University of Durham

EXAMINATION PAPER

May/June 2016 Examination code: PHYS2631-WE01

THEORETICAL PHYSICS 2

SECTION A. Classical Mechanics
SECTION B. Quantum Theory 2

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	AU $= 1.50 \times 10^{11}$ m
Parsec:	pc $= 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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SECTION A. CLASSICAL MECHANICS

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Explain what a holonomic constraint is. How many degrees of freedom are possessed by a system of M point masses, moving in three-dimensional space, subject to j independent holonomic constraints? [4 marks]

(b) Give an example of a mechanical system in stable equilibrium and an example of a mechanical system in unstable equilibrium. How do these systems evolve if perturbed slightly from equilibrium? [4 marks]

(c) For an underdamped, sinusoidally driven harmonic oscillator, explain what is meant by the steady-state solution and the transient solution. What are the oscillation frequencies of these two solutions? [4 marks]

(d) A mechanical system consists of three collinear equal point masses connected by two springs, with the masses constrained to move along a line. How many normal modes does this system have? Either describe the motion associated with the normal modes of this system or write down their mode vectors. [4 marks]

(e) Which of Kepler's laws of planetary motion are approximations, and which are exact? Explain your answer. [4 marks]

(f) Using the implicit transformation equations $p = \partial F/\partial q$ and $P = -\partial F/\partial Q$, and the properties of the Poisson bracket of two arbitrary functions J and K, where

$$\{J, K\} = \frac{\partial J}{\partial q}\frac{\partial K}{\partial p} - \frac{\partial J}{\partial p}\frac{\partial K}{\partial q},$$

determine whether or not the generating function $F = qe^Q$ produces a canonical transformation. [4 marks]

(g) What type of force is the centrifugal force and how do such forces arise? Where can the centrifugal force be safely ignored when considering motion on Earth? [4 marks]

(h) Euler's equations of motion for a rigid body are

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= N_1, \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= N_2, \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= N_3. \end{aligned}$$

Explain, briefly, which quantities the symbols in these equations represent, making clear what coordinate system is being used. [4 marks]

2. A bead slides without friction on a fixed wire that has the shape $y = f(x)$, where y is the vertical coordinate, in a uniform gravitational field of strength g.

   (a) Show that the Lagrangian of the system in terms of the generalised coordinate x, can be written as

   $$L = \frac{m\dot{x}^2}{2}(1 + f'^2) - mgf,$$

   where $f' = df/dx$. Use the Euler-Lagrange equation,

   $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,$$

   to show that

   $$(1 + f'^2)\ddot{x} + f'f''\dot{x}^2 + gf' = 0.$$

   [7 marks]

   (b) Find an expression for p, the canonically conjugate momentum to coordinate x, and use the Legendre transformation $H(p, x) = p\dot{x} - L(x, \dot{x})$ to find the Hamiltonian of the system, $H(p, x)$. Is the Hamiltonian equal to the total energy of this system? [4 marks]

   (c) As the Lagrangian contains no explicit time dependence, H is a constant of the motion.

      (i) Using this fact, or otherwise, solve for $\dot{x}(t)$ in terms of H_0, the initial value of H. [4 marks]

      (ii) For the case when the bead is initially at rest at x_0 and $f(x) = x$, determine an expression for the time taken for the bead to reach a position x_1, where $x_1 < x_0$. [5 marks]

3. In a classical model of a multi-electron atom, electrons of mass m are assumed to move in a modified electrostatic potential, with a potential energy given by

$$V(r) = -\frac{k}{r}e^{-r/a},$$

where k and a are positive constants and r is the distance from the centre of the atom.

   (a) Using spherical polar coordinates (r, θ, ϕ), write down an expression for the kinetic energy, T. Show that, for motion in a plane the Lagrangian can be written as
   $$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{k}{r}e^{-r/a},$$
   explaining your reasoning. [4 marks]

   (b) (i) Noting that ϕ is an ignorable coordinate, determine the associated constant of motion, J, and show that the total energy may be written as $E = (m/2)\dot{r}^2 + V_{\text{eff}}(r)$, where
   $$V_{\text{eff}}(r) = \frac{J^2}{2mr^2} - \frac{k}{r}e^{-r/a}.$$
   [4 marks]

   (ii) Sketch $V_{\text{eff}}(r)$ for the cases where a is either very large or very small. [4 marks]

   (c) Determine the condition on r/a for a stable circular orbit to exist. [8 marks]

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SECTION B. QUANTUM THEORY 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Explain the roles of eigenvalues and expectation values in a quantum mechanical measurement. [4 marks]

   (b) Define the uncertainty related to the measurement of an observable. State the uncertainty relation for two arbitrary observables and specify it for the case of two which are compatible. [4 marks]

   (c) Prove that two commuting Hermitian operators have the same eigenkets. You may assume that all eigenvalues of both operators are non-degenerate. [4 marks]

   (d) Determine the energy eigenvalues and hence the eigenvectors of the Hamiltonian given by

$$\hat{H} = H_{11}|1\rangle\langle 1| + H_{12}|1\rangle\langle 2| + H_{21}|2\rangle\langle 1| + H_{22}|2\rangle\langle 2| \ , \tag{1}$$

   where all elements H_{ij} are real numbers and where, in particular, $H_{12} = H_{21}$. [4 marks]

   (e) What is the Hamiltonian operator of the one-dimensional harmonic oscillator, expressed in terms of the creation and annihilation operators, $\hat{a}^\dagger$ and $\hat{a}$? What commutation relations are satisfied for the creation and annihilation operators? [4 marks]

   (f) Define the Pauli spin matrices, give their commutation relations and explicitly check one non-trivial relation. [4 marks]

   (g) If the operator of the total angular momentum squared, $\hat{J}^2$, has the eigenvalue $12\hbar^2$, which eigenvalues are possible for $\hat{J}_z$? [4 marks]

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5. The states $|jm\rangle$ are the joint eigenstates of the angular momentum operators $\hat{J}^2$ and $\hat{J}_z$.

   (a) What are the commutation relations between the various components $\hat{J}_{x,y,z}$ and between $\hat{J}_{x,y,z}$ and $\hat{J}^2$? [2 marks]

   (b) In which state $|jm\rangle$, for a given value of j, do the operators $\hat{J}_x$ and $\hat{J}_y$ have the smallest uncertainty? How large is it? Is there any state without any uncertainty for any of the three operators $\hat{J}_{x,y,z}$? If yes, which one? The uncertainty of the operator $\hat{J}_x$, ΔJ_x, is defined to be

   $$(\Delta J_x)^2 = \langle jm|\hat{J}_x^2|jm\rangle - \langle jm|\hat{J}_x|jm\rangle^2 \, .$$

   [10 marks]

   [Hint: Express $\hat{J}_x$ and $\hat{J}_y$ in terms of the ladder operators $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$.]

   (c) A frequently encountered Hamiltonian operator reads

   $$\hat{H} = A\,\hat{L}_z^2 + B\left(\hat{L}_x^2 + \hat{L}_y^2\right) ,$$

   where A and B are constants. What are its eigenvalues and its real eigenfunctions? [8 marks]

   [Hint: The following relationship between spherical harmonics may be used, if required:

   $$\begin{aligned} Y_{00}(\theta,\,\phi) &= \sqrt{\frac{1}{4\pi}}, \\ Y_{1\pm1}(\theta,\,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{\pm i\phi} \, . \end{aligned}$$

   The relation between positive and negative values of m is:

   $$Y_{lm}(\theta,\,\phi) = (-1)^m\,Y^*_{l-m}(\theta,\,\phi)\,.]$$

6. A box containing a single particle is divided into a left and a right compartment by a thin foil. The two orthonormal base kets $|L\rangle$ and $|R\rangle$ stand for the particle being in either the left or the right compartment, respectively. Hence, any state ket in this system can be decomposed as

$$|\psi\rangle = |R\rangle\langle R|\psi\rangle \; + \; |L\rangle\langle L|\psi\rangle \,.$$

Tunneling between them through the foil is characterised by a quantity Δ, with units of energy, such that the corresponding Hamiltonian reads

$$\hat{H} = \Delta \left[|L\rangle\langle R| + |R\rangle\langle L| \right] \,.$$

(a) Write the Hamiltonian in matrix form in the $|L, R\rangle$ basis. [2 marks]

(b) Find the normalised energy eigenkets and the corresponding energy eigenvalues. [4 marks]

(c) Find the state ket $|\psi(t)\rangle = \psi_L(t)|L\rangle + \psi_R(t)|R\rangle$ in the Schrödinger picture, if the components $\psi_{L,R} = \langle L, R|\psi(t=0)\rangle$ are known. [8 marks]

(d) Suppose the system is in the left compartment at time $t = 0$. Using the result of (c), find the respective probabilities that the particle is in the left or right compartment at time t. [6 marks]

