University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 042581/01

LEVEL 2 PHYSICS: FOUNDATIONS OF PHYSICS 2A

SECTION A. QUANTUM MECHANICS 2
SECTION B. ELECTROMAGNETISM

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

SECTION A. QUANTUM MECHANICS 2

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) A particle in an infinite square well between $0 < x < 1$ has energy $E_n = n^2 E_1$ associated with eigenfunctions $\psi_n(x) = \sqrt{2}\sin(n\pi x)$.

   A particle in the well has $\psi(x) = A[3\psi_1 + \psi_3]$. Use the orthonormal properties of the energy eigenfunctions to show that $A = 1/\sqrt{10}$. [2 marks]

   Write down the probability that the system energy is measured to be E_1. Hence calculate the expectation value of the energy, $\langle E\rangle$. Can any individual measurement of the energy give this value? Explain your answer. [2 marks]

   (b) A particle with arbitrary wavefunction $\psi(x)$ can be decomposed into a weighted sum of the energy eigenfunctions, so $\psi(x) = \sum_n c_n\psi_n(x)$. Multiply by $\psi_m^*(x)$ and integrate over all space to show that $c_m = \int \psi_m^*(x)\psi(x)dx$. [1 mark]

   For the infinite square well potential in (a), calculate c_1 and c_2 for $\psi(x) = \sqrt{30}x(1-x)$. Justify your answer by commenting on the symmetry of the wavefunction. [3 marks]

$$\left[\int_0^1 x\sin(n\pi x)dx = -\frac{\cos(n\pi)}{n\pi},\ \int_0^1 x^2\sin(n\pi x)dx = \frac{(2-\pi^2n^2)\cos(n\pi)-2}{\pi^3n^3}.\right]$$

   (c) Write down the momentum operator, p, in one dimension. Use this to show that $[x,p]\psi = i\hbar\psi$ for any wavefunction, ψ. [4 marks]

   (d) Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$ and $\langle p^2\rangle$ for an energy eigenfunction $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ of the infinite square well potential in (a). What value of n minimizes the product $\Delta x\Delta p$, where the uncertainty in any quantity, A, is given by $\Delta A = (\langle A^2\rangle - \langle A\rangle^2)^{1/2}$? [4 marks]

$$\left[\int_0^1 x\sin^2(n\pi x)dx = \frac{1}{4},\quad \int_0^1 x^2\sin^2(n\pi x)dx = \frac{1}{6} - \frac{1}{4\pi^2n^2}.\right]$$

   (e) In three dimensions, with a spherically symmetric potential, $V(r)$, the virial theorem can be written as $2\langle T\rangle = \langle r dV/dr\rangle$ where T is kinetic energy.

   The potential for hydrogen is $V(r) = -e^2/(4\pi\epsilon_0 r)$. Hence show that $\langle T\rangle = -\langle V\rangle/2 = -E_n$ for an atom in one of the eigenstates, ψ_{nlm}, with associated energy, E_n. [3 marks]

   Explain why it is generally more convenient to use the virial theorem rather than to calculate $\langle T\rangle$ directly. [1 mark]

   (f) The wavefunction of a hydrogen atom with $n = 2, l = 1, m = 1$ is

$$\psi_{211} = -\frac{1}{8\sqrt{\pi a^3}}\frac{r}{a}\exp(-r/2a)\sin\theta e^{i\phi},$$

   where a is the Bohr radius and the other symbols have their usual meanings. Write down the probability of finding the electron within the volume dV of r, θ and ϕ. Integrate this over θ and ϕ to form $D(r)dr$, where $D(r)$ is the radial distribution function. Calculate the radius at which $D(r)$ has a maximum. [4 marks]

$$\left[\int_0^\pi \sin^3(x)dx = \frac{4}{3}.\right]$$

(g) The hyperfine transition in hydrogen splits the ground state by an amount $(\Delta E)_\mathrm{H} \propto g_\mathrm{e}g_\mathrm{p}/(3m_\mathrm{p}m_\mathrm{e}a^3)$, where $g_\mathrm{e} = 2.00$ and $g_\mathrm{p} = 5.59$ are the spin g-factors for the electron and proton, and $a = 4\pi\epsilon_0\hbar/(\mu e^2)$ is the typical size of the orbit for an electron-proton (masses m_e and m_p) atom with reduced mass $\mu = m_\mathrm{e}m_\mathrm{p}/(m_\mathrm{e}+m_\mathrm{p})$.

Positronium is a hydrogen-like atom with the proton replaced by a positron (same mass and g-factor as the electron). Calculate the predicted hyperfine energy shift of positronium relative to that of hydrogen, $(\Delta E)_+/(\Delta E)_\mathrm{H}$. Give your answer to 2 significant figures. [4 marks]

(h) The infinite square well potential in (a) is perturbed by $H' = \alpha/\Delta$ for $1/2-\Delta/2 < x < 1/2+\Delta/2$, where α is the fine structure constant. The first order change in energy for each state is, $E_n^1 = \langle\psi_n^0|H'\psi_n^0\rangle$. Calculate E_1^1 and E_2^1 for this perturbation. What values do these energies tend to as $\Delta \to 0$? [4 marks]

$$\left[\int_a^b \sin^2(n\pi x)dx = \frac{2n\pi(b-a)+\sin(2n\pi a)-\sin(2n\pi b)}{4n\pi}.\right]$$

2. The time-independent Schrödinger equation for a rigid rotator can be written as

$$H\psi = \frac{1}{2m_\mathrm{e}a^2}L^2\psi = E\psi,$$

where H is the Hamiltonian for a particle of mass m_e which is free to move in θ and ϕ but is at fixed distance a from the origin. L^2 is the square of the total angular momentum operator which has the eigenvalue equation $L^2Y_{lm} = l(l+1)\hbar^2 Y_{lm}$, where Y_{lm} are spherical harmonics and l, m are integers with $l \geq 0$ and $-l \leq m \leq l$.

(a) Write down the general form for the energy levels of the rigid rotator. Calculate the degeneracy of the $l = 0$ and $l = 1$ energy levels. Hence write down the degeneracy of an arbitrary level, l. [4 marks]

(b) Write down the probability dP of finding an electron, which is constrained to be a rigid rotator, within volume $dV = \sin\theta d\theta d\phi$ of θ, ϕ, when the electron has the wavefunction

$$Y_{22} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta \exp(2i\phi).$$

Use the substitution $\mu = \cos\theta$ so $d\mu = -\sin\theta d\theta$ to calculate the probability of finding the electron in the region $\pi/3 \leq \theta \leq \pi/2$. Evaluate your answer to 3 significant figures. [4 marks]

(c) The ladder operator is given by

$$L_- = L_x - iL_y = -\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right).$$

Calculate L_-Y_{22} and show that your answer can be written in terms of $Y_{21} = N_{21}\sin\theta\cos\theta\exp(i\phi)$, where N_{21} is a normalization constant. [4 marks]

(d) Given the standard commutation relations for angular momentum components $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$ and $[L_z, L_x] = i\hbar L_y$, show that $[L_z, L_-] = -\hbar L_-$. Hence show that $L_z(L_-Y_{lm}) = (m-1)\hbar(L_-Y_{lm})$ for any general Y_{lm}. [4 marks]

(e) Given that $L^2 = L_x^2 + L_y^2 + L_z^2$, calculate the eigenvalues of L_+L_-, where $L_+ = L_x + iL_y$. [4 marks]

3. The three dimensional Schrödinger equation for an electron of mass $m_{\rm e}$ trapped in an isotropic harmonic potential is

$$-\frac{\hbar^2}{2m_{\rm e}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi+\frac{1}{2}m_{\rm e}\omega^2(x^2+y^2+z^2)\psi=E\psi,$$

where ω is a frequency.

(a) Use separation of variables, with $\psi_{n_x,n_y,n_z}(x,y,z)=X_{n_x}(x)Y_{n_y}(y)Z_{n_z}(z)$, to show that this is equivalent to three one-dimensional systems. Hence derive the energy eigenvalues E_{n_x,n_y,n_z}, given that the one dimensional harmonic oscillator has energy eigenvalues $E_n=(n+1/2)\hbar\omega$ for $n=0,1,2\ldots$ [5 marks]

(b) The system is perturbed by a potential $H'=\lambda x^2yz$ where λ is a constant. Use non-degenerate perturbation theory to calculate the energy shift of the ground state $E^1_{0,0,0}=\langle\psi^0_{0,0,0}|H'\psi^0_{0,0,0}\rangle$, where $\psi^0_{0,0,0}=(a/\pi)^{1/4}\exp\left(-ax^2/2\right)\exp\left(-ay^2/2\right)\exp\left(-az^2/2\right)$ for $a=m_{\rm e}\omega/\hbar$. [2 marks]

(c) The first excited state is triply degenerate with unperturbed wavefunctions

$$\psi^0_1=\psi^0_{0,0,1}=Az\exp\left(-ax^2/2\right)\exp\left(-ay^2/2\right)\exp\left(-az^2/2\right)$$
$$\psi^0_2=\psi^0_{0,1,0}=Ay\exp\left(-ax^2/2\right)\exp\left(-ay^2/2\right)\exp\left(-az^2/2\right)$$
$$\psi^0_3=\psi^0_{1,0,0}=Ax\exp\left(-ax^2/2\right)\exp\left(-ay^2/2\right)\exp\left(-az^2/2\right),$$

where $A=(2a)^{1/2}(a/\pi)^{3/4}$. Evaluate the matrix elements $W_{ij}=\langle\psi^0_i|H'\psi^0_j\rangle$ for the perturbation in (b), where i,j take values $1,2,3$ denoting each wavefunction. [6 marks]

(d) Solve the resulting matrix equation for all possible values of E^1

$$\begin{pmatrix} W_{11} & W_{12} & W_{13}\\ W_{21} & W_{22} & W_{23}\\ W_{31} & W_{32} & W_{33}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=E^1\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}.$$

Is the level still degenerate? Write down the three wavefunctions $\chi=\alpha\psi_1+\beta\psi_2+\gamma\psi_3$ which correspond to each possible energy. [7 marks]

$$\left[\int_{-\infty}^{+\infty}u\exp\left(-\alpha u^2\right)du=0,\quad \int_{-\infty}^{+\infty}u^2\exp\left(-\alpha u^2\right)du=\frac{1}{2}\left(\frac{\pi}{\alpha^3}\right)^{1/2}.\right]$$

SECTION B. ELECTROMAGNETISM

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) A solid conducting sphere has radius 1 m and total charge 10^{-10} Coulombs. Calculate the electric field outside of the sphere, 1 cm from the surface. [4 marks]

   (b) Write down the expression for the Poynting vector and explain its significance. [4 marks]

   (c) Starting from Maxwell's equations, derive an expression for the velocity of light in a dielectric. [4 marks]

$$\left[\text{Hint } : \underline{\nabla} \times (\underline{\nabla} \times \underline{C}) = -\nabla^2 \underline{C} + \underline{\nabla}(\underline{\nabla}.\underline{C}).\right]$$

   (d) Define *displacement current density* and briefly discuss why it is important in the context of Ampère's law. [4 marks]

   (e) Define the term *electric polarisation.* Clearly define all the terms you use and specify their units. [4 marks]

   (f) Explain why the magnetic susceptibility of a sample of conducting magnetic material is found to be frequency dependent in experiments. [4 marks]

   (g) A charged solid sphere of diameter 1 m, with charge density 6 C m^{-3}, moves through space at 5 m s^{-1}. Discuss what you expect the electric and magnetic fields to be at the centre of the sphere. [4 marks]

5. The wave equation for the electric field of an electromagnetic wave propagating in vacuum is given by

$$\nabla^2 \underline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}.$$

One possible solution for the wave equation within the interior of an infinitely long perfectly-conducting hollow square metallic tube (known as a waveguide), of width a, is

$$\underline{E} = \underline{E}_0 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi z}{a}\right) \exp[i(ky - \omega t)],$$

where x, y and z are the conventional Cartesian coordinates, ω is the angular frequency, k is the wavevector, $\underline{E}_0$ gives the magnitude and polarisation of the wave and the axis of the tube is parallel to the y-axis. Derive the dispersion relation for this wave. [3 marks]

Plot the form of the derived dispersion relation and obtain an expression for the minimum frequency at which this particular wave can propagate along the waveguide [6 marks]

Write down another solution to the wave equation which describes a different wave that can propagate through the waveguide with a lower minimum frequency. [2 marks]

A scientist directs a very low frequency electromagnetic wave into the entrance of a copper waveguide. Describe what you expect to happen. [3 marks]

Calculate an approximate value for the magnitude of the $\underline{E}$-field inside the waveguide described above given that $a = 1$ m and the electromagnetic wave is high frequency and carries one megawatt of power. [6 marks]

6. The general dispersion relation for an electromagnetic wave propagating in a non-magnetic, conducting medium is given by

$$k^2 = \mu_0 \varepsilon \omega^2 + i\omega\mu_0\sigma_N,$$

where k is the wavevector, ω is the angular frequency, σ_N is the electrical conductivity and $\varepsilon = \varepsilon_r \epsilon_0$ where ε_r is the relative permittivity. Liquid nitrogen on Neptune has a relative permittivity of 1.5 and an electrical conductivity of $3.5 \times 10^{-5}\ \Omega^{-1}\mathrm{m}^{-1}$. Can it be considered a good conductor at a frequency of 2 MHz? [4 marks]

Two spacemen on Neptune transmit 2 MHz electromagnetic waves to each other through liquid nitrogen. Calculate the difference in wavelength of such a 2 MHz electromagnetic wave when propagating through liquid nitrogen compared to that when propagating through vacuum. [7 marks]

Find the decrease in the amplitude of the electromagnetic waves after they have passed through 4 km of liquid nitrogen. [5 marks]

Unfortunately the electromagnetic waves the spacemen receive are too weak for them to communicate properly. Suggest two approaches that they could try that might improve communication and explain why you think each approach would be successful. [4 marks]