L2 Foundation of Physics 2B Optics 2019-20

Solutions to O.WP.3

February 13, 2020

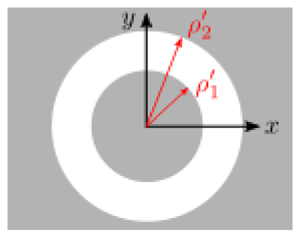
The field on-axis at a distance z downstream of a circularly symmetric aperture is given by

$$E^{(z)} = \frac{E_0 \mathrm{e}^{ikz}}{i\lambda z} \int_0^\infty f(\rho') \mathrm{e}^{ik\rho'^2/(2\pi)} 2\pi\rho' \mathrm{d}\rho' \ ,$$

where $f(\rho')$ is the aperture function. Let the aperture function $f(\rho')$ be a uniformly illuminated circular annulus with inner and outer radii ρ'_1 and ρ'_2 respectively.

(a) Sketch the aperture function in the $z = 0$ plane labelling ρ'_1 and ρ'_2.

$$\begin{aligned} f(\rho') &= 1 \quad \text{if} \quad \rho'_1 < \rho < \rho'_1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$



[1 mark sketch, 1 mark labels]

(b) Derive an expression for the field on-axis in the plane $z = z$ downstream of the annular aperture.

$$E^{(z)} = \frac{E_0 \mathrm{e}^{ikz}}{i\lambda z} \int_{\rho'_1}^{\rho'_2} \mathrm{e}^{ik\rho'^2/(2\pi)} 2\pi\rho' \mathrm{d}\rho' \ . \text{ [1 mark]}$$

Let $\xi^2 = k\rho'^2/(2z)$ then $2\xi\frac{\mathrm{d}\xi}{\mathrm{d}\rho'} = \frac{2k}{2z}\rho'$ $\quad \therefore 2\pi\rho'\mathrm{d}\rho' = 2\lambda z \xi \mathrm{d}\xi$. [2 marks]

So $E^{(z)} = E_0 \mathrm{e}^{ikz} \frac{\lambda z}{i\lambda z} \int_{\xi_1}^{\xi_2} \mathrm{e}^{i\xi^2} 2\xi \mathrm{d}\xi = -E_0 \mathrm{e}^{ikz} \left(\mathrm{e}^{i\xi_2^2} - \mathrm{e}^{i\xi_1^2} \right)$ using the hint [2 marks].

(c) Any aperture function with circular symmetry can be considered to be made up of a series of Fresnel zones. Write an expression for the electric field $E_2^{(z)}$ produced by the second Fresnel zone only in terms of E_0.

Boundary of the m^{th} Fresnel zone is $\rho' = \sqrt{\lambda m z}$, so the second zone lies between $\rho'_1 = \sqrt{\lambda z}$ and $\rho'_2 = \sqrt{2\lambda z}$. [2 marks] Thus

$$E_2^{(z)} = -E_0 \mathrm{e}^{ikz} \left(\mathrm{e}^{i2\pi} - \mathrm{e}^{i\pi} \right) = -E_0 \mathrm{e}^{ikz} (1 + 1) = -2E_0 \mathrm{e}^{ikz}. \text{ [1 mark]}$$

(d) Rewrite $E_2^{(z)}$ in terms of the electric field from the first Fresnel zone $E_1^{(z)}$.

For the 1st zone $\rho'_1 = 0$ and $\rho'_2 = \sqrt{\lambda z}$. So

$$E_1^{(z)} = -E_0 \mathrm{e}^{ikz} \left(\mathrm{e}^{i\pi} - 1\right) = -E_0 \mathrm{e}^{ikz} \left(-1 - 1\right) = 2E_0 \mathrm{e}^{ikz}. \text{ [2 marks]}$$

The contributions to the field from the first and second Fresnel zones are equal but out of phase $E_2^{(z)} = -E_1^{(z)}$.