University of Durham

EXAMINATION PAPER

Examination session:	Year:	Examination code:
May/June	2019	PHYS2581-WE01

Title:
Foundations of Physics 2A

Time allowed:	3 hours		
Additional material provided:	None		
Materials permitted:	None		
Calculators permitted:	Yes	Models permitted:	Casio fx-83 GTPLUS or Casio fx-85 GTPLUS
Visiting students may use dictionaries:	No		

Instructions to candidates:

- Attempt **all** questions. The short-answer questions at the start of each section carry 50% of the total marks for the paper. The remaining 50% of the marks are carried by the longer questions, which are equally weighted.
- The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK.**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.
- Slip your booklet for Section B inside your booklet for Section A, before they are collected by the invigilator.

Information

Section A: Quantum Mechanics 2
Section B: Electromagnetism

A list of physical constants is provided on the next page.

Revision:	

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

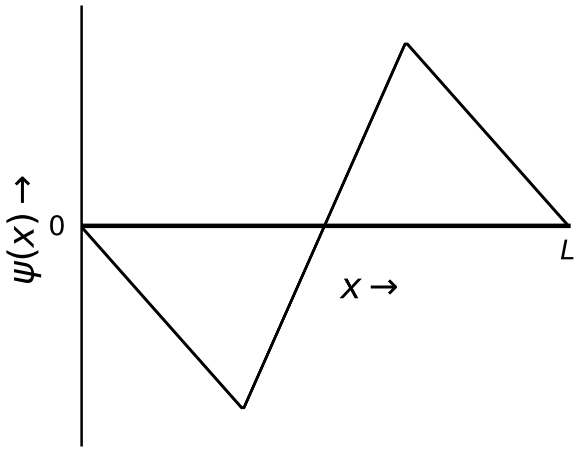
SECTION A: QUANTUM MECHANICS 2

1. (a) A particle in an infinite square well is described by the wave function

$$\psi(x) = \begin{cases} Ax^2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of the constant, A, such that the wavefunction is correctly normalized and find the probability of finding the particle at $x > 0.5$. [4 marks]

(b) A particle confined to the interval $0 < x < L$ by an infinite square well has the wavefunction $\psi(x)$ shown in the figure.



This wavefunction can be expressed by the expansion

$$\psi(x) = \sum_{n=1}^{\infty} c_n\,\phi_n(x),$$

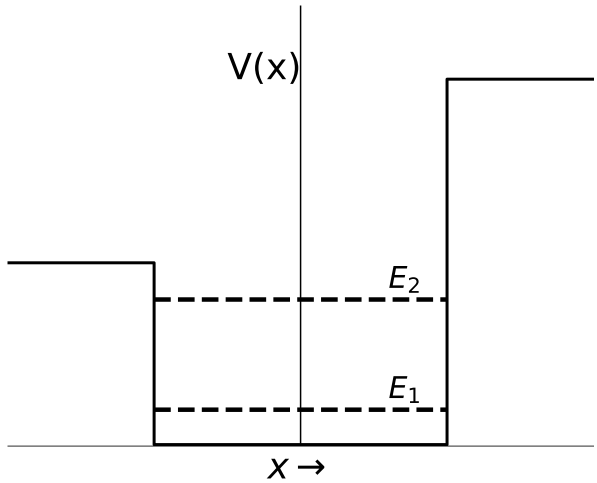
where $\phi_n(x) = \sqrt{\dfrac{2}{L}}\,\sin(n\pi x/L)$ are the usual eigenfunctions of the infinite square well. Which of the coefficents c_1, c_2, c_3, and c_4 are zero? Which has the largest magnitude? Which are positive and which are negative? Give reasons for your answers. [4 marks]

(c) The energy levels of a 2-dimensional simple harmonic oscillator are given by

$$E = \hbar\omega(n_x + n_y + 1),$$

where ω is a constant and n_x and n_y are non-negative integers. What are the degeneracies of the levels that have energy $E_1 = 2\hbar\omega$ and $E_2 = 3\hbar\omega$? [4 marks]

(d) The figure shows the form of a 1-dimensional potential well and the energies of its two lowest levels.



Sketch the wave functions of the ground and first exited state of the potential. Label important features. [4 marks]

(e) The unperturbed wavefunction of hydrogen in the ground state is

$$\psi^0_{100}(r,\theta,\phi) = (\pi a^3)^{-1/2} e^{-r/a},$$

where all symbols have their usual meanings. The hydrogen atom is then subject to an external electric field with a constant gradient in the z direction such that the perturbation of the Hamiltonian is $H' = \epsilon\,(r\cos\theta)^2$. Calculate the first order correction to the energy

$$E^1_1 = \langle \psi^0_{100} | H' | \psi^0_{100} \rangle.$$

[4 marks]

$$\left[\begin{array}{c} \text{Hint:} \\ \displaystyle\int_0^\infty r^n \exp(-r/b)\,dr = n!\; b^{n+1}, \\ \text{where } n \text{ is a non-negative integer and } b \text{ is a positive constant.} \end{array} \right]$$

(f) A particle of mass m confined in an infinite square well is described by a normalized wavefunction

$$\psi(x) = Ax(L-x) \quad \text{where} \quad A = \sqrt{30/L^5}$$

in the interval $0 < x < L$ and zero elsewhere. Calculate the expectation value of the energy, $\langle E \rangle$. [2 marks]

Compare your result with ground state energy $E_0 = \hbar^2\pi^2/(2mL^2)$ and comment on the sign and magnitude of the difference. [2 marks]

(g) A particle is a linear superposition

$$\psi = A(3\phi_1 - 4\phi_2)$$

where ϕ_1 and ϕ_2 are normalised eigenstates with energies E_1 and E_2 respectively. Find the value of the constant A and the probability that a measurement of the energy will yield the value E_2. [3 marks]

If the result of the first measurement is E_2, what is the probability that E_2 will be measured if a second measurement is made on the same system? [1 mark]

(h) Two energy eigenstates ψ_1 and ψ_2 are degenerate, implying any linear combination $\psi = \alpha\psi_1 + \beta\psi_2$ has the same energy eigenvalue E_0. A perturbation to the potential energy of the system breaks this degeneracy and modifies the energy levels by small amounts E^1 given by the solutions to the following matrix equation

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E^1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Find the two values of E^1 and the corresponding eigenvectors. [4 marks]

2. (a) The wave function for an electron in the ground state of a hydrogen atom is (in spherical polar coordinates)

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} \exp\left(-r/a\right),$$

where $a = (4\pi\epsilon_0\hbar^2)/(m_e e^2)$ is the Bohr radius. Find the expectation value of the radius, $\langle r \rangle$, for an electron in this state, expressing your answer in terms of the Bohr radius. [5 marks]

$$\left[\begin{array}{c} \text{Hint:} \\ \displaystyle\int_0^\infty r^n \exp(-r/b)\, dr = n!\ b^{n+1}, \\ \text{where } n \text{ is a non-negative integer and } b \text{ is a positive constant.} \end{array} \right]$$

(b) Given that the Coulomb potential energy of an electron in hydrogen can be written as

$$V(r) = \frac{-e^2}{4\pi\epsilon_0 r} = \frac{-\hbar^2}{m_e a r},$$

find the energy eigenvalue of the ground state of a hydrogen atom and show that it is consistent with

$$E = -\frac{\hbar^2}{2ma^2}\frac{1}{n^2} = \frac{-13.6\ \text{eV}}{n^2}$$

for the case of $n = 1$. [5 marks]

(c) The energy eigenfunctions of an electron in the hydrogen atom, ignoring spin, are conventionally written as $\psi_{nlm}(\underline{r})$. Explain the significance of the three labels n, l, and m. [3 marks]

How many degenerate states are there when $n = 3$? [3 marks]

(d) The radial wave function for an electron with wave function $\psi_{210}(\underline{r})$ is proportional to $r\exp(-r/2a)$. Without performing any integrals, explain whether you would expect $\langle r \rangle$ in this case to be smaller, equal or larger than that found in part (a)? [2 marks]

(e) An electron undergoes a transition from the state with wave function $\psi_{210}(\underline{r})$ to the ground state. Calculate the wavelength of the emitted photon. [3 marks]

(f) For an electron in the ground state of hydrogen find, as a multiple of the Bohr radius, the maximum radius, r_{max}, that would be allowable by classical energy conservation and the probability that a measurement of r will, in fact, yield a value larger than r_{max}. [9 marks]

$$\left[\text{Hint: Integrating by parts} \quad \int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx \quad \text{may be useful.}\right]$$

SECTION B: ELECTROMAGNETISM

3. (a) Briefly describe how the transmission and reflection coefficients associated with an electromagnetic wave crossing the boundary between two media are derived. [4 marks]

(b) Explain the basic principles of how a radio transmitter radiates electromagnetic waves. [4 marks]

(c) Show that the general vector identity

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{C}) = -\nabla^2 \underline{C} + \underline{\nabla}(\underline{\nabla}.\underline{C})$$

is true for the specific vector field $\underline{C} = x^2 y^2 \underline{\hat{\jmath}}$. [4 marks]

(d) The general dispersion relation for an electromagnetic wave propagating in a non-magnetic, conducting medium is given by

$$k^2 = \mu_0 \varepsilon \omega^2 + i\omega\mu_0\sigma_N,$$

where k is the wavevector, ω is the angular frequency, σ_N is the electrical conductivity and $\varepsilon = \varepsilon_r \varepsilon_0$ where ε_r is the relative permittivity. A material has a relative permittivity of 10 and an electrical conductivity of $2 \times 10^{-9}\ \Omega^{-1}\mathrm{m}^{-1}$. Can this material be considered a good conductor at a frequency of 10^9 Hz? [4 marks]

(e) A current of 1 Amp flows in the same direction through each of two infinitely long parallel wires that are 1 m apart. Calculate the magnitude and direction of the force on each wire. [4 marks]

(f) Provide a pictorial representation of the spatial variation of the fields of a plane-polarised electromagnetic wave propagating through vacuum. Your diagram should make clear the directions of the $\underline{E}$ field and the $\underline{B}$ field for the wave and the direction of propagation. [4 marks]

(g) Describe what a waveguide is and give two examples where waveguides are used. [4 marks]

4. (a) The London equation for the magnetic field produced by a superconductor is given by

$$\underline{B} = -\frac{m_{\rm e}}{n(T)e^2}\ \underline{\nabla} \times \underline{J},$$

where $\underline{B}$ is the magnetic field, $n(T)$ is the temperature dependent density of electrons that are paired and $\underline{J}$ is the current density. Show that this equation together with Maxwell's equations leads to the Meissner state in superconductors. [3 marks]

$$[\text{Hint}: \ \underline{\nabla} \times (\underline{\nabla} \times \underline{F}) = \ - \ \nabla^2 \underline{F} + \underline{\nabla}(\underline{\nabla} \cdot \underline{F}).]$$

(b) When an external magnetic field is applied parallel to the surface of a large superconducting cylinder, the magnitude of the magnetic field decreases exponentially inside the superconductor with distance from the surface. Find an expression for the characteristic length (known as the London penetration depth λ) for the exponential decrease. [3 marks]

(c) Consider applying a small magnetic field parallel to the axis of a cylindrical sample of a superconductor with radius a. Using the definition of magnetic susceptibility, find an approximate expression for the magnetic susceptibility of the superconductor, in the superconducting state, in terms of λ and a at low temperatures where $\lambda << a$. [8 marks]

(d) The density of paired electrons increases as the temperature decreases, from a value of zero at the critical temperature $T_{\rm c}$, to a value n_0 at zero temperature. (i) Provide a sketch of how you expect the measured susceptibility to change as a function of temperature, from zero temperature up to $T_{\rm c}$, and (ii) find an approximate expression for the derivative of the susceptibility with respect to temperature, valid at low temperatures. [8 marks]

(e) A scientist grinds the superconductor up into a fine powder such that even at low temperatures, the penetration depth remains small compared to the powder size. (i) Add to your sketch in (d), for the superconducting cylinder, a sketch of the susceptibility of the powder, that shows how you expect the grinding to change the temperature dependent signal from the superconductor and (ii) discuss the important features of the new sketch you have provided for the powder. [8 marks]