University of Durham

EXAMINATION PAPER

May/June 2011 Examination code: 043551/01 or 044191/01

LEVEL 3 PHYSICS: THEORETICAL PHYSICS
LEVEL 4 PHYSICS: THEORETICAL PHYSICS 4

SECTION A. QUANTUM MECHANICS
SECTION B. SPECIAL RELATIVITY AND ELECTROMAGNETISM

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Nuclear magneton:	$\mu_{\mathrm{N}} = 5.05 \times 10^{-27}\ \mathrm{J\,T^{-1}}$
Molar Gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$

SECTION A. QUANTUM MECHANICS

Answer Question 1 and **at least one** of Questions 2, 3 and 4.

1. (a) Give the definition of a Hermitian operator. [2 marks]

   Show that Hermitian operators have real eigenvalues. [2 marks]

   (b) Consider a linear harmonic oscillator in 1-dimension whose Hamiltonian is $\hat{H} = \frac{\hat{P}_x^2}{2m} + \frac{1}{2}m\omega_0^2\hat{Q}_x^2$, with mass m and frequency ω_0. $\hat{P}_x$ and $\hat{Q}_x$ are the position and momentum operators, respectively. We introduce $\lambda_0 \equiv \sqrt{\hbar/(m\omega_0)}$ and the operator $\hat{a} \equiv \frac{1}{\lambda_0\sqrt{2}}\hat{Q}_x + i\frac{\lambda_0}{\hbar\sqrt{2}}\hat{P}_x$. Show that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Here, n are the eigenvalues of the operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$ and $|n\rangle$ the corresponding eigenstates. [4 marks]

   (c) For the linear harmonic oscillator in (b), using $\hat{a}|0\rangle = 0$ show that the eigenfunction for the ground state is $\psi_0(x) = \mathcal{N}\exp(-x^2/(2\lambda_0^2))$, where $\mathcal{N}$ is a normalisation constant. [4 marks]

   (d) State the generalised uncertainty relation for two operators $\hat{A}$ and $\hat{B}$. [2 marks]

   Consider the angular momentum operator $\underline{\hat{L}}$ and a system which is in an eigenstate of $\hat{L}_z$, $|l, m\rangle$. What is the minimum uncertainty in a simultaneous measurement of $\hat{L}_x$ and $\hat{L}_y$? [2 marks]

   (e) Consider the Hamiltonian $\hat{H} = \underline{\hat{L}}^2/(2I)$, where I is a constant. Its normalized eigenstates are the eigenstates of $\hat{L}^2$ and $\hat{L}_z$ indicated as $|l\,, m\rangle$. A system is described by the state

   $$|\psi\rangle = \mathcal{N}\left(2|1\,,1\rangle + |1\,,-1\rangle + |2\,,0\rangle\right) \ ,$$

   with $\mathcal{N}$ a normalisation constant to be determined. What are the possible outcomes of a measurement of the energy and with what probability? [4 marks]

   (f) Consider a hydrogen atom. What are the eigenvalues of the Hamiltonian? Discuss the degeneracy of the corresponding eigenstates. [2 marks]

   If the hydrogen atom is put in a constant and uniform magnetic field, discuss qualitatively what the effect will be on the energies and on the degeneracy of the states. [2 marks]

(g) The Born approximation for the scattering amplitude for central potentials, $V(r)$, is $f^{\mathcal{B}}(\Omega) = -\frac{2\mu}{\hbar^2 q}\int_0^\infty dr\, rV(r)\sin(qr)$, where $\underline{q} = \underline{k} - \underline{k}'$. Here $|\underline{k}'| = |\underline{k}[$ and Ω indicates the angles θ and ϕ in spherical coordinates and μ is the reduced mass of the system.

Find the differential cross section $\frac{d\sigma}{d\Omega}$ for the screened Coulomb potential $V_{sC}(r) = -\frac{e^{-ra}}{r}$, with a constant. [2 marks]

[Hint: You can use the integral $\int_0^\infty dr\; e^{-ra}\sin(qr) = \frac{q}{q^2+a^2}$.]

Using the previous result, compute the differential cross section $\frac{d\sigma}{d\Omega}$ for the Coulomb potential $V_C(r) = -\frac{1}{r}$ and compare it with the exact result:

$$\frac{d\sigma}{d\Omega} = \frac{(2\mu)^2}{(2\hbar k)^4}\frac{1}{\sin^4(\theta/2)} \ ,$$

where $q = 2k\sin(\theta/2)$. [2 marks]

2. Three of the six quarks are the "up", "down" and "strange" quarks. Their states are denoted by $|u\rangle$, $|d\rangle$ and $|s\rangle$, respectively. They are orthogonal and normalised. A system composed of these objects is governed by the Hamiltonian

$$\hat{H} = \hbar\omega_0 \left(|u\rangle\langle d| + |d\rangle\langle u|\right) + \hbar\omega_1 |s\rangle\langle s| \ ,$$

where ω_0 and ω_1 are two constants.

(a) Determine the matrix representation of $\hat{H}$ in the $|u\rangle$, $|d\rangle$, $|s\rangle$ basis. What are its eigenvalues? [3 marks]

(b) Check that the eigenstates can be expressed in terms of $(|u\rangle + |d\rangle)$, $(|u\rangle - |d\rangle)$ and $|s\rangle$ and properly normalise them. [3 marks]

(c) At time $t = 0$, the system is described by the state $|\psi\rangle = |d\rangle$. What is the probability of finding it in the same state $|d\rangle$ at a later time t? [6 marks]

(d) Consider now a series of subsequent measurements of the state in (c) at times T, $2T$, $3T$,... nT, with n a natural number. By definition $\tau \equiv nT$. What is the probability of finding the system in the $|d\rangle$ state in every measurement, from the one at T to the one at nT? What is this probability in the limit $n \to \infty$ at fixed τ? [4 marks]

[Hint: The limit for $x \to \infty$ of $(\cos(1/x))^x$ is 1.]

(e) An observable A is described by the operator

$$\hat{A} = \hbar\omega_2 |u\rangle\langle u| + \hbar\omega_3 \left(|d\rangle\langle s| + |s\rangle\langle d|\right) \ ,$$

where ω_2 and ω_3 are two constants. Can the energy and A be measured simultaneously with infinite precision for a system in the state $|d\rangle$? [4 marks]

3. Consider a non-relativistic particle of mass m in a 3D delta-shell potential. The Hamiltonian is given by

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V}(r) \qquad \text{with} \qquad V(r) = -\frac{\hbar^2 g_\delta^2}{2m}\delta(r-a) \,,$$

where g_δ is a constant, r indicates the radial coordinate and a is a constant.

(a) Using the definition of the angular momentum, $\hat{L}_i = \sum_{jk} \epsilon_{ijk} \hat{Q}_j \hat{P}_k$, show that $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$. Here, $\hat{Q}_i$ and $\hat{P}_i$ are the position and momentum operators in the i-direction. What does the fact that $\hat{L}_x$ and $\hat{L}_y$ do not commute imply for their simultaneous measurement? [4 marks]

Consider bound states, i.e. states with negative energies. The eigenfunctions of the Hamiltonian can be expressed in spherical coordinates as $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$, where $Y_{lm}(\theta,\phi)$ are the spherical harmonics. The reduced radial equation for $u_{nl}(r) = rR_{nl}(r)$ is

$$\left(-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right) + V(r)\right) u_{nl}(r) = E_{nl}\, u_{nl}(r).$$

(b) Discuss the asymptotic behaviour of $u_{nl}(r)$ for $r \to 0$ and for $r \to \infty$. [6 marks]

(c) Consider $l = 0$. Write down the continuity condition for the wavefunction at $r = a$. [1 mark]

Derive the discontinuity relation for the derivative of the eigenfuntion, which for this potential reads

$$\frac{d}{dr}(R_{n0})_{r<a}(a) - \frac{d}{dr}(R_{n0})_{r>a}(a) = g_\delta^2 R_{n0}(a) \,.$$

In order to do so: (i) integrate in dr both sides of the reduced radial equation given above for $l = 0$ in the interval $[a-\Delta, a+\Delta]$. (ii) Then, take the limit for $\Delta \to 0$. [5 marks]

(d) For $l = 0$, we have that $(R_{n0})_{r<a}(r) = A(\exp(\alpha r) - \exp(-\alpha r))/r$ and $(R_{n0})_{r>a}(r) = B\exp(-\alpha r)/r$, where A and B are constants and $\alpha \equiv \sqrt{2m|E_0|/\hbar^2}$. By using the two boundary conditions at $r = a$ given in (c), discuss if there can be bound states, depending on the value of $g_\delta^2\, a$. [4 marks]

4. Work in the matrix representation with respect to the eigenbasis of the operator $\hat{S}_z$, given by $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Consider a free electron with mass m_e, charge $-e$ and spin 1/2.

   (a) Write the matrix representation of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$. [3 marks]

   (b) What are the possible outcomes of a measurement of S_x? Show that the corresponding eigenstates in the chosen matrix representation are

   $$|S_x = \pm\hbar/2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} .$$

   [3 marks]

   (c) S_x is measured for a large number N of electrons, all prepared in the same state. It is found that $\hbar/2$ is measured $N/3$ times and $-\hbar/2$ is measured $2N/3$ times. What is the state which describes the electrons just before the measurement? [3 marks]

At $t = 0$, the electron is in an eigenstate of $\hat{S}_x$ with eigenvalue $\hbar/2$. A constant uniform magnetic field $\vec{B} = (0, B, 0)$ is switched on. Focus only on the spin part. The relevant term of the Hamiltonian is

$$\hat{H}_S = 2\frac{\mu_B}{\hbar}\hat{\vec{S}} \cdot \vec{B} ,$$

where μ_B is the Bohr magneton.

   (d) Given that the eigenstates of S_y are $|S_y = \hbar/2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $|S_y = -\hbar/2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$, calculate the probability of finding the electron with its spin pointing in the positive z-direction at a later time t. [8 marks]

   (e) Compute σ_x^n with n even and n odd. Here σ_x is the first Pauli matrix. Using the definition of the exponential of an operator in terms of its Taylor expansion, show that $\exp(i\theta\sigma_x) = \cos\theta + i\sin\theta\sigma_x$, where θ is a constant angle. [3 marks]

   [Hint: You will need to use the fact that $\cos\theta = \sum_m (-1)^m \theta^{2m}/(2m)!$ and $\sin\theta = \sum_m (-1)^m \theta^{2m+1}/(2m+1)!$.]

SECTION B. SPECIAL RELATIVITY AND ELECTROMAGNETISM

Answer Question 5 and **at least one** of Questions 6, 7 and 8.

5. (a) State the definition of an inertial frame. Show that a frame moving with constant velocity with respect to an inertial frame is also an inertial frame. [4 marks]

   (b) The polarization tensor for a vector boson with mass m and 4-momentum p^μ is
$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2c^2}\right).$$
   Calculate $p_\mu T^{\mu\nu}$ and $\epsilon_{\alpha\beta\mu\nu}T^{\mu\nu}$. [4 marks]

   (c) A tau lepton with mass m_τ decays in its rest frame to a charged pion with mass m_π and a massless tau neutrino. Calculate the energy of the charged pion in the rest frame of the tau lepton. [4 marks]

   (d) State the definition of the 4-force f^μ. Use the definition to express the components of f^μ in terms of the energy E and three momentum $\underline{p}$ and their derivatives with respect to time. [4 marks]

   (e) How does the 4-potential A^μ transform under gauge transformations? Use this to show that the electric and magnetic fields do not change under a gauge transformation. [4 marks]

   (f) The 4-potential for a parallel-plate capacitor oriented normal to the z axis is $A^\mu = (Ez, 0, 0, 0)$ where E is the constant electric field between the plates and z the distance from the negatively charged plate. Show that, in a frame moving with velocity v in the x direction, there is a magnetic field in the y direction and calculate its magnitude. [4 marks]

   (g) Express the 0-component of the Maxwell equation
$$\partial_\mu F^{\mu\nu} = \frac{j^\nu}{c\epsilon_0}$$
   in terms of the electric and magnetic fields. [4 marks]

   [Hint: In terms of the electric and magnetic fields the field strength tensor is given by:
$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$
   ]

   (h) Using the covariant form of the inhomogeneous Maxwell equation derive the wave equation for the 4-potential A^μ using the Lorentz gauge. [4 marks]

6. Consider the decay of a particle X to two particles Y and Z. Show that the magnitude of the three-momentum of either Y or Z in the rest frame of X is

$$|\underline{p}_Y| = |\underline{p}_Z| = \frac{c}{2m_X}\sqrt{(m_X^2 - (m_Y + m_Z)^2)\,(m_X^2 - (m_Y - m_Z)^2)},$$

where m_X, m_Y and m_Z are the masses of X, Y and Z respectively. [8 marks]

The top quark decays in its rest frame S to the bottom quark and the W^+ boson. The three-momenta of the bottom quark and W^+ boson are in the (x, y) plane and the bottom quark moves at an angle θ with respect to the x axis.

Calculate the energies of the bottom quark and W^+ boson in a frame S' moving with velocity v with respect to S along the x axis in terms of v, θ, the top quark mass m_t and W^+ boson mass m_W. The mass of the bottom quark can be neglected. [6 marks]

The probability of the bottom quark having a value of θ between θ and $\theta + \mathrm{d}\theta$ is

$$\mathrm{d}P = \frac{1}{2}(1 - \cos\theta)\sin\theta \mathrm{d}\theta.$$

Calculate the average energy of the bottom quark and W^+ boson in S' and comment on the value you obtain for the sum of the average energies of the bottom quark and W^+ boson. [6 marks]

7. The wave 4-vector for a light wave with frequency f and wavevector $\underline{k}$ is

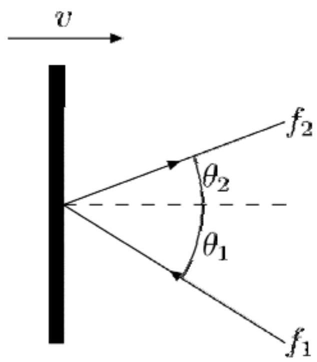
$$k^{\mu} = \left(\frac{2\pi f}{c}, \underline{k}\right).$$

Show that the plane wave $\Phi = \exp\left(ik^{\mu}x_{\mu}\right)$, where x^{μ} is the position 4-vector, is a solution of the wave equation $\partial^{\mu}\partial_{\mu}\Phi = 0$. [4 marks]

A light source in the frame S emits light with a frequency f at an angle θ with respect to the x-axis. Using the Lorentz transformation of the wave 4-vector, or otherwise, calculate the frequency measured by an observer in the frame S' moving with velocity $\underline{v}$ in the x direction with respect to S. [6 marks]

What is the frequency for $\theta = 0°$ and $\theta = 90°$. [2 marks]

A mirror with its normal in the x direction is at rest in S'. An observer in S sees a ray of light with frequency f_1 strike the mirror at an angle of incidence θ_1. The light is then reflected with frequency f_2 at an angle of reflection θ_2 as shown below.



By boosting the wave 4-vectors of the light into S' and using the normal relations for reflection in that frame, or otherwise, show that f_2/f_1 can be written in all of the following forms

$$\frac{f_2}{f_1} = \frac{\sin\theta_1}{\sin\theta_2} = \frac{c\cos\theta_1 + v}{c\cos\theta_2 - v} = \frac{c + v\cos\theta_1}{c - v\cos\theta_2}.$$

[8 marks]

8. Give the definition of the electric field $\underline{E}$ and the magnetic field $\underline{B}$ in terms of the scalar potential Φ, and the vector potential $\underline{A}$. [4 marks]

   State the definition of the field strength tensor in terms of the 4-potential $A^\mu = (\Phi, c\underline{A})$. [2 marks]

   Show that the field strength tensor can be written in terms of the electric and magnetic fields as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

   [6 marks]

   In the rest frame S' of a medium the charge density is ρ_0. The current density $\underline{J}'$, is related to the electric field $\underline{E}'$ by Ohm's law $J' = \sigma E'$ where σ is the conductivity.

   The frame S' moves with velocity $\underline{v}$ with respect to the frame S. In S the 4-current is

$$j^\mu = av^\mu + \frac{\sigma}{c} F^{\mu\nu} v_\nu,$$

   where a is a constant and $v^\mu = \gamma(c, \underline{v})$ is the 4-velocity of the medium with $\gamma = \left(1 - \frac{|\underline{v}|^2}{c^2}\right)^{-1/2}$.

   Compute a in terms of the quantities in S' by calculating $j^\mu v_\mu$ and using the Lorentz invariance of the Minkowski scalar product. [2 marks]

   [Hint: In the frame S', $j'^\mu = (\rho_0 c, \sigma\underline{E}')$]

   Calculate the 3-current density $\underline{J}$ in S in terms of the electric field $\underline{E}$, and the magnetic field $\underline{B}$, in S, and the relative velocity $\underline{v}$ between S' and S. Interpret your result. [6 marks]