Workshop 7 Solution

1. Before we consider the specific situation at hand we should remember the equations for the fields of point charge q in arbitrary motion given in lectures,

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}[(c^2-v^2)\vec{u}+\vec{R}\times(\vec{u}\times\vec{a})], \tag{1}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{R}\times\vec{E}(\vec{r},t). \tag{2}$$

Recall that $\vec{u} = c\hat{R} - \vec{v}$. Therefore, the Poynting vector can be written as,

$$\vec{S} = \frac{1}{\mu_0 c}[E^2\hat{R} - (\hat{R}\cdot\vec{E})\vec{E}]. \tag{3}$$

As in the lectures, the power radiated away from the the accelerating charged particle is given by,

$$P_{rad} = \lim_{R\to\infty}\int d\vec{A}\cdot\vec{S}. \tag{4}$$

Looking at this limit and noting that $d\vec{A}$ is proportional to $\vec{R}^2$, any term in $\vec{S}$ that goes $1\vec{R}^2$ will give a finite answer. This is why we consider only the radiation field,

$$E_{rad} = \frac{q}{4\pi\epsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}(\vec{R}\times(\vec{u}\times\vec{a})). \tag{5}$$

Naively, one might guess, using spherical symmetry arguments ($d\vec{A} = R^2 d\Omega$) that,

$$\frac{dP}{d\Omega} = \frac{1}{\mu_0 c}E_{rad}^2 R^2. \tag{6}$$

This however is wrong! As you've seen in the lecture course (pg 41) there are subtle corrections due to the doppler effect that you need to take into account. The equation is actually.

$$\frac{dP}{d\Omega} = (\frac{\vec{R}\cdot\vec{u}}{Rc})\frac{1}{\mu_0 c}E_{rad}^2 R^2 \tag{7}$$

In this question we are faced with a particle with acceleration colinear to its velocity. This yields,

$$\begin{aligned} \vec{u}\times\vec{a} &= c\hat{R}\times\vec{a} && (8)\\ \hat{R}\times(\hat{R}\times\vec{a}) &= \hat{R}a\cos(\theta)-\vec{a} && (9)\\ \vec{R}\cdot\vec{u} &= R(\hat{R}\cdot(c\hat{R}-\vec{v})) = \overset{c}{\cancel{v}}R(1-\beta\cos\theta) && (10) \end{aligned}$$

where θ is the angle between $\hat{R}$ and $\vec{a}$. and β is the usual v/c factor. So plugging all this in to [7] we get,

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{11}$$

To find the total power just integrate so,

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\sin\theta d\theta d\phi. \tag{12}$$

You can integrate over $d\phi$ trivially but the integral you're left with is tricky. Making the substitution $x\equiv\cos\theta$ and integrating by parts (twice), gives

$$P = \frac{\mu_0 q^2 a^2\gamma^6}{6\pi c} \tag{13}$$

where γ is the Lorentz factor.

2. If this particle is accelerating it will be radiating. From the worldline given in the question we can find the acceleration by differentiating with respect to time twice. Therefore,

$$\vec{a} = \frac{b^2c^2}{(b^2+c^2t^2)^{\frac{3}{2}}}\hat{x} \tag{14}$$

now using [13] and remembering that γ will also depent on v, one obtains

$$P = \frac{q^2 c}{6\pi\epsilon_0 b^2}. \tag{15}$$

For the second part of this question you should find that $F_{rad}=0$. Now this is an important statement when one considers Einstein's General Relativity which is built on the equivalence principle, which states that a gravitational force is indistinguishable from acceleration.