Book : p51-62.

Distributions: Classical (or distinguishable) particles we counted the number of microstates such that the order in which particles were arranged mattered ($N!$), e.g.

$$\Omega \rightarrow \frac{N!}{\prod_i n_i!} \rightarrow \text{classical Maxwell-Boltzmann distribution.}$$

Identical/Indistinguishable Particles - this is a purely quantum concept.

Quantum particles - either Fermion or Boson.

If we have two quantum particles then the wavefunction of the two-particle system is written $\psi(r_1, r_2)$. Let's swap the two particles, so we have system $\psi(r_2, r_1)$. How is $\psi(r_1, r_2)$ described in terms of $\psi(r_2, r_1)$?

Let S be a "swap" operator, i.e.

$$S\,\psi(r_1, r_2) = x\,\psi(r_2, r_1)$$

$$S^2\,\psi(r_1, r_2) = x^2\,\psi(r_1, r_2)$$

$$\Rightarrow x^2 = 1.$$

$$\Rightarrow x = \pm 1.$$

We're left with two situations:

$$\psi_F(r_1, r_2) = -\psi_F(r_2, r_1) \longleftarrow \text{Fermions}$$

$$\psi_B(r_1, r_2) = \psi_B(r_2, r_1). \longleftarrow \text{Bosons.}$$

[Aside: Anyons $x = e^{i\theta}$].

Put 2 Fermions at the same place i.e. Pauli exclusion.

$$\psi_F(r, r) = -\psi_F(r, r) \Rightarrow \boxed{\psi_F(r, r) = 0}$$

<u>Fermions</u> Each single particle state will be either occupied ($n_i = 1$) or unoccupied ($n_i = 0$). If state i is g_i-fold degenerate ~~at~~ with energy ε_i. We want the number of ways g_i slots can be divided into 2 piles - occupied or unoccupied.

One pile (the occupied) will have n_i ($n_i \le g_i$) and hence the other pile (the unoccupied) will have $g_i - n_i$ slots free.

Using the binomial distribution

$$\Omega(\varepsilon_i) = \frac{g_i!}{n_i!\,(g_i - n_i)!}$$

This is the contribution for the i^{th} state so for all states

$$\Omega_{FD} = \prod_i \frac{g_i!}{n_i!\,(g_i - n_i)!}$$

As previously done, we maximise entropy under the same constraints (ie constant energy and particle number) :

$$\frac{S}{k_B} = \ln\left[\prod_i \frac{g_i!}{n_i!\,(g_i-n_i)!}\right]$$

$$= \sum_i \left(\ln g_i! - \ln n_i! - \ln (g_i - n_i)!\right) \qquad \text{(log rules)}$$

$$= \sum_i \Big[(g_i \ln g_i - \underset{\sim}{g_i}) - (n_i \ln n_i - \underset{\sim}{n_i}) - \big((g_i - n_i)\ln(g_i - n_i) - \underline{(g_i - n_i)} \big) \Big] \quad \text{(Stirling)}$$

$$= \sum_i \left[g_i \ln g_i - n_i \ln n_i - (g_i - n_i)\ln(g_i - n_i) \right]$$

Now introduce the Lagrange multipliers :

$$\sum_i \left[g_i \ln g_i - n_i \ln n_i - (g_i - n_i)\ln(g_i - n_i) - \alpha n_i - \beta n_i \varepsilon_i \right]$$

and <u>maximise</u> this with respect to n_i i.e

$$\partial(\ldots) / \partial n_i = 0$$

$$0 = 0 - \ln n_i - 1 + \ln(g_i - n_i) + 1 - \alpha - \beta\varepsilon_i$$ (only ith term in sum survives).

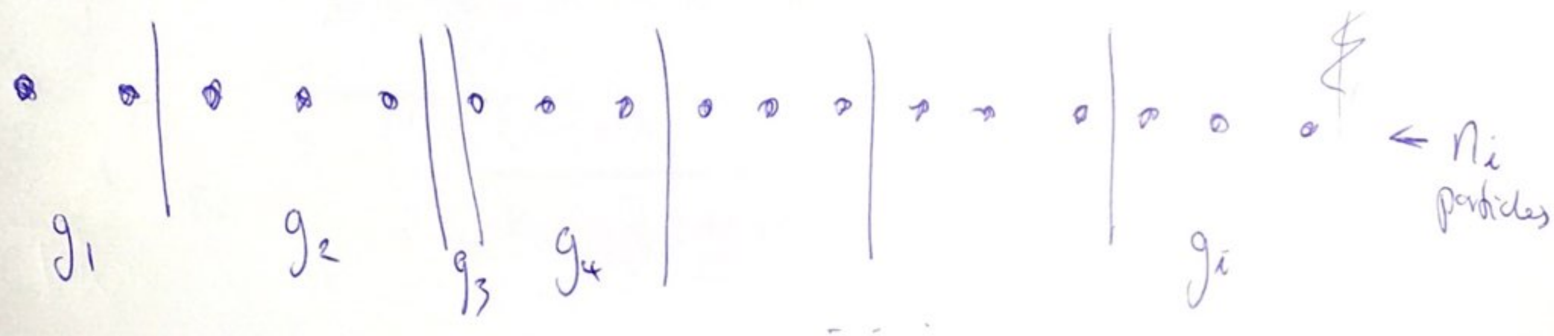
$$\Rightarrow \ln\left(g_i/n_i - 1\right) = \alpha + \beta\varepsilon_i$$

$$\Rightarrow n_i/g_i = f_{FD}(\varepsilon_i) = \frac{1}{e^{\alpha}e^{\beta\varepsilon_i} + 1}.$$

This is the Fermi-Dirac Distribution function.

<u>Bosons</u> How to count the number of microstates?

There is no limit on state occupancy, any single particle state can have any number of particles. Let's take a single energy ε_i with degeneracy g_i and we have n_i particles to distribute.



We have $g_i - 1$ and n_i particles and so we need the number of arrangements of the $n_i + (g_i - 1)$ objects into 2 piles.

So n_i particles of energy level ε_i in the g_i states

$$\Omega(\varepsilon_i) = \frac{(n_i + g_i - 1)!}{n_i!\,(g_i-1)!} \simeq \frac{(n_i+g_i)!}{n_i!\,g_i!}$$

This is for state i, so for all states we have

$$\Omega_{BE} = \prod_i \frac{(n_i+g_i)!}{n_i!\,g_i!}$$

So let's play the same game with entropy :

$$S/k_B = \ln\left[\prod_i \frac{(n_i+g_i)!}{n_i!\,g_i!}\right] = \sum_i \left(\ln(n_i+g_i)! - \ln n_i! - \ln g_i!\right) \quad \text{(log rules)}$$

(Stirling)

$$= \sum_i (n_i+g_i)\ln(n_i+g_i) - (n_i+g_i) - (n_i \ln n_i - n_i) - (g_i \ln g_i - g_i)$$

$$= \sum_i (n_i+g_i)\ln(n_i+g_i) - n_i \ln n_i - g_i \ln g_i$$

Lagrange to constrain particle number and energy:

$$\sum_i (n_i+g_i)\ln(n_i+g_i) - n_i \ln n_i - g_i \ln g_i - \alpha n_i - \beta n_i \varepsilon_i$$

maximise with $\partial(\ldots)/\partial n_i = 0$

$$0 = \ln(n_i + g_i) + 1 - \ln n_i - 1 - \alpha - \beta \varepsilon_i$$ (no other terms in the sum survives).

$$\Rightarrow \ln\left(g_i/n_i + 1 \right) = \alpha + \beta \varepsilon_i$$

hence $$n_i/g_i = f_{BE}(\varepsilon_i) = \frac{1}{e^{\alpha} e^{\beta \varepsilon_i} - 1}$$

This is the Bose - Einstein distribution function.