University of Durham

EXAMINATION PAPER

May/June 2016 Examination code: PHYS2611-WE01

MATHEMATICAL METHODS IN PHYSICS

SECTION A. Mathematical Methods part 1
SECTION B. Mathematical Methods part 2

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **any three** of the four optional questions. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

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Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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SECTION A. MATHEMATICAL METHODS PART 1

Question 1 is compulsory. Questions 2 and 3 are optional.

1. (a) Given the following:

$$\begin{array}{lll} a_{11} = 1, & a_{12} = -1, & a_{13} = 0, \\ a_{21} = -2, & a_{22} = 3, & a_{23} = 1, \quad \text{and} \quad b_1 = 1,\ b_2 = -1,\ b_3 = 4, \\ a_{31} = 2, & a_{32} = 0, & a_{33} = 4, \end{array}$$

evaluate (Einstein summation convention applies)

(i) $a_{1i}b_i$.

(ii) $a_{ji}a_{i1}b_j$.

[4 marks]

(b) Determine whether V, defined below, is a vector space. If V is not a vector space, state an axiom of vector spaces which does not hold.

(i) V is the set of vectors of the form $(x_1, x_2, 0)$ where x_1 and x_2 are real numbers. The operations in V are the usual addition of vectors and multiplication by a scalar.

(ii) V is the set of vectors of the form $(x_1, x_2, 1)$ where x_1 and x_2 are real numbers. The operations in V are the usual addition of vectors and multiplication by a scalar.

[4 marks]

(c) Consider the matrix

$$A = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Is this matrix symmetric, antisymmetric, orthogonal, and/or singular?

[4 marks]

(d) Calculate the Laplace transforms of the following functions

(i) $f(t) = H(t-4)$, where H is the Heaviside step function.

(ii) $f(t) = t^2\delta(t-2)$.

[4 marks]

(e) The Fourier series of a function $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right),$$

with

$$a_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi r x}{L}\right) dx, \qquad b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi r x}{L}\right) dx,$$

where L is the period and x_0 is arbitrary. Find the Fourier series of the function $f(x) = x$, for $-\pi < x \leq \pi$, repeating itself with period $L = 2\pi$ on the entire real axis.

[4 marks]

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(f) Evaluate the line integral $I = \int_C \underline{a} \cdot d\underline{r}$ where $\underline{a} = (x+z)\,\hat{\underline{i}} + z\,\hat{\underline{j}} + (x-y)\,\hat{\underline{k}}$ along the curve $\underline{r}(u) = u\,\hat{\underline{i}} + u^2\,\hat{\underline{j}} + u^3\,\hat{\underline{k}}$ with $0 \leq u \leq 2$.
[4 marks]

(g) Calculate the gradient of the following scalar functions

(i) $f(x,y,z) = e^{xyz}$.

(ii) $f(r) = (2+r^2)^{-1}$, where r is the modulus of the position vector $\underline{r}$.

[4 marks]

(h) Given the parametric expression

$$\underline{r}(u,v,\phi) = uv\cos\phi\,\hat{\underline{i}} + uv\sin\phi\,\hat{\underline{j}} + \frac{(u^2-v^2)}{2}\,\hat{\underline{k}},$$

calculate the volume element dV associated with such an expression.
[4 marks]

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2. (a) Two matrices A and B satisfy

$$A^2 = B^2 = 1, \quad AB = -BA, \quad [A, B] \equiv AB - BA = 2iC.$$

Prove that $C^2 = 1$ and that $[B, C] \equiv BC - CB = 2iA$.
[8 marks]

(b) Consider the matrix

$$A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

Is A diagonalisable? If it is, find the matrices D, S and S^{-1} such that $D = S^{-1}AS$. If it is not, explain why the diagonalisation of the matrix A is not possible.
[6 marks]

(c) Consider the functions

$$f(t) = \delta(t+2) + \delta(t-2), \qquad g(t) = \begin{cases} 2 & -2 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Use the convolution theorem to find the Fourier transform of the convolution integral $h = (f * g)$.
[6 marks]

3. (a) State Gauss' Theorem for a vector field $\underline{a}$ in volume V, enclosed by surface S. Verify this explicitly by computing both sides of the equation for

$$\underline{a} = x\,\hat{\underline{i}} + y\,\hat{\underline{j}} + z^2\,\hat{\underline{k}}$$

over a volume V, given by the region enclosed by a cylinder located around the z-axis with radius 2 and bounded by the planes $z = 2$ and $z = 4$.
[16 marks]

$$\left[\begin{array}{c}\text{Hint: Use cylindrical polar coordinates:}\\ \underline{r} = \rho\cos\phi\,\hat{\underline{i}} + \rho\sin\phi\,\hat{\underline{j}} + z\hat{\underline{k}},\ dV = \rho\,d\rho\,d\phi\,dz.\end{array}\right]$$

(b) Let S be a closed surface surrounding a region V. Using Gauss' Theorem show that the volume V enclosed by S is

$$\frac{1}{3}\int_S \underline{r}\cdot d\underline{S},$$

where $\underline{r}$ is the position vector.
[4 marks]

SECTION B. MATHEMATICAL METHODS PART 2

Question 4 is compulsory. Questions 5 and 6 are optional.

4. (a) Explain why $y(x) \propto e^{r\,x}$ (with r a constant) cannot be the solution of the equation:

$$x^2 y'' + x\; y' - y = 0.$$

Give the underlying principle and proof. [2 marks]

Solve this equation with an appropriate technique. [2 marks]

(b) The number n of individuals of a fictional species X evolves with time t and depends on the temperature, T. The time dependence is simple: the number of individuals either grows or decreases with time. The temperature dependence is more complicated. For some critical values of the temperature, the number of individuals is zero. For example, at $T = 0$, $n = 0$ and this is true for other values of the temperature.

Write the simplest equation that describes the evolution of this species. Solve it and discuss the meaning of your solution. [4 marks]

(c) Use the Wronskian method to solve the ordinary differential equation

$$3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = \; e^x,$$

bearing in mind that the homogeneous equation has two solutions e^x and $e^{x/3}$. [4 marks]

[Hint: If $y = k_1\, y_1 + k_2\, y_2$, then $k_1' = -\frac{h(x)}{W(x)} y_2$ and $k_2' = \frac{h(x)}{W(x)} y_1$, with $W(x)$ the Wronskian and $h(x)$ the inhomogeneous term normalised appropriately.]

(d) Which mathematical functions are the most suited to describe the temperature (and temperature variations) on the surface of the Earth? Write the temperature decomposition accordingly and explain your answer. Which mathematical tool should enable you to characterise the typical size of the hot and cold regions? [3 marks]

Give another example where such a tool is used and briefly explain why. [1 mark]

(e) Give the solution of the Legendre equation,

$$(1 - x^2)y'' - 2xy' + 2y = 0,$$

and characterise it completely. How do you interpret this solution when $x = \cos\theta$? [4 marks]

(f) Write the time-dependent Schrödinger equation for a free particle in terms of partial derivatives (assuming one spatial dimension). [2 marks]

The Schrödinger equation for the case considered above shares some similarities with the massless Klein-Gordon equation, given by

$$\frac{\partial^2}{\partial t^2}\psi - \frac{\partial^2}{\partial x^2}\psi = 0.$$

Solve this equation and determine the condition for the spatial part of your solution to be identical in form to that of the Schrödinger equation. [2 marks]

(g) Consider the equation

$$A\,x^2\,y'' + 8\;x\;y' + y = 0,$$

where A is a constant to be determined.

Which values of the constant A lead to a logarithmic solution? After choosing the lowest value for A, write the generic form of the solution. [4 marks]

5. (a) Solve the differential equation

$$x^2\, y'' + x\, (p+1)\, y' + 2y = 0$$

with the simplest technique. [4 marks]

(b) Writing a possible solution as

$$y = \sum a_n x^{n+\rho},$$

show that ρ can take two values and find the expressions for these values. [6 marks]

(c) After choosing the lowest value of ρ, determine the coefficients a_n that enable your power series solution to be the same as your findings in (a). [2 marks]

(d) Solve

$$x^2 y'' + 4\, x\; y' + 2y = e^x.$$

[6 marks]

(e) You have solved the equation

$$x^2 y'' + x(p+1)\; y' + 2y = 0$$

with two methods. State a third technique. Explain its advantage. [2 marks]

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6. A particle experiences a potential

$$V(r,\theta) = \left(1 - \frac{2r}{R}\cos\theta + \left(\frac{r}{R}\right)^2\right)^{-1/2},$$

where R is a constant.

(a) In order to express the potential as:

$$V(h,x) = \left(1 - 2\ h\ x + h^2\right)^{-1/2},$$

what must be definitions of h and x? [2 marks]

(b) Take the total derivative of V and show that the following identity is true:

$$\left(1 - 2hx + h^2\right)\frac{\partial V}{\partial h} = (x - h)\ V.$$

Explain why $V(h,x)$ is special. [6 marks]

(c) The potential can be written as a power series expansion

$$V = \sum_l h^l X_l(x).$$

Which of these terms are coefficients? [1 mark]

(d) Use this decomposition in power series to obtain a new equation which relates the different X_l. [5 marks]

(e) Using this decomposition in power series and the identities $X_0(x) = 1$, $X_1(x) = x$, compute $X_2(x)$ and $X_3(x)$. [4 marks]

(f) What do the X_l represent? [2 marks]

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