Mathematical Methods in Physics

Examination June 2019

Question 1

(a) (Analysis)

(i) No. For instance: there is no zero element or the set is not closed with respect to addition. [2 marks]

(ii) No. For instance: there are no inverse elements. [2 marks]

(b) (Evaluation)

(i) Linearly independent. The determinant is different from zero. [1 mark]

(ii) Linearly dependent. The third is the first multiply by $-1/2$. [1 mark]

(iii) Linearly dependent. Four three component vectors cannot be linearly independent. [1 mark]

(iv) Linearly independent. The determinant is different from zero. [1 mark]

(c) (Application)

The expression $A\underline{v} = \lambda\underline{v}$ leads to the equations $2 - y = \lambda$ and $2y - 2 = \lambda y$, which imply that $y = \pm\sqrt{2}$ and $\lambda = 2 \mp \sqrt{2}$ (signs correlated). [3 marks].

The third eigenvalue can be found because the trace of the matrix A, which is 6, is equal to the sum of the three eigenvalues. It follows that the third eigenvalue is 2. [1 mark]

(d) (Evaluation)

$$x = 2\,\cos\theta, \quad y = 3\,\sin\theta \quad \longrightarrow \quad dx = -2\,\sin\theta\,d\theta, \quad dy = 3\,\cos\theta\,d\theta.$$

[2 marks]

Hence

$$I = \int_0^{2\pi} \left(-\frac{4}{3}\cos\theta + \frac{9}{2}\sin\theta\right) d\theta = 0.$$

[2 marks]

(e) (Evaluation)

$$\frac{\partial \underline{r}}{\partial u} = \hat{\underline{i}} + 2u\,\hat{\underline{k}}, \qquad \frac{\partial \underline{r}}{\partial v} = \hat{\underline{j}}.$$

$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\,dv = (-2u\,\hat{\underline{i}} + \hat{\underline{k}})\,du\,dv.$$

[2 marks]

Hence

$$\underline{a} = uv\,\hat{\underline{i}} + u^2\,\hat{\underline{k}}, \qquad \underline{a}\cdot d\underline{S} = (-2u^2v + u^2)\,du\,dv$$

and

$$I = \int_{-1}^{1} du\,u^2 \int_{0}^{2} dv\,(1-2v) = \int_{-1}^{1} du\,u^2(-2) = -\frac{4}{3}.$$

[2 marks]

(f) (Knowledge)

The divergence theorem states that

$$\int_V (\nabla\cdot\underline{F})dV = \int_S \underline{F}\cdot d\underline{S}.$$

The integral on the left is the integral of the divergence of the vector field $\underline{F}$ over the volume enclosed by the surface $\mathcal{S}$. The integral of the right is the integral of the vector field $\underline{F}$ over the surface $\mathcal{S}$. The symbol ∇ is a vector differential operator and $d\underline{S}$ is a differential vector perpendicular to the surface $\mathcal{S}$. [4 marks]

(g) (Comprehension)

$$\begin{aligned}
&\frac{1}{(b^2-a^2)}\mathcal{L}\left[\frac{e^{iat}+e^{-iat}-e^{ibt}-e^{-ibt}}{2}\right](s)\\
&= \frac{1}{2(b^2-a^2)}\mathcal{L}[e^{iat}](s) + \mathcal{L}[e^{-iat}](s) - \mathcal{L}[e^{ibt}](s) - \mathcal{L}[e^{-ibt}](s)\\
&= \frac{1}{2(b^2-a^2)}\left(\frac{1}{(s-ia)} + \frac{1}{(s+ia)} - \frac{1}{(s-ib)} - \frac{1}{(s+ib)}\right)\\
&= \frac{1}{(b^2-a^2)}\left(\frac{s}{(s^2+a^2)} - \frac{s}{(s^2+b^2)}\right) = \frac{s}{(s^2+a^2)(s^2+b^2)}
\end{aligned}$$

[4 marks]

(h) (Evaluation)

$$\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} H(t)e^{i\alpha t-i\omega t}dt - \int_{-\infty}^{\infty} H(t-\pi)e^{i\alpha t-i\omega t}dt\right) = \frac{1}{\sqrt{2\pi}}\left(\int_{0}^{\infty} e^{i\alpha t-i\omega t}dt - \int_{\pi}^{\infty} e^{i\alpha t-i\omega t}dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{0}^{\pi} e^{it(\alpha-\omega)}dt\right) = \frac{1}{\sqrt{2\pi}}\left.\frac{e^{it(\alpha-\omega)}}{i(\alpha-\omega)}\right|_0^{\pi} = \frac{1}{\sqrt{2\pi}}\frac{e^{i\pi(\alpha-\omega)}-1}{i(\alpha-\omega)}$$

$$= e^{i\pi(\alpha-\omega)/2}\sqrt{\frac{2}{\pi}}\,\frac{\sin\pi(\alpha-\omega)/2}{(\alpha-\omega)}.$$

[4 marks]

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Question 2

(a) (Analysis)

(i) The force is not conservative since $\nabla \times \underline{F}_1 = -\hat{\underline{j}} - \hat{\underline{k}} \neq 0$. [2 marks]

A suitable parametrisation for the path is $\underline{r}(t) = t\,\hat{\underline{i}} + t\,\hat{\underline{j}} + t\,\hat{\underline{k}}$, [2 marks]

Then

$$I = \int_C \underline{F}_1 \cdot d\underline{r} = \int_0^1 \underline{F}_1 \cdot \frac{d\underline{r}}{dt}\,dt = \int_0^1 (2t+1)\,dt = 2.$$

[3 marks]

(ii) The force is conservative since $\nabla \times \underline{F}_2 = 0$. [2 marks]

The potential ϕ is:

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &\equiv xy^2 + z \quad \longrightarrow \quad \phi = \frac{(xy)^2}{2} + zx + f(y,z),\\
\frac{\partial \phi}{\partial y} &= x^2 y + \frac{\partial f}{\partial y} \equiv x^2 y + 1 \quad \longrightarrow \quad f(y,z) = y + g(z),\\
\frac{\partial \phi}{\partial z} &= x + \frac{dg}{dz} \equiv x \quad \longrightarrow \quad g(z) = 0.
\end{aligned}$$

Therefore $\phi = (xy)^2/2 + xz + y + c$. [3 marks]

Hence

$$I = \int_C \underline{F}_2 \cdot d\underline{r} = \phi(2, 1/2, 2) - \phi(1,1,2) = 1/2 + 4 + 1/2 - 1/2 - 2 - 1 = 3/2.$$

[2 marks]

Three marks for students that solve the integral without the use of the potential.

(b) (Application)

$$\begin{aligned}
((\underline{u} \times \underline{v}) \times \underline{w})_i &= \epsilon_{ij\,k}\,(\underline{u} \times \underline{v})_j\,\underline{w}_k\\
&= \epsilon_{ij\,k}\,\epsilon_{j\,lm}\,u_l v_m w_k = (\delta_{i\,m}\delta_{kl} - \delta_{i\,l}\delta_{km})\,u_l v_m w_k
\end{aligned}$$

[2 marks]

$$= u_k\,v_i\,w_k - u_i\,v_k\,w_k = (\underline{u} \cdot \underline{w})\,v_i - (\underline{v} \cdot \underline{w})\,u_i.$$

[3 marks]

(c) (Comprehension)

(i)

$$\int_0^t u^a\,(t-u)^b\,du = \int_0^t u^a\,t^b\,\left(1-\frac{u}{t}\right)^b\,du.$$

Setting $u/t = x$ it becomes

$$\int_0^1 (xt)^a\,t^b\,(1-x)^b\,t\,dx = t^{a+b+1}\int_0^1 x^a\,(1-x)^b\,dx.$$

[4 marks]

(ii) Using the convolution theorem for the Laplace transforms applied to h

$$\mathcal{L}[h] = \mathcal{L}[f]\mathcal{L}[g] = \frac{a!}{s^{a+1}}\frac{b!}{s^{b+1}} = \frac{a!\,b!}{s^{a+b+2}}.$$

[3 marks]

$$\mathcal{L}^{-1}[\bar{h}] = h(t) = \frac{a!\,b!}{(a+b+1)!}\,t^{a+b+1}.$$

[2 marks]

Then, we notice that

$$h(1) = \int_0^1 x^a\,(1-x)^b\,dx = \frac{a!\,b!}{(a+b+1)!}.$$

[2 marks]

Alternatively, we can take the result of part (c)(i) and apply the Laplace transform. Because the integral is independent of t, we get

$$\mathcal{L}[h] = \mathcal{L}[t^{a+b+1}](s)\int_0^1 x^a\,(1-x)^b\,dx = \frac{(a+b+1)!}{s^{a+b+2}}\int_0^1 x^a\,(1-x)^b\,dx.$$

Then, by applying the convolution theorem, the result follows.

(a) [Application]

$$\frac{2y}{x}\frac{dy}{dx} = 4y^2 + 3xy^2$$

Separate the variables and integrate,

$$\int \frac{2y}{y^2} dy = \int (4 + 3x) x dx$$

[1 mark]

$$2lny = \frac{4x^2}{2} + \frac{3x^3}{3} + c$$

[1 mark]

Take exponents,

$$y = Ae^{x^2 + \frac{x^3}{2}}$$

[1 mark]

Solve for A, given that $x = 1$ and $y = 2e^{3/2}$,

$$y = 2e^{x^2 + \frac{x^3}{2}}$$

[1 mark]

(b) [Application]

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3x^2 + x + 2,$$

Find the auxillary equation by setting $RHS = 0$ and substituting $y = Ae^{\lambda x}$ then dividing result by $y = Ae^{\lambda x}$,

$$\lambda^2 - 4\lambda + 3 = 0$$

[1 mark]

Solve for λ,

$$(\lambda - 1)(\lambda - 3) = 0$$

Thus, roots are found at $\lambda = 1$ and $\lambda = 3$. Both roots are real and unique, so now we know the form of the complementary function,

$$y_c(x) = Ae^x + Be^{3x}$$

[1 mark]

Given the form of the right hand side, the particular solution is of the form,

$$y_p(x) = ax^2 + bx + c$$

We sub this into the ODE and solve for the coefficients,

$$2a - 4(2ax + b) + 3(ax^2 + bx + c) = 3x^2 + x + 2$$

x^2 terms

$$3a = 3 \Rightarrow a = 1$$

x terms

$$-8a + 3b = 3b - 8 = 1 \Rightarrow b = 3$$

const. terms

$$2a - 4b + 3c = 3c - 10 = 2 \Rightarrow c = 4$$

Hence the particular solution is,

$$y_p(x) = x^2 + 3x + 4$$

[1 mark]

And the general solution is,

$$y(x) = Ae^x + Be^{3x} + x^2 + 3x + 4$$

[1 mark]

(c) [Application]

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 20x^3e^{4x}$$

Find the auxillary equation by setting $RHS = 0$ and substituting $y = Ae^{\lambda x}$ then dividing result by $y = Ae^{\lambda x}$,

$$\lambda^2 - 8\lambda + 16 = 0$$

Solve for λ,

$$(\lambda - 4)(\lambda - 4) = 0$$

Thus, roots are found at $\lambda = 4$. Both roots are real but repeat, so now we know the form of the complementary function,

$$y_c(x) = Ae^{4x} + Bxe^{4x}$$

[1 mark]

Now we calculate the Wronskian,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{4x} & xe^{4x} \\ 4e^{4x} & e^{4x} + 4xe^{4x} \end{vmatrix} = e^{8x} + 4xe^{8x} - 4xe^{8x} = e^{8x}$$

[1 mark]

Using the hint provided we can determine A and B,

$$A' = -\frac{h(x)}{W(x)}y_2 = -\frac{20x^3e^{4x}}{e^{8x}}xe^{4x} = -20x^4$$

$$B' = \frac{h(x)}{W(x)}y_1 = \frac{20x^3e^{4x}}{e^{8x}}e^{4x} = 20x^3$$

Integrating,

$$A = -\int 20x^4 dx = -4x^5 + c_1$$

$$B = \int 20x^3 dx = 5x^4 + c_2$$

[1 mark]

Thus the general solution is given by,

$$y = (-4x^5 + c_1)e^{4x} + (5x^4 + c_2)xe^{4x}$$
$$= (x^5 + c_2x + c_1)e^{4x}$$

[1 mark]

(d) [Knowledge]

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 56y = 0$$

is a Legendre equation.

[1 mark]

The general form is,

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - vy = 0$$

[1 mark]

The solution is a polynomial if $-v = l(l + 1)$, where l is an integer. Here $l = 7$, so the solution is a Legendre polynomial,

$$P_6(x) = a_1x + a_3x^3 + a_5x^5 + a_7x^7$$

[2 marks]

(e) [Application]

$$\bar{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt$$
$$= \int_0^\infty e^{at}e^{-st}dt$$

$$= \int_0^\infty e^{(a-s)t} dt$$
$$= \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^\infty$$
$$= \frac{e^{(a-s)\infty}}{a-s} - \frac{1}{a-s}$$
$$= \frac{1}{s-a}$$

[3 marks]

if $s > a$.

[1 mark]

(f) [Application]

Look for solutions of form $u(x,t) = f(x)g(t)$. Plug into equation,

$$f(x)g^{''}(t) = c^2 g(t) f^{''}(x)$$
$$\frac{f^{''}(x)}{f(x)} = \frac{1}{c^2}\frac{g''(t)}{g(t)}$$

Since the LHS is t-independent and RHS is x-independent can say that neither side depends on either variable, i.e. they are constant. So assume,

$$\frac{f^{''}(x)}{f(x)} = \frac{1}{c^2}\frac{g''(t)}{g(t)} = -K^2 < 0$$

where K^2 is assumed to be positive. So,

$$f^{''}(x) + K^2 f(x) = 0$$

[1 mark]

$$g^{''}(x) + K^2 c^2 g(t) = 0$$

[1 mark]

First equation has solution $f(x) = A\cos Kx + B\sin Kx$, second has solution $g(t) = C\cos Kct + D\sin Kct$. $f(x)g(t)$ is a solution of the wave equation.

$$u(x,t) = (A\cos Kx + B\sin Kx)(C\cos Kct + D\sin Kct)$$

[2 marks]

Other common solutions:

$$u(x,t) = (A\cos(K/c)x + B\sin(K/c)x)(C\cos Kt + D\sin Kt)$$
$$u(x,t) = (Ae^{iKx} + Be^{-iKx})(Ce^{iKct} + De^{-iKct})$$
$$u(x,t) = (Ae^{i(K/c)x} + Be^{-i(K/c)x})(Ce^{iKt} + De^{-iKt})$$

(g) [Knowledge]/[Application]

The equation is an Euler equation.

[1 mark]

$$2x^2y'' + 2xy' - 8y = 0$$

Begin by substituing $y = x^\lambda$, since $RHS = 0$ (could use $x = e^t$, but takes more time and effort). This gives,

$$y' = \lambda x^{\lambda-1}$$

$$y'' = (\lambda - 1)\lambda x^{\lambda-2}$$

Sub back into equation,

$$2\lambda(\lambda - 1)x^\lambda + 2\lambda x^\lambda - 8x^\lambda = 0$$

$$(2\lambda^2 - 2\lambda + 2\lambda - 8)x^\lambda = 0$$

$$(2\lambda^2 - 8)x^\lambda = 0$$

$$\Rightarrow 2\lambda^2 - 8 = 0$$

$$\lambda = \pm 2$$

Giving the solution,

$$y = c_1x^2 + c_2x^{-2}$$

[3 marks]

(a) [Knowledge]

$$\frac{ds}{dx} = \frac{\partial s}{\partial x}\frac{dx}{dx} + \frac{\partial s}{\partial y}\frac{dy}{dx} + \frac{\partial s}{\partial z}\frac{dz}{dx}$$

$$= \frac{ds}{dx} = \frac{\partial s}{\partial x} + \frac{\partial s}{\partial y}\frac{dy}{dx} + \frac{\partial s}{\partial z}\frac{dz}{dx}$$

[2 marks]

(b) [Application]

$$2x\frac{\partial u}{\partial x} - 8x^4\frac{\partial u}{\partial y} = 0$$

If

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

then

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}$$

If

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = 0$$

then

$$\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y} = -\frac{B}{A}$$

Hence,

$$\frac{dy}{dx} = \frac{B}{A} = -\frac{8x^4}{2x} = -4x^3$$

so,

[2 marks]

$$dy = -4x^3 dx$$

$$y = -x^4 + c$$

$$c = y + x^4$$

[1 mark]

So $u(x,y) = f(y + x^4)$. If $u = 5$ when $y = -x^4$ then if $g(0) = 0$,

$$u(x,y) = g(y + x^4) + 5$$

where f and g are arbitrary functions that must be differentiable once in x and y.

[1 mark]

(c) [Application]

(i)

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial xy} + C\frac{\partial^2 u}{\partial y^2} = 0$$

Assume a solution $u(x, y) = f(ax + by)$. We know that $\frac{\partial p}{\partial x} = a$ and $\frac{\partial p}{\partial y} = b$ where a and b are constants, as p is linear in x and y. So, since

$$\frac{\partial u}{\partial x} = \frac{\partial p}{\partial x}\frac{df(p)}{dp}$$

we can say

$$\frac{\partial u}{\partial x} = a\frac{df(p)}{dp}$$

$$\frac{\partial u}{\partial y} = b\frac{df(p)}{dp}$$

thus

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{d^2 f(p)}{dp^2}$$

[1 mark]

$$\frac{\partial^2 u}{\partial xy} = ab\frac{d^2 f(p)}{dp^2}$$

[1 mark]

$$\frac{\partial^2 u}{\partial y^2} = b^2\frac{d^2 f(p)}{dp^2}$$

[1 mark]

Substitute these back into original equation and factorise

$$(Aa^2 + Bab + Cb^2)\frac{d^2 f(p)}{dp^2} = 0$$

We can find a solution independent of $f(p)$ if we require that

$$Aa^2 + Bab + Cb^2 = 0$$

[2 marks]

(ii) Dividing by a^2 and solving the quadratic we find

$$\frac{b}{a} = \frac{-B \pm (B^2 - 4AC)^{1/2}}{2C}$$

[1 mark]

And so if λ_1 and λ_2 are equal to the solutions of the quadratic then

$$p_1 = x + \lambda_1 y$$

$$p_2 = x + \lambda_2 y$$

and so

$$u(x,y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

[2 marks]

(d) [Application]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Begin by identifying the quadratic to be solved,

$$Aa^2 + Bab + Cb^2 = 0$$

In this case $A = 1$, $B = 0$ and $C = 1$, so,

$$a^2 + b^2 = 0$$

[2 marks]

Divide by a^2 and sub in $\lambda = b/a$

$$1 + \lambda^2 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

[1 marks]

Therefore the complementory function is

$$u_c = f(x + iy) + g(x - iy)$$

[2 marks]

Integrate the x and y terms separately

$$\int\int 24x dx dx = \int 12x^2 + c_1 dx = 4x^3 + c_1 x + c_2$$

$$\int\int 12y^2 dy dy = \int 4y^3 + e_1 dy = y^4 + e_1 y + e_2$$

[2 marks]

By comparing these solutions to the original equation, we note that the terms in x and y and the constant terms are not necessary, hence a solution

$$u_p = 4x^3 + y^4$$

[1 mark]

Combining u_c and u_p to get the general solution

$$u(x,y) = f(x + iy) + g(x - iy) + 4x^3 + y^4$$

[1 mark]

(e) (i) [Knowledge]

Spatial: second order

[1 mark]

Temporal: first order

[1 mark]

(ii) [Knowledge]

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

[2 marks]

(iii) [Knowledge]

Length$^2\times$Time^{-1}

[1 mark]

(iv) [Comprehension]

For part (c), by assuming a solution of the form $u(x, y) = f(p)$, where p is an unknown linear function of x and y, $p(x, y) = ax + by$, we may be able to obtain a common factor $d^2f(p)/dp^2$ as the only appearance of $f(p)$ on the LHS. This eliminates the problem of having 3 types of differential, i.e. d/dx^2, d/dy^2 and $d/dxdy$. Then, because of the zero RHS, all reference to the form of $f(p)$ can be cancelled out.

The method used in (c) relies on having differentials of the same order, but the diffusion equation has a second order differential in space and a first order in time.

[1 mark]

This means that for a solution $u(x, y) = f(p)$ with $p = ax + by$, $f(p)$ cannot be cancelled out. i.e. you cannot obtain the following

$$(Aa^2 + Bab + Cb^2)\frac{d^2f(p)}{dp^2} = 0$$

[1 mark]