Quantum Theory - Worksheet 6

This worksheet contains more problems than you might be able to pass through in 50 minutes. Try to complete Problems 1 and 2 during the workshop. Use Problem 3 for further practice. Problem 4 is just for interest.

Problem 1

Recall the Born rule: Suppose you have a measuring apparatus which can tell you whether or not a certain quantum system is the state described by a given ket vector $|\phi\rangle$. Suppose that just before the measurement, this system is in the state described by the ket vector $|\psi\rangle$. Then, assuming that both $|\phi\rangle$ and $|\psi\rangle$ are normalized, the probability $\Pr(|\phi\rangle; |\psi\rangle)$ that you find the system to be in the state $|\phi\rangle$ is $|\langle\phi|\psi\rangle|^2$.

Consider a set of kets $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, ..., forming an orthonormal basis for the Hilbert space $|\psi\rangle$ belongs to. Show that

$$\sum_n \Pr(|\phi_n\rangle; |\psi\rangle) = 1.$$

Important note: If $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal, finding the system in the state $|\phi_1\rangle$ and finding it in the state $|\phi_2\rangle$ are two mutually exclusive events: the system cannot be both in state $|\phi_1\rangle$ and in state $|\phi_2\rangle$, since if it was in state $|\phi_1\rangle$ there would be a zero probability of finding it in state $|\phi_2\rangle$ (zero because $|\langle\phi_2|\phi_1\rangle|^2 = 0$ as $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal). Recall the addition rule of probabilities: If P_1 is the probability that event 1 occurs and P_2 the probability that event 2 occurs, then the probability that any one of these two events occurs is $P_1 + P_2$ when these two events are mutually exclusive. If no other event is mutually exclusive with both 1 and 2, then $P_1 + P_2 = 1$. (E.g, roll a dice: the probability of finding an even number if 1/2, the probability of finding an odd number is 1/2, the probability of finding an even number or an odd number if 1/2+1/2 since being even and being odd are mutually exclusive, and this latter probability is 1 as it should be since no integer is neither even nor odd.) The equation above is just a particular instance of this general result.

Problem 2

As is explained in some of our level 4 courses, an electromagnetic field can be treated as a quantum object and be described in terms of photons rather than classical waves. For example, consider an extremely weak electromagnetic field containing only one photon, and assume that this photon could be horizontally polarized or vertically polarized (i.e., that the electric field component of this field could be directed in a horizontal direction or a vertical direction). Skipping many details, we represent these two possibilities by two orthonormal ket vectors, $|H\rangle$ and $|V\rangle$ respectively. Thus $|H\rangle$ represents the state of the field when the photon is horizontally polarized and $|V\rangle$ the state of the field when the photon is vertically polarized. $\langle H|H\rangle = \langle V|V\rangle = 1$, and $\langle H|V\rangle = 0$. By the principle of superposition, any linear combination of $|H\rangle$ and $|V\rangle$ also represents a possible state of the field (in general, such linear combinations will represent states of elliptical polarization, but do not worry about this). Let $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$, where α and β are two complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

(a) Show that the norm of $|\psi\rangle$ is 1.

(b) Is the ket vector $i|\psi\rangle$ also normalized? Do $|\psi\rangle$ and $i|\psi\rangle$ represent the same quantum state?

(c) Let $|\phi\rangle = \beta^*|H\rangle - \alpha^*|V\rangle$. Show that $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.

(d) Suppose that you check, by a measurement, whether the field is in the state represented by the vector $|\phi\rangle$. What is the probability that you find it to be in that state if at the time of this measurement the field was (i) in the state $|H\rangle$, (ii) in the state $|\psi\rangle$?

(e) Assume that any state the photon could be in can be represented by an elements of the vector space spanned by $|H\rangle$ and $|V\rangle$.

  (i) Is it the case that $|\psi\rangle$ and $|\phi\rangle$ form a basis for this space for any values of α and β (as long as α and β are not both zero)?

  (ii) Let $|\eta\rangle$ be a certain element of this space. You are told that $|\eta\rangle$ is orthogonal to $|\phi\rangle$ and is not the zero vector. Why can you then be sure that $|\eta\rangle$ represents the same state as $|\psi\rangle$? [Hint: Recall Problem 1 and the reasoning for part (b) of this problem.]

(f) Now, consider a pair of photons (photon a and photon b), in the joint state described by the ket vector $|\Psi^-\rangle_{ab} = (|H\rangle_a|V\rangle_b - |V\rangle_a|H\rangle_b)/\sqrt{2}$.

  (i) Show that ${}_{ab}\langle\Psi^-|\Psi^-\rangle_{ab} = 1$.

  (ii) Suppose that polarization measurements are made on photons a and b. What is the probability of finding both photon a and photon b horizontally polarized? What is the probability of finding photon a horizontally polarized and photon b vertically polarized? What is the probability of finding photon a horizontally polarized irrespective of the polarization of photon b? (Perhaps you can immediately see what these probabilities should be, from the definition of $|\Psi^-\rangle_{ab}$; however, for the sake of the exercise, calculate them step by step, using the Born rule.)

Problem 3

(Do Problem 2 before this one.) Alice produces photon A in the state $\alpha|H\rangle_A + \beta|V\rangle_A$, and Bob produces photons B and C in the entangled state $|\Psi^-\rangle_{BC} = (|H\rangle_B|V\rangle_C - |V\rangle_B|H\rangle_C)/\sqrt{2}$. As in Problem 2, $|\alpha|^2 + |\beta|^2 = 1$. Having studied what quantum opticians call "quantum state teleportation", Alice and Bob try to "teleport" the state of photon A to you — the state of the photon, not the photon itself! To this effect, Alice sends photon A to Bob while Bob sends photon C to you and keeps photon B for himself. Then, by doing an appropriate measurement, Bob checks whether photons A and B are in the entangled state $|\Psi^-\rangle_{AB} = (|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B)/\sqrt{2}$. He is lucky and finds them to be in that state. Calling you on the phone, he tells you that you can be sure that photon C is in the state in which photon A was before that measurement.

(a) Why can Bob say so, and what was the probability of the result he found in that measurement? [Hint: Calculate the inner product of $(\alpha|H\rangle_A + \beta|V\rangle_A)|\Psi^-\rangle_{BC}$, (i) with $|\Psi^-\rangle_{AB}(\beta^*|H\rangle_C - \alpha^*|V\rangle_C)$, and (ii) with $|\Psi^-\rangle_{AB}(\alpha|H\rangle_C + \beta|V\rangle_C)$.]

(b) Any ket vector describing a joint state of photons A and B can written as a linear combination of the four orthonormal product states $|H\rangle_A|H\rangle_B$, $|H\rangle_A|V\rangle_B$, $|V\rangle_A|H\rangle_B$ and $|V\rangle_A|V\rangle_B$. These four product states thus form an orthonormal basis for the Hilbert space of the joint states of these two photons. Other choices of basis are of course possible, too. For example, the following entangled states also form an orthonormal basis for that Hilbert space:

$$\begin{aligned}
|\Psi^+\rangle_{AB} &= (|H\rangle_A|V\rangle_B + |V\rangle_A|H\rangle_B)/\sqrt{2},\\
|\Psi^-\rangle_{AB} &= (|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B)/\sqrt{2},\\
|\Phi^+\rangle_{AB} &= (|H\rangle_A|H\rangle_B + |V\rangle_A|V\rangle_B)/\sqrt{2},\\
|\Phi^-\rangle_{AB} &= (|H\rangle_A|H\rangle_B - |V\rangle_A|V\rangle_B)/\sqrt{2}.
\end{aligned}$$

(i) It might be that you immediately see that these four vectors are orthogonal to each other and that each of them is of unit norm. If not, calculate, e.g., ${}_{AB}\langle\Phi^+|\Phi^+\rangle_{AB}$ and ${}_{AB}\langle\Psi^+|\Phi^+\rangle_{AB}$. That they form a basis is established by the fact that they are linearly independent (since they are orthogonal to each other) and that there are four of them (which is the dimension of the space).

(ii) Write $|H\rangle_A|H\rangle_B$, $|H\rangle_A|V\rangle_B$, $|V\rangle_A|H\rangle_B$ and $|V\rangle_A|V\rangle_B$ in terms of $|\Psi^+\rangle_{AB}$, $|\Psi^-\rangle_{AB}$, $|\Phi^+\rangle_{AB}$ and $|\Phi^-\rangle_{AB}$. E.g.,

$$|H\rangle_A|H\rangle_B = (|\Phi^+\rangle_{AB} + |\Phi^-\rangle_{AB})/\sqrt{2}.$$

(iii) Note that

$$\begin{aligned}
(\alpha|H\rangle_A + \beta|V\rangle_A)|\Psi^-\rangle_{BC} = {}&\\
&\alpha|H\rangle_A|H\rangle_B|V\rangle_C/\sqrt{2} -\\
&\alpha|H\rangle_A|V\rangle_B|H\rangle_C/\sqrt{2} +\\
&\beta|V\rangle_A|H\rangle_B|V\rangle_C/\sqrt{2} -\\
&\beta|V\rangle_A|V\rangle_B|H\rangle_C/\sqrt{2}.
\end{aligned}$$

Use the results of part (b)(ii) to show that

$$\begin{aligned}
(\alpha|H\rangle_A + \beta|V\rangle_A)|\Psi^-\rangle_{BC} = {}&\\
&|\Psi^+\rangle_{AB}(-\alpha|H\rangle_C + \beta|V\rangle_C)/2 +\\
&|\Psi^-\rangle_{AB}(-\alpha|H\rangle_C - \beta|V\rangle_C)/2 +\\
&|\Phi^+\rangle_{AB}(\ \ \alpha|V\rangle_C - \beta|H\rangle_C)/2 +\\
&|\Phi^-\rangle_{AB}(\ \ \alpha|V\rangle_C + \beta|H\rangle_C)/2.
\end{aligned}$$

Is this result consistent with what your answer for part (a)?

Problem 4

By definition, the trace of a square matrix is the sum of its diagonal elements. It is possible to show that the trace of a product of two matrices does not depend on the order of the matrices in the product, even if these matrices do not commute. I.e., if P and Q are two $N \times N$ matrices, $\mathrm{Tr}(\mathsf{PQ}) = \mathrm{Tr}(\mathsf{QP})$.

Let us write

$$\mathsf{QP} - \mathsf{PQ} = i\hbar\mathsf{M},$$

where M is a $N \times N$ matrix. Use the information above to show that M cannot be the $N \times N$ unit matrix. [Hint: The unit matrix is the matrix whose diagonal components are all 1 and off-diagonal components are all 0.]

Note: This result shows that the commutation relation $[\hat{Q}, \hat{P}] = i\hbar$ between the position and momentum operators would not be possible if these operators were acting in a finite-dimensional Hilbert space.