Mathematical Methods II
PDF 10

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Key Points

- Partial differential equations
- Separation of variables (PDE)

Partial Differential Equations

- **PDEs:** A PDE is an equation relating an unknown function (the dependent variable) of two or more independent variables to its partial derivatives w.r.t those variables; most commonly those independent variables are space and time. We will largely be restricting ourselves to linear PDEs of order 2 and degree 1. Many of the ideas that we have explored with ODEs carry over into solving PDEs, so expect some parallels in the techniques used.

  One way of viewing the difference between an ODE and a PDE is that for an ODE we must assume that each variable has some dependence on the independent variable; whereas in a PDE, dependence is explicit.

  Let $u = u(x, y)$ be our dependent variable, where x and y are independent variables (i.e. one does not depend on the other). This generic function of two (or potentially more) variables is our function of interest. We would generally like to solve for this, just as we did for $y(x)$ for most of our ODE work.

  A reminder about notation: as partial derivatives can have more than one independent variable we must take care over notation, and be sure to distinguish between partial (e.g. ∂x) and total (e.g. dx) derivatives.

$$u_x = \frac{\partial u}{\partial x} \qquad\qquad u_y = \frac{\partial u}{\partial y}$$
$$u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad\qquad u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

$$u_{xy} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = u_{yx}$$

Remember, since x and y are independent of one another, when partially differentiating one w.r.t to the other it is treated as a constant.

$$y_x = \frac{\partial y}{\partial x} = 0 \qquad\qquad x_y = \frac{\partial x}{\partial y} = 0$$

For 2$^{\text{nd}}$ order and higher partial derivatives of more than one variable, the order of the derivatives does not matter.

e.g. PDF10.1 Show that $\partial^2 u/\partial x \partial y = \partial^2 u/\partial y \partial x$ if $u(x, y) = x^2 y^3$.

$$u_x = \frac{\partial u}{\partial x} = 2xy^3 \qquad\qquad u_y = \frac{\partial u}{\partial y} = 3x^2y^2$$
$$u_{yx} = \frac{\partial^2 u}{\partial y \partial x} = 6xy^2 \qquad\qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = 6xy^2$$

- **Total derivative:** As a reminder, we can express the total derivative in terms of partial derivatives

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

So for example, du/dx

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

and du/dt

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

- **General form and classification of 2$^{\text{nd}}$ order PDEs:** The general form of a 2$^{\text{nd}}$ order PDE with 2 variables and constant coefficients is

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y).$$

There is an interesting parallel here between 2$^{\text{nd}}$ order PDEs and conics. When analysing conics one looks at a quadratic equation and based on its terms one can determine if the equation is elliptic, parabolic or hyperbolic. e.g. For conical quadratic equations

$$x^2 - y^2 = 1 \qquad \text{is hyperbolic.}$$
$$x^2 - y = 0 \qquad \text{is parabolic.}$$

A similar approach works for 2$^{\text{nd}}$ order PDEs. In order to be 2$^{\text{nd}}$ order the coefficients of the 2$^{\text{nd}}$ order terms, a,b and c cannot all be zero. We can define the discriminant of these coefficients to be

$$D(x, y) = b^2 - 4ac.$$

The type of the PDE can then easily be assessed.

- If $D < 0$ then the equation is *elliptic*. e.g. the Laplace equation.
- If $D = 0$ then the equation is *parabolic*. e.g. the heat equation.
- If $D > 0$ then the equation is *hyperbolic*. e.g. the wave equation.

In general, elliptic equations describe processes in equilibrium, while the parabolic and hyperbolic equations model processes that change over time.

Some examples of common PDEs in physics are:

- The 1D wave equation
$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
where $u(x, t)$ is a function of the displacement of the wave and α is a constant.
- The 2D Laplace equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
where $u(x, y)$ is a twice differentiable real-valued function.
- The 3D heat equation
$$\frac{\partial u}{\partial t} = \alpha \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$
where $u(x, y, z, t)$ is the temperature and α is a the thermal diffusivity.

You may notice that the 'dimension' referred to in the name of these physical equations usually refers to spatial dimensions, excluding time.

Just as with ODEs, a good test of a solution $u(x, y)$ is to substitute it and its derivatives into a PDE as a check.

e.g. PDF10.2 Show that $u(x, t) = \sin \alpha t \sin x$ is a solution to the 1D wave equation
$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

We need to show that the LHS matches the right. Find the derivatives

$$u_t = \alpha \cos \alpha t \sin x \qquad u_x = \sin \alpha t \cos x$$
$$u_{tt} = -\alpha^2 \sin \alpha t \sin x \qquad u_{xx} = -\sin \alpha t \sin x$$

So, $LHS = \alpha^2 RHS$, which satisfies the equation.

- **Separation of Variables:** Just as with ODEs, if the variables of a PDE can be split up then we can solve it relatively easily.

First, let's assume u can be expressed as the product of two independent functions
$$u(x, t) = X(x)T(t)$$

Hence the partial derivatives of u are

$$u_x = X'T \qquad u_t = XT'$$
$$u_{xx} = X''T \qquad u_{tt} = XT''$$

Now if we substitute these derivatives into a PDE and separate the variables we will notice something interesting - the two sides of the equation *must be constant!*

e.g. PDF10.3 Consider the 1D heat equation

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$$

where k is a real consant. Show that it can be separated into two independent ODEs.

First, let's sub. our independent functions and their derivatives into the PDE

$$XT' = k^2 X'' T.$$

Next, separate the variables

$$\frac{X''}{X} = \frac{1}{k^2}\frac{T'}{T}.$$

Now, here's the trick. Notice that the LHS is only dependent on x and the RHS is only dependent on t. Well, if both sides are dependent on different variables, but are always equal then we can draw only one conclusion - despite appearances, they must each be constant. In a 1D problem such as this, they must in fact be equal to the same constant - *the separation constant.*

$$\frac{X''}{X} = \frac{1}{k^2}\frac{T'}{T} = \mu.$$

This allows us to form two, independent ODEs, which can be solved separately.

$$\begin{aligned} \frac{X''}{X} &= \mu \\ X'' - \mu X &= 0 \end{aligned} \qquad\qquad \begin{aligned} \frac{1}{k^2}\frac{T'}{T} &= \mu \\ T' - k^2\mu T &= 0 \end{aligned}$$

Once we have solutions for X and T, their product will give us the solution for u.

$$\begin{aligned} \lambda_1^2 - \mu &= 0 \\ \lambda_1 &= \pm\sqrt{\mu} \\ X &= c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x} \end{aligned} \qquad\qquad \begin{aligned} \lambda_2 - k^2\mu &= 0 \\ \lambda_2 &= k^2\mu \\ T &= c_3 e^{k^2\mu t} \end{aligned}$$

Hence the general solution is

$$u(x,t) = XT = \left(c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}\right) e^{k^2\mu t}$$

Try substituting this solution back into the PDE to confirm that it viable.

The logic behind this method holds for higher dimensions PDEs, but with the added complication that although $LHS = RHS = const$, if there are multiple terms on one side of the equation each may have its own constant.

e.g. PDF10.4 Consider the 3D heat equation.

$$\frac{\partial u}{\partial t} = k^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right].$$

This time $u = X(x)Y(y)Z(z)T(t)$. Now after substituting in the terms for X, Y, Z and T we get

$$XYZT' = k^2 \left[X''YZT + XY''ZT + XYZ''T \right].$$

After separating the variables by dividing by $k^2(XYZT)$ we are left with

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{k^2}\frac{T'}{T} = \mu$$

So once more we have $LHS = RHS = \mu$. But the LHS has multiple terms, they may each have a different value. Let's say then that

$$l + m + n = \mu$$

where

$$\frac{X''}{X} = l \qquad \frac{Y''}{Y} = m \qquad \frac{Z''}{Z} = n \qquad \frac{T'}{T} = k^2\mu$$

Each of these ODEs can be solved separately giving

$$X = c_1 e^{\sqrt{l}x} + c_2 e^{-\sqrt{l}x} \quad Y = c_3 e^{\sqrt{m}y} + c_4 e^{-\sqrt{m}y} \quad Z = c_5 e^{\sqrt{n}z} + c_6 e^{-\sqrt{n}z} \quad T = c_7 e^{k^2\mu t}$$

Hence the general solution is

$$u(x,y,z,t) = XYZT = \left(c_1 e^{\sqrt{l}x} + c_2 e^{-\sqrt{l}x}\right)\left(c_3 e^{\sqrt{m}y} + c_4 e^{-\sqrt{m}y}\right)\left(c_5 e^{\sqrt{n}z} + c_6 e^{-\sqrt{n}z}\right) e^{k^2\mu t}$$

It should be noted that separating the variables of a PDE in order to solve it is generally not possible. However, this method may be used on several common PDEs that occur in physics and engineering.