ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 1+2+3 (Rev 5)

1 Fabulous Science.

1.1 Maxwell's Equations and Classical Physics

FEYNMAN claims there are 7 equations that describe all of classical Physics

Maxwell's 4 equations:

From Coulomb's law:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \quad \text{(MI)} \qquad \textbf{1-1}$$

Given no magnetic monopoles have been observed:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{B}} = 0 \quad \text{(MII)} \qquad \textbf{1-2}$$

From Faraday's law of induction:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \frac{-\partial \boldsymbol{B}}{\partial t} \quad \text{(MIII)} \qquad \textbf{1-3}$$

From Ampere's law:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad \text{(MIV)} \qquad \textbf{1-4}$$

where the symbol $\underline{\boldsymbol{\nabla}}$ denotes the vector operator 'del':

$$\underline{\boldsymbol{\nabla}} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \qquad \textbf{1-5}$$

$\underline{\boldsymbol{E}}$: electric field (V m^{-1})
$\underline{\boldsymbol{B}}$: magnetic field – or flux density (T)
ρ total charge density (C m^{-3})
$\underline{\boldsymbol{J}}$: total current density (A m^{-2})

$c = 3 \times 10^8$ m s^{-1}, $\mu_0 = 4\pi \times 10^{-7} \mathrm{Hm}^{-1}$, $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}

Newton's law of motion:

$$\underline{\boldsymbol{F}} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \text{ where } \underline{\boldsymbol{p}} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \textbf{1-6}$$

Newton's law of Gravity:

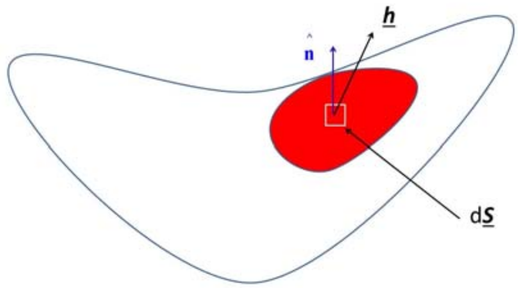
$$\underline{\boldsymbol{F}}_2 = -\frac{Gm_1 \mathrm{m}_2}{\mathrm{r}^2}\hat{\mathbf{r}}_{1\to 2} \qquad \textbf{1-7}$$

Force on a moving charge in a magnetic and electric field:

$$\underline{\boldsymbol{F}} = \mathrm{q}(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \qquad \textbf{1-8}$$

2 Vector Fields

2.1 The flux of a vector field



An arbitrary three-dimensional 'closed surface' is shown which is the surface of a single contiguous volume. The shaded surface is characterised as an 'open surface' because it does not enclose a volume.

$$\phi = \int \underline{\boldsymbol{h}}\,.\,\widehat{\boldsymbol{n}}\;\mathrm{d}S = \int \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} \tag{2-1}$$

If the area is a closed surface we can help the reader and make it explicit by writing a closed loop on the integral sign:

$$\phi = \oint \underline{\boldsymbol{h}}\,.\,\widehat{\boldsymbol{n}}\;\mathrm{d}S = \oint \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} \tag{2-2}$$

2.2 Gauss' (divergence) theorem

$$\oint \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} = \int \underline{\boldsymbol{\nabla}}\,.\,\underline{\boldsymbol{h}}\,\mathrm{d}V \tag{2-3}$$

where $\underline{\boldsymbol{h}}$ is any arbitrary vector field

2.3 Stoke's (curl) theorem

$$\oint \underline{\boldsymbol{h}}\,.\,\mathrm{d}\underline{\boldsymbol{l}} = \int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{h}})\mathrm{d}\underline{\boldsymbol{S}} \tag{2-4}$$

where $\underline{\boldsymbol{h}}$ is any arbitrary vector field.

2.4 Differential vector identities

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}) = \underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}}) - \nabla^2 \underline{\boldsymbol{A}} \tag{2-5}$$

$$\underline{\boldsymbol{\nabla}} \cdot (\underline{\boldsymbol{A}} \times \underline{\boldsymbol{B}}) = \underline{\boldsymbol{B}} \cdot (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}) - \underline{\boldsymbol{A}} \cdot (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}) \tag{2-6}$$

3 Maxwell I (From Coulomb's Law)

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \tag{3-1}$$

where ρ is the total charge density.

3.1 Coulomb's law for interacting charges and Gauss' law

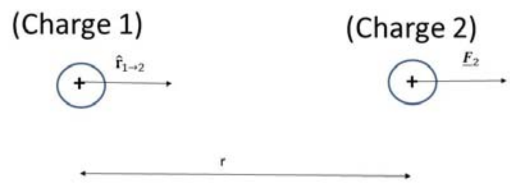


Figure 1 : Two positive charges interacting.

$$\underline{\boldsymbol{F}}_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}_{1\to 2} \qquad \textbf{3-2}$$

$$\underline{\boldsymbol{F}}_2 = q_2 \underline{\boldsymbol{E}}_1 \qquad \textbf{3-3}$$

$$\underline{\boldsymbol{E}}_1 = \frac{q_1}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}_{1\to 2} \qquad \textbf{3-4}$$

3.2 Deriving Gauss's law and Maxwell I from Coulomb's law

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \frac{\sum q}{\varepsilon_o} = \frac{1}{\varepsilon_o}\int \rho \mathrm{d}V \qquad \textbf{3-5}$$

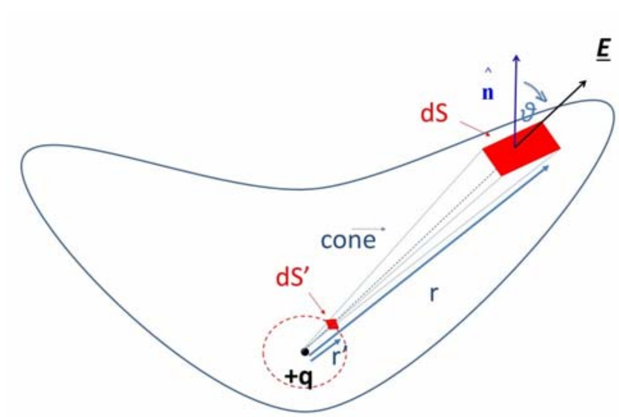


Figure 2 : The cone first passes through a sphere (S') centred about the charge and then through the surface of arbitrary shape (S). The areas bounded by the cone at the sphere and at the arbitrary surface are given by $\mathrm{d}\underline{\boldsymbol{S}}'$ and $\mathrm{d}\underline{\boldsymbol{S}}$ respectively.

We now use geometrical arguments and superposition of $\underline{\boldsymbol{E}}$-fields to find Gauss' law. Gauss' law is equivalent to Coulomb's law but can be applied to a collection of charges. Consider first the specific case of a sphere surrounding the charge at its centre. The electric field ($\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}')$) has constant magnitude over the surface of the sphere and is everywhere parallel to $\mathrm{d}\underline{\boldsymbol{S}}'$, (i.e. $\hat{\boldsymbol{n}}\,\mathrm{d}S'$) so we have**:**

$$\oint \underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}').\,\mathrm{d}\underline{\boldsymbol{S}}' = \left|E(r')\right| \oint \mathrm{d}S' \qquad \textbf{3-6}$$

$$= \frac{q}{4\pi\epsilon_0 (r')^2} 4\pi (r')^2 = \frac{\mathrm{q}}{\varepsilon_\mathrm{o}}$$

Consider now the flux through the elemental area d$\underline{\boldsymbol{S}}$ which is part of the surface of the arbitrary shape surrounding the charge. We have:

$$\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}).\,\hat{\boldsymbol{n}}\,\mathrm{d}S = |E(r)||\hat{\boldsymbol{n}}|\cos(\theta)\,\mathrm{d}S \qquad \textbf{3-7}$$

Since dS is a projection of dS' (because they are both bounded by the cone), we can relate them using:

$$\frac{dS'}{\pi(r')^2} = \frac{dS}{\pi r^2}\cos(\theta) \qquad \textbf{3-8}$$

Substituting Equation 3-8 into Equation 3-9 , we have :

$$\underline{\boldsymbol{E}}(\underline{\boldsymbol{r}}).\,\hat{\boldsymbol{n}}\,\mathrm{d}S = E(r)\frac{r^2}{(r')^2}dS' = \frac{\mathrm{q}}{4\pi\varepsilon_0 r^2}\frac{r^2}{(r')^2}dS' = \frac{\mathrm{q}}{4\pi\varepsilon_0 (r')^2}dS' \qquad \textbf{3-9}$$

Equation 3-9 shows that the flux through the surface dS is the same as the flux through dS'. Integrating both sides of Equation 3-9 gives:

$$\oint \underline{\boldsymbol{E}} \cdot \boldsymbol{d}\underline{\boldsymbol{S}} = \frac{\mathrm{q}}{\varepsilon_\mathrm{o}} \qquad \textbf{3-10}$$

Applying superposition to a collection of charges inside the arbitrary shaped surface:

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \frac{\sum q}{\varepsilon_\mathrm{o}} = \frac{1}{\varepsilon_\mathrm{o}}\int \rho \mathrm{d}V - \text{ Gauss'Law} \qquad \textbf{3-11}$$

Then, using the divergence (Gauss') theorem we can rewrite Gauss' law as:

$$\int \underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}}\,\mathrm{d}V = \frac{1}{\varepsilon_\mathrm{o}}\int \rho \mathrm{d}V \qquad \textbf{3-12}$$

Since the volume integrals are equal for any arbitrary volume (no matter how big or small), the integrands must be equal so:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0} \quad \text{Maxwell's 1st equation} \qquad \textbf{3-13}$$

3.3 Superposition of Fields

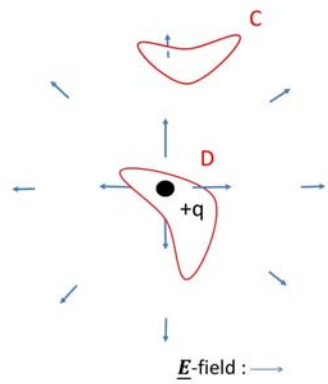


Figure 3 : An electric vector field found near a point source charge. Surface C does not enclose any charge so the net flux through the surface is zero. Surface C has zero net flux passing through it. Surface D has a net flux passing through it of magnitude $\frac{q}{\varepsilon_o}$.

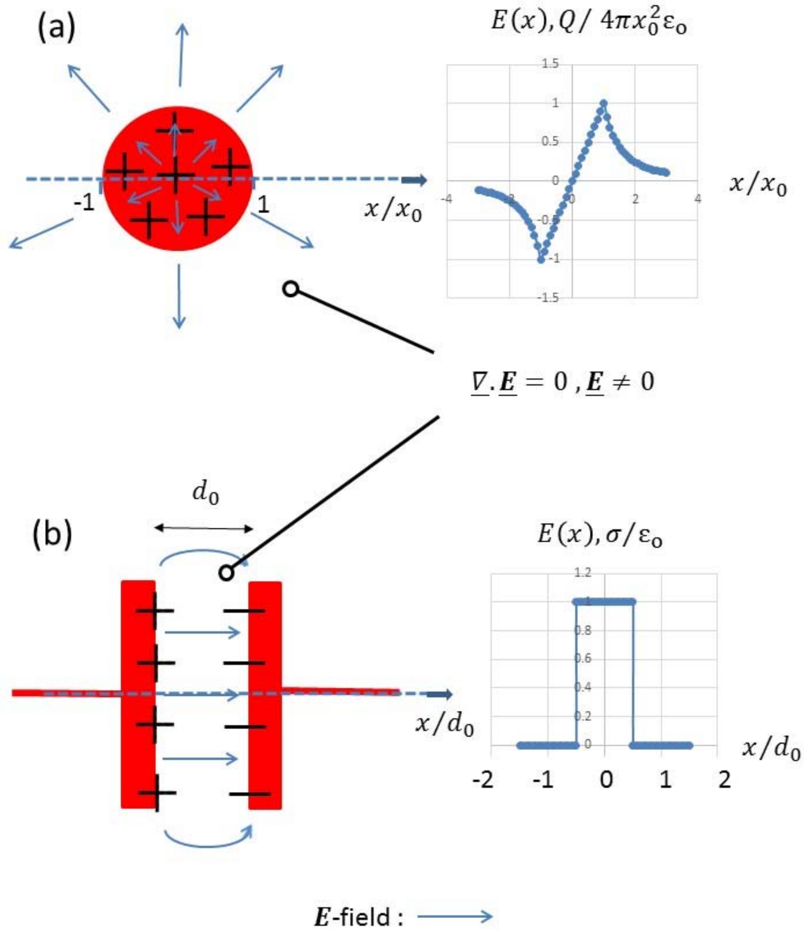


Figure 4 : Two charge configurations – a spherical volume of charge density and two charged capacitor plates.

The $\underline{\boldsymbol{E}}$-field resulting from any charge distribution gives $\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = 0$ in the local regions where there is no charge.

In the regions where there are no charges:

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad \textbf{3-14}$$

$$\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}^2\mathrm{E_r}) + \frac{1}{\mathrm{rsin\theta}}\frac{\partial}{\partial \theta}(\mathrm{E_\theta sin\theta}) + \frac{1}{\mathrm{rsin\theta}}\frac{\partial}{\partial \phi}(\mathrm{E}_\phi) = 0 \qquad \textbf{3-15}$$

4 Maxwell II (No magnetic monopoles)

4.1 Ampere's Law

$$\underline{\boldsymbol{F}}_2 = -\frac{\mu_0 \mathrm{I_1 I_2} L}{2\pi \mathrm{d}}\hat{\mathbf{r}}_{1\to 2} \qquad \textbf{3-16}$$

$$\underline{\boldsymbol{F}} = \mathrm{q}(\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \qquad \textbf{3-17}$$

$$\mathrm{I} = \mathrm{Q}_{\,L}\mathrm{v} \qquad \textbf{3-18}$$

where $\mathrm{Q}_{\,L}$ is the charge per unit length and v is the velocity of the charges or equivalently, for a length L,

$$\mathrm{I} = \frac{\sum q}{L}\mathrm{v} \qquad \textbf{3-19}$$

Where $\sum q$ isa the sum of all the charges in a length L .

$$\oint \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_o \mathrm{I} \quad \text{Ampere's Law} \qquad \textbf{3-20}$$

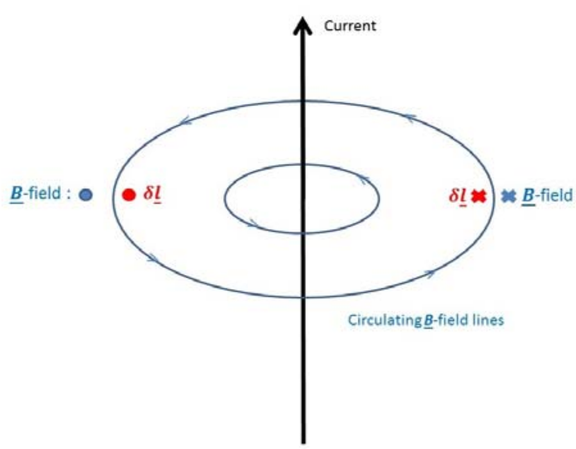


Figure 5 : The $\underline{\boldsymbol{B}}$-field outside a straight wire carrying a current. Note the $\underline{\boldsymbol{B}}$-field circulates the current

We simplify the vector integration in Ampere's law by taking advantage of symmetry to account for the magnitude and direction of $\underline{\boldsymbol{B}}$ so the (LHS) integral is a simple scalar integral and for a straight wire:

$$\mathrm{B} \times 2\pi \mathrm{r} = \mu_o \mathrm{I} \qquad \textbf{3-21}$$

$$\underline{\boldsymbol{B}} = \frac{\mu_o \mathrm{I}}{2\pi \mathrm{r}} \hat{\boldsymbol{\varphi}} \qquad \textbf{3-22}$$

4.2 Maxwell II from the Biot-Savart law

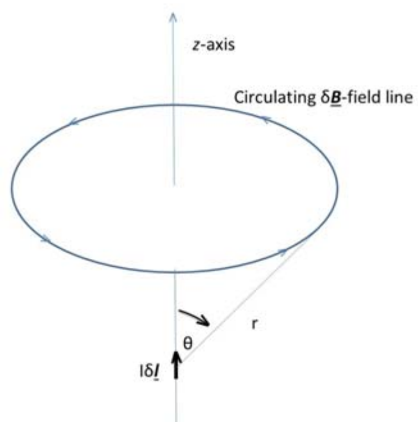


Figure 6 : The configuration associated with the Biot-Savart law. The small length of a current element ($\mathrm{Id}\underline{\boldsymbol{l}}$) produces a small circulating magnetic field ($\delta\underline{\boldsymbol{B}}$).

Ampère's law can also be written in the form that describes the small magnetic field ($\delta\underline{\boldsymbol{B}}$) produced by a small length of a current element ($\mathrm{Id}\underline{\boldsymbol{l}}$) where

$$\delta\underline{\boldsymbol{B}} = \frac{\mu_0 I}{4\pi r^2} \mathrm{d}\underline{\boldsymbol{l}} \times \hat{\boldsymbol{r}} = \frac{\mu_0 I}{4\pi r^2} \sin\Theta \, \mathrm{dl} \, \widehat{\boldsymbol{\phi}} \text{ - the Biot Savart law.} \qquad 3\text{-}23$$

Given that the divergence of $\delta\underline{\boldsymbol{B}}$ is a physical quantity, it's value does not depend on the coordinate system it is calculated in. More specifically, it doesn't matter whether $\underline{\boldsymbol{\nabla}}.\delta\underline{\boldsymbol{B}}$ is calculated in (Cartesian coordinates - not shown) spherical polar coordinates where

$$\underline{\boldsymbol{\nabla}}.\delta\underline{\boldsymbol{B}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \delta B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\delta B_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\delta B_\phi), \qquad 3\text{-}24$$

or in cylindrical polar coordinates where

$$\underline{\boldsymbol{\nabla}}.\delta\underline{\boldsymbol{B}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho(\delta B_\rho)) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\delta B_\phi) + \frac{\partial}{\partial \mathrm{z}}(\delta B_z) = 0\,. \qquad 3\text{-}25$$

In both cases, inserting equation **3-23** into either equation **3-24** or **3-25** shows that δB_r and δB_θ (spherical polar coordinates) or δB_ρ and δB_z (cylindrical coordinates) are both zero and that because δB_ϕ has no phi dependence, $\underline{\boldsymbol{\nabla}}.\,\delta\underline{\boldsymbol{B}} = 0$. It then follows from superposition that $\sum \delta\underline{\boldsymbol{B}} = \underline{\boldsymbol{B}}$, and hence for any configuration of currents

$$\underline{\boldsymbol{\nabla}}.\,\underline{\boldsymbol{B}} = \underline{\boldsymbol{\nabla}}.\sum \delta\underline{\boldsymbol{B}} = \sum \underline{\boldsymbol{\nabla}}.\,\delta\underline{\boldsymbol{B}} = 0. \qquad \text{3-26}$$

Equation 3-26 is Maxwell's 2nd equation given by

$$\underline{\boldsymbol{\nabla}}.\,\underline{\boldsymbol{B}} = 0 - \text{Maxwell's } 2^{nd} \text{ Equation}. \qquad \text{3-27}$$

The spatial variation of magnetic fields

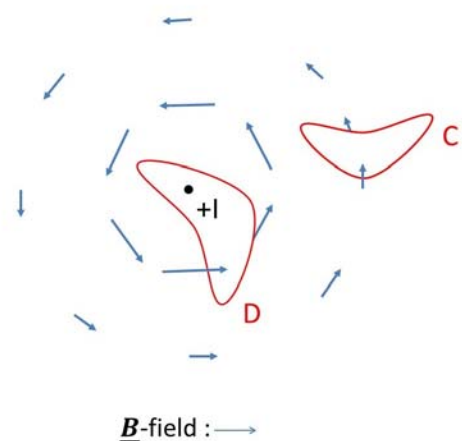


Figure 7 : The magnetic vector field, $\underline{\boldsymbol{B}}$, surrounding a current, I, flowing through a long straight wire coming out of the board. Although surface C encloses a source and surface D does not, the net flux through both surfaces is zero for this circulating vector field.

$$\int \underline{\boldsymbol{\nabla}}\,.\,\underline{\boldsymbol{B}}\,\mathrm{d}V = 0 \qquad \textbf{3-28}$$

$$\oint \underline{\boldsymbol{B}}\,.\,\mathrm{d}\underline{\boldsymbol{S}} = 0 \qquad \textbf{3-29}$$

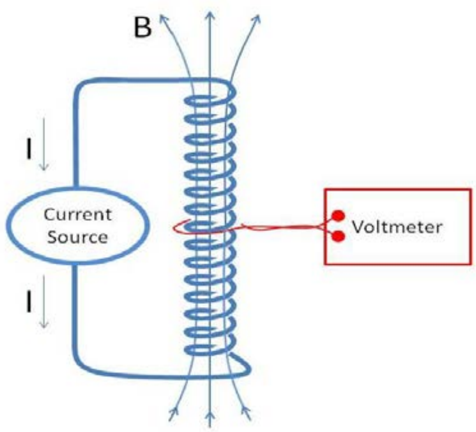
ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 4+5+6

5 Maxwell III (From Faraday's Law)

5.1 Faraday's fabulous experiments

5.1.1 Inductive Electromotive force



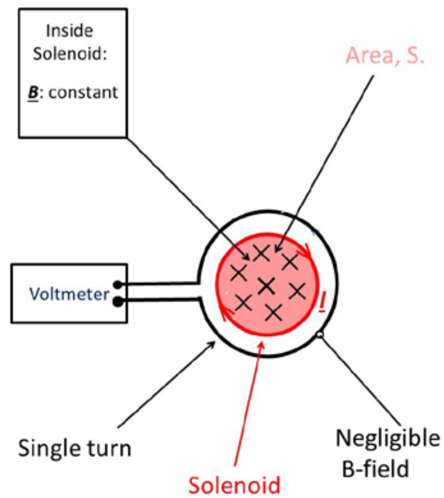
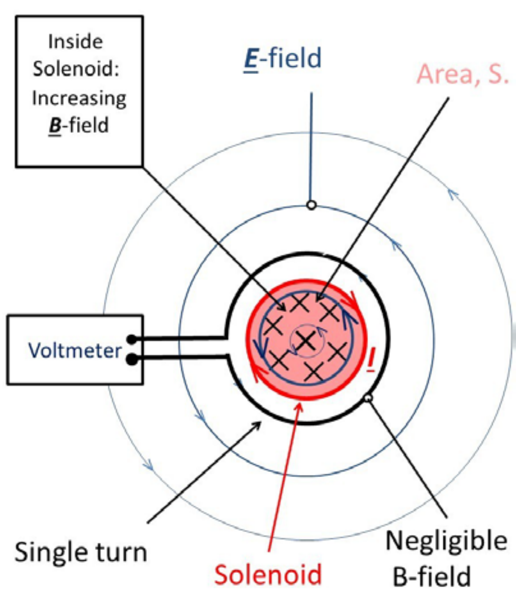


Figure 1 : LHS: A current source producing a constant magnetic field inside a long solenoid. A single loop passes round the outside of the solenoid and is attached to a voltmeter. There is negligible magnetic field outside the solenoid. The solenoid has a cross sectional area, *S*. Under these simple conditions, the voltage measured is zero. RHS: A plan view of the LHS.



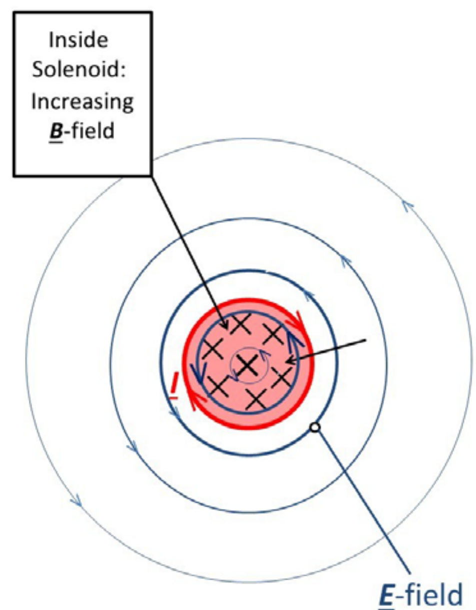


Figure 2 : LHS: A plan view of current source producing a steadily increasing magnetic field inside a long solenoid. A single loop passes round the outside of the solenoid and is attached to a voltmeter which measures a non-zero voltage. RHS: For clarity, the $\underline{\boldsymbol{E}}$-fields and $\underline{\boldsymbol{B}}$-fields (alone) are shown. Of course the fields are produced whether or not the voltmeter is present.

$$\mathrm{V} = \frac{\partial \underline{B}}{\partial t} \cdot \underline{\mathrm{S}} = \frac{\partial \phi_{\mathrm{B}}}{\partial \mathrm{t}} \quad \text{Faraday's Law of Induction} \tag{5-1}$$

5.1.2 Motional Electromotive force

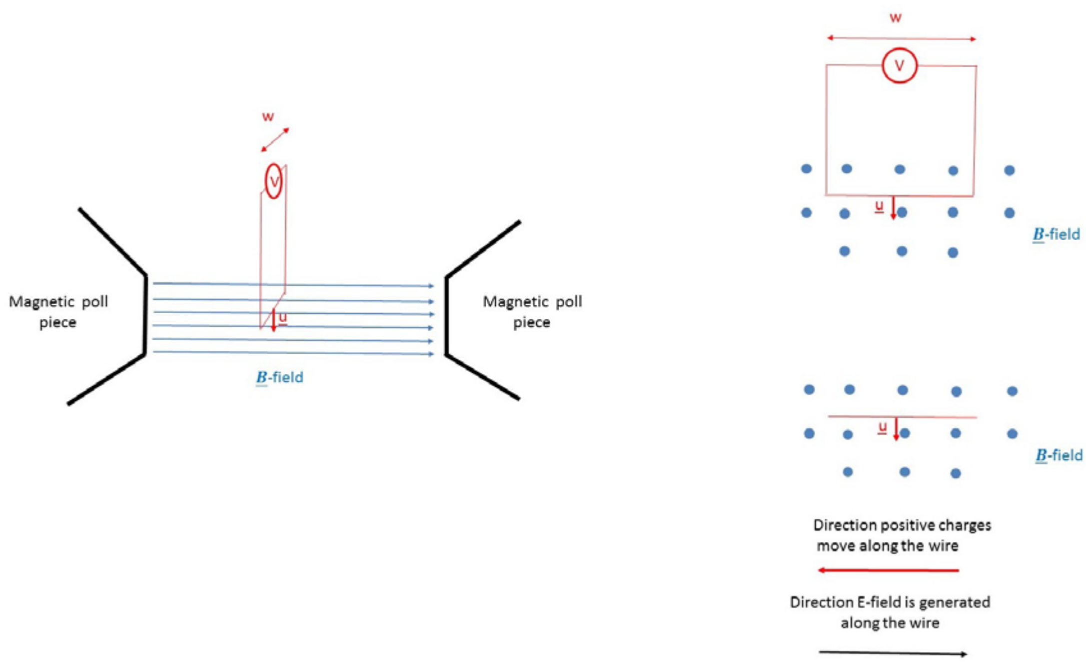


Figure 3 : LHS: A loop of wire moving with a velocity, $\underline{u}$, down through the homogenous field produced between two magnetic pole-pieces. The loop has a width w and is attached to a voltmeter. RHS (upper): A side view of the experiment looking from the RHS pole-piece along the field lines. RHS (lower): The forces on the (conventional) positive charges in the horizontal bottom part of the loop of wire.

$$V = \frac{\partial \phi_{\mathrm{B}}}{\partial \mathrm{t}} = B\frac{\mathrm{dS}}{\mathrm{dt}} = \mathrm{Buw} \tag{5-2}$$

$$\underline{\boldsymbol{F}} = \mathrm{q}\left(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}\right) \tag{5-3}$$

$$\underline{\boldsymbol{E}} = -\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}} \tag{5-4}$$

$$\mathrm{V} = -\mathrm{wE} = Buw \tag{5-5}$$

5.2 Using Faraday's Law to Find Maxwell III

$$\xi_t = -\frac{\partial \phi_B}{\partial t} \quad \text{Faraday's Law} \tag{5-6}$$

$\Uparrow$

Lenz's Law: the emf tries to oppose the change in flux.

$$\xi_t = \oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = -\mathrm{V} \tag{5-7}$$

$$\xi = \oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = -\frac{\partial \phi_B}{\partial t} = -\frac{\partial}{\partial t} \int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \tag{5-8}$$

Stokes' Theorem gives:

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\boldsymbol{l} = \int \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} . \mathrm{d}\underline{\boldsymbol{S}} \tag{5-9}$$

Therefore:

$$\int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = -\frac{\partial}{\partial t} \int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \tag{5-10}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \quad \text{Maxwell's 3rd Equation} \tag{5-11}$$

6 Maxwell IV (From Ampere's Law)

6.1 Maxwell's Correction to Ampere's Law

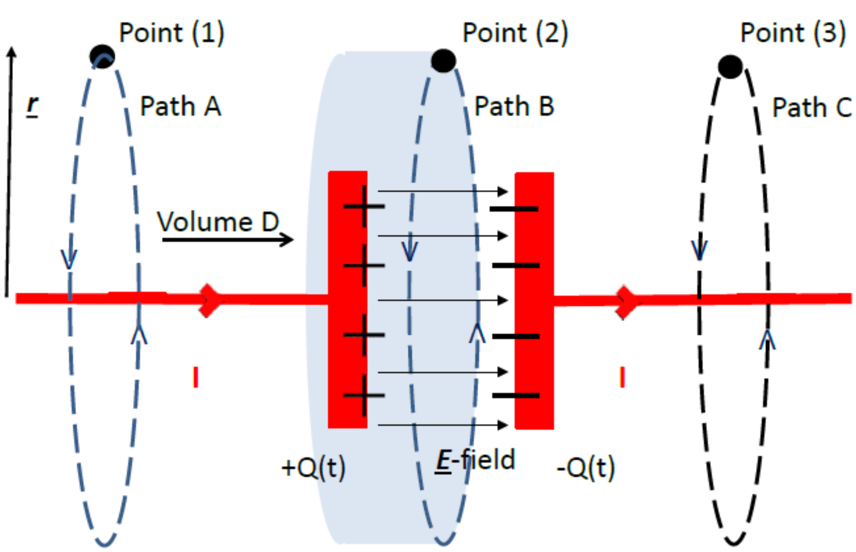


Figure 4 : There are paths A, B and C shown. The surfaces bounded by paths A and C each have a current I that threads through them. Path B has no current threading through it. Only an increasing E-field threads through the flat circular surface bounded by path B. The volume D, which is lightly coloured, includes one of the capacitor plates. There is a charge, +Q, on each of the capacitor plates.

$$\oint_{Path\ A} \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_o \mathrm{I} \qquad \textbf{6-1}$$

$$\oint_{\text{Path A}} \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_o \oint_{\text{Path A}} \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{6-2}$$

$$\oint_{\boldsymbol{D}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int_D \frac{\rho}{\varepsilon_o} \mathrm{dV} = \frac{Q}{\varepsilon_o} \qquad \textbf{6-3}$$

$$\mathrm{I} = \frac{\partial Q}{\partial \mathrm{t}} \qquad \textbf{6-4}$$

$$\oint_{\text{Closed surface.D}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int_{\text{Circular area bounded by path B}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{6-5}$$

$$=> \frac{\partial}{\partial \mathrm{t}} \int_{\text{Circular area bounded by path B}} \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \frac{1}{\varepsilon_o}\frac{\partial \mathrm{Q}}{\partial \mathrm{t}} = \frac{1}{\varepsilon_o}\mathrm{I} \qquad \textbf{6-6}$$

$$\oint \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_o \int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \mu_o \varepsilon_o \frac{\partial}{\partial \mathrm{t}} \int \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{6-7}$$

$$\int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \mu_o \int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \mu_o \varepsilon_o \frac{\partial}{\partial \mathrm{t}} \int \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{6-8}$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} - \text{Maxwell's } 4^{\text{th}} \text{ Equation} \qquad \textbf{6-9}$$

7 Transverse electromagnetic waves and charge conservation

7.1 Electromagnetic waves in a vacuum

If we assume that there are no charges or electrical currents

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = 0,$$
$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0,$$
$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}},$$
$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}}.$$

Taking the curl of Maxwell III

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial}{\partial \mathrm{t}} \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} \quad \textbf{7-1}$$

.

Substituting Maxwell IV gives,

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial \mathrm{t}^2} \quad \textbf{7-2}$$

.

Using the vector identity

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) = \underline{\boldsymbol{\nabla}}\big(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}\big) - \nabla^2 \underline{\boldsymbol{E}} \quad \textbf{7-3}$$

(Nb: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$).
which gives:

$$\underbrace{\underline{\boldsymbol{\nabla}}\big(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}\big)}_{\text{zero by Maxwell I}} - \nabla^2 \underline{\boldsymbol{E}} = -\mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial \mathrm{t}^2} \quad \textbf{7-4}$$

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial \mathrm{t}^2} \quad \textbf{7-5}$$

$$\mathrm{c} = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \approx 3.00 \times 10^8 \mathrm{ms}^{-1}.$$

$$v_{Maxwell} \approx 3.11 \times 10^8 \mathrm{ms}^{-1}$$

$$v_{Foucault} \approx 2.98 \times 10^8 \mathrm{ms}^{-1}$$

Taking the curl of Maxwell IV gives:

$$\nabla^2 \underline{\boldsymbol{B}} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{B}}}{\partial \mathrm{t}^2} \quad \textbf{7-6}$$

7.2 The vector nature of electromagnetic waves

Given that $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ are plane waves we can write them as the real or imaginary part of:

$$\underline{\boldsymbol{E}}(\mathrm{r}, \mathrm{t}) = \underline{\boldsymbol{E}}_o \exp i(\underline{\mathbf{k}} \cdot \underline{\boldsymbol{r}} - \omega \mathrm{t}) \quad \textbf{7-7}$$

$$\underline{\boldsymbol{B}}(\mathrm{r}, \mathrm{t}) = \underline{\boldsymbol{B}}_o \exp i(\underline{\mathbf{k}} \cdot \underline{\boldsymbol{r}} - \omega \mathrm{t}) \quad \textbf{7-8}$$

$\underline{\boldsymbol{E}}_{\mathrm{o}}$ and $\underline{\boldsymbol{B}}_{\mathrm{o}}$ are the polarization directions of $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ respectively and $\underline{\mathbf{k}}$ gives both the magnitude of the wavevector and the direction of propagation. There are non-trivial solutions (in vacuum) only if ω and $\underline{\boldsymbol{k}}$ are the same for $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$.

The solution to the wave equation (Equation **7-5**) for electromagnetic waves can be written:

$$\underline{\mathbf{k}} \cdot \underline{\mathbf{k}} = \mathrm{k}^2 = \omega^2 \mu_0 \varepsilon_0 \quad \textbf{7-9}$$

,

$$\Rightarrow v_{phase} = \frac{\omega}{k} = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \quad \textbf{7-10}$$

Substitute the plane wave solutions into Maxwell III

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = i\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} = i\omega \underline{\boldsymbol{B}} \quad \textbf{7-11}$$

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}}_o = \omega \underline{\boldsymbol{B}}_o \quad \textbf{7-12}$$

Similarly from Maxwell IV

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{B}}_o = -\mu_0 \varepsilon_0 \omega \underline{\boldsymbol{E}}_o \quad \textbf{7-13}$$

For plane waves:

- $\underline{\boldsymbol{E}}_o$, $\underline{\boldsymbol{k}}$, $\underline{\boldsymbol{B}}_o$ are perpendicular to each other
- $\underline{\boldsymbol{B}}_o$ is a factor *c* smaller than $\underline{\boldsymbol{E}}_o$
- $\underline{\boldsymbol{E}}_o \times \underline{\boldsymbol{B}}_o$ is in the direction of $\underline{\boldsymbol{k}}$ and gives the direction of propagation.

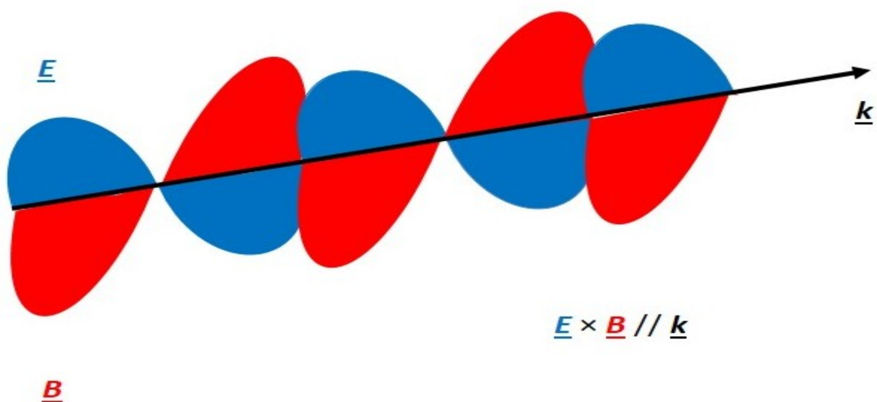


Figure 5 : An electromagnetic wave propagating through space showing the relative directions of $\underline{\boldsymbol{E}}$, $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{k}}$.

7.3 The continuity equation and charge conservation

Maxwell IV

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}.$$

Take the divergence of both sides,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \underline{\boldsymbol{\nabla}} \cdot \mu_0 \underline{\boldsymbol{J}} + \underline{\boldsymbol{\nabla}} \cdot \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t} \quad \textbf{8-14}$$

Use the vector identity

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = 0 \quad \textbf{8-15}$$

(note the similarity with vector algebra ($\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} = 0$)

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} = -\varepsilon_0 \frac{\partial}{\partial t} \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} \qquad \textbf{8-16}$$

.

Using

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0},$$

We find the differential continuity equation:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} = -\frac{\partial \rho}{\partial t}, \qquad \textbf{8-17}$$

where $\boldsymbol{J}$ is current density (charge flowing per unit area per unit time) and ρ is the charge density.

Take the volume integral of both sides,

$$\int \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} \mathrm{dV} = -\int \frac{\partial \rho}{\partial t} \mathrm{dV} \qquad \textbf{8-18}$$

We find the continuity equation in integral form:

$$\underbrace{\int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}}}_{\text{Charge flowing out through the surface every second}} = \underbrace{-\frac{\partial}{\partial t} \int \rho \mathrm{dV}}_{\text{Change in total charge inside a volume every second}} \qquad \textbf{8-19}$$

Hence the continuity equation is equivalent to conservation of charge. ⇒ Maxwell's equations require conservation of charge.

8 Maxwell and Einstein

8.1 The collapse of Newtonian/Galilean physics

a) Observer P- stationary with respect to positive charges

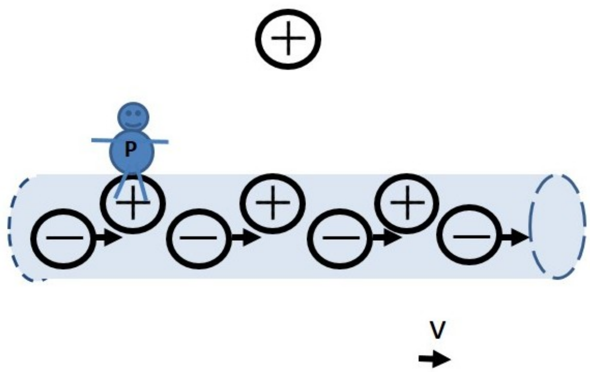


Figure 6 : Observer P loves alliteration and therefore sits on a positive charge. The positive test charge and the positive charges in the wire are stationary for Observer P. The negative charges in the wire move to the right with respect to observer P with a velocity v.

$$\underline{\boldsymbol{F}} = \mathrm{q}\left(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}\right) = 0$$

⇒ +ve test charge does not move

b) Observer N- stationary with respect to negative charges

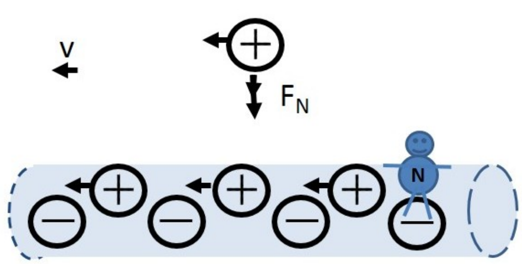


Figure 7 : Observer N also loves alliteration and therefore sits on a negative charge. The negative charges in the wire are stationary with respect to observer N. The positive test charge and the positive charges in the wire move to the left with velocity v.

$$\underline{\boldsymbol{F}} = \mathrm{q}(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \neq 0$$

⇒ the +ve test charge moves towards the wire

8.2 Einstein discovers relativity

- Assume that observer N sets the experiment up: , the wire is charge neutral (i.e. in the rest frame, of the battery) and there is a test charge moving in a magnetic field where:

$$I = n_l e v \quad \textbf{7-20}$$

n_l: number of charges per unit length.

$$\mathrm{B} = \frac{\mu_o \mathrm{I}}{2\pi \mathrm{r}} = \frac{\mu_o n_l e v}{2\pi \mathrm{r}} \quad \textbf{7-21}$$

r : distance from the wire to the test charge.

$$\underline{\boldsymbol{F}}_N = \mathrm{q}\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}} \Rightarrow \mathrm{F_N} = \frac{\mu_o q n_l e v^2}{2\pi r} \quad \textbf{7-22}$$

- For observer P (sitting on the +ve charge), there is a Lorentz contraction of the moving negative charge:

$$\lambda_{-ve} = -\gamma n_l e \quad \textbf{7-23}$$

where λ charge per unit length and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Observer P also says that in order that there is charge neutrality for observer N, in the rest frame for the positive charges:

$$\lambda_{+ve} = \frac{n_l e}{\gamma} \quad \textbf{7-24}$$

$$\lambda_{\mathrm{TOTAL}} = -\gamma n_l e \left(1 - \frac{1}{\gamma^2}\right) \quad \textbf{7-25}$$

$$\lambda_{\mathrm{TOTAL}} = -\gamma n_l e \frac{v^2}{c^2} \quad \textbf{7-26}$$

From Gauss' law:

$$\Rightarrow \mathrm{E_P} = \frac{-\lambda_{\mathrm{TOTAL}}}{2\pi\varepsilon_o r} = \frac{\gamma \mu_o n_l e v^2}{2\pi r} \quad \textbf{7-27}$$

$$\Rightarrow \mathrm{F_P} = \mathrm{qE_P} = \gamma \mathrm{qvB} = \gamma \mathrm{F_N} \quad \textbf{7-28}$$

Observer N is in the rest frame, so

$$\delta\tau_N = \gamma \delta\tau_P \quad \textbf{7-29}$$

Since, $\underline{\boldsymbol{F}}_N = \frac{\delta \underline{\boldsymbol{P}}_N}{\delta\tau_N}$, $\underline{\boldsymbol{F}}_P = \frac{\delta \underline{\boldsymbol{P}}_P}{\delta\tau_P}$ where P = relativistic momentum.

$$\Rightarrow \delta\underline{\boldsymbol{P}}_P = \delta\underline{\boldsymbol{P}}_N \quad \textbf{7-30}$$

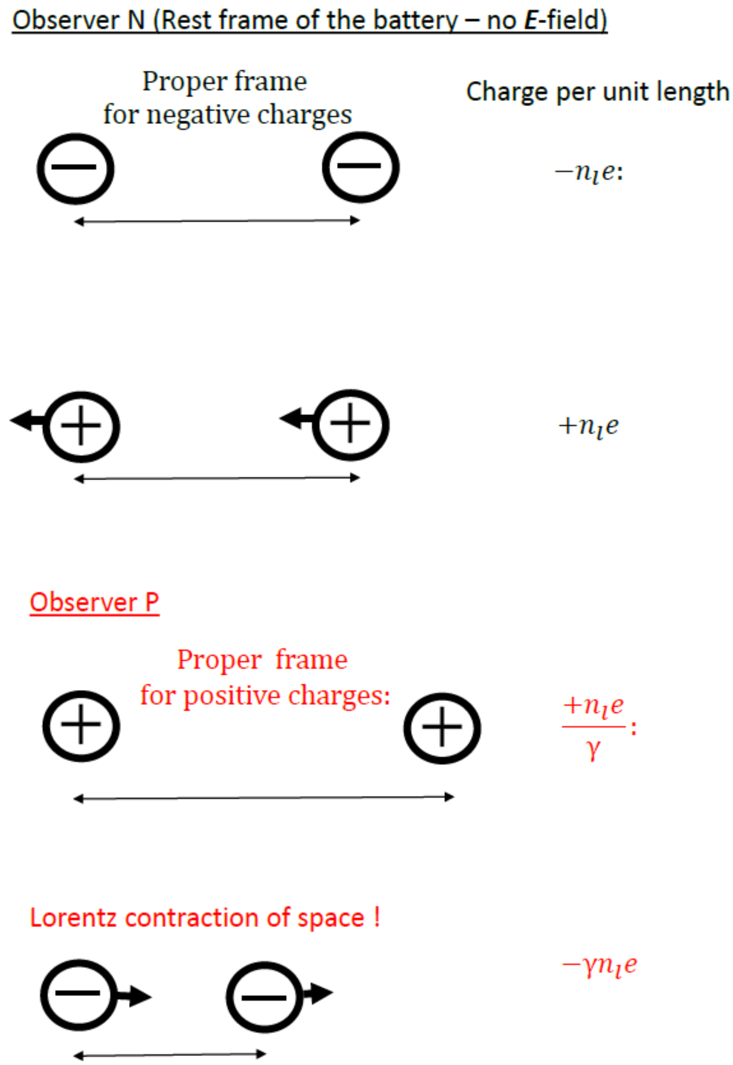


Figure 8 : A summary of the observations made by observer N (who is in the rest frame of the battery) and observer P.

- Coulomb's Law can be regarded as fundamental with magnetic forces and induced EMF as relativistic corrections.

Albert Einstein (1952)
"What lead me more or less directly to the special theory of relativity was the conviction that the electromotive force acting on a body in motion in a magnetic field was nothing other than the electric field"

ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 8 + 9 + 10 + 11 (Rev 7)

9 Conducting and Superconducting materials

9.1 Extensive and Intensive Properties of Conductors

$$V = IR \text{ – Ohm's Law} \qquad 9\text{-}1$$

Let's rewrite Ohm's law using the intensive variables $\underline{\boldsymbol{E}}$ (= V/L) the electric field and $\underline{\boldsymbol{J}}$ (= I/A) the current density.

$$\frac{V}{L} = \frac{I}{A}(RA/L) \qquad 9\text{-}2$$

Which gives:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{J}}\rho_n \text{ – Definition of resistivity} \qquad 9\text{-}3$$

where the resistivity, ρ_n, is given by $\rho_n = \frac{RA}{L}$.

9.1.1 Current Densities and Charge Densities in Conductors

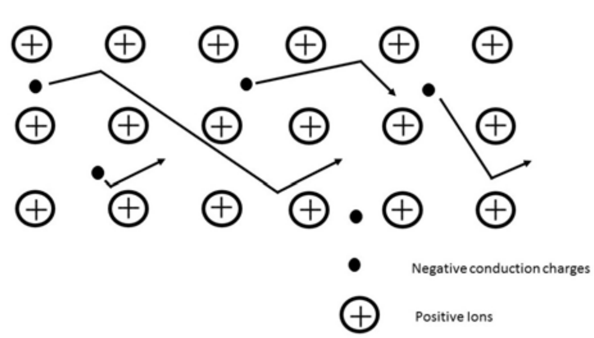
$$\underline{\boldsymbol{J}} = \sigma_n \underline{\boldsymbol{E}} \text{ – Definition of conductivity} \qquad 9\text{-}4$$

where the conductivity, σ_n, is given by $\sigma_n = 1/\rho_n$.

Using Ohm's law ($\underline{\boldsymbol{E}} = \underline{\boldsymbol{J}}\rho_n$), Maxwell I ($\underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0}$) and the continuity equation ($\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{J}} = -\frac{\partial \rho}{\partial t}$), we can use a dimensionality argument to find that the characteristic life-time, τ , is:

$$\tau = \rho_n \varepsilon_0 \qquad 9\text{-}5$$

9.2 Drude Model



The Drude model - The electric field accelerates the charges which then collide with the scattering sites.

In the Drude model, the charge-carriers accelerate for an average time τ - they then scatter and instanteously stop. The fraction of charge-carriers that scatter in a time ∂t is $\frac{\partial t}{\tau}$ and the remaining fraction that don't scatter is $(1 - \frac{\partial t}{\tau})$. Hence the momentum of the charge carriers at time $t + \partial t$ is given by:

$$p(t + \partial t) = \left(1 - \frac{\partial t}{\tau}\right) p(t) + \left(1 - \frac{\partial t}{\tau}\right) F(t)\, \partial t + \left\{F(t)\, \partial t \frac{\partial t}{\tau}\right\} \qquad 9\text{-}6$$

Writing $\partial p = p(t + \partial t) - p(t)$, gives:

$$\frac{\partial p}{\partial t} = -\frac{p(t)}{\tau} + F(t) + O(\partial t) \qquad 9\text{-}7$$

$$\underline{\boldsymbol{F}} = q(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}) \qquad 9\text{-}8$$

$|B| = \left|\frac{E}{c}\right|$ so $\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}} \ll \underline{\boldsymbol{E}}$ **.** Using F = qE and $p = mv$ gives:

$$m\frac{dv}{dt} = qE - \frac{mv}{\tau} \; - \; \text{Equation of motion for carriers} \qquad 9\text{-}9$$

where the $\frac{mv}{\tau}$ term can also be interpreted as a frictional damping term.

$$\underline{\boldsymbol{J}} = Nq\underline{\boldsymbol{v}} \qquad 9\text{-}10$$

where *N* : number of charge carriers per volume, *q* : charge on each carrier and $\underline{\boldsymbol{v}}$: velocity of the carriers.

$$m\frac{d\underline{\boldsymbol{J}}}{dt} = Nq^2\underline{\boldsymbol{E}} - \frac{m}{\tau}\underline{\boldsymbol{J}} \qquad 9\text{-}11$$

We now assume that we can rewrite the equation of motion in terms of complex variables where

$$m\frac{d\underline{\tilde{\boldsymbol{J}}}}{dt} = Nq^2\underline{\tilde{\boldsymbol{E}}} - \frac{m}{\tau}\underline{\tilde{\boldsymbol{J}}} \qquad 9\text{-}12$$

The trial solutions are both the form:

$$\underline{\tilde{\boldsymbol{J}}} \text{ and } \underline{\tilde{\boldsymbol{E}}} \propto \exp - i\omega t \qquad 9\text{-}13$$

$$\underline{\tilde{\boldsymbol{J}}} = \frac{Nq^2}{m(\tau^{-1} - i\omega)} \cdot \underline{\tilde{\boldsymbol{E}}} \qquad 9\text{-}14$$

Comparing this expression to the definition for conductivity : $\underline{\boldsymbol{J}} = \sigma_n \underline{\boldsymbol{E}}$ **,** gives the complex conductivity as:

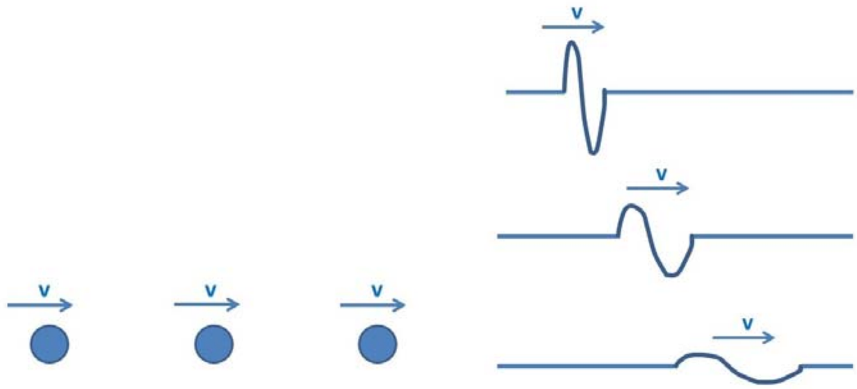
$$\tilde{\sigma}_n = \frac{Nq^2}{m(\tau^{-1} - i\omega)} \qquad 9\text{-}15$$

$$\underline{\boldsymbol{E}} = \mathrm{real}(\underline{\tilde{\boldsymbol{E}}}) = \mathrm{real}(\underline{\boldsymbol{E}}_0 \exp - i\omega t) = \underline{\boldsymbol{E}}_0 \cos(\omega t)\,. \qquad 9\text{-}16$$

$$\underline{\boldsymbol{J}} = \text{real}\left(\underline{\tilde{\boldsymbol{J}}}\right) = \text{real}\left(\frac{\mathrm{Nq}^2}{\mathrm{m}(\tau^{-1} - \mathrm{i}\omega)}\underline{\tilde{\boldsymbol{E}}}\right) = \text{real}\left(\frac{\mathrm{Nq}^2\tau(1 + \mathrm{i}\omega\tau)}{\mathrm{m}(1 + (\omega\tau)^2)}\underline{\tilde{\boldsymbol{E}}}\right)$$

$$= \underline{\boldsymbol{E}}_0 \frac{\mathrm{Nq}^2\tau}{\mathrm{m}(1+(\omega\tau)^2)}\{\cos(\omega\mathrm{t}) + \omega\tau\sin(\omega\mathrm{t})\}\ . \qquad 9\text{-}17$$

9.3 Dispersive and ballistic motion of waves



LHS: A ball moving ballistically – its shape does not change as it moves. RHS: A wave moving dispersively along a wire – it spreads out as it moves.

Consider a single complex infinite plane wave with a single wave-vector and frequency propagating in the *x*-direction. The wave-vector k points in the *x*-direction and we have

$$\phi(x,t) = \phi_o \exp \mathrm{i}(k \cdot x - \omega \mathrm{t}) \qquad 9\text{-}18$$

The full disturbance of a complex wavepacket (or profile) is

$$G(x,t) = \int_{-\infty}^{\infty} g(k)\exp\mathrm{i}(k \cdot x - \omega(k)t)\,\mathrm{d}k\ . \qquad 9\text{-}19$$

$$g(k) = \mathrm{A}\exp[-\left(\frac{k - \mathrm{k}_0}{\mathrm{k}_W}\right)^2] \qquad 9\text{-}20$$

$$\omega = \mathrm{f}(k) - \text{ Definition of a Dispersion Relation} \qquad 9\text{-}21$$

Using Taylor's Theorem

$$\omega = \omega_o + \alpha(k - \mathrm{k}_o) + \frac{\beta}{2}(k - \mathrm{k}_o)^2 + \ldots\ , \qquad 9\text{-}22$$

where $\omega_o = \omega(\mathrm{k}_o)$ is the angular frequency of the most important component wave and

$$\alpha = \left(\frac{\partial \omega}{\partial k}\right)_{k=\mathrm{k_o}} \text{ and } \beta = \left(\frac{\partial^2 \omega}{\partial k^2}\right)_{k=\mathrm{k_o}} \qquad 9\text{-}23$$

Substituting into Equation 9-19, the general solution for the wave is

$$G(x,t) = \int_{-\infty}^{\infty} \mathrm{A}\exp[-\left(\frac{k-\mathrm{k_0}}{\mathrm{k}_W}\right)^2]\exp i(k\cdot x - [\omega_o + \alpha(k-\mathrm{k_o}) + \frac{\beta}{2}(k - \mathrm{k_o})^2 + \ldots]t)\,\mathrm{d}k \qquad 9\text{-}24$$

The second order solution to Equation 9-24 is

$$G(x,t) = \mathrm{A}\mathrm{k}_W\pi^{\frac{1}{2}}.\exp i(\mathrm{k_o}\cdot x - \omega_o t).\exp[-\frac{\mathrm{k}_W^2\pi^2(x-\alpha t)^2}{1+\mathrm{k}_W^4(\pi\beta t)^2}]\,. \qquad 9\text{-}25$$

$$\exp\left[-\frac{\mathrm{k}_W^2\pi^2(\lambda_{\mathrm{FWHM}}/2)^2}{1+\mathrm{k}_W^4(\pi\beta t)^2}\right] = \frac{1}{2}, \qquad 9\text{-}26$$

which can be rearranged to give

$$\lambda_{\mathrm{FWHM}} = 2\left\{\frac{ln2(1+\mathrm{k}_W^4(\pi\beta t)^2)}{\mathrm{k}_W^2\pi^2}\right\}^{\frac{1}{2}}. \qquad 9\text{-}27$$

<u>Important general results that you will need to remember are:</u>

1:

$$\mathrm{v_{phase}} = f\lambda = \frac{\omega}{k_{real}} \qquad 9\text{-}28$$

2:

$$\mathrm{v_{group}} = \alpha = \left.\frac{\partial\omega}{\partial k_{real}}\right|_{k=\mathrm{k_o}} \qquad 9\text{-}29$$

3:

$$\frac{\partial^2\omega}{\partial k_{real}^2} = \beta = 0 \qquad 9\text{-}30$$

$$\frac{\partial^2\omega}{\partial k_{real}^2} = \beta \neq 0 \Rightarrow \text{dispersive} \qquad 9\text{-}31$$

9.4 Electromagnetic waves propagating through metals (short scattering time)

$$\sigma_n = \frac{\mathrm{Nq^2\tau}}{\mathrm{m}} \qquad 9\text{-}32$$

Taking the curl of Maxwell III gives:

$$\underline{\nabla} \times \underline{\nabla} \times \underline{\boldsymbol{E}} = -\frac{\partial}{\partial \mathrm{t}} \underline{\nabla} \times \underline{\boldsymbol{B}} \qquad 9\text{-}33$$

Substituting Ohm's law ($\underline{\boldsymbol{J}} = \sigma_n \underline{\boldsymbol{E}}$) into Maxwell IV gives:

$$\underline{\nabla} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} = \mu_0 \sigma_N \underline{\boldsymbol{E}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad 9\text{-}34$$

$$\underline{\nabla} \times \underline{\nabla} \times \underline{\boldsymbol{E}} == -\mu_0 \sigma_N \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} - \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad 9\text{-}35$$

Using the vector identity

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\boldsymbol{E}}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{\boldsymbol{E}}) - \nabla^2 \underline{\boldsymbol{E}} \qquad 9\text{-}36$$

And Maxwell I for a conducting medium, $\underline{\nabla}.\underline{\boldsymbol{E}} = 0$, gives

$$\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \sigma_N \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad 9\text{-}37$$

This equation is of the form of a travelling wave travelling along the x-direction. We assume a trial solution is of the :

$$\underline{\widetilde{\boldsymbol{E}}}(\mathrm{r}, \mathrm{t}) = \underline{\boldsymbol{E}_o} \exp i(kx - \omega \mathrm{t}) \qquad 9\text{-}38$$

$$\mathrm{k}^2 = \mu_0 \varepsilon_0 \omega^2 + i\omega \mu_0 \sigma_N \qquad 9\text{-}39$$

i) For a highly insulating material, $\sigma_n \rightarrow 0$:

$$\mu_0 \varepsilon_0 \omega^2 \gg \mu_0 \sigma_N \omega \qquad 9\text{-}40$$

$$\mathrm{k} = \sqrt{\mu_0 \varepsilon_0}\,\omega \ \ - \text{ dispersion relation for an insulator} \qquad 9\text{-}41$$

The phase velocity is $v_{phase} = f\lambda = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$

The propagation is ballistic ($\frac{\partial^2 \omega}{\partial k^2}$=0)

ii)

$$\mathrm{k}^2 = i\omega \mu_0 \sigma_n \qquad 9\text{-}42$$

Given:

$$\sqrt{i} = \frac{1 + i}{\sqrt{2}} \qquad 9\text{-}43$$

$$\mathrm{k} = (1 + i)\sqrt{\frac{\omega \mu_0 \sigma_n}{2}} = \mathrm{k}_{real} + i\mathrm{k}_{imaginary} \qquad 9\text{-}44$$

where $k_{real} = \sqrt{\frac{\omega \mu_0 \sigma_n}{2}}$ and $k_{imaginary} = \sqrt{\frac{\omega \mu_0 \sigma_n}{2}}$.

The solution for the E-field inside the good conductor is found by simply substituting k back into the plane wave equation so that for a good conductor:

$$\underline{\tilde{\boldsymbol{E}}} = \underline{\boldsymbol{E}}_{\mathrm{o}} \exp\left[\mathrm{i}\left(\left(\mathrm{k}_{\mathrm{real}} + i\mathrm{k}_{\mathrm{imaginary}}\right)x - \omega t\right)\right]$$

$$= \underline{\boldsymbol{E}}_{\mathrm{o}} \exp\mathrm{i}(\mathrm{k}_{\mathrm{real}}\mathrm{x} - \omega \mathrm{t}) \exp(-\mathrm{k}_{\mathrm{imaginary}}x) \quad \text{9-45}$$

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_{\mathrm{o}} \cos\left(\frac{x}{\delta} - \omega \mathrm{t}\right) \exp\left(-\frac{x}{\delta}\right) \quad \text{9-46}$$

$$\delta = \sqrt{2/\omega\mu_0\sigma_{\mathrm{n}}} \approx 4\ \mathrm{nm} \quad \text{9-47}$$

9.5 Electromagnetic waves propagating through low density plasmas (long scattering time)

$$\sigma_{\mathrm{n}} = \lim_{\omega \gg \tau^{-1}} \frac{\mathrm{Nq}^2}{\mathrm{m}(\tau^{-1} - \mathrm{i}\omega)} = \lim_{\omega \gg \tau^{-1}} \frac{-\mathrm{Nq}^2}{\mathrm{m_e i}\omega} = +\frac{\mathrm{iNq}^2}{\mathrm{m_e}\omega} \quad \text{9-48}$$

$$\mathrm{k}^2 = \mu\varepsilon\omega^2 + \mathrm{i}\omega\mu\sigma_{\mathrm{N}} \quad \text{9-49}$$

Since the plasma is not contained in a magnetic or dielectric media (i.e. $\varepsilon_{\mathrm{r}} = 1$ and $\mu_{\mathrm{r}} = 1$), the dispersion relation can be written,

$$\mathrm{k}^2 = \mu_0\varepsilon_0\omega^2 + \mathrm{i}\omega\mu_0 \frac{\mathrm{iNe}^2}{\mathrm{m_e}\omega} = \mu_0\varepsilon_0\omega^2 - \frac{\mu_0\mathrm{Ne}^2}{\mathrm{m_e}} \quad \text{9-50}$$

$$\mathrm{k}^2 = \frac{\omega^2}{\mathrm{c}^2}\left\{1 - \left(\frac{\omega_p}{\omega}\right)^2\right\} \quad \text{9-51}$$

where the (angular) plasma frequency is given by,

$$\omega_{\mathrm{p}} = \left(\frac{\mathrm{Ne}^2}{\mathrm{m_e}\varepsilon_{\mathrm{o}}}\right)^{\frac{1}{2}} \quad \text{9-52}$$

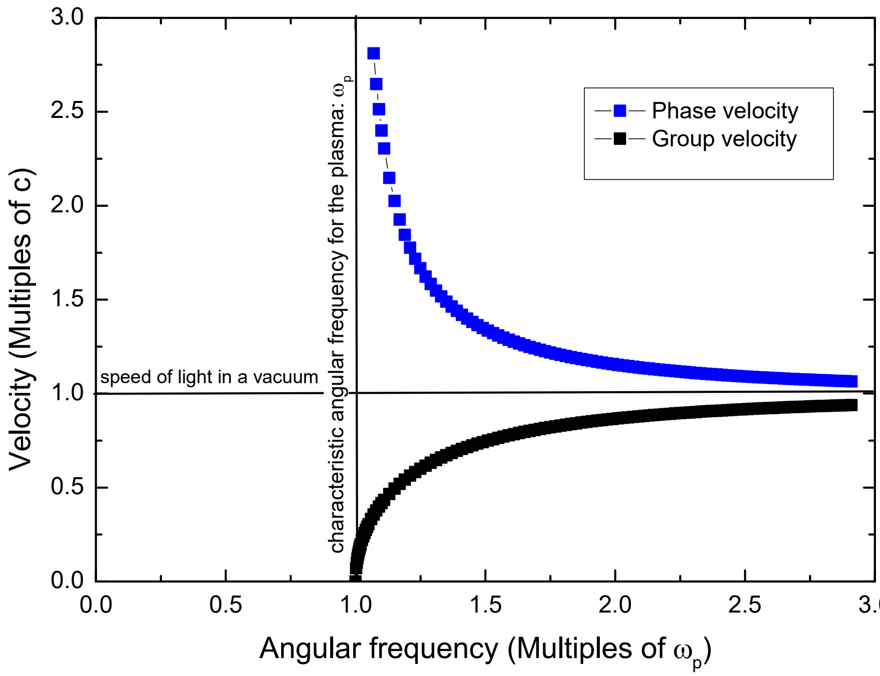
The dispersion relation has two distinct regimes:

i) $\omega > \omega_{\mathrm{p}}$; k is real and the electromagnetic waves propagate without attenuation through the plasma.

ii) $\omega < \omega_{\mathrm{p}}$; k is imaginary and there is an attenuated propagation of the electromagnetic waves.

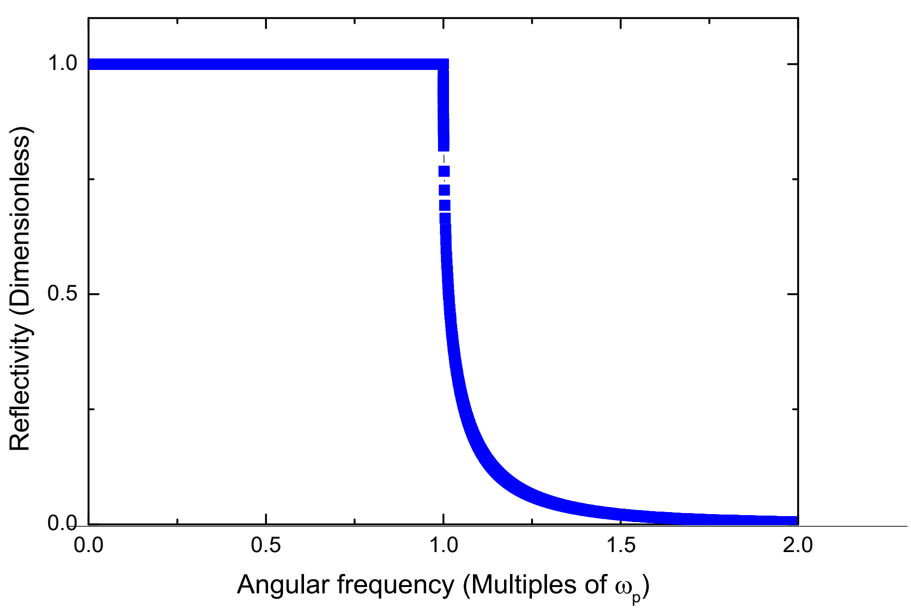
$$\mathrm{v}_{\mathrm{phase}} = \frac{\mathrm{c}}{\sqrt{1 - \left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2}} \quad (> c \ \text{ for } \ \omega > \omega_{\mathrm{p}}) \quad \text{9-53}$$

$$\mathrm{v}_{\mathrm{group}} = \mathrm{c}\sqrt{1 - \left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2} \quad (< c \ \text{ for } \ \omega > \omega_{\mathrm{p}}) \quad \text{9-54}$$



The velocity of an electromagnetic wave in a plasma as a function of angular frequency. Electromagnetic waves propagating through the plasma are attenuated at angular frequencies below the angular plasma frequency (ω_p).

9.5.1 Reflectance of metals



The typical reflectance versus angular frequency of an electromagnetic wave for a metal.

Elemental metal:	Li	Na	K	Rb
λ (Calculated from $\lambda = 2\pi c/\omega_p$) :	1550	2090	2870	3220
λ (Experimental data) :	1550	2100	3150	3400

9.6 Superconducting Materials

a) Zero resistivity

$$\mathrm{m}\underline{\boldsymbol{a}} = \mathrm{m}\frac{\partial \underline{\boldsymbol{v}}}{\partial \mathrm{t}} \ . \qquad \text{9-55}$$

Using $\underline{\boldsymbol{F}} = \mathrm{q}\underline{\boldsymbol{E}}$ and $\underline{J} = \mathrm{Nq}\underline{\boldsymbol{v}}$, we find the first London equation,

$$\underline{\boldsymbol{E}} = \mu_0 \lambda_L^2 \frac{\partial \underline{\boldsymbol{J}}}{\partial \mathrm{t}} \quad (1^{\text{st}} \text{ London Equation}), \qquad \text{9-56}$$

where the London penetration depth , λ_L, is given by $\lambda_L = (\frac{\mathrm{m_e}}{\mu_0 \mathrm{N_s} e^2})^{\frac{1}{2}}$ and N_s is the density of superelectrons.

b) The Meissner state – exclusion of magnetic flux

The 2nd London equation is given by,

$$\underline{\boldsymbol{B}} = -\mu_0 \lambda_L^2 \underline{\nabla} \times \underline{\boldsymbol{J}} \ . \ (2^{\text{nd}} \text{ London Equation}). \qquad \text{9-57}$$

Substituting Maxwell's 4th equation for $\underline{J}$, where $(\partial \underline{\boldsymbol{E}}/\partial \mathrm{t} = 0)$, into the second London equation gives

$$\underline{\boldsymbol{B}} = -\lambda_L^2 \underline{\nabla} \times \underline{\nabla} \times \underline{\boldsymbol{B}} \qquad \text{9-58}$$

Using the vector relation:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\boldsymbol{B}}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{\boldsymbol{B}}) - \nabla^2 \underline{\boldsymbol{B}} \qquad \text{9-59}$$

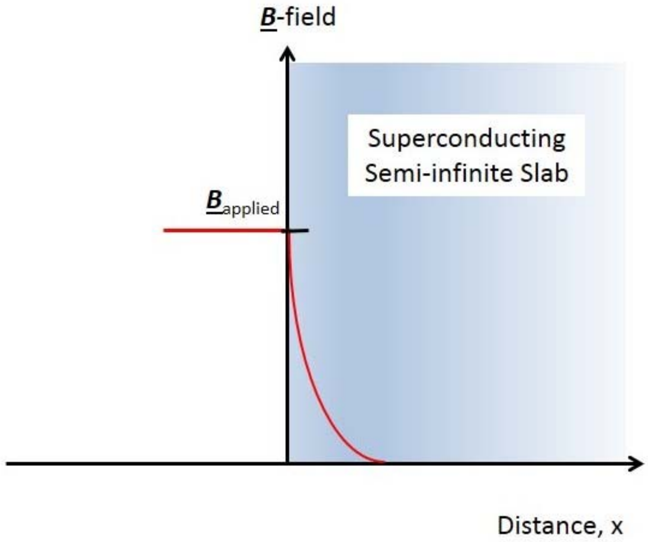
gives

$$\nabla^2 \underline{\boldsymbol{B}} = \frac{1}{\lambda_L^2} \underline{\boldsymbol{B}} \qquad \text{9-60}$$

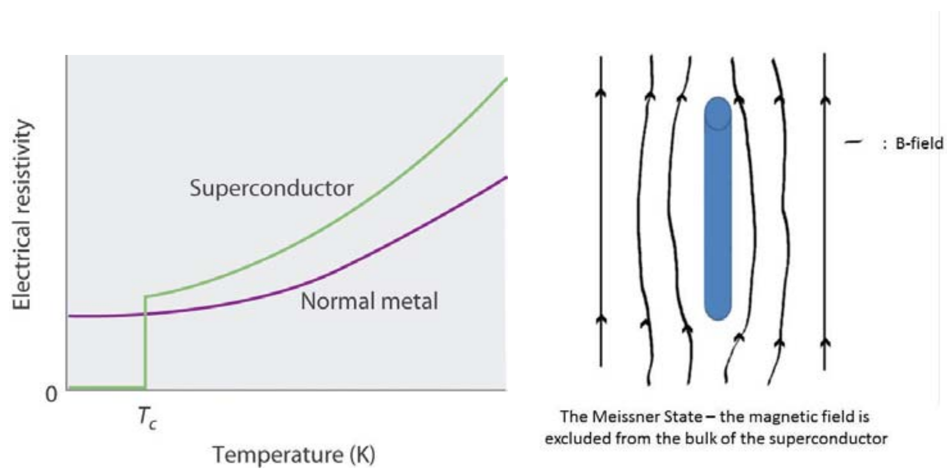
For a semi-infinite slab, the equation has solutions of the form:

$$\underline{\boldsymbol{B}}(x) = \underline{\mathbf{B}}_0 \exp(-x/\lambda_L\,) \quad \text{for} \quad x > 0\ , \qquad \text{9-61}$$

which describes the Meissner effect.



The magnetic field profile for a semi-infinite slab of superconductor in an applied field ($\underline{B}_{\mathrm{applied}}$). The field decays exponentially at the surface to zero over a characteristic distance λ_L .



The two fundamental properties of superconductors. (LHS) At the critical temperature, T_C, the material becomes superconducting and the resistivity drops to zero. (RHS) The Meissner state in low fields. The magnetic field is excluded from the bulk of all metallic superconductors in low applied fields.

ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 12+13+14+15 (Rev 4)

10 Dielectrics

10.1 Microscopic properties of dielectrics

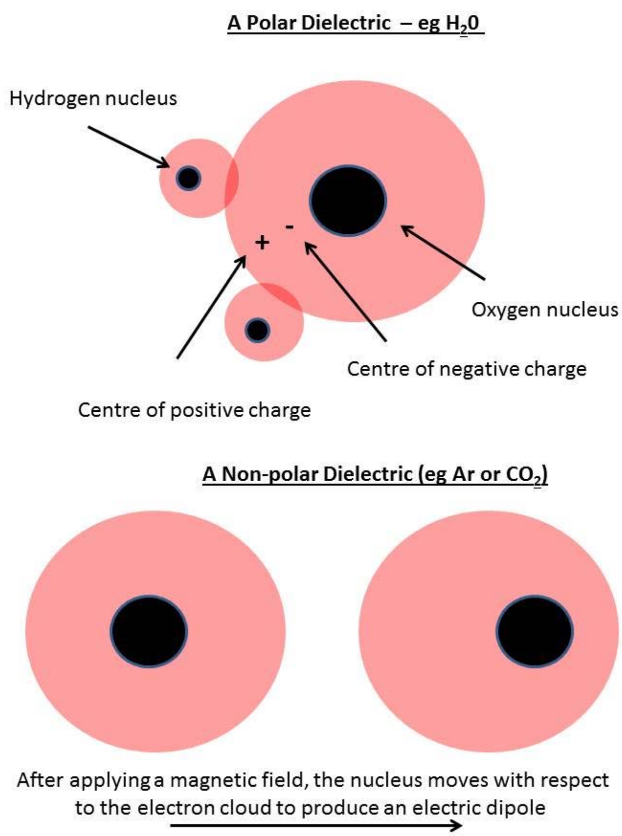


Figure 1 : A polar dielectric and a non-polar dielectric

Dielectrics produce their own electric field in response to an applied electric field – hence the name ‘dielectric’ : 2nd electric field.

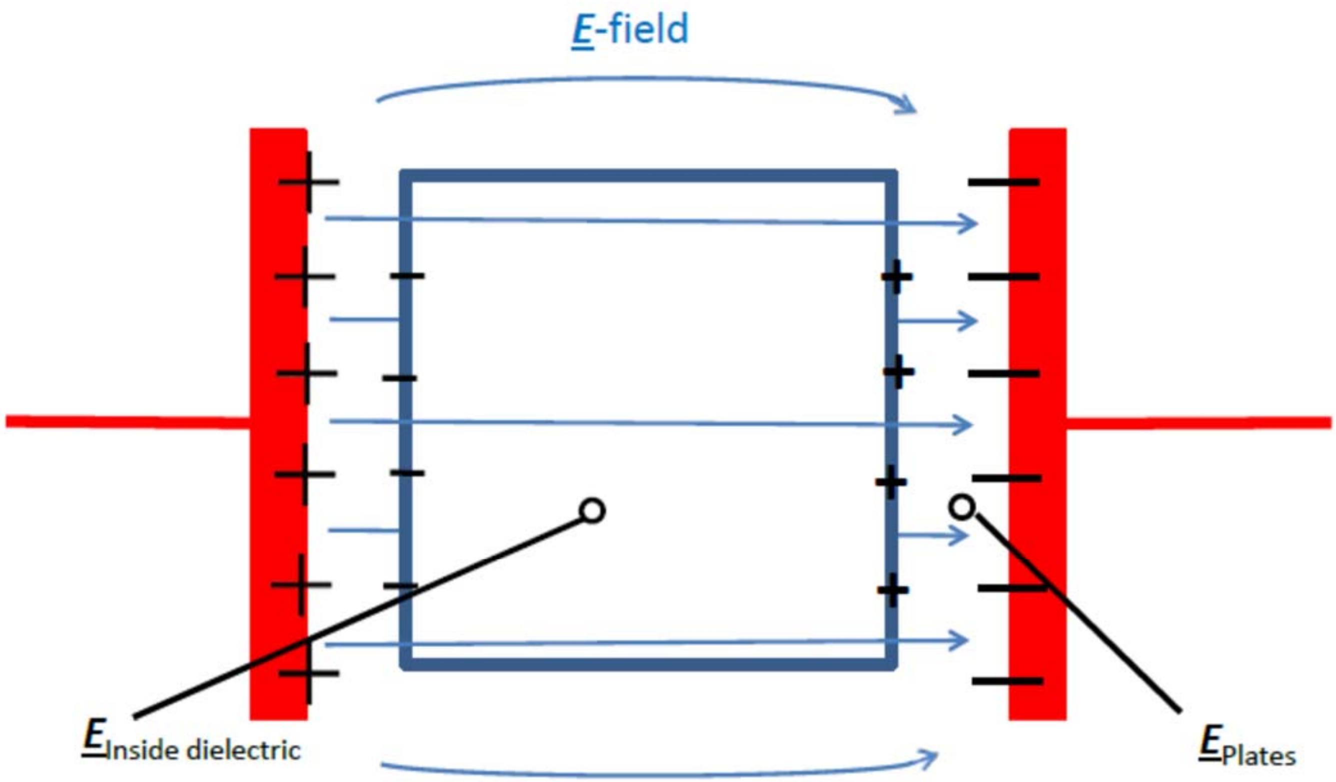


Figure 2 : A polarized cube of dielectric between two capacitor plates. The dielectric field opposes the applied electric field. The net $\underline{\boldsymbol{E}}$-field inside the dielectric is lower than the applied field.

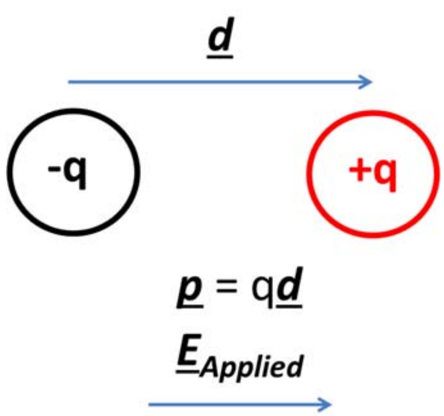


Figure 3 : An electric dipole moment.

The fundamental response of a dielectric:

$$\underline{\boldsymbol{p}} = q\underline{\boldsymbol{d}} \; - \text{ Definition of the electric dipole moment } (\boldsymbol{p}) \qquad 10\text{-}1$$

$$\underline{\boldsymbol{P}} = \mathrm{N}\underline{\boldsymbol{p}} - \text{ Definition of the polarisation, } \boldsymbol{P} \ (\mathrm{Cm}^{-2}) \qquad 10\text{-}2$$

where N is the number of dipoles per unit volume(m^{-3}).

$$\underline{\boldsymbol{P}} = \varepsilon_o(\varepsilon_r - 1)\underline{\boldsymbol{E}} - \text{ Definition of the relative dielectric constant or permittivityy } (\varepsilon_r) \qquad 10\text{-}3$$

$$\underline{\boldsymbol{P}} = \varepsilon_o\chi_e\underline{\boldsymbol{E}} - \text{Definition of the electric susceptibility, } \chi_e\text{, of the medium} \qquad 10\text{-}4$$

Note that as either ε_r or χ_e increases, $\underline{P}$ increases for a given $\underline{E}$-field => the material is a stronger dielectric

10.2 Current density and charge density in dielectrics

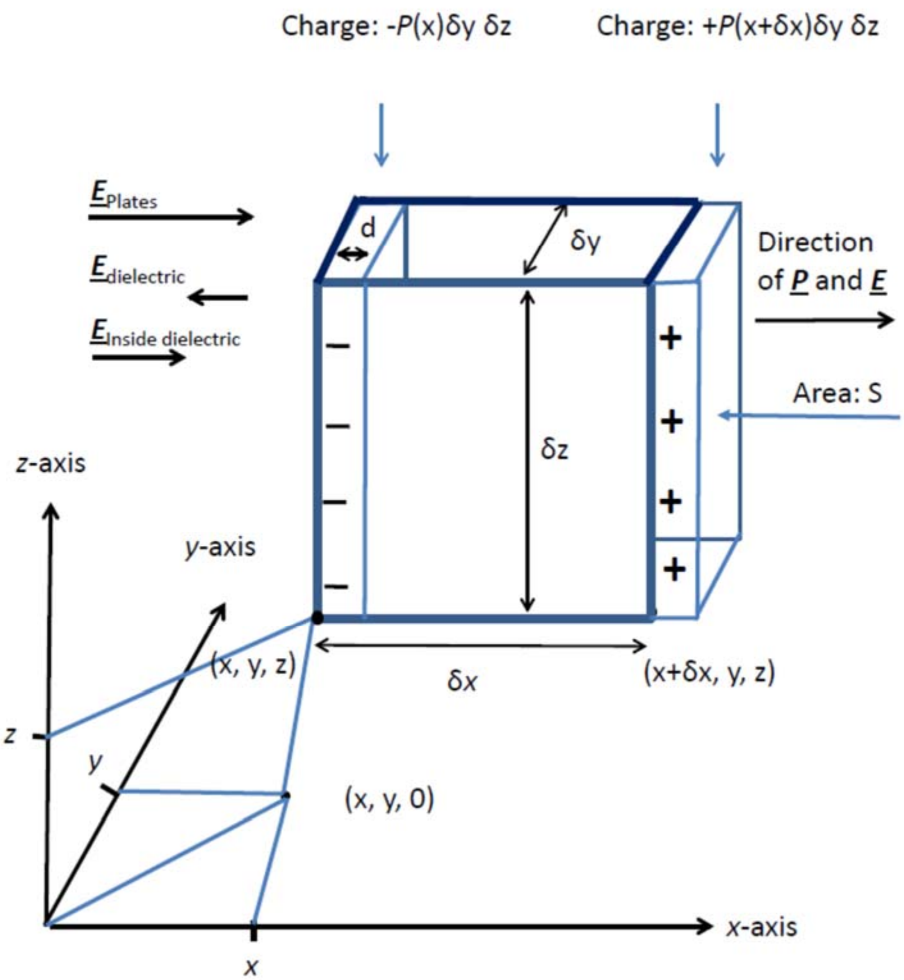


Figure 4 : A cube of dielectric in an $\underline{\boldsymbol{E}}$-field consisting of displaced +ve charge.

$$Q_{+ve} = \underbrace{Nq}_{\text{charge/volume}} \times \underbrace{Sd}_{\text{Volumeof+vecharge}} \quad 10\text{-}5$$

$$\sigma = \frac{Q_{+ve}}{S} = Nqd = P \quad 10\text{-}6$$

$$\sigma = \underline{\boldsymbol{P}} \cdot \hat{\mathbf{n}} \quad 10\text{-}7$$

The positive charge that moves into the cube from the neighbouring cube is (c.f. $\sigma = P \cdot \hat{n}$) at x is given by :

$$Q_{In}(x) = P(x)\Delta y \Delta z \quad 10\text{-}8$$

$$Q_{Out}(x + \Delta x) = P(x + \Delta x)\Delta y \Delta z \quad 10\text{-}9$$

$$Q_{net} = -[P(x + \Delta x) - P(x)]\Delta y \Delta z = -\frac{\partial P}{\partial x}\Delta x \Delta y \Delta z \quad 10\text{-}10$$

$$\rho_b = -\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z} = -\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{P}} \quad 10\text{-}11$$

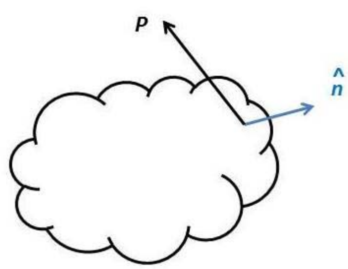


Figure 5 : An arbitrary volume of polarized material

If the polarisation changes with time, the moving charges are equivalent to a current. The current or charge flowing out per second ($\frac{\partial Q_{Out}}{\partial t}$) of an arbitrary volume is:

$$\frac{\partial Q_{Out}}{\partial t} = -\int \frac{\partial \rho_{Inside}}{\partial t} dV = -\int \frac{\partial \rho_b}{\partial t} dV = \frac{\partial}{\partial t}\int \underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{P}} dV = \frac{\partial}{\partial t}\int \underline{\boldsymbol{P}} \cdot d\underline{\boldsymbol{S}} = \int \underline{\boldsymbol{J}}_b \cdot d\underline{\boldsymbol{S}} \quad 10\text{-}12$$

Hence the current density is :

$$\underline{\boldsymbol{J}}_{\mathrm{b}} = \frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}} \qquad \text{10-13}$$

10.3 Microscopic description of dielectrics

$$\mathrm{m}\frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{q}E - \frac{\mathrm{m}v}{\tau} - \mathrm{m}\omega_0^2 x \qquad \text{10-14}$$

$$-\omega^2\tilde{x}\mathrm{m} = \mathrm{q}\tilde{E} + \frac{\mathrm{i}\omega\mathrm{m}\tilde{x}}{\tau} - \mathrm{m}\omega_0^2\tilde{x} \qquad \text{10-15}$$

$$\mathrm{m}\left(\omega_0^2 - \omega^2 - \frac{\mathrm{i}\omega}{\tau}\right)\tilde{x} = \mathrm{q}\tilde{E} \qquad \text{10-16}$$

Using Euler's equation

$$\frac{\tilde{x}}{\tilde{E}} = \frac{\mathrm{q}\left(\omega_0^2 - \omega^2 + \frac{\mathrm{i}\omega}{\tau}\right)}{\mathrm{m}\left((\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)} \qquad \text{10-17}$$

Writing the definition of polarization ($\widetilde{\mathrm{P}}$) and permittivity ($\tilde{\varepsilon}_r$) in complex form:

$$\tilde{P} = (\tilde{\varepsilon}_r - 1)\varepsilon_0\tilde{E} = \mathrm{Nq}\tilde{x} \qquad \text{10-18}$$

Which gives:

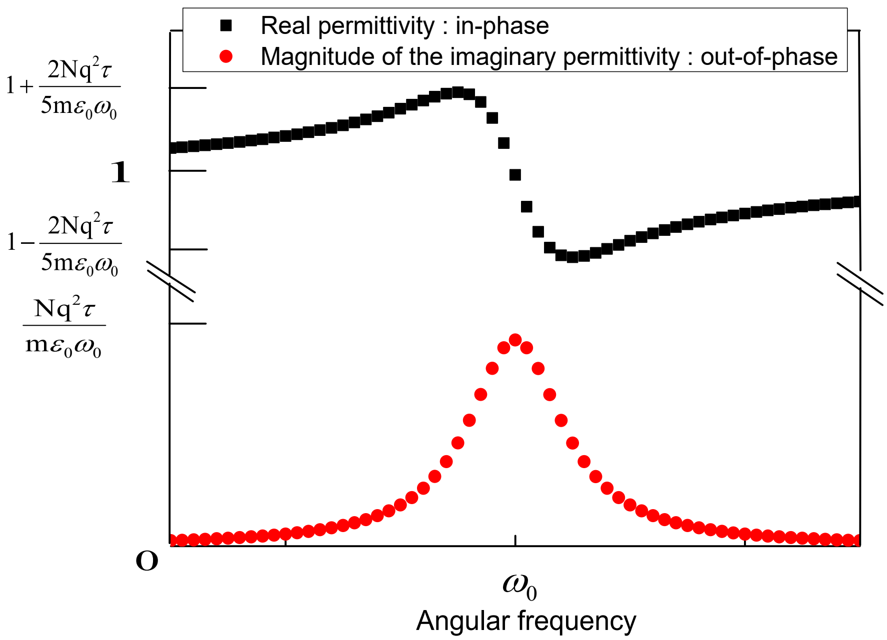
$$\tilde{\varepsilon}_r = \varepsilon_{\mathrm{real}} + i\varepsilon_{\mathrm{imaginary}} = 1 + \frac{\mathrm{Nq}}{\varepsilon_0}\frac{\tilde{x}}{\tilde{E}} \qquad \text{10-19}$$

Hence the real ($\varepsilon_{\mathrm{real}}$) and imaginary parts ($\varepsilon_{\mathrm{imaginary}}$) of the relative permittivity are:

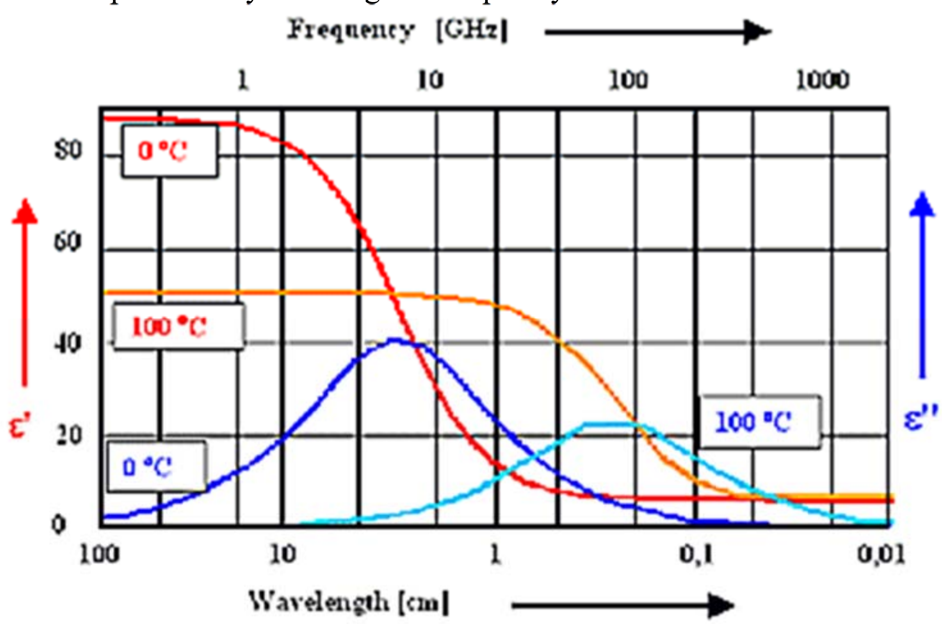
$$\varepsilon_{\mathrm{real}} = 1 + \frac{\mathrm{Nq}^2}{\mathrm{m}\varepsilon_0}\frac{(\omega_0^2 - \omega^2)}{\left((\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)} \qquad \text{10-20}$$

and

$$\varepsilon_{\mathrm{imaginary}} = \frac{\mathrm{Nq}^2}{\mathrm{m}\varepsilon_0\tau}\frac{\omega}{\left((\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right)} \qquad \text{10-21}$$



The variation of relative permittivity with angular frequency near a resonance.



The variation of real and imaginary permittivity with frequency for water at 0 ºC and 100 ºC.

10.4 The auxiliary field D.

We can write Maxwell I :

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_o} = \frac{\rho_{\text{free}} + \rho_{\text{dielectric}}}{\varepsilon_o} = \frac{\rho_{\text{free}} - \underline{\nabla} \cdot \underline{P}}{\varepsilon_o} \qquad 10\text{-}22$$

This leads to a definition for the electric displacement field $\underline{\boldsymbol{D}}$ where:

$$\underline{\boldsymbol{D}} = \varepsilon_o \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}} \ - \text{definition of } \underline{\boldsymbol{D}} \qquad 10\text{-}23$$

So Maxwell I becomes:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{D}} = \rho_{\text{free}} \qquad 10\text{-}24$$

$\underline{D}$ is useful shorthand commonly used in many calculations (no new Physics):

$$\underline{\boldsymbol{D}} = \varepsilon_o \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}} - \text{the defintion of } \underline{\boldsymbol{D}}\text{, the electric displacement field.} \qquad 10\text{-}25$$

11 Magnetic Materials`

11.1 Microscopic properties for magnetic materials - Ampere's model

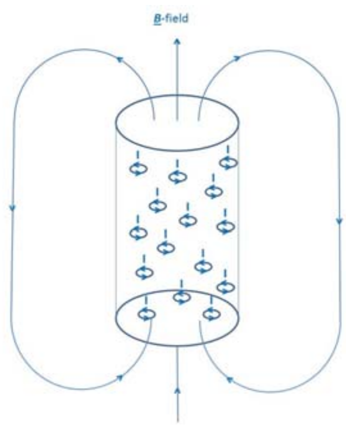


Figure 6 : The local and macroscopic fields produced by a magnetic material

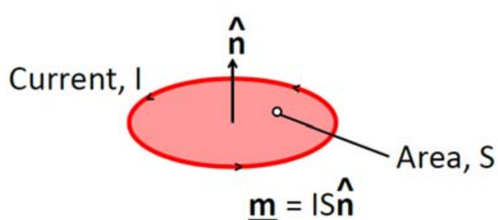


Figure 7 : A magnetic dipole moment.

The fundamental response of a magnetic material:

$$\underline{\boldsymbol{m}} = \mathrm{I}\mathrm{S}\hat{\boldsymbol{n}} - \text{ Definition of the magnetic dipole moment } (\boldsymbol{m}) \tag{11-1}$$

where I is the current flowing around a loop of area S.

$$\underline{\boldsymbol{M}} = \mathrm{N}\underline{\boldsymbol{m}} - \text{ Definition of the magnetization } (\boldsymbol{M}) \tag{11-2}$$

where N is the number of magnetic dipoles per unit volume.

11.2 Currents densities in magnetic materials

a) Bulk current density

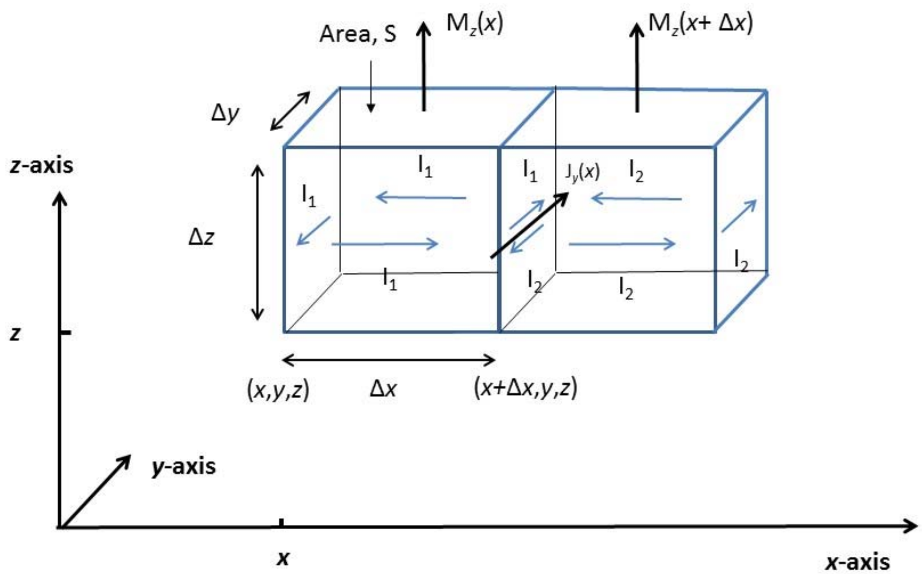


Figure 8 : Two magnetized cubes next to each other. The magnetization in each cube points in the z-direction because the current circulates in the x-y plane

$$m_z = M_z(x).\delta\mathrm{x}.\delta\mathrm{y}.\delta\mathrm{z} = I_1(x)A = I_1(x)\delta\mathrm{x}.\delta\mathrm{y} \tag{11-3}$$

$$I_1 = M_z\Delta\mathrm{z} \tag{11-4}$$

The current along the interface between the cubes (I_{net})

$$I_{\mathrm{net}} = M_z.\Delta z - (M_z(x+\Delta x).\Delta z = -\Delta M_z.\Delta z \tag{11-5}$$

$$I_{\text{net}} = -\frac{\Delta M_z}{\Delta x} \cdot \Delta x . \Delta z \qquad 11\text{-}6$$

$$J_y = \frac{I_{net}}{\Delta x . \Delta z} = -\frac{\Delta M_z}{\Delta x} \qquad 11\text{-}7$$

$$J_y = -\frac{\partial M_z}{\partial \mathrm{x}} \qquad 11\text{-}8$$

There is also a contribution to J_y if M_x varies, given by

$$J_y = \frac{\partial M_x}{\partial z} \qquad 11\text{-}9$$

.

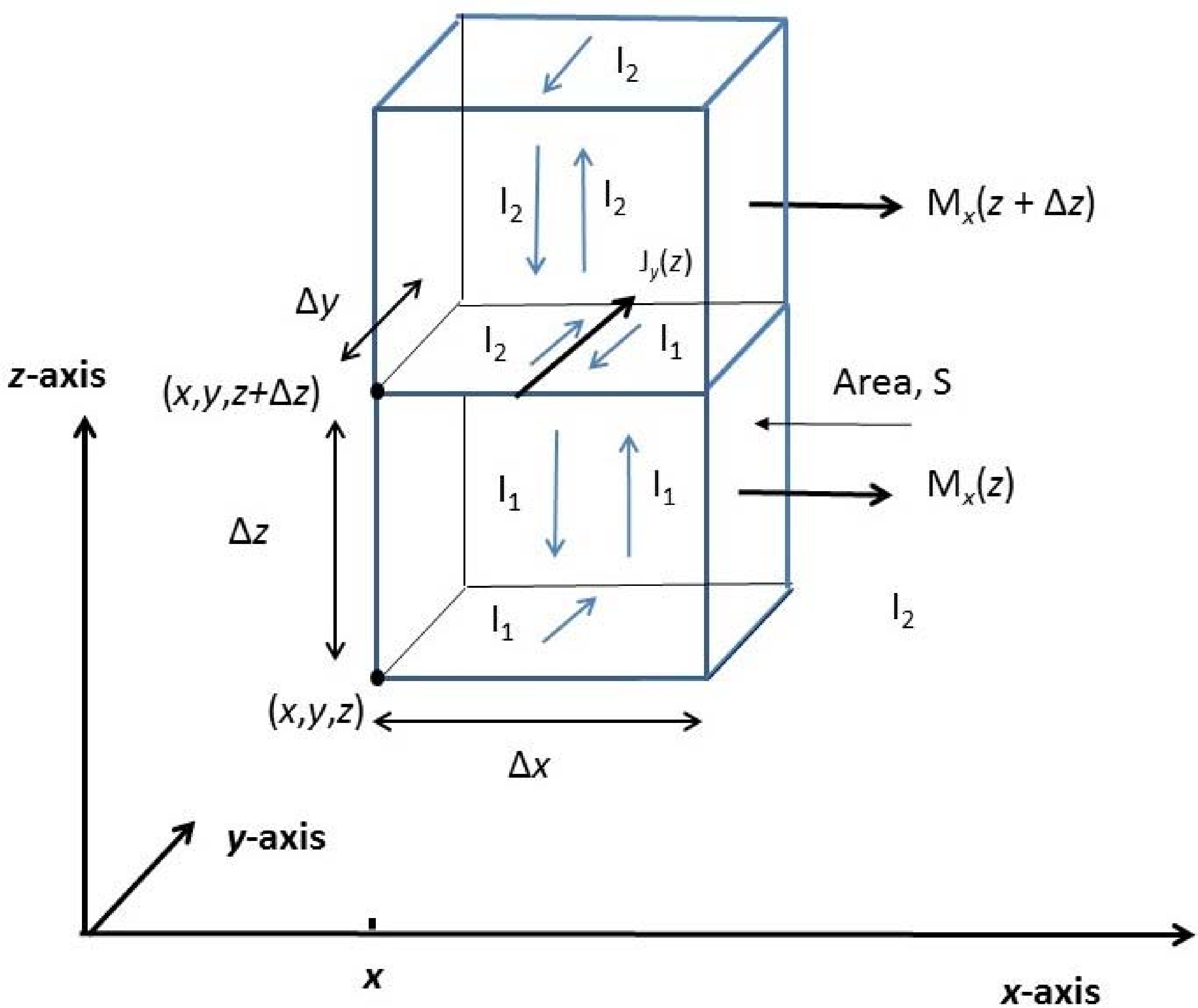


Figure 9 : Two magnetized cubes next to each other. The magnetization in each cube points in the x-direction because the current circulates in the *y-z* plane

Hence in 3D,

$$\underline{\boldsymbol{J}} = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \qquad 11\text{-}10$$

b) Surface Current Density

In a magnetized cylinder, the circulating currents in the bulk of the material cancel. The field from the material comes entirely from the circulating surface current.

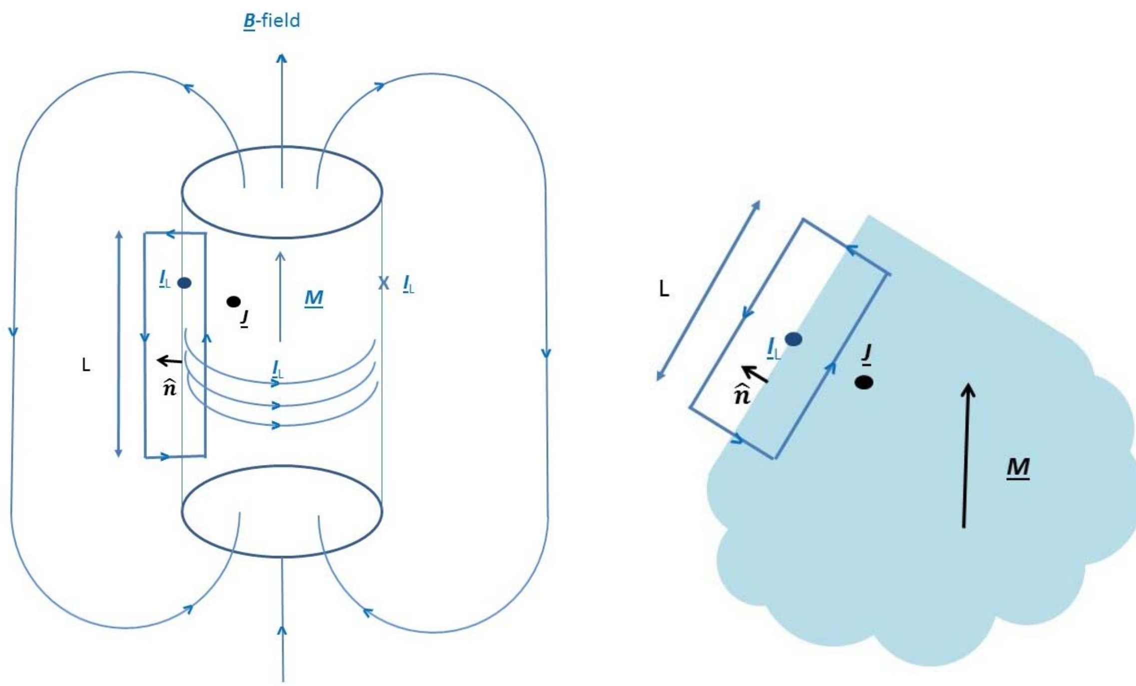


Figure 10 : The current flowing at the surface of a cylinder and an arbitrary shape. There is no bulk current density if the magnetisation is uniform.

For a uniformly magnetized material, there is a discontinuous change in $\underline{\boldsymbol{M}}$ at the surface. The current through the rectangle shown in the figure (normal to the board) can equally well be considered as a surface current per unit length or a bulk current ($\underline{\boldsymbol{J}}$) where

$$\int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = I_L L \qquad 11\text{-}11$$

Using

$$\underline{\boldsymbol{J}} = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \qquad 11\text{-}12$$

we have

$$\int \underline{\boldsymbol{J}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int \underline{\boldsymbol{M}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = ML = I_L L \qquad 11\text{-}13$$

$$\Rightarrow I_L = M \qquad 11\text{-}14$$

More generally

$$\Rightarrow \underline{\boldsymbol{I}}_{\mathrm{L}} = \underline{\boldsymbol{M}} \times \hat{\mathbf{n}} \qquad 11\text{-}15$$

11.3 The auxiliary field, H

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad 11\text{-}16$$

The total current density ($\underline{\boldsymbol{J}}$) through an ILIH material is

$$\underline{\boldsymbol{J}} = \underline{\boldsymbol{J}}_{\mathrm{free}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \qquad 11\text{-}17$$

where:

$$\underline{\boldsymbol{J}}_{\mathrm{free}} = \underline{\boldsymbol{J}}_{\mathrm{Ohms\ law}} + \underline{\boldsymbol{J}}_{\mathrm{Experimentalist}} \qquad 11\text{-}18$$

Maxwell IV can be rearranged as

$$\underline{\boldsymbol{\nabla}} \times \left(\underline{\boldsymbol{B}} - \mu_0 \underline{\boldsymbol{M}}\right) = \mu_0 \underline{\boldsymbol{J}}_{\mathrm{free}} + \mu_0 \frac{\partial}{\partial \mathrm{t}} \left(\underline{\boldsymbol{P}} + \varepsilon_0 \underline{\boldsymbol{E}}\right) \qquad 11\text{-}19$$

This leads to a definition for the magnetic field strength, $\underline{\boldsymbol{H}}$, defined by

$$\underline{\mathbf{B}} = \mu_o \left[\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}\right] - \text{ Definition of the magnetic field strength } (\boldsymbol{H}) \qquad 11\text{-}20$$

So we can rewrite MIV in terms of $\underline{\boldsymbol{D}}$, $\underline{\boldsymbol{H}}$ and ρ_{free}.

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}} = \underline{\mathbf{J}}_{\mathrm{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial \mathrm{t}} \qquad 11\text{-}21$$

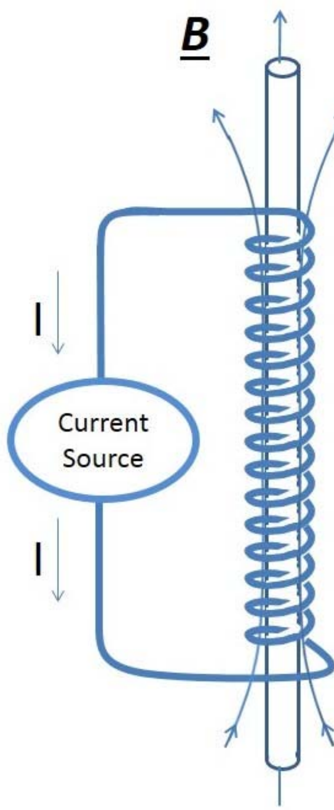


Figure 11 : A magnetized cylinder. The net field ($\underline{\boldsymbol{B}}$) is the sum of the magnetic field produced by the coil and the magnetic field produced by the material.

For a magnetized cylinder

$$\underline{\boldsymbol{B}}_{\text{net}} = \underline{\boldsymbol{B}}_{\text{applied}} + \mu_o \underline{\boldsymbol{M}} \qquad 11\text{-}22$$

$$\underline{\boldsymbol{M}} = \chi_H \underline{\boldsymbol{H}} - \text{Definition of the magnetic susceptibility } (\chi_H) \qquad 11\text{-}23$$

$$\underline{\boldsymbol{B}} = \mu_o \mu_r \underline{\boldsymbol{H}} - \text{ Definition of relative permeability } (\mu_r): \qquad 11\text{-}24$$

Note that μ_r can be 10^6.

Summary: Hence if we know $\underline{\boldsymbol{M}}$, we know the surface current per unit length $\underline{\boldsymbol{M}} \times \hat{\mathbf{n}}$ and the bulk current density $\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}}$.

12 The general dispersion relation

12.1 Aide-memoire

We can re-write Maxwell's equations using the definitions and derivations we have made (no new Physics):

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{D}} = \rho_{\mathrm{free}} \qquad 12\text{-}1$$

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0 \qquad 12\text{-}2$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} \qquad 12\text{-}3$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}} = \underline{\mathbf{J}}_{\mathrm{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial \mathrm{t}} \qquad 12\text{-}4$$

where $\underline{\boldsymbol{D}} = \varepsilon_0 \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$, and $\underline{\boldsymbol{B}} = \mu_0(\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}})$

and $\rho = \rho_{\mathrm{free}} - \underline{\boldsymbol{\nabla}}.\underline{\boldsymbol{P}}$ and $\underline{\boldsymbol{J}} = \underline{\boldsymbol{J}}_{\mathrm{free}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}}$

12.2 Propagation of transverse electromagnetic waves in materials

12.2.1 The general dispersion relation for an Infinite Linear-Isotropic-Homogeneous(ILIH) media

We can derive the general dispersion relation using Maxwell's fourth equation is:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad 12\text{-}5$$

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho_{\mathrm{free}} + \rho_{\mathrm{bound}}}{\varepsilon_0} = \frac{\rho_{\mathrm{free}}}{\varepsilon_0} - (\varepsilon_{\mathrm{r}} - 1)\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} \qquad 12\text{-}6$$

Rearranging, this gives:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = \frac{\rho_{\mathrm{free}}}{\varepsilon_{\mathrm{r}} \varepsilon_0} = 0 \qquad 12\text{-}7$$

Maxwell IV can be rewritten:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \left(\sigma_{\mathrm{n}} \underline{\boldsymbol{E}} + \frac{\partial \underline{\boldsymbol{P}}}{\partial \mathrm{t}} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{M}} \right) + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad 12\text{-}8$$

(Do not get confused: σ_{n} is the electrical conductivity from Ohm's law (Siemens or Ω^{-1}). σ is the surface charge density ($\mathrm{C.\ m^{-2}}$).

This can be rearranged as

$$\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{B}} - \mu_0 \underline{\boldsymbol{M}}) = \mu_0 \sigma_{\mathrm{n}} \underline{\boldsymbol{E}} + \mu_0 \frac{\partial}{\partial \mathrm{t}} (\underline{\boldsymbol{P}} + \varepsilon_0 \underline{\boldsymbol{E}}) \qquad 12\text{-}9$$

Using the materials properties definitions for ε_r , $\underline{\boldsymbol{H}}$, μ_r:

$$\underline{\boldsymbol{P}} = \varepsilon_o(\varepsilon_r - 1)\underline{\boldsymbol{E}} \qquad 12\text{-}10$$

$$\underline{\mathbf{B}} = \mu_o\left[\underline{\boldsymbol{H}} + \underline{\boldsymbol{M}}\right] = \mu_o\mu_r\underline{\boldsymbol{H}} \qquad 12\text{-}11$$

Maxwell IV can be written:

$$\underline{\boldsymbol{\nabla}} \times \frac{\underline{\mathbf{B}}}{\mu_r} = \mu_0\sigma_n\underline{\boldsymbol{E}} + \mu_0\frac{\partial}{\partial t}\left(\varepsilon_r\varepsilon_0\underline{\boldsymbol{E}}\right) \qquad 12\text{-}12$$

We substitute this into the curl of Maxwell's 3rd equation to find:

$$\underline{\boldsymbol{\nabla}} \times \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}\right) = -\frac{\partial}{\partial t}\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = -\frac{\partial}{\partial t}\left[\mu_o\mu_r\left(\sigma_n\underline{\boldsymbol{E}} + \varepsilon_o\varepsilon_r\frac{\partial \underline{\boldsymbol{E}}}{\partial t}\right)\right] \qquad 12\text{-}13$$

Hence using the vector identity $\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}) = \underline{\boldsymbol{\nabla}}\left(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}}\right) - \nabla^2\underline{\boldsymbol{E}}$ gives,

$$-\nabla^2\underline{\mathbf{E}} = -\underbrace{\mu_o\mu_r}_{\mu}\sigma_n\frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \underbrace{\mu_o\mu_r}_{\mu}\underbrace{\varepsilon_o\varepsilon_r}_{\varepsilon}\frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} \qquad 12\text{-}14$$

$$\Rightarrow \nabla^2\underline{\boldsymbol{E}} - \mu\sigma_n\frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \mu\varepsilon\frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} = 0 - \text{ Wave equation} \qquad 12\text{-}15$$

Equally taking the curl of MIV

$$\nabla^2\underline{\boldsymbol{B}} - \mu\sigma_n\frac{\partial \underline{\boldsymbol{B}}}{\partial t} - \mu\varepsilon\frac{\partial^2 \underline{\boldsymbol{B}}}{\partial t^2} = 0 \;\; - \text{ Wave equation} \qquad 12\text{-}16$$

We can substitute a plane wave solution into the wave equation.

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o\text{expi}\left(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t\right) \;\; - \text{ plane wave equation} \qquad 12\text{-}17$$

Hence we find the dispersion relation (which is by definition the relationship between ω and k).

$$-k^2 + i\omega\mu\sigma_n + \mu\varepsilon\omega^2 = 0 \qquad 12\text{-}18$$

$$k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma_n$$

$$- \text{General dispersion relation for an infinite, linear, isotropic, homogenous medium}$$

12.2.2 Electromagnetic waves in conducting/dielectric/magnetic materials

We now use the general dispersion relation to calculate the approximate decay length, wave-vector and wavelength of a wave of frequency ~ 10^{15} Hz propagating through tungsten - ρ_n = 1 mΩ.cm. Tungsten is neither magnetic nor a dielectric.

Solution

$$\nabla^2 \underline{\boldsymbol{E}} - \mu\sigma_n \frac{\partial \underline{\boldsymbol{E}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \underline{\boldsymbol{E}}}{\partial t^2} = 0$$

Assume that the plane wave has the form of $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \text{expi}(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t)$, then substitute it into the wave equation,

$$\Rightarrow (ik)^2 \underline{\boldsymbol{E}} - \mu\sigma_n(-i\omega)\underline{\boldsymbol{E}} - \mu\varepsilon(-i\omega)^2 \underline{\boldsymbol{E}} = 0$$

Using

$$k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma_n.$$

Given $f{\sim}10^{15}$Hz, $\rho_n = 1 \times 10^{-5}\Omega$m or $\sigma_n = \frac{1}{\rho_n} = 10^5\ \Omega^{-1}m^{-1}$, $\mu = \mu_0$, and $\varepsilon = \varepsilon_0$

$$k^2 = 4\pi \times 10^{-7} \times 8.85 \times 10^{-12} \times (2\pi \times 10^{15})^2 + i(2\pi \times 10^{15}) \times 4\pi \times 10^{-7} \times 10^5$$

$$k^2 = 4.4 \times 10^{14} + i7.9 \times 10^{14}.$$

Note: Tungsten cannot be characterised as either a good insulator or a good conductor

Draw Argand diagram:

$$\tan\phi = \frac{7.9}{4.4} \Rightarrow \phi = 60.88^\circ, \qquad \phi/2 = 30.44^\circ$$

$$|k| = [(4.4 \times 10^{14})^2 + (7.9 \times 10^{14})^2]^{\frac{1}{4}} = 3.0 \times 10^7$$

hence,

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \text{expi}((2.59 \times 10^7 + i1.52 \times 10^7)x - \omega t)$$

$$= \underline{\boldsymbol{E}}_o \text{expi}(2.59 \times 10^7 x - \omega t)\exp(-1.52 \times 10^7 x)$$

Taking either the real (or imaginary) part:

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \cos(2.59 \times 10^7 x - \omega t)\exp(-1.52 \times 10^7 x)$$

The wave vector is the real part of k, $2.59 \times 10^7 m^{-1}$ and the wavelength is $\lambda = \frac{2\pi}{2.59\times10^7} \approx 2.4 \times 10^{-7}$m.
The decay length is the reciprocal of imaginary part of k, $\delta = \frac{1}{k_{imaginary}} = \frac{1}{1.52\times10^7} \approx 66$nm.

ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 16+17 (Rev 3)

13 Radiation and communication – Poynting vector

13.1 The Poynting Vector

The direction of propagation for an electromagnetic waves is $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}$. Poynting suggested that the instantaneous power per unit area for electromagnetic waves is given by $\underline{\boldsymbol{N}}$ where,

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \qquad 13\text{-}1$$

[units E = $\mathrm{Vm^{-1}}$, H = $\mathrm{Am^{-1}}$, N = $\mathrm{VAm^{-2}} = \mathrm{Wm^{-2}}$].

General proof:

$$P_{radiation} = \int \underline{\boldsymbol{N}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = \int \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}\right) \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad 13\text{-}2$$

$$P_{radiation} = \int \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{N}} \mathrm{dV} = \int \underline{\boldsymbol{\nabla}} \cdot \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}\right) \mathrm{dV} \qquad 13\text{-}3$$

$$\underline{\boldsymbol{\nabla}} \cdot \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}\right) = \underline{\boldsymbol{H}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}\right) - \underline{\boldsymbol{E}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}}\right) \qquad 13\text{-}4$$

$$P_{radiation} = \int \underline{\boldsymbol{H}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}}\right) \mathrm{dV} - \int \underline{\boldsymbol{E}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}}\right) \mathrm{dV} \qquad 13\text{-}5$$

From Maxwell's equations for an infinite linear-isotropic-homogeneous medium we have:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} = -\mu_o \mu_r \frac{\partial \underline{\boldsymbol{H}}}{\partial \mathrm{t}} \qquad 13\text{-}6$$

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}} = \underline{\boldsymbol{J}}_{\mathrm{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial \mathrm{t}} = \underline{\boldsymbol{J}}_{\mathrm{free}} + \varepsilon_r \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \qquad 13\text{-}7$$

and

$$P_{radiation} = \int \underline{\boldsymbol{H}} \cdot \left(-\mu_o \mu_r \frac{\partial \underline{\boldsymbol{H}}}{\partial \mathrm{t}}\right) \mathrm{dV} - \int \underline{\boldsymbol{E}} \cdot \left(\underline{\boldsymbol{J}}_{\mathrm{free}} + \varepsilon_r \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}}\right) \mathrm{dV} \qquad 13\text{-}8$$

$$P_{radiation} = -\frac{\partial}{\partial \mathrm{t}} \int \left(\frac{1}{2}\mu_r \mu_0 \mathrm{H}^2 + \frac{1}{2}\varepsilon_r \varepsilon_0 \mathrm{E}^2\right) \mathrm{dV} - \int \underline{\boldsymbol{E}} \cdot \underline{\boldsymbol{J}}_{\mathrm{free}} \mathrm{dV} \qquad 13\text{-}9$$

The derivation is a self-consistent statement of the conservation of energy so $\boldsymbol{N}$ correctly represents the instantaneous power per unit area.

13.2 Maxwell's equations rewritten using potentials

For time independent problems, we use;

$$\underline{\boldsymbol{E}} = -\underline{\boldsymbol{\nabla}}\mathrm{V} \qquad 13\text{-}10$$

where V is the electrostatic potential.

In general, the electric potential, V , is defined through,

$$\underline{\boldsymbol{E}} = -\underline{\boldsymbol{\nabla}}V - \frac{\partial \underline{\boldsymbol{A}}}{\partial \mathrm{t}} \;\; \text{(Definition of electric potential, V).} \qquad 13\text{-}11$$

Similarly, the magnetic vector potential, $\underline{\boldsymbol{A}}$, is defined as,

$$\underline{\boldsymbol{B}} = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}} \text{ (Defintion of magnetic vector potential)} \qquad 13\text{-}12$$

Consider Maxwell II,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{B}} = 0 \;\; \text{since } \underline{\boldsymbol{\nabla}} \cdot \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}\right) = 0 \qquad 13\text{-}13$$

Consider Maxwell III:

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = \underline{\boldsymbol{\nabla}} \times \left(-\underline{\boldsymbol{\nabla}}\mathrm{V} - \frac{\partial \underline{\boldsymbol{A}}}{\partial \mathrm{t}}\right) = -\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}}\mathrm{V} - \frac{\partial}{\partial \mathrm{t}}\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}$$

$$= -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} \text{ since } \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}}\mathrm{V} \;=\; 0 \text{ – note similarity with vector algebra.} \qquad 13\text{-}14$$

Hence, if we rewrite $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{E}}$ in terms of $\underline{\boldsymbol{A}}$ and V, Maxwell II and III are *automatically solved*. Maxwell's four equations are thus reduced to two equations:

$$-\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}}\mathrm{V} - \frac{\partial}{\partial \mathrm{t}}\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = \frac{\rho}{\varepsilon_o} \;-\; \text{Maxwell I} \qquad 13\text{-}15$$

$$\underline{\boldsymbol{\nabla}} \times \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{A}}\right) = \mu_o \underline{\boldsymbol{J}} + \mu_o \varepsilon_o \frac{\partial}{\partial \mathrm{t}}\left[-\underline{\boldsymbol{\nabla}}\mathrm{V} - \frac{\partial \underline{\boldsymbol{A}}}{\partial \mathrm{t}}\right] - \text{Maxwell IV} \qquad 13\text{-}16$$

The Lorentz condition constrains $\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}}$,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{A}} = -\mu_o \varepsilon_o \frac{\partial \mathrm{V}}{\partial \mathrm{t}} \qquad 13\text{-}17$$

Maxwell I becomes

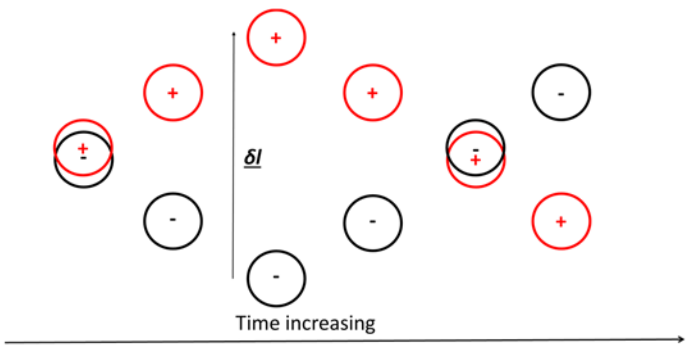
$$-\nabla^2 \mathrm{V} + \mu_o \varepsilon_o \frac{\partial^2 \mathrm{V}}{\partial \mathrm{t}^2} = \frac{\rho}{\varepsilon_o} - \text{Wave equation for V} \qquad 13\text{-}18$$

and Maxwell IV becomes

$$-\nabla^2 \underline{\boldsymbol{A}} + \mu_o \varepsilon_o \frac{\partial^2 \underline{\boldsymbol{A}}}{\partial t^2} = \mu_o \underline{\boldsymbol{J}} \text{ - Wave equation for } \underline{\boldsymbol{A}} \qquad 13\text{-}19$$

13.3 Hertzian dipole/ oscillating electric dipole

$$I(t)\underline{\boldsymbol{\delta l}} = \omega \underline{\boldsymbol{p}}(t) \qquad 13\text{-}20$$



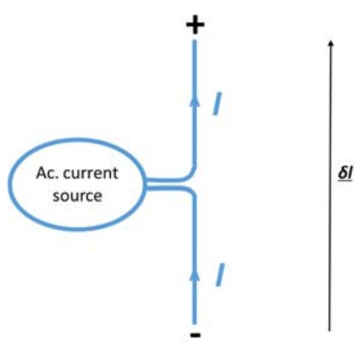


Figure 1 : A Hertzian dipole which is an alternating electric dipole moment. The dipole is equivalent to a length of wire in which a time dependent alternating current is flowing.

We can start to solve the wave equation for $\underline{\boldsymbol{A}}$ by considering (static) solutions for:

$$-\nabla^2 \underline{\boldsymbol{A}} = \mu_o \underline{\boldsymbol{J}} \qquad 13\text{-}21$$

From electrostatics, using $\underline{\boldsymbol{E}} = -\underline{\boldsymbol{\nabla}} V$, we know the solution for:

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{E}} = -\nabla^2 \mathrm{V} = \frac{\rho}{\varepsilon_o} \qquad 13\text{-}22$$

is

$$\mathrm{V} = \frac{\mathrm{Q}}{4\pi\varepsilon_o r} \text{ where } Q = \int \rho \, dV \qquad 13\text{-}23$$

By inspection, the static solution for $\underline{\boldsymbol{A}}$ is:

$$\underline{\boldsymbol{A}} = \frac{\mu_o I \underline{\boldsymbol{\delta l}}}{4\pi r} \qquad \text{where } I\underline{\boldsymbol{\delta l}} = \int \underline{\boldsymbol{J}}\, dV \tag{13-24}$$

Recognising that if we set $\underline{\boldsymbol{J}}$ to be zero, we find oscillatory solutions for the complex wave equation for $\underline{\widetilde{\boldsymbol{A}}}$**.** Hence the general solution to the complex wave equation for $\underline{\widetilde{\boldsymbol{A}}}$ is

$$\underline{\widetilde{\boldsymbol{A}}} = \frac{\mu_o I_0 \underline{\boldsymbol{\delta l}}}{4\pi r} e^{i(kr - wt)} \tag{13-25}$$

where $\tilde{I}(t) = I_0 e^{-i(wt)}$. In spherical coordinates $\underline{\boldsymbol{\delta l}} = \delta \mathrm{l}\,(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$. Using $\underline{\widetilde{\boldsymbol{B}}} = \underline{\boldsymbol{\nabla}} \times \underline{\widetilde{\boldsymbol{A}}}$ so we have (leaving it as a homework problem)

$$\underline{\tilde{\boldsymbol{B}}}(r,\theta,\phi) = \left(\tilde{B}_{\mathrm{r}}\hat{\mathbf{r}} + \tilde{B}_\theta\hat{\boldsymbol{\theta}} + \tilde{B}_\phi\hat{\boldsymbol{\phi}}\right) = \frac{\mu_o I_0 \delta \mathrm{l}}{4\pi r^2}(1-ikr)\sin(\theta)e^{i(kr-\omega t)}\hat{\boldsymbol{\phi}} \qquad \text{13-26}$$

$$\underline{\boldsymbol{B}}_{\mathrm{Near}} = \frac{\mu_o I_0 \delta \mathrm{l}}{4\pi r^2}\sin(\theta)\cos(kr-\omega t)\hat{\boldsymbol{\phi}} \qquad \text{13-27}$$

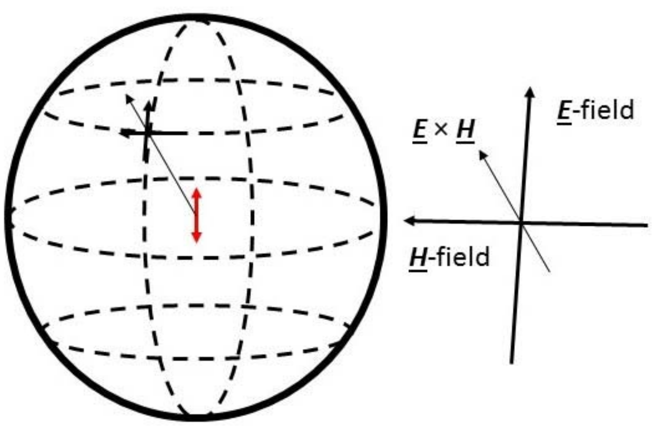
$$\underline{\boldsymbol{B}}_{\mathrm{Far}} = \frac{\mu_o I_0 \delta \mathrm{l}}{4\pi r}k\sin(\theta)\sin(kr-\omega t)\hat{\boldsymbol{\phi}} \qquad \text{13-28}$$

$$\underline{\tilde{\boldsymbol{E}}} = \frac{i}{\mu_0\varepsilon_0\omega}\,\underline{\boldsymbol{\nabla}}\times\underline{\tilde{\boldsymbol{B}}} \qquad \text{13-29}$$

$$\begin{aligned}\underline{\tilde{\boldsymbol{E}}} = \frac{\mu_o I_0 \delta \mathrm{l}}{2\pi}\frac{\omega(i+kr)\cos(\theta)}{k^2r^3}e^{i(kr-\omega t)}\hat{\mathbf{r}} \\ + \frac{\mu_o I_0 \delta \mathrm{l}}{4\pi}\frac{\omega(i+kr-ik^2r^2)\sin(\theta)}{k^2r^3}e^{i(kr-\omega t)}\hat{\boldsymbol{\theta}}\end{aligned} \qquad \text{13-30}$$

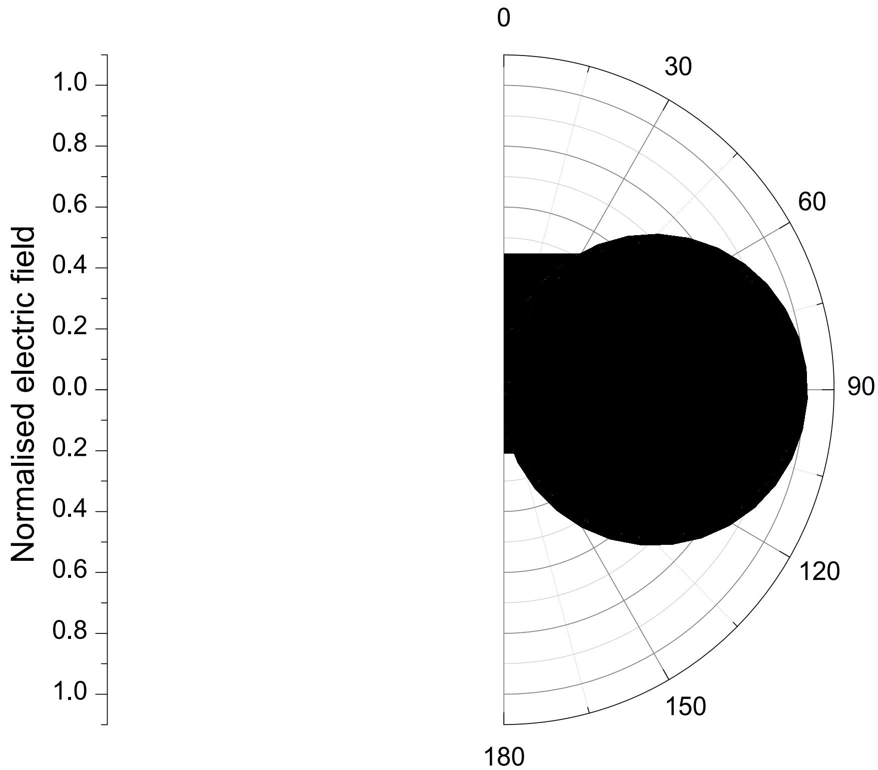
$$\underline{\boldsymbol{E}}_{\mathrm{Near}} = -\frac{\mu_o I_0 \delta \mathrm{l}}{2\pi r^3}\frac{\omega}{k^2}\cos(\theta)\sin(kr-\omega t)\hat{\boldsymbol{r}} - \frac{\mu_o I_0 \delta \mathrm{l}}{4\pi r^3}\frac{\omega}{k^2}\sin(\theta)\sin(kr-\omega t)\hat{\boldsymbol{\theta}} \qquad \text{13-31}$$

$$\underline{\boldsymbol{E}}_{\mathrm{Far}} = \frac{\mu_o I_0 \delta \mathrm{l}}{4\pi r}\omega\sin(\theta)\sin(kr-\omega t)\hat{\boldsymbol{\theta}} \qquad \text{13-32}$$



E points up and down, B left to right horizontally

Figure 2 : The relative directions of the radiation fields of a Hertzian dipole. The red arrow at the centre is the Hertzian dipole. The radiating fields are tangential to the surface of a sphere centred on the dipole. For an isotropic medium the $\underline{\boldsymbol{H}}$-field is parallel to the $\underline{\boldsymbol{B}}$-field.



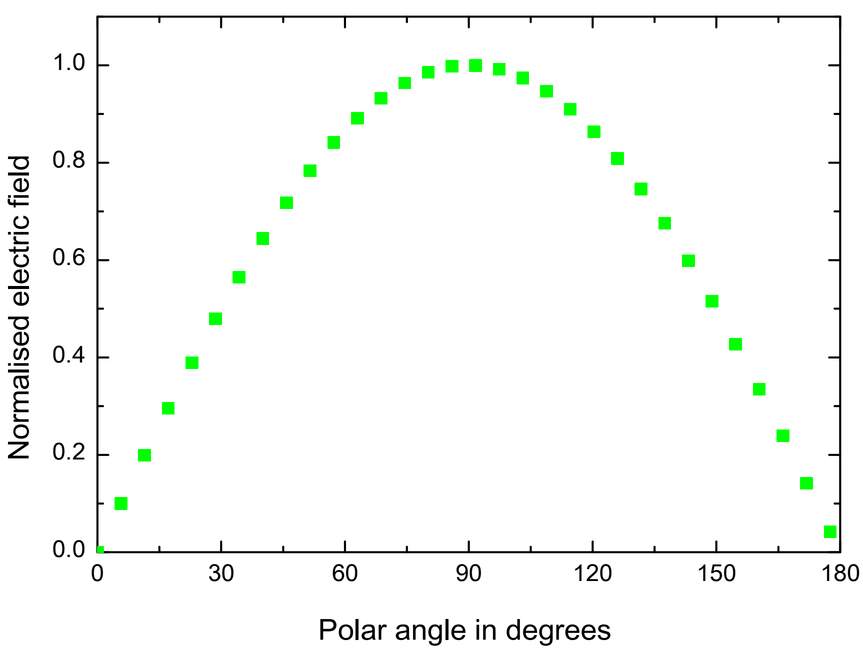
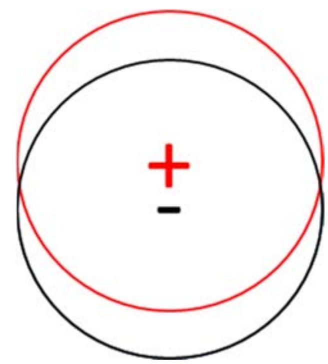
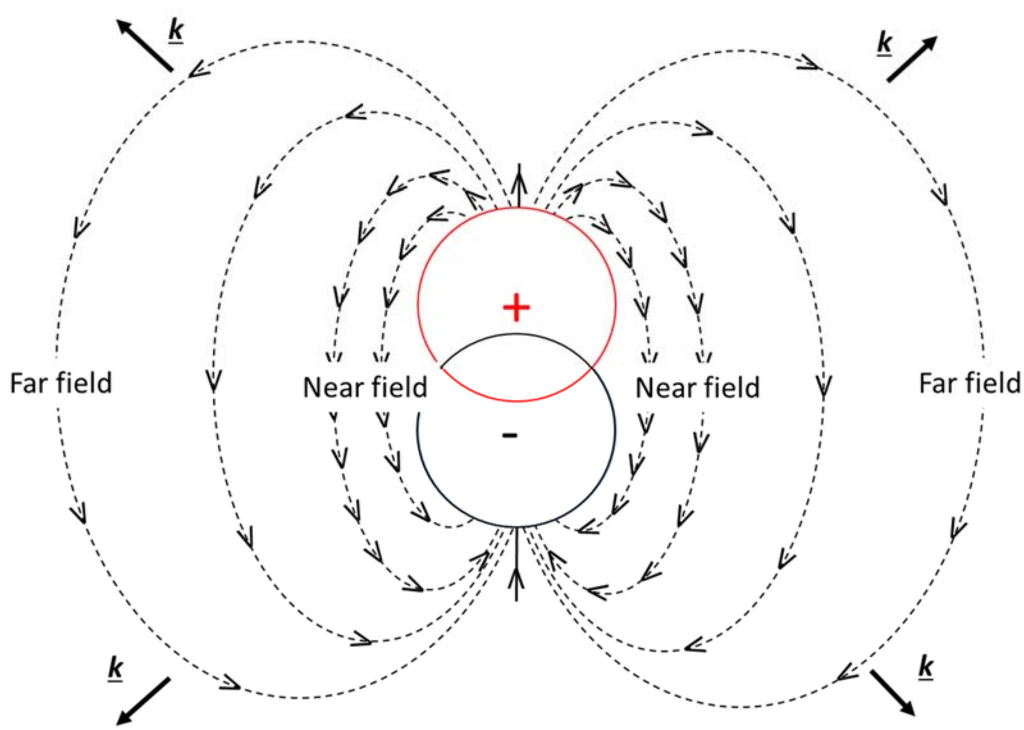


Figure 3 : The polar diagram for a Hertzian dipole. The solid curved line (i.e. circular) gives the relative strengths of the radiation field at different points on the surface of a sphere centred on the dipole. The dipole lies along the z-axis and the field is independent of the azimuthal angle. There is no radiation produced along the z-axis.

(i)



(ii)



(iii)

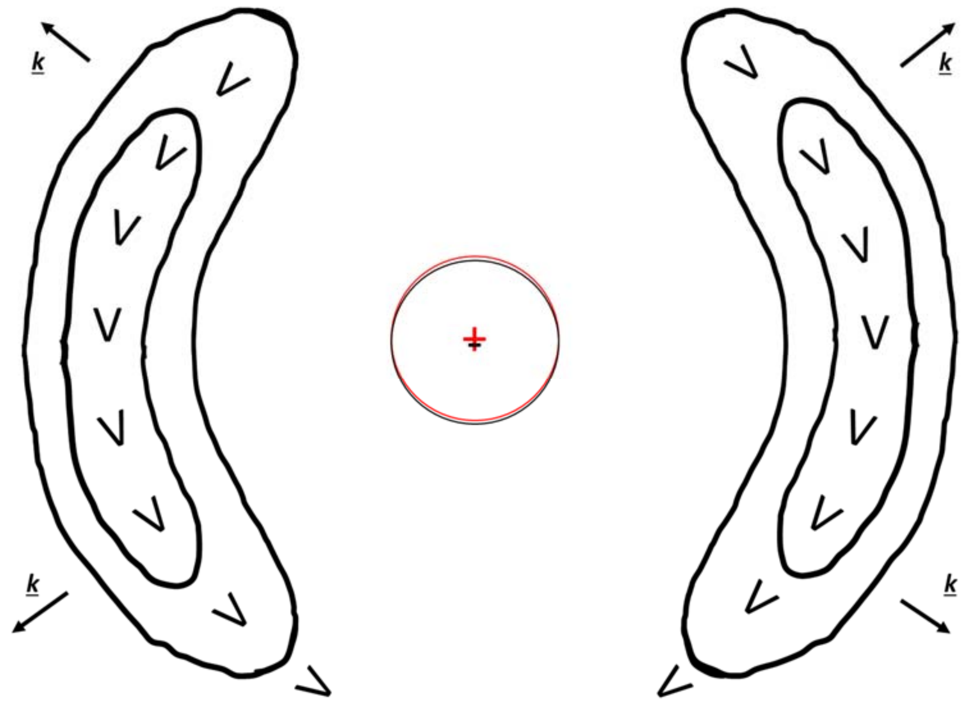


Figure 4 : The production of the first half of a wave in snapshots. We consider the initial state where the current is at its maximum value as the two charges pass each other. We chose not to draw all fields prior to the initial state. (i) The charges separate and there are E-fields (not shown) very close to the charges. (ii) The v-shaped chevrons show the direction of the electric field. For any snapshot in time, the charges behave like an extended dipole with both near field and far field terms contributing to the total E-field. (iii) The first half wave separates and radiates from the dipole. The solid lines are contours. The E-field points in the positive theta direction throughout the half-wave.

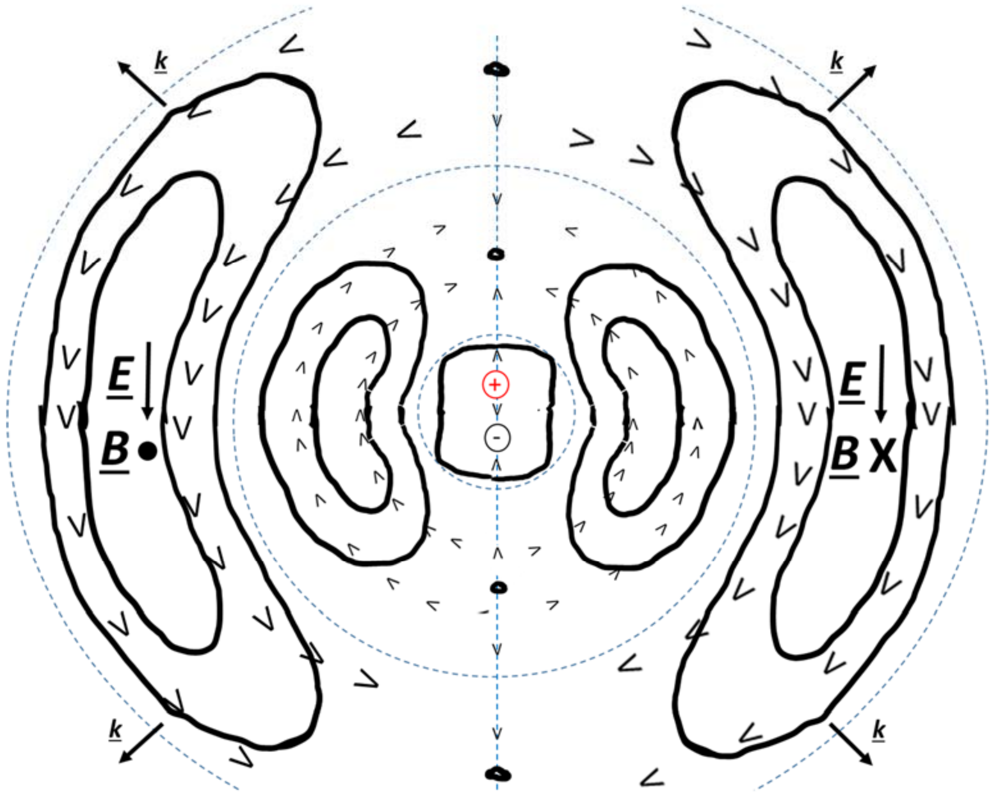
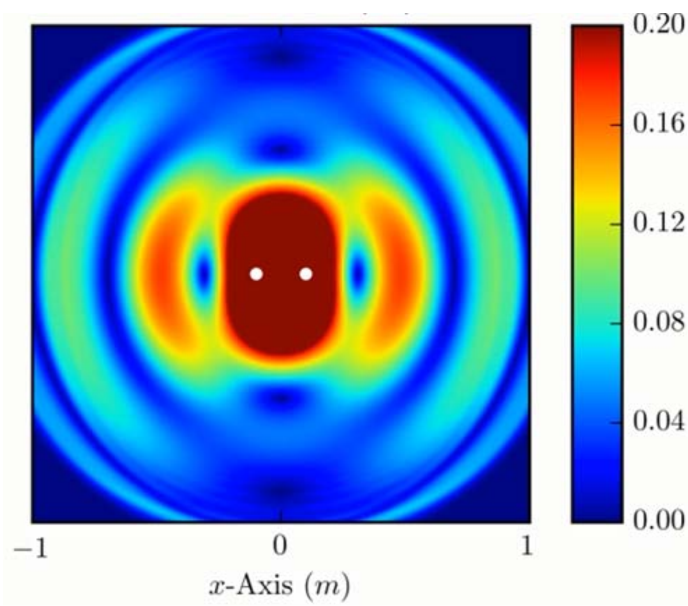


Figure 5 : The E-fields and B-fields in the far field. The complete wave is shown propagating away. The solid lines are contours. The E-field points in the direction of the v-shaped chevrons. Each magnetic field line is a circle with a dipole along its axis. The cross product E×B is directed away from the dipole (i.e. in the r-direction) at all points in space where and the right-hand screw rule for E×B gives the direction of B with respect to E.



A visualisation produced by Hooper:
http://community.dur.ac.uk/superconductivity.durham/Teaching%20Materials.html

13.3.1 The angular dependence of the power radiated by a Hertzian dipole

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \tag{13-33}$$

The angular dependence of time averaged power per unit area is:

$$\underline{\boldsymbol{N}}_{\text{time averaged}}(\theta) = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} = \frac{1}{2}\frac{\mu_0 c {I_0}^2 (\delta l)^2 \sin^2\theta}{4r^2\lambda^2}\underline{\hat{\boldsymbol{r}}} \tag{13-34}$$

13.3.2 The time averaged power (P_{Total}) radiated by a Hertzian dipole.

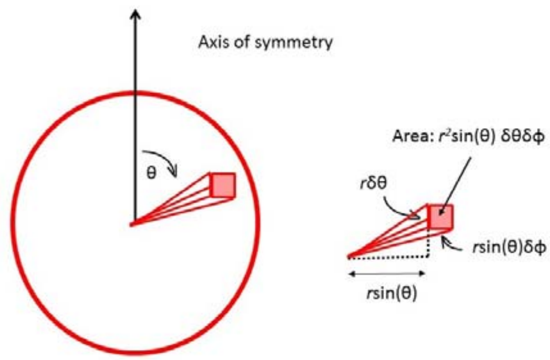


Figure 6 : An element of area in spherical coordinates.

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \tag{13-35}$$

$$\begin{aligned} P_{\text{Total}} &= \int \frac{1}{2}\frac{\mu_0 c {I_0}^2 (\delta l)^2 \sin^2\theta}{4r^2\lambda^2} r^2 \sin\theta d\theta d\phi \\ &= \frac{\mu_0 c {I_0}^2 (\delta l)^2}{8\lambda^2}\int_0^{\pi} \sin^3\theta d\theta \int_0^{2\pi} d\phi \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \frac{4}{3}\dots\dots\dots 2\pi \end{aligned} \tag{13-36}$$

$$P_{\text{Total}} = \frac{\mu_0 c\pi}{3}\left(\frac{\delta l}{\lambda}\right)^2 {I_0}^2 \tag{13-37}$$

<u>Antennae:</u>

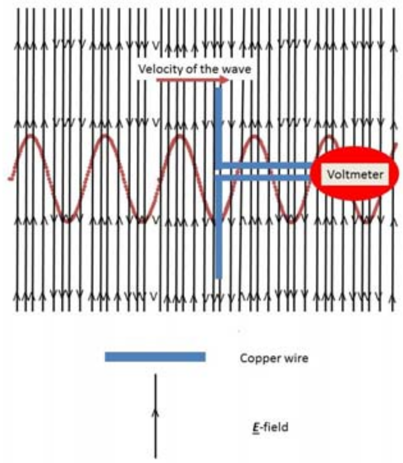


Figure 7 : An electric dipole antenna for detecting electromagnetic waves. The alternating electric field of the incoming wave produces an alternating current in the antenna.

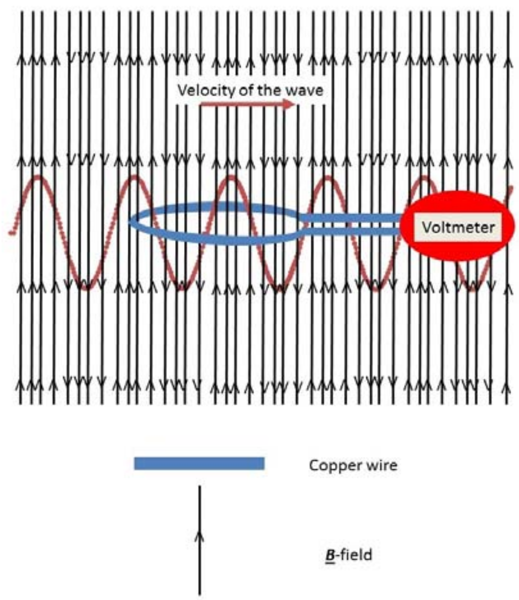


Figure 8 : A loop antenna for detecting electromagnetic radiation. The alternating magnetic flux through the loop due to the magnetic field of the radiation induces an alternating current in the loop.

ELECTROMAGNETISM

Professor D P Hampshire – Summary notes for lectures 18+19+20 (Rev 3)

14 Electromagnetic fields and waves crossing interfaces

Pauli: "God made the bulk, but the devil has the surfaces"

14.1 Boundary Conditions across the interfacial plane between two dielectrics

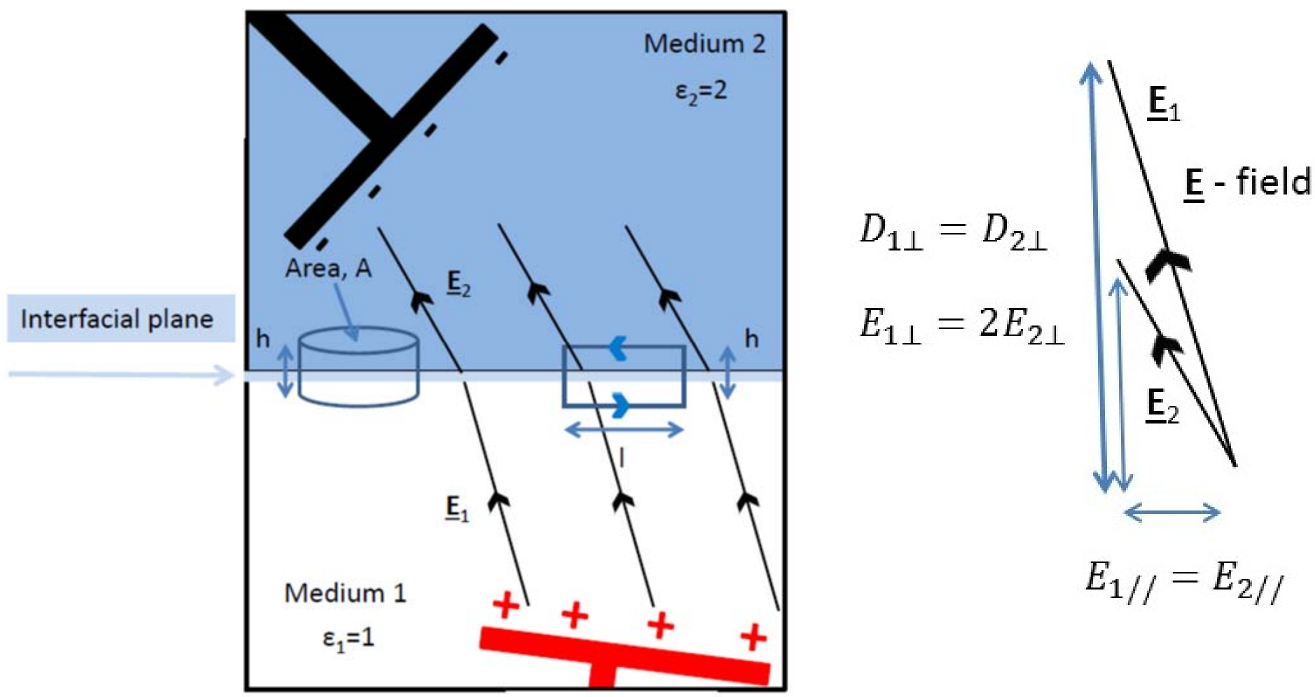


Figure 1 : (LHS) Two charged capacitor plates immersed in two dielectrics. The direction and magnitude of a dc. electric field is different from one side of the dielectric interface to the other. The arrows give the direction of the dc. E-fields on either side of the interfacial plane (Nb. This is not a ray diagram). (RHS). Maxwell's equations require that the displacement field orthogonal to the interfacial plane is continuous (i.e. $D_{1\perp} = D_{2\perp}$ or equivalently $\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$) and that the electric field parallel to the interfacial plane is continuous (i.e. $E_{1//} = E_{2//}$).

A) For an insulating material Maxwell I

$$\underline{\nabla} \cdot \underline{D} = \rho_{free} = 0 \qquad 14\text{-}1$$

$$\oint \underline{\boldsymbol{D}}_1 \cdot d\underline{\boldsymbol{S}} = -\oint \underline{\boldsymbol{D}}_2 \cdot d\underline{\boldsymbol{S}} \Rightarrow D_{1\perp} = D_{2\perp} \qquad 14\text{-}2$$

or equivalently

$$\varepsilon_1 E_{1_\perp} = \varepsilon_2 E_{2\perp}. \quad (\underline{\boldsymbol{D}} = \varepsilon_o \varepsilon_r \underline{\boldsymbol{E}}) \tag{14-3}$$

B) Using Maxwell III (consider path B)

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} \tag{14-4}$$

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = -\frac{\partial}{\partial \mathrm{t}} \int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \tag{14-5}$$

$$\mathrm{E}_{1//}\mathrm{l} - \mathrm{E}_{2//}\mathrm{l} = -\frac{\partial \phi}{\partial \mathrm{t}} = 0 \tag{14-6}$$

The loop area tends to zero

$$\Rightarrow \mathrm{E}_{1//} = \mathrm{E}_{2//} \tag{14-7}$$

$\mathrm{D}_\perp$(or equivalently $\varepsilon_r \mathrm{E}_\perp$) and $\mathrm{E}_{//}$ are continuous across the interfacial plane between two dielectrics.

14.2 Boundary Conditions across the interfacial plane between two magnetic materials

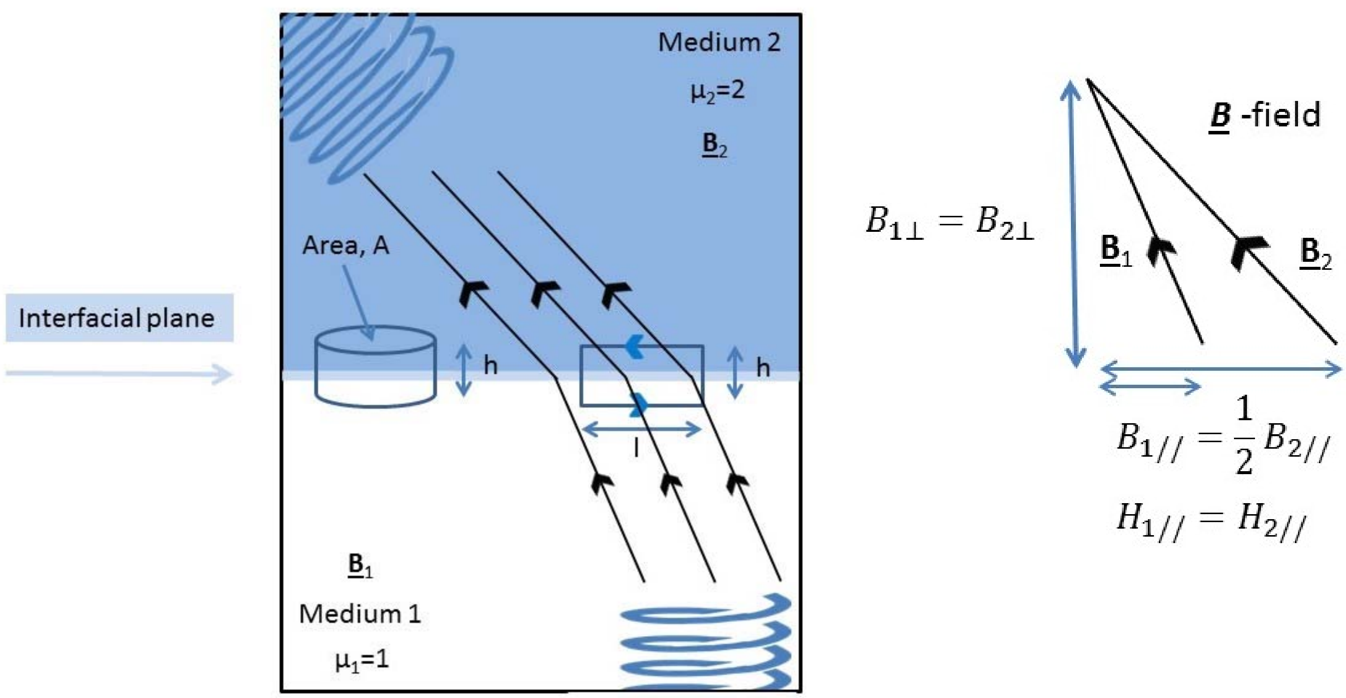


Figure 2 Two magnets immersed in two magnetic materials. The direction and magnitude of a dc. magnetic field from opne side of the magnetic interface to the other. The arrows give the direction of the dc. B-field on either side of the interfacial plane (Nb. This is <u>not</u> a ray diagram). (RHS). Maxwell's equations require that the magnetic field orthogonal to the interfacial plane is continuous (i.e. $\mathrm{B}_{1_\perp} = \mathrm{B}_{2\perp}$) and that the field strength parallel to the interfacial plane is continuous (i.e. $\mathbf{H}_{1//} = \mathbf{H}_{2//}$ or equivalently $\mathrm{B}_{1//}/\mu_1 = \mathrm{B}_{2//}/\mu_2$)).

Maxwell II

$$\int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = 0 \Rightarrow \mathrm{B}_{1\perp} = \mathrm{B}_{2\perp} \tag{14-8}$$

Maxwell IV

$$\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}}_{\text{total}} + \mu_0 \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial \mathrm{t}} \tag{14-9}$$

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{H}} = \underline{\boldsymbol{J}}_{\mathbf{free}} + \frac{\partial \underline{\boldsymbol{D}}}{\partial \mathrm{t}} \tag{14-10}$$

$$\int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \oint \underline{\boldsymbol{H}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \underbrace{\int \underline{\boldsymbol{J}}_{\text{free}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \int \frac{\partial \boldsymbol{\nabla}.\underline{\boldsymbol{D}}}{\partial \mathrm{t}} \cdot \mathrm{dV}}_{\substack{-\text{no free current density} \\ -\text{no free charge density}}} \tag{14-11}$$

$$\Rightarrow \mathrm{H}_{1//}\mathrm{l} = \mathrm{H}_{2//}\mathrm{l} \quad \Rightarrow \mathrm{H}_{1//} = \mathrm{H}_{2//} \tag{14-12}$$

$\therefore$ $\mathrm{B}_{\perp}$, $\mathrm{H}_{//}$ are continuous across the boundary.

14.3 The Laws of geometrical optics

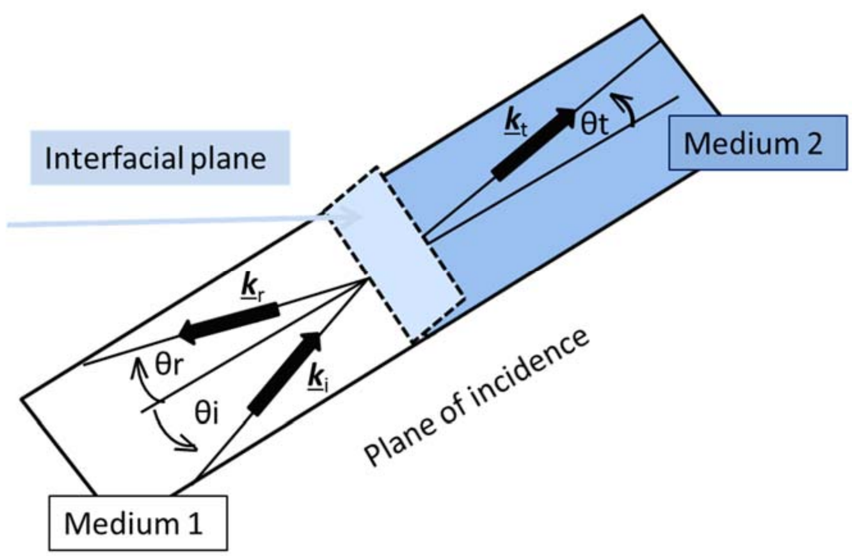


Figure 3 : An electromagnetic wave incident on an interface – it is partially reflected and partially transmitted.

$$\underline{\tilde{E}}_i(\underline{r}, t) = \underline{\mathbf{E}}_{0i}\exp i(\underline{\mathbf{k}}_i \cdot \underline{r} - \omega_i t) \qquad 14\text{-}13$$

$$\underline{\tilde{E}}_r(\underline{r}, t) = \underline{\mathbf{E}}_{0r}\exp i(\underline{\mathbf{k}}_r \cdot \underline{r} - \omega_r t) \qquad 14\text{-}14$$

$$\underline{\tilde{E}}_t(\underline{r}, t) = \underline{\mathbf{E}}_{0t}\exp i(\underline{\mathbf{k}}_t \cdot \underline{r} - \omega_t t) \qquad 14\text{-}15$$

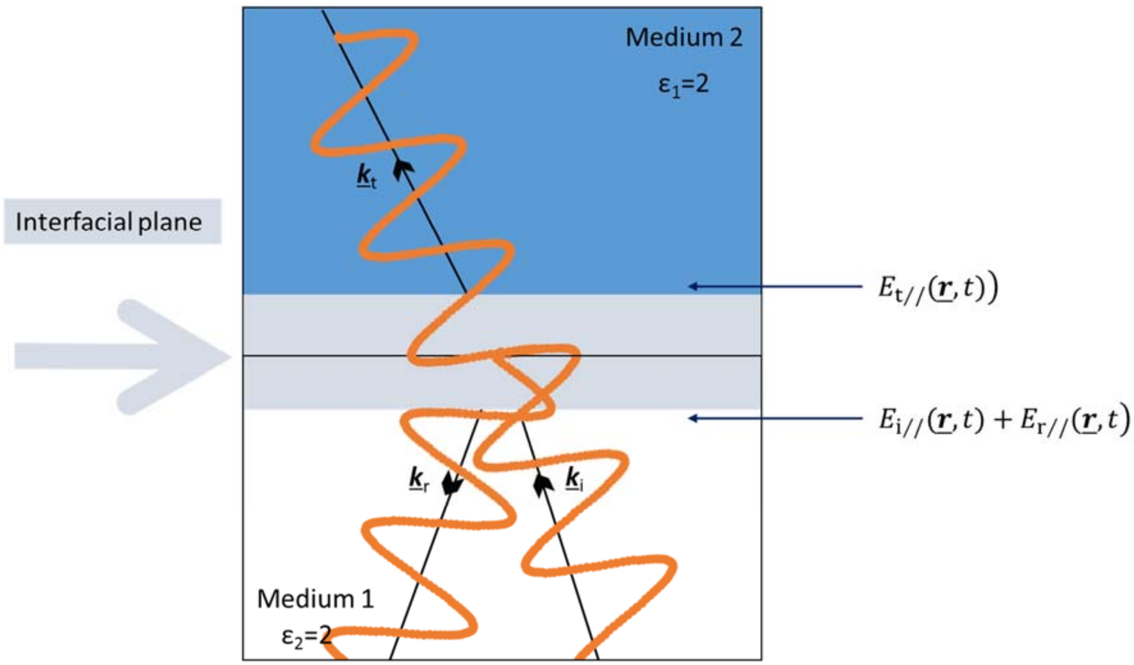


Figure 4 : An electromagnetic wave incident on an interface. After the incident and reflected waves have left the interfacial plane, they superimpose in medium 1. The transmitted wave is the only wave in medium 2.

Across the interfacial plane, the general condition for the continuity of $E_{//}$ (i.e E parallel to the interface) requires:

$$E_{i//}(\underline{r}, t) + E_{r//}(\underline{r}, t) = E_{t//}(\underline{r}, t) \qquad 14\text{-}16$$

a) $\underline{r} = 0$

$$E_{0i//}e^{-i\omega_i t} + E_{0r//}e^{-i\omega_r t} = E_{0t//}e^{-i\omega_t t} \qquad 14\text{-}17$$

where

$E_{0i//}$, $E_{0r//}$, $E_{0t//}$ are all real constants 14-18

$$E_{0i//}\cos\omega_r t + E_{0r//}\cos\omega_i t = E_{0t//}\cos\omega_t t \qquad 14\text{-}19$$

$$\Rightarrow \omega_i = \omega_r = \omega_t \qquad 14\text{-}20$$

b) $t = 0$

$$E_{0i//}e^{i\underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}}} + E_{0r//}e^{i\underline{\mathbf{k}}_r\cdot\underline{\boldsymbol{r}}} = E_{0t//}e^{i\underline{\mathbf{k}}_t\cdot\underline{\boldsymbol{r}}} \qquad 14\text{-}21$$

$$\Rightarrow \underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_r\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_t\cdot\underline{\boldsymbol{r}} \qquad 14\text{-}22$$

Consider reflection (i.e. $\underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_r\cdot\underline{\boldsymbol{r}}$):

$$(\underline{\mathbf{k}}_i - \underline{\mathbf{k}}_r)\cdot r = 0 \qquad 14\text{-}23$$

$$\Rightarrow (\underline{\mathbf{k}}_i - \underline{\mathbf{k}}_r)\ \text{ is normal to the interface} \qquad 14\text{-}24$$

$$k_{//i} = k_{//r} \qquad 14\text{-}25$$

$$|\underline{\mathbf{k}}_i| = |\underline{\mathbf{k}}_r| \qquad 14\text{-}26$$

$$\Rightarrow \theta_i = \theta_r \ - \ \text{1st law of geometrical optics: Angle of incidence} = \text{Angle of reflection.} \qquad 14\text{-}27$$

Consider transmission (i.e. $\underline{\mathbf{k}}_i\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_t\cdot\underline{\boldsymbol{r}}$):

$$(\underline{\mathbf{k}}_i - \underline{\mathbf{k}}_t)\cdot\underline{\boldsymbol{r}} = 0 \qquad 14\text{-}28$$

$$\Rightarrow k_{//i} = k_{//t} \qquad 14\text{-}29$$

$$k_i\sin\theta_i = k_t\sin\theta_t \qquad 14\text{-}30$$

Using the definition of the refractive index, $n_i = \frac{ck_i}{\omega}$ and $n_t = \frac{ck_t}{\omega}$.

$$\Rightarrow \frac{n_i}{n_t} = \frac{k_i}{k_t} = \frac{\sin\theta_t}{\sin\theta_i} - \ \text{2nd law of geometrical optics} \ - \ \text{Snell's law.} \qquad 14\text{-}31$$

Note that Snell's law is valid for all polarisations.

From Snells' law, one can if $n_i > n_t$, given $\sin\theta_t \le 1$

$$\sin\theta_i = \frac{n_t}{n_i} - \ \text{is the condition for the critical angle.} \qquad 14\text{-}32$$

Further analysis shows one has an attenuated (evanescent) waves.

Since we can also write down $\underline{\mathbf{k}}_{\mathrm{r}} \cdot \underline{\boldsymbol{r}} = \underline{\mathbf{k}}_{\mathrm{t}} \cdot \underline{\boldsymbol{r}}$

$$(\underline{\mathbf{k}}_{\mathrm{r}} - \underline{\mathbf{k}}_{\mathrm{t}}) \cdot \underline{\boldsymbol{r}} = 0 \qquad \text{14-33}$$

which gives

$$\mathrm{k_r sin\theta_r = k_t sin\theta_t} \qquad \text{14-34}$$

The constraints on the components of the incident, reflected and transmitted wavevectors parallel to the interfacial plane leads and on the magnitude of the wavevectors in both media leads to:

- 3rd law of geometrical optics. The incident, reflected and transmitted wave are all in the plane of incidence:

14.4 Fresnel's equations - E-field normal to the plane of incidence.

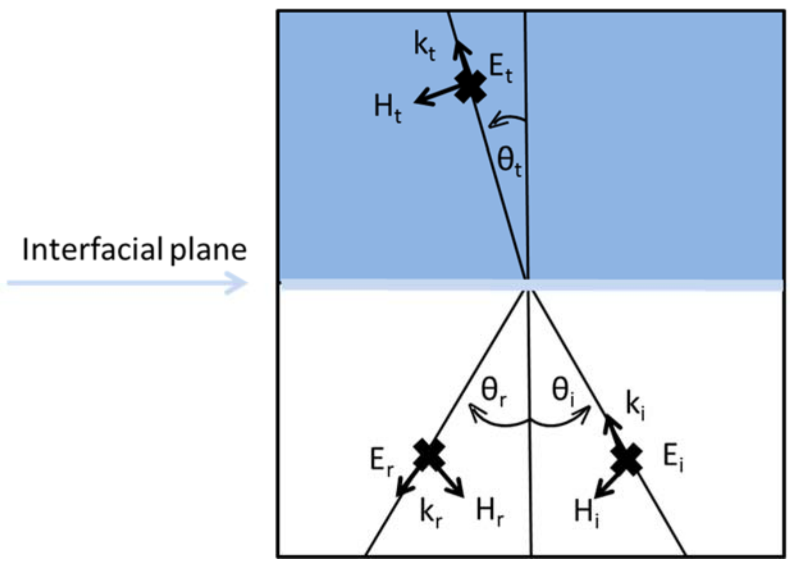


Figure 5 :An electromagnetic wave incident on an interfacial plane where $\underline{\boldsymbol{E}}$ is normal to the plane of incidence.

The continuity of $\underline{\boldsymbol{E}}$ parallel to interfacial plane gives (with $\underline{\boldsymbol{r}} = 0$, $t = 0$)

$$\mathrm{E_{0i} + E_{0r} = E_{0t}}\ (1) \qquad \text{14-35}$$

The continuity of $\underline{\boldsymbol{H}}$ parallel to the interfacial plane gives:

$$\mathrm{H_{0i}cos\theta_i - H_{0r}cos\theta_r = H_{0t}cos\theta_t}\ (2) \qquad \text{14-36}$$

Using $\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} \Rightarrow \underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = \omega \underline{\boldsymbol{B}}$:

$$B = \frac{E}{v_{phase}} \quad 14\text{-}37$$

The refractive index, *n*, by definition is:

$$n = \frac{c}{v_{phase}} = \sqrt{\varepsilon_r \mu_r} \quad 14\text{-}38$$

Using $\underline{\boldsymbol{B}} = \mu_r \mu_0 \underline{\boldsymbol{H}} = \mu \underline{\boldsymbol{H}}$:

$$H = \frac{En}{c\mu}. \quad (3) \quad 14\text{-}39$$

$$(n_i E_{0i} - n_r E_{0r})\cos\theta_i = n_t E_{0t}\cos\theta_t \quad (4) \quad 14\text{-}40$$

$$\frac{E_{0r}}{E_{0i}} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} \quad 14\text{-}41$$

$$\frac{E_{0t}}{E_{0i}} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t} \quad 14\text{-}42$$

Fresnel's first 2 equations –***E*** normal to plane of incidence.

14.5 Fresnel's equations - E-field parallel to the plane of incidence

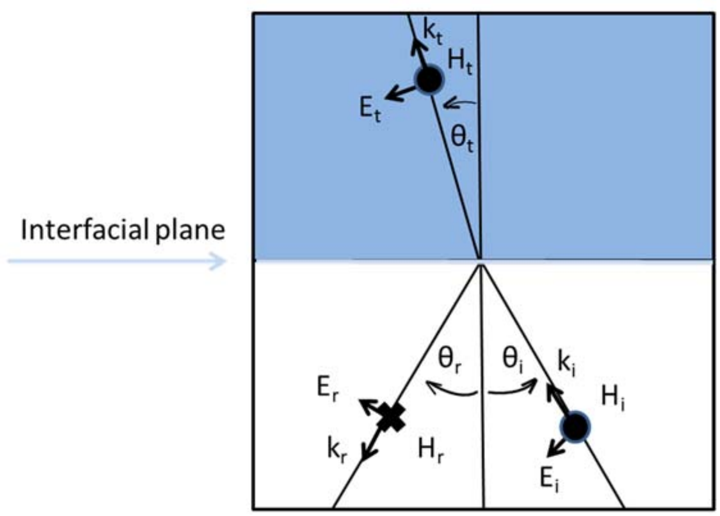


Figure 6 : An electromagnetic wave incident on an interfacial plane where ***E*** is parallel to the plane of incidence

The continuity equations for $H_{//}$ and $E_{//}$ are now,

$$(E_{0i} + E_{0r})\cos\theta_i = E_{0t}\cos\theta_t \quad 14\text{-}43$$

and

$$H_{0i} - H_{0r} = H_{0t} \tag{14-44}$$

Substituting in for H_0:

$$\Rightarrow n_i(E_{0i} - E_{0r}) = n_t E_{0t} \tag{14-45}$$

Again eliminating E_{OT}, E_{OR} in turn gives,

$$\frac{E_{or}}{E_{oi}} = \frac{n_i\cos\theta_t - n_t\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \tag{14-46}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \tag{14-47}$$

Fresnel's equations for $E_{//}$ to the plane of incidence

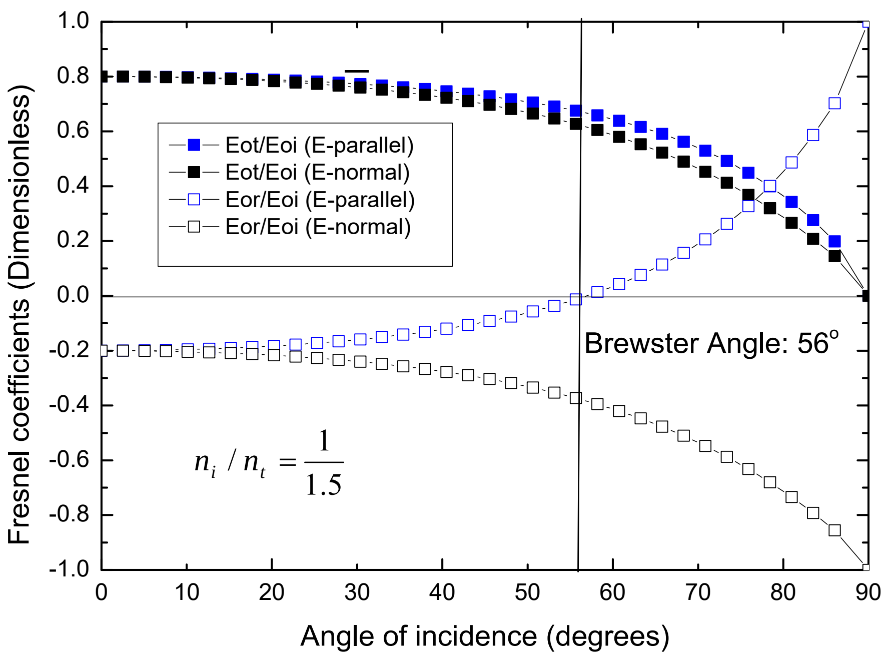


Figure 7 : The Fresnel coefficients as a function of the angle of incidence for the ***E*** - field parallel and normal to the plane of incidence. The Brewster angle is 56 degrees for an air/glass boundary

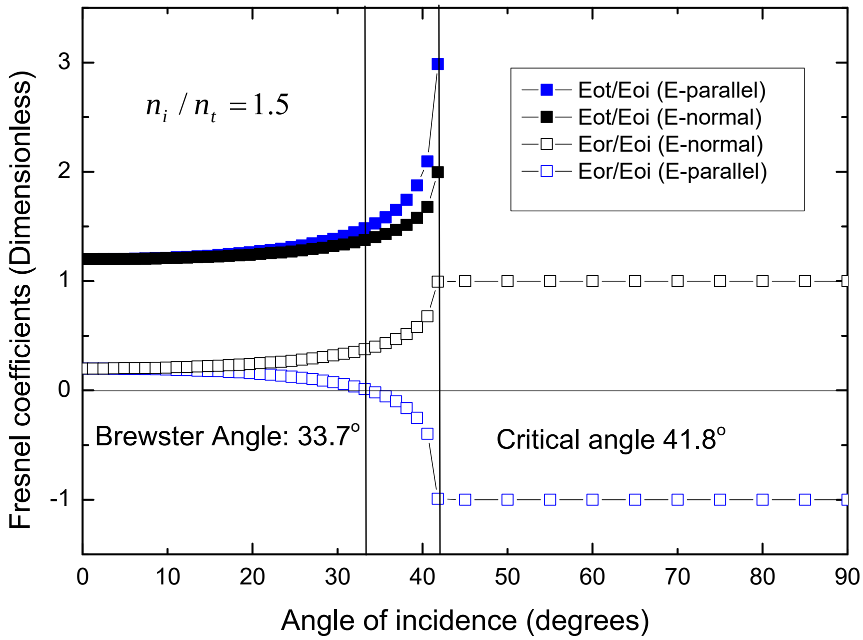


Figure 8 : The Fresnel coefficients as a function of the angle of incidence for the ***E*** - field parallel and normal to the plane of incidence. The Brewster angle is 33.7 degrees for an glass/air boundary. Above the critical angle of 41.8 degrees, there is no transmission.

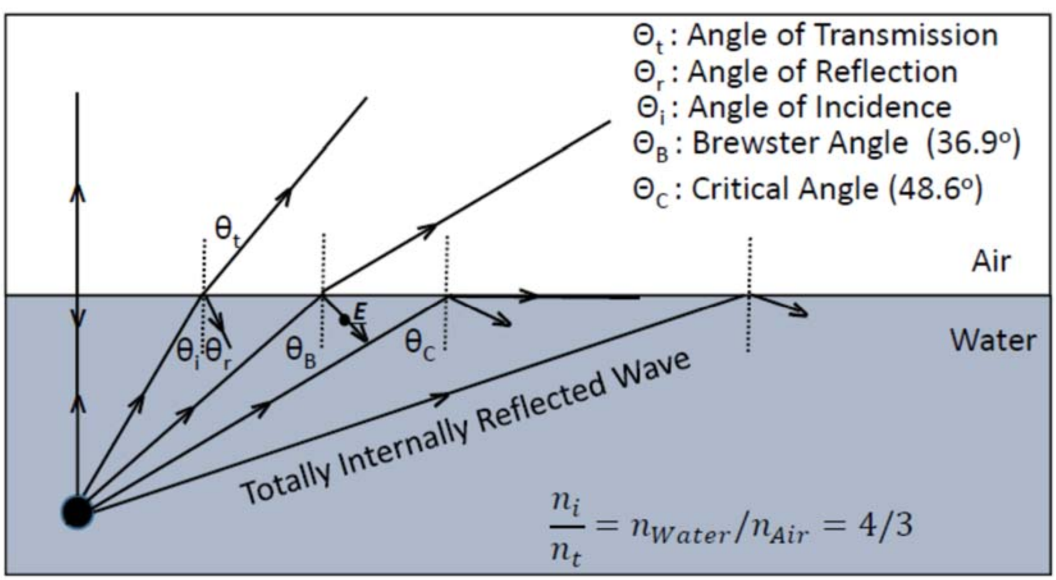


Figure 9 : The reflection and transmission of light into the air that has originated in the water. The Brewster angle is 36.9 degrees for an glass/air boundary so the light that is reflected is polarized normal to the angle of incidence. Above the critical angle of 48.6 degrees, there is no transmission.

ELECTROMAGNETISM

Professor D P Hampshire – Detailed notes for the final lectures (20200310)

14 Static Fields and Electromagnetic waves crossing interfaces

Physicists have developed some beautiful ideas and theories to describe the properties of materials. However if we want to do something useful with these materials, at some point we have to address the complexity of interfaces. External interfaces (i.e. surfaces) are complicated enough, but if we think about internal interfaces such as grain boundaries, or interfaces between two different phases of materials, or interfaces between two different materials in a composite, we soon appreciate that there is huge amount of work to be done in materials physics on interfaces. Pauli emphasised the importance and complexity of interfaces when he famously said: 'God made the bulk, but the devil has been given the surfaces'. Fortunately, because Maxwell's equations are correct at every *point* in space and time, they provide the tools to describe interfaces. Here we derive Fresnel's equations that illuminate: why we can see through glass but not through a brick wall, how polaroid glasses and telescopes work, why the light from rainbows is polarised,…. and indeed that all three laws of geometrical optics follow on from Maxwell's equations.

14.1 Static Boundary Conditions across the interfacial plane between two dielectrics

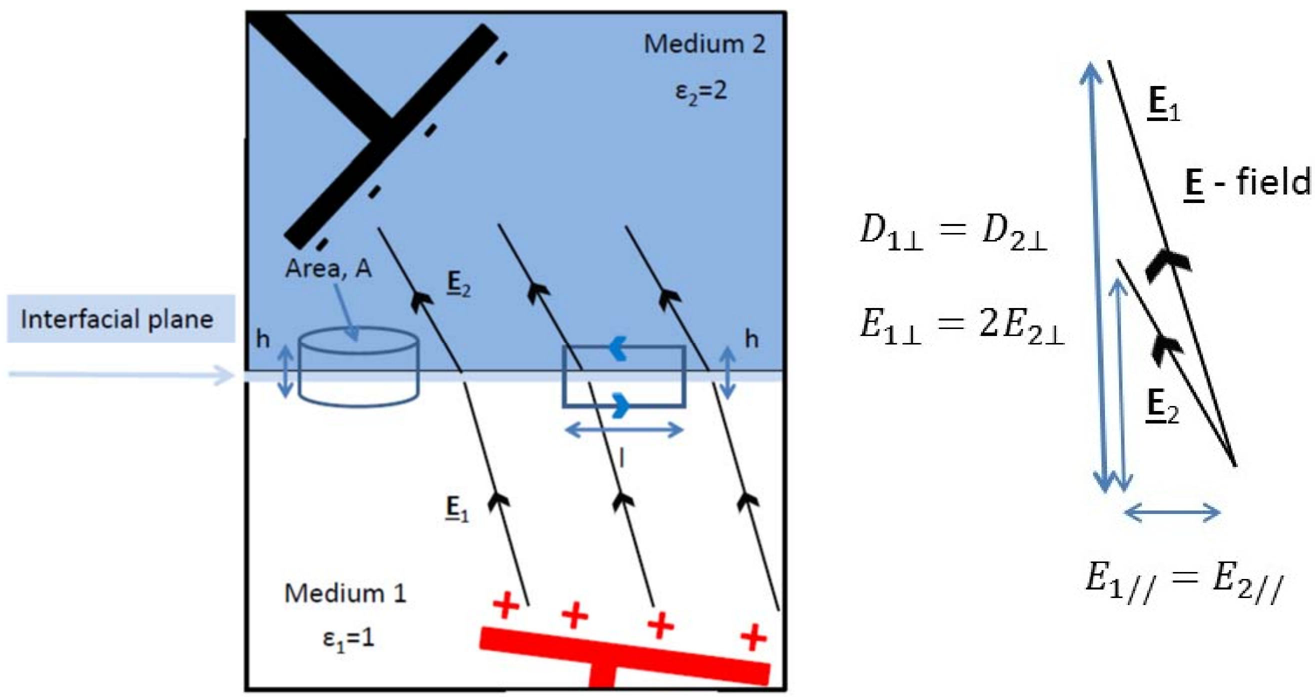


Figure 1 : (LHS) Two charged capacitor plates immersed in two different dielectrics. Both the direction and magnitude of the static electric field is different in the two media. The arrows give the direction of the static ***E***-fields on either side of the interfacial plane (Nb. This is not a ray diagram). (RHS). Maxwell's equations require that the displacement field orthogonal to the interfacial plane is continuous (i.e. $D_{1\perp} = D_{2\perp}$ or equivalently $\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$) and that the electric field parallel to the interfacial plane is continuous (i.e. $E_{1//} = E_{2//}$).

For an insulating dielectric material, Maxwell I can be written: $\underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{D}} = \rho_{\mathrm{free}}$. If we integrate both sides of Maxwell I over the volume of the cylinder shown in Figure 1 we have:

$$\int_{Cylinder} \underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{D}}\ \mathrm{dV} = 0 \qquad \textbf{14-1}$$

This can be rewritten as a surface integral using the divergence theorem. In the limit that we let the height of the cylinder, h, tend to zero, the only non-zero surface integrals are those for the circular surfaces at the top and bottom of the cylinder where:

$$\int_{Circle\ Medium\ 1} \underline{\boldsymbol{D}}_1 \cdot \mathrm{d}\underline{\boldsymbol{S}} + \int_{Circle\ Medium\ 2} \underline{\boldsymbol{D}}_2 \cdot \mathrm{d}\underline{\boldsymbol{S}} = 0, \qquad \textbf{14-2}$$

Therefore

$$\mathrm{D}_{1_\perp} = \mathrm{D}_{2\perp} \Rightarrow \varepsilon_1 \mathrm{E}_{1_\perp} = \varepsilon_2 \mathrm{E}_{2\perp}. \qquad \textbf{14-3}$$

where we have used $\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$.

Similarly, taking the surface integral of Maxwell III over the surface bounded by the rectangular path shown in Figure 1. gives:

$$\int_{Rectangle} \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = -\int_{Rectangle} \frac{\partial \underline{\boldsymbol{B}}}{\partial \mathrm{t}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{14-4}$$

This can be rewritten using Stoke's theorem:

$$\oint \underline{\boldsymbol{E}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = -\frac{\partial}{\partial \mathrm{t}} \int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} \qquad \textbf{14-5}$$

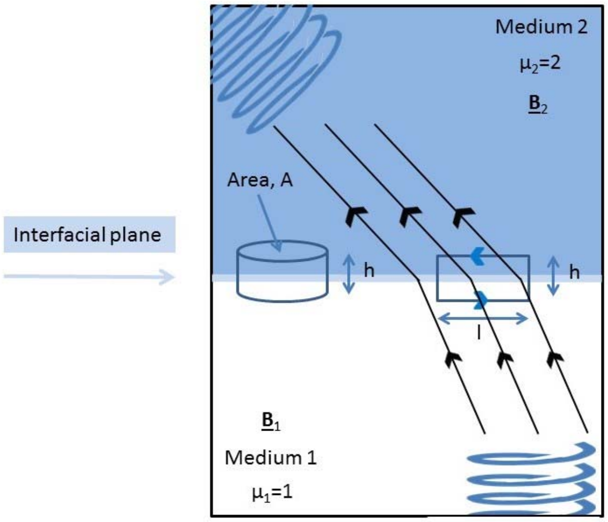
In the limit that the height of the rectangular path, h, tends to zero, the magnetic flux through the rectangle tends to zero and the only non-zero terms in the path integral are those along on the two long lengths of the rectangle. This gives:

$$\mathrm{E}_{1//}\mathrm{l} - \mathrm{E}_{2//}\mathrm{l} = 0 \qquad \textbf{14-6}$$

$$\Rightarrow \mathrm{E}_{1//} = \mathrm{E}_{2//} \qquad \textbf{14-7}$$

Hence we have constraints on each of the orthogonal components of electric field: $\mathrm{D}_\perp$(or equivalently $\varepsilon_r \mathrm{E}_\perp$) and $\mathrm{E}_{//}$ are continuous across the interfacial plane between two dielectrics.

14.2 Static Boundary Conditions across the interfacial plane between two magnetic materials



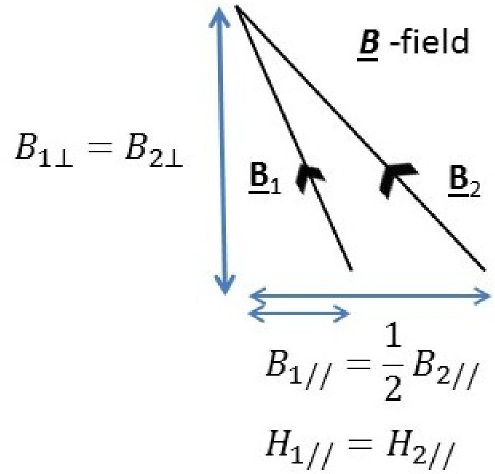


Figure 2: Two magnets immersed in two different magnetic materials. The direction and magnitude of the static magnetic field is different on the two sides of the magnetic interface. The arrows give the direction of the static $\underline{\boldsymbol{B}}$-field on either side of the interfacial plane. (RHS). Maxwell's equations require that the magnetic field orthogonal to the interfacial plane is continuous (i.e. $\mathbf{B_{1_\perp}} = \mathbf{B_{2\perp}}$) and that the field strength parallel to the interfacial plane is continuous (i.e. $\mathbf{H_{1//}} = \mathbf{H_{2//}}$ or $\mathbf{B_{1//}}/\boldsymbol{\mu_1} = \mathbf{B_{2//}}/\boldsymbol{\mu_2}$).

Integrating Maxwell II over volume of the cylinder shown in Figure 2 leads to:

$$\int_{Cylinder} \underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{B}}\ \mathrm{dV} = 0 \qquad \textbf{14-8}$$

Using the divergence theorem, and (again) considering the limit that the height, h, of the cylinder tends to zero, leaves only surface integrals over the circular surfaces so:

$$\int_{Circle\ Medium\ 1} \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \int_{Circle\ Medium\ 2} \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{S}} = 0 \qquad \textbf{14-9}$$

Therefore

$$\mathrm{B}_{1\perp} = \mathrm{B}_{2\perp} \qquad \textbf{14-10}$$

Similarly, taking the surface integral of Maxwell IV $\nabla \times \underline{H} = \underline{J}_{\text{free}} + \partial \underline{D}/\partial \mathrm{t}$ over the surface bounded by the rectangular path shown in Figure 2. gives:

$$\int (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{H}}) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \underbrace{\int \underline{\boldsymbol{J}}_{\text{free}} \cdot \mathrm{d}\underline{\boldsymbol{S}} + \int \frac{\partial \boldsymbol{\nabla}.\underline{\boldsymbol{D}}}{\partial \mathrm{t}} \cdot \mathrm{dV}}_{\substack{-\text{no free current density} \\ -\text{no free charge density}}} \qquad \textbf{14-11}$$

Using Stoke's theorem, this can be rewritten:

$$\oint \underline{\boldsymbol{H}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = 0 \qquad \text{14-12}$$

Taking the limit that h tends to zero gives

$$\mathrm{H}_{1//}\mathrm{l} = \mathrm{H}_{2//}\mathrm{l} \quad \Rightarrow \mathrm{H}_{1//} = \mathrm{H}_{2//} \qquad \textbf{14-13}$$

Hence we have constraints on each of the orthogonal components of the magnetic field: $\mathrm{B}_{\perp}$and $\mathrm{H}_{//}$ are continuous across the boundary.

14.3 The Laws of geometrical optics (for Electromagnetic Waves)

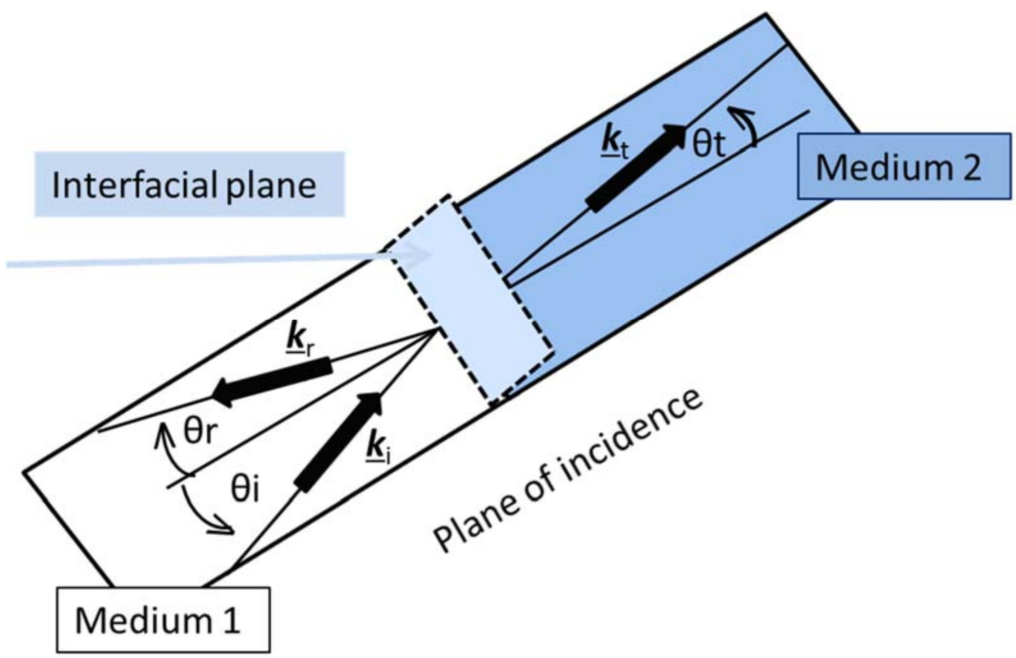


Figure 3 : An electromagnetic wave in medium 1 incident on medium 2. The incident wave is partially reflected and partially transmitted.

Let's make no assumptions about the polarization, direction of propagation or the angular frequency of the incident, reflected and transmitted waves. We describe them in the most general form using:

$$\underline{\boldsymbol{E}}_{\mathrm{i}}(\underline{\boldsymbol{r}},t) = \underline{\mathbf{E}}_{0\mathrm{i}}\exp i(\underline{\mathbf{k}}_{\mathrm{i}} \cdot \underline{\boldsymbol{r}} - \omega_{\mathrm{i}} t) \tag{14-14}$$

$$\underline{\boldsymbol{E}}_{\mathrm{r}}(\underline{\boldsymbol{r}},t) = \underline{\mathbf{E}}_{0\mathrm{r}}\exp i(\underline{\mathbf{k}}_{\mathrm{r}} \cdot \underline{\boldsymbol{r}} - \omega_{\mathrm{r}} t) \tag{14-15}$$

$$\underline{\boldsymbol{E}}_{\mathrm{t}}(\underline{\boldsymbol{r}},t) = \underline{\mathbf{E}}_{0\mathrm{t}}\exp i(\underline{\mathbf{k}}_{\mathrm{t}} \cdot \underline{\boldsymbol{r}} - \omega_{\mathrm{t}} t) \tag{14-16}$$

where the subscripts i, r and t denote the incident, reflected and transmitted waves respectively. Note that the terms that describe the polarization of the three waves (i.e. $\underline{\mathbf{E}}_{0\mathrm{i}}$, $\underline{\mathbf{E}}_{0\mathrm{r}}$ and $\underline{\mathbf{E}}_{0\mathrm{t}}$) are real constant vectors. The terms that describe the wavevectors of the three waves (i.e. $\underline{\mathbf{k}}_{\mathrm{i}}$, $\underline{\mathbf{k}}_{\mathrm{r}}$, and $\underline{\mathbf{k}}_{\mathrm{t}}$) are constant vectors that can in principle also be complex, to incorporate phase information and decay.

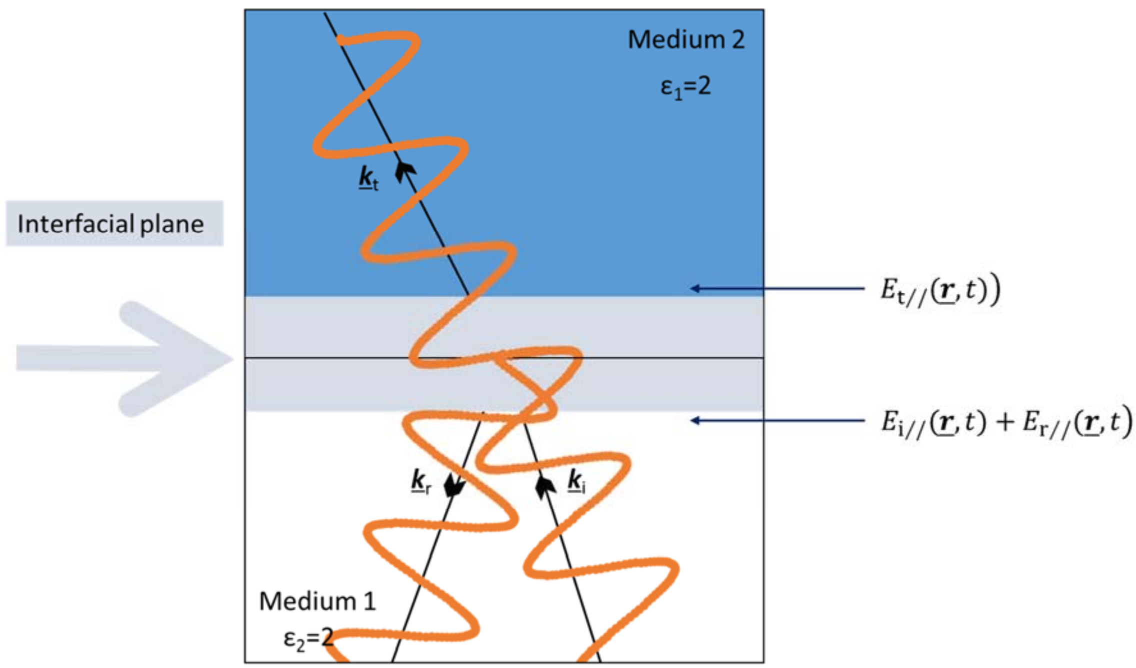


Figure 4 : An electromagnetic wave incident on an interface. After the incident and reflected waves have left the interfacial plane[1], they superimpose in medium 1. The transmitted wave is the only wave in medium 2.

In the interfacial plane, Eqn (**14-7**) leads to:

$$E_{\mathrm{i}//}(\underline{\boldsymbol{r}},t) + E_{\mathrm{r}//}(\underline{\boldsymbol{r}},t) = E_{\mathrm{t}//}(\underline{\boldsymbol{r}},t) \tag{14-17}$$

[1] Figure 4 shows the optical interface with a finite thickness. The incident wave propagates into the body of the interface. The material within the interface interacts with the incident wave and radiates (or reflects or refracts) new waves into both media. The fraction of the incident wave that passes through the interfacial plane, and the wave radiating out into medium 2 from the material inside the interfacial plane, combine to form the transmitted wave. The wave radiating out into medium 1 from the material inside the interfacial plane in the reflected wave. In applying Eqn **14-17**, we apply the boundary conditions just outside an infinitesimally thin interfacial plane as shown in Figure 4 . There are two waves in medium 1 and just one wave in medium 2.

for all $\underline{\boldsymbol{r}}$ in the interfacial plane at all t. The factors $\mathrm{E}_{0\mathrm{i}//}$, $\mathrm{E}_{0\mathrm{r}//}$, and $\mathrm{E}_{0\mathrm{t}//}$ are the components of $\underline{\boldsymbol{E}}_{\mathrm{i}}(\underline{\boldsymbol{r}}, t)$, $\underline{\boldsymbol{E}}_{\mathrm{r}}(\underline{\boldsymbol{r}}, t)$ and $\underline{\boldsymbol{E}}_{\mathrm{t}}(\underline{\boldsymbol{r}}, t)$ resolved in the direction parallel to the interface. This approach does not need to consider what happens inside the interfacial plane. The LHS of the equation is the net $\underline{\boldsymbol{E}}$-field parallel to the interfacial plane in medium 1. The RHS is the net $\underline{\boldsymbol{E}}$-field parallel to the interfacial plane in medium 2 as required by the continuity equation (Eqn **14-7**)[2].

For simplicity, we assume the origin of the coordinate system is in the interfacial plane, so that $\underline{\boldsymbol{r}}$ in all cases is a vector parallel to the interfacial plane and consider two special cases of Eqn. **14-17**:

First Case when r = 0.

When r = 0, Eqn (**14-17**) becomes:

$$\mathrm{E}_{0\mathrm{i}//}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{i}}\mathrm{t}} + \mathrm{E}_{0\mathrm{r}//}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{r}}\mathrm{t}} = \mathrm{E}_{0\mathrm{t}//}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{t}}\mathrm{t}} \qquad \textbf{14-18}$$

where

$$\mathrm{E}_{0\mathrm{i}//}, \quad \mathrm{E}_{0\mathrm{r}//}, \quad \mathrm{E}_{0\mathrm{t}//} \quad \text{are all real constants} \qquad \textbf{14-19}$$

Taking the real part of Equation **14-18** gives:

$$\mathrm{E}_{0\mathrm{i}//}\cos\omega_{\mathrm{r}}\mathrm{t} + \mathrm{E}_{0\mathrm{r}//}\cos\omega_{\mathrm{i}}\mathrm{t} = \mathrm{E}_{0\mathrm{t}//}\cos\omega_{\mathrm{t}}\mathrm{t} \qquad \textbf{14-20}$$

Since this must be true at all times,

$$\Rightarrow \omega_{\mathrm{i}} = \omega_{\mathrm{r}} = \omega_{\mathrm{t}} \qquad \textbf{14-21}$$

Hence the incident, reflected and transmitted waves all have the same (angular) frequency.

Second Case when t = 0.

When t = 0, Eqn (**14-17**) becomes:

$$\mathrm{E}_{0\mathrm{i}//}\mathrm{e}^{\mathrm{i}\underline{\mathbf{k}}_{\mathrm{i}}\cdot\underline{\boldsymbol{r}}} + \mathrm{E}_{0\mathrm{r}//}\mathrm{e}^{\mathrm{i}\underline{\mathbf{k}}_{\mathrm{r}}\cdot\underline{\boldsymbol{r}}} = \mathrm{E}_{0\mathrm{t}//}\mathrm{e}^{\mathrm{i}\underline{\mathbf{k}}_{\mathrm{t}}\cdot\underline{\boldsymbol{r}}} \qquad \textbf{14-22}$$

This condition is true for all $\underline{\boldsymbol{r}}$ in the interfacial plane alone. Therefore:

$$\Rightarrow \underline{\mathbf{k}}_{\mathrm{i}}\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_{\mathrm{r}}\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_{\mathrm{t}}\cdot\underline{\boldsymbol{r}} \qquad \textbf{14-23}$$

i) Consider reflection (i.e. $\underline{\mathbf{k}}_{\mathrm{i}}\cdot\underline{\boldsymbol{r}} = \underline{\mathbf{k}}_{\mathrm{r}}\cdot\underline{\boldsymbol{r}}$) :

$$(\underline{\mathbf{k}}_{\mathrm{i}} - \underline{\mathbf{k}}_{\mathrm{r}})\cdot\underline{\boldsymbol{r}} = 0 \qquad \textbf{14-24}$$

$$\Rightarrow (\underline{\mathbf{k}}_{\mathrm{i}} - \underline{\mathbf{k}}_{\mathrm{r}}) \text{ is normal to the interface} \qquad \textbf{14-25}$$

[2] Those students who turn up to my lectures to hear me witter on about the common misunderstandings that can occur in learning electromagnetism for the first time may hear more than once that Equation **14-17** is a *boundary constraint* on the variables which in this case is of the form 'incident variable + reflected variable' = 'transmitted variable'. It can be contrasted with a *law of conservation* (eg conservation of momentum) which can be written in the form 'sum of incoming properties' = 'sum of outgoing properties'.

Then, $$k_{//i} = k_{//r}$$ 14-26

Since the phase velocities are also the same,

$$|\underline{\mathbf{k}}_i| = |\underline{\mathbf{k}}_r|$$ 14-27

$$\Rightarrow \theta_i = \theta_r :$$ 14-28

1st law of geometrical optics: Angle of incidence = Angle of reflection.

ii) Consider transmission (i.e. $\underline{\mathbf{k}}_i \cdot \underline{\boldsymbol{r}} = \underline{\mathbf{k}}_t \cdot \underline{\boldsymbol{r}}$):

$$(\underline{\mathbf{k}}_i - \underline{\mathbf{k}}_t) \cdot \underline{\boldsymbol{r}} = 0$$ 14-29

$$\Rightarrow k_{//i} = k_{//t}$$ 14-30

$$k_i \sin\theta_i = k_t \sin\theta_t$$ 14-31

Using the definition of the refractive index, $n_i = \frac{ck_i}{\omega}$ and $n_t = \frac{ck_t}{\omega}$.

$$\Rightarrow \frac{n_i}{n_t} = \frac{k_i}{k_t} = \frac{\sin\theta_t}{\sin\theta_i},$$ 14-32

2nd law of geometrical optics - Snell's law.

iii) Since we can also write down (i.e. $\underline{\mathbf{k}}_r \cdot \underline{\boldsymbol{r}} = \underline{\mathbf{k}}_t \cdot \underline{\boldsymbol{r}}$):

$$(\underline{\mathbf{k}}_r - \underline{\mathbf{k}}_t) \cdot \underline{\boldsymbol{r}} = 0$$ 14-33

And

$$k_r \sin\theta_r = k_t \sin\theta_t$$ 14-34

3rd law of geometrical optics. The incident, reflected and transmitted wave are all in the plane of incidence:

14.4 Fresnel's equations – dielectric materials

If light passes from one material to another, scientists want to know how much of the light is reflected and how much is transmitted. For example, we want to distinguish the metal in a (silvered) mirror, where most of the light is reflected, from glass, where most of the incident light is transmitted.

For any interface there are four Fresnel equations. One pair of equations quantifies the fractional $\underline{\boldsymbol{E}}$-field reflected (i.e. E_{0r}/E_{0i}) and the fractional $\underline{\boldsymbol{E}}$-field that is transmitted (i.e. E_{0t}/E_{0i}) when the $\underline{\boldsymbol{E}}$-field is polarized normal to the plane of incidence. The other pair of equations describe the fractional $\underline{\boldsymbol{E}}$-fields when the $\underline{\boldsymbol{E}}$-field is polarized parallel to the plane of incidence.

Fresnel's equations can be used for an incident electromagnetic wave polarized at any an arbitrary angle with respect to the plane of incidence. We can resolve the incident wave into two component waves that have their polarizations parallel and orthogonal to the plane of incidence. We can then apply the relevant pair of Fresnel's equations for each component polarization and calculate the component reflected and transmitted waves. Then we can add the resultant component polarized waves together to find the net waves that are reflected and transmitted.

14.4.1 - *E*-field normal to the plane of incidence.

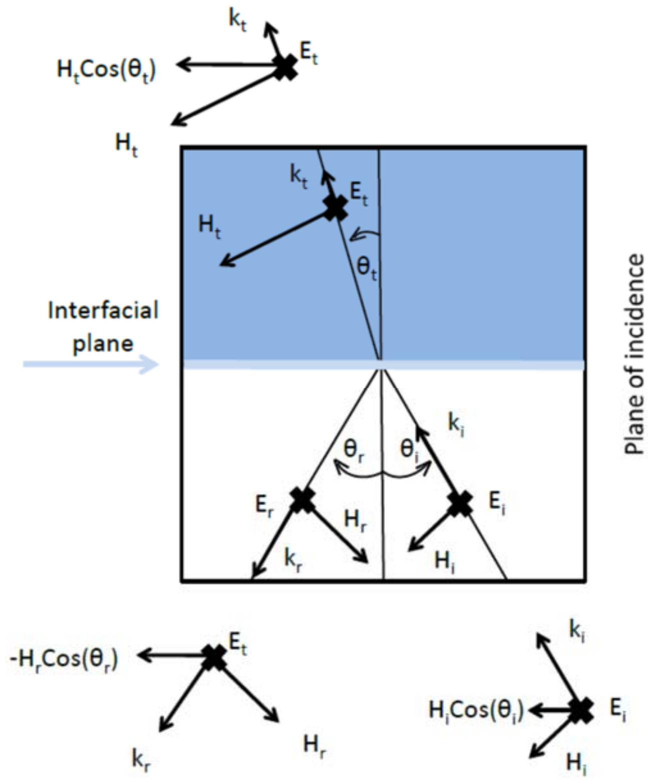


Figure 5 :An electromagnetic wave incident on an interfacial plane where ***E*** is normal to the plane of incidence. By definition: the plane of incidence is the plane that contains the propagation vector of the incoming wave and the reflected wave.

In drawing Figure 5, there are a number of different possible conventions for the relative directions of ***E*** and ***H*** in the reflected and transmitted waves relative to the incident wave (and they are all used in the various text-books). Make sure when you draw this figure that ***E*** × ***H*** is in the direction of ***k*** using the right-hand screw rule and that ***E*** and ***H*** are orthogonal to the direction of propagation[3]. We will use the arbitrary convention that the direction of ***E*** for the reflected and transmitted waves are broadly in similar directions to ***E*** in the incident wave (cf Figure 5 and Figure 6).

The continuity of ***E*** parallel to interfacial plane (ie.Eqn (**14-7**)) gives (with ***r*** = 0, *t* = 0)

$$E_{0i} + E_{0r} = E_{0t} \qquad (1) \qquad \textbf{14-35}$$

The continuity of ***H*** parallel to the interfacial plane (i.e. Eqn (**14-13**)) gives:

[3] If you find yourself reading a textbook that has figures where ***E*** × ***H*** is not in the direction of ***k*** (i.e. not in the direction of propagation) as some are won't to do, I recommend that you laugh heartily, tear any sections that mention electromagnetic waves out of said textbook and throw them into a bin. You may then as a service to the community, ask those of your friends and colleagues who are also studying science, to join you while you escort the bin to the tip. I wouldn't recommend following the items from the tip to the incinerator, because that might be considered excessive. https://www.youtube.com/watch?v=78b67l_yxUc

$$H_{0i}\cos\theta_i - H_{0r}\cos\theta_r = H_{0t}\cos\theta_t. \quad (2)$$ **14-36**

In Figure 5, we have explicitly shown how each of the terms in Eqn. **14-36** occurs.

We know that $\theta_i = \theta_r$ and that ω = constant. Using Maxwell III, $\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{E}} = -\partial\underline{\boldsymbol{B}}/\partial t \Rightarrow \underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = \omega\underline{\boldsymbol{B}}$ and the definitions of the refractive index $n = \frac{c}{v_{phase}}$, and the permeability $\underline{\boldsymbol{B}} = \mu_r\mu_0\underline{\boldsymbol{H}} = \mu\underline{\boldsymbol{H}}$, we can relate the magnitude of $\underline{\boldsymbol{H}}$ to the magnitude of $\underline{\boldsymbol{E}}$ for each of the incident, reflected or transmitted waves where:

$$B = \frac{E}{v_{phase}} \Rightarrow H = \frac{En}{c\mu}. \quad (3)$$ **14-37**

Substituting (3) into (2), and assuming the media are non-magnetic $\mu = \mu_0$[4]

$$(n_i E_{0i} - n_r E_{0r})\cos\theta_i = n_t E_{0t}\cos\theta_t \quad (4)$$ **14-38**

Then combining (1) and (4) to eliminate first E_{ot} to find E_{0r}/E_{0i}, and then E_{or} to find E_{0t}/E_{0i}, we obtain the first pair of Fresnel's equations for the polarization $\underline{\boldsymbol{E}}$ normal to plane of incidence:

$$\frac{E_{0r}}{E_{0i}} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$ **14-39**

$$\frac{E_{0t}}{E_{0i}} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$ **14-40**

14.4.2 - *E*-field parallel to the plane of incidence

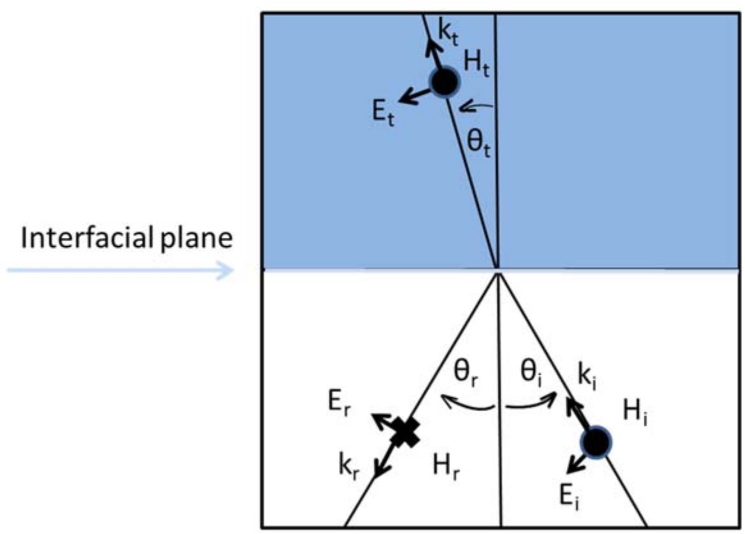


Figure 6 : An electromagnetic wave incident on an interfacial plane where $\underline{\boldsymbol{E}}$ is parallel to the plane of incidence

[4] Many of the interesting optical properties that occur in everyday life occur are associated with either water or glass which are both common insulating dielectrics. Most common magnetic materials are electrically conducting so their optical properties are dominated by their conductivity rather than their magnetic permeability. Hence we only consider *non-magnetic,* insulating, dielectric materials which simplifies the form of Fresnel's equations.

The continuity equations for $E_{//}$ and $H_{//}$ are now,

$$(E_{0i} + E_{0r})\cos\theta_i = E_{0t}\cos\theta_t \quad \textbf{14-41}$$

and

$$H_{0i} - H_{0r} = H_{0t} \quad \textbf{14-42}$$

Again we can rewrite this equation by rewriting the H-fields in terms of the equivalent ***E***-fields. Substituting Eqn (**14-37**) into Eqn (**14-42**) gives:

$$\Rightarrow n_i(E_{0i} - E_{0r}) = n_t E_{0t} \quad \textbf{14-43}$$

We again eliminate E_{0t}, E_{0i} in turn to find the second pair of Fresnel's equations for $E_{//}$ to the plane of incidence:

$$\frac{E_{0r}}{E_{0i}} = \frac{n_i\cos\theta_t - n_t\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \quad \textbf{14-44}$$

$$\frac{E_{0t}}{E_{0i}} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \quad \textbf{14-45}$$

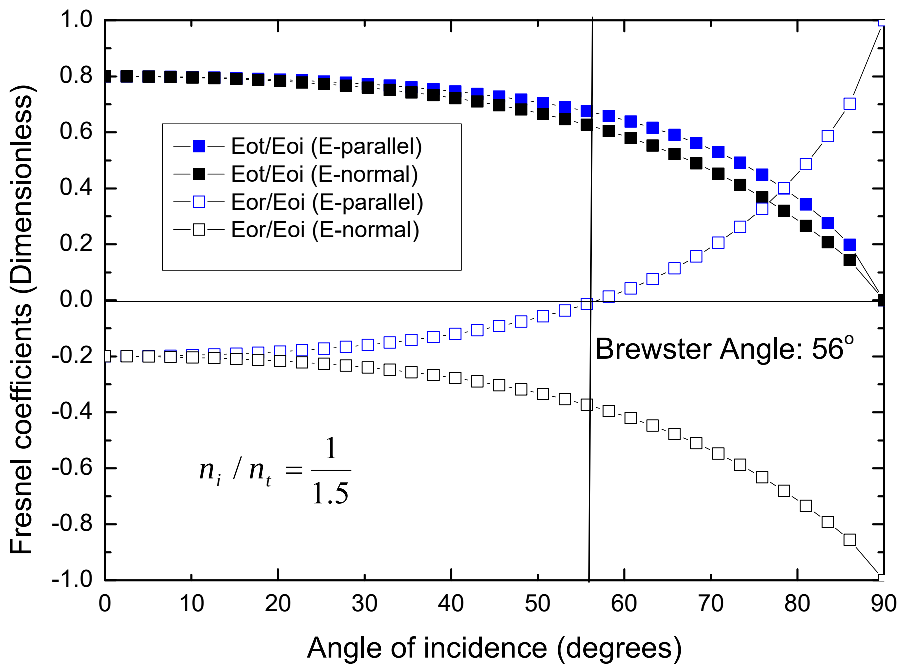


Figure 7 : The Fresnel coefficients as a function of the angle of incidence for ***E*** parallel and normal to the plane of incidence. These data are obtained by using each of Fresnel's equations in turn together with Snell's law. The Brewster angle is 56 degrees for an air/glass boundary. Note that the coefficients are independent of polarisation when the direction of propagation is normal to the interface.

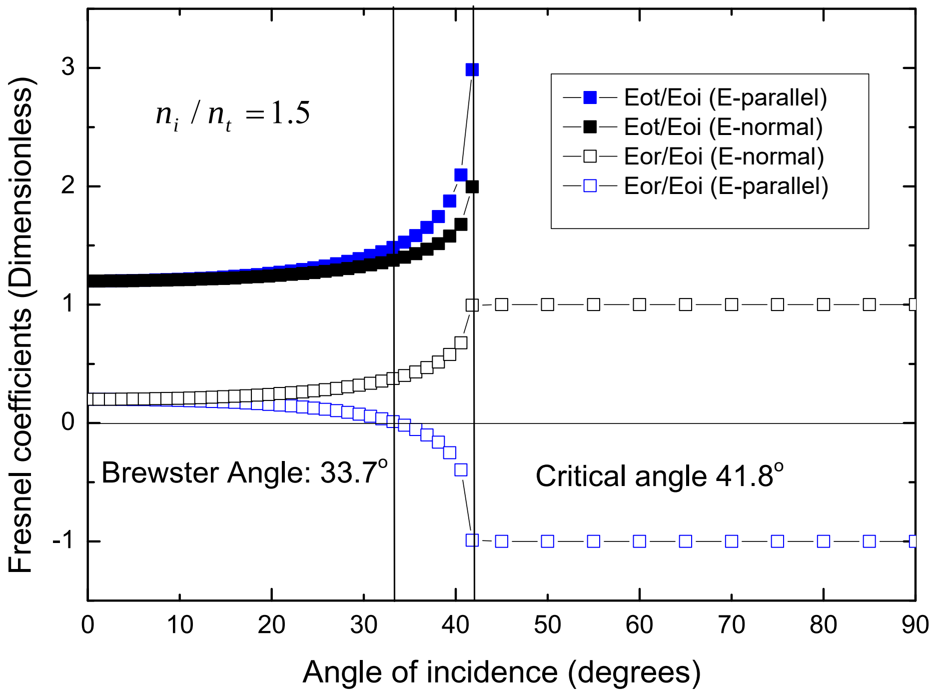


Figure 8 : The Fresnel coefficients as a function of the angle of incidence for the ***E*** - field parallel and normal to the plane of incidence. The Brewster angle is 33.7 degrees for an glass/air boundary. Above the critical angle of 41.8 degrees, there is total internal reflection. Note that the coefficients are independent of polarisation when the direction of propagation is normal to the interface.

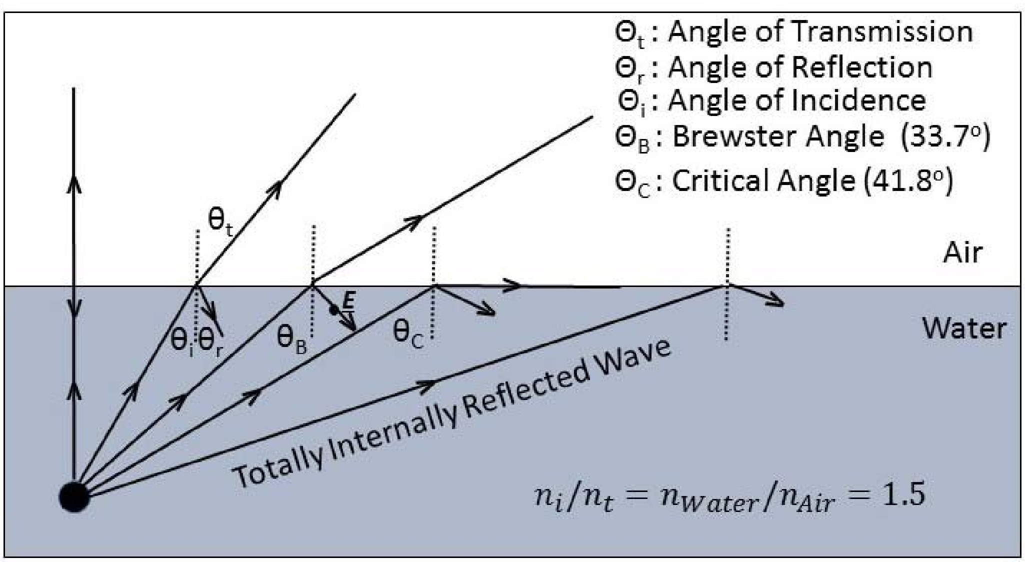


Figure 9 : The reflection and transmission of light into the air that has originated in the water. The Brewster angle is 36.9 degrees for a water/air boundary so the light that is reflected is polarized normal to the angle of incidence. Above the critical angle of 48.6 degrees, there is no transmission.

Brewster Angle : The reflection coefficient for the $\underline{\boldsymbol{E}}$-field parallel to the plane of incidence is zero and the reflected light is fully polarized.

$$\tan\theta_B = \frac{n_t}{n_i} \qquad \textbf{14-46}$$

Critical Angle : Above the the critical angle, there is only total internal reflection of the wave.

$$\sin\theta_C = \frac{n_t}{n_i} \qquad \textbf{14-47}$$

14.5 Coefficients of Reflection and Transmission

We can use Fresnel's equations and Poynting's vector to calculate the fraction of the incident power that is transmitted across the interface and the fraction of the incident power that is reflected from the interface.

For a dielectric that is non-magnetic and non-conducting, we can use $B = \frac{E}{v_{phase}}$ and $v_{phase} = \sqrt{\frac{1}{\varepsilon\mu_0}}$ so $H = \sqrt{\frac{\varepsilon}{\mu_0}}E$. We then utilize the Poynting vector ($\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}$) which gives the instantaneous power per unit area perpendicular to the direction of travel, to calculate the time-averaged power per unit area incident on the interface, N_i:

$$N_i = \frac{1}{2}E_{Oi}H_{Oi}\cos\theta_i = \frac{1}{2}\left(\frac{\varepsilon_1}{\mu_0}\right)^{\frac{1}{2}}E_{Oi}{}^2\cos\theta_i \qquad \textbf{14-48}$$

Note that the factor ½ comes from time averaging $\cos^2\omega t$, and the factor $\cos\theta_i$ occurs because the interface is not perpendicular to the direction the wave is travelling.

Similarly, for the reflected wave:

$$N_r = \frac{1}{2}\left(\frac{\varepsilon_1}{\mu_0}\right)^{\frac{1}{2}}E_{Or}{}^2\cos\theta_r \qquad \textbf{14-49}$$

and for the transmitted wave:

$$N_t = \frac{1}{2}\left(\frac{\varepsilon_2}{\mu_0}\right)^{\frac{1}{2}}E_{Ot}{}^2\cos\theta_t \qquad \textbf{14-50}$$

The reflection coefficient is defined as the fraction of the incident power that is reflected so :

$$R = \frac{N_r}{N_i} = \left(\frac{E_{Or}}{E_{Oi}}\right)^2 \qquad \textbf{14-51}$$

Similarly, the transmission coefficient is defined as the fraction of the incident power that is transmitted so:

$$T = \frac{N_t}{N_i} = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2}\frac{E_{Ot}{}^2}{E_{Oi}{}^2}\frac{\cos\theta_t}{\cos\theta_i} \qquad \textbf{14-52}$$

$$= \frac{n_2}{n_1}\frac{E_{0t}{}^2}{E_{0i}{}^2}\frac{\cos\theta_t}{\cos\theta_i} \quad \textbf{14-53}$$

Using Fresnel's equations (in either polarisation), we find the requirement that there is conservation of energy so:

$$R+T = 1. \quad \textbf{14-54}$$

14.6 Fresnel's equations for highly conducting materials – normal incidence

We have already derived the dispersion relation for a (non-magnetic, non-dielectric) highly conducting dielectric. It is:

$$\mathrm{k} = (1+\mathrm{i})\sqrt{\frac{\omega\mu\sigma_n}{2}} = \frac{1+\mathrm{i}}{\delta} \quad \textbf{14-55}$$

where $\delta = \sqrt{\frac{2}{\omega\mu\sigma_n}}$, $\underline{\tilde{\boldsymbol{E}}}_t(\underline{\boldsymbol{r}},t) = \underline{\mathbf{E}}_{0t}\mathrm{expi}\left(\frac{z}{\delta} - \omega t\right)\exp -\frac{z}{\delta}$ and $\underline{\tilde{\boldsymbol{B}}}_t(\underline{\boldsymbol{r}},t) = \underline{\mathbf{B}}_{0t}\mathrm{expi}\left(\frac{z}{\delta} - \omega t\right)\exp -\frac{z}{\delta}$

Using Maxwell III, $\underline{\boldsymbol{\nabla}} \times \underline{\tilde{\boldsymbol{E}}} = -\partial\underline{\tilde{\boldsymbol{B}}}/\partial t$, gives $\underline{\mathbf{k}} \times \underline{\tilde{\boldsymbol{E}}} = \omega\,\partial\,\underline{\tilde{\boldsymbol{B}}}/\partial t$ which leads to the relationship between the magnitude and polarization of $\underline{\tilde{\boldsymbol{B}}}$ and $\underline{\tilde{\boldsymbol{E}}}$in the transmitted wave where:

$$\mathrm{B}_{0t} = \frac{(1+i)}{\omega\delta}\mathrm{E}_{0t} \quad \textbf{14-56}$$

When the incident, reflected and transmitted wave are all at normal incidence, the continuity equations for $H_{//}$ and $E_{//}$ are simply:

$$(\mathrm{E}_{0i} + \mathrm{E}_{0r}) = \mathrm{E}_{0t} \quad \textbf{14-57}$$

and

$$\mathrm{H}_{0i} - \mathrm{H}_{0r} = \mathrm{H}_{0t} \quad \textbf{14-58}$$

The latter condition (Eqn.**14-58**) is rewritten using Eqn (**14-56**) as:

$$\frac{\mathrm{E}_{0i}}{c} - \frac{\mathrm{E}_{0r}}{c} = \frac{(1+i)}{\omega\delta}\mathrm{E}_{0t} \quad \textbf{14-59}$$

Solving the two equations (Eqns **14-57** and **14-59**) with three unknowns means we can (again) find solutions that are ratios of the unknowns where:

$$\frac{\mathrm{E}_{0r}}{\mathrm{E}_{0i}}\left\{1 + \frac{c}{\omega\delta} + i\frac{c}{\omega\delta}\right\} = 1 - \frac{c}{\omega\delta} - i\frac{c}{\omega\delta} \quad \textbf{14-60}$$

and from the general definition of the reflection coefficient, $R = \frac{N_r}{N_i}$, we have to first order (for small δ):

$$R = \left|\frac{\mathrm{E}_{0\mathrm{r}}}{\mathrm{E}_{0\mathrm{i}}}\right|^2 = 1 - \frac{2\omega\delta}{c} \qquad \textbf{14-61}$$

Also the transmission coefficient $T = \frac{N_t}{N_i}$ is:

$$T = \frac{2\omega\delta}{c} = 1 - R \qquad \textbf{14-62}$$

In the optical range, for highly conducting metals $2\omega\delta/c \approx 0.03$, so we use metals for mirrors where typical reflection coefficients achieved are R ~ 0.97.

Worked examples (from workshop)

Questions

1. Can you use Fresnel's equations to explain why Polaroid glasses have chains of conducting polymers running in the horizontal direction across them ?

2. The light seen when a rainbow is formed has been reflected by water droplets. The light is reflected at an angle close to the Brewster angle. Hence most of the light from a rainbow is polarized. Using Fresnel's equations, describe the polarization of the light from a rainbow Have you ever looked at a rainbow through a polarizer or polarized glasses ?

Answers

In order to answer either Qn. 1 and Qn. 2, you need to notice from Equation **14-44** (and equivalently Figure 7) that at the Brewster angle there is no light that is reflected with an ***E***-field parallel to the plane of incidence. This means that the only light that is reflected is polarised with its ***E***-field orthogonal to the plane of incidence.

1. The light that is vertically polarized is not reflected at Brewster's angle (i.e. R = 0 for light polarized in the plane of incidence). The light that forms the glare is horizontally polarized. The conducting polymers remove the horizontally polarized light. Hence the Polaroid glasses remove glare.

2. Fresnel's equations show that at the Brewster angle, the ***E***-field parallel to the plane of incidence is completely transmitted. Only the light E-field normal to the plane of incidence produces a reflection off the back of each droplet. Hence the light is horizontally polarized from a rainbow at the top of the rainbow and vertically polarized at the ground (if the arc is the typical 180 degrees).

If you get the chance to go to Niagara Falls in the USA, you can find places where there is so much splashing that there are water droplets all around you. When the sun is 'high in the sky', on a sunny day you can see rainbows with a ~ 360 degree arc, which is fun.

Yes is better than No. If No, please have a look at: https://www.youtube.com/watch?v=Efszf7tc5JU

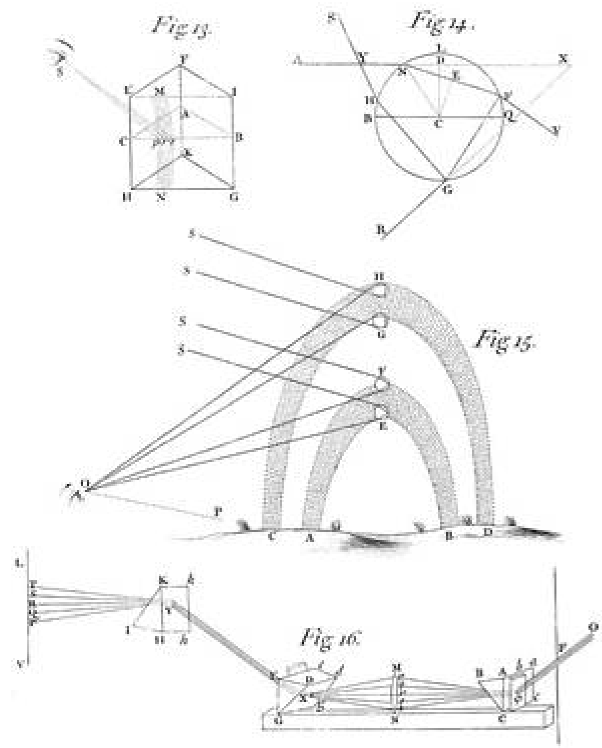


Figure 10 : Newton's drawings of a primary rainbow (1 internal reflection) and a secondary rainbow (2 internal reflections).

15 Waveguides.

A very efficient way of transmitting large amounts of power is using electromagnetic waves in waveguides. Waveguides are hollow conducting tubes made of highly conducting materials – often copper but also in some cases superconductors. Waveguides that have a rectangular cross-section are the most straightforward to describe mathematically. These are the only type considered here.

15.1 Waves propagating in hollow rectangular waveguides

Consider a wave propagating through a rectangular hollow waveguide of internal dimensions a (x-direction) and b (y-direction).

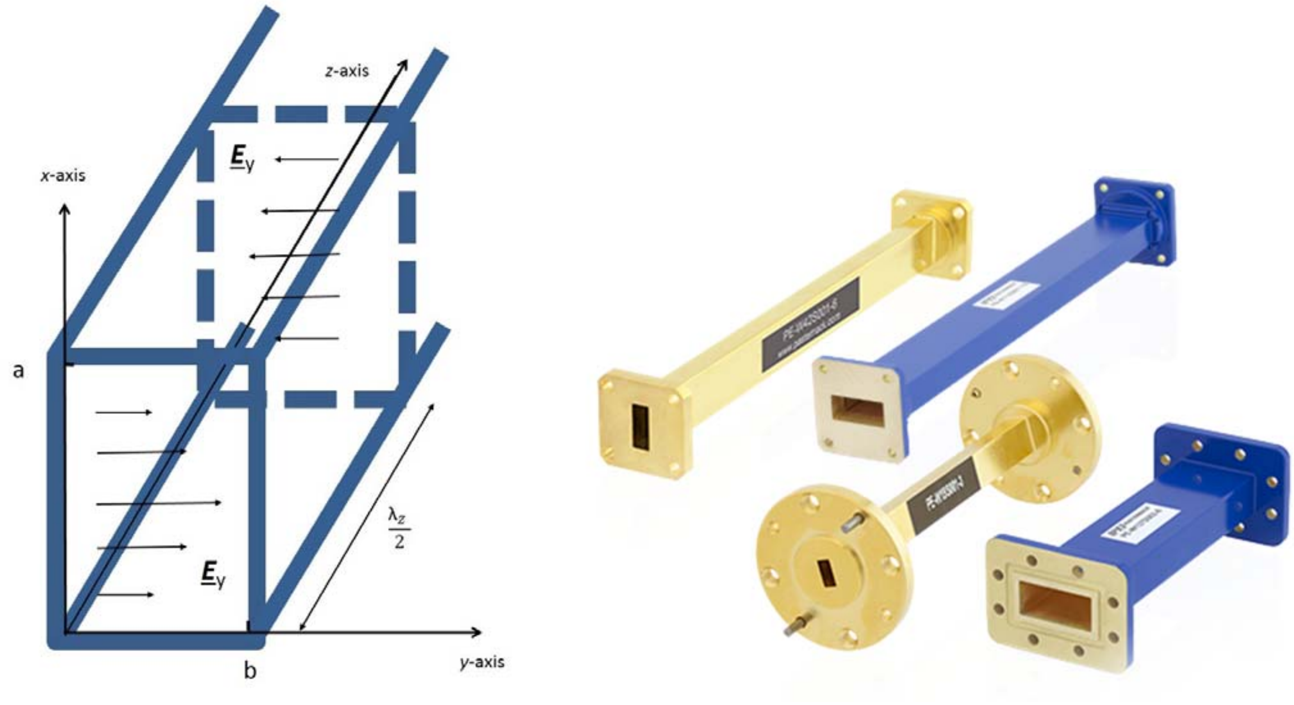


Figure 11 : (LHS) A schematic of a hollow waveguide carrying a TE (transverse electric) wave. Transverse electromagnetic waves cannot exist in hollow waveguides. (RHS) Some waveguides from the internet that can be purchased commercially.

If the waveguide is a good conductor, we can assume that for steady state solutions (i.e. no dissipation) there is no $\underline{\boldsymbol{E}}$-field parallel to the edges of the waveguide. Maxwell's equations lead to the wave-equation which (in complex form) can be written:

$$\nabla^2 \underline{\widetilde{\boldsymbol{E}}} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{\widetilde{\boldsymbol{E}}}}{\partial t^2} \qquad \textbf{15-1}$$

For propagation in the z-direction we can first consider transverse $\underline{\boldsymbol{E}}$-fields (i.e. transverse to the direction of propagation) and for simplicity take them to be in the y-direction only. For steady state propagation, a possible solution is of the form [5]:

$$\underline{\widetilde{\boldsymbol{E}}}(\underline{\boldsymbol{r}}, t) = \hat{\mathbf{j}} \mathrm{E}_{0y} \exp i(\mathrm{k}z - \omega t) \sin(\mathrm{k}_x x) \qquad \textbf{15-2}$$

For steady state propagating solutions, the constraint that the component of $\underline{\boldsymbol{E}}$-field parallel at the internal boundaries of the waveguide must be zero requires:

$$\mathrm{k}_x a = m\pi \qquad \textbf{15-3}$$

These solutions must satisfy the wave equation so:

[5] Some text-books write k in Eqn. **15-2** as k_z but I have found that this can lead to confusion because the physical interpretation of k is quite different to k_x (and k_y).

$$k^2 + k_x^2 = \frac{\omega^2}{c^2} \quad \textbf{15-4}$$

which gives:

$$k^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} \text{ - Dispersion relation} \quad \textbf{15-5}$$

Similar to a plasma, we can identify a lowest cut-off waveguide angular frequency (where k = 0) given by,

$$\omega_c = \frac{\pi c}{a} \quad \textbf{15-6}$$

We can identify two regimes,

i) $\omega > \omega_c$; k is real and the EM wave can propagate without attenuation through the waveguide.
ii) $\omega < \omega_c$; k is imaginary and there is an attenuated solution.

The phase velocity is given by:

$$v_{phase} = \frac{c}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \quad (> c \text{ for } \omega > \omega_c) \quad \textbf{15-7}$$

The group velocity,

$$v_{group} = c\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \quad (< c \text{ for } \omega > \omega_c) \quad \textbf{15-8}$$

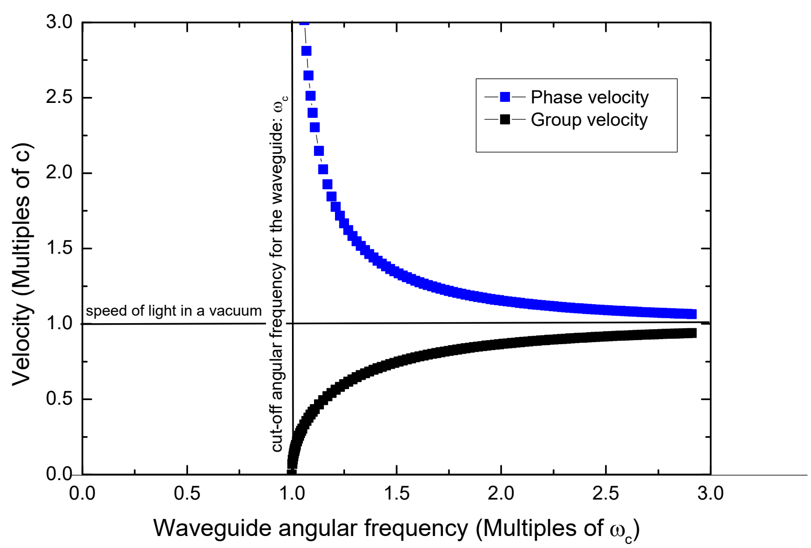


Figure 12 : The velocity of a TE (Transverse electric) wave or TM (Transverse magnetic) wave in a waveguide as a function of angular frequency. Waves are reflected off the waveguide and absorbed by the conducting walls at frequencies below the cut-off angular frequency (ω_c).

For a propagating wave, we can use Maxwell III, $\underline{\boldsymbol{\nabla}} \times \underline{\widetilde{\boldsymbol{E}}} = -\partial \underline{\widetilde{\boldsymbol{B}}}/\partial t$, to find the magnetic field associated with the $\underline{\boldsymbol{E}}$-field given by Eqn. 15-2

$$\underline{\boldsymbol{\nabla}} \times \underline{\widetilde{\boldsymbol{E}}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & \mathrm{E}_y & 0 \end{vmatrix} \tag{15-9}$$

$$= -\hat{\mathbf{i}}\, ik\mathrm{E}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\sin(\mathrm{k}_x x) + \hat{\mathbf{k}}\, k_x\mathrm{E}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\cos(\mathrm{k}_x x)$$

Integrating Maxwell III with respect to time gives:

$$\underline{\widetilde{\boldsymbol{B}}} = \frac{1}{i\omega}\underline{\boldsymbol{\nabla}} \times \underline{\widetilde{\boldsymbol{E}}} \tag{15-10}$$

$$\underline{\widetilde{\boldsymbol{B}}} = -\hat{\mathbf{i}}\frac{\mathrm{E}_{0y}}{\mathrm{c}}\sqrt{1 - \left(\frac{\omega_\mathrm{c}}{\omega}\right)^2}\,\mathrm{expi}(\mathrm{kz} - \omega t)\sin(\mathrm{k}_x x) - \hat{\mathbf{k}}\, i\frac{\omega_\mathrm{c}}{\mathrm{c}\omega}\mathrm{E}_{0y}\mathrm{expi}(\mathrm{kz} - \omega t)\cos(\mathrm{k}_x x) \tag{15-11}$$

Equations **15-2** and **15-11** give the $\underline{\boldsymbol{E}}$-field and $\underline{\boldsymbol{B}}$-field associated with the travelling wave. The $\underline{\boldsymbol{E}}$-field is transverse to the direction of propagation as is the x-component of B. However interestingly, the z-component of B is a longitudinal component. These two equations describe a transverse electric (TE) wave[6]. It has an $\underline{\boldsymbol{E}}$-field that is solely transverse as the description TE suggests. However the TE has a $\underline{\boldsymbol{B}}$-field that has components that are both transverse and parallel to the direction of propagation (!).
Equally we can find mathematical solutions for a transverse magnetic (TM) wave that can propagate in a wave-guide. Such waves have a $\underline{\boldsymbol{B}}$-field that is only transverse but both a transverse and a longitudinal electric fields.
Some scientists like to visualise TE and TM waves as propagating at an angle to the z-axis (bouncing of the walls) as a useful way to explain the longitudinal components of the $\underline{\boldsymbol{B}}$-field and $\underline{\boldsymbol{E}}$-field.

[6] The cognoscenti would describe a wave given by Equations **15-2** and **15-11** as an electromagnetic wave and a transverse electric wave. It could not correctly be called a transverse electromagnetic wave. Just as there are engineers who spend their time optimizing the designs of transmitter antennae and receiver antennae for different locations, there are others who use their imagination and skill to optimize the design of waveguides for various applications.