ELECTROMAGNETISM - Workshop 8th Set (Qns)
Optics and Fresnel's Equations

Professor D P Hampshire – 2nd Year Physics Lecture Course

The material for this workshop is split into three parts. Part I gives some background material. Part II gives some worked examples. Part III gives some additional unseen questions.

1 Background Material

1.1 The Laws of Optics (Independent of Polarisation)

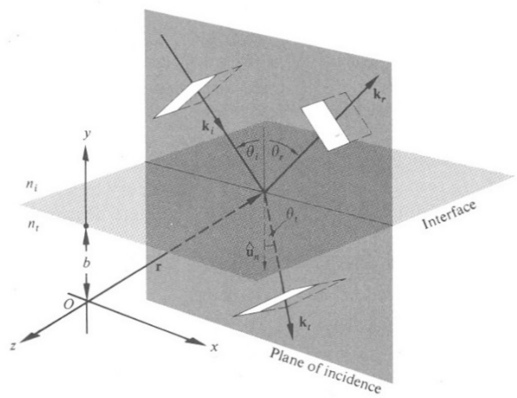


Figure 1 The reflection and transmission of waves at an oblique incidence.

1. The angle of incident is equal to the angle of reflection,

$$\theta_i = \theta_r \qquad \textbf{1-1}$$

2. Snell's Law,

$$\Rightarrow \frac{n_i}{n_t} = \frac{k_i}{k_t} = \frac{\sin\theta_t}{\sin\theta_i} \qquad \textbf{1-2}$$

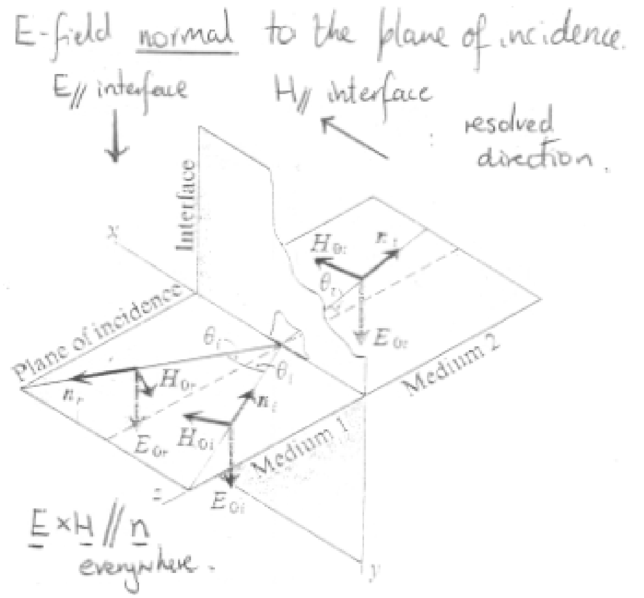
(Note: The refractive index of water is ~ 4/3 and that of Crown glass for reading glasses is ~ 1.5).

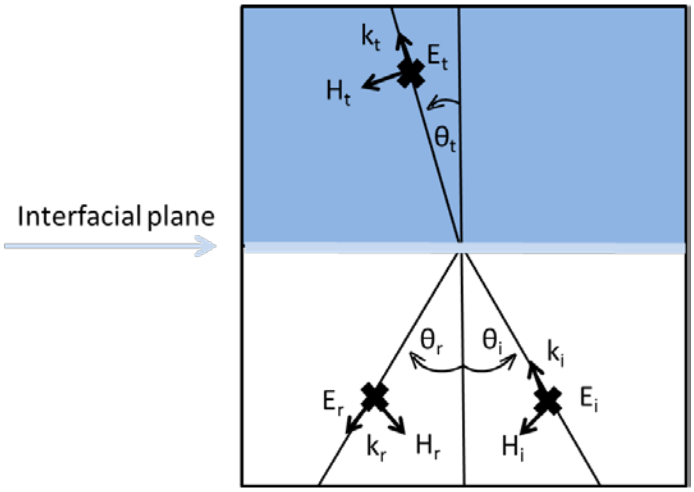
3. The incident, reflected and transmitted wave vector lie in the plane of incidence which is normal to the interfacial plane.

1.2 Fresnel's Equations

Fresnel's equations provide the framework for calculating the waves reflected from a boundary and refracted across a boundary.

1.2.1 E-field normal to the plane of incidence



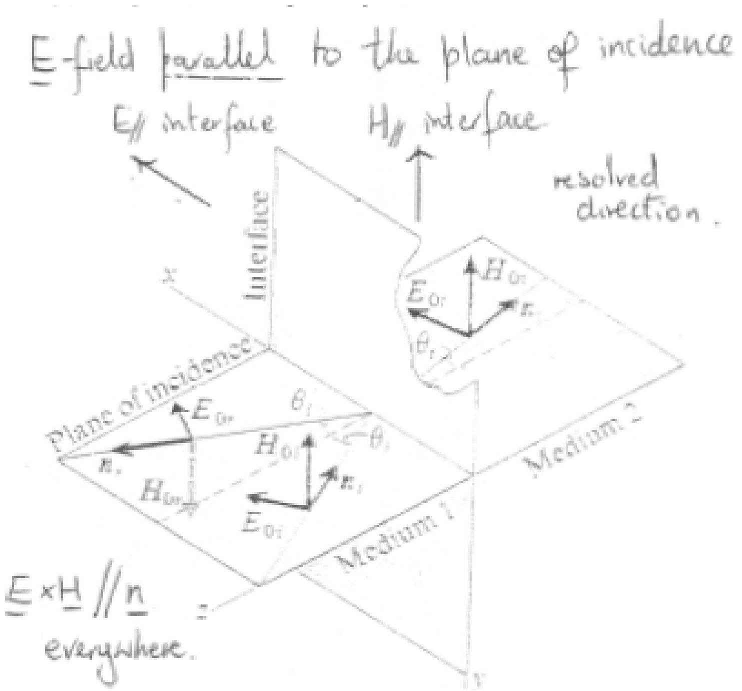


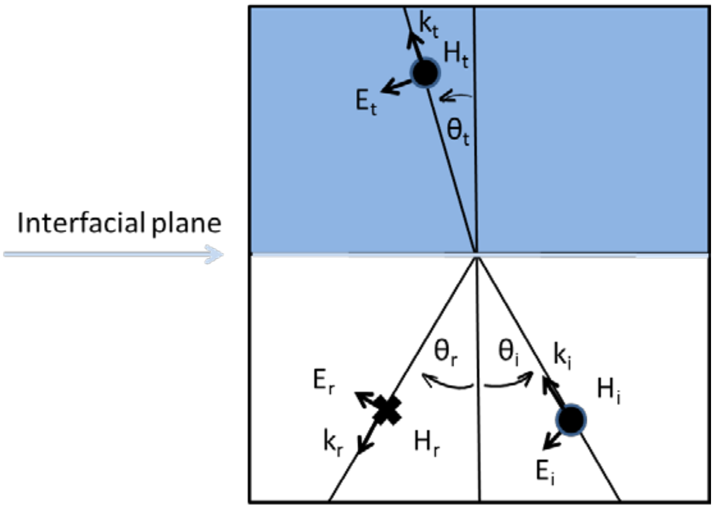
Fresnel's equations - E-field normal to the plane of incidence

$$\frac{E_{0r}}{E_{0i}} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} \qquad \textbf{1-3}$$

$$\frac{E_{0t}}{E_{0i}} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t} \qquad \textbf{1-4}$$

1.2.2 E-field parallel to the plane of incidence





Fresnel's equations - E-field parallel to the plane of incidence

$$\frac{E_{or}}{E_{oi}} = \frac{n_i \cos\theta_t - n_t \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} \qquad \textbf{1-5}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} \qquad \textbf{1-6}$$

E_{oi}, E_{or} *and* E_{ot} : incident, reflected and transmitted waves,
n_i and n_t : refractive index of medium 1 and 2 respectively,

....

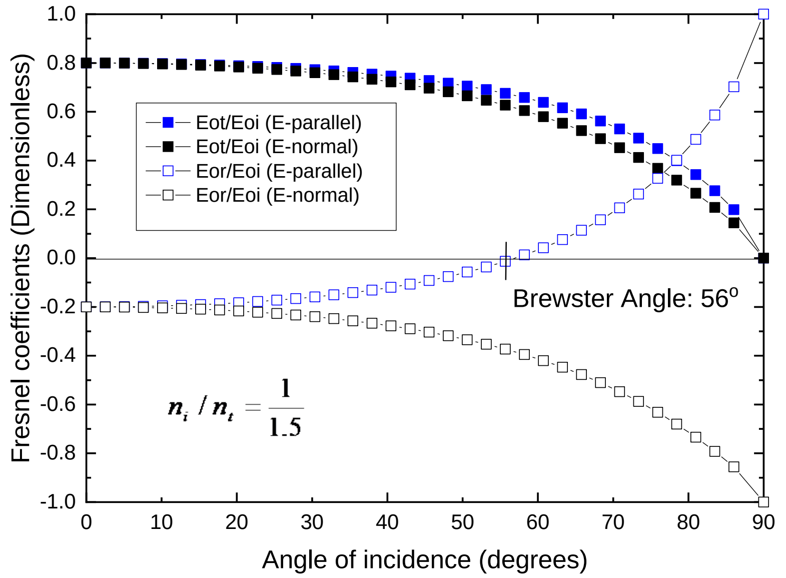
Brewster Angle : The reflection coefficient for the ***E***-field parallel to the plane of incidence is zero and the reflected light is fully polarized.

$$\tan\theta_B = \frac{n_t}{n_i} \qquad \textbf{1-7}$$

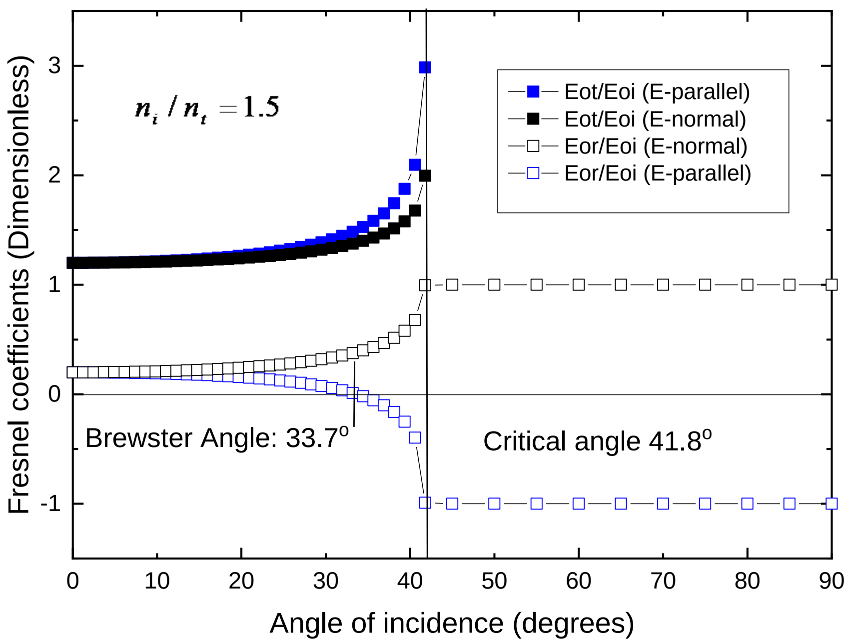
Critical Angle : Above the critical angle, there is only total internal reflection of the wave.

$$\sin\theta_C = \frac{n_t}{n_i} \qquad \textbf{1-8}$$

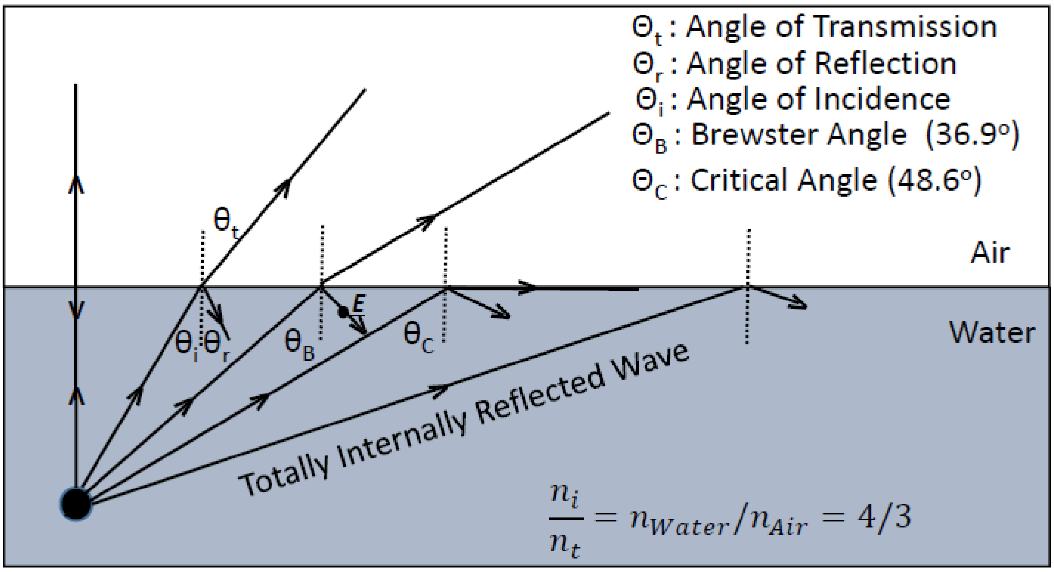
1.2.3 Transmission and reflection in air-glass, glass-air and water-air.



The Fresnel coefficients as a function of the angle of incidence for the ***E*** - field parallel and normal to the plane of incidence. The Brewster angle is 56 degrees for an air/glass boundary. Note that the coefficients are independent of polarisation when the direction of propagation is normal to the interface.



The Fresnel coefficients as a function of the angle of incidence for the ***E***-field parallel and normal to the plane of incidence. The Brewster angle is 33.7 degrees for a glass/air boundary. Above the critical angle of 41.8 degrees, there is internal reflection. Note that the coefficients are independent of polarisation when the direction of propagation is normal to the interface.



The reflection and transmission of light into the air that has originated in the water. The Brewster angle is 36.9 degrees for an glass/air boundary so the light that is reflected is polarized normal to the angle of incidence. Above the critical angle of 48.6 degrees, there is no transmission.

1.3 Transmission and Reflection coefficients

We can find reflection and transmission coefficients that are consistent with conservation of energy using the Poynting vector (***N*** : the instantaneous value of the power per unit area):

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \qquad \textbf{1-9}$$

for a dielectric (non-magnetic, non-conducting)

Using $B = \frac{E}{v_{phase}}$, $v_{phase} = \sqrt{\frac{1}{\varepsilon\mu_0}}$. The average power per unit area incident on the interface (note – not the perpendicular to the direction of travel), N_i, is:

$$N_i = \frac{1}{2}\left(\frac{\varepsilon_1}{\mu_0}\right)^{\frac{1}{2}} E_{0i}{}^2 \cos\theta_i \qquad \textbf{1-10}$$

The factor ½ comes from time averaging $\cos^2\omega t$..

The reflection coefficient is defined as

$$R = \frac{N_r}{N_i} = \left(\frac{E_{0r}}{E_{0i}}\right)^2, \qquad \textbf{1-11}$$

and the transmission coefficient is defined as

$$\mathrm{T} = \frac{\mathrm{N_t}}{\mathrm{N_i}} = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2} \frac{\mathrm{E_{Ot}}^2}{\mathrm{E_{Oi}}^2} \frac{\cos\theta_t}{\cos\theta_i} = \frac{n_2}{n_1} \frac{\mathrm{E_{Ot}}^2}{\mathrm{E_{Oi}}^2} \frac{\cos\theta_t}{\cos\theta_i} . \qquad \mathbf{1\text{-}12}$$

Using Fresnel's equations (in either polarisation), we find:

$$R + T = 1 \qquad \mathbf{1\text{-}13}$$

2 Worked examples

2.1 Questions

1. Using Fresnel's equations (in either polarisation), find the transmission and reflection coefficients when electromagnetic waves are incident normal to the interface (i.e. $\underline{\boldsymbol{k}}_i$ is normal to the interfacial plane). Assume that both media are non-magnetic materials.

If a plane wave travels from water, $n = \frac{4}{3}$, to air, $n = 1$, find the numerical value of the reflection (R) and transmission (T) coefficients and prove that $R + T = 1$.

2. Can you use Fresnel's equations to explain why Polaroid glasses have chains of conducting polymers running in the horizontal direction across them ?

3. The light seen when a rainbow is formed has been reflected by water droplets. The light is reflected at an angle close to the Brewster angle. Hence most of the light from a rainbow is polarized. Using Fresnel's equations, describe the polarization of the light from a rainbow Have you ever looked at a rainbow through a polarizer or polarized glasses ?

4. A plane wave in vacuum, with its $\underline{\boldsymbol{E}}$-field polarized normal to the plane of incidence, has an electric field amplitude $E_o = 10^{-4}$ Vm^{-1} and is incident at an angle of 30^o on an interface with a non-magnetic LIH (linear-isotropic-homogeneous) material with a refractive index of 2.6. What is the amplitude of the E-field of the reflected wave?

5. By using Snell's Law, show that, when the E-field is parallel to the plane of incidence and the media is nonmagnetic, the relevant Fresnel equation can be rewritten:

$$\frac{E_{or}}{E_{oi}} = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \qquad \mathbf{2\text{-}1}$$

You may use:

$$\sin A \cos B = \frac{1}{2}\sin(A + B) + \frac{1}{2}\sin(A - B) \qquad \mathbf{2\text{-}2}$$

And

$$\tan\frac{1}{2}(\alpha + \beta) = \frac{\sin(\alpha) + \sin(\beta)}{\cos(\alpha) + \cos(\beta)} \qquad \mathbf{2\text{-}3}$$

6. We can produce polarised light using the concepts developed by Brewster. When light is incident at Brewster's angle, the reflected light is polarized. Use the formula in question 3 to show that

$$\tan\theta_B = \frac{n_t}{n_i} \qquad \textbf{2-4}$$

where θ_B is the Brewster's angle.

7. At what angle is the light reflected from water polarized (for water $n = 1.5$) ?

8. Light travels from a material which has refractive index of 3 to air. Find the critical angle.

2.2 Answers

1. From the first law of optics, the angle of reflection is equal to the angle of incidence. From Snell's Law, we have:

$$\mathrm{n_i \sin\theta_i = n_t \sin\theta_t} \qquad \textbf{2-5}$$

If $\theta_i = 0^o$, then $\theta_r = \theta_t = 0$ too. We are given $\mu_i = \mu_t = \mu_o$

Using Fresnel's equations for either polarisation and substituting these angles into the equations, we have

$$\frac{E_{or}}{E_{oi}} = \frac{n_i - n_t}{n_i + n_t} \qquad \textbf{2-6}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2n_i}{n_i + n_t} \qquad \textbf{2-7}$$

Using the expression for the transmission coefficient gives

$$T = \frac{n_t}{n_i}\left(\frac{E_{ot}}{E_{oi}}\right)^2 = \frac{4n_i n_t}{(n_i + n_t)^2} \qquad \textbf{2-8}$$

and the reflection coefficient is

$$R = \left(\frac{E_{or}}{E_{oi}}\right)^2 = \left(\frac{n_i - n_t}{n_i + n_t}\right)^2. \qquad \textbf{2-9}$$

For the given values; $n_i = \frac{4}{3}$ and $n_t = 1$;

the reflection coefficient, $R = \left(\frac{E_{or}}{E_{oi}}\right)^2 = \left(\frac{\frac{4}{3} - 1}{\frac{4}{3} + 1}\right)^2 \approx 0.02$, and the transmission coefficient, $T = \frac{n_t}{n_i}\left(\frac{E_{ot}}{E_{oi}}\right)^2 = \frac{4\times\frac{4}{3}\times 1}{\left(\frac{4}{3}+1\right)^2} \approx 0.98$. Note that $R+T=1$.

2. The light that is vertically polarized is not reflected at Brewster's angle (i.e. R = 0 for light polarized in the plane of incidence). The conducting polymers remove the horizontally polarized light. Hence the Polaroid glasses remove glare.

3. Fresnel's equations show that E parallel to the plane of incidence is completely transmitted. Only the E-field normal to the plane of incidence produces a reflection. Hence the light is horizontally polarized from a rainbow at the top of the rainbow and horizontally polarized at the bottom if the arc is 180 degrees.
Yes is better than No.

4. First using Snell's Law to find θ_t,

$$n_i \sin\theta_i = n_t \sin\theta_t \qquad \textbf{2-10}$$

Substituting $n_i = 1, n_t = 2.6, \theta_i = 30^o$, then $\theta_t \approx 11^o$. Then, using Fresnel's equation when the polarization is normal the plane of incidence,

$$\frac{E_{0r}}{E_{0i}} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} = \frac{\cos 30 - 2.6\cos 11}{\cos 30 + 2.6\cos 11} = -4.9\times10^{-5}\ \mathrm{Vm^{-1}} \qquad \textbf{2-11}$$

The negative sign indicates that the reflected wave is π out of phase with the incident wave.

5. From Fresnel's equation,

$$\frac{E_{or}}{E_{oi}} = \frac{n_i\cos\theta_t - n_t\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad \textbf{2-12}$$

Using Snell's Law, $\frac{n_t}{n_i} = \frac{\sin\theta_i}{\sin\theta_t}$.

$$\frac{E_{or}}{E_{oi}} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i} \qquad \textbf{2-13}$$

Using the identity for $\sin A\cos B$ gives:

$$\frac{E_{or}}{E_{oi}} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \qquad \textbf{2-14}$$

Using:

$$\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)} = \frac{[\sin(\alpha)-\sin(\beta)]}{[\cos(\alpha)+\cos(\beta)]}\frac{[\cos(\alpha)+\cos(\beta)]}{[\sin(\alpha)+\sin(\beta)]} = \frac{[\sin(\alpha)-\sin(\beta)]}{[\sin(\alpha)+\sin(\beta)]} \qquad \textbf{2-15}$$

Hence:

$$\frac{E_{or}}{E_{oi}} = -\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)} \qquad \textbf{2-16}$$

6. In general, light can be considered as composed of E-fields with the polarization consisting of two components – parallel and normal to the plane of incidence. In order to polarise the reflected light, one of the E-field components must be vanish. Consider Fresnel's equations

$$r_{//} = \frac{E_{or}}{E_{oi}} = -\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)} \Leftarrow \text{parallel to the plane of incidence} \qquad \textbf{2-17}$$

$$r_{\perp} = \frac{E_{or}}{E_{oi}} = \frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)} \Leftarrow \text{normal to the plane of incidence} \qquad \textbf{2-18}$$

It is seen that $r_{\perp}$ cannot be made to vanish, while $r_{//}$ can. So this means at Brewster's angle, $r_{//}=0$, and this happens when $\tan(\theta_i+\theta_t)\to\infty$ i.e. $\theta_i+\theta_t=\frac{\pi}{2}$.

Using Snell's Law;

$$\frac{n_t}{n_i} = \frac{\sin\theta_i}{\sin\theta_t} = \frac{\sin\theta_i}{\sin(\frac{\pi}{2}-\theta_i)} = \frac{\sin\theta_i}{\cos\theta_i} = \tan\theta_i \qquad \textbf{2-19}$$

Therefore,

$$\tan\theta_B = \frac{n_t}{n_i} \qquad \textbf{2-20}$$

7. From $$\tan\theta_B = \frac{n_t}{n_i},$$

substituting $n_t=1.5$ and $n_i=1$ (air), $\tan\theta_B=1.5$ and $\theta_B=\arctan 1.5\approx 56^o$.

8. Using Snell's Law $$\frac{n_t}{n_i} = \frac{\sin\theta_i}{\sin\theta_t}.$$

At critical angle, the transmitted angle is 90°, $\frac{1}{3}=\frac{\sin\theta_c}{\sin 90}=\sin\theta_c$ and $\theta_c=\arcsin\frac{1}{3}\approx 19.5^o$

3 Unseen problems

1. Write down the general expression for the transmission and reflection coefficients. Consider an electromagnetic wave that travels from air, n =1, to glass, $n=1.5$ where the angle of incidence is 45 degrees and the E-field is normal to the plane of incidence. You may assume that both the air and glass are non-magnetic. Find the numerical values of the reflection (R) and transmission (T) coefficients and hence prove there is conservation of energy so that $R + T = 1$.

2. Write down the boundary conditions at a non-conducting interface.
When electromagnetic radiation impinges at normal incidence on the planar interface between a particular combination of two non-magnetic, non-conducting dielectric media it is found that the reflection and transmission coefficients are identical. Derive an expression from which the possible values for the ratio $\frac{n_1}{n_2}$ (where $n_1$andn_2 are the refractive indices of two media) can be obtained.
If one of the media has a relative permittivity of 9.0, what are the possible combinations of refractive indices for medium 2?

3. Imagine that there are two non-conducting materials such that one has $\mu = 2.0 \times 10^{-6}\text{Hm}^{-1}$ and $\varepsilon = 5.0 \times 10^{-11}\text{Fm}^{-1}$ and the other has $\mu = 1.25 \times 10^{-6}\text{Hm}^{-1}$ and $\varepsilon = 2.0 \times 10^{-11}\text{Fm}^{-1}$. What is the reflection coefficient at the normal incidence at the planar interface between these materials?

4. Given that $n_1 = 3$ and $n_2 = 5$, find the critical angle and state the condition for the critical angle to occur.

5. How exactly can polarized light be produced by using a block of non-magnetic ILIH (infinite linear isotropic homogeneous) material with refractive index n = 3.5? The transmitted E-field is larger than the incident E-field in configurations where the incident angle is less than the critical angle. Explain from a physical perspective why such a result is reasonable.

6. By using Snell's Law, show that, when the E-field is parallel to the plane of incidence and the media is nonmagnetic, the relevant Fresnel equation can be rewritten:

$$\frac{E_{\text{ot}}}{E_{\text{oi}}} = \frac{2\sin\theta_{\text{t}}\cos\theta_{\text{i}}}{\sin(\theta_{\text{i}} + \theta_{\text{t}})\cos(\theta_{\text{i}} - \theta_{\text{t}})} \qquad \textbf{3-1}$$

You may use:

$$\sin(A + B) = \sin A \cos\text{B} + \cos A \sin\text{B} \qquad \textbf{3-2}$$

and

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \qquad \textbf{3-3}$$

7. Light travels from a material which has refractive index of 6 to air. Find the critical angle.

8. A sailor is looking for a shipwreck at the bottom of a 100 m deep, beautifully clear ocean. Assuming the water is unpolluted by plastic from short-sighted and irresponsible people, and that it has a refractive index of 4/3, over what area of sea would you expect the sailor to be able to see the shipwreck?

9. Experimental - Put an object at the bottom of a clear glass of water and from the observations you choose to make, calculate the permittivity of water.

