Mathematical Methods II
Weekly problem set 4

(1) Find all the singular points of the following equations

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \tfrac{m^2}{1-x^2}\right] y = 0 \quad \text{(associated Legendre equation)}$$
$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \quad \text{(Bessel equation)}$$

and classify them. (Do not forget to consider potential singularities at infinity). Here ℓ, m and ν are constants.

Solution

Associated Legendre:

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] y = 0$$

First, divide by $1-x^2 = (1+x)(1-x)$ to match the canonical form

$$y'' - \frac{2xy'}{(1+x)(1-x)} + \left[\frac{\ell(\ell+1)}{(1+x)(1-x)} - \frac{m^2}{(1+x)^2(1-x)^2}\right] y = 0$$

Identify $p(x)$ and $q(x)$

$$p(x) = -\frac{2xy'}{(1+x)(1-x)}$$
$$q(x) = \frac{\ell(\ell+1)}{(1+x)(1-x)} - \frac{m^2}{(1+x)^2(1-x)^2}$$

Clearly $x \pm 1$ are singular points (lead to $\div 0$). Since $\lim_{x\to\pm1}(x\pm1)p(x)$ and $\lim_{x\to\pm1}(x\pm1)^2q(x)$ are finite $x \pm 1$ are regular singular points.

To check for singularities at ∞ we can change variable $x = 1/\omega$ and study the limit $\omega \to 0$. Change variables:

$$y' = -\frac{1}{x^2}\frac{dy}{d\omega} = -\omega^2\frac{dy}{d\omega}$$
$$y'' = \frac{d^2}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{d\omega}{dx}\frac{dy}{d\omega}\right] = \frac{d}{dx}\left[-\frac{1}{x^2}\frac{dy}{d\omega}\right]$$
$$= \frac{2}{x^3}\frac{dy}{d\omega} + \frac{1}{x^4}\frac{d^2y}{d\omega^2} = 2\omega^3\frac{dy}{d\omega} + \omega^4\frac{d^2y}{d\omega^2} = \omega^3\left(2\frac{dy}{d\omega} + \omega\frac{d^2y}{d\omega^2}\right)$$

Substituting into the ODE

$$\left(1-\frac{1}{\omega^2}\right)\omega^3\left[2\frac{dy}{d\omega} + \omega\frac{d^2y}{d\omega^2}\right] + \frac{2}{\omega}\omega^2\frac{dy}{d\omega} + \left[\ell(\ell+1) - \frac{m^2}{(1-\frac{1}{\omega^2})}\right] y = 0$$

$$\omega^2(\omega^2-1)\frac{d^2y}{d\omega^2} + 2\omega^3\frac{dy}{d\omega} + \left[\ell(\ell+1) + \frac{m^2\omega^2}{(1-\omega^2)}\right] y = 0$$

Canonical form:

$$\frac{d^2y}{d\omega^2} + \frac{2\omega}{\omega^2-1}\frac{dy}{d\omega} + \left[\frac{\ell(\ell+1)}{\omega^2(\omega^2-1)} - \frac{m^2}{(\omega^2-1)^2}\right] y = 0$$

Testing limit as $\omega \to 0$ we see it is a regular singular point.

So associated Legendre equation has a 3 regular singular points, $x = 1$, $x = -1$ and $x = \infty$ and no irregular singular points.

Bessel:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

In canonical form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

$$p(x) = \frac{1}{x}$$

$$q(x) = 1 - \frac{\nu^2}{x^2}$$

Clearly $x = 0$ is a singular point.

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{x}{x} = 1$$

$$\lim_{x\to 0} x^2q(x) = \lim_{x\to 0} \frac{x^2}{x} = 0$$

Thus x is a regular singular point.

To study $x \to \infty$ make the change of variable $x = 1/\omega$. The equatoin becomes

$$\frac{1}{\omega^2}\left[\omega^3\left(2\frac{dy}{d\omega} + \omega\frac{d^2y}{d\omega^2}\right)\right] + \frac{1}{\omega}\left(-\omega^2\frac{dy}{d\omega}\right) + \left(\frac{1}{\omega^2} - \nu^2\right)y$$

$$= \omega^2\frac{d^2y}{d\omega^2} + \omega\frac{dy}{d\omega} + \left(\frac{1}{\omega^2} - \nu^2\right)y = 0$$

In canonical form

$$\frac{d^2y}{d\omega^2} + \frac{1}{\omega}\frac{dy}{d\omega} + \left(\frac{1}{\omega^4} - \frac{\nu^2}{\omega^2}\right)y = 0$$

So

$$p = 1/\omega$$

$$q = \frac{1}{\omega^4} - \frac{\nu^2}{\omega^2}$$

$$\lim_{\omega\to 0} \omega^2 q = \infty$$

$\omega = 0$ is an irregular singularity. Therefore the Bessal equation has a regular singularity at 0 and an irregular singularity at ∞.

(2) The aim of this question it to find two power series solutions about $x = 0$ of the differential equation

$$(1 - x^2)y'' - 3xy' + \lambda y = 0, \tag{1}$$

where λ is a constant and construct a general solution. Begin by checking that $x = 0$ is an ordinary point.

This series terminates at a particular value of λ, for order $n = N$. Deduce the value of λ for which the corresponding power series becomes a finite N-th order polynomial $y_N(x)$ - i.e. express λ in terms of N.

Construct two polynomial solutions, U_2 and U_3 that terminate at the x^2 and x^3 terms, respectively. (Please note that these are not two solutions to the same ODE as they will require different values for λ, and therefore cannot be combined to construct a general solution). Show by substitution that U_2 and U_3 satisfy their respective differential equations.

Solution

$$(1 - x^2)y'' - 3xy' + \lambda y = 0$$

In canonical form

$$y'' - \frac{3x}{(1 - x^2)}y' + \frac{\lambda}{(1 - x^2)}y = 0$$

$$p = \frac{3x}{(1-x^2)}$$

$$q = \frac{\lambda}{(1-x^2)}$$

For solutions as Taylor series

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$y' = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}$$

Plugging these into the equation

$$\sum_{n=0}^{\infty} \left[(1-x^2)n(n-1)a_n x^{n-2} - 3xna_n x^{n-1} + \lambda a_n x^n\right]$$

$$= \sum_{n=0}^{\infty} \left[n(n-1)a_n x^{n-2} - n(n-1)a_n x^n - 3na_n x^n + \lambda a_n x^n\right] = 0$$

Now shift the index of the x^{n-2} term and factorise in x^n

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3na_n + \lambda a_n\right] x^n = 0$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - \left(n(n+2) - \lambda\right) a_n\right] x^n = 0$$

So our recursion relation is

$$a_{n+2} = \frac{n(n+2) - \lambda}{(n+2)(n+1)} a_n$$

This splits for odd and even numbers. Solution 1 found when $a_0 = 1, a_1 = 0$ (all odd a_n will vanish). Solution 2 found when $a_0 = 0, a_1 = 1$ (all even a_n will vanish). We can deduce from the recursion relation that it will terminate $(= 0)$ when

$$\lambda = N(N+2)$$

As a_N is zero, a_{N+2} and so on will also be zero.

For U_2 $N = 2$ (so, $\lambda = 2(2+2) = 8$), $a_0 = 1$, $a_1 = 0$, as we want a degree 2 polynomial with even terms, so

$$a_2 = \frac{n(n-2) - \lambda}{(n+2)(n+1)} \times a_0 = \frac{0 - 2(2+2)}{(0+2)(0+1)} \times 1 = -\frac{8}{2} = -4$$

So, $U_2 = 1 - 4x^2$.

For U_3 $N = 3$ (so, $\lambda = 3(3+2) = 15$), $a_0 = 0$, $a_1 = 1$, as we want a degree 3 polynomial with odd terms, so

$$a_3 = \frac{n(n-2) - \lambda}{(n+2)(n+1)} \times a_1 = \frac{1(1+2) - 3(3+2)}{(1+2)(1+1)} \times 1 = -\frac{12}{6} = -2$$

So, $U_3 = x - 2x^3$.

Subbing U_2 and U_3 into the ODE individually should give $RHS = 0$.