Mathematical Methods in Physics

Weekly Problems 7

7.1

The position vector $\mathbf{r} = x\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of a mass $(m = 1)$ subject to a central force $\mathbf{F} = f(r)\mathbf{r}$ satisfies the following equation of motion:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} = f(r)\mathbf{r} \ ,$$

where $f(r)$ is a scalar function of the modulus r of $\mathbf{r}$. Show that in this case the quantity $\mathbf{L}$ below is conserved.

$$\mathbf{L} = \mathbf{r} \times \frac{d\mathbf{r}}{dt} \ .$$

7.2

a) Find a parametric representation of the straight line represented by

$$x + y + z = 1, \quad y - z = 0.$$

b) What curve is represented by the following parametric expression?

$$\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{t}\,\mathbf{k}.$$

Write down the equation of the curve.

7.3

Find the gradient of the following functions

a) $f_2 = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{r}$ is the position vector and $\mathbf{c} = c_x\,\mathbf{i} + c_y\,\mathbf{j} + c_z\,\mathbf{k}$ is a constant vector.

b) $f_4 = e^{-\alpha r}/r$, where α is a scalar and r is the modulus of the position vector.

[*Hint: In part d) you could use the chain rule,* $\nabla(\psi(r)) = \psi'(r)\nabla r$, *together with the result* $\nabla r = \mathbf{r}/r$ *seen in the lecture.*]

7.4

The origin of the cartesian coordinates is at the Earth's centre. The moon is on the z-axis, a fixed distance R away (centre-to-centre distance). The tidal force exerted by the moon on a particle at the Earth's surface (point x, y, z) is given by

$$\mathbf{F} = -GMm\,\frac{x}{R^3}\,\mathbf{i} - GMm\,\frac{y}{R^3}\,\mathbf{j} + GMm\,\frac{z}{R^3}\,\mathbf{k}.$$

Find the potential ϕ that yields this tidal force.
[*Hint: Since the exercise is dealing with a force use* $\mathbf{F} = -\nabla\phi$.]