Lecture 10. The Expansion of the Universe in General Relativity

10.1 Introduction

Gravity is unique among the four forces in that it acts universally on all objects regardless of composition. Another way of saying this is that gravitational and inertial masses are equivalent.

In the absence of gravitational (and other external) forces the trajectories of all objects are straight lines. In the presence of a gravitational field all objects follow the same paths, but now these paths are curved.

In Einstein's description of gravity, this behaviour is the result of space being curved. Thus in the presence of a gravitational field the objects are still travelling along the shortest 'distance' between two points, but because space-time is curved these paths are no longer straight lines.

10.2 Metrics

In normal flat space-time, the proper separation of two events[1], $\mathrm{d}s$, is given by

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2,$$

and is invariant, i.e., its value is the same for all observers, regardless of their motions. This equation can be rewritten, more compactly, as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu,$$

where $(\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3) = (c\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ and the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Note that in our convention, Greek letters are indices running over $(0, 1, 2, 3)$.

Einstein proposed the idea that in the presence of mass(-energy) space-time becomes curved and thus the metric is deformed, so that $g_{\mu\nu}$ becomes a function of space-time, $g_{\mu\nu}(x^\alpha)$.

The curvature of space-time is determined by how the metric varies as a function of position. It is measured by the 'Ricci curvature tensor' $R_{\mu\nu}$, which is a complicated sum over expressions such as

$$\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} g_{\mu\nu}(x).$$

[1] An event is described by its time and position, or a 4-dimensional coordinate (t, x, y, z).

10.3 The Friedmann-Robertson-Walker metric

If one takes the very specialised case of a homogeneous isotropic universe, then it can be shown that the metric can be written in the following form, where the space sector is described now in spherical polar coordinates,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}\tilde{r}^2}{1-k\tilde{r}^2} + \tilde{r}^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2)\right]. \tag{1}$$

This is the **Friedmann-Robertson-Walker (FRW)** metric, the most general metric describing a homogeneous, isotropic, expanding universe, hence respecting the cosmological principle.

In the FRW metric, $a(t)$ is the scale factor, or expansion factor (see Lecture 2). This metric describes flat, open or closed geometries depending on the value of k:

- A critical-density universe ($\Omega = 1$, $k = 0$) is flat and infinite.
- A low-density universe ($\Omega < 1$, $k < 0$) is open and infinite.
- A high-density universe ($\Omega > 1$, $k > 0$) is closed and finite.

By a change of variables $\tilde{r} \to S_k(r)$ the metric expression given in Eq. (1) may also be written in the following form[2],

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + S_k(r)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{2}$$

where

$$S_k(r) = \begin{cases} \sin(\sqrt{k}\,r)/\sqrt{k} & \text{if } (k > 0) \\ r & \text{if } (k = 0). \\ \sinh(\sqrt{-k}\,r)/\sqrt{-k} & \text{if } (k < 0) \end{cases} \tag{3}$$

Both expressions of the metric are equivalent, but it is this second form that we will use in the next lecture.

The **Example** later will illustrate how the k, which here describes the curvature of *space*, is the same k that appeared as the integration constant in our Newtonian derivation of the Friedmann equation (see Cosmology Part I).

10.4 Einstein's Equation

The theory of General Relativity quantifies the connection between the distribution of mass-energy and the curvature of space-time, as well as the dynamical effect of curved space-time. Its fundamental equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda\, g_{\mu\nu},$$

[2] In Cosmology I, e.g., Lecture 2, r was used to denote the *physical* distance in contrast to the *comoving* distance x. Here, r stands for the *radial* comoving distance since using x (which implies *Cartesian* coordinates) would be improper. The meaning, nevertheless, should always be either explicitly stated or clear from the context.

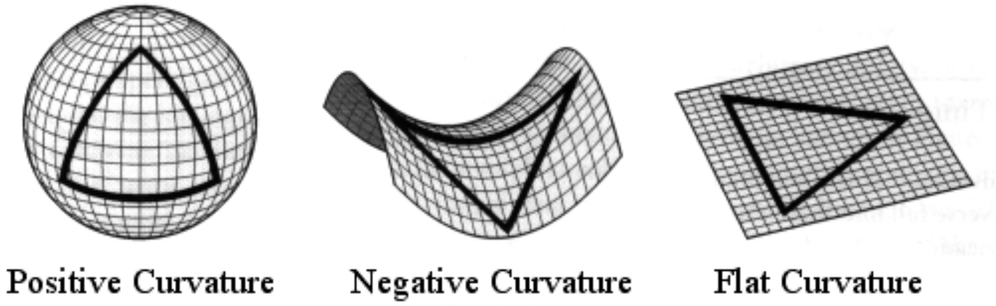

Figure 1: An illustration of the spatial curvature in an Friedmann-Robertson-Walker (FRW) universe. *left*: a closed universe with positive spatial curvature ($k > 0$) where the three inner angles of a triangle sum to larger than 180 degrees; *middle*: open universe with negative spatial curvature ($k < 0$) where the three inner angles of a triangle sum to smaller than 180 degrees; *right*: flat universe with zero spatial curvature $k = 0$ which is a Euclidean space. Illustrations are 2-dimensional for simplicity.

is known as **Einstein's Field Equation**, some times loosely described as 'cuvature = matter'.

Here $T_{\mu\nu}$ is the stress-energy tensor, and $R_{\mu\nu}$, R are the Ricci tensor and Ricci scalar quantifying space-time curvature. For a chosen coordinate system, each term in this equation is a symmetric 4×4 matrix, and so in general there are 10 independent equations.

In the special case with the homogeneous and isotropic FRW metric, $T^{\mu\nu} = \mathrm{diag}(\rho c^2, P, P, P)$ and the number of independent equations reduces to 2. These are

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G\rho(t)}{3} + \frac{\Lambda}{3}, \tag{4}$$

which is the **Friedmann Equation**, and

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi GP}{c^2} + \Lambda. \tag{5}$$

Subtracting the Friedmann equation from this gives the **acceleration equation** (which does not contain $(\dot{a}/a)^2$). Therefore if we wished we could work backwards from these equations to derive the **fluid equation**

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0. \tag{6}$$

The result is that both Newtonian theory and General Relativity predict exactly the same evolution for the homogeneous isotropic universe, but General Relativity is more general while Newtonian theory has only limited applications in cosmology.

Note the Λ term in the **Einstein Equation** was inserted by Einstein solely so that a cosmological model could be constructed in which the universe was static and not expanding. Einstein later called it his greatest blunder when it became clear that the universe *is* expanding.

Key Takeaway Points of Lecture 10

• The expansion of the Universe is governed by the Friedman equation, which is a natural prediction of General Relativity

• A metric allows to define and measure distances in a space(time).

• The only metric satisfying the cosmological principle (Lecture 2), namely that the Universe is homogeneous and isotropic on large scales, is the Friedmann-Robertson-Walker (FRW) metric:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + S_k(r)^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right],$$

where

$$S_k(r) \equiv \begin{cases} \sin(\sqrt{k}\,r)/\sqrt{k} & \text{if } (k>0) \\ r & \text{if } (k=0), \\ \sinh(\sqrt{-k}\,r)/\sqrt{-k} & \text{if } (k<0) \end{cases}$$

and $a(t)$ is the scale factor or expansion factor.

• The parameter k above, while it appears also in the Newtonian derivation of the Friedmann equation, has a different meaning there than here. It characterises the curvature of space.

• General Relativity, therefore, is the foundation of modern cosmology.

Tips

Note that the Friedmann equation is some times written as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G\rho(t)}{3} + \frac{\Lambda}{3},$$

while in other occasions as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G\rho(t)}{3},$$

i.e., without the Λ term explicitly included. In the latter case, we view Λ as but a special 'matter species', and the quantity $\rho(t)$ there contains contributions from all matter species:

$$\rho(t) = \rho_{\text{matter}} + \rho_{\text{radiation}} + \rho_\Lambda + \cdots,$$

with

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}.$$