University of Durham

EXAMINATION PAPER

May/June 2012 Examination code: 043541/01 or 044131/01

LEVEL 3 PHYSICS: ASTROPHYSICS
LEVEL 4 PHYSICS: ASTROPHYSICS 4

SECTION A. PLANETARY SYSTEMS
SECTION B. COSMOLOGY

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. PLANETARY SYSTEMS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Describe the principal characteristics of the long and short period comets. From where in the Solar System are they thought to have originated? [4 marks]

   (b) Calculate the transfer time for a satellite, via a Hohmann transfer orbit, going between a circular orbit about the Earth's equator with a period of 90 minutes, and a geostationary orbit with a period of 1436 minutes. Give the answer in minutes. [4 marks]

   (c) Draw a diagram showing the locations of all five Lagrange points relative to two massive bodies, with a mass ratio of thirty to one, orbiting each other in a circular orbit. State for each Lagrange point whether it is stable or not. [4 marks]

   (d) The Roche radius of a planet of radius R is given by the formula:

$$r_R = 2.44\Big(\frac{\rho_p}{\rho_m}\Big)^{1/3} R.$$

   Explain the physical significance of the Roche radius and define the symbols ρ_p and ρ_m. Calculate the Roche radius of Saturn in km with respect to its giant moon Titan. Assume Titan has a uniform density of 1880 kg m^{-3}. [4 marks]

   [Mass of Saturn $M_{\text{Saturn}} = 5.7 \times 10^{26}$ kg]

   (e) The Moon has a moment of inertia factor, α_p of 0.39. What does this value imply about the internal structure of the Moon and the size of any hypothetical iron core? [4 marks]

   (f) Make an estimate for the amount of energy stored in the Earth's magnetic field. At the Earth's surface the typical field strength is 4×10^{-5} T. State any assumptions that you need to make. [4 marks]

   [Radius of the Earth $R_{\text{E}} = 6371$ km]

   (g) Show by applying the principle of hydrostatic equilibrium to a plane parallel isothermal atmosphere made of an ideal gas in a uniform gravitational field, that pressure is given by: $P \propto \exp(-h/H)$, where h is altitude, and H is a constant. Give the value of H for an atmosphere made of molecular hydrogen at 300 K at the Earth's surface. [4 marks]

   (h) The dwarf planet Eris is in an orbit with maximum and minimum distances from the Sun of 98 AU and 38 AU respectively. Estimate the period and eccentricity of its orbit. [4 marks]

2. (a) Write down the Kepler equation and define each variable with the help of a clearly labelled diagram. Use the same diagram to define the true anomaly. [4 marks]

   (b) The Oort spacecraft is sent initially on a parabolic orbit passing close by the Sun and is attached to and surrounded by a light and highly reflective heat shield. Assuming the heat shield radiates like a blackbody and tumbles rapidly, estimate how close the spacecraft can approach the Sun if the effective temperature of the heat shield cannot exceed 600 K. Assume that the shield is spherical and has an albedo of 0.96. [6 marks]

   (c) When the spacecraft reaches perihelion, at a distance of 8×10^6 km from the centre of the Sun, a rocket motor is fired for a short time using all of the fuel and accelerating the spacecraft in the direction of its velocity vector as seen in the solar frame. Taking the exhaust velocity of the rocket to be 3 km s^{-1}, and the spacecraft to be initially 75% fuel by mass, calculate to two significant figures the speed of the spacecraft in the future when it is very far from the Sun. [8 marks]

   (d) The final kinetic energy of the spacecraft in the solar frame for the situation described in part (c) is much higher than would be the case if the rocket had been fired much later when the spacecraft was already far from the Sun. Explain why that is. [2 marks]

3. (a) Describe qualitatively how seismic waves are used to probe the Earth's interior structure and give an account of the main results. Include a labelled diagram illustrating the interior structure of the Earth as part of your answer. [6 marks]

(b) (i) Modelling the Earth as a uniform sphere of radius R and density ρ, show by integrating the equation of hydrostatic equilibrium from the centre to the surface that this gives a difference in pressure of $2\pi G\rho^2R^2/3$. [4 marks]

(ii) The Earth rotates slowly about an axis with angular speed Ω. Show that if the equation of hydrostatic equilibrium is integrated from the centre to the surface in the equatorial plane, in the rotating frame of the Earth, then the difference in pressure is given instead by
$2\pi G\rho^2R^2/3 - \Omega^2\rho R^2/2$. [2 marks]

(c) The Earth is not spherical but oblate. Show that if the Earth is modelled as a uniform density but slightly oblate spheroid, with a polar axis of $R(1-\epsilon)$ and equatorial axes of $R(1+\epsilon/2)$ then hydrostatic equilibrium is possible for a special value of ϵ. Express this value of ϵ in terms of Ω, R, and the surface gravity g for the spherical Earth model. Make the simplifying assumption that the gravitational field for this model is the same as for the spherical Earth model. Assume ϵ is small and ignore second and higher order terms in ϵ. [6 marks]

[Hint: Use the results for part b(i) and b(ii) and require the pressure difference between the centre of the Earth and the surface of the Earth to be the same at the poles as at the equator.]

(d) When the calculation in part (c) is done properly by accounting for the change in the gravitational field caused by the change in shape of the model Earth, the resulting value for ϵ is significantly larger than is actually measured for the Earth. Why is that? [2 marks]

4. (a) Explain what gravitational microlensing is and how it can be used to detect exoplanets. Include in your answer a clearly labelled diagram to show how microlensing works. [6 marks]

   (b) The Kepler satellite discovers a planet that transits a particular star every 7.20 days. The transit itself lasts 3.62 hours measured from first to last contact, and the light curve suggests the radius of the planet is a thirtieth that of the star and that the planet transits across the diameter of the stellar disc precisely. Assume the planet is on a circular orbit about the star and is much less massive than the star. Show using only the information above that the mean density of the star is approximately 1410 kg m^{-3}. [6 marks]

   (c) Follow up spectroscopic measurements of the reflex motion of the star reveal that the star shows a sinusoidal variation in radial velocity of amplitude 4.8 m s^{-1}. By assuming the star is very similar to the Sun, estimate the mass and size of the planet, and show that its mean density would be close to 1660 kg m^{-3}. [6 marks]

   [Radius of the Sun = 696000 km.]

   (d) Describe the likely interior structure of the exoplanet by analogy with the closest matching planet(s) in the Solar System. [2 marks]

SECTION B. COSMOLOGY

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) The present-day value of the Hubble parameter is $75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Express this in units of year^{-1} and $\mathrm{km\,hour^{-1}\,light\text{-}year^{-1}}$. [4 marks]

   (b) Define the following quantities in terms of the expansion factor a, its time derivatives and fundamental constants: redshift (z); the Hubble parameter (H); critical density ($\rho_{\rm crit}$); and the deceleration parameter (q_0). [4 marks]

   (c) The Friedmann equation for a matter-dominated flat universe (with negligible cosmological constant) is

$$H^2 = \frac{8\pi G}{3}\rho,$$

   where H is the Hubble parameter and ρ the matter density. Show that the age of the universe at redshift z is

$$t(z) = \frac{2}{3H_0}(1+z)^{-3/2},$$

   where H_0 is the present-day value of the Hubble parameter. [4 marks]

   (d) The Friedmann equation for an open matter-dominated universe (with negligible cosmological constant) is

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2},$$

   where H is the Hubble parameter, ρ is the matter density, a is the expansion factor and k is a constant.

   Show that as $a \to 0$, the matter density approaches the critical density. [4 marks]

   (e) In a matter-dominated, critical density universe (with negligible cosmological constant), the flux received from an object of luminosity L emitting isotropically at redshift z is given by

$$f = \frac{L}{4\pi r^2 (1+z)^2}$$

   where r is the comoving distance to redshift z. Explain this result and give the reason for the $(1+z)^2$ factor in the denominator. [4 marks]

   (f) Consider a flat universe with present-day matter density equal to one tenth of the critical density and with the remaining energy density in the form of a vacuum energy density. Assuming the vacuum has an equation of state $P_{\rm vac} = -\rho_{\rm vac}c^2$, use the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

   where a is the expansion factor, P is the pressure and ρc^2 the total energy density, to calculate the redshift at which the expansion of the universe began to accelerate. [4 marks]

(g) During inflation, the universe has a large (positive) vacuum energy density ($\rho_{\mathrm{vac}} c^2$). Assume that just prior to inflation the universe has a temperature of $10^{40}\,\mathrm{K}$ and that inflation continues for $10^{-34}\,\mathrm{s}$ with $\rho_{\mathrm{vac}} c^2 = 10^{108}\,\mathrm{GeV\,m^{-3}}$. Estimate the temperature of the universe just before the end of inflation. [4 marks]

[Hint: assume that the energy density is at all times dominated by relativistic particles and that g_*, the effective number of bosonic degrees of freedom remains constant.]

6. (a) Outline the early history of the universe as it cools from $kT \sim 10\,\mathrm{GeV}$, explaining the following events and placing them in the correct order: "quark-hadron phase transition","recombination", "nucleosynthesis", "matter-radiation equality". [8 marks]

(b) The reactions that convert neutrons to protons freeze out at a temperature of T_{fo}. Given that the resulting ratio, R, of neutrons to protons is 0.16, estimate T_{fo} (ignore free neutron decay). [3 marks]

(c) The fraction, f_D, of these neutrons that end up locked in deuterium nuclei can be shown to be given by

$$f_D = 1.8 \times 10^{-6} \Omega_{b,0}^{-1.4}$$

where $\Omega_{b,0}$ is the present baryon density expressed in units of the critical density.

Assuming that all of the other primordial neutrons end up locked in ^{4}He nuclei, and ignoring free neutron decay, show that the primordial ratio of the number of deuterium to hydrogen nuclei, ^{2}D/H, is $f_D R/(1-R)$. [4 marks]

[Hint: note that the fraction of neutrons locked into deuterium is very much smaller than the number locked into ^{4}He.]

(d) Hence, stating any assumptions you make, estimate $\Omega_{b,0}$ if a deuterium to hydrogen abundance ratio of $^2\mathrm{D/H} = 2.5 \times 10^{-5}$ is measured in a high-redshift quasar absorption system. [5 marks]

[The rest masses of the proton and neutron are $m_p = 1.6726 \times 10^{-27}\,\mathrm{kg}$ and $m_n = 1.6749 \times 10^{-27}\,\mathrm{kg}$.]

7. (a) Briefly describe three problems with the hot Big Bang cosmological model. State why these problems can be solved by postulating a period of inflation when the universe has a temperature of $T \sim 10^{40}\,\mathrm{K}$. [9 marks]

(b) When $z > 1000$, the universe contains a hot ionized plasma which has a sound speed of $c/\sqrt{3}$. At $z = 1000$ the plasma recombines and the emitted photons travel freely to an observer who sees them as the Cosmic Microwave Background (CMB). Assume that the universe is flat and matter dominated (and that the cosmological constant is negligible), and take the present-day value of the Hubble parameter, H_0, to be $75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

(i) If $z > 1000$, show that maximum co-moving distance travelled by the sound wave is

$$l_{\mathrm{sound}} = \frac{2c}{H_0\sqrt{3}}(1+z)^{-1/2}$$

Hence calculate the maximum comoving distance over which a sound wave can propagate prior to $z = 1000$. [7 marks]

(ii) Given that the comoving distance from an observer at redshift $z = 0$ to the plasma at redshift $z = 1000$ is approximately $2c/H_0$, calculate the maximum angular scale of features of the CMB produced by sound waves that began propagating at the Big Bang. Express your answer in degrees. [4 marks]

8. (a) Briefly describe three pieces of evidence that convince astronomers that the universe must contain much more mass than is associated with stars. [6 marks]

(b) (i) Gas in hydrostatic equilibrium in the gravitational potential of a cluster of galaxies obeys the relation

$$\frac{1}{\rho_g}\frac{dP}{dr} = -\frac{GM}{r^2}.$$

State the meaning of the symbols used in the equation. [3 marks]

(ii) The thermal pressure of the X-ray plasma that fills a galaxy cluster is

$$\frac{\rho_g k_B T}{0.6 m_p}$$

(where T is the temperature at radius r) and its X-ray emissivity (luminosity per unit volume) is

$$4.8 \times 10^{-25} \left(\frac{\rho_g}{10^{-21}\,\mathrm{kg\,m^{-3}}}\right)^2 \left(\frac{T}{10^7\,\mathrm{K}}\right)^{1/2} \mathrm{J\,s^{-1}\,m^{-3}}.$$

A particular galaxy cluster is observed to have a temperature profile

$$T(r) = 1.0 \times 10^7 \left(\frac{r}{1\,\mathrm{Mpc}}\right)^{-1} \mathrm{K}$$

and X-ray emissivity profile

$$\epsilon_X(r) = 2.0 \times 10^{-30} \left(\frac{r}{1\,\mathrm{Mpc}}\right)^{-2} \mathrm{J\,s^{-1}\,m^{-3}}$$

Determine the gas density and pressure at $r = 1\,\mathrm{Mpc}$ and hence determine total mass within this radius. [8 marks]
[Hint: note that $\frac{dP}{dr} = \frac{P}{r}\frac{d\ln P}{d\ln r} = \alpha\frac{P}{r}$ if $P \propto r^{\alpha}$.]

(iii) A measurement of the cluster mass based on the Virial Theorem suggests that the total mass is larger than determined from the X-ray data. Discuss two assumptions that might account for the discrepancy. [3 marks]