Theoretical Physics 2019/20 — Solution of Problem QT2.4

(a) $\hat{a}^\dagger\hat{a}|\psi_n\rangle = \sqrt{n}\,\hat{a}^\dagger|\psi_{n-1}\rangle = \sqrt{n}\sqrt{n}|\psi_n\rangle = n|\psi_n\rangle$. [2 marks]

(b) First, we show that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$:

$$\begin{aligned}\hat{a}|\alpha\rangle &= \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,\hat{a}|\psi_n\rangle\\ &= \exp(-|\alpha|^2/2)\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,\sqrt{n}|\psi_{n-1}\rangle\\ &= \exp(-|\alpha|^2/2)\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{(n-1)!}}\,|\psi_{n-1}\rangle.\end{aligned}$$

(The summation over n runs from 1 to ∞, not 0 to ∞, because $\hat{a}|\psi_0\rangle = 0$ (the zero vector), and therefore the term in $n = 0$ contributes nothing. We have also used the fact that $n! = n \times (n-1)!$ to simplify the ratio $\sqrt{n}/\sqrt{n!}$ into $1/\sqrt{(n-1)!}$.)

To converge to the expected result, we set $m = n - 1$ and sum over m rather than over n. Since n starts at 1, m starts at 0. This gives

$$\begin{aligned}\hat{a}|\alpha\rangle &= \exp(-|\alpha|^2/2)\sum_{m=0}^{\infty}\frac{\alpha^{m+1}}{\sqrt{m!}}\,|\psi_m\rangle\\ &= \alpha\,\exp(-|\alpha|^2/2)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\,|\psi_m\rangle = \alpha|\alpha\rangle.\end{aligned}$$

Then, $\langle\alpha|\alpha\rangle = 1$:

$$\begin{aligned}\langle\alpha|\alpha\rangle &= \exp(-|\alpha|^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{m!}}\,\frac{\alpha^n}{\sqrt{n!}}\,\langle\psi_m|\psi_n\rangle\\ &= \exp(-|\alpha|^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{m!}}\,\frac{\alpha^n}{\sqrt{n!}}\,\delta_{mn}\\ &= \exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\\ &= \exp(-|\alpha|^2)\,\exp(|\alpha|^2) = 1.\end{aligned}$$

[1 mark for $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and 1 mark for $\langle\alpha|\alpha\rangle = 1$.]

(c) $\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha\,\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha\,\langle\alpha|\hat{a}|\alpha\rangle^* = \alpha\,\alpha^*\langle\alpha|\alpha\rangle^* = |\alpha|^2$.

$\langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha\,\langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger|\alpha\rangle$. This can be simplified further, using once again $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, by replacing the product $\hat{a}\hat{a}^\dagger$ by $\hat{a}^\dagger\hat{a} + \hat{I}$:

$$\hat{a}\hat{a}^\dagger|\alpha\rangle = \hat{a}^\dagger\hat{a}|\alpha\rangle + |\alpha\rangle = \alpha\,\hat{a}^\dagger|\alpha\rangle + |\alpha\rangle.$$

Thus $\langle\alpha|\hat{n}^2|\alpha\rangle = \alpha^2\langle\alpha|(\hat{a}^\dagger)^2|\alpha\rangle + \alpha\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^2(\alpha^*)^2 + \alpha\alpha^* = |\alpha|^4 + |\alpha|^2$.

[1 mark for the expectation value of $\hat{n}$ and 1 mark for the expectation value of $\hat{n}^2$.]

(d) Since $\langle\alpha|\alpha\rangle = 1$, $P(n) = |\langle\psi_n|\alpha\rangle|^2$.

$$\begin{aligned}\langle\psi_n|\alpha\rangle &= \exp(-|\alpha|^2/2)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\,\langle\psi_n|\psi_m\rangle \\ &= \exp(-|\alpha|^2/2)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\,\delta_{nm} \\ &= \exp(-|\alpha|^2/2)\,\frac{\alpha^n}{\sqrt{n!}}.\end{aligned}$$

Therefore

$$P(n) = \left|\exp(-|\alpha|^2/2)\,\frac{\alpha^n}{\sqrt{n!}}\right|^2 = \exp(-|\alpha|^2)\,\frac{|\alpha|^{2n}}{n!}.$$

[2 marks]

(e) By definition, the variance of this distribution of probability, $(\Delta n)^2$, is $\langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2$. I.e., using the results of Part (c), $|\alpha|^4 + |\alpha|^2 - |\alpha|^4 = |\alpha|^2$. [2 marks]

Note: The variance found in Part (e) is equal to the mean of this probability distribution. In fact, the variance of a Poisson distribution is always equal to its mean.