University of Durham
EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2019	PHYS3621-WE01

Title:
Foundations of Physics 3A

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Attempt **all** questions. The short-answer questions at the start of each section carry 50% of the total marks for the paper. The remaining 50% of the marks are carried by the longer questions, which are equally weighted.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklets for Sections B and C, in order, inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Quantum Mechanics 3
Section B: Nuclear and Particle Physics part 1
Section C: Nuclear and Particle Physics part 2

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A: QUANTUM MECHANICS 3

1. (a) Consider two particles that do not interact with each other. We ignore spin and we let one particle be described by the single-particle wavefunction $\phi_1(\underline{r})$ and the other by the single-particle wavefunction $\phi_2(\underline{r})$. The two wavefunctions ϕ_1 and ϕ_2 are orthonormal. Give the wavefunction for the two particles $\Psi(\underline{r}, \underline{r}')$ when the two particles are (i) distinguishable, (ii) fermions, (iii) bosons.

   Taking the modulus square of $\Psi(\underline{r}, \underline{r})$ or otherwise, explain briefly the effective "exchange force" that appears to be acting between two identical particles, when these particles are described by a spatially symmetric or a spatially antisymmetric wavefunction.

   [4 marks]

   (b) (i) A time-independent Hamiltonian for an atom, H_0, is perturbed by a time-dependent Hamiltonian term $H'(t)$, such that $H'(t) = 0$ for $t < 0$. The atom for which $H_0 + H'(t)$ is the total Hamiltonian, is in the eigenstate ψ_b of H_0 (with energy E_b) for $t < 0$. In first order time dependent perturbation theory, the probability $P_{b\to a}(t)$ that the atom is found in the eigenstate ψ_a of H_0 (with energy E_a), at time $t > 0$ is given by the following equation:

$$P_{b\to a} = \frac{1}{\hbar^2} \left| \int_0^t dt' H'_{ba}(t') \exp\left(i\omega_{ba} t'\right) \right|^2 .$$

   Give the definition of the quantities $H'_{ba}(t')$, ω_{ba} appearing in this equation in terms of the eigenstates and energy levels of H_0.

   (ii) Assuming that there is no time-dependent perturbation acting on the atom and that the atomic state ψ_b can decay only by spontaneous emission to state ψ_a, with $E_b > E_a$, state how the corresponding Einstein A-coefficient (A_{ba}) is related to the lifetime τ of the state ψ_b.

   [4 marks]

   (c) A hydrogen atom is placed in a time-dependent homogeneous electric field $\underline{E}(t) = E(t)\,\hat{\underline{z}}$, where $\hat{\underline{z}}$ is the unit vector along the positive z-axis. Write the corresponding time-dependent Hamiltonian that describes the atomic electron in the electric field. The time-dependent Hamiltonian should have the form of a scalar potential energy.

   The electric field causes the transition of the $(n = 2, l = 1, m = 0)$ state of the hydrogen atom to another state characterised by the quantum numbers (n', l', m'). Give the possible values of m' and explain how you obtain them. You may use $[L_z, z] = 0$, where L_z is the z-component of the angular momentum operator.

   [4 marks]

   (d) Without any derivation, explain briefly why the $(n = 2, l = 1, m = 0)$ state of atomic hydrogen has a shorter lifetime than the $(n = 2, l = 0, m = 0)$ state in the absence of an external perturbation.

   [4 marks]

(e) As a mechanism for downward transitions in an atom, spontaneous emission competes with thermally stimulated emission (blackbody radiation). Show that at $T = 300$ K thermal stimulation dominates for frequencies well below 5×10^{12}Hz, whereas spontaneous emission dominates for frequencies well above 5×10^{12}Hz.

The spontaneous emission rate between initial state ψ_b and final state ψ_a is $A_{ba} = \omega_0^3 |\mathcal{P}|^2/(3\pi\,\epsilon_0 \hbar c^3)$ with $\mathcal{P}$ the matrix element of the electric dipole moment between the initial and final states, and $\hbar\,\omega_0 > 0$ the energy difference between the two energy levels.The emission rate stimulated by thermal (blackbody) radiation is $R_{b\to a} = \pi\,|\mathcal{P}|^2 \rho(\omega_0)/(3\epsilon_0\hbar^2)$, where $\rho(\omega_0)$ is the energy density of the thermal radiation at frequency ω_0 and

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1}.$$

You do not have to evaluate $\mathcal{P}$.

[4 marks]

2. The wavefunction for the ground state of the electron in a hydrogen atom is $\psi_{100}(r,\theta,\phi) = R_{10}(r)\,Y_{00}(\theta,\phi)$ where $R_{10}(r) = 2\,a^{-3/2}\exp(-r/a)$, $Y_{00}(\theta,\phi) = (4\,\pi)^{-1/2}$ and a is the Bohr radius.

You may use that the volume element in spherical coordinates is $r^2\,dr\,\sin\theta\,d\theta\,d\phi$ and $z = r\,\cos\theta$, $x = r\,\sin\theta\,\cos\phi$, $y = r\,\sin\theta\,\sin\phi$, $\int_0^\infty r^k\,\exp(-\alpha\,r)\,dr = k!/\alpha^{k+1}$.

(a) Explain briefly why the expectation value $\langle\psi_{100}|\underline{r}|\psi_{100}\rangle$ vanishes.
[3 marks]

The Hamiltonian of an electron in a three-dimensional harmonic oscillator potential is $H_\omega = T + V_\omega$, with ground state energy E_ω, where

$$T = -\frac{\hbar^2}{2\,m\,r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right) + \frac{L^2}{2\,m\,r^2}, \qquad V_\omega = \frac{1}{2}\,m\,\omega^2\,r^2.$$

L^2 is the square of the angular momentum operator.

(b) Using the hydrogenic wavefunction above, show that the expectation values of T, V_ω are $\langle\psi_{100}|T|\psi_{100}\rangle = \hbar^2/(2\,m\,a^2)$, $\langle\psi_{100}|V_\omega|\psi_{100}\rangle = (3/2)\,m\,\omega^2\,a^2$.
[4 marks]

(c) Explain whether the energy difference $\langle\psi_{100}|H_\omega|\psi_{100}\rangle - E_\omega$ is positive, negative or zero. Using a in $\langle\psi_{100}|H_\omega|\psi_{100}\rangle$ as a variational parameter, obtain an approximation for E_ω. Compare with the exact value of E_ω.
[4 marks]

(d) Consider the negative hydrogen ion H^-, formed when a hydrogen atom captures a second electron. Ignoring electronic repulsion and taking spin into account, write the ground state wavefunction $\Psi(\mathbf{r}_1,\sigma_1;\mathbf{r}_2,\sigma_2)$ of H^- and give its energy. Discuss the symmetry of the *space part* of Ψ under exchange of the two electrons. Explain if the *spin part* of Ψ is symmetric under exchange, or antisymmetric, or if it can be either of the two. The ground state energy of the hydrogen atom is -13.61 eV.
[4 marks]

(e) Show that the spin part of Ψ is an eigenstate of the z component of the total spin, $S_z = S_{1z} + S_{2z}$, and give the corresponding eigenvalue.
[2 marks]

(f) Write the electronic Hamiltonian for the H^- ion (including electronic repulsion) and estimate the magnitude of the electronic repulsion energy, given that the ground state energy of H^- is -14.36 eV.
[3 marks]

SECTION B: NUCLEAR AND PARTICLE PHYSICS part 1

3. (a) The semi-empirical mass formula for the mass of a nucleus with N neutrons, Z protons and atomic number $A = Z + N$ is given by:

$$M(A,Z) = NM_n + ZM_p + Z\,m_e - a_V A + a_s A^{2/3} + a_c \frac{Z^2}{A^{1/3}} + a_a \frac{(N-Z)^2}{4A} + \frac{\delta}{A^{1/2}}$$

with

$$\delta = \begin{cases} -\delta_p & \text{even-even nucleus} \\ 0 & \text{odd-even nucleus} \\ +\delta_p & \text{odd-odd nucleus} \end{cases}$$

Sketch the mass of an isobar as a function of Z for an odd and for an even value of A. Which nuclei can have both a β^+ and β^- decay? [4 marks]

(b) The order of the four lowest shells in the nuclear shell model for protons and neutrons is $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$. What are the predicted spin and parity of the following nuclei? [4 marks]

$${}^{14}_{6}\mathrm{C}\,, \quad {}^{15}_{8}\mathrm{O}$$

(c) What are all the values for the orbital angular momentum of the final state that allow the following decays if the interaction conserves parity?

(i) $2^+ \to 1^+ 0^-$

(ii) $0^- \to 1^+ 1^-$

J^P in the reactions above represents a particle with spin J and intrinsic parity P. [4 marks]

4. (a) Sketch the region of observed nuclei in a $N-Z$ plot where N is the number of neutrons and Z the number of protons. Explain why in spontaneous fission the daughter particles are often accompanied by free neutrons. [4 marks]

(b) Assuming the ideal ratio between the number of neutrons and protons is given by the function $r(A)$, show that if a large-A nucleus with this ideal ratio decays into two identical daughter particles with the ideal neutron-proton ratio and free neutrons, then the expected number of free neutrons is given by: [6 marks]

$$A\frac{r(A)-r(A/2)}{1+r(A)}\ .$$

(c) The first three levels of the $^{27}_{14}$Si nucleus are given in the following table

Energy [keV]	J^P
0	$5/2^+$
781	$1/2^+$
957	$3/2^+$

Give a list of all E1, E2, M1 and M2 transitions you expect with their energies. [6 marks]

(d) Using the shell model and given the order of the first nuclear shells:

$$1s_{1/2},\ 1p_{3/2},\ 1p_{1/2},\ 1d_{5/2},\ 2s_{1/2},\ 1d_{3/2}$$

give an explanation for the spins and parities of the 781 keV and 957 keV levels of $^{27}_{14}$Si. How would you explain a $1/2^-$ level? [4 marks]

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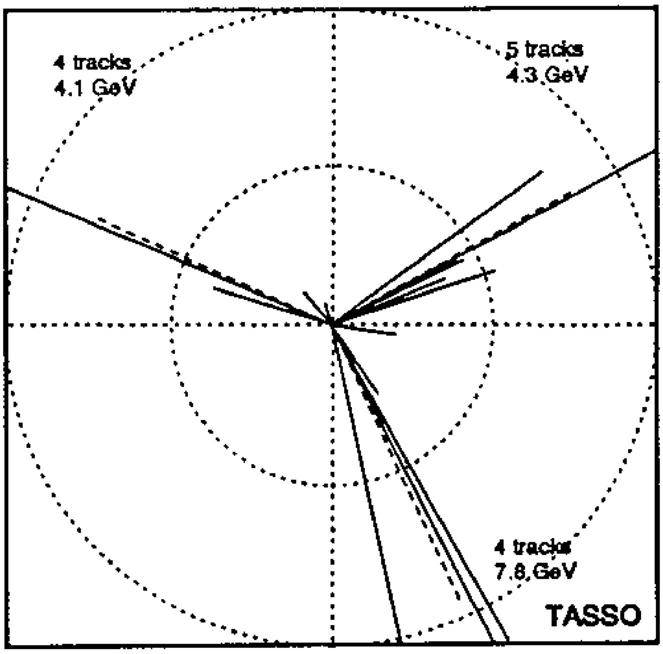
SECTION C: NUCLEAR AND PARTICLE PHYSICS part 2

5. (a) The top quark is the heaviest Standard Model particle with a mass of $m_t = 173.3$ GeV$/c^2$. The top quark is unstable and can decay into a specific gauge boson and a fermion. List the possible decays and draw the leading Feynman diagrams for these decays. State which decay is the most likely. [4 marks]

   (b) The width of the top quark is $\Gamma_t = 1.35$ GeV$/c^2$. Calculate the lifetime of the top quark in seconds, using $\hbar = 6.58 \times 10^{-25}$ GeV s. [4 marks]

   (c) Calculate the momentum of a visible photon that has a wavelength of $\lambda = 500$ nm. Calculate the velocity of an electron with the same momentum (consider the electron to be non-relativistic and the electron mass is given by $m_e = 511\,\mathrm{keV}/c^2 = 9.11 \times 10^{-31}$ kg). Calculate the energy of the photon and the kinetic energy of the electron in eV, with $1\,\mathrm{eV} = 1.6 \times 10^{-19}$ J and $hc = 1240\,\mathrm{eV}$ nm. [4 marks]

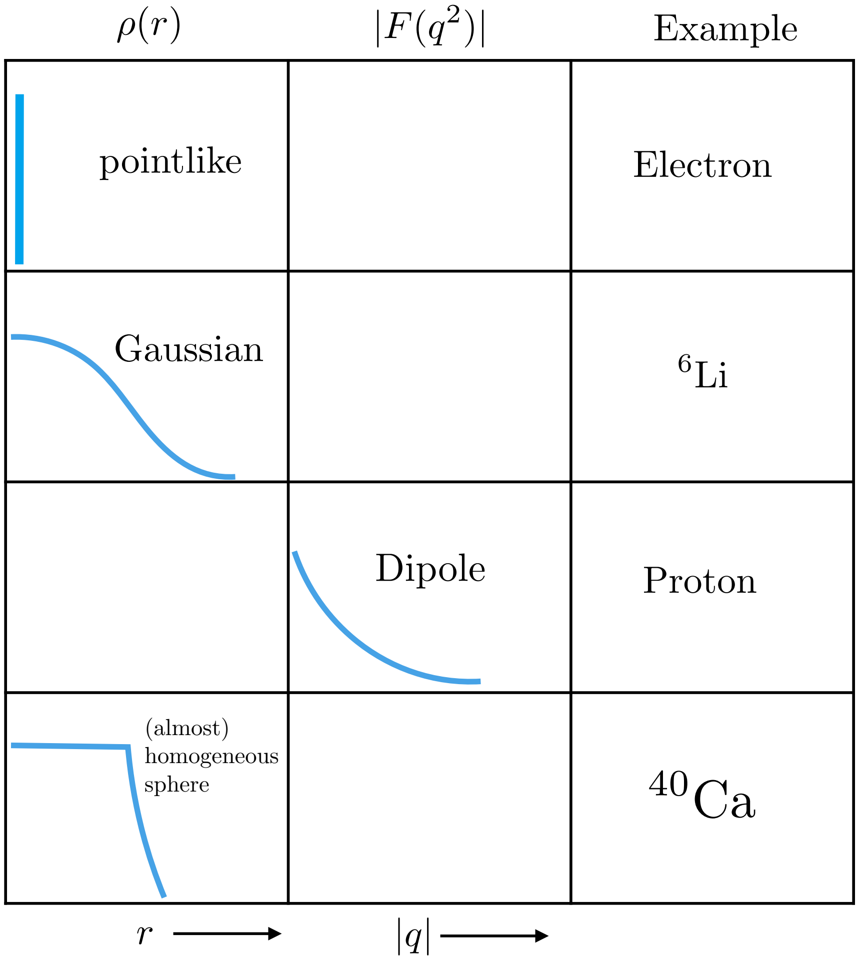
   (d) The figure below shows an event display of a three-jet final state observed in an electron-positron collider. Draw the Feynman diagram that leads to this three-jet final state. Explain why this picture is considered evidence for the existence of the gauge bosons of the strong force that couple to colour charged particles. [4 marks]



   (e) Show that one can write the form factor $F(\underline{q})$ for radially symmetric charge distributions $\rho(\underline{x}) = \rho(|\underline{x}|) \equiv \rho(r)$ as in the formula below. You can orient the coordinate system such that $\underline{q} \cdot \underline{x} = |\underline{q}| r \cos\theta$ where θ is the angle between the z axis and $\underline{x}$. [4 marks]

$$F(\underline{q}) = \int e^{i\underline{q}\cdot\underline{x}} \rho(\underline{x}) d^3x = 4\pi \int \rho(r) \frac{\sin|\underline{q}|r}{|\underline{q}|r} r^2 dr \,.$$

(f) The figure below shows the form factors for different radially symmetric charge distributions. Complete the table. [4 marks]



(g) Explain why an energetic photon γ in vacuum cannot decay into an electron and a positron $\gamma \to e^+e^-$. Prove your answer using energy and momentum conservation, where the energy of the photon is given by $E_\gamma = p_\gamma c$ and the energy of the electron is $E_-^2 = p_e^2c^2 + m_e^2c^4$, and analogously the energy of the positron reads $E_+^2 = p_p^2c^2 + m_p^2c^4$. [4 marks]

6. Imagine that you have performed an experiment and measured the cross sections for the processes

$$p + N \to \mu^+\mu^- + \text{anything}\,,$$
$$\pi^+ + N \to \mu^+\mu^- + \text{anything}\,,$$
$$\pi^- + N \to \mu^+\mu^- + \text{anything}\,,$$

where p is a proton, π^+, π^- are charged pions and N is a target nucleus with equal numbers of protons and neutrons.

(a) In the first step we consider the partonic process $q_i\bar{q}_i \to \mu^+\mu^-$. Assume that the beam is energetic enough so that we can neglect the masses of the muons and hadrons in these processes, $\hat{s} \gg m_\mu^2, m_q^2$, where $\hat{s}$ is the partonic center of mass energy, m_μ is the muon mass and m_q the quark mass. The electromagnetic cross section for muon pair production in quark-antiquark annihilation is then given by

$$\sigma(q_i\bar{q}_i \to \gamma \to \mu^+\mu^-) = \frac{4\pi}{3\hat{s}}\alpha^2 N_C\, Q_i^2\,,$$

in which Q_i is the charge of the quark q_i, α is the fine-structure constant and $N_C = 3$ is the number of colors. Using the equation above, how do you expect the differential cross section $d\sigma(q\bar{q} \to \mu^+\mu^-)/dm$ to scale with the invariant mass of the muon pair m? [2 marks]

(b) Assume that there are only valence quarks in the hadrons involved in the proton and pion scattering off the nucleus N (no x-dependence). Show that

$$\frac{d\sigma_{pN}(s,m)}{dm} : \frac{d\sigma_{\pi^+N}(s,m)}{dm} : \frac{d\sigma_{\pi^-N}(s,m)}{dm} \;=\; 0:1:4\,,$$

where s is the hadronic center of mass energy. [4 marks]

(c) Explain how the fact that the measurement to leading order agrees with the result of part (b) supports the theory of the quark substructure of the pions. [2 marks]

(d) In the presence of the Z boson, there are two additional contributions to the cross section $\sigma(q_i\bar{q}_i \to \mu^+\mu^-)$ beyond the one given in part (a). For $\hat{s} \ll M_Z^2$ these additional terms scale as

$$\sigma_Z(q_i\bar{q}_i \to \mu^+\mu^-) \propto \frac{1}{2M_Z^2}\alpha^2\,, \quad \sigma_{\gamma Z}(q_i\bar{q}_i \to \mu^+\mu^-) \propto \frac{\hat{s}}{4M_Z^4}\alpha^2\,,$$

in which M_Z is the mass of the Z boson. Draw the Feynman diagram for the Z exchange, write down the Z boson propagator and explain the scaling of the two additional contributions to the cross section. Why are there 3 contributions in total, even though there are only 2 diagrams (for a given quark q_i)? [4 marks]

(e) Explain whether there are additional contributions to $\sigma(q_i\bar{q}_i \to \mu^+\mu^-)$ from the other gauge bosons in the Standard Model: the gluons or the W^-, W^+. [2 marks]

(f) The scattering cross sections can be used to determine the sea quark content of the proton and the neutron. One can do this by replacing $q \to (1-\epsilon)q + \epsilon\bar{q}$ in the proton and neutron and repeat the calculation of part (b). Use this ansatz to show that for $\epsilon = 0.01$

$$\frac{d\sigma_{pN}(s,m)}{dm} : \frac{d\sigma_{\pi^+N}(s,m)}{dm} : \frac{d\sigma_{\pi^-N}(s,m)}{dm} \;=\; 0.17 : 1 : 3.85.$$

[6 marks]