**(a,application)**
Normalization requires

$$\int_0^1 \psi^* \psi \, dx = 1 \qquad \text{with} \qquad \psi = Ax^2(1-x)$$

$$1 = A^2 \int_0^1 x^4(1-x)^2 dx = A^2 \int_0^1 (x^4 - 2x^5 + x^6) dx = A^2 \left[ \frac{x^5}{5} - \frac{x^6}{3} + \frac{x^7}{7} \right]_0^1$$

$$= A^2 \left( \frac{1}{5} - \frac{1}{3} + \frac{1}{7} \right) = A^2 \left( \frac{21 - 35 + 15}{105} \right) = A^2 \frac{1}{105}$$

Hence $A = \sqrt{105}$ [2 marks]
Probability that $x > 1/2$ is

$$\int_{1/2}^1 \psi^* \psi \, dx = 105 \left[ \frac{x^5}{5} - \frac{x^6}{3} + \frac{x^7}{7} \right]_{1/2}^1 = 105 \left( \frac{1}{105} - \frac{1}{32 \times 5} + \frac{1}{64 \times 3} - \frac{1}{128 \times 7} \right) = 1 - \left( \frac{21}{32} - \frac{35}{64} + \frac{15}{128} \right)$$

$$= 1 - \frac{84 - 70 + 15}{128} = 1 - \frac{29}{128} = 0.773$$

[2 marks]

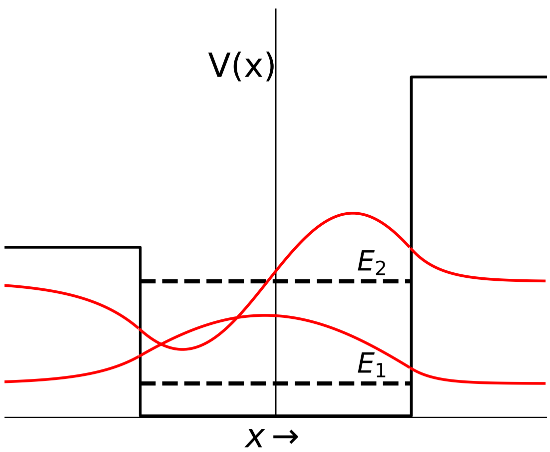
**(b,analysis)**
The wavefunction in the figure is anti-symmetric about $x = L/2$ hence the coefficients of the symmetric eigenfunctions

$$c_1 = c_3 = 0$$

[2 marks]
The wavefunction looks most like $\phi_2$ but with the sign flipped and so $c_2$ is negative and of the largest magnitude. [1 mark]
$c_4$ is positive as the peaks of $\psi$ are closer to together than those of $\phi_2$ [1 mark]

**(c,analysis)**
For $E_1 = 2\hbar\omega$ you can have either $n_x = 1$ and $n_y = 0$ or $n_x = 0$ and $n_y = 1$ and so the degeneracy is two. [2 marks]
For $E_1 = 3\hbar\omega$ you can have either $n_x = 1$ and $n_y = 1$ or $n_x = 2$ and $n_y = 0$ or $n_x = 0$ and $n_y = 2$ and so the degeneracy is three. [2 marks]

**(d,analysis)**



The sketch should show the following features.
Ground state has one turning point and the first excited state two turning points. [2 marks]
The wavefunctions should decay to zero more gradually on the left than the right. [1 mark]
The turning points should be slightly displaced to the left relative to the symmetric case. [1 mark]

**(e,application)**
The perturbation to the energy is given by

$$E_1^1 = \langle \psi_{100}^0 | H' \psi_{100}^0 \rangle = (\pi a^3)^{-1} \int_0^\infty \int_0^{2\pi} \int_0^\pi e^{-r/a} \epsilon \, (r\cos\theta)^2 \, e^{-r/a} r^2 \sin\theta \, d\theta d\phi dr$$

[1 mark]

$$E_1^1 = \frac{\epsilon}{\pi a^3} \, 2\pi \int_0^\infty r^4 e^{-2r/a} dr \int_0^\pi \cos^2\theta \, \sin\theta \, d\theta$$

$$E_1^1 = \frac{2\epsilon}{a^3} \int_0^\infty r^4 e^{-2r/a} dr \int_0^\pi \cos^2\theta \, \sin\theta \, d\theta$$

$$E_1^1 = \frac{2\epsilon}{3a^3} \left( 4! \left(\frac{a}{2}\right)^5 \right) \left[ -\cos^3\theta \right]_0^\pi = -\frac{a^2\epsilon}{2} \left[ \cos^3\theta \right]_0^\pi = -\frac{a^2\epsilon}{2} \left[ -1 - 1 \right] = a^2\epsilon$$

[3 marks]

**(f,analysis)**
The energy expectation value is given by

$$\langle E \rangle = \int \psi^* H \psi dx = \int \psi^* \left( \frac{-\hbar^2}{2m} \right) \frac{d^2}{dx^2} \psi \, dx$$

$$\langle E \rangle = A^2 \int_0^L x(L-x) \left( \frac{2\hbar^2}{2m} \right) dx = \frac{30}{L^5} \frac{\hbar^2}{m} \left[ \frac{Lx^2}{2} - \frac{x^3}{3} \right]_0^L = \frac{30\hbar^2}{mL^5} \left( \frac{L^3}{2} - \frac{L^3}{3} \right) = \frac{5\hbar^2}{mL^2}$$

[2 mark]

The ground state energy is $E_1 = (\pi^2/2)\hbar^2/mL^2 = 4.934\,\hbar^2/mL^2$.
$\langle E \rangle > E_1$ as it must be as you can't have energy less than the ground state [1 mark]
But also the difference is very small as $\psi(x) = Ax(l-x)$ is very similar in shape to the ground state $\phi_1 \propto \sin(\pi x/L)$ [1 mark]

**(g,analysis)**
Given

$$\psi = A(3\phi_1 - 4\phi_2)$$

Conservation of probability requires $9A^2 + 16A^2 = 1$ ie. $A = 1/5$ [1 mark]
Hence the probability if measuring $E_2$ is $(-4/5)^2 = 16/25 = 64\%$ [2 marks]
If $E_2$ is measured the wave function has collapsed to become $\phi_2$ and subsequent measurements will also measure $E_2$ with 100% probability. [1 mark]

**(h,application)**
To find the eigenvalues for

$$\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

we solve

$$\begin{vmatrix} 2-E & 3 \\ 2 & 1-E \end{vmatrix} = (1-E)(2-E) - 6 = E^2 - 3E - 4 = 0$$

$$E = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4 \quad \text{or} \; -1$$

[2 marks]

Sub in for $E = 4$ we have $-2\alpha + 3\beta = 0$ i.e. eigenvector is

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

[1 mark]

Sub in for $E = -1$ we have $3\alpha + 3\beta = 0$ i.e. eigenvector is

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

[1 mark]

**(a,application)**
The expectation value is given by

$$\langle r \rangle = 4\pi \int \psi * \, r \, \psi \, r^2 \, dr$$

[2 marks]

$$\langle r \rangle = \frac{4\pi}{\pi a^3} \int_0^\infty r^3 e^{-2r/a} \, dr$$

[1 mark]

which using the given standard integral

$$\langle r \rangle = \frac{4}{a^3} 3! \left(\frac{a}{2}\right)^4 = \frac{3a}{2}$$

[2 marks]

**(b,analysis)**
Sub into the Schödinger equation

$$\frac{-\hbar^2}{2mr^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + V(r)\psi = E\psi$$

[2 marks]

$$\frac{-\hbar^2}{2mr^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial e^{-r/a}}{\partial r} \right) - \frac{\hbar^2}{mar}, e^{-r/a} = Ee^{-r/a}$$

$$\frac{-\hbar^2}{2mr^2} \frac{\partial}{\partial r} \left( (-r^2/a) e^{-r/a} \right) - \frac{\hbar^2}{mar} e^{-r/a} = Ee^{-r/a}$$

[1 mark]

$$\frac{\hbar^2}{2mr^2} \left( (2r/a - r^2/a^2) e^{-r/a} \right) - \frac{\hbar^2}{mar} e^{-r/a} = Ee^{-r/a}$$

$$E = \frac{\hbar^2}{mar} - \frac{\hbar^2}{2ma^2} - \frac{\hbar^2}{mar} = -\frac{\hbar^2}{2ma^2} \qquad \text{QED}$$

[2 marks]

**(c,knowledge)**
Information below in [] not required to get the mark.
The $n$ is the principal or radial quantum number and determines the energy of the state [1 mark]
The $l$ is the total angular momentum quantum number [such that $L^2\psi = l(l+1)\hbar^2\psi$.] [1 mark]
The $m$ is the magnetic quantum number and labels the $z$-component of the angular momentum [$L_z\psi = m\hbar\psi$.] [1 mark]
$l < n$ and $-l \le m \le l$
Hence we have $l = 0$ with $m = 0$
$l = 1$ with $m = -1, 0, 1$
$l = 2$ with $m = -2, -1, 0, 1, 2$
I.E. degeneracy equals 1+3 +5= 9 (or $n^2 = 9$) [3 marks]

**(d,synthesis)**
$\langle r \rangle > 3a/2$ as

$$\langle r \rangle = 4\pi \int \psi * \, r \, \psi \, r^2 \, dr \propto 4\pi \int r^3 r^2 e^{-2r/a} \, dr$$

is weighted to larger $r$ by the extra $r^2$ factor. [2 marks]

Alternatively, could say it has to be larger as it is not the ground state and the ground state is the lowest energy and hence most compact state.

**(e,application)**
Change in energy of the bound electron is

$$\Delta E = -13.6 \text{ eV} \left(1 - \frac{1}{2^2}\right) = -10.2 \text{ eV}$$

[1 mark]

Hence the emitted photon must carry this energy away

$$10.2 \text{ eV} = h\nu = hc/\lambda$$

$$\lambda = \frac{hc}{10.2 \text{ eV}} = \frac{6.63 \times 10^-34 \times 3 \times 10^8}{10.2 \times 1.60 \times 10^{-19}} = 1.21(5) \times 10^{-7} \text{ m} = 121 \text{ or } 122 \text{ nm}$$

[2 marks]

**(f,analysis)**
Classically the kinetic energy can't be negative and so $r_{\text{max}}$ is given by $E = V(r_{\text{max}})$ [1 mark]

$$E = -\frac{\hbar^2}{2ma^2} = V(r_{\text{max}}) = \frac{-\hbar^2}{mar_{\text{max}}},$$

$$\Rightarrow \quad r_{\text{max}} = 2a$$

[2 marks]

The probability of measuring $r > 2a$ is given by

$$P(r > 2a) = 4\pi \int_{2a}^{\infty} \psi^* \psi \, r^2 \, dr$$

[1 mark]

$$P(r > 2a) = \frac{4\pi}{\pi a^3} \int_{2a}^{\infty} r^2 e^{-2r/a} \, dr$$

[1 mark]

Integrate by parts

$$P(r > 2a) = \frac{4}{a^3} \left( \left[ r^2 \left(\frac{a}{-2}\right) e^{-2r/a} \right]_{2a}^{\infty} - \int_{2a}^{\infty} 2r \left(\frac{a}{-2}\right) e^{-2r/a} \, dr \right)$$

[1 mark]

and again

$$P(r > 2a) = \frac{4}{a^3} \left( \left[ r^2 \left(\frac{a}{-2}\right) e^{-2r/a} \right]_{2a}^{\infty} + a \left[ r \left(\frac{a}{-2}\right) e^{-2r/a} \right]_{2a}^{\infty} - a \int_{2a}^{\infty} \left(\frac{a}{-2}\right) e^{-2r/a} \, dr \right)$$

[1 mark]

$$P(r > 2a) = \frac{4}{a^3} \left( \left[ r^2 \left(\frac{a}{-2}\right) e^{-2r/a} \right]_{2a}^{\infty} + a \left[ r \left(\frac{a}{-2}\right) e^{-2r/a} \right]_{2a}^{\infty} + \frac{a^2}{2} \left[ \left(\frac{a}{-2}\right) e^{-2r/a} \right]_{2a}^{\infty} \right)$$

$$P(r > 2a) = \frac{4}{a^3} \frac{1}{4} a \left( e^{-2(2a)/a} \left(a^2 + 2a(2a) + 2(2a)^2\right) \right) = e^{-4}(1 + 4 + 8) = 13e^{-4} = 0.248 = 23.8\%$$

[2 marks]

# Electromagnetism

Professor Hampshire June 19 Qn. 1

a) Fresnel's equations are derived by requiring that Maxwell's equations are met at all points in space and time across the interface between the two media. More specifically that the continuity of $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{H}}$ are met across the interface.

[4 marks – Comprehension]

b) A radio transmitter is an arrangement of conductors that conduct an A.C. current. The A.C. current causes charges to accelerate and produce the electromagnetic waves that are transmitted.

[4 marks – Comprehension]

c)

$$\underline{\boldsymbol{C}} = x^2y^2\hat{\boldsymbol{j}}$$

$$LHS:\ \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{C}} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2y^2 & 0 \end{vmatrix} = 2xy^2\,\hat{\boldsymbol{k}},\ \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{C}} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 2xy^2 \end{vmatrix} = 4xy\hat{\boldsymbol{\imath}} - 2y^2\,\hat{\boldsymbol{j}}$$

RHS,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{C}} = 2x^2y,\ +\underline{\boldsymbol{\nabla}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{C}}) = 4xy\hat{\boldsymbol{\imath}} + 2x^2\hat{\boldsymbol{\jmath}}$$

$$-\nabla^2\underline{\boldsymbol{C}} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)x^2y^2\hat{\boldsymbol{j}}\ = -(2y^2 + 2x^2)\hat{\jmath}$$

LHS = RHS = $4xy\hat{\boldsymbol{\imath}} - 2y^2\,\hat{\boldsymbol{j}}$ as required

[4 marks – Application]

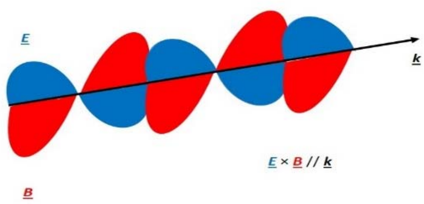
d) For a good conductor $\sigma_N\mu_0\omega \gg \mu_0\epsilon\omega^2$ => $\sigma_N \gg \epsilon\omega$ .
In this case: $\sigma_N = 2 \times 10^{-9}\ \Omega^{-1}\mathrm{m}^{-1}$and $\epsilon\omega = \epsilon_0\epsilon_r \cdot 2\pi f = 8.85 \times 10^{-12}. 10. 2\pi. 10^9 =$ $0.55\ \Omega^{-1}\mathrm{m}^{-1}$ so $\sigma_N \ll \epsilon\omega$ and the material is a poor conductor

[4 marks – Application]

e) The Lorentz force equation leads to $F = BIL$ where $B$ is the magnetic field each wire experiences from the other wire, $I$ is the current in the wire and $L$ is the length of wire on which the force acts. Ampere's law gives $B = \mu_0 I/2\pi r$ where symbols have their usual meaning. Hence the force per unit length $F/L = \mu_0/2\pi = 2 \times 10^{-7}\mathrm{N.m^{-1}}$

[4 marks – Application]

f)




$\underline{\boldsymbol{E}} \perp \underline{\boldsymbol{B}} \perp \underline{\boldsymbol{k}}$ and $(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}})//\underline{\boldsymbol{k}}$

[4 marks – Comprehension]

g) A waveguide is material (eg glass or metal tube) that confines an electromagnetic wave to propagate in one direction and that generates very little energy loss. Examples: (i) a copper tube for guiding microwaves to heat a fusion plasma, (ii) an optical fibre for communications.

[4 marks – Application]

# Electromagnetism

Professor Hampshire June 19 Qn. 2

a) Given : $\underline{B} = -\mu_0\lambda^2\, \underline{\nabla} \times \underline{J}$ and $\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{B}) - \nabla^2\underline{B}$. Substituting in the curl of Maxwell's equation $\underline{\nabla} \times \underline{B} = \mu_0\underline{J}$ gives $\underline{B} = -\lambda^2\, \underline{\nabla} \times \underline{\nabla} \times \underline{B}$ . Using the vector identity gives :

$$\nabla^2\underline{B} = \frac{1}{\lambda^2}\underline{B}$$

This equation has exponential solutions where $\underline{B}(\mathrm{x}) = \underline{\mathrm{B}}_0\exp(-x/\lambda)$ - Meissner state.

[3 marks – Comprehension]

b) Substituting into the differential equation gives:

$$\lambda = (\mathrm{m_e}/\mu_0 n \mathrm{e}^2)^{1/2}$$

[3 marks – Comprehension]

c) The susceptibility $\chi = \mathrm{M}/H$. For a cylinder $\chi = \mu_0\mathrm{M}/B_{applied}$ and $\mathrm{M} = \mathrm{I}A/\mathrm{V}$

Maxwell's equation gives :

$$\partial B/\partial x = B_{applied}/\lambda = \mu_0 J = \mu_0 I/L\lambda$$

where I is the current flowing around the surface of a length L of the cylinder. [2 Marks - Synthesis]

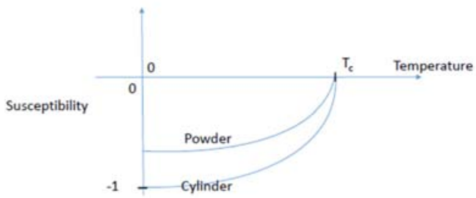
The screening current flows over a distance $\lambda$ so

$$\mathrm{M} = \mathrm{I}\,(A/V) = -(L\lambda B_{applied}/\lambda\,\mu_0)(\pi(a-\lambda)^2/\pi a^2 L)$$ [2 Marks - Synthesis]

Substituting into these equations gives:

$$\chi = \mu_0\mathrm{M}/B_{applied} = -\,(a-\lambda)^2/a^2 \approx -1 + 2\lambda/a$$ [4 Marks - Analysis]

d)




[2 Marks – Synthesis + 4 later]

$n = n_0(1 - T/T_c)$ where $T_c$ is the critical temperature [2 Marks - Synthesis]

Hence:

$$\chi \approx -1 + 2\lambda/a \approx -1 + 2(\mathrm{m_e}/\mu_0 n_0(1 - T/T_c)\mathrm{e}^2)^{1/2}/a$$

Differentiating gives, for low temperatures:

$$\partial\chi/\partial T \approx \frac{2}{a}\left(\frac{\mathrm{m_e}}{\mu_0 n_0 \mathrm{e}^2}\right)^{\frac{1}{2}}\frac{1}{2}\frac{(1-T/T_c)^{-\frac{3}{2}}}{T_c} = \frac{1}{aT_c}\left(\frac{\mathrm{m_e}}{\mu_0 n_0 \mathrm{e}^2}\right)^{\frac{1}{2}}$$ [4 Marks – Synthesis]

e) Addition of powder line to sketch [4 marks – Synthesis]

After powdering the sample, each powder particle of the superconductor is much less well screened. This general argument holds at all temperatures. Hence this leads to a reduction of the diamagnetic signal at all temperatures.

The critical temperature of the supeconductor is unchanged.

[4 marks – Synthesis]