Relativistic Electrodynamics, Workshop 2

Exercise 1

The explicit form of a certain contravariant tensor of rank two, $F^{\mu\nu}$, is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

Use the general rule for raising and lowering indices find the explicit expression of the corresponding mixed tensor $F^{\mu}{}_{\nu}$ and the covariant tensor $F_{\mu\nu}$.

Exercise 2

Consider two rockets A and B of rest length $L_0 = 1$m travelling in opposite directions along the x-axis towards each other with a tiny shift in the y-direction so they do not collide. Each sees the other approach it with speed u. According to A, when the tail of B passed the tip of A, a missile was fired from the tail of A towards B. It will clearly miss due to length contraction of B as seen by A. (Ignore the time it takes the missile to travel across the off-set in the trajectories of the rockets).

But analysed in the frame of B, the rocket A will be length contracted. Analyse what this means for any conclusion B might arrive at, on whether the missile fired by A will hit B.

Choose the two frames of the rockets such that their origin is at the tip of each rocket. It will be most educational to assign spacetime coordinates to five events in each frame: tip of A passes tip of B (set it to $(0, \underline{0})$ for both frames), tail of B passes tip of A, missile is fired, tip of B passes tail of A and finally the two tails pass each other. It will be useful to know that if an event occurs at either end of either rockets its spatial coordinates are that rocket frame since the tips are always at $x = 0$, $x' = 0$ and the tails are at $x = -1$ and $x' = +1$. It should also be easy to find time elapsed between tip of your rocket passing my tip and my tail since I am of unit length and you are moving at speed u. Ditto for tail. As a check of your coordinates you may want to see that the spacetime interval between any event and $(0, \underline{0})$ comes out same for both.

Solutions

Exercise 1

First of all we note that writing

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} \tag{1}$$

is nothing but a convenient way to write $F^{00} = 0; F^{01} = -E_x; F^{02} = -E_y;\ \ldots$ $F^{32} = cB_x; F^{33} = 0$. Thus, how we call the labels in such an equation is irrelevant.

According to the general rule of raising and lowering indices we have

$$F^{\mu}{}_{\nu} = g_{\nu\rho} F^{\mu\rho} = F^{\mu\rho} g_{\nu\rho} = F^{\mu\rho} g_{\rho\nu} = (F \cdot g)^{\mu}{}_{\nu}$$

In the second to last step we used $g_{\nu\rho} = g_{\rho\nu}$ since the metric is symmetric, i.e. invariant under exchange of rows and columns $g = g^T$ (T stands for transpose). Performing the matrix multiplication (or simply the sums) we get

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \tag{2}$$

We can also obtain this without doing any calculation by noting $F^{\mu}{}_{0} = F^{\mu 0}$ and $F^{\mu}{}_{i} = -F^{\mu i}$ with $i \in \{1, 2, 3\}$. Since the second label denotes the columns we see immediately that columns 1,2 and 3 (but not 0) get a minus sign in going from $F^{\mu\nu}$ to $F^{\mu}{}_{\nu}$. However, using $F^{\mu}{}_{\nu} = g_{\nu\rho} F^{\mu\rho}$ is more general since this will also be valid for a more complicated metric, showing up in general relativity.

For the second part we note

$$F_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} F^{\rho\sigma} = g_{\mu\rho} F^{\rho\sigma} g_{\sigma\nu} = (g \cdot F \cdot g)_{\mu\nu}$$

where F is the matrix given in eq.(1) Doing the matrix multiplications we find

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

Again, we could get this simply by noting $F_{00} = F^{00}$; $F_{ij} = (-1)^2 F^{ij}$, but $F_{i0} = -F^{i0}$; $F_{0i} = -F^{0i}$, where $i, j \in \{1, 2, 3\}$.

We could also write

$$F_{\mu\nu} = g_{\mu\rho} F^{\rho}{}_{\nu} = (g \cdot F)_{\mu\nu}$$

where F is now the matrix given in eq.(2).

Exercise 2

Let $\gamma = 1/\sqrt{1-(u/c)^2}$. In the frame of A (travelling in positive x-direction) the other rocket has length L_0/γ and velocity $-u$. The coordinates in the rest-frame of A for the five events are then:

$$\begin{aligned}
\text{Tip of } B \text{ passes tip of } A :& (t, \underline{x}) = (0, \underline{0}) && (3)\\
\text{Tail of } B \text{ passes tip of } A :& (L_0/(u\gamma), \underline{0}) && (4)\\
\text{Missile is fired from tail of } A :& (L_0/(\gamma u), -L_0) && (5)\\
\text{Tip of } B \text{ passes tail of } A :& (L_0/u, -L_0) && (6)\\
\text{Tail of } B \text{ passes tail of } A :& (L_0/u + L_0/(\gamma u), -L_0). && (7)
\end{aligned}$$

Note that because γ is greater than 1, the missile is fired before the tip of B reaches the tail of A, and the missile misses rocket B according to A.

Let us know analyse the situation from Rocket B, which moves with velocity $-u$ relative to A. We therefore find

$$x_B = \gamma(x_A + u\, t_A) \tag{8}$$

$$t_B = \gamma(t_A + \frac{u}{c^2}\, x_A) \tag{9}$$

The space-time coordinates for the five events observed by A are therefore in the frame of B:

$$\begin{aligned}
\text{Tip of } A \text{ passes tip of } B :& (0, \underline{0}) && (10)\\
\text{Tip of } A \text{ passes tail of } B :& (L_0/u, L_0) && (11)\\
\text{Missile is fired from tail of } A :& (L_0/u - \gamma u L_0/c^2, L_0 - \gamma L_0) && (12)\\
\text{Tail of } A \text{ passes tip of } B :& (L_0/(\gamma u), 0) && (13)\\
\text{Tail of } A \text{ passes tail of } B :& (L_0/u + L_0/(\gamma u), L_0). && (14)
\end{aligned}$$

We can now compare the times when B sees the missile being fired (M) and the tail of A passing the tip of B (T):

$$t_T = \frac{L_0}{u\gamma} \qquad t_M = \frac{L_O}{u\gamma}\left(\gamma - \frac{\gamma^2 u^2}{c^2}\right). \tag{15}$$

If we set $z = u/c$ we can rewrite the last bracket as

$$\frac{1}{\sqrt{1-z^2}} - \frac{z^2}{1-z^2} = \frac{\sqrt{1-z^2} - z^2}{1-z^2}. \tag{16}$$

Since the square root in the numerator is always less than 1, the numerator as a whole is always less than the denominator, and therefore the expression is less than one. Therefore, in the frame of B, $t_T > t_M$. According to B, the missile fires before the tail of A passes the tip of B! Therefore the missile also misses according to B. This does not contradict both A and B agreeing that the other rocket is Lorentz contracted and therefore shorter than their own rocket.