ELECTROMAGNETISM

Level 2 Physics problems – Foundations of physics 2

Solution 2 Cycle 2 Version 1

Professor D P Hampshire – 2nd Year Physics Lecture Course

Information underlined or indicated by red text is required for marks to awarded. The mark scheme is a guide and solutions should not be considered to be unique. Marks are awarded for correct relevant Physics.

1. Evaluate each term separately, starting inside brackets and working way outwards,

a) LHS,

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5yz & 2y^2x & -z \end{vmatrix} = 5y\hat{\mathbf{j}} + (2y^2 - 5z)\hat{\mathbf{k}}$$

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 5y & (2y^2 - 5z) \end{vmatrix} = 4y\hat{\mathbf{i}}$$ 1-1

RHS,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = 4yx - 1$$

$$\underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}}) = 4y\hat{\mathbf{i}} + (4x)\hat{\mathbf{j}}$$

$$\nabla^2 \underline{\boldsymbol{A}} = 4x\hat{\mathbf{j}}$$

Both sides together,

$$=> \underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}}) - \nabla^2 \underline{\boldsymbol{A}} = 4y\hat{\mathbf{i}} = \underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}})$$ 1-2

LHS equals RHS so relationship holds.

b) LHS,

$$fg = x^3yz^2 - xy^2z$$

$$\underline{\boldsymbol{\nabla}}(fg) = (3x^2yz^2 - y^2z)\hat{\mathbf{i}} + (x^3z^2 - 2xyz)\hat{\mathbf{j}} + (2x^3yz - xy^2)\hat{\mathbf{k}}$$ 1-3

RHS,

$$\underline{\boldsymbol{\nabla}} g = (2xz)\hat{\mathbf{i}} - \hat{\mathbf{j}} + (x^2)\hat{\mathbf{k}} \text{ and } \underline{\boldsymbol{\nabla}} f = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

$$f\underline{\boldsymbol{\nabla}} g = (2x^2yz^2)\hat{\mathbf{i}} - (xyz)\hat{\mathbf{j}} + (x^3yz)\hat{\mathbf{k}}$$

$$g\underline{\nabla}f = (x^2yz^2 - y^2z)\hat{\mathbf{i}} + (x^3z^2 - xyz)\hat{\mathbf{j}} + (x^3yz - xy^2)\hat{\mathbf{k}}$$

$$=> f\underline{\nabla}g + g\underline{\nabla}f$$

$$= (3x^2yz^2 - y^2z)\hat{\mathbf{i}} + (x^3z^2 - 2xyz)\hat{\mathbf{j}} + (2x^3yz - xy^2)\hat{\mathbf{k}}$$ 1-4

LHS equals RHS so relationship holds.

c) LHS,

$$\underline{\nabla}f = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

$$\underline{\nabla} \times \underline{\nabla}f = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\hat{\mathbf{i}} - (y - y)\hat{\mathbf{j}} + (z - z)\hat{\mathbf{k}} = 0$$ 1-5

LHS equals RHS so relationship holds.

1 mark for each part. Both sides of equation must be calculated explicitly.

d)

$$(\underline{\nabla} \times \underline{A}) = 5y\hat{\mathbf{j}} + (2y^2 - 5z)\hat{\mathbf{k}}, \text{ as in part (a)}$$

$$\underline{B} \cdot (\underline{\nabla} \times \underline{A}) = -(2y^2z^2 - 5z^3 - 2x^2y^2 + 5x^2z)$$ 1-6

1 mark for correct function for 1-6
[Qn 1: 4 marks total]

2. From (0,0,0)→(1,0,0): y = 0, dy = 0, z = 0, dz = 0

$$=> \int \underline{A}.d\underline{r} = \int (x\hat{\mathbf{i}}) \cdot (dx\hat{\mathbf{i}}) = \int_0^1 xdx = \frac{1}{2}$$ 2-1

From (1,0,0)→(1,1,0): x = 1, dx = 0, z = 0, dz = 0

$$=> \int \underline{A}.d\underline{r} = \int (\hat{\mathbf{i}} + y^2\hat{\mathbf{k}}) \cdot (dy\hat{\mathbf{j}}) = 0$$ 2-2

From (1,1,0)→(1,1,1): x = 1, dx = 0, y = 1, dy = 0

$$=> \int \underline{A}.d\underline{r} = \int (\hat{\mathbf{i}} + 2z\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (dz\hat{\mathbf{k}}) = \int_0^1 dz = 1$$ 2-3

So by summing each line segment the total line integral $= \frac{1}{2} + 1 = \frac{3}{2}$ 2-4

1 mark if 2-1, 2-2, 2-3 and 2-4 all correct.

The line integral will be independent of path only if $\underline{\boldsymbol{A}} = \underline{\boldsymbol{\nabla}} f$ i.e. $\underline{\boldsymbol{A}}$ is the gradient of some scalar function f. Integrating the three components of $\underline{\boldsymbol{A}}$ gives:

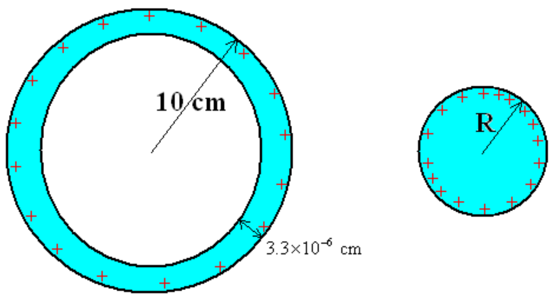
$$\int A_x dx = \frac{x^2}{2} + g(y,z)$$
$$\int A_y dy = y^2 z + h(x,z)$$
$$\int A_z dz = y^2 z + m(x,y)$$

So, if $f = y^2 z + \frac{x^2}{2}$, then $\underline{\boldsymbol{\nabla}} f = \underline{\boldsymbol{A}}$. 2-5

=> The integral is independent of path.

1 mark if statement to similar effect of 2-5 is written.
[Qn 2: 2 marks total]

3. Assumption: all charge is distributed on the outer surface of the drop. The potential at the surface is equivalent to the potential that would be due to a point charge of the same magnitude located at the centre of the drop.



The potential a distance r from a charge q is given by,

$$\phi = \frac{q}{4\pi\varepsilon_0 r}$$

Originally, $\phi = 100$ V and $r = 0.2$ m 3-1

$$=> q = 4\pi\varepsilon_0 r\phi = 2.22 \times 10^{-9}\ \mathrm{C}$$

1 mark for correct charge

Volume at start,

$$V = \frac{4}{3}\pi((0.2)^3 - (0.2 - 3.3 \times 10^{-8})^3)$$
$$= 1.659 \times 10^{-8}\ \mathrm{m}^3$$

Volume at end = volume at start = $V = {}^4/_3\, \pi R^3$

$$=> R = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = 1.582 \times 10^{-3}\ \mathrm{m}$$

So the potential of the new drop, $\phi = \frac{q}{4\pi\varepsilon_0 R} = 1.2 \times 10^4$ V

1 mark for correct final potential 3-2

[Qn 3: 2 marks total]

4. The energy of a sphere is given by,

$$dU = \int \underline{\boldsymbol{F}} \cdot d\underline{\boldsymbol{S}} = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

The potential energy is given by,

$$V = \frac{dU}{q_2} = \frac{q_1}{4\pi\varepsilon_0 r} = 5 \times 10^3\ \mathrm{V}$$

By rearrangement can determine q_1 4-1

$$=> q_1 = (5 \times 10^3)(1.1 \times 10^{-10})(0.1) = 5.5 \times 10^{-8}\ \mathrm{C}$$

Using Gauss' law,

$$\int \underline{\boldsymbol{E}}.d\underline{\boldsymbol{S}} = \sum \frac{Q}{\varepsilon_0} => E = \frac{5 \times 10^3}{0.1} = 5 \times 10^4\ \mathrm{Vm}^{-1}$$ 4-2

1 mark for correct charge and electric field

At 20 cm:

$$V = 2.5 \times 10^3\ \mathrm{V}, E = 1.25 \times 10^4\ \mathrm{Vm}^{-1}$$ 4-3

1 mark for correct potential and electric field

[Qn 4: 2 marks total]

Total for all questions 10 marks