Mathematical Methods II
Lecture 16

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Key Points

- General solutions to PDEs
- 2nd order PDEs

General Solutions to PDEs (ctud)

- **1st order PDEs:** Recalling the general form for a 1st order PDE

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y),$$

we have looked at solutions for:

i) $A = 0$ or $B = 0$,

ii) $C = 0$ and $R = 0$,

iii) $R = 0$,

iv) no terms are zero.

The first scenario something of a special case. The second and third scenarios are variations on solving homogeneous PDEs with and without a term in u, which can be extended to inhomogeneous problems in the fourth scenario, as discussed in Lecture 15. The techniques used up to now allow us to come up with a solution to essentially any 1st order PDE with 2 independent variables.

- **2nd order PDEs:** Let's move on to higher orders. A given PDE could potentially be of any order and contain any number of independent variables. However, we are going to restrict ourselves to 2nd order PDEs with just two independent variables. This is still a very broad class of equations so we are going to place another greatly simplifying restriction on our PDEs: our coefficients will be constant. The general equation for a 2nd order PDEs with two independent variables and constant coefficients is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x,y),$$

or in Lagrange notation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = R(x, y).$$

As we have already noted, a number of important physical relationships fall into this category, such as:

- the 1D wave equation,
- the 1D diffusion/heat equation,
- the 1D Schrodinger equation,
- the 2D Laplace equation.

But we're going to simplify again and only consider equations where $D = E = F = 0$, i.e.

$$Au_{xx} + Bu_{xy} + Cu_{yy} = R(x, y),$$

starting initially with the homogeneous problem, where $R = 0$ also. Notice this equation only contains 2$^{\text{nd}}$ order terms, so the following technique will not work for equations that include additional 1$^{\text{st}}$ order terms or terms in u. In other words, this method will work for the wave equation and Laplace equation, but not the diffusion equation or Schrodinger equation.

Let's begin as we did previously, by looking for a solution of the form $u(x, y) = f(p)$ where $p = p(x, y)$. For 1$^{\text{st}}$ order PDEs we were able to produce an expression that could be factorised for $f(p)$, i.e.

$$\left(A\frac{\partial p}{\partial x} + B\frac{\partial p}{\partial y}\right)\frac{df(p)}{dp} = 0.$$

This allowed us to eliminate all references to the particular form of $f(p)$. We reasoned that $f'(p) = 0$ only led to a trivial solution, so we can set the bracket equal to 0 instead and find that p must be constant since it is a function of two variables, but $f(p)$ is only a function of one. This restricted the form of p, hence restricting $f(p)$ and ultimately $u(x, y)$.

We would like to do something similar this time. Since we are dealing only with 2$^{\text{nd}}$ order terms and assuming a solution of $u(x, y) = f(p)$, by analogy with the earlier technique we would like to arrive at an equation that has a common term of $d^2f(p)/dp^2$. Let's consider our derivatives again,

$$u(x, y) = f(p) \qquad\qquad u(x, y) = f(p)$$

$$\frac{\partial u}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x} \qquad\qquad \frac{\partial u}{\partial y} = \frac{df(p)}{dp}\frac{\partial p}{\partial y}$$

These are our 1$^{\text{st}}$ order derivatives. Remember, our goal is to eliminate references to $f(p)$ from our PDE so that we don't need to consider its particular form. If we differentiate the above derivatives to obtain 2$^{\text{nd}}$ order derivatives, the product rule is going to leave us with terms in both 1$^{\text{st}}$ order and 2$^{\text{nd}}$ order derivatives of $f(p)$. Hence, factorising $f(p)$ out via a single term that we can set equal to zero will not be possible.

This can be remedied if we require that

$$\frac{\partial p}{\partial x} = a \quad \text{and} \quad \frac{\partial p}{\partial y} = b \qquad \Rightarrow \qquad \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y^2} = 0,$$

where a and b are constants.

If the partial derivatives of p are constant then integrating the total derivative leads us directly to the form of p

$$\begin{aligned} dp &= \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \\ dp &= a dx + b dy \\ p &= ax + by. \end{aligned}$$

i.e. p must be a linear function of x and y, so our solution will be of the form

$$u = f(ax + by).$$

Let's quickly look at the effect this has on the derivatives, using the x derivatives as an example

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{df(p)}{dp} \frac{\partial p}{\partial x} = a \frac{df(p)}{dp} \\ \frac{\partial^2 u}{\partial x^2} &= a \frac{\partial}{\partial x} \frac{df(p)}{dp} = a \frac{d^2 f(p)}{dp^2} \frac{\partial p}{\partial x} = a^2 \frac{d^2 f(p)}{dp^2} \end{aligned}$$

So if we assume a solution of this form then we can evaluate our 1st and 2nd order derivatives ready to be substituted into the equation

$$\frac{\partial u}{\partial x} = a \frac{df(p)}{dp} \qquad\qquad \frac{\partial u}{\partial y} = b \frac{df(p)}{dp}$$

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{d^2 f(p)}{dp^2} \qquad \frac{\partial^2 u}{\partial x \partial y} = ab \frac{d^2 f(p)}{dp^2} \qquad \frac{\partial^2 u}{\partial y^2} = b^2 \frac{d^2 f(p)}{dp^2}$$

Subbing these into the PDE

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

$$A a^2 \frac{d^2 f(p)}{dp^2} + B ab \frac{d^2 f(p)}{dp^2} + C b^2 \frac{d^2 f(p)}{dp^2} = 0$$

$$\left(A a^2 + B ab + C b^2\right) \frac{d^2 f(p)}{dp^2} = 0.$$

This is the form we were looking for. We can ignore the solution to this equation where $d^2 f(p)/dp^2 = 0$. Since the solution to any 2nd order derivative that equals zero is just a first order polynomial and $u = f(ax+by)$, then this leads us to the solution $u = kx+ly+m$. This is a trivial solution, since it means that each individual 2nd order derivative in our PDE is zero, which will obviously give a $RHS = 0$.

Instead we can find a solution independent of the form of $f(p)$ if we require that the constants a and b satisfy the equation

$$Aa^2 + Bab + Cb^2 = 0.$$

This is a quadratic equation, which we can solve. First divide by a^2

$$A + B\frac{b}{a} + C\frac{b^2}{a^2} = 0 \quad \rightarrow \quad A + B\lambda + C\lambda^2 = 0$$

Thus the factor b/a is given by the two solutions of

$$\frac{b}{a} = \lambda = \frac{-B \pm (B^2 - 4AC)^{1/2}}{2C}$$

If we call these two ratios λ_1 and λ_2 then any functions of the two variables p_1 and p_2

$$p_1 = x + \lambda_1 y \qquad\qquad p_2 = x + \lambda_2 y$$

will be solutions of the original PDE. Absorbing the a term into the λ terms is fine, since only the relative weighting of x and y in p is important. The particular form of the solution can be adjusted based on the given problem. You might consider then that the we have arrived at a solution of the form $u = f(p)$ where $p = x + (b/a)y$, which is equivalent to our assumption of $p = ax + by$.

Finally, as p_1 and p_2 are in general not the same, the general solution is given by

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

where f and g are arbitrary functions. This is a very simple solution, as in order to find find u we merely need to examine the coefficients of the derivatives.

e.g. 16.1 The general solution to the 1D wave equation: Find the general solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

First, we identify the coefficients as $A = 1$, $B = 0$ and $C = -1/c^2$. λ_1 and $\lambda 2$ are solutions to the following equation

$$A + B\lambda + C\lambda^2 = 0,$$
$$1 - \frac{\lambda^2}{c^2} = 0.$$

So λ_1 and λ_2 are given by

$$\lambda = \pm c.$$

Or using the quadratic formula

$$\begin{aligned}\lambda &= \frac{-B \pm (B^2 - 4AC)^{1/2}}{2C}\\ &= \frac{\pm(4/c^2)^{1/2}}{-2/c^2}\\ &= \frac{\pm 2/c}{-2/c^2}\\ &= \pm c\end{aligned}$$

Hence, $\lambda_1 = c$ and $\lambda_2 = -c$, thus the solution contains the forms of

$$p_1 = x + \lambda_1 t = x + ct, \qquad\qquad p_2 = x + \lambda_2 t = x - ct,$$

giving a general solution of

$$u(x, y) = f(x + ct) + g(x - ct),$$

where f and g are arbitary functions.

- **Limitations of the method:** Above we mentioned that this method will work for the wave equation and Laplace equation, but not the diffusion equation or the Schrodinger equation. We can see that this is because the latter have 1$^{\text{st}}$ order derivatives as well as 2$^{\text{nd}}$ order. So $f(p)$ cannot be eliminated. e.g. Let's examine the wave equation and diffusion equation to find an equation of the form

$$\left(Aa^2 + Bab + Cb^2\right) \frac{d^2 f(p)}{dp^2} = 0.$$

By substituting the derivatives in terms of a and b we found earlier into these equations, we find

The wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

$$\left(a^2 - \frac{1}{c^2}b^2\right)\frac{d^2 f(p)}{dp^2} = 0,$$

The diffusion equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{k^2}\frac{\partial u}{\partial t} = 0,$$

$$a^2\frac{d^2 f(p)}{dp^2} - \frac{1}{k^2}b\frac{df(p)}{dp} = 0.$$

Clearly $f(p)$ can only be eliminated for the wave equation, by assuming the bracket is equal to zero. Hence, this method will not work for the diffusion equation.