Theoretical Physics 2019/20 — Solution of Problem QT2.2

(a) $\mathbf{B}\cdot\mathbf{S}$ is simply the dot product of these two vectors. As $\mathbf{B} = B_x\hat{\mathbf{x}} + B_z\hat{\mathbf{z}}$ (there is no y-component since B_y is assumed to be zero),

$$\mathbf{B}\cdot\mathbf{S} = \frac{\hbar}{2}\left[B_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + B_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix}B_z & B_x\\ B_x & -B_z\end{pmatrix}.$$

Given that $H = -\gamma\,\mathbf{B}\cdot\mathbf{S}$, Eq. (1) of the question follows.

[3 marks for this part of the problem.]

(b) First, we show that $H\chi_a = E_a\chi_a$ and find E_a:

$$\begin{aligned} H\chi_a &= -\frac{\gamma\hbar}{2}\frac{1}{\sqrt{k^2+1}}\begin{pmatrix}B_z & B_x\\ B_x & -B_z\end{pmatrix}\begin{pmatrix}-k\\ 1\end{pmatrix}\\ &= -\frac{\gamma\hbar}{2}\frac{1}{\sqrt{k^2+1}}\begin{pmatrix}-kB_z + B_x\\ -kB_x - B_z\end{pmatrix}. \end{aligned}$$

Now, we note that

$$\begin{aligned} -kB_z + B_x &= -\frac{B_x}{B_z + |\mathbf{B}|}\,B_z + B_x\\ &= \frac{-B_xB_z + B_zB_x + |\mathbf{B}|B_x}{B_z + |\mathbf{B}|} = k|\mathbf{B}|, \end{aligned}$$

and, similarly,

$$\begin{aligned} -kB_x - B_z &= -\frac{B_x}{B_z + |\mathbf{B}|}\,B_x - B_z\\ &= -\frac{B_x^2 + B_z^2 + |\mathbf{B}|B_z}{B_z + |\mathbf{B}|}\\ &= -\frac{|\mathbf{B}|^2 + |\mathbf{B}|B_z}{B_z + |\mathbf{B}|} = -|\mathbf{B}|\,\frac{|\mathbf{B}| + B_z}{B_z + |\mathbf{B}|} = -|\mathbf{B}|. \end{aligned}$$

Thus

$$H\chi_a == -\frac{\gamma\hbar}{2}\frac{1}{\sqrt{k^2+1}}\begin{pmatrix}k|\mathbf{B}|\\ -|\mathbf{B}|\end{pmatrix} = \frac{\gamma\hbar|\mathbf{B}|}{2}\frac{1}{\sqrt{k^2+1}}\begin{pmatrix}-k\\ 1\end{pmatrix} = \frac{\gamma\hbar|\mathbf{B}|}{2}\chi_a.$$

We see that χ_a is indeed an eigenvector of H and $E_a = \gamma\hbar|\mathbf{B}|/2$.

The calculation of χ_b is almost identical. It yields $E_b = -\gamma\hbar|\mathbf{B}|/2$.

[4 marks for part (b)]

(c) First, χ_a and χ_b are both normalized:

$$(\chi_a, \chi_a) = \frac{1}{k^2+1}\begin{pmatrix}-k & 1\end{pmatrix}\begin{pmatrix}-k\\ 1\end{pmatrix} = \frac{1}{k^2+1}(k^2+1) = 1,$$

and likewise $(\chi_b, \chi_b) = 1$. Second χ_a and χ_b are orthogonal:

$$(\chi_a, \chi_b) = \frac{1}{k^2+1}\begin{pmatrix}-k & 1\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix} = \frac{1}{k^2+1}(-k+k) = 0.$$

That these two vectors are orthogonal can also be deduced from the fact that the matrix H is Hermitian: eigenvectors of a Hermitian matrix belonging to different eigenvalues are always orthogonal (E_a can be equal to E_b only if $|\mathbf{B}| = 0$, but here $|\mathbf{B}| \neq 0$ since the question says that $B_x \neq 0$). Third, χ_a and χ_b are non-zero and linearly independent, and any set of two linearly independent 2-component column vectors forms a basis since this vector space is 2-dimensional. (In fact, a basis can always be constructed from the eigenvectors of a Hermitian operator, at least in the case where the vector space is finite-dimensional.)

[3 marks for part (c)]