ELECTROMAGNETISM – Examples Class 7th Set (Qns)
Pointing Vector and Radiation/Communication

Professor D P Hampshire – 2nd Year Physics Lecture Course

The material for this examples class is split into three parts. Part I gives some background material. Part II gives some worked examples. Part III gives some additional unseen questions.

Background Material

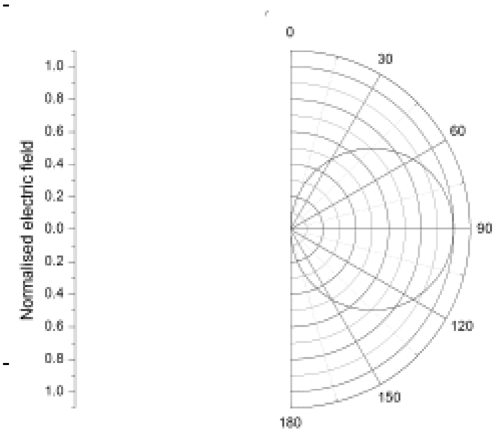
1. The Poynting vector:

The direction of propagation for electromagnetic waves is $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}$, Poynting's vector. $\underline{\boldsymbol{N}}$, gives the instantaneous power per unit area for electromagnetic waves and is given by,

$$\underline{\boldsymbol{N}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} \qquad 1\text{-}1$$

There are some definitions that provide 'figures of merit' that determine whether something is well designed for a particular application. For example one might need a highly directional transmitter that does not it waste most of its power output by radiating EM waves towards an unpopulated region or off into space.

2. The polar diagram for a Hertzian dipole.



The solid curved lines give the relative strengths of the radiation field at different points on the surface of a sphere centred on the dipole. The dipole lies along the z-axis and the field is independent of the azimuthal angle.

3. Radiation resistance R_r

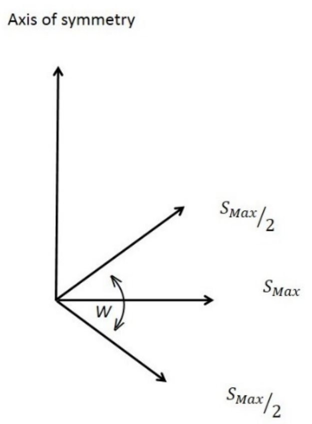
The definition of the radiation resistance R_r is:

$$P_{Time-averaged} = \frac{1}{2} I_0^2 R_r \qquad \text{1-2}$$

where $P_{Time-averaged}$ is the time-averaged power transmitted. For a half-wave Hertzian dipole $R_r = 197\Omega$

4. Beam Width W

By definition, the beam width is the angle between the half- power directions.



The beam-width, W , for a half-wave Hertzian dipole is π/2 (90^0).

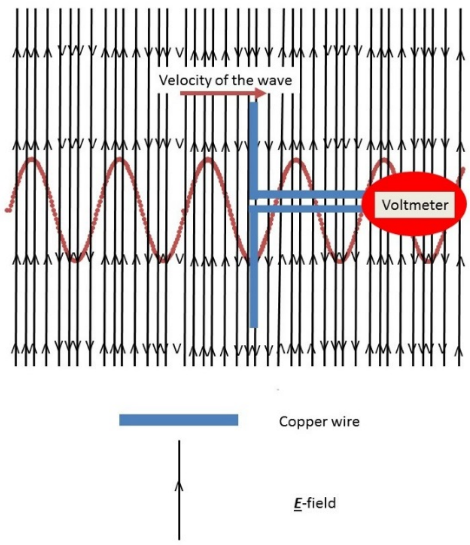
5. Directivity, D

By definition the directivity, D, is given by:

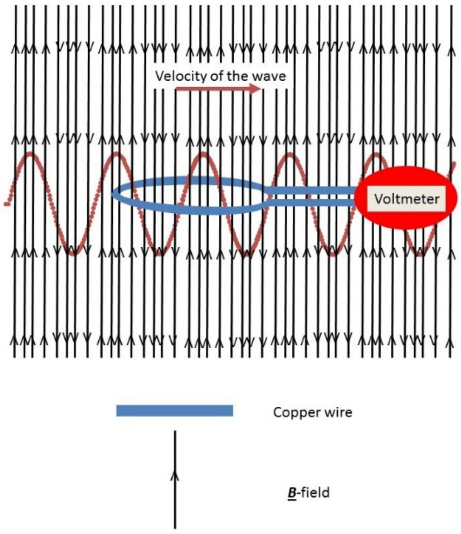
$$D = \frac{\text{Power per unit area in the peak direction}}{\text{Total power radiated in all directions/Total area}} \qquad \text{1-3}$$

For a half-wave Hertzian dipole D=3/2

6. Antennae:



An electric dipole antenna for detecting electromagnetic waves. The alternating electric field of the incoming wave produces an alternating current in the antenna.



A loop antenna for detecting electromagnetic radiation. The alternating magnetic flux through the loop due to the magnetic field of the radiation induces an alternating current in the loop.

1 Worked examples

If you are asked to write down equations or definitions in an exam, make sure that every symbol you use is properly defined using vector scalar notation and scientific English.

1.1 Questions

1. In free space, the electric field of an electromagnetic wave is given by: $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_o \cos(\mathrm{ky} - \omega\mathrm{t})$ where $\underline{\boldsymbol{E}}_o = 50\hat{\mathbf{i}}\ \mathrm{Vm}^{-1}$. Find the average power crossing a square of area 1 m^2 in the plane y = constant and in the plane x = constant.

2. The electromagnetic wave propagating in free space at distances far from a transmitter designed by a group of 2nd year students from Durham, is given in spherical coordinates by: $\underline{\boldsymbol{E}}(r,t) = \frac{10}{r}\sin^2\theta\cos\left(\omega\frac{r}{c} - \omega t\right)\hat{\boldsymbol{\Phi}}$ when an ac. current with a peak value, I_o, of 1 A flows through it.
Note: Integrate $\int_0^{\pi}\sin^5\theta.\mathrm{d}\theta = 16/15$

(i) Find the time-averaged power transmitted by the antenna.
(ii) What is the radiation resistance, R_r , for the antenna where: $P_{Average} = \frac{1}{2}I_0^2R_r$ and $P_{Average}$ is the time-averaged power radiated ?
(iii) What is the beam, width, W, for the antenna? The beam width is the angle subtended between the two half-power directions.
(iv) What is the value of the directivity? The directivity, D, is given by

$$\frac{\text{Power per unit area in the peak direction}}{\text{Total power radiated in all directions/Total area}}$$

1.2 Answers

1. $B_0 = \mu_0 H_0 = \frac{E_0}{c} \quad P = N_{av}1 = \frac{E_0 H_0}{2} = \frac{1}{2}\frac{E_0^2}{\mu_0 c}$

(i) Hence we have $P = \frac{50^2}{2\left(4\pi\times10^{-7}\cdot3\times10^{8}\right)} = 3.3$ W

(ii) 0

2. (i)

$$\underline{\boldsymbol{E}}(r,t) = \frac{10}{r}\sin^2\theta\cos\left(\omega\frac{r}{c} - \omega t\right)\widehat{\boldsymbol{\Phi}} \text{ and } |B| = |E|/c$$

$$\left|\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}\right| = \frac{100}{r^2 c\mu_0}\sin^4\theta\cos^2\left(\omega\frac{r}{c} - \omega t\right)$$

$$P_{Total} = \iint \frac{100}{r^2 c\mu_0}\sin^4\theta\cos^2\left(\omega\frac{r}{c} - \omega t\right)\cdot r^2\sin\theta d\theta d\phi$$
$$= \frac{100}{c\mu_0}\int_0^{\pi}\sin^5\theta d\theta\int_0^{2\pi}d\phi\left|\cos^2\left(\omega\frac{r}{c} - \omega t\right)\right| = \frac{100}{377}\frac{16}{15}2\pi\cdot\frac{1}{2} = 0.889 \text{ W}$$

(ii) R_r = 2×0.889/1= 1.78 Ω

(iii) $\sin^4\theta = 1/2 \Rightarrow \theta = 57^0 \rightarrow W = 2(90 - \theta) = 66^0$

(iv) $D = \frac{100}{r^2 377}\cos^2\left(\omega\frac{r}{c} - \omega t\right)4\pi r^2 / \frac{100}{377}\frac{16}{15}2\pi\frac{1}{2} = 15.4\pi/16.2\pi = 1.875$

2 Unseen problems

1. How would you prevent a very sensitive instrument from being affected by (i) radio waves (ii) dc. electric fields and (iii) dc. magnetic fields.

2. Consider two very large parallel plates that are both in the x-y plane with a surface charge density of +σ and –σ on each of them and a charge Q between the plates that is initially stationary with respect to the plates. The two plates and the charge are all moving at a high velocity v in the x-direction in a laboratory.
The relativistic vector field transformation for electric fields and magnetic fields are given by:

$E_x = E_x^{/} \quad E_y = \gamma(E_y^{/} + vB_z^{/}) \;\; E_z = \gamma(E_z^{/} - vB_y^{/})$

$B_x = B_x^{/} \quad B_y = \gamma(B_y^{/} - \frac{v}{c^2}E_z^{/}) \;\; B_z = \gamma(B_z^{/} + \frac{v}{c^2}E_y^{/})$

where $\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}$ and the primes relate to the E and B fields in the moving frame and the non-primes those in the stationary laboratory frame.
Calculate the E-fields and B-fields in both the frame of the plates and the laboratory frame. Show that the change in relativistic momentum $(F\cdot dt)$ for the charge Q is the same for both frames.

3. Write down a formula for the energy density associated with a constant electric field. Write down a formula for the energy density associated with a constant magnetic field.

4. Write down a formula for the time-averaged power per unit area associated with an electromagnetic field propagating in free space?

5. A plane EM wave in free space has an associated electric field amplitude of $E_o = 1.5 \times 10^{-3}\ \mathrm{Vm^{-1}}$. What is the amplitude of the associated H_o field and what is the value of N_{av} for this wave?

6. In free space, the magnetic field of an electromagnetic wave is given by: $\underline{\boldsymbol{B}} = \underline{\boldsymbol{B}}_{\mathrm{o}} \cos(kx - \omega t)$ where $\underline{\boldsymbol{B}}_{\mathrm{o}} = 2\,\hat{\mathbf{j}}$ nT. Find the average power crossing a square of area 1 m^2 in the plane x = constant.

7. A 10 m section of an infinitely straight wire with a circular cross-section of radius 1 mm is carrying a current of 1.5 A in the z direction in otherwise free space. The resistance of the 10 m length of wire is 2 Ω.
(a) What is the power dissipation due to Joule heating in the 10 m section of wire?
(b) What is the size and direction of the H field just at the surface of the wire?
(c) What is the size and direction of the E field in the wire?
(d) What is the value of the Poynting vector at the surface of the wire?
(e) Make use of the answers to (d) to calculate the total power flowing into the 10 m section of the wire from the EM fields.

8. The E-field of an electromagnetic wave propagating in free space at distances far from an antenna designed by a group of 2nd year students from Durham, is given in spherical coordinates by: $\underline{\boldsymbol{E}}(r,t) = \frac{10}{r}\sin\theta\ \cos\left(\omega\frac{r}{c} - \omega t\right)\hat{\boldsymbol{\phi}}$. when an ac. current with a peak value, I_o, of 1 Amp flows through it.
Note: Integrate $\int_0^{\pi} \sin^3\theta \cdot \mathrm{d}\theta = 4/3$

(i) Find the average power transmitted by the antenna.
(ii) What is the radiation resistance, R_r, for the antenna where: $P = \frac{1}{2} I_o^2 R_r$ and P is the total power radiated ?
(iii) What is the beam, width, W, for the antenna? The beam width is the angle subtended between the half-power directions.
(iv) What is the value of the directivity? The directivity, D, is given by
$$\frac{\text{Power per unit area in the peak direction}}{\text{Total power radiated in all directions/Total area}}$$

9. A coaxial cable is composed of a long straight metallic wire of radius a surrounded by a concentric cylindrical metallic sheath of inner radius b with air in between. An electromagnetic wave traveling within the cable has electric and magnetic fields given by;
$$\underline{\boldsymbol{E}} = \frac{Ve^{i(kz-\omega t)}}{r\cdot\ln(\frac{b}{a})}\hat{\mathbf{r}} \qquad , \qquad \underline{\boldsymbol{B}} = \frac{Ve^{i(kz-\omega t)}}{rc\cdot\ln(\frac{b}{a})}\hat{\boldsymbol{\phi}},$$
where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are unit vector of the cylindrical coordinate system, and z is the direction of propagation, along the length of the cable. V is a real constant.

Sketch the pattern of $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ fields at the time t = 0 in the plane z = 0.

By using the general equation $\underline{\boldsymbol{N}}_{av} = {}^{1}/_{2}\,\mathrm{Re}\left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^{*}\right)$ and integrating over the cross section of the cable in the air-filled region obtain an expression for the time-averaged power flow along the coaxial cable.

10. A small circular loop of wire of area a, in the x-y plane carries a current $I_0\cos\omega t$ radiates a vector potential distant from the loop given by :

$$\underline{\boldsymbol{A}}(r,t) = \frac{\mu_0 \omega a^2 I_0 \sin\theta}{4cr}\,\mathrm{expi}\left(\omega\frac{r}{c} - \omega t\right)\hat{\boldsymbol{\phi}}$$

Calculate the far-field E-field and B-field and hence using Poynting vector (or otherwise) show that the average power radiated is given by:

$$P_{Total} = \frac{\mu_0 \pi \omega^4 a^4 I_0^2}{12c^3}.$$