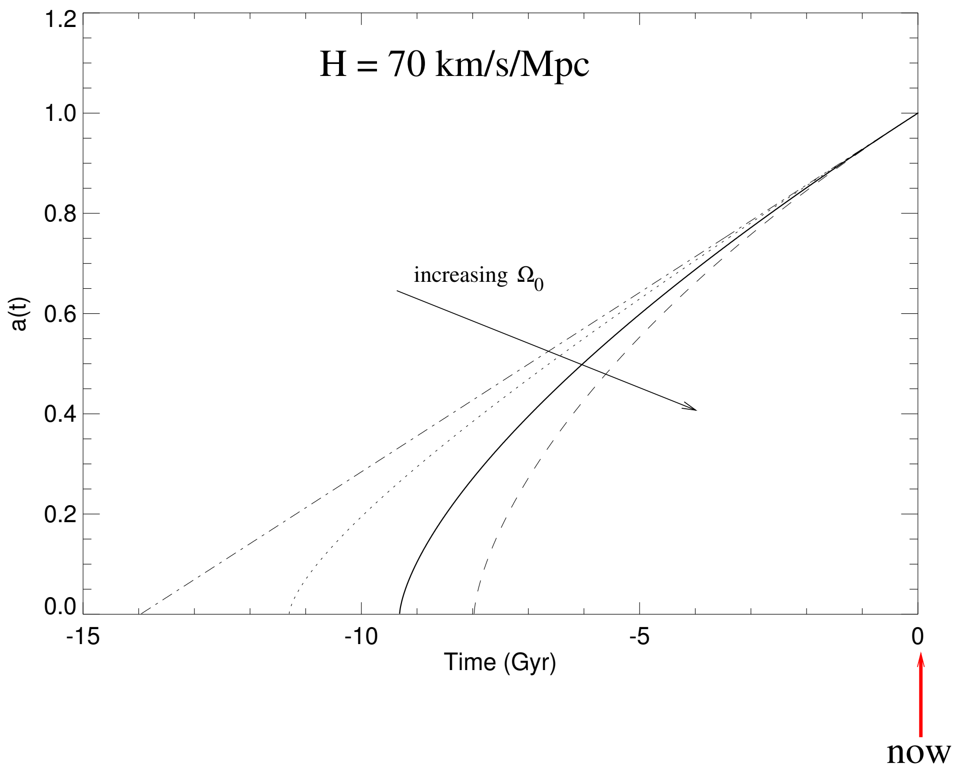

Figure 4: Calculating the age of the Universe. The plot shows the expansion factor a as a function of cosmic time t for values of $\Omega_0 = 0, 0.3, 1, 2$ (from left to right). The difference in time from $a = 0$ to $a = 1$ gives the current age of the Universe for that particular model.

Examples

3.1 The Friedmann equation for a universe with zero cosmological constant is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2},$$

where ρ is the density, a the expansion factor and k is a constant. If the universe is mass-dominated and has

$$\rho_0 = 2\rho_{\mathrm{crit},0} = \frac{3\mathrm{H}_0^2}{4\pi G}$$

today, where H_0 is the present value of the Hubble parameter, calculate by what factor the universe will continue to expand before stopping and then collapsing.

3.2 Calculate the rate of change of the expansion rate (i.e. $\ddot{a}$) for a Universe with $H_0 = 75$ km s^{-1} Mpc^{-1} and $\Omega_0 = 0.3$. Express the result in units of km s^{-1} Mpc^{-1} yr^{-1}.