ELECTROMAGNETISM - Workshop 9th Set (Qns)
Waveguides and Jefimenko Equations

Professor D P Hampshire – 2nd Year Physics Lecture Course

The material for this workshop is split into three parts. Part I gives some background material. Part II gives some worked examples. Part III gives some additional unseen questions.

1 Background Material

1.1 Waveguides

A waveguide, as the name suggests, restricts waves to travel within a guide. This means that as the waves propagate, they do not to follow the inverse square law. The most simple waveguide is a highly conducting pipe, typically made of copper. Optical fibres are also waveguides. They are the basis of much of our communications technology and are typically a flexible, transparent fibre of a dielectric such as doped glass or plastic. State-of-the-art fibres can transmit a petabit (1000 terabit) per second over 200 km equivalent to sending 5,000 two hour long high-definition videos every second.

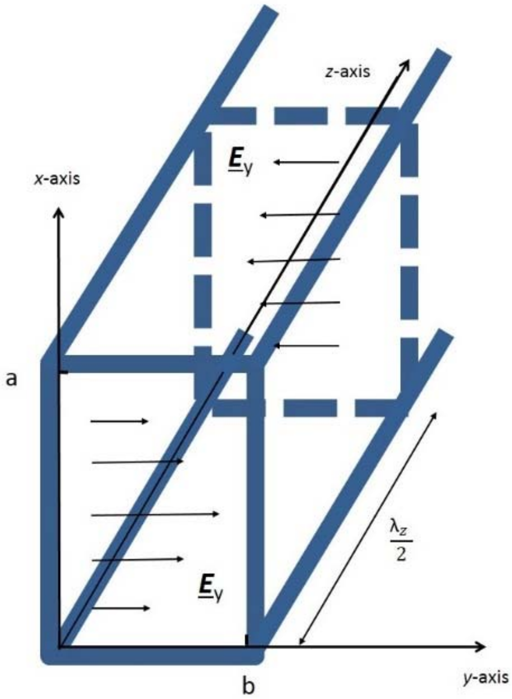

Figure 1 : A hollow waveguide carrying a TE (transverse electric) wave.

Transverse electromagnetic waves cannot exist in hollow waveguides. A transverse electric (TE) wave has an $\underline{\boldsymbol{E}}$-field that is transverse to the axis of the waveguide but a $\underline{\boldsymbol{B}}$-field that has components that are both transverse and parallel to the direction of propagation (!). A transverse magnetic (TM) wave has a $\underline{\boldsymbol{B}}$-field that is transverse to the axis of the waveguide but both a transverse and a longitudinal electric field.

Some text books talk about visualising these TE and TM waves as propagating at an angle to the z-axis, and bouncing or reflecting of the walls as an explanation for the longitudinal component of the $\underline{\boldsymbol{E}}$-field and $\underline{\boldsymbol{B}}$-field.

2 Worked examples

2.1 Questions

1. Maxwell's equations give the following wave-equation for electromagnetic waves propagating through a vacuum:

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad 2\text{-}1$$

For a TE wave of the form:

$$\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}, t) = \hat{\mathrm{j}} \mathrm{E}_{0y} \exp i(\mathrm{kz} - \omega t) \sin(\mathrm{k}_x x) \qquad 2\text{-}2$$

Calculate the associated magnetic field.

2. For a TM wave of the form:

$$\underline{\boldsymbol{B}}(\underline{\boldsymbol{r}}, t) = \hat{\mathrm{j}}\mathrm{B}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\sin(\mathrm{k}_x x) \qquad \text{2-3}$$

Calculate the associated electric field.

2.2 Answers

1. From the first law of optics, the angle of reflection is equal to the angle of incidence. From Snell's Law, we have:

Using Maxwell III:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \frac{-\partial \underline{\boldsymbol{B}}}{\partial t} \quad \text{(MIII)} \qquad \text{2-4}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & \mathrm{E}_y & 0 \end{vmatrix} \qquad \text{2-5}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\hat{\mathbf{i}} ik\mathrm{E}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\sin(\mathrm{k}_x x) + \hat{\mathbf{k}} k_x\mathrm{E}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\cos(\mathrm{k}_x x) \qquad \text{2-6}$$

Integrating Maxwell III with respect to time gives:

$$\underline{\boldsymbol{B}} = \frac{1}{i\omega}\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} \qquad \text{2-7}$$

so

$$\underline{\boldsymbol{B}} = -\hat{\mathbf{i}}\frac{k\mathrm{E}_{0y}}{\omega}\mathrm{expi}(\mathrm{kz} - \omega t)\sin(\mathrm{k}_x x) - \hat{\mathbf{k}}\frac{ik_x}{\omega}\mathrm{E}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\cos(\mathrm{k}_x x) \qquad \text{2-8}$$

The first component is a transverse component, the second a longitudinal component.

2. Using Maxwell IV:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0\varepsilon_0\frac{\partial \underline{\boldsymbol{E}}}{\partial t} \quad \text{(MIII)} \qquad \text{2-9}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = -\hat{\mathbf{i}} ik\mathrm{B}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\sin(\mathrm{k}_x x) + \hat{\mathbf{k}} k_x\mathrm{B}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\cos(\mathrm{k}_x x) \qquad \text{2-10}$$

Integrating Maxwell IV with respect to time gives:

$$\underline{\boldsymbol{E}} = \frac{i}{\omega\mu_0\varepsilon_0}\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} \qquad 2\text{-}11$$

so

$$\underline{\boldsymbol{E}} = \hat{\mathbf{i}}\frac{kB_{0y}}{\omega\mu_0\varepsilon_0}\mathrm{expi}(\mathrm{kz}-\omega t)\sin(\mathrm{k}_x x) + \hat{\mathbf{k}}\frac{ik_x B_{0y}}{\omega\mu_0\varepsilon_0}\mathrm{expi}(\mathrm{kz}-\omega t)\cos(\mathrm{k}_x x) \qquad 2\text{-}12$$

The first component is a transverse component, the second a longitudinal component.

3 Unseen problems

1. Starting from Maxwell's equations, derive the following wave-equation for electromagnetic waves propagating through a vacuum:

$$\nabla^2\underline{\boldsymbol{E}} = \mu_0\varepsilon_0\frac{\partial^2\underline{\boldsymbol{E}}}{\partial \mathrm{t}^2} \qquad 3\text{-}1$$

It is known that one possible solution for the wave equation within the interior of an infinitely long hollow square metallic tube (known as a waveguide) of width a is:

$$\mathrm{E}_x = \mathrm{E}_0\sin\left(\frac{\pi x}{\mathrm{a}}\right)\sin\left(\frac{\pi y}{\mathrm{a}}\right)\mathrm{expi}(\mathrm{kz}-\omega t) \qquad 3\text{-}2$$

$$\mathrm{E}_y = \mathrm{E}_z = 0 \qquad 3\text{-}3$$

where the z-axis is parallel to the tube and E_0 is a constant. Given that a = 10^{-2} m, find the minimum frequency that can propagate along the waveguide.
Hint: Derive the dispersion relation and find the condition that the wavevector k is real.

Write down another solution which describes a new wave that can propagate thorough the waveguide. Discuss whether the photons associated with your proposed solution have higher or lower energy than the original one considered above.

2. 'Post hoc ergo propter hoc' (or what does physics, and more specifically what do Maxwell's equations, tell us about Causality):
i) When we write down Newton's second law, $\underline{\boldsymbol{F}} = \mathrm{m}\underline{\boldsymbol{a}}$, does the equation demonstrate that the acceleration is caused by the force ?
ii) Do Maxwell's equations demonstrate that charges and currents cause electric and magnetic fields ?
iii) If we write Maxwell's equations in the form of Jefimenko's equations (see below), does that demonstrate that charges and currents cause electric and magnetic fields ?

3. Consider two very large parallel plates that are both in the x-y plane with a surface charge density of +σ and −σ on each of them and a charge Q between the plates that is initially stationary with respect to the plates. The two plates and the charge are all moving at a high velocity v in the x-direction in a laboratory.

The relativistic vector field transformation for electric fields and magnetic fields are given by:

$$E_x = E_x^{/} \quad E_y = \gamma(E_y^{/} + vB_z^{/}) \;\; E_z = \gamma(E_z^{/} - vB_y^{/}) \qquad 3\text{-}4$$

$$B_x = B_x^{/} \quad B_y = \gamma(B_y^{/} - \frac{v}{c^2}E_z^{/}) \;\; B_z = \gamma(B_z^{/} + \frac{v}{c^2}E_y^{/}) \qquad 3\text{-}5$$

where $\gamma = 1/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$ and the primes relate to the E and B fields in the moving frame and the non-primes those in the stationary laboratory frame. Calculate the E-fields and B-fields in both the frame of the plates and the laboratory frame. Show that the change in relativistic momentum $(F \cdot dt)$ for the charge Q is the same for both frames.

Page 413 – Griffiths.

4. A Durham Physics student is on holiday in Italy and takes a day-trip out into the bay of Naples to see fish in their natural habitat through the bottom of a glass flat-bottomed boat. At the end of the trip one of the student's fellow holiday makers was very disappointed at seeing no fish, and said to the captain 'I've seen fish more clearly in a fish and chip shop'. The student suggested to the captain that the thickness of the sheet of glass at the bottom of the boat was too thick. Fortunately the student had completed the physics course at Durham University and so had developed a good physical understanding of Maxwell's equations and what they can be used for. Not surprisingly, the student was able to derive the transmission coefficient for normal incidence of light, T, for the water-glass-air configuration in the boat. Show it is of the form,

$$T^{-1} = \frac{1}{4n_w}(1+n_w)^2 + \frac{(n_w^2-n_g^2)(1-n_g^2)}{n_g^2}\sin^2\left(\frac{n_g\omega d}{c}\right). \qquad 3\text{-}6$$

where n_w and n_g are the refractive indices of the water and the glass, ω is the typical angular frequency of light and d is the thickness of the flat sheet of glass. The student suggested to the captain that there were in fact a lot of fish under the boat, but that the thickness of the glass at the bottom of the boat, d, needed to be optimized by making it $\frac{\pi c}{n_g\omega}$ m thick. Was the technical information from the student correct ? Given the student saw no fish that day, was the student being helpful? Was the student's fellow holiday maker being a little rude ? What would you recommend the student says to the captain?

Hint: Mathematica shows that

$$\text{Solve}[\{x + w == y + z, x - w == ay - az, y(\text{Cos}[d] + I * \text{Sin}[d]) + z(\text{Cos}[d] - I * \text{Sin}[d]) == (\text{Cos}[e] + I * \text{Sin}[e]), ay(\text{Cos}[d] + I * \text{Sin}[d]) - az(\text{Cos}[d] - I * \text{Sin}[d]) == b(\text{Cos}[e] + I * \text{Sin}[e])\}, \{x, y, z, w\}]$$

Leads to

$$|x|^2 = \frac{(a^2(1+b)^2\text{Cos}^2[d] + (a^2+b)^2\text{Sin}^2[d])}{4a^2}$$

Mathematica also gives:

$$(a^2+b)^2 - (a(1+b))^2 = (a^2-1)(a^2-b^2)$$

5. Page 413 – Griffiths: An antenna radiating at 5 GHz is to be surrounded by a dielectric shield of relative permittivity 5 . Using Equation 3-6 , what is the optimum thickness of the protective shield ?

4 Jefimenko Equations (Not examinable)

In lectures we found that if you rewrite Maxwell's equations in terms of the magnetic vector potential, $\underline{\boldsymbol{A}}$, and the electric potential, V , and use the Lorentz condition to constrain $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}}$, where

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = -\mu_o \varepsilon_o \frac{\partial \mathrm{V}}{\partial \mathrm{t}} \qquad \text{4-1}$$

Maxwell I becomes

$$-\nabla^2 \mathrm{V} + \mu_o \varepsilon_o \frac{\partial^2 \mathrm{V}}{\partial \mathrm{t}^2} = \frac{\rho}{\varepsilon_o} - \text{Wave equation for V} \qquad \text{4-2}$$

and Maxwell IV becomes

$$-\nabla^2 \underline{\boldsymbol{A}} + \mu_o \varepsilon_o \frac{\partial^2 \underline{\boldsymbol{A}}}{\partial \mathrm{t}^2} = \mu_o \underline{\boldsymbol{J}} \text{ - Wave equation for } \underline{\boldsymbol{A}} \qquad \text{4-3}$$

We found steady state oscillatory solutions for the complex wave equation for $\underline{\widetilde{\boldsymbol{A}}}$ where

$$\underline{\widetilde{\boldsymbol{A}}} = \frac{\mu_o I_0 \, \partial \underline{\boldsymbol{l}}}{4\pi r} e^{i(kr - wt)} \text{ where } I \, \partial \underline{\boldsymbol{l}} = \int \underline{\boldsymbol{J}} \, d\tau \qquad \text{4-4}$$

where τ is the volume[1]. Lorenz found the so called 'retarded solutions' to the two wave equations of the form:

$$V(\underline{\boldsymbol{r}}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{\eta} \rho_r' \mathrm{d}\tau' \qquad \text{4-5}$$

and

$$\underline{\boldsymbol{A}}(\underline{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi} \int \frac{1}{\eta} \underline{\boldsymbol{J}}_r' \mathrm{d}\tau' , \qquad \text{4-6}$$

where $\underline{\boldsymbol{\eta}} = \underline{\boldsymbol{r}} - \underline{\boldsymbol{r}}'$ and $\mathrm{d}\tau'$ denotes integrating over the primed spatial variables of the charge densities while the unprimed spatial variables remain constant. The variable ρ_r' and $\underline{\boldsymbol{J}}_r'$ are subject to the constraint of the conservation of charge. The charge density, ρ_r', and current density, $\underline{\boldsymbol{J}}_r'$, are calculated at the retarded time t_r where $t_r = t - \eta/c$. From the definitions of the potentials and using some careful algebra associated with delta functions, one finds the Jefimenko equations [14] which are the general solutions to Maxwell's four equations where

$$\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}, t) = \frac{1}{4\pi\varepsilon_0} \left\{ \int \frac{\hat{\boldsymbol{\eta}}}{\eta^2} \rho_r' \mathrm{d}\tau' + \int \frac{\hat{\boldsymbol{\eta}}}{\eta c} \frac{\partial \rho_r'}{\partial t} \mathrm{d}\tau' - \int \frac{1}{\eta c^2} \frac{\partial \underline{\boldsymbol{J}}_r'}{\partial t} \mathrm{d}\tau' \right\} \qquad \text{4-7}$$

and

$$\underline{\boldsymbol{B}}(\underline{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi} \left\{ \int \underline{\boldsymbol{J}}_r' \times \frac{\hat{\boldsymbol{\eta}}}{\eta^2} \mathrm{d}\tau' + \int \frac{\partial \underline{\boldsymbol{J}}_r'}{\partial t} \times \frac{\hat{\boldsymbol{\eta}}}{\eta c} \mathrm{d}\tau' \right\}, \qquad \text{4-8}$$

These two Jefimenko equations are the generalisations of Coulomb's law and the Biot-Savart law.

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[1] The symbol V is reserved for electric potential.