Mathematical Methods II
Lecture 14

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Key Points

- General solutions to PDEs
- 1st order PDEs

General Solutions to PDEs

- **General form of solution:** In general PDE's are not separable and cannot be solved using the method of separation of variables. We require a more general method of solving them.

  For a given PDE we can try to produce a general solution by seeking a function made from a combination of the independent variables in the PDE. For the function $u(x, y)$ we would seek a solution $u(p)$ where $p = p(x, y)$. That is to say we can express a general solution $u(x, y)$ as a function of a specific combination of x and y. e.g. a PDE might have the solution
  $$u(x, y) = f(x^2 + y^2).$$
  In this example we are stating that the solution to our PDE is a function of $p = x^2 + y^2$. This may not seem terribly useful or enlightening, but when combined with boundary conditions we can be more specific and arrive at a particular solution to a given PDE.

- **1$^{\text{st}}$ order PDEs (1 derivative):** Initially, let's consider 1$^{\text{st}}$ order PDEs containing two independent variables. The most general form for this type of PDE is
  $$A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} + C(x, y)u = R(x, y)$$
  where $A(x, y)$, $B(x, y)$, $C(x, y)$ and $R(x, y)$ are given functions. If either $A = 0$ or $B = 0$ then the equation can simply be solved as a 1$^{\text{st}}$ order linear ODE. Recall that the form of this type of equation is
  $$\frac{\partial u}{\partial x} + P(x, y)u = Q(x, y) \qquad\qquad \frac{\partial u}{\partial y} + P(x, y)u = Q(x, y)$$

and that it can be solved using an integrating factor (that depends on which independent variable is being used in the derivatives).

$$\mu(x,y) = e^{\int P(x,y)dx} \qquad\qquad \mu(x,y) = e^{\int P(x,y)dy}$$

with a solution given by

$$u = \frac{1}{\mu(x,y)} \int \mu(x,y) Q(x,y) dx \qquad\qquad u = \frac{1}{\mu(x,y)} \int \mu(x,y) Q(x,y) dx.$$

e.g. 14.1 1$^{\text{st}}$order linear PDE: Find the general solution $u(x,y)$ of

$$x\frac{\partial u}{\partial x} + 3u = x^2.$$

There are no y derivatives here, but since u is a function of x and y, this is still a 1$^{\text{st}}$ order linear PDE rather than simplfying to an ODE. However it may be solved in the same way.

We can begin by dividing by x to arrive at the canonical form

$$\frac{\partial u}{\partial x} + \frac{3u}{x} = x.$$

This gives us the form of a 1$^{\text{st}}$ order linear equation. After identifying $P(x) = 3/x$ and $Q(x) = x$ this can be solved by using an integrating factor

$$\mu(x) = e^{\int P(x)dx} = e^{\int (3/x)dx} = e^{3\ln x} = x^3$$

Multiplying through by μ we find

$$\mu\frac{\partial u}{\partial x} + \mu\frac{3u}{x} = \mu x$$

Recalling that the integrating factor is used so that we can apply the following relation

$$\frac{\partial}{\partial x}(\mu u) = \mu\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial x}u = \mu\frac{\partial u}{\partial x} + \mu P u$$

where

$$\frac{\partial \mu}{\partial x} = \mu P \qquad \text{since} \qquad \frac{\partial}{\partial x}e^{\int P dx} = P(x)e^{\int P dx}.$$

Applying this relation to the LHS, the equation becomes

$$\frac{\partial}{\partial x}(\mu u) = \mu x$$

$$\frac{\partial}{\partial x}(x^3 u) = x^4$$

Integrating

$$x^3 u = \frac{x^5}{5} + f(y)$$

where $f(y)$ is some unknown function of y instead of a constant, as y would be considered a contant when partially differentiating u w.r.t x. Hence

$$u(x,y) = \frac{x^2}{5} + \frac{f(y)}{x^3}$$

In our solution we have been able to specify the relation u has to x, but $f(y)$ remains unknown without boundary conditions to give us more information. This would be considered the most general solution to this problem.

- **1^{st} order PDEs (2 derivatives):** The method above is fine for PDE's containing partial derivatives w.r.t only one independent variable. However this method will not work for an equation that contains both $\partial/\partial x$ and $\partial/\partial y$. Let's consider a special case, where $C(x,y) = R(x,y) = 0$, i.e.

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = 0$$

Again, let's look for a solution of the form $u(x,y) = f(p)$, where p is some unknown combination of x and y. Consider the derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial x} = \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f(p)}{\partial y} = \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y} = \frac{df(p)}{dp}\frac{\partial p}{\partial y}$$

Let's sub these into our PDE

$$A(x,y)\frac{df(p)}{dp}\frac{\partial p}{\partial x} + B(x,y)\frac{df(p)}{dp}\frac{\partial p}{\partial y} = 0$$

$$\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]\frac{df(p)}{dp} = 0$$

Let's consider two equations at this point:

1) For non-trivial p we can assume $df(p)/dp \neq 0$ and so the bracket is zero giving

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0.$$

2) If we require that p is constant then the total derivative of p is zero

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0.$$

Notice that these two equations are very similar. In fact, they become the same if we require that

$$dx = A(x,y) \qquad\qquad dy = B(x,y)$$

$$\frac{dx}{A(x,y)} = 1 \qquad\qquad \frac{dy}{B(x,y)} = 1$$

Hence,

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}.$$

p is a constant that relates x to y for our given PDE and so is the constant of integration of this equation. Hence p can be found by integrating this expression and setting the constant of integration (or a multiple of the constant) equal to p.

e.g. 14.2 1$^{\text{st}}$order PDE with 2 derivatives and boundary conditions: For

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0,$$

find (i) the solution that takes the value $2y+1$ on the line $x = 1$, and (ii) a solution that has the value 4 at the point (1,1).

So, we're seeking a solution of the form $u(x,y) = f(p)$. We now know that $u(x,y)$ will be constant along lines of (x,y) that satisfy

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}$$

$$\frac{dx}{x} = \frac{dy}{-2y}.$$

Integrating we get

$$\begin{aligned}
\int \frac{1}{x}dx &= -\frac{1}{2}\int \frac{1}{y}dy \\
\ln x &= -\frac{1}{2}\ln y + c \\
\ln x &= \ln y^{-1/2} + c \\
x &= e^{\ln y^{-1/2}} e^{c} \\
x &= Cy^{-1/2} \\
C &= xy^{1/2}
\end{aligned}$$

This constant gives us p, some function of x and y. Simply to avoid fractional powers in this case we shall say $p = C^2$, hence

$$p = x^2 y$$

Thus the general solution to the PDE is

$$u(x,y) = f(x^2 y)$$

where f is an arbitrary function. Before going on to apply our boundary conditions, let's try one or two functions as a test.

– $u = x^2y$: Our derivatives are

$$u_x = 2xy \qquad\qquad u_y = x^2$$

Subbing into the PDE

$$xu_x - 2yu_y = x(2xy) - 2y(x^2) = 2x^2y - 2x^2y = 0$$

we get $RHS = 0$ as required by the PDE.

– $u = \sin(x^2y)$: Our derivatives are

$$u_x = 2xy\cos(x^2y) \qquad\qquad u_y = x^2\cos(x^2y)$$

Subbing into the PDE

$$xu_x - 2yu_y = x[2xy\cos(x^2y)] - 2y[x^2\cos(x^2y)] = 2x^2y\cos(x^2y) - 2x^2y\cos(x^2y) = 0$$

we get $RHS = 0$ as required by the PDE.

We have our general solution. Let's find our particular solutions by applying the respective boundary conditions.

(i) $\underline{u = 2y + 1 \text{ on the line } x = 1}$: Test the general solution at the BC,

$$f(x^2y)|_{x=1} = f(y) = 2y + 1.$$

Comparing the argument of the function to the RHS we find

$$f(z) = 2z + 1,$$

where $z = p|_{x=1} = y$. The most general solution at this BC is therefore given by replacing the z (which is p defined only at the BC) with p (which is defined for all x and y)

$$u = f(p) = 2p + 1$$

i.e.

$$u(x, y) = f(x^2y) = 2(x^2y) + 1.$$

Note that we do not need to add a $g(p)$ term to this solution because it is fully specified, since at the boundary $z = y$ and y is undetermined (i.e. z, and hence p, can take any arbitrary value). This means we can find a solution that applies for all values of y and therefore all values of p, without the need for an additional function. Effectively, there is only one way to write down the solution, an additional g term would be zero for all y and therefore all p.

(ii) $\underline{u = 4 \text{ at the point } (1,1)}$: Test the general solution at the BC,

$$f(x^2y)|_{x=1,y=1} = f(1) = 4.$$

Comparing the argument of the function to the RHS we find that there are many possible interpretations of this relation for $z = p|_{x=1,y=1} = 1$, including,

$$f(z) = z + 3,$$
$$f(z) = 4z,$$
$$f(z) = 4.$$

If we replace the z's with p's as before we find

$$f(p) = f(x^2y) = x^2y + 3,$$
$$f(p) = f(x^2y) = 4x^2y,$$
$$f(p) = f(x^2y) = 4.$$

Clearly we have a lot of freedom to choose a solution, since our BC here only has to apply to a single point rather than a continuum of points along a line or a curve. Our value for z is a constant ($z = 1$); it has no relation to our independent variables and so $f(z)$ (and $f(p)$) can only be determined for a single value of z (or p). This means there are many forms the solution could take. Thus, our corresponding general solutions require an additional arbitrary term $g(p)$, which can account (albeit in an unknown way) for p at all values of x and y. They are

$$u(x, y) = g(x^2y) + x^2y + 3,$$
$$u(x, y) = g(x^2y) + 4x^2y,$$
$$u(x, y) = g(x^2y) + 4,$$

where $g(x^2y) = g(p)$ is an arbitrary function subject to the condition $g(1) = 0$. Any one of these is acceptable as a general solution to the boundary value problem. Clearly g can have various forms and still obey the boundary conditions. For the first general solution shown here substituting the following three forms of $g(p)$ into it will return the 3 particular solutions shown above, in order,

$$g(x^2y) = 0,$$
$$g(x^2y) = 3(x^2y) - 3,$$
$$g(x^2y) = 1 - x^2y,$$

though the only restriction on g is $g(1) = 0$, so in fact any of these forms will do for any of the general solutions. With additional or more comprehensive boundary conditions we could specify its precise form. Check these forms for g are acceptable by subbing them into the general solutions and testing that $u(1, 1) = 4$.

Applying boundary conditions can require a little thought. Remember, if you're not totally convinced that your solution works then check it by both i) subbing it into the PDE and ii) subbing the boundary conditions into the solution. It must satisfy both the original PDE and the given BCs to be a valid solution.