Problem 1

(a) $$\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix} = \begin{pmatrix} 2\times\sqrt{2} + i\sqrt{2}\times(-2i) \\ -i\sqrt{2}\times\sqrt{2} + 3\times(-2i) \end{pmatrix} = \begin{pmatrix} 4\sqrt{2} \\ -8i \end{pmatrix}.$$

(b) $$\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix} = \begin{pmatrix} 4\sqrt{2} \\ -8i \end{pmatrix} = 4\begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix}.$$

Hence this column vector is indeed an eigenvector of this matrix. The corresponding eigenvalue is 4.

$$\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ i \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ i \end{pmatrix} = 1\begin{pmatrix} \sqrt{2} \\ i \end{pmatrix}.$$

Hence this column vector is also an eigenvector of this matrix. The corresponding eigenvalue is 1.

(c) The inner product of these two column vectors is

$$\begin{pmatrix} \sqrt{2} & i \end{pmatrix}^* \begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -i \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix} = \sqrt{2}\times\sqrt{2} + (-i)\times(-2i)$$
$$= 2 - 2 = 0.$$

Note the complex conjugation of the row vector!

Note: These calculations illustrate a general property of "Hermitian matrices", i.e., matrices which are equal to their complex-conjugated transpose matrix: such matrices have real eigenvalues and eigenvectors belonging to different eigenvalues are orthogonal.

Worksheet 2 - Problem 2

(i) $$S_n = \sin\theta\cos\phi\,\frac{\hbar}{2}\sigma_x + \sin\theta\sin\phi\,\frac{\hbar}{2}\sigma_y + \cos\theta\,\frac{\hbar}{2}\sigma_z$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

where in the last step we have used $e^{i\phi} = \cos\phi + i\sin\phi$ and $e^{-i\phi} = \cos\phi - i\sin\phi$.

Now, $$S_n \chi_\uparrow = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta\cos\theta/2 + \sin\theta\sin\theta/2 \\ (\sin\theta\cos\theta/2 - \cos\theta\sin\theta/2)\, e^{i\phi} \end{pmatrix}$$

Since $\cos\theta = \cos^2\theta/2 - \sin^2\theta/2$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$,

$$\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} = \cos^3\theta/2 - \sin^2\theta/2\cos\theta/2 + 2\sin^2\theta/2\cos\theta/2 = \cos\theta/2$$

$$\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\theta/2 = 2\sin\theta/2\cos^2\theta/2 - \cos^2\theta/2\sin\theta/2 + \sin^3\theta/2 = \sin\theta/2$$

Hence $$S_n\chi_\uparrow = \frac{\hbar}{2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{pmatrix} = \frac{\hbar}{2}\chi_\uparrow.$$

Thus $\chi_\uparrow$ is an eigenstate of S_n with eigenvalue $\frac{\hbar}{2}$.

Likewise,

$$S_n \chi_\downarrow = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}\begin{pmatrix}\sin\theta/2 \\ -\cos\frac{\theta}{2} e^{i\phi}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta\sin\theta/2 - \sin\theta\cos\theta/2 \\ (\sin\theta\sin\theta/2 + \cos\theta\cos\theta/2)e^{i\phi}\end{pmatrix}$$

Since $\cos\theta\sin\theta/2 - \sin\theta\cos\theta/2 =$
$\cos^2\theta\sin\theta/2 - \sin^3\theta/2 - 2\sin\theta/2\cos^2\theta/2 = -\sin\theta/2$

$\sin\theta\sin\theta/2 + \cos\theta\cos\theta/2 =$
$2\sin^2\theta/2\cos\theta/2 + \cos^3\theta/2 - \sin^2\theta/2\cos\theta/2 = \cos\theta/2,$

$$S_n \chi_\downarrow = \frac{\hbar}{2}\begin{pmatrix}-\sin\theta/2 \\ \cos\theta/2\, e^{i\phi}\end{pmatrix} = -\frac{\hbar}{2}\chi_\downarrow .$$

Thus $\chi_\downarrow$ is an eigenstate of S_n with eigenvalue $-\frac{\hbar}{2}$.

$\chi_\uparrow$ and $\chi_\downarrow$ are orthogonal to each other:

$$\langle\chi_\uparrow|\chi_\downarrow\rangle = \begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi}\end{pmatrix}\begin{pmatrix}\sin\theta/2 \\ -\cos\frac{\theta}{2}e^{i\phi}\end{pmatrix} = \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0.$$

note the complex conjugation in the "bra" vector.

$\chi_\uparrow$ and $\chi_\downarrow$ are normalized to unity:

$$\langle\chi_\uparrow|\chi_\uparrow\rangle = \begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi}\end{pmatrix}\begin{pmatrix}\cos\theta/2 \\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix} = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

$$\langle\chi_\downarrow|\chi_\downarrow\rangle = \begin{pmatrix}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}e^{-i\phi}\end{pmatrix}\begin{pmatrix}\sin\theta/2 \\ -\cos\frac{\theta}{2}e^{i\phi}\end{pmatrix} = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1.$$

(b) $\hat{n} \equiv \hat{x}$ for $\theta = \pi/2$ and $\phi = 0$. For the x-direction, we thus have

$$\chi_\uparrow = \begin{pmatrix} \cos \pi/4 \\ \sin \pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$, which the eigenstate of S_x with eigenvalue $\frac{\hbar}{2}$

$$\chi_\downarrow = \begin{pmatrix} \sin \pi/4 \\ -\cos \pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$, which is the eigenstate of S_x with eigenvalue $-\frac{\hbar}{2}$.

$\chi_\uparrow$ and $\chi_\downarrow$ are not orthonormal to α and β, the eigenstates of S_z :

$$\langle \alpha | \chi_\uparrow \rangle = (1 \quad 0) \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \neq 0$$

$$\langle \alpha | \chi_\downarrow \rangle = (1 \quad 0) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \neq 0$$

$$\langle \beta | \chi_\uparrow \rangle = (0 \quad 1) \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \neq 0$$

$$\langle \beta | \chi_\downarrow \rangle = (0 \quad 1) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = -\frac{1}{\sqrt{2}} \neq 0.$$

(c) $S_y = \frac{\hbar}{2} \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

The expectation value required is

$$\frac{1}{\sqrt{2}} (1, -1) \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} (1, -1) \frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ i \end{pmatrix}$$

$$= \frac{\hbar}{4} (i - i) = 0.$$

The second part of (c) requires a little thinking:

By definition of the expectation value, this quantity is, in the present case,

$(\frac{\hbar}{2}) \times$ (probability that the particle is in the eigenstate of S_y with eigenvalue $\hbar/2$) +
$(-\hbar/2) \times$ (probability that the particle is in the eigenstate of S_y with eigenvalue $-\hbar/2$),

since S_y has only $\hbar/2$ and $-\hbar/2$ as eigenvalues. (That, in general, S_n has no other eigenvalues that $\hbar/2$ and $-\hbar/2$, can be seen from the fact that a 2×2 matrix has at most two eigenvalues.) Given the result of the first part of (c), this sum is zero. Hence the two probabilities are equal, and since they must sum to 1, each one is equal to 1/2.

(d) The column vector representing the state of spin up is $\begin{pmatrix}1\\0\end{pmatrix}$. Hence, the probability of finding $X_{n\uparrow}$ is

$$\left|\left(\cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\, e^{-i\phi}\right)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \cos^2\theta/2.$$

note the complex conjugation in the row vector

Likewise, the probability of finding $X_{n\downarrow}$ is $\sin^2\theta/2$.

Worksheet 2 Problem 3.

$\chi_\uparrow(t)$ satisfies the T.D.S.E. $i\hbar \frac{\partial \chi_\uparrow(t)}{\partial t} = H\chi_\uparrow(t)$

with $\chi_\uparrow(t=0) = \cos\frac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix}$. ⊛

$H = -\frac{\hbar\gamma B}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ The eigenstates of H are $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ with energies $E_\uparrow = -\frac{\hbar\gamma B}{2}$, $E_\downarrow = +\frac{\hbar\gamma B}{2}$.

The T.D. state will be from ⊛:

$$\chi_\uparrow(t) = \cos\frac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix}e^{-iE_\uparrow t/\hbar} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix}e^{\frac{-iE_\downarrow t}{\hbar}}$$

$$= \cos\frac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix}e^{i\gamma Bt/2} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix}e^{-i\gamma Bt/2}$$

$$= e^{i\gamma Bt/2}\left[\cos\frac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\theta}{2}e^{i(\phi-\gamma Bt)}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

which is what we wanted for $\omega = -\gamma B$.

Problem 4

$$\vec{S}^2 = \vec{S}\cdot\vec{S} = S_x S_x + S_y S_y + S_z S_z$$

$$= \frac{\hbar^2}{4}\left(\sigma_x\sigma_x + \sigma_y\sigma_y + \sigma_z\sigma_z\right)$$

where σ_x, σ_y, σ_z are the familiar Pauli matrices.

Explicitely:

$$\vec{S}^2 = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$