Theoretical Physics 2019/20 — Solution of Problem QT2.7

(a) We need to invert the relations between $\hat{a}$, $\hat{a}^\dagger$, $\hat{x}$ and $\hat{p}$ given in the question, namely

$$\hat{a} = (2\hbar m\omega)^{-1/2}\,(m\omega\hat{x} + i\hat{p}), \qquad \hat{a}^\dagger = (2\hbar m\omega)^{-1/2}\,(m\omega\hat{x} - i\hat{p}).$$

This can be done simply by adding or subtracting these two equations. Adding them gives

$$\hat{a} + \hat{a}^\dagger = (2\hbar m\omega)^{-1/2}\,(2m\omega\hat{x}),$$

while substracting the second from the first gives

$$\hat{a} - \hat{a}^\dagger = (2\hbar m\omega)^{-1/2}\,(2i\hat{p}).$$

The results quoted in the question follow. [2 marks]

(b) This is an easy step. Since $\hat{x} = [2\hbar/(m\omega)]^{1/2}\,\hat{S}$, $\langle\alpha|\hat{x}|\alpha\rangle = [2\hbar/(m\omega)]^{1/2}\,\langle\alpha|\hat{S}|\alpha\rangle$, and therefore, since $\langle\alpha|\hat{S}|\alpha\rangle = \mathrm{Re}\,\alpha$, $\langle\alpha|\hat{x}|\alpha\rangle = [2\hbar/(m\omega)]^{1/2}\,\mathrm{Re}\,\alpha$. The result for $\langle\alpha|\hat{p}|\alpha\rangle$ is obtained in the same way. [1 mark]

(c) (i) Since $[\hat{a}_\mathrm{H}(t), \hat{H}_\mathrm{H}(t)] = \hat{U}(0,t)\,[\hat{a}, \hat{H}]\,\hat{U}^\dagger(0,t)$, the first step is to calculate $[\hat{a}, \hat{H}]$. We are told that $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$ (a result worth remembering). Since $\hat{a}$ commutes with the 1/2 term, we should simply calculate $[\hat{a}, \hbar\omega\hat{a}^\dagger\hat{a}]$ (the calculation makes use of the commutator given in the question, $[\hat{a}, \hat{a}^\dagger] = 1$):

$$\begin{aligned}[\hat{a}, \hbar\omega\hat{a}^\dagger\hat{a}] &= \hbar\omega\left(\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{a}\right)\\ &= \hbar\omega\left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right)\hat{a}\\ &= \hbar\omega\hat{a}.\end{aligned}$$

Thus $[\hat{a}_\mathrm{H}(t), \hat{H}_\mathrm{H}(t)] = \hbar\omega\,\hat{U}(0,t)\,\hat{a}\,\hat{U}^\dagger(0,t)$. Since $\hat{a}_\mathrm{H}(t) = \hat{U}(0,t)\,\hat{a}\,\hat{U}^\dagger(0,t)$, we obtain $[\hat{a}_\mathrm{H}(t), \hat{H}_\mathrm{H}(t)] = \hbar\omega\,\hat{a}_\mathrm{H}(t)$, as stated in the question. [2 marks]

(ii) The simplest way of showing that $\hat{a}_\mathrm{H}(t) = \hat{a}\,\exp(-i\omega t)$ is to check that this Heisenberg operator satisfies that differential equation as well as the initial condition $\hat{a}_\mathrm{H}(t=0) = \hat{a}$. The latter is obvious: since $\exp(0) = 1$, $\hat{a}_\mathrm{H}(t) = \hat{a}$ at $t = 0$. For the former, let us first rewrite the equation as

$$\frac{\mathrm{d}\hat{a}_\mathrm{H}}{\mathrm{d}t} = \frac{\hbar\omega}{i\hbar}\,\hat{a}_\mathrm{H}(t) = -i\omega\,\hat{a}_\mathrm{H}(t).$$

Replacing $\hat{a}_\mathrm{H}(t)$ by $\hat{a}\,\exp(-i\omega t)$ in this equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}\,\exp(-i\omega t) = -i\omega\,\hat{a}\,\exp(-i\omega t).$$

I.e., since $\hat{a}$ does not depend on time,

$$\hat{a}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\exp(-i\omega t) = \hat{a}\,[-i\omega\,\exp(-i\omega t)],$$

which is true. Hence $\hat{a}_\mathrm{H}(t) = \hat{a}\,\exp(-i\omega t)$ is indeed a solution of that initial condition problem. [2 marks]

Now, you may wonder, could there be another solution? If $\hat{a}_{\rm H}(t)$ was an ordinary function rather than an operator, you know that the answer would be no: the general solution of the equation $\mathrm{d}y/\mathrm{d}t = -i\omega y(t)$ is $C\exp(-i\omega t)$ where C is an arbitrary constant, and if you impose the condition that $y(t)$ at t_0 is some given y_0 then C is no longer arbitrary but must be equal to y_0 — there is one and only solution of that initial value problem. However, $\hat{a}_{\rm H}(t)$ is an operator, not an ordinary function... In fact, that $\hat{a}_{\rm H}(t)$ is not an ordinary function changes nothing here, the solution of the differential equation is still unique once an initial condition has been specified. Here is a proof: Suppose that $\hat{a}_{\rm H}(t) = \hat{a}\exp(-i\omega t) + \hat{O}(t)$, where $\hat{O}(t)$ is some unknown operator. It is easy to see that this operator must be such that $\mathrm{d}\hat{O}/\mathrm{d}t = -i\omega\hat{O}(t)$ with $\hat{O}(t=0) = 0$. Hence, for δt small enough, $\hat{O}(t+\delta t) = \hat{O}(t) - i\omega\hat{O}(t)\delta t = (1 - i\omega\delta)\hat{O}(t)$. In particular, setting $t = 0$, $\hat{O}(\delta t) = (1-i\omega\delta)\hat{O}(0) = 0$. But then $\hat{O}(2\delta t) = (1-i\omega\delta)\hat{O}(\delta t) = 0$, $\hat{O}(3\delta t) = (1 - i\omega\delta)\hat{O}(2\delta t) = 0$, etc., and we see that $\hat{O}(t)$ must be zero at all values of t. Hence there is no other solution than $\hat{a}\exp(-i\omega t)$. [No mark for this, but congratulations if you had thought about the issue of uniqueness.]

The corresponding result for $\hat{a}^\dagger_{\rm H}(t)$ can be obtained simply by taking the adjoint of $\hat{a}_{\rm H}(t)$. In general, the adjoint of the product of an operator $\hat{A}$ by a scalar c is $c^*\hat{A}^\dagger$. Thus the adjoint of $\hat{a}^\dagger_{\rm H}(t)$ is $[\exp(-i\omega t)]^*\hat{a}^\dagger$, which is $\hat{a}^\dagger\exp(i\omega t)$. [1 mark]

(iii) Using results obtained previously,

$$\begin{aligned}
\langle x\rangle(t) &= \langle\alpha|\hat{x}_{\rm H}(t)|\alpha\rangle \\
&= [2\hbar/(m\omega)]^{1/2}\,\Big(\langle\alpha|\hat{a}_{\rm H}(t)|\alpha\rangle/2 + \langle\alpha|\hat{a}^\dagger_{\rm H}(t)|\alpha\rangle/2\Big) \\
&= [2\hbar/(m\omega)]^{1/2}\,\big[\langle\alpha|\hat{a}|\alpha\rangle\exp(-i\omega t) + \langle\alpha|\hat{a}^\dagger|\alpha\rangle\exp(i\omega t)\big]/2.
\end{aligned}$$

The right-hand side of this equation can be worked out as in Question 2 of the Problem Test: Since $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\langle\alpha|\hat{a}|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha$. Moreover, $\langle\alpha|\hat{a}^\dagger|\alpha\rangle = (\langle\alpha|\hat{a}|\alpha\rangle)^* = \alpha^*\langle\alpha|\alpha\rangle^* = \alpha^*$. Therefore

$$\begin{aligned}
\langle x\rangle(t) &= [2\hbar/(m\omega)]^{1/2}\,[\alpha\exp(-i\omega t) + \alpha^*\exp(i\omega t)]/2 \\
&= [2\hbar/(m\omega)]^{1/2}\,[|\alpha|\exp(i\arg\alpha)\exp(-i\omega t) + |\alpha|\exp(-i\arg\alpha)\exp(i\omega t)]/2 \\
&= [2\hbar/(m\omega)]^{1/2}\,|\alpha|\,(\exp[i(\arg\alpha - \omega t)] + \exp[-i(\arg\alpha - \omega t)])/2 \\
&= [2\hbar/(m\omega)]^{1/2}\,|\alpha|\,\mathrm{Re}\,\exp[i(\arg\alpha - \omega t)] \\
&= [2\hbar/(m\omega)]^{1/2}\,|\alpha|\,\cos(\arg\alpha - \omega t).
\end{aligned}$$

Likewise,

$$\begin{aligned}
\langle p\rangle(t) &= (2\hbar m\omega)^{1/2}\,[\alpha\exp(-i\omega t) - \alpha^*\exp(i\omega t)]/(2i) \\
&= (2\hbar m\omega)^{1/2}\,[|\alpha|\exp(i\arg\alpha)\exp(-i\omega t) - |\alpha|\exp(-i\arg\alpha)\exp(i\omega t)]/(2i) \\
&= (2\hbar m\omega)^{1/2}\,|\alpha|\,(\exp[i(\arg\alpha - \omega t)] - \exp[-i(\arg\alpha - \omega t)])/(2i) \\
&= (2\hbar m\omega)^{1/2}\,|\alpha|\,\mathrm{Im}\,\exp[i(\arg\alpha - \omega t)] \\
&= (2\hbar(m\omega)^{1/2}\,|\alpha|\,\sin(\arg\alpha - \omega t).
\end{aligned}$$

[2 marks]