Chapter 5

Dynamics of galactic discs

CO §24.3

The stars in the Milky Way disc are on (almost) circular orbits, with gravity balancing centripetal acceleration. Given that most of the light of the disc comes from its central parts, we would expect the circular velocity in the outer parts of the disc to fall with distance as appropriate for Keplerian motion. We should also be able to compute how velocities of stars in the solar neighbourhood depend on direction. Observations do not follow these expectations at all, which leads to the startling conclusion that most of the mass in the Milky Way is invisible.

5.1 Differential rotation (CO p. 917)

5.1.1 Keplerian rotation

The velocity of a test mass m in circular motion around a point mass M ($M \gg m$) at distance R is

$$\frac{V^2}{R} = \frac{GM(< R)}{R^2}\,. \tag{5.1}$$

Applied to planets in the solar system, we find that the period increases as

$$P = \frac{2\pi R}{V} = \frac{2\pi R^{3/2}}{(GM)^{1/2}}\,.$$

The mass M need not be a point mass: Newton's theorem guarantees this equation is also correct for an extended spherical mass distribution, provided we use the mass $M(< R)$ enclosed by the orbit; any mass outside R does not contribute to the gravitational force:

$$V^2 = \frac{G\,M(< R)}{R}\,.$$

To describe the motion of stars in the MW, we would like to apply this equation to a disc, with $M(< R)$ the disc mass enclosed by the orbit. However, Newton's theorem does not apply there, since obviously a disc is not spherical. Fortunately, the error is not very large.

So applying this to the MW's disc, we observe the following. Most of the *light* in the MWs disc+bulge is interior to the Sun's orbit. If this means that also most of the *mass* is enclosed, than in the previous equation $M(< R)$ remains constant for $R \geq R_\odot$, where $R_\odot$ is the distance Sun-MW centre (because (almost) all of the mass is interior to $R_\odot$). In that case, we expect that $V^2 \propto R^{-1}$ (for $R \geq R_\odot$), just as we found for planets: Newton's law, applied to the motion of stars in the outskirts of the disc, predicts that rotation speed falls with distance as $V \propto 1/R^{-1/2}$ - just as is the case of planets in the Solar system. We can test this assertion by studying the motion of stars in the solar neighbourhood, by measuring 'Oort's constants'.

The curve $V(R)$ (rotation speed as a function of distance to the centre) is called the **rotation curve** of the galaxy.

5.1.2 Oort's constants (CO p. 908-913)

Assume all disc stars are on circular orbits, with circular velocity $V(R)$ for a star at distance R from the MW centre. An observer moves on a circular orbit with radius R_0 and circular speed $V_0 \equiv V(R = R_0)$. They measure the line-of-sight speed, V_r, and the tangential speed, V_t, for a star at distance d with galactic coordinates $(b = 0, l)$ (see Fig.5.1). The orbit of the star has radius R, and the star moves with circular speed $V(r)$.

The rotation curve $V(R)$ can be inferred by measuring $V_r(d, l)$ and $V_t(r, l)$ as follow. First, use trigonometry to show that

$$\begin{aligned} V_r &= V\cos(\alpha) - V_0\sin(l) \\ V_t &= V\sin(\alpha) - V_0\cos(l)\,. \end{aligned} \qquad (5.2)$$

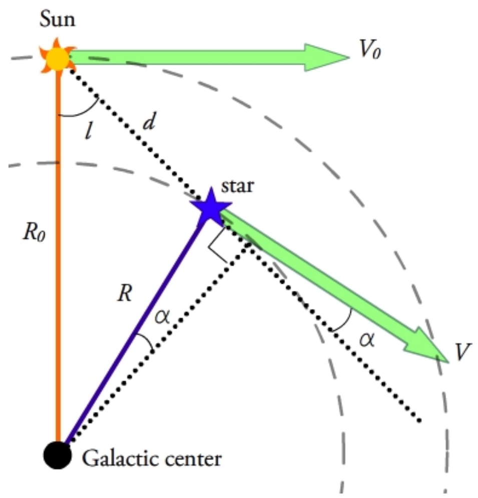

Figure 5.1: (Taken from wikipedia) The observer at the Sun is moving on a circular orbit with velocity V_0 and radius R_0. The observed star is at distance d and has galactic longitude l (with $b = 0$ since it is in the disc). The star is on a circular orbit with radius R and speed V.

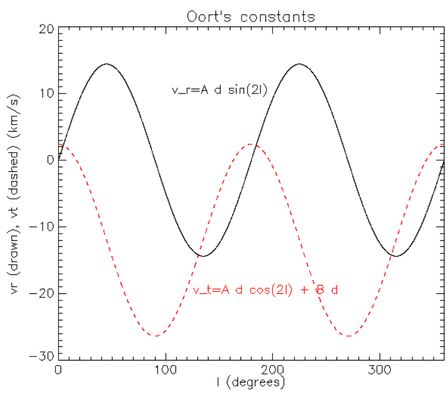

Figure 5.2: Line-of-sight velocity, V_r, and tangential velocity, V_t, of a star at distance $d = 1$ kpc, as a function of its galactic longitude, l.

Again using trigonometry in the indicated right-angled triangle,

$$\begin{aligned}
d + R\sin(\alpha) &= R_0\cos(l) \\
R\cos(\alpha) &= R_0\sin(l) \\
R_0 &= d\cos(l) + R\cos(\beta) \approx d\cos(l) + R \text{ when } d \ll R_0\,, \qquad (5.3)
\end{aligned}$$

where β is the angle Sun-MW Centre-Star. The last step assumes that $d \ll R_0$ so that the angle $\beta \approx 0$: we restrict the analysis to *nearby* stars[1]. A little algebra yields

$$\begin{aligned}
V_r &= (\Omega - \Omega_0)R_0\sin(l) \\
V_t &= (\Omega - \Omega_0)R_0\cos(l) - \Omega\, d\,, \qquad (5.4)
\end{aligned}$$

where $\Omega_0 \equiv V_0/R_0$ is the *angular velocity* of the Sun, and $\Omega \equiv V/R$ the angular velocity of the star. For *nearby stars*, we can expand $\Omega(R)$ in Taylor series around $R = R_0$, keeping only the first terms,

$$\Omega(R) \approx \Omega(R_0) + \frac{d\Omega}{dR}|_{R=R_0}(R - R_0)\,.$$

Notice that

$$\frac{d\Omega}{dR} = \frac{d}{dR}\left(\frac{V}{R}\right) = \frac{1}{R}\frac{dV}{dR} - \frac{V}{R^2}\,.$$

Now define **Oort's constants** A **and** B by

$$\begin{aligned}
A &\equiv -\frac{1}{2}\left[\frac{dV}{dR}|_{R=R_0} - \frac{V_0}{R_0}\right] \approx 14.4 \pm 1.2 \mathrm{km\,s^{-1}\,kpc^{-1}} \\
B &\equiv -\frac{1}{2}\left[\frac{dV}{dR}|_{R=R_0} + \frac{V_0}{R_0}\right] \approx -12.0 \pm 2.8 \mathrm{km\,s^{-1}\,kpc^{-1}}\,. \qquad (5.5)
\end{aligned}$$

Some juggling yields

$$\begin{aligned}
V_r &= A\,d\,\sin(2l) \\
V_t &= A\,d\,\cos(2l) + Bd\,. \qquad (5.6)
\end{aligned}$$

[1]Notice that this is not true in the figure!

and the results are plotted in Fig. 5.2. For example a star toward the galactic centre (or anti-centre; $l = 0$ and $l = 180^o$ respectively), has $V_r = 0$ and $V_t = (A + B)d$.

Jan Oort[2] measured (V_r, V_t) for stars as function of l and d, and inferred $A \approx 14.4 \pm 1.2\mathrm{km\,s^{-1}\,kpc^{-1}}$ and $B = -12.0 \pm 2.8\mathrm{km\,s^{-1}\,kpc^{-1}}$.

What do we expect for these constants, if light traces mass in the MW disc? For such a 'Keplerian disc', $V \equiv V_0(R_0/R)^{1/2}$, hence $dV/dR = -(1/2)V/R$. Therefore we expect

$$\begin{aligned} A_{\mathrm{Kepler}} &= \frac{3}{4}\frac{V_0}{R_0} \approx 19.4\mathrm{km\ s^{-1}\,kpc^{-1}} \\ B_{\mathrm{Kepler}} &= -\frac{1}{4}\frac{V_0}{R_0} \approx -6.5\mathrm{km\ s^{-1}\,kpc^{-1}}\,, \end{aligned} \tag{5.7}$$

where the numerical values use $V_0 \approx 220\mathrm{km\ s^{-1}}$ and $R_0 \approx 8\mathrm{kpc}$, as measured for the Sun.

Our expected values for A and B are clearly inconsistent with the values measured by Oort. In particular, let's compare the values of expected and measured gradient,

$$\frac{dV}{dR} = -(A + B)$$

which are

$$\begin{aligned} \frac{dV}{dR}|_{\mathrm{measured}} &= -(A + B) = -2.4 \pm 3.0\,\mathrm{km\ s^{-1}\,kpc^{-1}} \\ \frac{dV}{dR}|_{\mathrm{Keplerian}} &= -\frac{1}{2}\frac{V}{R} = -13.0\,\mathrm{km\ s^{-1}\,kpc^{-1}}\,. \end{aligned} \tag{5.8}$$

As expected, the Keplerian circular velocity drops $\propto R^{-1/2}$ and therefore $dV/dR < 0$. However, the *measured value is consistent with zero*: **the Milky Way's rotation curve is flat, *i.e.* $V(R) \approx$ constant**.

Of course in a real galaxy stars are not *exactly* on circular orbits, and each star has a small *peculiar* velocity with respect to the perfect circular motion $V_c(R)$. A standard of rest that moves on an exact circular orbit is called the 'local standard of rest'[3]: the speed of the Sun is ~ 16 km s^{-1} with

[2]Note that these are values appropriate for the Sun.
[3]Note this is not an inertial system

Figure 5.3: Prof Vera Rubin was pivotal in establishing that the rotation curves of spiral galaxies are flat in their outskirts, thereby unambiguously demonstrating that galaxies are dominated by dark matter.

respect to its local standard of rest.

Oort also discovered a small number of stars with very large deviations from the expectation given by Eqs. (5.6), which he called *high velocity stars*. He correctly identified these with stars belonging to the halo: the high velocity is because the halo does not rotate, whereas the disc, and the Sun with it, rotates with a speed ~ 220 km s^{-1}.

5.1.3 Rotation curves measured from HI 21-cm emission

The HI 21-cm emission line can be used to measure the velocity of the gas from its Doppler shift, and hence the rotation curve of the gas disc[4]. To see how, use Eqs. (5.2) to show that V_r has a *maximum*

$$V_{r,\max} = V - V_0 \sin(l)\,, \tag{5.9}$$

(which occurs for $\alpha = 0$ and for which $d = R_0 \cos(l)$ and $R = R_0 \sin(l)$). Therefore

$$\frac{dV_{r,\max}(l)}{dl} = \frac{dV(R)}{dR}\,\frac{dR}{dl} - V_0\,\cos(l)\,, \tag{5.10}$$

which can be simplified, using $R = R_0 \sin(l)$, to

$$\frac{dV(R)}{dR} = \frac{dV_{r,\max}(l)}{dl}\,/(R_0\,\cos(l)) + V_0/R_0\,. \tag{5.11}$$

[4]You can attempt this in L4, using the radio dish on the physics building.

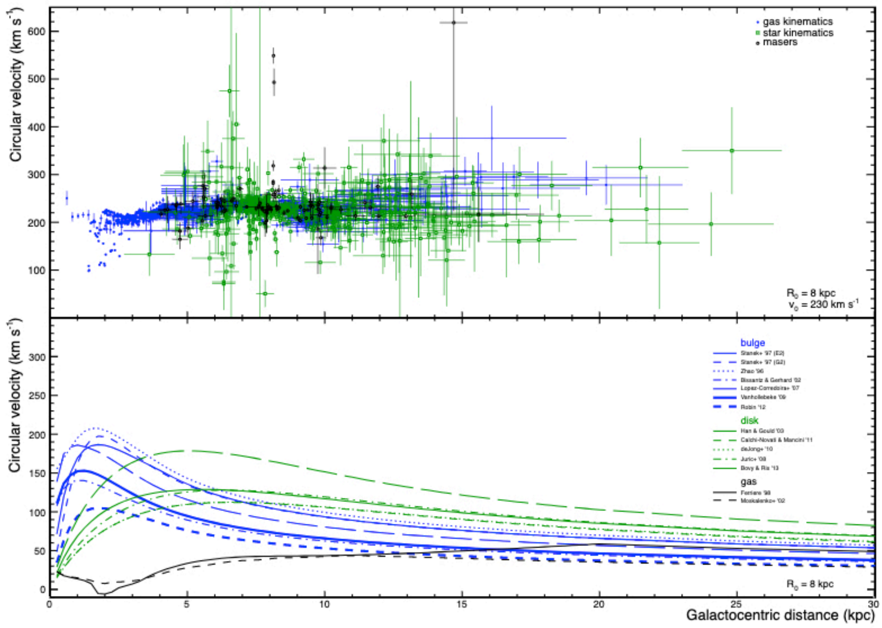

Figure 5.4: Compilation of data (top) and models (bottom) for the Milky Way's rotation curve. Fig 1 from Iocco et al., arXiv:1502.03821

So we don't need to measure the distance to the H I clouds, simply measure their maximum speed as a function of l. The 21-cm analysis confirmed Oort's measurements: the MW's rotation curve near the Sun is essentially flat.

It is much easier though to measure rotation curves in other spiral galaxies, by simply measuring the Doppler shift as function of distance to the centre. Such measurements are now available for tens of spirals, and they all show flat rotation curves in their outskirts; the MW is definitely not unusual in this respect. Prof Vera Rubin, Fig. 5.3 pioneered such measurements. Figure 5.4 shows a recent compilation of measurements of the MW's rotation curve.

5.2 Rotation curves and dark matter (CO p. 914)

A flat rotation curve, $V \sim$ const, rather than the Keplerian expectation, $V \propto R^{-1/2}$, implies that the MW's mass is **not** all concentrated within the solar circle, but is more extended. To find the shape of the density distribution that gives rise to a flat rotation curve, take the derivative[5] with respect to R of

$$V^2\,R = GM\,, \tag{5.12}$$

for V is constant:

$$\begin{aligned} V^2 &= G\frac{dM}{dR} \\ \rho(R) &= \frac{V^2}{4\pi G R^2}\,. \end{aligned} \tag{5.13}$$

Hence a spherical distribution of mass, with $\rho(R) \propto 1/R^2$, gives rise to a flat rotation curve. But the observed light distribution in the MW is very different from this. This suggests three equally astonishing alternatives,

1. The mass-to-light ratio of stars in spirals conspires such that, although the light is very much centrally concentrated, the mass in stars is not. But stellar populations do not seem to vary significantly between the centre and the outskirts. However there may be unseen gas for example in the outskirts of the Milky Way providing the required $\rho(R) \propto 1/R^2$ density.

2. The Milky Way contains invisible matter, which does not emit, nor absorb light.

3. Gravity does not behave as $1/R^2$ on galactic scales. If this were true, then our reasoning above is simply not valid. The theory of **Mo**dified **N**ewtonian **D**ynamics (MOND) is able to provide very good fits to measured rotation curves with a small modification of gravity that cannot be probed in other regimes.

The currently favoured option is (2), namely that the MW, and other galaxies, contains invisible dark matter. More evidence for this later, including the fact that this matter cannot be baryonic[6] in nature.

[5] Show that $dM/dR = 4\pi\rho(R)\,R^2$ in a spherically symmetric density distribution.

[6] Baryons are subatomic particles made out of three quarks, such as protons and neutrons. The dark matter has to be composed of something else, hence cannot be in the form of faint stars, planets, rocks, or past exam papers.

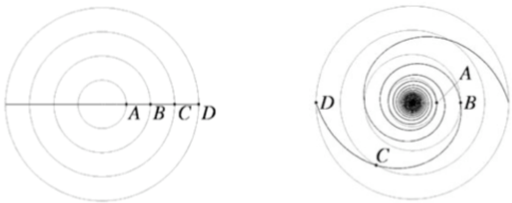

Figure 5.5: A spiral pattern made out of stars will rapidly wind-up in a disc undergoing differential rotation.

5.3 Spiral arms (CO §25.3)

There is quite a variety in spiral patterns. For example, contrast the relatively poorly defined pattern in a *flocculent* spiral galaxy to the well-defined structure of a *grand design* spiral galaxy. Maybe the physical processes that shape these are different?

However, we first need to discuss a problem. Consider two stars, A and B, which are in the same spiral arm but at distances R and $2R$ from the centre, respectively. Assume further that the galaxy has a flat rotation curve. Since the stars move with the same speed (flat rotation curve), the period of the orbit of star B is twice that of star A, $P_B = 2P_A$: after half a period of B, A will have completed a full period. As a consequence of this 'differential rotation', the spiral arm winds up. This means that we expect spiral arms to be very tightly wound - but that is not what we see. This is called the *winding problem*, see the illustration of Fig. 5.5.

The resolution of the paradox is the realisation that spiral arms are **not** material structures (meaning stars remain in spiral arms all the time). Rather, they are a pattern with stars (and gas) moving in and out of spiral arms. The pattern spins with some angular speed, which may be different from the angular speed of the stars. The *density wave* theory of spiral arms states that, when a disc of stars and gas is disturbed, a spiral pattern is induced. To make an analogy: if you disturb a guitar string by plucking it, a series of standing waves is induced. Similarly, disturb a galactic disc, and a spiral pattern is excited. A flocculent pattern may result from a cooling instability in the gas, a grand design pattern from a tidal perturbation caused by another galaxy passing close by.

Figure 5.6: The spiral pattern in NGC 1566, rotating clock wise.. The dark clouds are the location where gas enters the arm from the inside, gets compressed and starts making the stars. By the time many stars have formed, the cluster has overtaken the arm, and we see the shiny new clusters on the 'outside'of the arm.

If the spiral pattern spins with constant angular velocity Ω, say, the the tangential speed of the arm at distance R is $V_t = \Omega\, R$, which clearly increases with R. This implies there is a critical radius R_c, given by $\Omega R_c = V$, such that for $R < R_c$, stars overtake the spiral pattern (because $V > V_t$), and for $R > R_c$, the spiral pattern overtakes stars (because $V < V_t$.) The consequence of this can be seen in Fig. 5.6, where the spiral pattern spins clockwise, and we are looking at the region where $R < R_c$. Gas enters the spiral arms from the inside, gets compressed and makes stars in the dark clouds. It takes a while to make these stars, by which time the gas has overtaken the spiral arm. This is why we see the shiny new clusters on the outside of the arm. (At larger R, it would be the other way around. In real galaxies, V tends to decrease close to the centre: this means that there is another, smaller co-rotation radius - where $V = V_t$- closer in.)

5.4 Summary

After having studied this lecture, you should be able to

- Derive the rotation curve for a Keplerian disc
- Derive the equations for the radial and tangential velocity of stars on circular orbits in a disc in differential rotation, and derive expressions for Oort's constants.
- Compute Oort's constants A and B for a Keplerian disc. Explain how A and B are measured in the MW.
- Explain how the 21-cm emission line can be used to estimate the rotation curve of the MW.
- Explain why both Oort's constants, and the rotation curve measured from 21-cm emission, suggest the presence of dark matter in the outer parts of the MW.
- Describe why spiral arms cannot all be material structures by explaining the winding problem.
- Discuss solutions to the winding problem.
- Explain the density wave theory of spiral arms.