# Mathematical Methods II
# Lecture 13

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## Key Points

- Separation of variables (PDE)
- Sinusoidal vs hyperbolic solutions

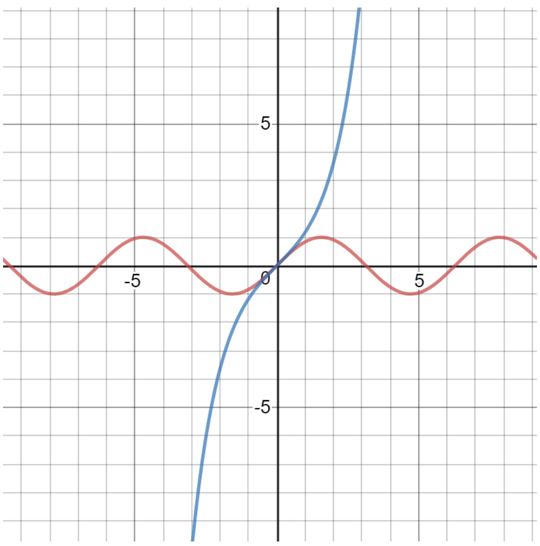
## Separation of Variables for PDE's (ctud)

- **Checking BC's that tend to $\infty$:** We have already shown that imposing different boundary conditions on our equations can describe different physical scenarios. Checking BC's in the limit where a variable tends to infinity can also be very information, telling us where the solution exists and what its nature may be.

  For instance, consider the exponential forms of the sine and hyperbolic sine functions

  $$sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad\qquad sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

  The forms of these equations are very similar. Sometimes we may have difficulty telling immediately if a solution to an equation is sinusoidal or hyperbolic, especially if our solutions are complicated. Here are $\sin(x)$ (red) and $\sinh(x)$ (blue):



Of course, knowing the nature of the solution is important. If we explore the value of our solution in the limit as one of our variables tends to infinity we can determine if it is sinusoidal or hyperbolic.

**e.g. 13.1** Use the method of separation of variables to obtain a solution for the 1D diffusion equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

that tends to zero for all $x$ as $t \to \infty$. $k$ is the diffusivity, a positive, real constant. Assume a separation constant of $\mu$. Determine whether the solution in this limit is sinusoidal or hyperbolic.

We have two independent variables, $x$ and $t$ so we assume a solution of the form

$$u(x,t) = X(x)T(t).$$

Substituting $X$, $T$ and their derivatives into the PDE and dividing by $kXT$ we obtain

$$\frac{X''}{X} = \frac{T'}{kT} = \mu$$

This leads to two ODEs

$$X'' - \mu X = 0 \qquad\qquad T' - \mu k T = 0$$

which have solutions

$$X = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x} \qquad\qquad T = Ce^{\mu k t}$$

giving a combined solution of

$$u = XT = \left(Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}\right)e^{\mu k t}.$$

With this general solution in place we can apply the boundary conditions. We want $u \to 0$ as $t \to \infty$. Hence $\mu k < 0$. This means the time term will be moved to the denominator and as time goes on the denominator of the solution will increase and $u$ will tend to zero for all $x$, which would not be the case if $\mu k = 0$ or $\mu k > 0$.

Since $k$ is real and positive we know that $\mu$ is a real, negative number and that the solution to $u$ is sinusoidal in $x$, as we have complex exponents from square rooting $\mu < 0$. The equation is therefore not hyperbolic.

**e.g. 13.2** Use the method of separation of variables to obtain for the 1D diffusion equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

a solution that tends to zero for all $x$ as $t \to \infty$. $k$ is the diffusivity, a positive, real constant. This time assume the separation constant is $-\mu^2$. Determine whether the solution in this limit is sinusoidal or hyperbolic.

We have two independent variables, $x$ and $t$ so we assume a solution of the form

$$u(x,t) = X(x)T(t).$$

Substituting $X$, $T$ and their derivatives into the PDE and dividing by $kXT$ we obtain

$$\frac{X''}{X} = \frac{T'}{kT} = -\mu^2$$

Choosing to square $\mu$ helps because experience tells us we are going to square-root it when examining the roots of a 2nd order ODE. Choosing to make it negative helps as we will immediately produce a sin and cos based solution to the 2nd order ODE. Both of these decisions are based on foreknowledge; they are not necessary, but make solving this particular problem a little easier. This leads to two ODEs

$$X'' + \mu^2 X = 0 \qquad\qquad T' + \mu^2 kT = 0$$

which have solutions

$$X = A\cos\mu x + B\sin\mu x \qquad\qquad T = Ce^{-\mu^2 kt}$$

giving a combined solution of

$$u = XT = \frac{A\cos\mu x + B\sin\mu x}{e^{\mu^2 kt}}.$$

With this general solution in place we can apply the boundary conditions. We want $u \to 0$ as $t \to \infty$. Hence $\mu^2 k > 0$. This means as time goes on the denominator of the solution will increase and $u$ will tend to zero for all $x$, which would not be the case if $\mu^2 k = 0$ or $\mu^2 k < 0$.

Since $k$ is real and positive we know that $\mu$ is a real non-zero number and that the solution to $u$ is sinusoidal in $x$, not hyperbolic. This time there is not need to consult the exponential forms for sin and sinh, the nature of the solution is obvious from inspection.

**e.g. 13.3 MM2 2014 Q4(f):** Solve the partial differential equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + 4\frac{\partial f(x,y)}{\partial y} = 0$$

where $x$, $y$ are the Cartesian coordinates. [2 marks]

Determine whether or not your solution is physical if we impose the condition that $f(x,y)$ tends to zero when $y \to \infty$. [2 marks]

Assume a solution of $f = XY$, giving

$$\frac{X''}{X} = -4\frac{Y'}{Y} = m$$

If $m > 0$, i.e. m is positive

$$X = Ae^{\sqrt{m}x} + Be^{-\sqrt{m}x} \qquad\qquad Y = Ce^{-my/4}$$

Giving a hyperbolic solution (due to real exponents)

$$f(x,y) = \left(Ae^{\sqrt{m}x} + Be^{-\sqrt{m}x}\right) e^{-my/4}$$

If $m = -n$ where $n > 0$. i.e. m is negative

$$X = Ae^{i\sqrt{n}x} + Be^{-i\sqrt{n}x} \qquad\qquad Y = Ce^{ny/4}$$

Giving a sinusoidal solution (due to imaginary exponents)

$$f(x,y) = \left(Ae^{i\sqrt{n}x} + Be^{-i\sqrt{n}x}\right) e^{ny/4}$$

[2 marks]

Checking our boundary conditions we see that only one solution satisfies the condition that $f(x,y) \to 0$ when $y \to \infty$; the solution for $m > 0$, due to the negative exponent in the $y$ term. [2 marks]

**e.g. 13.4 MM2 2015 Q5:** Consider a vibrating string.

(a) What would be the equation of motion describing the string's oscillations in a plane, with only one polarisation state, if they were undamped with time? [2 marks]

**Solution**

$$k^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

(b) In this context, what is the simplest interpretation of the following equation ($u$ is a function of time and space and $c$, $k$ are two constants):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial t}$$

[2 marks]

**Solution** The first order $du/dt$ term implies there is dissipation with time.

(c) What is the best method to solve this equation? [2 marks]

**Solution** Separation of variables with $u = X(x)T(t)$

(d) Show that if $u$ vanishes at $x = 0$ and $x = L$ the spatial dependence of the function $u$ is given by $u \propto \sin(Ax)$ where $A$ is a constant that you need to determine. [6 marks]

**Solution**

$$T''X = c^2 TX'' - kT'X$$

$$\frac{T''}{T} + k\frac{T'}{T} = c^2 \frac{X''}{X} = \pm\mu^2$$

Meaning that

$$c^2 \frac{X''}{X} = \pm\mu^2$$

Consider $-\mu^2$:

$$X'' + \frac{\mu^2}{c^2} X = 0$$

$$\lambda^2 + \frac{\mu^2}{c^2} = 0$$
$$\lambda^2 = -\frac{\mu^2}{c^2}$$
$$\lambda = \pm i\frac{\mu}{c}$$
Complex roots imply the solution is sinusoidal
$$X = d_1 e^{i\mu x/c} + d_2 e^{-i\mu x/c} = a\sin\left(\frac{\mu}{c}x\right) + b\cos\left(\frac{\mu}{c}x\right)$$

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Aside: To explain why we include both a sin and a cos term try adding sin and cos in exponential form
$$\begin{aligned}\sin x + \cos x &= \frac{e^{ix} - e^{-ix}}{2i} + \frac{e^{ix} + e^{-ix}}{2} \\ &= \frac{-ie^{ix} + ie^{-ix}}{2} + \frac{e^{ix} + e^{-ix}}{2} \\ &= \frac{1}{2}\left(e^{ix} - ie^{ix} + e^{-ix} + ie^{-ix}\right) \\ &= \frac{1-i}{2}e^{ix} + \frac{1+i}{2}e^{-ix} \\ &= d_1 e^{ix} + d_2 e^{-ix}\end{aligned}$$

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Consider $-\mu^2$:
$$X'' - \frac{\mu^2}{c^2}X = 0$$
$$\lambda^2 - \frac{\mu^2}{c^2} = 0$$
$$\lambda^2 = \frac{\mu^2}{c^2}$$
$$\lambda = \pm\frac{\mu}{c}$$
Complex roots imply the solution is hyperbolic
$$X = d_1 e^{\mu x/c} + d_2 e^{-\mu x/c} = a\sinh\left(\frac{\mu}{c}x\right) + b\cosh\left(\frac{\mu}{c}x\right)$$
Applying the boundary condition $u(0,t) = 0$ we see that $b = 0$ as both $\cos(0) = \cosh(0) = 1$.

Applying $u(L,t) = 0$ we realise that the solution must be $\sin(Ax)$, not $\sinh(Ax)$ as $\sin(AL)$ can have a value of zero at a distance $L$ but $\sinh(AL) \neq 0$.

So the solution must be
$$X = a\sin\left(\frac{\mu}{c}x\right)$$
Giving the constant $A = \mu/c = 2\pi/L$, guaranteeing $X = a\sin(0) = 0$ at the BC's. So $u \propto \sin(Ax)$ as required.

(e) Find the time dependence of $u$ [4 marks]

**Solution** Using

$$\frac{T''}{T} + k\frac{T'}{T} = -\mu^2$$

$$T'' + kT' + \mu^2 T = 0$$

Examining the roots we find

$$\lambda^2 + k\lambda + \mu^2 = 0$$

$$\lambda_\pm = \frac{-k \pm \sqrt{k^2 - 4\mu^2}}{2}$$

Therefore

$$T = me^{\lambda_+ t} + ne^{\lambda_- t}$$

(f) Assume that the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - k\frac{\partial u}{\partial t}$$

is a correct description of the vibrating string. Discuss the form of the time evolution of the vibration as a function of $k$. [4 marks]

**Solution** The vibrations are thus described by

$$u = a \sin\left(\frac{\mu}{c}x\right)\left[me^{\lambda_+ t} + ne^{\lambda_- t}\right]$$

with

$$\lambda_\pm = \frac{-k \pm \sqrt{k^2 - 4\mu^2}}{2}$$

When $k = 2\mu$, $\lambda_\pm < 0$ and the vibration is exponentially damped with time.

When $k > 2\mu$, the term in $e^{\lambda_- t}$ decreases with time while the other exponential vanishes so the solution is damped.

When $k < 2\mu$, the exponentials contain an imaginary part but the oscillations are damped.

At last when $k = 0$, there is no damping, just oscillations.