CM1: The Atwood machine

A monkey of mass M climbs at a known, irregular, rate, $\dot{\phi}(t)$, up a light, inextensible rope. $\phi(t)$, the distance the monkey has climbed along the rope, is a specified function of time and therefore not a dynamical variable. The rope goes over a light, frictionless pulley with a mass m at the other end. The monkey and mass are at heights $z_1(t)$ and $z_2(t)$ respectively above the ground, both of which are positive. Initially the monkey rests at the end of the rope and is held at height h.

1. Write down the kinetic and potential energies of this system, and hence determine its Lagrangian in terms of z_1, z_2, $\dot{z}_1$ and $\dot{z}_2$.

2. By considering the heights of the end points of the inextensible rope, or otherwise, write down a rheonomic constraint relating the two generalised coordinates z_1 and z_2. State why the constraint is not scleronomic.

3. Show that the Lagrangian for the system can be written as

$$L = (1/2)[M\dot{z}_1^2 + m(\dot{\phi} - \dot{z}_1)^2] - g[Mz_1 + m(\phi - z_1)].$$

4. Using the Euler-Lagrange equation, determine an expression for the vertical acceleration of the monkey, $\ddot{z}_1$. Making clear what initial conditions you have used, integrate the expression for $\ddot{z}_1$ to find $z_1(t)$.

5. In the case $M = 2m$ and $\ddot{\phi}(t) = 5g/3$, how long will it take the monkey to reach height $2h$?