University of Durham

EXAMINATION PAPER

May/June 2013 Examination code: 043631/01

LEVEL 3 PHYSICS: FOUNDATIONS OF PHYSICS 3B

SECTION A. MODERN OPTICS 3
SECTION B. STATISTICAL PHYSICS
SECTION C. MAGNETIC MATERIALS

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

CALCULATORS: The following types ONLY may be used: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A. MODERN OPTICS 3

Answer Question 1 and **either** Question 2 **or** Question 3.

1. (a) The Fourier transforms of $g(x)$ and $h(x)$ are

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{i}k_x x}\mathrm{d}x \quad \text{and} \quad H(k_x) = \int_{-\infty}^{\infty} h(x)\mathrm{e}^{-\mathrm{i}k_x x}\mathrm{d}x \ ,$$

respectively. Give expressions for the Fourier transforms of: (i) $g(2x)$, (ii) $g(x) + h(x/2)$, (iii) $g(x + d) + g(x - d)$ and (iv) $g(x)h(x)$. [4 marks]

(b) The Fourier transform of the two-dimensional function,

$$\mathrm{circ}\left(\frac{\rho}{D}\right) = \begin{cases} 1 & \rho \ \leq \ D/2 \\ 0 & \rho \ > \ D/2 \end{cases}$$

is $\pi(D/2)^2\mathrm{jinc}(k_\rho D/2)$, where $\rho^2 = x^2 + y^2$ and $k_\rho^2 = k_x^2 + k_y^2$. Find the two dimensional Fourier transform of the function,
$f(x, y) = \mathrm{circ}(\rho/D)\cos(\beta x)$. [4 marks]

(c) Give approximate expressions for the far-field angular spread of (i) light downstream of a single slit of width a, and (ii) for a Gaussian beam with beam waist a. Estimate the spot size (i.e., the beam radius) of a laser beam on the Moon produced by a green laser pointer with wavelength 532 nm and a beam waist $w_0 = 1$ mm on Earth. The distance to the Moon is 384,000 km. [4 marks]

(d) The propagation of a monochromatic light field over a distance z is given by the Fresnel integral

$$\mathcal{E}^{(z)} = \frac{1}{\mathrm{i}\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}^{(0)}\mathrm{e}^{\mathrm{i}kr}\mathrm{d}x'\mathrm{d}y' \ ,$$

where $\mathcal{E}^{(0)}$ is the input field, and $r = [(x - x')^2 + (y - y')^2 + z^2]^{1/2}$ is the distance between the points (x', y') and (x, y) in the input and output planes, respectively. Re-write an expression for r using the paraxial approximation and use this result to re-write the Fresnel integral in the form of a Fourier transform. [4 marks]

(e) In a $4f$ spatial filter set up, an aperture with the shape of the letter d in the input plane is illuminated normally by uniform monochromatic light. Sketch the intensity distribution in the Fourier plane. What filter is required to change a d into an o? [4 marks]

(f) A mask is punctured with 3 holes in an equilateral triangle with one side horizontal. Sketch the far–field intensity distribution along the horizontal and vertical axes if the mask is illuminated normally by uniform monochromatic light. [4 marks]

2. Sketch the $4f$ set-up used for optical spatial filtering. [6 marks]

   The field in the Fourier plane has the functional form gauss$(\rho/w)\otimes$comb$_4(x/d)$, where

$$\text{gauss}(x) = \mathrm{e}^{-x^2} \ , \quad \text{comb}_N\left(\frac{x}{d}\right) = \sum_{n=-(N-1)/2}^{(N-1)/2} \delta\left(x - nd\right) \ ,$$

   $\rho^2 = x^2 + y^2$, w and d are both distances with $w \ll d$. Sketch the intensity distribution along the x and y axes. [4 marks]

   Sketch the intensity distribution along the x and y axes in the output plane. [3 marks]

   A mask is placed in the Fourier plane that blocks the central two intensity maxima. Sketch the intensity distribution in the output plane along the x axis. How does the maximum intensity compare to the case without the mask? [3 marks]

   Next the mask is changed to only block one of the central 'spots'. Draw phasor diagrams corresponding to positions where the intensity in the output plane is 1/9th of the maximum. [4 marks]

3. The equation for a Gaussian beam with wavelength $\lambda = 2\pi/k$ and beam waist w_0 is

$$\mathcal{E}^{(z)}(x,y,z) \quad = \quad \mathcal{E}_0 \frac{w_0}{w} \mathrm{e}^{\mathrm{i}kz} \mathrm{e}^{-\mathrm{i}\alpha} \mathrm{e}^{\mathrm{i}k\rho^2/2R} \mathrm{e}^{-\rho^2/w^2} \ ,$$

where $\rho^2 = x^2 + y^2$, the beam radius $w = w_0(1 + z^2/z_\mathrm{R}^2)^{1/2}$, the radius of wavefront curvature $R = z + z_\mathrm{R}^2/z$ with $z_\mathrm{R} = \pi w_0^2/\lambda$, and α is the Gouy phase.

Sketch on the same axes the intensity as a function of position along the x-axis at $z = 0$ and $z = z_R$. Indicate w_0 on the plot. [4 marks]

Sketch the beam radius as a function of z and indicate the wavefront curvature at $z = 0$, $z = z_\mathrm{R}$ and $z > z_\mathrm{R}$. [5 marks]

Plot the radius of wavefront curvature as a function of z and indicate where it is a minimum. [2 marks]

A laser cavity is formed from two identical mirrors with radius of curvature R_m. Find an expression for the beam waist at the position of optimal stability. [5 marks]

Give an expression for the optimal cavity length in terms of the Rayleigh range and comment, briefly, on the mirror curvature. [2 marks]

Assuming that the beam waist remains constant draw a sketch to indicate what happens if one mirror is moved either (i) farther away from, or (ii) closer to the other. [2 marks]

SECTION B. STATISTICAL PHYSICS
Answer Question 4 and **either** Question 5 **or** Question 6.

4. (a) There are five molecules in an otherwise empty lecture theatre. Using the appropriate distribution calculate both
(i) how many different arrangements there are with two molecules in the front third of the room and
(ii) the probability that at any given time there are two of them in the front third of the room. [4 marks]

(b) Explain why ^{3}He and ^{4}He are composite fermions and bosons, respectively. [4 marks]

(c) Write expressions for the Bose-Einstein and Fermi-Dirac distributions. Define a condition for the classical limit, where the Boltzmann distribution provides an accurate description of both bosons and fermions. [4 marks]

(d) The two lowest-lying energy levels of a hydrogen atom are $E_0 = -13.6$ eV and $E_1 = -3.4$ eV. At what temperature would you expect to find 10^4 fewer hydrogen atoms in the first excited state than in the ground state? [4 marks]

(e) Is a negative spin temperature *'hotter'* (i.e., higher energy) or *'colder'* than a positive spin temperature? Suggest how a negative spin temperature might be produced in an experiment. [4 marks]

(f) Briefly describe how the technique of simulated annealing may be used to find the global minimum energy of a large particle system with many variables. At a given temperature, T, what is the probability of escaping a local minimum of depth ΔE? [4 marks]

5. (a) Show using the equipartition theorem, or otherwise, that the heat capacity, C_V, of a classical monatomic gas of N particles at a temperature T is

$$C_V = \frac{3}{2} N k_{\mathrm{B}}.$$

[2 marks]

(b) Sketch a graph of the variation of the heat capacity of a diatomic gas with temperature. Use this graph to explain why the experimentally measured ambient temperature heat capacities of most diatomic gases are substantially different to that predicted by the equipartition theorem. [6 marks]

(c) The vibration of a diatomic molecule can be modelled as a 1D harmonic oscillator with energy levels

$$\varepsilon_i = (i + 1/2)\hbar\omega.$$

Show that the vibrational contribution to the heat capacity of a system of N molecules of a diatomic gas is given by

$$C_{vib} = N k_{\mathrm{B}} \frac{x^2 e^x}{(e^x - 1)^2} \qquad \text{where} \qquad x = \frac{\hbar\omega}{k_{\mathrm{B}} T}.$$

Use this to estimate the vibrational contribution to the heat capacity of a system of N molecules of a diatomic gas at both low temperatures ($T \to 0$) and at high temperatures ($T \to \infty$). [8 marks]

(d) The physicists in Flatworld exist in a two dimensional universe. What would be their observed heat capacity of (i) a gas of N atoms and (ii) a gas of N diatomic molecules? [4 marks]

6. (a) Define the chemical potential of a system containing a large number of weakly interacting particles. What limiting values does the chemical potential have at low temperatures for (i) fermions and (ii) bosons? [4 marks]

(b) Consider N lithium atoms trapped inside a 3D harmonic potential well. The energy levels are $\varepsilon_i = (i + 3/2)\hbar\omega$ with degeneracies $g_i = \frac{1}{2}(i + 1)(i + 2)$. What is the chemical potential at absolute zero, i.e. $T = 0$, for 50 ^{6}Li atoms with spin 1/2? Express your answer in units of $\hbar\omega$. [4 marks]

(c) For ^{6}Li there are approximately 4.6 x 10^{22} conduction electrons per cm^3, which behave approximately as a free electron gas. Calculate the Fermi wave vector and the Fermi energy (in eV) of free electrons in ^{6}Li metal. [4 marks]

(d) In 2005 a dilute ^{6}Li vapour was cooled to produce a molecular condensate that displayed quantised vortices and superfluidity. Given that ^{6}Li is a composite fermionic particle explain how Bose-Einstein condensation could be observed. [2 marks]

This transition happens at the Bose-Einstein condensation temperature, T_{B}, given by

$$\frac{N}{V}\left(\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T_{\mathrm{B}}}\right)^{\frac{3}{2}} = 2.612$$

where V is the volume of the condensate of N particles, and m is the mass of the particles. Calculate the Bose-Einstein condensation temperature (T_{B}) and the de Broglie wavelength (λ) of ^{6}Li$_2$ molecules given that approximately 10^5 lithium molecules were observed to Bose-Einstein condense in a sphere of radius 20 μm. [6 marks]

SECTION C. MAGNETIC MATERIALS
Answer Question 7 and **either** Question 8 **or** Question 9.

7. (a) Use Hund's rules to find the values of the total spin, S, the total orbital angular momentum, L, and the total orbital angular momentum, J, of an isolated holmium ion, Ho^{3+} which has 10 electrons in the $4f$ shell. [4 marks]

   (b) Briefly describe the nature of superexchange and indicate the type of magnetic ordering that often results from this exchange mechanism. [4 marks]

   (c) Sketch the temperature dependence of the magnetic susceptibility of a Type I superconductor when $\mu_o H < \mu_o H_\mathrm{C}$ and when $\mu_o H > \mu_o H_\mathrm{C}$ assuming that only the atomic diamagnetic response is present in the normal states. Indicate the critical temperature on your sketch [4 marks].

8. (a) Sketch the variation of the potential energy with applied magnetic field of an atomic magnetic moment with $J = S = 1$ and $L = 0$, where J is the total angular momentum, S is the total spin and L is total orbital angular momentum. Include the variations of all states in your sketch and label each state with the appropriate value for m_J, the quantum number associated with the z component of the total angular momentum. [4 marks]

(b) It can be shown that, for a solid of atoms with $J = 1$, the magnetization at a magnetic flux density B and temperature T is given by,

$$M = Ng\mu_{\mathrm{B}} \frac{\sum\limits_{m_J=-1}^{m_J=1} m_J \exp(m_J g \mu_{\mathrm{B}} B/(k_{\mathrm{B}}T))}{\sum\limits_{m_J=-1}^{m_J=1} \exp(m_J g \mu_{\mathrm{B}} B/(k_{\mathrm{B}}T))}$$

where N is the number of atoms per unit volume and μ_{B} is the Bohr magneton where $\mu_{\mathrm{B}} = 9.27 \times 10^{-24}$ J T^{-1}. Use the expression for M to show that, at small values of B and/or high values of T, the magnetic susceptibility follows Curie's Law, $\chi = C/T$ where C is Curie's constant. [5 marks]

(c) Calculate the Curie constant for the solid given that it has a simple cubic crystal structure with a lattice constant of 0.3 nm. [4 marks]

(d) It is found that the measured susceptibility vanishes at a temperature of 500 K. Use this information to calculate the diamagnetic susceptibility of the solid. Assuming that Langevin's theory of diamagnetism is applicable to this solid calculate the root mean square distance between an electron and the nucleus in an atom given that the atomic number is 110. State another possible origin for the diamagnetism in this solid. [7 marks]

9. (a) Sketch the variation of the spontaneous magnetization, M_s, of a ferromagnet as a function of temperature. Indicate the Curie temperature and spontanteous magnetization at $T = 0$ K. [4 marks]

(b) A ferromagnetic material, with Curie temperature $T_C = 100$ K, consists of atoms with total spin $S = 1/2$ and total orbital angular momentum $L = 0$. It can be shown that the magnetization of the solid has the form $M = M_{sat}\tanh(\mu_{\mathrm{B}}B/k_{\mathrm{B}}T)$ where M_{sat} is the saturated magnetization value and the other symbols have their usual meanings. Modify this expression to describe the spontaneous component of the magnetization, for $T < T_C$, by using Weiss' molecular field theory and taking the molecular field constant to be $N_W = 3k_{\mathrm{B}}T_C/[S(S+1)g^2\mu_{\mathrm{B}}^2\mu_0 N]$ where g is the Landé - g factor, N is the number of atoms per unit volume and μ_{B} is the Bohr magneton where $\mu_{\mathrm{B}} = 9.27 \times 10^{-24}$ J T^{-1}. [4 marks]

(c) Calculate the value of the spontaneous magnetization at $T = 0$ K given that the ferromagnet has a face centred cubic crystal structure with a conventional unit cell lattice constant of 0.5 nm. [4 marks]

(d) At temperatures just below T_C, the spontaneous magnetization has the form,

$$M_s^2 = 3M_{sat}^2[t+1]^3[\frac{1}{t+1} - 1]$$

where $t = (T - T_C)/T_C$. Use this expression to demonstrate that, for very small values of t, $M_s \propto (T_C - T)^{1/2}$. [4 marks]

(e) The spontaneous magnetization of the material mentioned in part (b) is measured at 99.9 K to be 24 A m^{-1} while at 99.5 K the value is 41 A m^{-1}. Use these data to determine whether the molecular field result of part (d) is a correct description of the spontaneous magnetization of the ferromagnet very close to T_C. [4 marks]