Quantum Theory - Worksheet 1

Problem 1

It is mentioned in Part 1 of the course notes, in relation to the Stern Gerlach experiment, that the column vectors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

are eigenvectors of the matrix

$$S_z = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}, \tag{2}$$

and that the column vectors

$$\chi'_+ = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \chi'_- = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \tag{3}$$

are eigenvectors of the matrix

$$S'_z = \begin{pmatrix} 0 & -\hbar/2 \\ -\hbar/2 & 0 \end{pmatrix}. \tag{4}$$

(a) Show, by multiplying $\chi_\pm$ by the matrix S_z and $\chi'_\pm$ by the matrix S'_z, that these column vectors are indeed eigenvectors of these matrices. What are the corresponding eigenvalues?

(b) Show that χ'_+ and χ'_- are orthonormal (i.e., that they both have unit norm and that they are orthogonal to each other).

Problem 2

Consider the matrix

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha \end{pmatrix},$$

where α is real and β is complex and non-zero.

(a) Find the eigenvalues of this matrix, and for each of these eigenvalues find an eigenvector.

(b) Show that these eigenvectors are orthogonal.

(c) Show that they form a basis for the vector space of 2-component column vectors. [What you are asked to do here is to show, by direct calculation, that any vector of the form

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

can be written as a linear combination of the two eigenvectors found in (a). If you are a keen mathematician, you may have immediately understood that these two eigenvectors form a basis because they are orthogonal and non-zero, that non-zero orthogonal vectors are linearly independent, that the dimension of this vector space is 2, and that a set of two linearly independent vectors belonging to a vector space of dimension 2 is always a basis for this vector space.]

(d) Show that, by contrast, the eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

do not span this vector space.

Note: The key mathematical difference between these two matrices is that the first one is Hermitian and the second one isn't. [Recall that a Hermitian matrix is a matrix equal to its conjugate transpose.] The eigenvectors of Hermitian matrices always span the space of the vectors they act on. Non-Hermitian matrices do not have this property.

Problem 3

Recall that the spherical harmonics $Y_{lm}(\theta, \phi)$, are functions of the polar angle θ and ϕ, that the index l (the orbital angular momentum quantum number in Quantum Mechanics) is a non-negative integer ($l = 0, 1, 2, \ldots$), and that for each value of l the index m can have any integer value between $-l$ and l. Recall, also, that these functions are orthonormal in the sense that

$$\int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi \, Y^*_{lm}(\theta, \phi)\, Y_{l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'}. \tag{5}$$

We will see later in the course (and perhaps you already know) that any eigenfunction of the angular momentum operator $\mathbf{L}^2$ can be written as a linear combination of spherical harmonics of a same value of l. Consider two such linear combinations for $l = 1$,

$$\begin{aligned} f(\theta, \phi) &= c_{-1}\, Y_{1-1}(\theta, \phi) + c_0\, Y_{10}(\theta, \phi) + c_1\, Y_{11}(\theta, \phi), \\ g(\theta, \phi) &= d_{-1}\, Y_{1-1}(\theta, \phi) + d_0\, Y_{10}(\theta, \phi) + d_1\, Y_{11}(\theta, \phi), \end{aligned}$$

where the coefficients c_m and d_m are complex numbers.

(a) Show that the inner product of these two functions, defined as the integral

$$\int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi \, f^*(\theta, \phi)\, g(\theta, \phi),$$

is $c^*_{-1} d_{-1} + c^*_0 d_0 + c^*_1 d_1$.

(b) Show that the same result is also obtained by taking the inner product of the column vectors formed by the coefficients c_m and d_m,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_{-1} \\ d_0 \\ d_1 \end{pmatrix}.$$