The unperturbed wavefunctions of Hydrogen including spin are $\psi^0_{n,l,m,m_s} = R_{nl}Y_{lm}\chi_\pm$, where χ_+ (spin up) and χ_- (spin down) are the common eigenfunctions of the spin operators S^2 and S_z, with eigenvalues $s(s+1)\hbar^2 = 3\hbar^2/4$ and $m_s\hbar$ for $m_s = \pm 1/2$, respectively. Similarly, Y_{lm} are the common eigenfunctions of L^2 and L_z with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$, and ψ^0_{n,l,m,m_s} are the eigenfunctions of H^0 with eigenvalues E_n.

Spin-orbit coupling in Hydrogen gives rise to a perturbation of $H'_{\rm so} \propto \underline{L}\cdot\underline{S}$ where $\underline{L}$ and $\underline{S}$ are the orbital and electron spin angular momentum operators, respectively. H^0, L^2, L_z, S^2 and S_z all commute with each other.

(a) $\underline{L}\cdot\underline{S} = L_xS_x + L_yS_y + L_zS_z$. Show that this can be rewritten using the ladder operators $L_\pm = L_x \pm iL_y$ and $S_\pm = S_x \pm iS_y$ as $\underline{L}\cdot\underline{S} = (L_+S_- + L_-S_+)/2 + L_zS_z$. [2 marks]

(b) Ladder operators raise and lower their associated angular momentum quantum number by unity, so $S_-\chi_+ = a\chi_-$ and $S_+\chi_- = a\chi_+$ where $a = \hbar/\sqrt{2}$, and $L_\pm Y_{lm} = A_{lm\pm}Y_{l,m\pm1}$ where $A_{lm\pm} = \hbar\sqrt{l(l+1) - m(m\pm1)}$. All angular momenta have a maximum value beyond which the upwards ladder operator gives zero, and a minimum below which the downwards ladder operator gives zero. Use this together with the definition of $\underline{L}\cdot\underline{S}$ in terms of ladder operators in (a) above to show that

$$\underline{L}\cdot\underline{S}\ \psi^0_{2,1,-1,1/2} = \frac{\hbar^2}{2}(\psi^0_{2,1,0,-1/2} - \psi^0_{2,1,-1,1/2}).$$

Are the unperturbed energy eigenfunctions of Hydrogen also eigenfunctions of $\underline{L}\cdot\underline{S}$? [5 marks]

(c) Use $\underline{L}\cdot\underline{S} = L_xS_x + L_yS_y + L_zS_z$, together with the the standard commutation relations for any general angular momenta $[J_x, J_y] = i\hbar J_z$, $[J_y, J_z] = i\hbar J_x$, $[J_z, J_x] = i\hbar J_y$, (i.e., these relations hold for the components of $\underline{L}$ and $\underline{S}$ as well as $\underline{J}$) to show that $[\underline{L}\cdot\underline{S}, L_z] \neq 0$ and $[\underline{L}\cdot\underline{S}, S_z] \neq 0$ but that $[\underline{L}\cdot\underline{S}, J_z] = [\underline{L}\cdot\underline{S}, L_z + S_z] = 0$.

Which set of quantum numbers allow the effect of the perturbation to be calculated using non-degenerate perturbation theory? [3 marks]