(a) A holonomic constraint is one that can be described by an equation of the form $f(q_1, q_2, ..., q_N, t) = 0$, where there are $N$ generalised coordinates. It is independent of the generalised velocities. **[2 marks,B]**
This system has $N = 3M - j$ degrees of freedom. **[2 marks,B]**

(b) A mechanical system in stable equilibrium would be a pendulum hanging beneath its pivot point. Unstable equilibrium would be the same pendulum above its pivot point. (Other good examples will be acceptable.) **[2 marks,B]**
When perturbed from stable equilibrium, a system will oscillate about the equilibrium position, whereas a pertubation from unstable equilibrium leads to the system accelerating away from that equilibrium configuration. **[2 marks,B]**

(c) The transient solution oscillates with the frequency of the undriven oscillator and has an exponentially decaying amplitude. The steady-state solution persists and oscillates with the driving frequency. **[4 marks,B]**

(d) This 3 DoF system will have 3 normal modes. One involves a coherent translation of all masses, $(1, 1, 1)$. A second would have the two end masses moving in antiphase, with the central mass static, $(1, 0, -1)$. The final mode has the two end masses oscillating in phase with the central mass in antiphase and with twice the amplitude, $(1, -2, 1)$. **[4 marks,U]**

(e) Kepler's first law is an approximation, as seen by the precession of the perihelion of Mercury's orbit. Kepler's second law is a consequence of the conservation of angular momentum, which is exact. Kepler's third law assumes that the mass of the planet is negligible with respect to that of the Sun. This is an approximation, albeit a good one. **[4 marks,B]**

(f) $p = e^Q$, $P = -qe^Q$, hence $P = -pq$ and $Q = \ln p$. **[2 marks, U]**
The Poisson bracket $\{Q, P\} = 0 - (1/p)(-p) = 1$, hence $F$ produces a canonical transformation. **[2 marks, U]**

(g) The centrifugal force is an inertial/fictitious/pseudo force. Such forces arise when considering dynamics in a non-inertial, i.e. accelerating, reference frame. **[2 marks,B]**
It disappears on the axis of rotation, so can be safely ignored at the poles of the Earth. **[2 marks,B]**

(h) 1, 2 and 3 represent directions along the principal axes of the rigid body. $I_n$ is the principal moment of inertia associated with rotations around the $n$th principal axis. **[2 marks,B]**
$\omega_n$ is the component of the instantaneous angular velocity along the $n$ axis, and $\dot{\omega}_n$ is its time derivative. $N$ represents an applied external torque. **[2 marks,B]**

(a) [7 marks total] **(Unseen)**
Using $\dot{y} = \dot{x}df/dx$, and taking the mass of the bead as $m$, the kinetic energy is $T = [m\dot{x}^2(1 + f'^2)]/2$. **[3 marks]**
The potential energy can be written as $V = mgf(x)$.
Hence,

$$L = T - V = \frac{m\dot{x}^2}{2}(1 + f'^2) - mgf.$$

**[2 marks]**

Applying the Euler-Lagrange equation implies

$$(1 + f'^2)\ddot{x} + 2f'f''\dot{x}^2 - f'f''\dot{x}^2 + gf' = 0,$$

from which the required result follows. **[2 marks]**

(b) [4 marks total] **(Unseen)**
$p = \partial L/\partial\dot{x} = m(1 + f'^2)\dot{x}$, from which it can be shown that

$$H(p, x) = \frac{p^2}{2m(1 + f'^2)} + mgf.$$

**[3 marks]**

This is the total energy, i.e. $H = T + V$. **[1 mark]**

(c) (i) [4 marks total] **(Unseen)**
Either take $H$ and substitute in the expression for $p$ in terms of $\dot{x}$, or just use the conservation of energy if these weren't successfully found in previous part to give $(m/2)\dot{x}^2(1 + f'^2) + mgf = H_0$, from which it follows that

$$\dot{x} = \pm\sqrt{\frac{2(H_0 - mgf)}{m(1 + f'^2)}}.$$

**[4 marks]**

(ii) [5 marks total] **(Unseen)**
(This part can be addressed with Level 1 techniques if the previous expression wasn't acquired.) For the case where $f = x$, $f' = 1$. Therefore,

$$\pm\int_{x_0}^{x_1}\sqrt{\frac{1+1}{2(c - x)}}dx = \sqrt{g}(t_1 - t_0),$$

where $c = H_0/(mg)$. **[2 marks]**
As $t_1 - t_0 > 0$, this sets the sign and

$$t_1 - t_0 = (2/\sqrt{g})[\sqrt{c - x}]_{x_0}^{x_1}.$$

Noting that $H_0 = mgx_0$, so $c = x_0$, this leads to

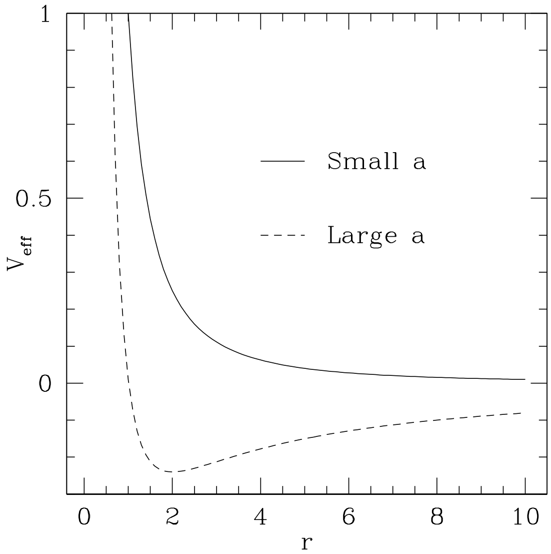
$$t_1 - t_0 = 2\sqrt{(x_0 - x_1)/g}.$$

**[3 marks]**

(a) [4 marks total] **(1 Bookwork, 3 Unseen)**
The kinetic energy is $T = (m/2)(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2)$. If the motion takes place in a plane, then we can choose our coordinates such that $\theta = \pi/2$, in which case $\dot{\theta} = 0$ and $\sin\theta = 1$, from which the required Lagrangian $(= T - V)$ follows. **[4 marks]**

(b) (i) [4 marks total] **(Unseen)**
The associated constant of motion is the canonically conjugate momentum, defined as $J = \partial L/\partial\dot{\phi} = mr^2\dot{\phi}$. Eliminating $\dot{\phi}$ from $E = T + V$ leads to the required expression for $V_{\rm eff}$. **[4 marks]**

(ii) [4 marks total] **(Unseen)**




**[4 marks]**

(c) [8 marks total] **(Unseen)**
For a stable circular orbit to exist, $dV_{\rm eff}/dr = 0$ and $d^2V_{\rm eff}/dr^2 > 0$. $dV_{\rm eff}/dr = 0$ occurs at $r = r_c$, where

$$-\frac{J^2}{mr_c^3} + \frac{k}{r_c}e^{-r_c/a}\left(\frac{1}{r_c} + \frac{1}{a}\right) = 0,$$

i.e.

$$\frac{J^2}{mr_c^2} = ke^{-r_c/a}\left(\frac{1}{r_c} + \frac{1}{a}\right).$$

**[4 marks]**

$$\frac{d^2V_{\rm eff}}{dr^2} = \frac{3J^2}{mr^4} - \frac{k}{r}e^{-r/a}\left(\frac{2}{r^2} + \frac{2}{ar} + \frac{1}{a^2}\right).$$

evaluating this at $r = r_c$ and requiring that $d^2V_{\rm eff}/dr^2 > 0$ for a stable orbit leads to

$$\frac{3J^2}{mr_c^4} - \frac{k}{r_c}e^{-r_c/a}\left(\frac{2}{r_c^2} + \frac{2}{ar_c} + \frac{1}{a^2}\right) > 0.$$

Defining $x = r_c/a$, and using the expression for $J^2/(mr_c^2)$ to eliminate $J$ leads to $x^2 - x - 1 < 0$, from which the physically sensible solution is $r_c/a < (1+\sqrt{5})/2$. **[4 marks]**

**Part (d) is unseen, other parts are bookwork.**

(a) In most cases, the state vector of a system is a linear combination of eigenkets, with respective coefficients. A measurement then involves repeating the measurement for a large number of systems, which are described by the same linear combination of eigenkets. When doing a quantum mechanical measurement, the state vector of the system being measured collapses to one of the eigenkets, with the corresponding eigenvalue being measured for this single system. Repeating the measurement, statistically, with a large numbers of systems, the final result will be the expectation value, which constitutes itself as an average of the eigenvalues, weighted with the relative occurence of the respective eigenvector. [4 marks]

(b) The uncertainty related to the measurement of an observable $A$ represented by the operator $\hat{A}$ is given by

$$\Delta\hat{A} = \sqrt{\left\langle(\hat{A} - \langle\hat{A}\rangle)^2\right\rangle}\,.$$

The uncertainty relation between two such observables reads

$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2}\left|\langle[\hat{A},\,\hat{B}]\rangle\right|\,.$$

Two observables are compatible if the operators representing them commute, implying that their uncertainty relation vanishes. [4 marks]

(c) Eigenkets of two hermitean operators $\hat{A}$ and $\hat{B}$:

$$\hat{A}|a_i\rangle = a_i|a_i\rangle \quad \text{and} \quad \hat{B}|b_i\rangle = b_i|b_i\rangle$$

If $\hat{A}$ and $\hat{B}$ commute, then

$$\hat{A}(\hat{B}|a_i\rangle) = \hat{B}(\hat{A}|b_i\rangle) = a_i(\hat{B}|a_i\rangle)$$

end therefore $\hat{B}|a_i\rangle$ must be an eigenket of $\hat{A}$ with eigenvalue $a_i$. This implies that $\hat{B}|a_i\rangle$ can differ from $|a_i\rangle$ only by some constant factor, and hence $|a_i\rangle$ is also an eigenket of $\hat{B}$. The same can be repeated the other way around. [4 marks]

(d) Eigenvalues follow from characteristic equation,

$$\det\begin{pmatrix} H_{11}-\lambda & H_{12} \\ H_{12} & H_{22}-\lambda \end{pmatrix} = 0$$

$$\longrightarrow \quad \lambda_{1,2} = \frac{H_{11}+H_{22}}{2} \pm \frac{H_{11}-H_{22}}{2}\sqrt{1+\frac{4H_{12}^2}{(H_{11}-H_{22})^2}}\,.$$

[2 marks] The eigenvectors are given by solutions of

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \begin{pmatrix} \lambda_{i1} \\ \lambda_{i2} \end{pmatrix} = \lambda_i \begin{pmatrix} \lambda_{i1} \\ \lambda_{i2} \end{pmatrix}$$

$$\longrightarrow \quad \lambda_{i2} = \frac{\lambda_i - H_{11}}{H_{12}} \lambda_{i1} \longrightarrow \vec{\lambda}_i = \begin{pmatrix} H_{12} \\ \lambda_i - H_{11} \end{pmatrix} .$$

[2 marks]

(e) The Hamiltonian reads

$$\hat{H} = \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) .$$

[2 marks] The commutation relations read

$$[\hat{a},\, \hat{a}] = [\hat{a}^\dagger,\, \hat{a}^\dagger] = 0 \quad \text{and} \quad [\hat{a},\, \hat{a}^\dagger] = 1 .$$

[2 marks]

(f) Pauli matrices $\sigma_{x,y,z}$ are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[2 marks] A non-trivial commutator is

$$\begin{aligned} [\sigma_x,\, \sigma_y] &= \sigma_x \sigma_y - \sigma_y \sigma_x \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

as given by the general commutator identity

$$[\sigma_i,\, \sigma_j] = 2i\epsilon_{ijk}\sigma_k .$$

[2 marks]

(g) If the eigenvalues of $\hat{J}^2$ are given by $j(j+1)\hbar^2$, the eigenvalues of $\hat{J}_z$ are given by $m\hbar$ with $m \in \{-j,\, -j+1,\, \ldots,\, j-1,\, j\}$. In the case at hand, $j = 3$. This means that the eigenvalues of $J_z$ are $m \in \{-3,\, -2,\, \ldots, 2,\, 3\}$. [4 marks]

(a) [Bookwork, 2 marks]

$$\begin{aligned}\left[\hat{J}_i,\, \hat{J}_i\right] &= 0\\ \left[\hat{J}_i,\, \hat{J}_j\right] &= i\hbar\epsilon_{ijk}\hat{J}_k\\ \left[\hat{J}_i,\, \hat{J}^2\right] &= 0\,.\end{aligned}$$

(b) The uncertainty of an operator reads

$$(\Delta J_x)^2 = \langle jm|\hat{J}_x^2|jm\rangle - \langle jm|\hat{J}_x|jm\rangle^2\,.$$

[Bookwork, 4 marks] Use the ladder operators $\hat{J}_\pm$,

$$\begin{aligned}\hat{J}_x &= \frac{\hat{J}_+ + \hat{J}_-}{2}\\ \hat{J}_y &= \frac{\hat{J}_+ - \hat{J}_-}{2i}\,,\end{aligned}$$

with

$$\langle jm|\hat{J}_\pm|jm\rangle = 0$$

and

$$\langle jm|\hat{J}_\pm^2|jm\rangle = 0\,.$$

[Unseen, 2 marks] This leaves

$$\begin{aligned}(\Delta J_x)^2 &= \langle jm|\hat{J}_x^2|jm\rangle - \langle jm|\hat{J}_x|jm\rangle^2\\ &= \left\langle jm\left|\frac{\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+}{4}\right|jm\right\rangle\\ &= \left\langle jm\left|\frac{2\left(\hat{J}^2 - \hat{J}_z^2\right)}{4}\right|jm\right\rangle = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right]\end{aligned}$$

and identical for $\hat{J}_y$.

[Unseen, 2 marks] The smallest uncertainty thus emerges for $|m| = j$ and is given by

$$\Delta\hat{J}_{x,y} = \hbar\sqrt{\frac{j}{2}}\,.$$

[Unseen, 2 marks] From above it is clear that this state would be $|\psi\rangle = |00\rangle$.

(c) [Unseen, 2 marks] Following the logic above for the angular momentum and applying it to the orbital angular momentum, we can rewrite

$$\begin{aligned}\hat{H} &= A\,\hat{L}_z^2 + B\,\left(\hat{L}_x^2 + \hat{L}_y^2\right) \\ &= B\hat{L}^2 + (A-B)\hat{L}_z^2\,.\end{aligned}$$

This allows us to read off the eigenvalues as

$$E_{lm} = \hbar^2\left[Bl(l+1) + (A-B)m^2\right]\,.$$

[Bookwork, 2 marks] In general, the eigenfunctions of such a system are the spherical harmonics.

[Unseen, 4 marks] Using the properties of the spherical harmonics under complex conjugation, $Y_{l-m} = (-1)^m Y_{lm}^*$ it is simple to construct eigenfunctions as suitable linear combinations:

$$\begin{aligned}\hat{Y}_{lm}^{(+)} &= \frac{1}{N_{lm}^{(+)}}\left[Y_{lm} + (-1)^m Y_{l-m}\right] \\ \hat{Y}_{lm}^{(-)} &= \frac{i}{N_{lm}^{(-)}}\left[Y_{lm} - (-1)^m Y_{l-m}\right]\,,\end{aligned}$$

where the $N$ are suitable normalisation factors.

(a) [Unseen, 2 marks]

$$(\hat{H})_{i,j} = \begin{pmatrix} \langle L|H|L\rangle & \langle L|H|R\rangle \\ \langle R|H|L\rangle & \langle R|H|R\rangle \end{pmatrix} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} .$$

(b) [Unseen, 4 marks] Energy eigenkets and eigenvalues:

$$|E_\pm\rangle = \frac{1}{\sqrt{2}} \left(|L\rangle \pm |R\rangle\right) \quad \text{with} \quad E_\pm = \pm\Delta \,.$$

(c) [Unseen, 4 marks] In the energy eigenbasis we have $|\psi(t)\rangle = \psi_+(t)|E_+\rangle + \psi_-(t)|E_-\rangle$ with

$$\psi_\pm(t=0) = \frac{1}{\sqrt{2}} \left(\psi_L \pm \psi_R\right) \,.$$

In this basis the Hamiltonian is diagonal, with elements $\pm\Delta$.

[Unseen, 4 marks] This leads to

$$\psi_\pm(t) = \psi_\pm(t=0) \exp\left(\mp \frac{i\Delta t}{\hbar}\right)$$

and with

$$\psi_{L,R}(t) = \frac{1}{\sqrt{2}} \left[\psi_+(t) \pm \psi_-(t)\right]$$

we arrive at

$$\begin{aligned}
\psi_{L,R}(t) &= \frac{1}{2}\left[(\psi_L + \psi_R) \exp\left(-\frac{i\Delta t}{\hbar}\right) \pm (\psi_L - \psi_R) \exp\left(+\frac{i\Delta t}{\hbar}\right)\right] \\
\psi_L(t) &= \psi_L \cos\frac{\Delta t}{\hbar} - i\psi_R \sin\frac{\Delta t}{\hbar} \\
\psi_R(t) &= \psi_R \cos\frac{\Delta t}{\hbar} - i\psi_L \sin\frac{\Delta t}{\hbar}
\end{aligned}$$

(d) [Unseen, 3 marks] At time $t = 0$: $|\psi(t)\rangle = |L\rangle$, therefore $\psi_L(0) = 1$ and $\psi_R(0) = 0$. This implies that

$$\mathcal{R}(t) = |\langle R|\psi(t)\rangle|^2 = |\psi_R(t)|^2 = \sin^2\frac{\Delta t}{\hbar}$$

[Unseen, 3 marks]
and similarly

$$\mathcal{L}(t) = |\langle L|\psi(t)\rangle|^2 = |\psi_L(t)|^2 = \cos^2\frac{\Delta t}{\hbar} \,.$$