4 Quantum states and the Dirac notation

4.1 Quantum states and ket vectors

The whole of Quantum Mechanics is based on a few fundamental rules. The first of them can be expressed as follows:

> Each state of a quantum system can be described by a vector belonging to a Hilbert space.

Recall that Hilbert spaces are vector spaces and that their elements are called vectors. At the beginning of the course we have seen an example of system in which the vectors describing the states of interest were square-integrable functions of r, θ and ϕ, and an example where they were 2-component column vectors. States of quantum systems can be represented in a more general way by what Dirac called ket vectors. These vectors are usually denoted by the symbol $|\ldots\rangle$, with $\ldots$ standing for whatever label would be used to identify the particular vector (e.g., $|\psi\rangle$, $|n\rangle$, $|\uparrow\rangle$, $|x,+\rangle$, etc.).[1] Ket vectors are often called kets, in short, or state vectors. Depending on the system, they can in turn be represented by a function of r, θ and ϕ, or by a 2-component column vector, or by some other mathematical object appropriate for the problem at hands.

We stress that ket vectors themselves do not depend on specific coordinates. Only their representations in terms of wave functions do. They do not depend on the choice of basis vectors either, contrary to the column vectors representing them. Ket vectors are vectors in their own right, however vectors belonging to an abstract Hilbert space rather than a Hilbert space of functions or column vectors.

Take, for example, the non-relativistic Hamiltonian of atomic hydrogen given by Eq. (1.1) of Section 1.2 of these notes,

$$H = -\frac{\hbar^2}{2\mu}\,\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\,\frac{1}{r}, \tag{4.1}$$

and the equation

$$H\psi_{nlm}(r,\theta,\phi) = E_n\psi_{nlm}(r,\theta,\phi) \tag{4.2}$$

defining the energy levels and energy eigenfunctions of that Hamiltonian. This operator H acting on wave functions representing possible quantum states of a

[1] The zero ket vector is usually represented by the numeral 0 rather than by a $|\ldots\rangle$ symbol. E.g., the combination $|\psi\rangle + 0$ represents the sum of the vector $|\psi\rangle$ with the zero vector, not the sum of the vector $|\psi\rangle$ with the number 0, and $|\psi\rangle + 0 = |\psi\rangle$ for any $|\psi\rangle$.

hydrogen atom corresponds to an operator $\hat{H}$ acting on ket vectors representing the same quantum states. Passing to this other formulation, Eq. (4.2) becomes

$$\hat{H}|n,l,m\rangle = E_n|n,l,m\rangle, \tag{4.3}$$

where $\hat{H}$ is a certain operator acting on the vectors $|n,l,m\rangle$ (operators are discussed in Part 3 of these notes). Neither $|n,l,m\rangle$ nor $\hat{H}$ are given by some combinations of the variable r, θ and ϕ. However, they can be *represented* by such combinations — i.e., by the wave functions $\psi_{nlm}(r,\theta,\phi)$ and the operator H of Eq. (4.1) — in the sense that there is a one-to-one correspondence between the kets $|n,l,m\rangle$ and the wave functions $\psi_{nlm}(r,\theta,\phi)$. (In the mathematical terminology of Section 2.11, one would say that the Hilbert space inhabited by these ket vectors is isomorphic to the one inhabited by these wave functions.) In particular, if the ket vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ are represented by the wave functions $\psi_a(r,\theta,\phi)$ and $\psi_b(r,\theta,\phi)$, then their linear combination $\alpha|\psi_a\rangle + \beta|\psi_b\rangle$ is represented by the wave function $\alpha\psi_a(r,\theta,\phi) + \beta\psi_b(r,\theta,\phi)$.

The same also applies to inner products: Inner products of ket vectors can be calculated as inner products of the wave functions or column vectors representing these ket vectors. Suppose, for instance, that the ket vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ describe certain states of a linear harmonic oscillator and correspond to the wave functions $\psi_a(x)$ and $\psi_b(x)$. The inner product of these two wave functions is the integral

$$\int_{-\infty}^{\infty} \psi_a^*(x)\,\psi_b(x)\,\mathrm{d}x.$$

Because of the correspondence between the ket vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ and the wave functions $\psi_a(x)$ and $\psi_b(x)$, the inner product of the latter, (ψ_a,ψ_b), is equal to the inner product of the former, $\langle\psi_a|\psi_b\rangle$:

$$\langle\psi_a|\psi_b\rangle = \int_{-\infty}^{\infty} \psi_a^*(x)\,\psi_b(x)\,\mathrm{d}x. \tag{4.4}$$

(The notation introduced in Part 2 of the notes for denoting the inner product of vectors is not used for ket vectors: We denote the inner product of a ket $|\psi_a\rangle$ with a ket $|\psi_b\rangle$ by the symbol $\langle\psi_a|\psi_b\rangle$, not by $(|\psi_a\rangle,|\psi_b\rangle)$.) If $|\psi_a\rangle$ and $|\psi_b\rangle$ described quantum states of atomic hydrogen instead, and corresponded to the wave functions $\psi_a(r,\theta,\phi)$ and $\psi_b(r,\theta,\phi)$, we would have had, similarly,

$$\langle\psi_a|\psi_b\rangle = (\psi_a,\psi_b) = \int_0^{\infty} \mathrm{d}r\, r^2 \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi\, \psi_a^*(r,\theta,\phi)\,\psi_b(r,\theta,\phi). \tag{4.5}$$

The same applies to column vectors: If the ket vectors $|\chi\rangle$ and $|\chi'\rangle$ describe spin states and are represented by the column vectors

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{and} \qquad \chi' = \begin{pmatrix} a' \\ b' \end{pmatrix}, \tag{4.6}$$

then

$$\langle\chi|\chi'\rangle = (\chi,\chi') = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix}. \tag{4.7}$$

The correspondence between ket vectors and wave functions or column vectors goes both ways. The discussion above is phrased in terms of ket vectors being represented by wave functions or column vectors, but one can say equally well that wave functions or column vectors are represented by the corresponding ket vectors. (Seen in that way, symbols such as $|\psi\rangle$ and $|\chi\rangle$ may be thought of as being merely a simplified notation for the corresponding wave functions or column vectors. However, it is useful to keep in mind that these symbols actually refer to vectors in their own right.)

☞ The rule stated at the beginning of this section relates the states of a quantum system to wave functions or some other vectors. But what do one mean by "states" in this context?

The meaning may seem almost obvious at first sight. For example, it is an extremely well established experimental fact that an atom may behave quite differently when exposed to a laser beam than when left alone, although it is still the same atom (same electrons, same nucleus). It is natural to say that the atom is in a different state when exposed to the laser beam than when left alone. The rule says that each of these states can be described by a wave function or some other vector, and the same for any other state the atom could be in.

Digging a little deeper, however, one hits a difficulty with defining precisely what a "state" is in this context. What the issue is is perhaps best explained by a simple example. Take a classical system consisting of a mass hanging from a spring, and suppose that this mass moves only in the vertical direction (i.e., that it does not swing laterally like a pendulum). One could say that the position and the momentum of this mass define its state of motion: to know them is to know the amplitude, frequency and phase of its oscillation, and in fact anything that one might want to know about how the mass moves. The trajectory of this mass can be represented by the function $z(t)$ describing how its z-coordinate varies as a function of time. Someone knowing this function could predict, with complete accuracy, where the mass is at any given time. Position measurements on several identical oscillators in exactly the same state of motion would return exactly the same results if done at exactly the same time (within experimental error, of course, but this limitation is not fundamental). The same can be said for a system of many particles. In Classical Mechanics, in general, a state is defined by the positions and momenta of all its constituents.

The situation is quite different in the case of a quantum oscillator. A measurement of the position of the mass would only return a random

result in this case (random within the distribution of probability determined by the wave function). If position measurements were made simultaneously on several identical oscillators, all prepared in exactly in the same way, then a different result would normally be found for each of these oscillators (even if the measurements were so accurate and precise than the experimental error was negligible). In these circumstances, it would make little sense to say the mass follows a certain trajectory. Whether its position and momentum could be taken as defining its state is altogether questionable, too, since its position and momentum cannot be both assigned precise values at the same time, neither experimentally nor theoretically (recall the uncertainty relation).

Hence, what the rule stated at the beginning really means becomes rather unclear if one goes beyond the intuitive notion of state mentioned above. In fact, this issue touches to the philosophy and interpretation of Quantum Mechanics and is still a matter of controversy.

☞ What is meant by the word system in "quantum system" also deserves some scrutiny. Briefly, in this context, a system is something which is a self-contained entity as far as describing its state is concerned. For example, an atom may often be regarded as an independent quantum system — hence one can talk about its ground state, etc. — but an atom forming part of a molecule can't. It is rarely possible to assign an individual wave function or state vector to each of the different parts of a multipartite system.

4.2 Bra vectors

The symbol $\langle\psi|\phi\rangle$ representing the inner product of the kets $|\psi\rangle$ and $|\phi\rangle$ is called a bracket. This inner product can be viewed as a combination of a ket vector $|\phi\rangle$ with what Dirac called a "bra vector", $\langle\psi|$. (Bra and ket vectors are so named, of course, because their combinations are represented by brackets.) For consistency, bra and ket vectors must obey the following rules:

- To each ket vector $|\psi\rangle$ there must be a bra vector $\langle\psi|$, and to each bra vector $\langle\psi|$ there must be a ket vector $|\psi\rangle$ (in other words, bra and ket vectors are in one-to-one correspondence).

- Ket vectors correspond to wave functions or to column vectors:

$$|\psi\rangle \quad \longleftrightarrow \quad \psi(x), \quad \begin{pmatrix} a \\ b \end{pmatrix}.$$

- Bra vectors correspond to complex conjugated wave functions or to complex conjugated row vectors:

$$\langle\psi| \quad \longleftrightarrow \quad \psi^*(x), \quad \begin{pmatrix} a^* & b^* \end{pmatrix}.$$

- If the ket vectors $|\psi\rangle$ and $|\phi\rangle$ correspond to the bra vectors $\langle\psi|$ and $\langle\phi|$, then the ket vector $|\psi\rangle + |\phi\rangle$ must correspond to the bra vector $\langle\psi| + \langle\phi|$.

- If c is a complex number and the ket vector $|\psi\rangle$ corresponds to the bra vector $\langle\psi|$, then the ket vector $c|\psi\rangle$ corresponds to the bra vector $c^*\langle\psi|$. (Note that the factor multiplying $\langle\psi|$ is the *complex conjugate* of c.)

> ☞ A linear combination $c_1|\psi_1\rangle + c_2|\psi_2\rangle$ of ket vectors thus corresponds to a linear combination $c_1^*\langle\psi_1| + c_2^*\langle\psi_2|$ of bra vectors for any kets $|\psi_1\rangle$ and $|\psi_2\rangle$ and any complex numbers c_1 and c_2. Such correspondences are said to be antilinear (anti because of the complex conjugation of c_1 and c_2).
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> ☞ A bra vector $\langle\psi|$ defines a mapping from ket vectors to complex numbers such that any ket vector $|\phi\rangle$ is mapped to the complex number $\langle\psi|\phi\rangle$ and that any linear combination $c_1|\phi_1\rangle + c_2|\phi_2\rangle$ is mapped to the complex number $c_1\langle\psi|\phi_1\rangle + c_2\langle\psi|\phi_2\rangle$. Such mappings are called linear functionals. Mathematically, bra vectors are best seen as being linear functionals on the vector space of ket vectors. The set of all linear functionals on a vector space V is also a vector space, called the dual of V. The one-to-one correspondence between ket and bra vectors mentioned above is guaranteed by a theorem of functional analysis, the Riesz Representation Theorem.

4.3 Operators and the Dirac notation

As we have just seen, quantum states can be represented not only by wave functions or column vectors of complex numbers, as befits the problem considered, but also by "ket vectors" belonging to an abstract Hilbert space. We have also seen that each operator acting on wave functions or column vectors has a counterpart in term of an operator acting on ket vectors. To take the same example as in Section 4.1, the Hamiltonian operator

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}, \tag{4.8}$$

which acts on wave functions depending on the space coordinates r, θ and ϕ corresponds to an operator $\hat{H}$ acting on ket vectors. In particular, the eigenvalue equation

$$H\psi_{nlm}(r,\theta,\phi) = E_n\psi_{nlm}(r,\theta,\phi) \tag{4.9}$$

becomes

$$\hat{H}|n,l,m\rangle = E_n|n,l,m\rangle \tag{4.10}$$

when expressed in terms of ket vectors rather than wave functions. One can say that the wave functions $\psi_{nlm}(r,\theta,\phi)$ represent the ket vectors $|n,l,m\rangle$ and that the differential operator H represents the operator $\hat{H}$. More generally, if a ket vector $|\psi\rangle$ is represented by, e.g., a wave function $\psi(r,\theta,\phi)$ and an operator $\hat{A}$ acting on $|\psi\rangle$ is represented by an operator A acting on $\psi(r,\theta,\phi)$, then the ket vector $\hat{A}|\psi\rangle$ is represented by the function $A\psi(r,\theta,\phi)$. For example, Eq. (3.71) defining the adjoint of an operator would read

$$\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle^* \tag{4.11}$$

if written in terms of ket vectors $|\phi\rangle$ and $|\psi\rangle$ rather than in terms of generic vectors v and w.

Given this correspondence, the matrix elements of an operator acting on ket vectors can be calculated as the matrix elements of the corresponding operator acting on wave functions or column vectors. That is to say, if the two orthonormal vectors u_i and u_j (wave functions or column vectors) represent the orthonormal ket vectors $|u_i\rangle$ and $|u_j\rangle$ and the operator A acting on u_i and u_j represents the operator $\hat{A}$ acting on $|u_i\rangle$ and $|u_j\rangle$, then the inner product of $|u_i\rangle$ with $\hat{A}|u_j\rangle$ — i.e., $\langle u_i|\hat{A}|u_j\rangle$ — is nothing else than the inner product of u_i with Au_j:

$$\langle u_i|\hat{A}|u_j\rangle = (u_i, Au_j). \tag{4.12}$$

For example, when written in terms of ket vectors, Eq. (3.20) reads

$$\mathsf{A} = \begin{pmatrix} \langle u_1|\hat{A}|u_1\rangle & \langle u_1|\hat{A}|u_2\rangle & \cdots & \langle u_1|\hat{A}|u_N\rangle \\ \langle u_2|\hat{A}|u_1\rangle & \langle u_2|\hat{A}|u_2\rangle & \cdots & \langle u_2|\hat{A}|u_N\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_N|\hat{A}|u_1\rangle & \langle u_N|\hat{A}|u_2\rangle & \cdots & \langle u_N|\hat{A}|u_N\rangle \end{pmatrix}. \tag{4.13}$$

Thus calculations involving operators acting on ket vectors can generally be reduced to calculations involving operators acting on wave functions or column vectors if this would be necessary for obtaining the result.

The reduction is often not necessary, though. Consider, for example, the following problem of quantum optics: calculate the matrix element $\langle\alpha|\hat{a}_+|\alpha\rangle$, where α is a complex number, the ket vector $|\alpha\rangle$ is a normalized eigenvector of the operator $\hat{a}_-$ with eigenvalue α ($\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\alpha\rangle = 1$), and $\hat{a}_+$ and $\hat{a}_-$ are ladder operators ($\hat{a}_+ = \hat{a}_-^\dagger$). This matrix element can be calculated immediately without passing to a representation in terms of wave functions or column vectors: In view of Eq. (4.11), of the fact that $\hat{a}_+^\dagger = (\hat{a}_-^\dagger)^\dagger = \hat{a}_-$ and of the assumptions that $\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\alpha\rangle = 1$,

$$\langle\alpha|\hat{a}_+|\alpha\rangle = \langle\alpha|\hat{a}_+^\dagger|\alpha\rangle^* = \langle\alpha|\hat{a}_-|\alpha\rangle^* = (\alpha\langle\alpha|\alpha\rangle)^* = \alpha^*. \tag{4.14}$$

Two important notes:

1. The inner product $\langle\phi|\psi\rangle$ is in general a complex number, and $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$.

2. The bra vector conjugate to the ket vector $\hat{A}|\psi\rangle$ can be written as $\langle\psi|\hat{A}^\dagger$. (Recall that any ket vector $|\psi\rangle$ has a conjugate bra vector $\langle\psi|$, see Section 4.2 above.)

☞ That $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ simply follows from the axioms of the inner product. To understand why the bra conjugate to $\hat{A}|\psi\rangle$ can be written as $\langle\psi|\hat{A}^\dagger$, consider the ket vector $|A\psi\rangle = \hat{A}|\psi\rangle$ and the inner product $\langle\phi|A\psi\rangle$, where $|\phi\rangle$ is arbitrary. Note that

$$\langle\phi|A\psi\rangle = \langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle^* \tag{4.15}$$

and also that

$$\langle\phi|A\psi\rangle = \langle A\psi|\phi\rangle^*. \tag{4.16}$$

Thus $\langle A\psi|\phi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle$ for any $|\phi\rangle$, which shows that the bra $\langle A\psi|$ can be replaced by $\langle\psi|\hat{A}^\dagger$ in any calculation.

☞ Take, for example, $\langle\alpha|\hat{a}_-^\dagger$. ($\langle\alpha|$ is the bra conjugate to the ket $|\alpha\rangle$ and as above $\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle$.) Since $\alpha^*\langle\alpha|$ is the bra conjugate to the ket $\alpha|\alpha\rangle$, we see that $\langle\alpha|\hat{a}_-^\dagger = \alpha^*\langle\alpha|$. In other words, $\langle\alpha|$ is a *left* eigenvector of $\hat{a}_-^\dagger$, in the same way as $|\alpha\rangle$ is a *right* eigenvector of $\hat{a}_-$. The words left and right can be taken as defining the direction in which the operator acts: in $\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle$, the operator $\hat{a}_-$ acts "on the right" on $|\alpha\rangle$, whereas in $\langle\alpha|\hat{a}_-^\dagger = \alpha^*\langle\alpha|$ the operator $\hat{a}_-^\dagger$ acts "on the left" on $\langle\alpha|$.

The above calculation of the matrix element $\langle\alpha|\hat{a}_+|\alpha\rangle$ could thus be done quickly by letting $\hat{a}_+$ act "on the left" on $\langle\alpha|$: since $\hat{a}_+ = \hat{a}_-^\dagger$,

$$\langle\alpha|\hat{a}_+ = \langle\alpha|\hat{a}_-^\dagger = \alpha^*\langle\alpha| \tag{4.17}$$

and therefore $\langle\alpha|\hat{a}_+|\alpha\rangle = \alpha^*\langle\alpha|\alpha\rangle = \alpha^*$.

There is nothing unusual with operators having left eigenvectors. For example, it is easy to verify that

$$\begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.18}$$

and also that

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 1 & 1 \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{4.19}$$

The first of these two equations shows that the column vector it figures is a right eigenvector of this square matrix, and the second that the corresponding row vector is a left eigenvector of the conjugate transpose of this matrix (its adjoint). Note that the left and right eigenvectors are different kinds of vectors (the former is a row vector, the latter a column vector), in the same way as the left eigenvector of $\hat{a}^\dagger$ and the right eigenvector of $\hat{a}$ are different kinds of vectors (the former is a bra vector, the latter a ket vector).

We will use the Dirac notation from now on, for simplicity, unless we would talk specifically about operators acting on wave functions or column vectors. However, all the results stated normally apply to any Hilbert space, not just spaces of ket vectors.

4.4 The Principle of Superposition

The Principle of Superposition is a fundamental principle of Quantum Mechanics. It postulates that if the vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ represent physically possible states of a quantum system, then any linear combination of these two vectors also represents a physically possible state of that system.

Take, for example, an atom of hydrogen. This atom can be in the ground state, whose time-dependent wave function is $\psi_{100}(r,\theta,\phi)\exp(-iE_1t/\hbar)$. It can also be in the 2s state, whose time-dependent wave function is $\psi_{200}(r,\theta,\phi)\exp(-iE_2t/\hbar)$. Therefore an atom of hydrogen can also (at least in principle) be in a linear superposition of these two states, e.g., in a state whose wave function is given by the equation

$$\Psi(r,\theta,\phi,t) = \frac{1}{\sqrt{2}}\,\psi_{100}(r,\theta,\phi)\exp(-iE_1t/\hbar) + \frac{1}{\sqrt{2}}\psi_{200}(r,\theta,\phi)\exp(-iE_2t/\hbar). \tag{4.20}$$

An atom in this state is, in a sense, both in the 1s state and in the 2s state. If checked, this atom could be found to be in the ground state or, with the same probability, to be in the 2s state. Note that the state described by $\Psi(r,\theta,\phi,t)$ is neither the 1s state nor the 2s state. Some of its physical properties are quite different; for instance, it is not difficult to see that the probability density $|\Psi(r,\theta,\phi,t)|^2$ varies in time whereas $|\psi_{100}(r,\theta,\phi)\exp(-iE_1t/\hbar)|^2$ and $|\psi_{200}(r,\theta,\phi)\exp(-iE_2t/\hbar)|^2$ don't.

As a second example, let us consider the spin states of the silver atoms in Stern's and Gerlach's experiment. As mentioned in Section 1.2 of these notes, these spin states can be represented by 2-component column vectors, e.g.,

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}. \tag{4.21}$$

Later in the course, we will see that conventionally these two column vectors represent a state of spin up (χ_+) or spin down (χ_-) in the z-direction. We can make linear combinations of χ_+ and χ_- to form new column vectors, for instance

$$\chi_a = \frac{1}{\sqrt{2}}\,\chi_+ + \frac{1}{\sqrt{2}}\,\chi_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}. \tag{4.22}$$

Since χ_+ and χ_- represent possible spin states of an atom of silver, by the Principle of Superposition χ_a also represents a possible spin state of that system. In fact, χ_a represents a state of "spin up" in the x-direction (not the z-direction, this will be explained later in the course). Moreover, introducing vectors representing states of "spin up" and "spin down" in the y-direction, respectively

$$\chi_{y+} = \begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} \quad \text{and} \quad \chi_{y-} = \begin{pmatrix}i/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}, \tag{4.23}$$

we can write χ_a not only as a linear combination of the vectors χ_+ and χ_- but also as a linear combination of the vectors χ_{y+} and χ_{y-}: as can be checked easily,

$$\chi_a = \frac{1}{2}(1-i)\chi_{y+} + \frac{1}{2}(1-i)\chi_{y-}. \tag{4.24}$$

An atom of silver in the spin state described by χ_a can thus be understood as being in a state of "spin up" in the x-direction, or as being in both the state of spin up and the state of spin down in the z-direction, or also in both the state of "spin up" and the state of "spin down" in the y-direction. These three descriptions are equivalent, and each one is as good as the other two, even though they may seem contradictory.

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