Mathematical Methods in Physics

Weekly Problems 6. Solution

6.1

$$\delta(4t^2-1) = \frac{\delta(t-1/2)}{4} + \frac{\delta(t+1/2)}{4},$$

therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(4t^2-1)\, e^{it}\, dt &= \frac{1}{4}\left(\int_{-\infty}^{\infty} \delta(t-1/2)\, e^{it}\, dt + \int_{-\infty}^{\infty} \delta(t+1/2)\, e^{it}\, dt\right) \\
&= \frac{1}{4}\left(e^{i/2} + e^{-i/2}\right) = \frac{\cos 1/2}{2}. \qquad \boxed{\text{3 marks}}
\end{aligned}$$

6.2

$$\mathcal{L}[f(t-a)H(t-a)](s) = \int\limits_0^{\infty} f(t-a)H(t-a)\, e^{-st} dt.$$

Set $t-a=u$, then the integral becomes

$$\int\limits_{-a}^{\infty} f(u)H(u)\, e^{-s(u+a)}\, du = \int\limits_0^{\infty} f(u)\, e^{-s(u+a)}\, du = e^{-sa}\bar{f}(s).$$

$\boxed{\text{2 marks}}$

6.3

a)

$$\mathcal{L}[H(t-2)(t-2)^3](s) = e^{-2s}\mathcal{L}[t^3](s) = 6\,\frac{e^{-2s}}{s^4},$$

$\boxed{\text{1 mark}}$

b)

$$\mathcal{L}[2e^{3t}\sin 3t](s) = 2\mathcal{L}[\sin 3t](s-3) = \frac{6}{(s-3)^2+9},$$

$\boxed{\text{1 mark}}$

c) Using Linearity:

$$\mathcal{L}[4te^{-t} + t\cosh 3t](s) = 4\mathcal{L}[te^{-t}](s) + \mathcal{L}[t\cosh 3t](s).$$

Then, using, for instance, Exponential Multiplication for the first term and Polynomial Multiplication for the second (see Lecture Summary) together with the Laplace transforms provided, we have

$$\begin{aligned} 4\mathcal{L}[te^{-t}](s) + \mathcal{L}[t\cosh 3t](s) &= 4\mathcal{L}[t](s+1) - \frac{d}{ds}\mathcal{L}[\cosh 3t](s) \\ &= \frac{4}{(s+1)^2} - \frac{d}{ds}\left(\frac{s}{s^2-9}\right) \\ &= \frac{4}{(s+1)^2} + \frac{s^2+9}{(s^2-9)^2}. \end{aligned}$$

$\boxed{\text{1 mark}}$

d) Using partial fraction decomposition

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{(s+1)}\right](t) = 1 - e^{-t}.$$

$\boxed{\text{2 marks}}$