Workshop 6: Hamiltonian Mechanics

1. Two particles of different mass, m_1 and m_2, are connected by a massless spring of spring constant k and equilibrium length d. The system lies on a horizontal, frictionless, table and may both oscillate and rotate. Use the definitions of the centre of mass position and relative coordinate,
$$\underline{R} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{M}, \quad \underline{r} = \underline{r}_2 - \underline{r}_1$$
respectively, where $M = m_1 + m_2$, to show that
$$\frac{1}{2}M\dot{\underline{R}}^2 + \frac{1}{2}\mu\dot{\underline{r}}^2 = \frac{1}{2}m_1\dot{\underline{r}}_1^2 + \frac{1}{2}m_2\dot{\underline{r}}_2^2,$$
where $\mu = m_1m_2/M$ is the reduced mass.
Hence show that the Lagrangian can be written as
$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{k}{2}(r-d)^2,$$
where r and ϕ represent the relative separation of the two masses in polar coordinates. Find the Hamiltonian of the system and Hamilton's equations of motion.

2. Suppose a bug of mass m is crawling on a turntable rotating arbitrarily around an axis perpendicular to its plane. The bug's polar coordinates relative to the turntable are r and ϕ, whereas in the inertial, lab frame, they are $r_{\text{lab}} = r$ and $\phi_{\text{lab}} = \phi + \theta(t)$, where $\theta(t)$ is the angle between the two coordinate systems and the turntable rotates anticlockwise when viewed from above.

   In mixed coordinates, the Lagrangian of the bug is
$$L = \frac{m}{2}v_{\text{lab}}^2 - V(r,\phi),$$
where v_{lab} is the speed of the bug in the lab frame and $V(r,\phi)$ represents an arbitrary potential, expressed in terms of its polar coordinates.

   (a) Why should we use v_{lab}^2 and not v^2 in the Lagrangian?
   (b) Substitute the rotating coordinates into the expression for the lab kinetic energy in the Lagrangian then find the canonically conjugate momenta p_r and p_ϕ.
   (c) Calculate the bug's Hamiltonian in terms of r, ϕ, p_r, p_ϕ. Prove that, for arbitrary variation of θ with time,
$$H = H_{\text{lab}} - \dot{\theta}p_\phi,$$
   where H_{lab} is the bug's Hamiltonian if $\dot{\theta} = 0$.