Workshop 9: Rigid Body Motion

1. A rigid body is turning about a fixed point O, and $Oxyz$ are Cartesian axes. The components of velocity of the particle with coordinates $(1, 0, 0)$ are $(0, 2, 5)$. Find the component in the direction of the x axis of the velocity of the particle with coordinates $(0, 0, 1)$.

2. (a) For a uniform density cylinder of height $2h$, radius R and total mass M, the inertia tensor is defined as

$$I_{\alpha\beta} = \int_{\text{volume}} dxdydz\rho(x,y,z)(|\underline{r}|^2\delta_{\alpha\beta} - r_\alpha r_\beta),$$

where ρ is the mass density, $|\underline{r}|^2 = x^2 + y^2 + z^2$, and α and β run over the three Cartesian coordinates of vector $\underline{r}$ (i.e. $r_1 \equiv x$, $r_2 \equiv y$ and $r_3 \equiv z$). Show that, with the z axis lined up with the symmetry axis of the cylinder, the inertia tensor about the centre of mass can be written as

$$\hat{I}_{\text{CoM}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{xx} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix},$$

where $I_{xx} = M(R^2/4 + h^2/3)$ and $I_{zz} = MR^2/2$.

A victorious table tennis player celebrated by throwing their bat into the air. The subsequent rotational motion of the table tennis bat can be understood by approximating it as two connected cylinders - a long, thin cylinder for the handle, and a short, fat cylinder for the blade with which the ball is typically hit - and calculating the inertia tensor of this compound object. Assume that the blade has radius R, height $2h \ll R$ and mass M, while the handle has radius $u \ll R$, height R and mass M. The bat is constructed by attaching one end of the handle to the circular edge of the blade such that the axis of symmetry of the handle passes through the centre of mass of the blade.

For the remainder of the question, use 'body' coordinates such that axis 1 is along the axis of symmetry of the handle and axis 2 is parallel to the symmetry axis of the blade.

(b) Where is the centre of mass of the bat?

Using the displaced axis theorem,

$$\hat{I} = \hat{I}_{\text{CoM}} + M\hat{A},$$

where $A_{\alpha\beta} = |\underline{R}_C|^2\delta_{\alpha\beta} - R_{C,\alpha}R_{C,\beta}$, and $\underline{R}_C$ is the vector representing the centre of mass position vector of either the handle or the blade with respect to the centre of mass position of the whole bat, determine the inertia tensors for the handle and the blade about the centre of mass of the bat. Hence find the inertia tensor for the bat with respect to rotations about its centre of mass.

(c) Rank the principal moments of inertia of the bat about the 1, 2 and 3 axes (I_1, I_2 and I_3 respectively) in order of increasing size.

The initial angular velocity given to the bat is $\underline{\omega} = (0, \omega_2, \omega_3)$, where $\omega_2 \gg \omega_3 > 0$. Describe the subsequent rotational motion of the bat.