Mathematical Methods in Physics

Workshop 7

7.1

Given that following results

$$\mathcal{L}[c](s) = \frac{c}{s}, \quad \mathcal{L}[e^{at}](s) = \frac{1}{s-a}, \quad \mathcal{L}[\cos at](s) = \frac{s}{s^2+a^2}, \quad \mathcal{L}[\sin at](s) = \frac{a}{s^2+a^2},$$

where c and a are constants, compute the inverse Laplace transform of the following functions

a)

$$\bar{f}(s) = \frac{1}{s(4+s^2)} \ .$$

b)

$$\bar{f}(s) = \frac{1}{(s^2-1)(s+3)};$$

c)

$$\bar{f}(s) = \frac{3s+2}{(s^2+2s+2)};$$

[*Hint: Use convolution theorem in a), partial fraction decomposition in b) and notice that in c) the function can be rewritten as* $f(s) = (3(s+1)-1)/(s+1)^2+1$.]

7.2

Consider the two functions

$$\phi(\mathbf{r}) = -\frac{(\mathbf{c}\cdot\mathbf{r})}{r^3}, \qquad \mathbf{a}(\mathbf{r}) = -\frac{(\mathbf{c}\times\mathbf{r})}{r^3}$$

where $\mathbf{c}$ is a constant vector and r is the modulus of the position vector $\mathbf{r}$, i.e. $r = |\mathbf{r}|$. Use the following properties of grad, div and curl

$$\begin{aligned}
\nabla(\phi\psi) &= \psi\nabla\phi + \phi\nabla\psi \\
\nabla\cdot(\phi\,\mathbf{a}) &= \nabla\phi\cdot\mathbf{a} + \phi(\nabla\cdot\mathbf{a}), \quad \nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}) \\
\nabla\times(\phi\,\mathbf{a}) &= (\nabla\phi)\times\mathbf{a} + \phi(\nabla\times\mathbf{a}), \quad \nabla\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\nabla\cdot\mathbf{a})\mathbf{b} - (\mathbf{a}\cdot\nabla)\mathbf{b} + (\nabla\cdot\mathbf{b})\mathbf{a},
\end{aligned}$$

to compute $\nabla\phi$, $(\nabla\cdot\mathbf{a})$, $(\nabla\times\mathbf{a})$ in terms of $\mathbf{c}$ and $\mathbf{r}$, and r.
[*Hint:* $\nabla(r) = \mathbf{r}/r$, $\nabla(1/r) = -\mathbf{r}/r^3$, $\nabla\cdot\mathbf{r} = 3$, $\nabla\times\mathbf{r} = 0$, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.]

7.3

Show that

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\nabla \cdot \mathbf{b})\mathbf{a}.$$

Consider only one component. The first step is:

$$(\nabla \times (\mathbf{a} \times \mathbf{b}))_i = \epsilon_{ijk} \nabla_j (\mathbf{a} \times \mathbf{b})_k = \ldots$$

[*Hint:* $\epsilon_{ijk}\, \epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.]

7.4

The equation for the current $I(t)$ in an RLC (resistence, inductance, capacitance) circuit for zero initial charge is

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)d\tau = \nu(t).$$

This equation represents the fact that at any instant the sums of the potential differences around a circuit loop must be zero (Kirchhoff's law, conservation of energy.) Solve this equation using Laplace transforms when $L = 2$, $R = 3$, $C = 1/3$, $\nu(t) = 3\cos t$, for zero initial current and charge. In order to do so you need the following result

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{\bar{f}(s)}{s}.$$

[*Hint: For calculating the inverse Laplace transform use the results provided in question 7.1.*]