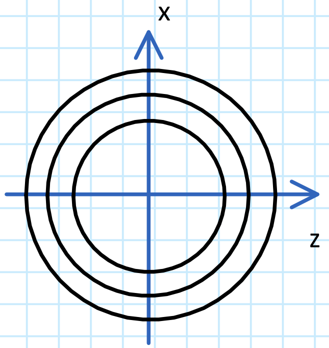
Spherical waves, paraxial approximation and lenses

09 January 2020 15:07

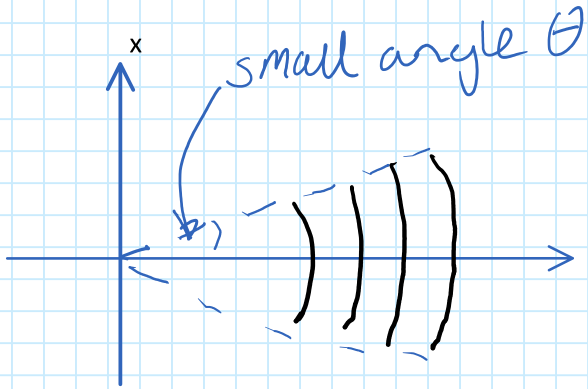
In the last lecture we introduced the concepts of **phase, phasors, wavefronts, plane waves** and **spatial frequency**.
In this lecture we will introduce

- Spherical waves F2F 2.12
- Paraxial Approximation F2F 2.13 2.14
- Wavefront curvature and lenses F2F 2.15 2.16 2.18

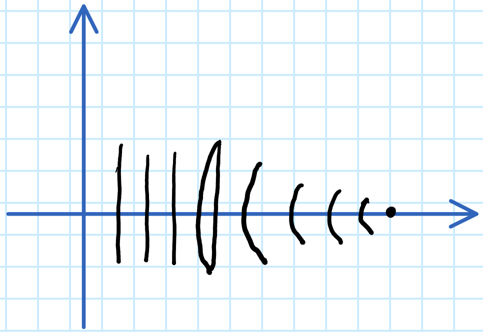
Key concepts:



Waves emitted from a point source have **spherical wavefronts**



Often only interested in:
Close to "optical axis"
Far from source
Paraxial approximation



Lenses change the wavefront curvature
By changing the phase of the wave

Spherical waves

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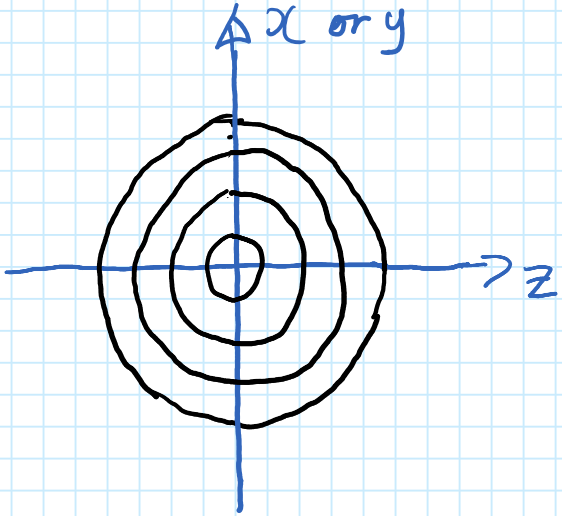
Source at the origin $\underline{r}=0$



WAVEFRONTS ARE SPHERICAL

Scalar spherical wave is $E = \dfrac{E_s}{ikr} e^{i(kr - \omega t)}$

NOTES:

- Approximate solution to scalar wave equation if $r \gg \lambda$
- $1/r$ ensures energy conserved $\left(I \propto |E|^2 \propto 1/r^2\right)$
- $1/k$ ensures E and E_s have same units
- $1/i$ is a choice of phasor angle @ $r=0$: (see {2} 2.12)

Spherical waves and the paraxial approximation

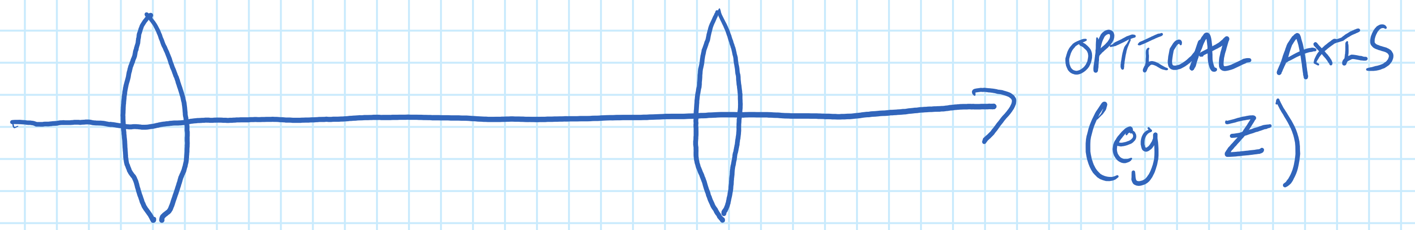
14 January 2020 12:03

What we need is a way to treat curved wavefronts propagating along a well-defined axis (e.g. z)

The approach we will use is known as the PARAXIAL ("near to the axis") APPROXIMATION

Paraxial spherical waves can be used to treat the **propagation** of any scalar field (see lectures on diffraction)

Thinking of systems like eg telescope / microscope



(NB modern microscopes collect light from a large angle
$\Rightarrow$ PARAXIAL APPROX breaks down)

The Paraxial Regime

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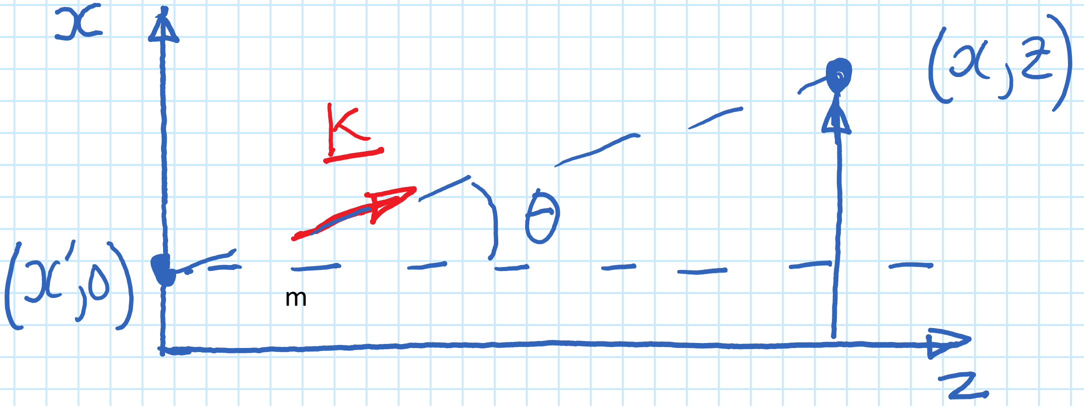

Approximations:

① SMALL ANGLE (θ SMALL)

$\therefore \sin\theta \approx \theta \quad \cos\theta \approx 1 - \frac{\theta^2}{2}$

② Approx ① implies $k_x, k_y \ll k_z$

$$\therefore \quad k_z = \underbrace{\left(k^2 - k_x^2 - k_y^2\right)^{1/2}}_{\text{(EXACT)}} \approx k - \frac{k_x^2 + k_y^2}{2k}$$

The Paraxial regime

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(3) We also assume (x', y') ; $(x, y) < z$

PARAXIAL MEANS "CLOSE TO AXIS"

Distance r' between points $(x', y', 0)$ & (x, y, z)

$$r' = \left[z^2 + (x - x')^2 + (y - y')^2\right]^{1/2}$$

is replaced by the PARAXIAL DISTANCE

$$\boxed{r_p' = z + \frac{(x - x')^2 + (y - y')^2}{2z}}$$

Paraxial spherical waves

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We can use these results to write down an expression for a scalar spherical wave in the paraxial approximation

Spherical wave $E = \frac{E_s}{ikr} e^{i(kr-\omega t)}$

Now $r \longrightarrow r_p$ & $1/r \longrightarrow 1/z$ (ignore ωt term in following)

$$\boxed{\text{PARAXIAL SPHERICAL WAVE} \quad E = \frac{E_s}{ikz} e^{ikr_p} = \frac{E_s}{ikz} e^{ikz} e^{\frac{ik\rho^2}{2z}} \quad \text{where } \rho = (x^2+y^2)^{1/2}}$$

- e^{ikz} term is same as plane wave along z
- $e^{\frac{ik\rho^2}{2z}}$ term is WAVEFRONT CURVATURE radius of curvature $= z$

Application:lenses

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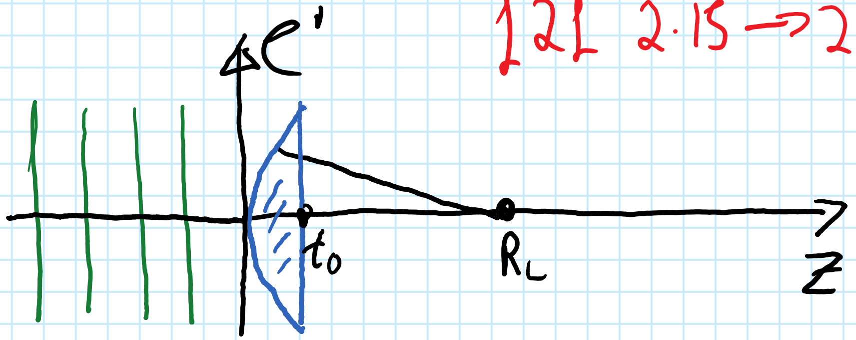

§2.1 2.15 → 2.18

PLANO-CONVEX LENS

- side radius of curvature R_L
- other side flat
- refractive index n
- thickness t_0 at midpoint

At $(\rho', 0)$ lens has thickness

$$t = \sqrt{R_L^2 - \rho'^2} - R_L + t_0$$

PARAXIAL APPROX $\quad t \simeq t_0 - \dfrac{\rho'^2}{2R_L}$

Plane wave incident on lens $E_i = E^{(0)} e^{ikz}$

Lenses continued

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Immediately after lens $E^{(L)}(\rho', t_0) = E^{(0)} \underbrace{e^{ik(t_0 - t(\rho'))}}_{\text{AIR in GAP}} \underbrace{e^{inkt}}_{\text{GLASS}}$

Paraxial approx $E^{(L)}(\rho', t_0) = E^{(0)} e^{ikt_0} e^{i(n-1)kt}$

$$= E^{(0)} e^{inkt_0} e^{-\frac{i(n-1)k\rho'^2}{2R_L}}$$

THIN LENS APPROX SET THIS TERM = 1

LET $f = \frac{R_L}{n-1}$ FOCAL LENGTH

Lenses continued

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THEN $E^{(L)} = E^{(0)} e^{-\frac{ik\rho'^2}{2f}}$

PARAXIAL SPHERICAL WAVE

Converges to a focus at $z = f$

$$\boxed{\text{LENS IMPRINTS A SPATIALLY VARYING PHASE QUADRATIC IN } \rho'}$$

$\Rightarrow$ PLANE $\underset{\text{LENS}}{\Longleftrightarrow}$ SPHERICAL

(reverse process spherical $\longrightarrow$ plane called collimation)

Lenses for imaging

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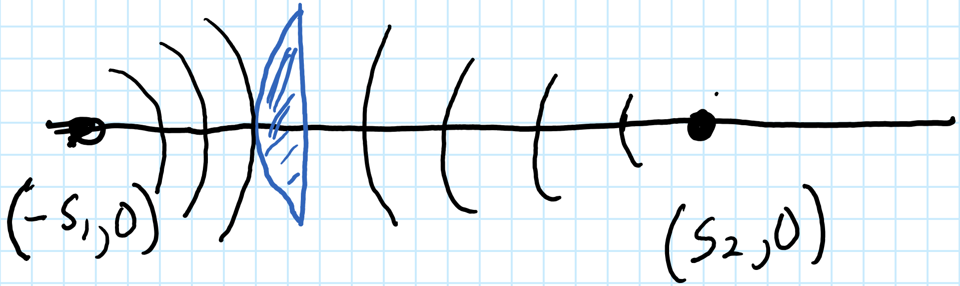
Take a spherical wave from an object point $(-s_1,0)$ as the input



Input at $z=0$ is $E^{(0)} = \dfrac{Es_1}{iks_1} e^{iks_1} e^{\frac{ik\rho'^2}{2s_1}}$

PARAXIAL SPHERICAL WAVE

origin $(s_1,0)$

After lens $E^{(1)} = E^{(0)} e^{-\frac{ik\rho'^2}{2f}} = \dfrac{Es_1}{iks_1} e^{iks_1} \underbrace{e^{\frac{ik\rho'^2}{2s_1}} e^{-\frac{ik\rho'^2}{2f}}}$

Lenses imaging continued

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We write $E^{(i)} = \frac{E s_2}{iks_2} e^{-iks_2} e^{-\frac{ike'^2}{2s_2}}$

equating gives

$$\frac{E s_2}{iks_2} e^{-iks_2} = \frac{E s_1}{iks_1} e^{iks_1}$$

and new radius of curvature s_2 obeys

$$\boxed{\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f} \qquad \text{THIN LENS EQUATION}}$$