(a) Probability of being in the volume between $r = 0$ to $r_{\rm p}$ is

$$\int_0^{r_{\rm p}} \psi^* \psi \, 4\pi r^2 dr = \int_0^{r_{\rm p}} R^* R \, r^2 dr = \int_0^{rp} 4a^{-3} e^{-2r/a} r^2 dr \approx \frac{4}{a^3} \left[ \frac{r^3}{3} \right]_0^{r_{\rm p}} = \frac{4}{3} \frac{r_{\rm p}^3}{a^3}$$

$$= \frac{4}{3} \left( 1.9 \times 10^{-5} \right)^3 = 9.1 \times 10^{-15}$$

[2 marks]

(b) We need $\langle r \rangle$ for each of $n, l, m$ of (2,0,0), (2,1,-1), (2,1,0) and (2,1,1)

$$\langle r \rangle_{nlm} = \int \int \int r R_{nl}^2 \, Y_{lm}^* Y_{lm} \, r^2 \sin\theta d\theta d\phi = \int r^3 R_{nl}^2 \, dr$$

so this can only depend on $n$ and $l$ not $m$ and so we just have two cases to evaluate

$$\begin{aligned}
\langle r \rangle_{200} &= \int r^3 \frac{1}{2a^3} \left( 1 - \frac{r}{2a} \right)^2 e^{-r/a} dr = \frac{1}{2a^3} \int r^3 \left( 1 - \frac{r}{a} + \frac{r^2}{4a^2} \right) e^{-r/a} dr \\
&= \frac{1}{2a^3} \left[ \int r^3 e^{-r/a} dr - \frac{1}{a} \int r^4 e^{-r/a} dr + \frac{1}{4a^2} \int r^5 e^{-r/a} dr \right] \\
&= \frac{1}{2a^3} \left[ \frac{3!}{(1/a)^4} - \frac{1}{a} \frac{4!}{(1/a)^5} + \frac{1}{4a^2} \frac{5!}{(1/a)^6} \right] \\
&= \frac{1}{2a^3} (6a^4 - 24a^4 + 30a^4) = \frac{a}{2} 12 = 6a.
\end{aligned}$$

[2 marks]

$$\begin{aligned}
\langle r \rangle_{21m} &= \int r^3 \frac{1}{24a^3} \frac{r^2}{a^2} e^{-r/a} dr = \frac{1}{24a^5} \int r^5 e^{-r/a} dr = \frac{1}{24a^5} \frac{5!}{(1/a)^6} \\
&= \frac{a}{24} (5.4.3.2) = 5a.
\end{aligned}$$

$\Rightarrow$ Higher $l$ lead to smaller radii. [2 marks]

(c) For $n, l$=2,0 , the probability of being in the volume from $r = 0$ to $r_{\rm p}$ is

$$\int_0^{r_{\rm p}} \frac{1}{2} a^{-3} \left( 1 - \frac{r}{2a} \right)^2 e^{-r/a} \, r^2 dr \approx \frac{1}{2a^3} \int_0^{r_{\rm p}} r^2 dr = \frac{1}{2a^3} \frac{r_{\rm p}^3}{3} = \frac{(r_{\rm p}/a)^3}{6} = 1.1 \times 10^{-15}$$

[2 marks]

For $n, l$=2,1 , the probability of being in the volume from $r = 0$ to $r_{\rm p}$ is

$$\int_0^{r_{\rm p}} \frac{1}{24a^3} \left( \frac{r}{a} \right)^2 e^{-r/a} \, r^2 dr \approx \frac{1}{24a^5} \frac{r_{\rm p}^5}{5} = \frac{(r_{\rm p}/a)^5}{120} = 2.0 \times 10^{-26}$$

[2 marks]