

5 Interaction with magnetic fields

This part of the notes covers all the examinable material concerning the interaction of quantum systems with magnetic fields, apart from calculations done in workshops or as homework.

There is much more to say about this topic than could fit in the course. Interested readers are directed to Bransden and Joachain [Section 12.2 from Eq. (12.26) to Eq. (12.44) and from Eq. (12.64a) to the end, and Section 12.13] and to Griffiths [Chapter 6, on DUO] for further information.

5.1 Magnetic dipole moment and spin

In this section: The magnetic dipole moment operator; its relationship with the spin operator, the gyromagnetic ratio, the Bohr magneton and the electron's g -factor; the Larmor frequency; Landau levels.

Spin-1/2 particles such as electrons, protons and neutrons have an intrinsic magnetic dipole moment, i.e., a magnetic dipole moment which does not depend on their orbital angular momentum. These particles therefore couple to external magnetic fields. This magnetic dipole moment is represented by an operator \mathcal{M}_s proportional to the spin operator \mathbf{S} :

$$\mathcal{M}_s = \gamma \mathbf{S}. \quad (5.1)$$

The constant γ appearing in this equation is called the gyromagnetic ratio (or more properly, the gyromagnetic ratio for spin — there is also an orbital angular momentum gyromagnetic ratio, of which more in Note 2 below). What it is in terms of fundamental constants depends on the type of the particle. For electrons,

$$\gamma = -g_s \mu_B / \hbar, \quad (5.2)$$

where g_s is a number close to 2 and

$$\mu_B = \frac{e\hbar}{2m}. \quad (5.3)$$

Here m denotes the mass of the electron and e its charge in absolute magnitude. The constant μ_B is called the Bohr magneton. The constant g_s , which is dimensionless, is called the g -factor of the electron. Dirac's theory predicts that $g_s = 2$. However, this result neglects various effects described by Quantum Electrodynamics, which bring its true value to 2.002..., in agreement with experiment.

Protons and neutrons also have a magnetic dipole moment proportional to \mathbf{S} . For these nucleons, however, the magnetic moment is usually written in terms of

the nuclear magneton (the same as the Bohr magneton with the mass of the electron replaced by the mass of the proton in the denominator). Protons, neutrons and electrons have different g -factors, too. Overall, the magnetic moments of protons and neutrons is three orders of magnitude smaller than that of electrons.

Due to its intrinsic dipole moment, exposing the electron to a magnetic field \mathcal{B} adds the potential energy term $-\mathcal{M}_s \cdot \mathcal{B}$ to its Hamiltonian. This interaction has an important consequence: In a few words, and loosely speaking, when a spin-1/2 particle such as an electron, a proton or a neutron is submitted to a static magnetic field, the direction in which the particle's spin “points” rotates about the direction of the field at an angular frequency $-\gamma\mathcal{B}$ proportional to the strength of the field. As found in Problem 3 of Worksheet 2, for an electron this angular frequency is $g_s\omega_L$ where ω_L is the Larmor angular frequency,

$$\omega_L = \mu_B \mathcal{B} / \hbar. \quad (5.4)$$

Note 1: The rotation of a magnetic dipole moment under the effect of a magnetic field is called Larmor precession. The terms Larmor frequency and gyromagnetic ratio are used differently in different fields of Physics. By “Larmor angular frequency” (or Larmor frequency, in brief), many authors refer to the angular frequency of the precession, which is species-dependent. Here, instead, we follow Bransden and Joachain and take the angular frequency defined by Eq. (5.4) as being *the* Larmor angular frequency [see, e.g., Chapter 34 of the Feynman Lectures on Physics for the relevance of this particular frequency]. However, we depart from Bransden and Joachain in regards to what we call gyromagnetic ratio: these authors (and others) refer to the dimensionless g factor as the gyromagnetic ratio, whereas we use this term in its more commonly accepted meaning, which is to refer to the factor γ appearing in Eq. (5.1). The gyromagnetic ratio is traditionally called the magnetogyric ratio in Chemistry.

Note 2: A magnetic field may change not only the spin state of an electron but also its spatial wave function. To see this, let us write down the Hamiltonian for the case of a free electron interacting with a constant magnetic field (by free electron, we mean here an electron which is not submitted to any other force than that exerted by the magnetic field). We will use the minimal-coupling formulation introduced in Part 2 of the course notes.

A magnetic field \mathcal{B} can always be written in terms of a vector potential \mathbf{A} as $\mathcal{B} = \nabla \times \mathbf{A}$. For a static (time-independent) and uniform (the same everywhere) magnetic field acting alone (no electric field), the scalar potential can be taken to be zero everywhere and the vector potential to be given by the following equation:

$$\mathbf{A}(\mathbf{r}) = \mathbf{B} \times \mathbf{r}/2. \quad (5.5)$$

(That $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})$ when $\mathbf{A}(\mathbf{r})$ is given by Eq. (5.5) is not difficult to check by direct calculation.) The Hamiltonian of a free electron in a static magnetic field is then given the equation

$$H = \frac{1}{2m}[-i\hbar\nabla + e\mathbf{A}(\mathbf{r})]^2 - \mathcal{M}_s \cdot \mathbf{B}. \quad (5.6)$$

Using standard identities of vector calculus, one can reduce this Hamiltonian to the following form:

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \mathcal{M} \cdot \mathbf{B} + \frac{e^2}{8m}[\mathbf{B}^2 r^2 - (\mathbf{B} \cdot \mathbf{r})^2], \quad (5.7)$$

where $\mathcal{B} = |\mathbf{B}|$, $r = |\mathbf{r}|$, and

$$\mathcal{M} = -(\mu_B/\hbar)(\mathbf{L} + g_s\mathbf{S}) \quad (5.8)$$

with \mathbf{L} the angular momentum operator. Usually one neglects the difference between the true value of g_s and 2, and define the total magnetic dipole moment to be $\mathcal{M} = -(\mu_B/\hbar)(\mathbf{L} + 2\mathbf{S})$. Further, if one takes the z -axis of the system of coordinates to be along \mathbf{B} , then H takes on the form

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{\mu_B}{\hbar}(L_z + 2S_z)\mathcal{B} + \frac{e^2}{8m}(x^2 + y^2)\mathcal{B}^2. \quad (5.9)$$

The second term in the right-hand side shifts the energy of the electron depending on its magnetic quantum numbers m_l and m_s . The third term is a 2-dimensional harmonic potential which confines the electron in the x - and y -directions. Since H does not contain any potential energy term depending on the coordinate z , the electron is free to move in the direction of the magnetic field (the z -direction).

After calculations, one finds that the eigenenergies of the Hamiltonian H of Eq. (5.9) are given by the following formula:

$$E = \frac{\hbar^2 k^2}{2m} + \hbar\omega_L(2r + 2m_s + 1), \quad (5.10)$$

where m_s is the spin magnetic quantum number of the spin state the electron is in ($m_s = 1/2$ for spin-up, $-1/2$ for spin down), r is a non-negative integer ($r = 0, 1, \dots$), and ω_L is the Larmor angular frequency defined by Eq. (5.4). The first term in the right-hand side of Eq. (5.10) is contributed by the free motion in the z -direction, which is not quantized: the wave number k can take any positive or negative value. We see that for a given value of k and of m_s , the electron has discrete energy levels separated by $2\hbar\omega_L$. These levels are called Landau levels. They are labelled by the quantum number r .

Incidentally, Eq. (5.8) associates the orbital angular momentum of the electron with a magnetic dipole moment equal to $\gamma_L \mathbf{L}$ with $\gamma_L = -\mu_B/\hbar$. The constant γ_L plays the role of a gyromagnetic ratio for the orbital angular momentum.

5.2 The Zeeman effect

In this section: How the energy levels of atomic hydrogen split under the effect of a weak static magnetic field; the Landé g -factor; the strong field Zeeman effect.

We now consider the case of an atom of hydrogen initially in the ground or a low excited state and exposed to a static magnetic field. It turns out that the shift and the splitting of the energy levels caused by this interaction can be adequately described by perturbation theory even for the strongest magnetic fields that can be created in laboratory. One can then distinguish three cases, in the approximation where the hyperfine structure of the energy levels is neglected:

- In weak magnetic fields, each fine structure level (identified by a principal quantum number n , an orbital angular momentum quantum number l and a total angular momentum quantum number j) splits into as many magnetic sublevels as there are possible values of m_j for the value of j of that sublevel (recall that for a given j , m_j can take any value between $-j$ and j by integer step). This is the weak field Zeeman effect, also known as the anomalous Zeeman effect for historical reasons. The shifts of each component with respect to the zero-field position is given by the formula

$$\Delta E_{m_j} = g\mu_B \mathcal{B} m_j, \quad (5.11)$$

where g is a number called the Landé g -factor. One finds, after calculations,

$$g = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}. \quad (5.12)$$

- In magnetic fields so strong that $|\Delta E_{m_j}|$ is larger than the fine-structure splitting of the levels, the energy levels of hydrogen are given by the equation

$$E_{nm_l m_s} = E_n + \mu_B(m_l + 2m_s)\mathcal{B}. \quad (5.13)$$

This is the strong field Zeeman effect.

- The structure of the spectrum is more complex between the weak field regime and the strong field regime. The shift of the spectral lines with \mathcal{B} in this transition regime is referred to as the Paschen-Back effect.

Note 1: As done in Bransden and Joachain, we have ignored the the hyperfine structure of the energy levels. Should the hyperfine structure be taken into account, which is often necessary in typical experiments involving not-too-strong magnetic fields, then one finds that Eq. (5.11) applies only for sufficiently strong fields.