University of Durham
EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2018	PHYS2631-WE01

Title:
Theoretical Physics 2

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper.
- Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** the answers that are not to be marked.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Classical Mechanics
Section B: Quantum Theory 2

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A: CLASSICAL MECHANICS

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) A mechanical system consists of three point masses connected by three massless, rigid rods to form an equilateral triangle. How many degrees of freedom does this system have? Write down one constraint equation that is relevant for this system. [4 marks]

   (b) The displacement, x, of a damped oscillator satisfies the following differential equation:
$$\ddot{x} + b\dot{x} + \frac{k}{m}x = 0.$$
   Find the auxiliary equation and determine an expression for b in the case that the oscillator is critically damped. [4 marks]

   (c) For an underdamped, sinusoidally driven harmonic oscillator, describe the characteristics of its steady-state and transient solutions in terms of their amplitude and frequency. [4 marks]

   (d) What are the normal modes of oscillation for a system of coupled oscillators, and why is it important that these oscillations are small? [4 marks]

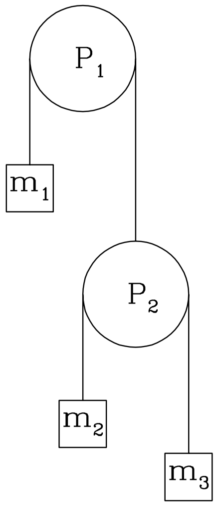
   (e) A system described by a single generalised coordinate z has kinetic and potential energies given by $T = m(\dot{z} + v)^2/2$ and $V = mgz$ respectively, where m, v and g are constants. Use the following Legendre transformation of the Lagrangian to find the Hamiltonian, H, for the system: $H(p, z) = p\dot{z} - L(z, \dot{z})$, where p is the canonically conjugate momentum to variable z. Is the Hamiltonian equal to the total energy? [4 marks]

   (f) Using the implicit transformation equations $q = -\partial F/\partial p$ and $P = -\partial F/\partial Q$, and the properties of the Poisson bracket of two arbitrary functions J and K, where
$$\{J, K\} = \frac{\partial J}{\partial q}\frac{\partial K}{\partial p} - \frac{\partial J}{\partial p}\frac{\partial K}{\partial q},$$
   determine whether or not the generating function $F = -pQ^2$ produces a canonical transformation. [4 marks]

   (g) What type of force is the Euler force and how do such forces arise? Why is the Euler force usually neglected when considering motion on Earth? [4 marks]

   (h) The moment of inertia of a uniform sphere, of mass M and radius R, for rotations about an axis through its centre of mass is $I_0 = 2MR^2/5$. Write down the inertia tensor for this same sphere relative to a point on its surface, using as one of the coordinates the direction between the centre of the sphere and that point on the surface. [4 marks]

2. The compound Atwood machine consists of three point masses connected via two light, inextensible strings draped over two massless, frictionless pulleys, as shown in the diagram. Pulley P_1 has its centre fixed, while pulley P_2 and the three masses are free to move vertically within a uniform gravitational field of strength g.



(a) (i) Write down the kinetic and potential energies of the system using the heights above the ground of the three masses, y_1, y_2 and y_3, as dynamical variables. [2 marks]

(ii) Determine a constraint equation connecting the heights of the three masses and hence show that the Lagrangian can be written, in terms of y_1 and y_2, as

$$L = \frac{1}{2}\left[m_1\dot{y}_1^2 + m_2\dot{y}_2^2 + m_3(2\dot{y}_1 + \dot{y}_2)^2\right] - g\left[m_1 y_1 + m_2 y_2 - m_3(2y_1 + y_2)\right].$$

[5 marks]

(b) Using the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0,$$

where $i = 1, 2$, show that

$$\frac{\ddot{y}_1}{g} = \frac{4m_2m_3 - m_1(m_2 + m_3)}{4m_2m_3 + m_1(m_2 + m_3)}.$$

[7 marks]

(c) Describe the motion of the system after it is released from rest if:

(i) $m_1 = 2m, m_2 = m_3 = m$, [2 marks]

(ii) $m_1 = 4m, m_2 = 3m, m_3 = m$. [4 marks]

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3. The Coriolis and centrifugal accelerations appearing to act on a body in a rotating reference frame are given by $\underline{a}_{\mathrm{Cor}} = -2\underline{\omega} \times \dot{\underline{r}}$ and $\underline{a}_{\mathrm{cent}} = -\underline{\omega} \times (\underline{\omega} \times \underline{r})$ respectively. $\underline{\omega}$ represents the angular velocity of the rotating frame and $\underline{r}$ is the displacement of the body from any point on the axis of rotation.

   A person throws a stone with speed v_0 radially outwards from the Earth's surface at a latitude λ in the northern hemisphere. You may ignore air resistance and assume that the stone moves in a uniform gravitational field of strength g.

   (a) By neglecting the inertial accelerations, show that the time taken to reach the maximum height, and the maximum height are

   $$t_{\mathrm{max}} \approx \frac{v_0}{g} \quad \text{and} \quad h_{\mathrm{max}} \approx \frac{v_0^2}{2g}$$

   respectively. Explain why it is justified to ignore the inertial accelerations. [6 marks]

   (b) In which directions does the Coriolis acceleration act on the stone when it is moving (i) away from the surface, and (ii) toward the surface? [2 marks]

   (c) How far away from the launch position does the stone land, and in which direction? Give your answer in terms of $\omega, \lambda, h_{\mathrm{max}}$ and g. [8 marks]

   (d) A stone dropped from a helicopter hovering a height h over the ground-based stone-thrower lands a distance

   $$d \approx \frac{\omega \cos \lambda}{3} \sqrt{\frac{8h^3}{g}}$$

   east of the stone-thrower. Compare this with your answer to part (c), assuming that $h = h_{\mathrm{max}}$, and explain any difference with reference to the orbital angular momentum of the two stones. [4 marks]

SECTION B: QUANTUM THEORY 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) (i) Give an example of a linear operator and explain why this operator is linear. [2 marks]
(ii) What is the definition of the adjoint of a linear operator? [2 marks]

(b) By way of an example or otherwise, explain the difference between the eigenvalue of an observable and the expectation value of an observable. [4 marks]

(c) (i) What is the defining relation of a unitary operator? [2 marks]
(ii) Show that if $\hat{A}$ is a unitary operator, $\hat{A}|\phi\rangle = |\phi'\rangle$ and $\hat{A}|\psi\rangle = |\psi'\rangle$, then $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$ so that the inner products of states are preserved under a unitary transformation. [2 marks]

(d) How does the Heisenberg picture (or Heisenberg representation) differ from the Schrödinger picture (or Schrödinger representation) with respect to the time dependence of operators and of quantum states? How can an operator in the Schrödinger picture be transformed into its counterpart in the Heisenberg picture? [4 marks]

(e) Find the matrix representation of an observable $\hat{O}$ in the basis of the orthonormal states $|a\rangle$ and $|b\rangle$, given that $\hat{O}$ acts on $|a\rangle$ and $|b\rangle$ as follows:

$$\hat{O}|a\rangle = |b\rangle, \qquad \hat{O}|b\rangle = |a\rangle.$$

Hence find the eigenvalues of $\hat{O}$. [4 marks]

(f) If the square of the total angular momentum operator, $\underline{\hat{J}}^2$, has the eigenvalue $6\hbar^2$, which eigenvalues are possible for $\hat{J}_z$? [4 marks]

(g) Two systems (1) and (2) in the states characterized by $|j^{(1)}, m^{(1)}\rangle$ and $|j^{(2)}, m^{(2)}\rangle$ are combined to form a new system in the state $|J, M\rangle$. What ranges of values are allowed for the quantum numbers J and M of the combined system if $m^{(1)} = -m^{(2)}$? [4 marks]

5. This question uses algebraic methods to obtain the eigenvalues of $\hat{L}_z$, the z-component of the orbital angular momentum operator $\underline{\hat{L}}$. The calculation is based on the fact that the position operators for the x- and y-directions and the corresponding momentum operators can be written in terms of the lowering and raising operators $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_x^\dagger$ and $\hat{a}_y^\dagger$ as

$$
\begin{aligned}
\hat{x} &= \alpha(\hat{a}_x + \hat{a}_x^\dagger), \quad \hat{p}_x = \frac{\hbar}{2i\alpha}\,(\hat{a}_x - \hat{a}_x^\dagger), \\
\hat{y} &= \alpha(\hat{a}_y + \hat{a}_y^\dagger), \quad \hat{p}_y = \frac{\hbar}{2i\alpha}\,(\hat{a}_y - \hat{a}_y^\dagger),
\end{aligned}
$$

where α is a length.

(a) (i) Express $\hat{L}_z$ in terms of the operators $\hat{x}$, $\hat{y}$, $\hat{p}_x$ and $\hat{p}_y$. [2 marks]

(ii) It is clear from the above that $\hat{L}_z$ could be written in terms of the operators $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_x^\dagger$ and $\hat{a}_y^\dagger$. Suppose that you would want to express the x-component of $\underline{\hat{L}}$ in a similar way. Which lowering and raising operators would you use? [3 marks]

(b) Let $\hat{b}_1 = (\hat{a}_x + i\hat{a}_y)/\sqrt{2}$ and $\hat{b}_2 = (\hat{a}_x - i\hat{a}_y)/\sqrt{2}$. These two operators and their adjoints satisfy the commutation relation of lowering and raising operators.

(i) Show that $[\hat{b}_1, \hat{b}_1^\dagger] = 1$. [8 marks]

(ii) Express $\hat{a}_x$ and $\hat{a}_y^\dagger$ in terms of $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_1^\dagger$ and $\hat{b}_2^\dagger$. [3 marks]

(iii) One can show that $\hat{L}_z = \hbar(\hat{b}_2^\dagger\hat{b}_2 - \hat{b}_1^\dagger\hat{b}_1)$ when this operator is expressed in terms of the $\hat{b}$-operators. Derive the spectrum of $\hat{L}_z$ from this result. [4 marks]

6. The z-component of the spin operator, $\hat{S}_z$, is represented by the matrix

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the usual representation of the spin states of a spin-1/2 particle. Consider an electron in a spin state $|\chi\rangle$ given in this representation by the column vector

$$\begin{pmatrix} \sqrt{3}/2 \\ i/2 \end{pmatrix}.$$

(a) Show that $|\chi\rangle$ is normalized. [3 marks]

(b) Suppose that you measure the z-component of the spin on this electron.
(i) What values could you get? [3 marks]
(ii) What is the probability of each? [4 marks]
(iii) Calculate the uncertainty $[\,\langle\chi|\hat{S}_z^2|\chi\rangle - \langle\chi|\hat{S}_z|\chi\rangle^2\,]^{1/2}$. [4 marks]

(c) Suppose that you have an apparatus which can measure the component of the spin of an electron in an arbitrary direction of polar angles θ ($0 \leq \theta \leq \pi$) and ϕ ($0 \leq \phi \leq 2\pi$). The corresponding spin operator is represented by the matrix

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \exp(-i\phi)\sin\theta \\ \exp(i\phi)\sin\theta & -\cos\theta \end{pmatrix}.$$

You decide to set $\phi = \pi/2$. Calculate the value you should choose for θ if you want that a measurement on an electron prepared in the state $|\chi\rangle$ gives a positive value with a probability of 1. [6 marks]