CM4: The Atwood machine (again)

An Atwood machine consists of two point masses, m_1 and m_2, attached via a massless rope into which a massless spring, of spring constant k, is inserted. The rope, which remains tight, is placed over a frictionless, massless pulley of radius r. When the spring is unstretched the combined length of the rope and spring is $l + \pi r$. You may assume that there is a uniform gravitational acceleration, g.

1. By defining the zero of height and potential energy to be a distance $l/2$ below the pulley centre, and the extension of the spring as x, write down a constraint equation relating the heights of the two masses, y_1 and y_2, to the extension of the spring.

2. Write down expressions for the kinetic (T) and potential (V) energies of the system using the constraint equation to eliminate x. Hence show that the Lagrangian for the system is

$$L = \frac{1}{2}(m_1\dot{y}_1^2 + m_2\dot{y}_2^2) - (m_1y_1 + m_2y_2)g - \frac{k}{2}(y_1 + y_2)^2.$$

3. Using the Euler-Lagrange equation for a generalised coordinate q,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

   determine second order coupled linear differential equations for the two generalised coordinates y_1 and y_2.

4. For the case $m_1 = m_2 = m$, use these coupled equations to determine the equilibrium extension of the spring.

5. Use the matrix formulation of the equations of motion

$$(\hat{v} - \omega^2\hat{\tau})\underline{b} = 0,$$

   where

$$\tau_{jk} = \frac{1}{2}\frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k}\bigg|_{\dot{q}_j,\dot{q}_k=0}, \qquad v_{jk} = \frac{1}{2}\frac{\partial^2 V}{\partial q_j \partial q_k}\bigg|_{q_j,q_k=0},$$

   and the trial solution is of the form $\underline{q} = \underline{b}e^{i\omega t}$, to find the normal mode frequencies and normalised mode vectors of the system considered in 4.

6. Describe the motions associated with the normal modes.