Quantum Theory - Worksheet 7

Problem 1

You have encountered the orbital angular momentum operator $\hat{\mathbf{L}}$ earlier this academic year. $\hat{\mathbf{L}}$ is a geometrical vector whose x-, y- and z-components, respectively $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$, are Hermitian operators. These three operators do not commute with each other. Instead

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z,$$
$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x,$$
$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y.$$

Note that the last two of these commutation relations can be obtained from the equation $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$ by making the cyclic permutation $x \to y$, $y \to z$, $z \to x$.

(a) We denote the dot product ot $\hat{\mathbf{L}}$ with itself by the symbol $\hat{\mathbf{L}}^2$. Explicitly, $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. Show that the above commutation relations imply that $\hat{\mathbf{L}}^2$ commutes with $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$. (You only need to show that $[\hat{\mathbf{L}}^2, \hat{L}_x] = 0$; that $[\hat{\mathbf{L}}^2, \hat{L}_y]$ and $[\hat{\mathbf{L}}^2, \hat{L}_z]$ are also zero follows by symmetry.)

(b) The eigenvalues of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ are, respectively, $m\,\hbar$ ($m = 0, \pm 1, \pm 2, \ldots$) and $l(l+1)\,\hbar^2$ ($l = 0, 1, 2, \ldots$). If the orthonormal vectors $|l, m\rangle$ are joint eigenvectors of these two operators,

$$\hat{\mathbf{L}}^2 |l, m\rangle = l(l+1)\,\hbar^2 |l, m\rangle,$$
$$\hat{L}_z |l, m\rangle = m\,\hbar |l, m\rangle,$$

and $\langle l', m' | l, m\rangle = \delta_{l'l}\delta_{m'm}$, with $l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l-1, l$. [For a general angular momentum operator $\hat{\mathbf{J}}$, the eigenvalues or $\hat{\mathbf{J}}^2$ would be $j(j+1)\,\hbar^2$ where j is an integer or a half-integer ($j \geq 0$); however, only integer values of j are possible if $\hat{\mathbf{J}}$ is an *orbital* angular momentum operator.]

(i) Consider eigenvectors $|l, m\rangle$ such that $\hat{\mathbf{L}}^2 |l, m\rangle = 12\,\hbar^2 |l, m\rangle$. What are the possible values of m for these eigenvectors?

(ii) Does $\hat{L}_z$ have eigenvectors that are not eigenvectors of $\hat{\mathbf{L}}^2$?

(iii) Could $\hat{\mathbf{L}}^2$ also have eigenvectors in common with $\hat{L}_x$ or $\hat{L}_y$? (Justify your answer.)

(c) In Classical Mechanics, the angular momentum of a particle is the cross product of its position vector and its momentum vector: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where

$$\mathbf{L} = L_x\,\hat{\mathbf{x}} + L_y\,\hat{\mathbf{y}} + L_z\,\hat{\mathbf{z}},$$
$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}},$$
$$\mathbf{p} = p_x\,\hat{\mathbf{x}} + p_y\,\hat{\mathbf{y}} + p_z\,\hat{\mathbf{z}},$$

with $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ unit vectors in the x-, y- and z-directions. Show that

$$L_x = y p_z - z p_y,$$
$$L_y = z p_x - x p_z,$$
$$L_z = x p_y - y p_x.$$

(Note that making the cyclic permutation $x \to y$, $y \to z$, $z \to x$ transforms L_x into L_y, L_y into L_z and L_z into L_x.)

(d) One can pass from the classical angular momentum vector to the quantum mechanical orbital angular momentum operator in position representation by replacing x, y, z, p_x, p_y and p_z by the operators. This gives the three operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ defined as follows:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y,$$
$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z,$$
$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.$$

(The same equations as in Part (c), but here $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$, $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ are operators.)

(i) Using these equations and the commutation relations $[\hat{x}, \hat{p}_x] = i\hbar$ etc., show that $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$.

(ii) Calculate the commutator $[\hat{L}_z, \hat{y}]$, using the expression of $\hat{L}_z$ quoted above.

Problem 2

Consider two states, state a and state b, described by the ket vectors $|a\rangle$ and $|b\rangle$, respectively. Suppose that these two states are also described, in the position representation, by the wave functions $\psi_a(x)$ and $\psi_b(x)$. Thus

$$\langle a | b\rangle = \int_{-\infty}^{\infty} \psi_a^*(x)\psi_b(x)\,\mathrm{d}x.$$

Passing to the momentum representation transforms $\psi_a(x)$ and $\psi_b(x)$ into the wave functions $\phi_a(p)$ and $\phi_b(p)$ such that

$$\phi_a(p) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \exp(-ipx/\hbar)\psi_a(x)\,\mathrm{d}x,$$
$$\phi_b(p) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \exp(-ipx/\hbar)\psi_b(x)\,\mathrm{d}x.$$

Show that

$$\langle a | b\rangle = \int_{-\infty}^{\infty} \phi_a^*(p)\phi_b(p)\,\mathrm{d}p.$$

The upshot is that the probabilities predicted by quantum mechanics do not depend on whether the states are described in the position representation or in the momentum representation.

Problem 3

Consider an operator $\hat{a}$ such that $[\hat{a}, \hat{a}^\dagger] = 1$, where $\hat{a}^\dagger$ is the adjoint of $\hat{a}$, and a vector $|\alpha\rangle$ such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where α is a number (possibly complex) and $\langle\alpha|\alpha\rangle = 1$. Also, let $\hat{S} = (\hat{a}+\hat{a}^\dagger)/2$ and $\hat{D} = (\hat{a}-\hat{a}^\dagger)/(2i)$. Show that $\langle\alpha|\hat{S}^2|\alpha\rangle = (\mathrm{Re}\,\alpha)^2 + 1/4$ and $\langle\alpha|\hat{D}^2|\alpha\rangle = (\mathrm{Im}\,\alpha)^2 + 1/4$, as was stated in Question 2 of the progress test.