University of Durham

EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2019	PHYS3661-WE01

Title:
Theoretical Physics 3

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Attempt **all** questions. The short-answer questions at the start of each section carry 50% of the total marks for the paper. The remaining 50% of the marks are carried by the longer questions, which are equally weighted.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Relativistic Electrodynamics
Section B: Quantum Theory 3

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_\mathrm{B} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

SECTION A: RELATIVISTIC ELECTRODYNAMICS

1. (a) Show that the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval. [4 marks]

(b) Calculate the speed of a particle (relative to c) if its relativistic kinetic energy is equal to twice its rest-mass m times c^2. [4 marks]

(c) A polarization tensor for a photon is

$$T^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n}\right),$$

where p^μ and n^μ are lightlike four-vectors. Calculate $T^\mu{}_\mu$ and $T^{\mu\nu}T_{\mu\nu}$. [4 marks]

(d) Compute $a^\mu v_\mu$, where a^μ is the 4-acceleration and v^μ is the 4-velocity of a given particle. [4 marks]

(e) Three identical point charges of charge q are arranged at the corners of an equilateral triangle. Calculate the value measured for $\underline{E} \cdot \underline{B}$ in the centre of the triangle within the inertial frame that moves with velocity v along one of the sides of the triangle. [4 marks]

(f) Use the covariant form of the inhomogeneous Maxwell equation to derive the wave equation in vacuum for the 4-potential A^μ in the Lorenz gauge. [4 marks]

(g) Write the gauge transformation of the 4-potential A^μ in contra-variant form. Use this to show that the elements of the field strength tensor are unchanged under a gauge transformation. [4 marks]

(h) The Lienard-Wiechert potential of a point charge q with 4-velocity u^μ is

$$A^\mu = \frac{q}{4\pi\epsilon_0}\frac{u^\mu}{u^\nu R_\nu},$$

where R_ν is the 4-distance between the observer and the point charge. The right-hand side of the expression must be evaluated at the retarded time t_{ret}. Evaluate this expression in the instantaneous rest frame of the point charge. Comment on why the result is as expected. [4 marks]

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2. Lienard's relativistic generalisation of Larmor's formula for the power radiated from an accelerated, charged, point-like particle can be written as

$$\mathcal{P} = \frac{d\mathcal{W}}{dt} = \frac{\gamma^2 q^2}{6\pi\epsilon_0 m^2 c^3}\left[\left|\frac{d\underline{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right], \tag{1}$$

where q and m are the charge and mass of the particle, $p = |\underline{p}|$ is the magnitude of the relativistic three-momentum $\underline{p} = \gamma m \underline{v}$ of the particle, $\beta = v/c$ where v is speed of the particle, and γ is the standard relativistic factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Consider a particle of charge $q > 0$ travelling at relativistic speed along the x-axis (starting at $x = -\infty$) towards the origin of the coordinate system. There is a point-charge $q' > 0$ fixed at $\underline{r}' = (x, y) = (0, b)$. The Coulomb field from the fixed charge is

$$\underline{E}(\underline{r}) = \frac{q'}{4\pi\epsilon_0}\frac{\hat{\underline{R}}}{R^2},$$

where $\underline{R} = \underline{r} - \underline{r}'$ and $\hat{\underline{R}}$ as usual denotes a unit vector in the direction of $\underline{R}$. During the scattering, the incident particle will emit electromagnetic radiation due to the acceleration it experiences. In the following you can assume that the impact parameter b and the velocity v are both so large that the particle travels in a straight line.

(a) Sketch the scattering including the fixed charge, and a segment of the movement of the scattering charge. The impact parameter b must be indicated on the sketch, along with the axes of the coordinate system. [2 marks]

b) Use the Lorentz force to write $(d\underline{p}/dt)$ in terms of the $\underline{E}$ and q. [4 marks]

c) Consider a small infinitesimal period of time Δt. Write $(\Delta \underline{p})^2$ in terms of Δt, and by keeping just the first term in Δt show that $(\Delta \underline{p}^2/\Delta t) = 2\underline{p}\cdot\underline{F}$, where $\underline{F}$ is the force applied to the particle. Use this to deduce that $(dp/dt) = \hat{\underline{p}}\cdot\underline{F}$. [4 marks]

The electric field leads to a perturbation of the movement with constant velocity $\underline{v} \approx \underline{\beta} c$. The lowest order effect in $\underline{E}$ is obtained by keeping the explicit factors of γ and β constant in Eq. (1), and evaluating the terms of $d\underline{p}/dt$ and dp/dt.

d) Using the approximations above, show that the total energy $\mathcal{W}$ radiated during the scattering from $t = -\infty$ to $t = \infty$ is given by

$$\mathcal{W} = \frac{\gamma^2 q^4 q'^2}{192\pi^2\epsilon_0^3 m^2 c^4 b^3 \beta}\left(1 - \frac{\beta^2}{4}\right).$$

[10 marks]

$$\left[\text{Hint: } \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3} \quad ; \quad \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)^2}\frac{x^2}{x^2+b^2}\,dx = \frac{\pi}{8b^3}\right]$$

Consider now a separate setup, with point-charges of $q'/2$ fixed at both $(x, y) = (0, b)$ and $(x, y) = (0, -b)$, and a scattering charge as before.

e) Sketch the scattering including the fixed charges, and indicate the direction of the total force acting on the scattering particle. [4 marks]

f) Using the same approximations as above, calculate the total energy $\mathcal{W}$ radiated during this scattering. [6 marks]

SECTION B: QUANTUM THEORY 3

3. (a) A beam of particles scatters from a target in a scattering experiment conducted over the time interval $t_i \le t \le t_f$. The incident flux is $F(t)$, and the total cross-section is σ. Derive an expression for the number of scattering events. [4 marks]

(b) The spherical Bessel functions of the second kind are defined by the Rodrigues formulae,

$$n_l(r) \;=\; (-1)^{l+1}\, r^l \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^l \frac{\cos r}{r}\,.$$

Derive the expression for $n_1(r)$ and expand it in powers of $1/r$ at small r to the order $(1/r)^0$. [4 marks]

(c) Assume that at large r the wave-function for a system in a centrally symmetric potential $V(r)$ is given by,

$$\psi(r,\theta) \;=\; N\, e^{ikr\cos\theta} \;+\; \frac{a}{1+b\sin^2\theta/2}\,\frac{e^{ikr}}{r}\,,$$

where r is the distance from the origin, θ is the polar angle and N, k, a and b are constants. Calculate the total cross-section σ for the scattering described by $\psi(r,\theta)$. [4 marks]

[Hint: $\sin\theta \;=\; 2\sin\theta/2\,\cos\theta/2$.]

(d) Consider a four-vector current $j^\mu(x)$ given by

$$j^\mu \;=\; \frac{i}{2m}\left(\phi^*\partial^\mu\phi \;-\; (\partial^\mu\phi^*)\phi\right)$$

where $\phi(x)$ is a scalar field of mass m. Determine whether the current j^μ is conserved if $\phi(x)$ satisfies the Klein-Gordon equation. Briefly explain why $\phi(x)$ cannot be interpreted as the wave function for a relativistic particle with the probability density given by j^0. [4 marks]

(e) Simplify the expression

$$(i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi(x)$$

using the gamma-matrix algebra. Hence prove that any solution of the Dirac equation is automatically a solution to the Klein-Gordon equation. [4 marks]

(f) Write down the relativistic equation describing the propagation of free electrons in quantum electrodynamics. What is the relation between the energies and momenta of the plane-wave solutions to this equation? Are there negative energy solutions and why is this problematic when interactions between electrons and photons are included? [4 marks]

(g) Consider an n-dimensional Hilbert space with the orthonormal basis $|\psi_1\rangle$, $|\psi_2\rangle, \ldots, |\psi_n\rangle$. Write down the density matrix ρ for the system which is in the pure state $|\psi_2\rangle$ and compute its von Neumann entropy $S = -\mathrm{tr}(\rho\log\rho)$. Another system has the density matrix $\tilde\rho$ which is proportional to the unit matrix in this basis, $\tilde\rho \;=\; \frac{1}{n}\mathrm{diag}(1,1,\ldots,1)$. Compute the von Neumann entropy for $\tilde\rho$ and comment on your result. [4 marks]

4. A particle of mass m is scattered by the potential $V(r)$ of the form,

$$V(r) \;=\; V_0\,\tanh^2(\mu r),$$

where V_0 and μ are positive constants.

(a) Sketch the potential $V(r)$ as a function of r and prove that the initial momentum p of the scattered particle with total energy E is given by $p = \sqrt{2m(E-V_0)}$. Show that the time-independent Schrödinger equation in this potential can be written as

$$\left(\Delta + k^2 - U\left(r\right)\right)\Psi_k\left(\underline{r}\right) = 0,$$

where $k = p/\hbar$ and $\lim_{r\to\infty} U(r) = 0$. Give an explicit expression for $U\left(r\right)$. [6 marks]

(b) Use the separation of variables approach and the expression for the Laplace operator in spherical polar coordinates,

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2\left(\theta,\phi\right)}{\hbar^2 r^2},$$

to separate the Schrödinger equation into equations for the radial and the angular components of the wave-function. State which functions solve the equation for the angular part and give the corresponding eignevalues of the operator $L^2\left(\theta,\phi\right)$. [5 marks]

(c) Now set $V_0 = \hbar^2\mu^2/m$ and, assuming that $k^2 > 2\mu^2$, show that the equation for the s-wave radial component of the wave-function can be written as,

$$\left(\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} + \eta^2 + \frac{2}{\cosh^2(\rho)}\right)R(\rho) = 0,$$

where η is a real constant. Determine η. [4 marks]

(d) Using two particular solutions of the equation above,

$$R_{\pm}(\rho) = \frac{e^{\pm i\eta\rho}}{\rho}\left(\tanh(\rho) \mp i\eta\right),$$

find the general solution of the s-wave radial equation. Impose the appropriate boundary condition at the origin and hence find the radial wave-function up to a total normalisation factor. [5 marks]

(e) What is the corresponding solution of the s-wave angular component of the wave-function? [2 marks]

(f) Show that the wave-function in the s-wave approximation, $\Psi_{k,0}(\underline{r})$, can be written for large r in the form,

$$\lim_{r\to\infty}\Psi_{k,0}(r) \;=\; \frac{C}{r}\,\sin\left(kr + \delta_0(k)\right),$$

where C is a normalisation constant, and compute the scattering phase $\delta_0(k)$. [4 marks]

(g) Use your result to express the s-wave cross-section σ_0 as a function of the kinetic energy of the initial particle, given that

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0(k).$$

[4 marks]