CM1 Solutions: The Atwood Machine

1. **(2 marks total)** The kinetic energy, $T = (1/2)M\dot{z}_1^2 + (1/2)m\dot{z}_2^2$.
   The potential energy, $V = Mgz_1 + mgz_2$ with any arbitrary constant added.
   Hence the Lagrangian is $L = T - V = (1/2)M\dot{z}_1^2 + (1/2)m\dot{z}_2^2 - (Mgz_1 + mgz_2)$.

2. **(2 marks total)** The sum of the heights of the end points of the rope is constant. This implies that

   $$(z_1 - \phi(t)) + z_2 = c$$

   where c is a constant.
   This is a rheonomic, not a scleronomic constraint, because there is an explicit time dependence in $\phi(t)$.

3. **(2 marks total)** Substituting $z_2 = c - z_1 + \phi$ into the Lagrangian derived in 1 yields, ignoring an uninteresting constant term and noting that $\dot{z}_2 = \dot{\phi} - \dot{z}_1$, $L = (1/2)[M\dot{z}_1^2 + m(\dot{\phi} - \dot{z}_1)^2] - g[Mz_1 + m(\phi - z_1)]$.

4. **(3 marks total)** Using the Euler-Lagrange equation,

   $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_1}\right) - \frac{\partial L}{\partial z_1} = 0,$$

   gives

   $$\ddot{z}_1 = \frac{m\ddot{\phi} - (M-m)g}{M+m}.$$

   Integrating once with respect to time gives

   $$\dot{z}_1 = \frac{m\dot{\phi} - (M-m)gt}{M+m},$$

   where $\dot{z}_1(0) = 0$ and $\dot{\phi}(0) = 0$ have been used.
   Integrating again with respect to time gives

   $$z_1 = h + \frac{m\phi - (M-m)gt^2/2}{M+m},$$

   where $z_1(0) = h$ and $\phi(0) = 0$ have been used.

5. **(1 mark)** If $\ddot{\phi} = 5g/3$, then $\phi = 5gt^2/6$. Substituting this, with $M = 2m$ and $z_1 = 2h$ into the answer to 4 yields $2h = h + (5/6 - 1/2)gt^2/3$, from which $t = 3\sqrt{h/g}$.