Foundations of Physics 2B/3C   2014/2015

Thermodynamics – Summer Examinations 2015

Short Questions – Solutions - Vetted

Parts of solution in **bold** minimum expected/key ideas students need to have written.

a) Total differential of the enthalpy after simplifying with the first law $dU = TdS - pdV$
$$dH = dU + pdV + Vdp = TdS + Vdp.$$
Hence, changing the entropy S or pressure p will change the enthalpy, so $\boldsymbol{H = H(S, p)}$. In an **isobaric process, $dp = 0$** so $\boldsymbol{dH = TdS = \delta Q}$ which represents a **heat interaction.**

[2 marks – bw]

The total differential can also be written as
$$dH = \left(\frac{\partial H}{\partial S}\right)_p dS + \left(\frac{\partial H}{\partial p}\right)_S dp$$

Comparing terms between the two $\left(\frac{\partial H}{\partial S}\right)_p = T$ and $\left(\frac{\partial H}{\partial p}\right)_S = V$. The enthalpy is a function of state, so is an exact differential (equivalence of second derivatives), so
$$\left(\frac{\partial^2 H}{\partial p \partial S}\right) = \left(\frac{\partial^2 H}{\partial S \partial p}\right) \Rightarrow \left(\frac{\partial (\partial H/\partial S)_p}{\partial p}\right)_S = \left(\frac{\partial (\partial H/\partial p)_S}{\partial S}\right)_p \Rightarrow \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$

[2 marks – bw]

b) The Nernst Heat Theorem says that the **entropy change** of any system reaction when the **temperature is close to absolute zero is itself zero**.   [1 mark – bw]

The most general definition of heat capacity is $\boldsymbol{C_\alpha = T \left(\frac{\partial S}{\partial T}\right)_\alpha}$.   [1 mark – bw]

The entropy change is given by
$$\Delta S = \int_{T_1}^{T_2} \frac{C_\alpha}{T} dT = C_\alpha [\ln T_2 - \ln T_1]$$

Now $\boldsymbol{\lim_{T_1 \to 0} \ln T_1 = -\infty}$. Thus, the **entropy change must blow-up as $T \to 0$ unless $C_\alpha \equiv 0$**.   [2 marks – bw]

c) The partition function is defined as $\boldsymbol{Z = \sum_{n=0}^{\infty} \exp(-\varepsilon_n / k_B T)}$.   [1 mark – bw]

In this case it becomes, and after using the geometric progression identity,
$$Z = \exp\left(-\frac{\hbar\omega}{2k_B T}\right) \sum_{n=0}^{\infty} \exp\left(-\frac{\hbar\omega n}{k_B T}\right) = \frac{\exp\left(-\frac{\hbar\omega}{2k_B T}\right)}{1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)}.$$

[2 marks - prob]

After multiplying top and bottom by $\boldsymbol{\exp\left(\frac{\hbar\omega}{2k_B T}\right)}$ and recalling $\boldsymbol{\sinh x = \frac{1}{2}(e^x - e^{-x})}$

$$Z = \frac{1}{\exp\left(\frac{\hbar\omega}{2k_BT}\right) - \exp\left(-\frac{\hbar\omega}{2k_BT}\right)} = \frac{1}{2\sinh\left(\frac{\hbar\omega}{2k_BT}\right)}.$$

[1 mark - prob]

d) The Boltzmann distribution is

$$n_j = A\exp\left(-\frac{\varepsilon_j}{k_BT}\right)$$

[1 mark – bw]

The two expressions for the energy level populations are

$$100 = A\exp\left(-\frac{10\text{ meV}}{k_BT}\right) \quad , \quad 800 = A\exp\left(-\frac{40\text{ meV}}{k_BT}\right)$$

Eliminating for A and simplifying leads to

$$\frac{100}{800} = \exp\left(-\frac{(10-40)\text{ meV}}{k_BT}\right) \quad \Rightarrow \quad k_BT\ln\left(\frac{1}{8}\right) = 30\text{ meV}$$

Hence, $T = -167\text{ K}$.

[2 marks – prob]

A negative temperature! This is because we have a **population inversion** – more particles in the higher energy (excited) state (e.g. a laser). [1 mark – prob]

Foundations of Physics 2B/3C   2014/2015

Thermodynamics – Summer Examinations 2015

Long Question 1 Solution – Post Committee

Parts of solution in **bold** minimum expected/key ideas students need to have written.

a) A small change to the entropy, S, is given in terms of a differential change to the heat interaction via $TdS = \delta Q$. This allows calculation of the entropy change between two states $\Delta S = \int \frac{\delta Q}{T}$.

Statistically, it is related to the **number of microstates** in the system, Ω via $S = k_B \ln \Omega$. In a **reversible process**, the **entropy change** around a **closed cycle is zero**, whilst it is non-zero in an irreversible process. The Clausius inequality is expressed by $\oint \frac{\delta Q}{T} \leq 0$.

The **higher the entropy change** of a process, the **lower the quality of its energy**, which means that less energy is available for work. The irreversibility of a process is given by $I = T_0 \Delta S_{Universe}$, where T_0 is the temperature of the lowest available heat reservoir.

[4 marks – bw, one per point]

b) The heat supplied to the water to boil it from the reservoir is the same as that "lost" to the room on cooling. Using the heat capacity formalism

$$\delta Q = mc_p dT \quad \Rightarrow \quad Q_{water} = \int_{293}^{368} mc_p dT = 75\, c_p \text{ J.}$$

The entropy changes of the reservoir and room are thus (as they are at constant temperatures)

$$\Delta S_{res} = -\frac{Q_{water}}{368} \quad ; \quad \Delta S_{room} = +\frac{Q_{water}}{293}.$$

[1 mark –prob]

The water has equal and opposite entropy changes on heating/cooling, given by

$$\Delta S_{water}^{\uparrow} = mc_p \int_{293}^{368} \frac{dT}{T} = c_p \ln\left(\frac{368}{293}\right) \quad ; \quad \Delta S_{water}^{\downarrow} = c_p \ln\left(\frac{293}{368}\right)$$

[1 mark –prob]

The total entropy change is thus,

$$\Delta S_{Universe} = \Delta S_{water}^{\uparrow} + \Delta S_{water}^{\downarrow} + \Delta S_{res} + \Delta S_{room} = 75\, c_p \left[\frac{1}{293} - \frac{1}{368}\right]$$

$$= 219 \text{ J K}^{-1}.$$

[1 mark – prob]

c) Applying the **reciprocity theorem to** $\left(\frac{\partial S}{\partial T}\right)_V$ gives

$$\left(\frac{\partial S}{\partial T}\right)_V = -\left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_S.$$

Using the **Maxwell Relation on the first term**, and working its derivative out for an ideal gas ($pV = RT$) shows $\left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V}$.

[2 marks – prob]

For an adiabatic, $pV^\gamma = \text{const}$, which can be expressed $RTV^{\gamma-1} = \text{const}$. [1 mark – bw]
Differentiating,

$$(\gamma - 1)\left(\frac{\partial V}{\partial T}\right)_S V^{\gamma-2} = -\frac{\text{const}}{RT^2} \Rightarrow (\gamma - 1)\left(\frac{\partial V}{\partial T}\right)_S \frac{1}{V} = -\frac{\text{const}}{RTV^{\gamma-1}}\frac{1}{T} \Rightarrow \left(\frac{\partial V}{\partial T}\right)_S$$

$$= -\frac{V}{(\gamma - 1)T}.$$

[1 mark – prob]

Now $\gamma = C_p/C_V$ and $C_p = C_V + R$. Thus, $(\gamma - 1) = R/C_V$ [1 mark – prob]

Combining altogether,

$$\left(\frac{\partial S}{\partial T}\right)_V = -\frac{R}{V} \times -\frac{V}{(R/C_V)T} = \frac{C_V}{T}.$$

[1 mark – prob]

d) The number of ways to arrange N particles in j states with n_j per state is

$$\Omega = \frac{N!}{\prod_j n_j!}$$

[1 mark – bw]

$$\Omega_{Before} = \frac{(n_1 + n_2)!}{n_1! n_2!} \quad ; \quad \Omega_{After} = \frac{(n_1 + n_2)!}{(2n_1)!(n_2 - n_1)!}$$

The entropies are therefore

$$S_{Before} = k_B[\ln(n_1 + n_2)! - \ln(n_1)! - \ln(n_2)!]$$
$$= k_B[(n_1 + n_2)\ln(n_1 + n_2) - (n_1 + n_2) - n_1\ln(n_1) + n_1 - n_2\ln(n_2) + n_2]$$
$$S_{After} = k_B[\ln(n_1 + n_2)! - \ln(2n_1)! - \ln(n_2 - n_1)!]$$
$$= k_B[(n_1 + n_2)\ln(n_1 + n_2) - (n_1 + n_2) - 2n_1\ln(2n_1) + 2n_1 - (n_2$$
$$- n_1)\ln(n_2 - n_1) + (n_2 - n_1)]$$

[2 marks – prob]

$$\Delta S = k_B[(n_1 + n_2)\ln(n_1 + n_2) - (n_2 - n_1)\ln(n_2 - n_1) - 2n_1\ln(2n_1)$$
$$- (n_1 + n_2)\ln(n_1 + n_2) + n_1\ln(n_1) + n_2\ln(n_2)].$$
$$= k_B[n_2\ln(n_2) - n_2\ln(n_2 - n_1) + n_1\ln(n_1) - n_1\ln(4n_1^2) + n_1\ln(n_2 - n_1)]$$
$$= k_B\left[n_2\ln\left(\frac{n_2}{n_2 - n_1}\right) + n_1\ln\left(\frac{n_1(n_2 - n_1)}{4n_1^2}\right)\right]$$
$$= k_B\left[n_2\ln\left(\frac{n_2}{n_2 - n_1}\right) + n_1\ln\left(\frac{(n_2 - n_1)}{4n_1}\right)\right].$$

[2 marks – prob]

For the specific case given in the problem the expression becomes

$$\Delta S = k_B\left[3N_A\ln\left(\frac{3N_A}{2N_A}\right) + N_A\ln\left(\frac{2N_A}{4N_A}\right)\right] = R\left[\ln\left(\frac{3}{2}\right)^3 + \ln\left(\frac{1}{2}\right)\right] = R\ln\left(\frac{27}{16}\right)$$
$$= 4.35 \text{ J K}^{-1}.$$

[2 marks – prob]

Foundations of Physics 2B/3C   2014/2015

Thermodynamics – Summer Examinations 2015

Long Question 2 Solution – Final Check

Parts of solution in **bold** minimum expected/key ideas students need to have written.

a) The engine goes through a cycle. **Heat is taken in** to the **working fluid** from a **hot reservoir**, which then **does some work**. **Waste heat is rejected to a low temperature reservoir**, before the **cycle returns to its start point** and runs again.

[4 marks bw - one per point]

b)

i) The ideal gas law, $pV = nRT$, tells us that if the **pressure doubles** at constant volume, then **so must the temperature**. The **isothermal expansion** takes place at $2T_i$ and will have **shape $p \propto \frac{1}{V}$** heat must be taken in. Similarly, the volume after the second thermodynamic process must be **twice the original volume** if the temperature at that point is double the initial temperature but at the original pressure. Finally, **decreasing the volume**, decreases the temperature and causes **heat to be rejected**.
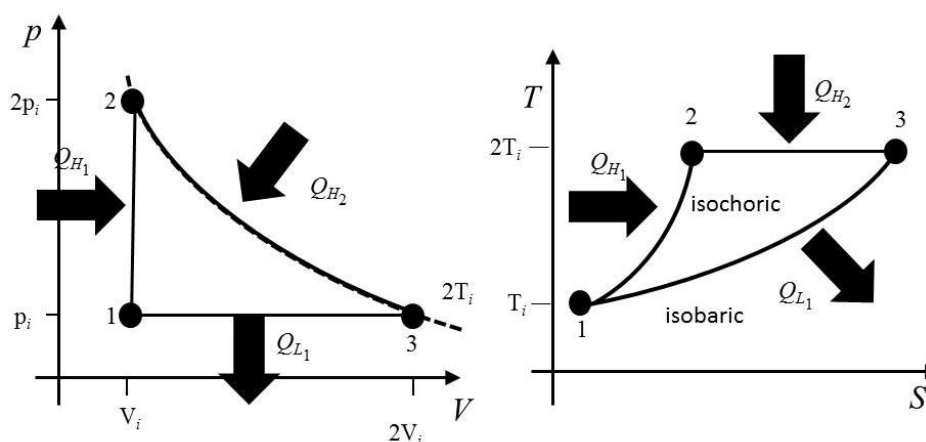
[3 marks - prob]

On the TS diagram the isochoric and isobaric shapes can be calculated from the general definition of their heat capacities, $C_\alpha = T(\partial S / \partial T)_\alpha$ as

$$S_{isochoric} = \int \frac{C_V dT}{T} = C_V \ln T + \text{const} \Rightarrow \ln T = \frac{S}{C_V} - \frac{\text{const}}{C_V} \Rightarrow T = A \exp\left(\frac{S}{C_V}\right)$$

$$S_{isobaric} = \int \frac{C_p dT}{T} = C_p \ln T + \text{const.} \Rightarrow T = B \exp\left(\frac{S}{C_p}\right)$$

We don't know the initial entropy, so can only **calculate the entropy changes** here, and see that they are exponential in shape. $C_p = R + C_V$ **so isochoric is steeper than the isobaric.**

[2 marks – prob]

[2 marks – prob, need axis and appropriate values labelling]

ii) Efficiency is given by

$$\eta = \frac{|\text{Work}|}{\text{Heat In}} = \frac{|Q_H - |Q_L||}{Q_H}.$$

[1 mark – bw]

The heat taken in during the isochoric process is $Q_{H_1} = C_V dT = C_V T_i > 0$.

Along the isotherm at $T = 2T_i$ $U = U(T)$ only for an ideal gas, so from the first law $\delta Q = -\delta W$. Therefore,

$$Q_{H_2} = -\int_{V_i}^{2V_i} (-pdV) = \int_{V_i}^{2V_i} \frac{2RT_i}{V} dV = 2RT_i \ln 2 > 0.$$

The heat rejected on the isobaric is $Q_L = C_p dT = -C_p T_i < 0$.

[2 marks - prob]

Combining, and using the fact that $C_p = R + C_V$.

[1 mark – bw]

$$\eta = \frac{2RT_i \ln 2 + c_V T_i - c_p T_i}{2RT_i \ln 2 + c_V T_i} = \frac{R[2 \ln 2 - 1]}{c_V + 2R \ln 2}$$

For a diatomic gas $\gamma = C_p / C_V = 7/5$ so $C_V = 5/7(R + C_V) \Rightarrow c_V = 5R/2$ so

$$\eta = \frac{R[\ln 4 - 1]}{R(5/2 + \ln 4)} = 9.9\ \%$$

A Carnot cycle would operate between reservoirs at $2T_i$ and T_i and would have efficiency

$$\eta_{Carnot} = 1 - \frac{T_L}{T_H} = 1 - \frac{T_i}{2T_i} = 50.0\ \%$$

[2 marks - prob]

c) The first law is a statement of energy conservation and $\delta Q = -\delta W$ around a closed cycle (exact differential). The engine takes in heat $6Q$ rejecting $-Q$ and doing $-5Q$ work, so round the cycle $5Q = -(-5Q)$ so the first law is satisfied.   [1 mark – prob]

The second law can be expressed by the **Clausius Inequality**, $\oint \frac{\delta Q}{T} \leq 0$. Calculating this we find,

$$\frac{4Q}{4T} + \frac{2Q}{2T} - \frac{Q}{T} = \frac{Q}{T} > 0.$$

This **violates the Clausius inequality**, and hence the second law, so the **engine cannot work** even though the first law is satisfied!   [2 marks – prob]

Examination Questions May/June 2015
Foundations of Physics 2B, Condensed Matter Physics, Q4 (5 parts):
SOLUTIONS

Bookwork   a. Sketch shown in diagram. The spacing of each of the families of planes is given by:

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

Problem   where a is the lattice constant (0.5 nm). Substituting in the ($h\,k\,l$) values gives: 0.35 nm and 0.20 nm
2 marks for diagram and 1 mark for each answer.
[4 marks]

Bookwork   b. For optical modes the two atoms in the basis move out of phase and can couple to EM radiation. For acoustic modes the two atoms in the basis move in phase and propagate as sound waves. For the optical mode (using the long wavelength

Problem   approximation) we have $\omega \cong \sqrt{2C\left(\frac{1}{M_1} + \frac{1}{M_2}\right)}$ Substituting in

the values gives ω = 2.076 × 10^{13} rad s^{-1} and E = 13.7 meV (as $E = \hbar\omega$) **[4 marks]**

Problem   c. For copper (which is a group 11 element) assume there is one free electron per atom (from the outer shell). The atomic density is then the same as the free electron density. Density = 8950 / (63.5 × 1.67×10^{-27}) = 8.44×10^{28} m^{-3} free electrons. The Drude conductivity formula

(from the Drude model) is $\sigma = \frac{ne^2\tau}{m_0}$ which gives $\tau = \frac{\sigma m_0}{ne^2}$. Using the data given we get τ =

2.52×10^{-14} s. **[4 marks]**

Bookwork   d. The effective mass is related to the energy wavevector

dispersion relation by $m_{eff} = \hbar^2 \left[\frac{d^2 E(k)}{dk^2}\right]^{-1}$. In other

words the effective mass is the inverse curvature, or second derivative, of E with respect to k. At $k = 0$ the effective mass is small and positive. As the wavevector increases the effective mass increases until at the mid-point of the Brillouin zone it diverges. As the wavevector k continues to increase the effective mass is negative and returns to a low constant negative value at the Brillouin zone boundary, as shown in the diagram. **[4 marks]**
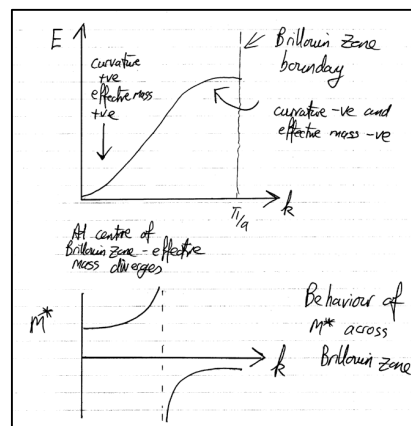
Bookwork   e. The hydrogen atom is considered analogous to the behaviour of shallow impurities in semiconductors. A donor consists of a single positively charged ion in the semiconductor crystal with one additional electron which is free to move throughout the crystal. When the electron is free to move it is in the conduction band therefore the donor level is below the bottom of the conduction band by the equivalent of the donor binding energy. The expression for the hydrogen atom is modified by the effective mass of the electron (rather than the free mass) and the relative permittivity of the semiconductor (as

Problem   a dielectric medium). The expression then becomes $E = -\frac{e^4 m_e^*}{2\left(4\pi\epsilon_0\epsilon_r\hbar\right)^2}$

which can be written as $E = R_H \frac{m_e^*}{m_0}\frac{1}{\epsilon_r^2}$ where R_H is the hydrogen Rydberg

(-13.6 eV). Substituting in the values gives 3.47 meV below the conduction band edge, as shown on the diagram. **[4 marks]**.
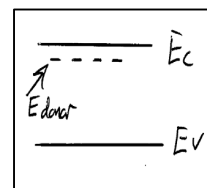
Examination Questions May/June 2015
Foundations of Physics 2B, Condensed Matter Physics, Q5
SOLUTION

Bookwork

a) A stable bond is composed of both a repulsive and attractive component in equilibrium. The repulsive component arises from the quantum interaction as a consequence of the Pauli Exclusion Principle. When two electron wavefunctions overlap spatially this interaction causes the energy to increase thereby creating a repulsive force. The exact functional form of this force is difficult to calculate but a common approximation is to use r^{-12}.

The attractive component is a consequence of different types of interaction and gives rise to the different types of bonds. In a covalent bond there is sharing of the electrons from adjacent atoms. The bond is a consequence of the wavefunction symmetry. A stable bonds has an asymmetric spin and a symmetric electron density with a non-zero density of electrons between atoms. This does not occur when the electron density wavefunction is asymmetric (with symmetric spin). **[4 marks]**

Problem

b) The Morse Potential is: $V(r) = V_0\left[\exp\left(\frac{-2(r-r_o)}{a}\right) - 2\exp\left(\frac{-(r-r_0)}{a}\right)\right]$
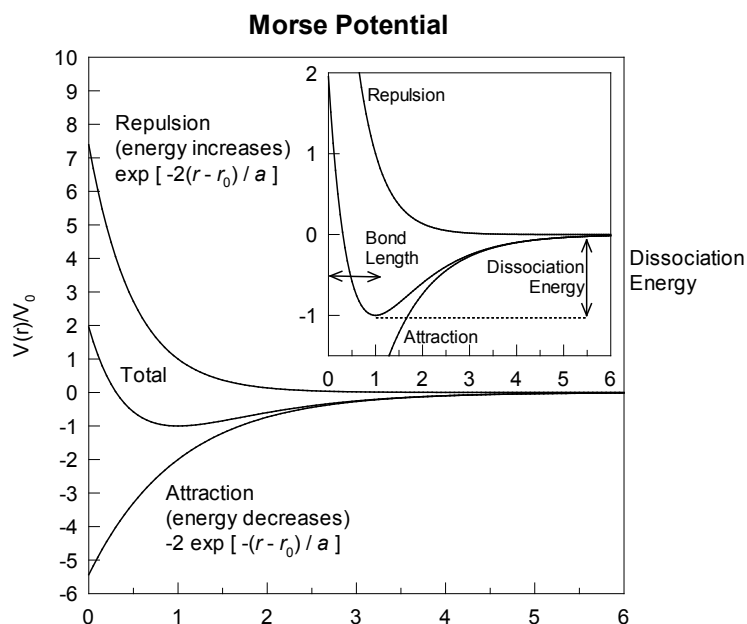
Morse Potential

[5 marks for graph]
[5 marks for labels]

Equilibrium separation occurs at $r = r_0$ (i.e. at $r/r_0 = 1$)   (a is a measure of the width of the potential well)

Problem

c) The restoring force for small displacements from equilibrium position $r_0 = r$. We have a displacement $x = r - r_0$.

$$V(x) = V_0\left[\exp\left(\frac{-2x}{a}\right) - 2\exp\left(\frac{-x}{a}\right)\right] \text{ and}$$

$$F(x) = -V_0\left[\frac{-2}{a}\exp\left(\frac{-2x}{a}\right) + \frac{2}{a}\exp\left(\frac{-x}{a}\right)\right] \text{ giving } F(x) = \frac{2V_0}{a}\left[\exp\left(\frac{-2x}{a}\right) - \exp\left(\frac{-x}{a}\right)\right]$$

Problem

As x is very small we can expand the exponential terms:

$$F(x) = \frac{2V_0}{a}\left[\left(1 - \frac{2x}{a} + \frac{2x^2}{a^2} + \ldots\ldots\right) - \left(1 - \frac{x}{a} + \frac{x^2}{2a^2} + \ldots\right)\right]$$

We then neglect all terms of x^2 and higher giving $F(x) = \frac{-2V_0}{a^2}x = \frac{-2V_0}{a^2}(r - r_0)$. **[4 marks]**

The resulting motion will be simple harmonic motion as the force is in the opposite direction to the displacement and proportional to the displacement – both key features of SHM. **[2 marks]**

Examination Questions May/June 2015
Foundations of Physics 2B, Condensed Matter Physics, Q6
SOLUTION

Bookwork a) The energy density of states function $n(E)$ describes the number of available energy states, which can be occupied by electrons, per unit energy range. It can be used to determine the density of free electrons in a metal. The function varies as $n(E) \propto E^{1/2}$ in 3D as shown in the diagram. **[4 marks]**
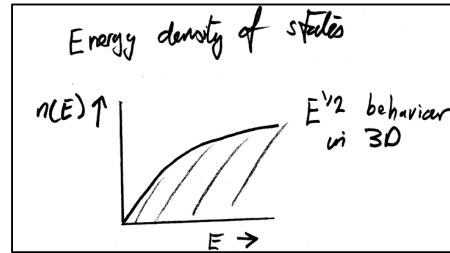
Bookwork b) The Fermi energy is the energy of the highest occupied state in a metal (which is in its ground state). In a metal all energy states below the Fermi energy are occupied and all energy states above the Fermi energy are vacant. Note that at finite temperatures there is a thermal broadening of the transition from probability 1 to probability 0. The width of the transition is approximately $2k_B T$ around E_F. The diagram shows the behaviour of the Fermi-Dirac function at zero and finite temperatures. **[4 marks]**
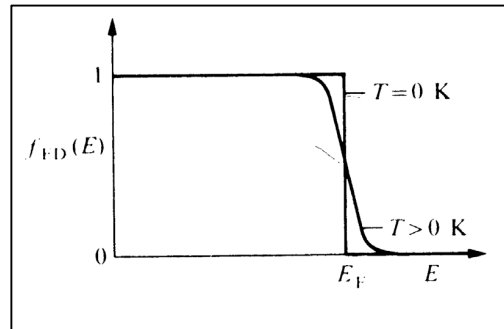
Problem c) The Fermi energy is determined from the free electron density: $E_F = \dfrac{\hbar^2 k_F^2}{2m_e}$ where the Fermi wavevector is given by $k_F = \left(3\pi^2 n\right)^{1/3}$ The Fermi wavevector can be determined from the Fermi sphere. Substituting in the electron density gives $k_F = \left(3\pi \times 6 \times 10^{28}\right)^{1/3}$ = 8.27×10⁹ m⁻¹ giving an energy of $E_F = \dfrac{\hbar^2 \times \left(8.27 \times 10^9\right)^2}{2m_e}$ = 4.17×10⁻¹⁹ J = 2.6 eV. The Fermi velocity is determined by $v_F = \dfrac{p}{m_e} = \dfrac{\hbar k_F}{m_e}$ = 9.57 × 10⁵ ms⁻¹ **[5 marks]**

Problem d) The thermal energy and velocity of electrons at 300K is predicted by the classical equipartition theorem: $\tfrac{1}{2} m v^2 = \tfrac{3}{2} k_B T$. This gives 25.8 meV for the thermal energy and $v = \sqrt{\dfrac{3k_B T}{m_e}}$ for the thermal velocity = 1.17×10⁵ ms⁻¹. This velocity is 8 times less than the Fermi velocity and the Fermi energy is 100 times larger than the thermal energy. This arises because of the large concentration of free electrons which pushes up the Fermi wavevector and Fermi energy. The electrons are subject to the Pauli exclusion principle and therefore increasing electron density requires them to occupy higher energy states in the free electron model as non localised free electron wavefunctions (plane waves) will overlap. This pushes up the energy of electrons at the Fermi surface. **[4 marks]**

Problem e) When the metal is cooled down from 300 K the metal will contract slightly this will produce a slightly higher electron density which will increase the Fermi wavevector and the Fermi energy. The transition from probability 1 to probability 0 will also occur over a slightly narrower energy range as $k_B T$ is reduced. **[3 marks]**

Q.7 Modern Optics

(a) [BOOKWORK]

(i) $G(u) + H(u)$ [1]

(ii) $G(u) e^{-i2\pi u_0 d} + a H(au)$ [1]

(iii) $G(u) \otimes H(u)$ [1]

(iv) $\frac{1}{2}\left[\delta\left(u - \frac{1}{\lambda_0}\right) + \delta\left(u + \frac{1}{\lambda_0}\right)\right]$ [1]

(b) [UNSEEN]

$$e^{-\rho^2/w^2} = e^{-(x^2+y^2)/w^2} = e^{-x^2/w^2} e^{-y^2/w^2} = g(x) h(y) \quad [2]$$

cartesian separable!

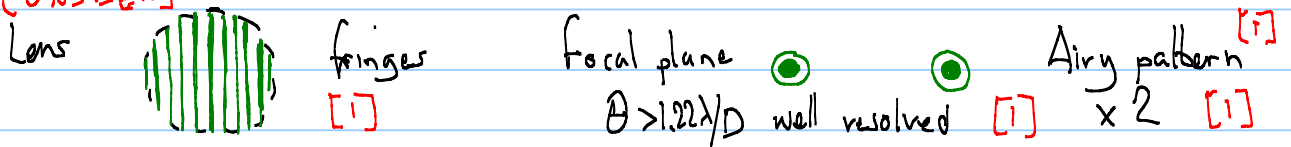By completing the square $\mathcal{F}[e^{-x^2/w^2}](k_x) = \sqrt{\pi} w \, e^{-k_x^2 w^2/2}$

$\mathcal{F}[gauss(\rho/w)](k_x, k_y) = \pi w^2 e^{-(k_x^2 + k_y^2)w^2/2}$ [2]

(c) [UNSEEN]

[1]   [1]

Slit with a, wire $2a$ so slit diffraction pattern is wider [1]

(d) [UNSEEN]

Lens   fringes   focal plane ⊙ ⊙ Airy pattern [1]

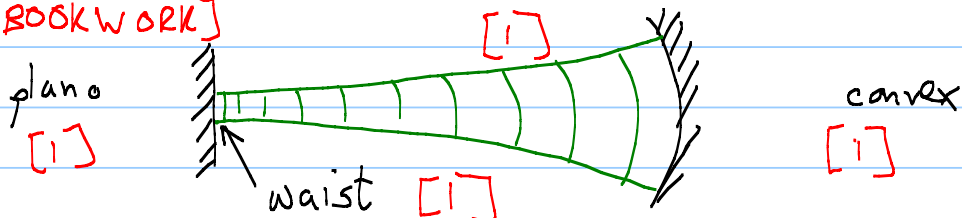[1] $\theta > 1.22\lambda/D$ well resolved [1] × 2 [1]

(e) [UNSEEN]

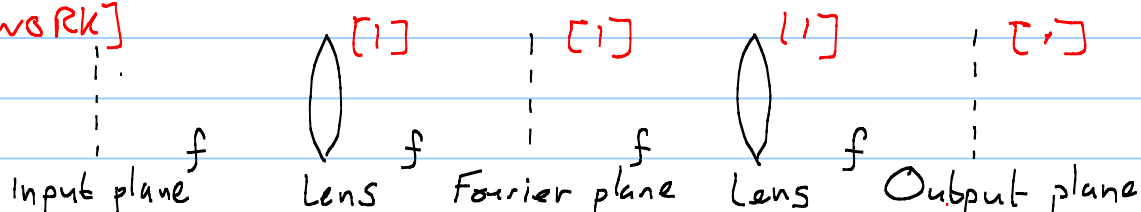Laser beam size at distance z: $w = \frac{\lambda}{\pi w_0} z = \frac{10^{-6} \, 10^4}{\pi \, 10^{-3}} \sim 3 \, m$ [1] [1]

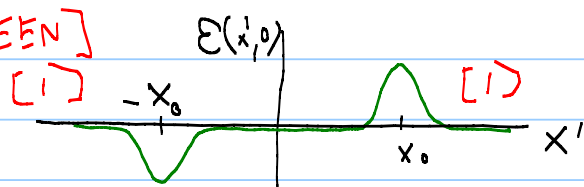assume resolution set by beam size [1] $\sim 3 \, m$ [1]

(f) [BOOKWORK]

plano   [1]   waist [1]   convex   [1]

Q.8 Modern Optics.

(a) [BOOKWORK]

[1]   [1]   [1]   [1]

Input plane f Lens f Fourier plane f Lens f Output plane

(b) [UNSEEN]

[1] $E(x',0)$ [1]

(c) [BOOKWORK] $\mathcal{F}[f(x',y')](k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y') e^{i(k_x x' + k_y y')} dx' dy'$ [1]   [1]

(d) [UNSEEN]

$\mathcal{F}\left[gauss\left(\frac{\rho'}{w_0}\right)\right] = \pi w_0^2 \, gauss\left(\frac{k\rho w_0}{2}\right)$ [1]

$\mathcal{F}\left[\delta(x'-x_0) - \delta(x'+x_0)\right] = -2i \sin k_x x'$ [1]

Using convolution theorem [1] and $k_x = kx/f$, $k_y = ky/f$ [1]

$E^{(f)} = \frac{2 E_0 e^{i2kf}}{\lambda f} \pi w_0^2 \, e^{-k^2(x^2+y^2)/4f^2} \sin\frac{kx'x}{f}$

(e) Ignoring the arbitrary global phase

Gauss envelope   [1]

sin [1]
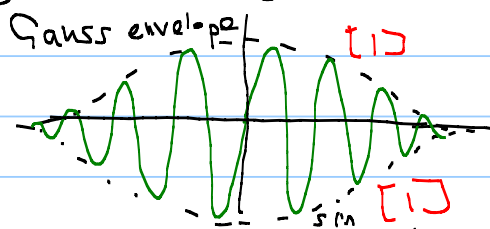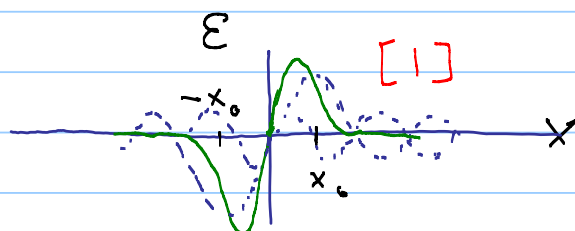
(f) Filtered field is $E^{(f)}$ $rect\left(\frac{x}{f\lambda/2x_0}\right)$ [2]

(g) We recover the input field ('flipped') convolved with   [1]

a sinc [1] from the filter

$E^{(out)} = E_0 f(-x,-y) \otimes \frac{f\lambda}{2x_0} sinc\frac{(kx/f)(f\lambda/2x_0)}{2} = \frac{f\lambda E_0}{2x_0} f(-x,-y) \otimes sinc\left(\frac{\pi x}{x_0}\right)$ [1]

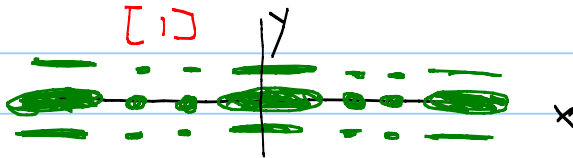1st zero at $x = x_0$

E

[1]

Q.9 Modern Optics

(a) [BOOKWORK] Fraunhofer observed in paraxial limit [1] where all transverse dimensions x', y' x, y are much less than propagation distance z [1]
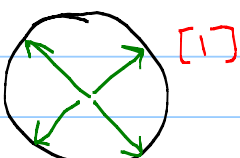
Scalar for polarized light [1] and paraxial limit. [1]

(b) [BOOKWORK] In the far field the field is given by a Fourier transform [1] of the input, i.e., $E(x, y) = \dfrac{E_0 e^{ikz}}{i\lambda z} \mathcal{F}[f(x', y')](k_x, k_y)$ [1]

(c) [UNSEEN] Writing $f(x', y') = g(x') h(y') * m(x')$ we use cartesian separability [1] and the convolution theorem [1] to write that $\mathcal{F}[f] = \mathcal{F}[g]\mathcal{F}[h]\mathcal{F}[m]$.

Using $\mathcal{F}[\text{rect}(x/a)] = a\,\text{sinc}(k_x a/2) \dots$ etc

$\mathcal{F}[f(x', y')] = 2ab\,\text{sinc}\left(\dfrac{k_x a}{2}\right)\text{sinc}\left(\dfrac{k_y b}{2}\right)\left[\cos\left(\dfrac{k_x d}{2}\right) + \cos\left(\dfrac{3k_x d}{2}\right)\right]$ [1]

[BOOKWORK] $I = \dfrac{I_0}{\lambda^2 z^2}\left|\mathcal{F}[f(x', y')]\right|^2$ [1]

(d) [UNSEEN]

sinc in y [1]

[1]

b) a pattern compressed in [1]

2 subsidiary max^m. [1]

(e) [UNSEEN] At a position of zero field phasor must sum to zero.

For comb4 4 phasors 2 moving at $\pm\frac{1}{2}k_x d$ and 2 at $\pm 3\frac{1}{2}k_x d$

So 1st zero

 [1]

 [1] central zero

(f) [UNSEEN] The field at the principal maxima is proportial to the number of phasors [1] and the intensity is proportional to the (field)^2. So relative Intensity $\dfrac{5^2}{4^2} = \dfrac{25}{16}$ [1]

(g) [UNSEEN] There are now 5 phasors, one is fixed and the others move as before. [1] To get a sum equal to [1] zero we can take phasor at $0, \pm 60, \pm 180°$