Restricted Perceptions and Regime Switches *

(Preliminary and incomplete, do not cite or distribute)

Tolga Ozden[†], Rafael Wouters[‡]

February 9, 2020

Abstract

We consider the estimation of Markov-switching DSGE models under adaptive learning (AL) to study the interaction between expectations and the business cycle over the Zero Lower Bound (ZLB) episode. We assume that regime shifts by monetary policy are not directly observable by agents, instead they indirectly learn about regime switches and economic conditions based on past observations. A novel feature of this setup is the persistence amplification and accelerated learning around sharp regime switches, illustrated in a basic Fisherian setup. We then use a variant of the well-known Kim & Nelson (1999) filter for Bayesian estimation of MS-DSGE models under constant gain adaptive learning, which we apply to the baseline 3-equation NKPC and workhorse Smets-Wouters (2007) models using U.S data. We find that (i) AL models outperform the Rational Expectations (RE) benchmark in all cases and the regime-switching RE model in most cases, (ii) the switch to ZLB episode is typically followed by a sharp jump in the learning parameters, suggesting a fast adoption of the new environment, (iii) the impulse responses in both RE and AL models typically change in the same direction with the switch to ZLB episode, but the magnitudes may vary. In particular, we find that the proportional change in fiscal multipliers over the ZLB period is smaller under AL compared to the REE benchmark. This suggests that standard models may overestimate the impact of a fiscal expansion over the ZLB period following the 2007-08 financial crisis.

JEL Classification: E37; E65; C11; C32 .

Keywords: Adaptive Learning; Markov-Switching; Bayesian Estimation of DSGE Models; Zero Lower Bound.

*The research project is funded by the National Bank of Belgium under *the Research Program for Young Researchers*. The views expressed in this paper are those of the authors and do not necessarily reflect the views of the National Bank of Belgium or any other institution to which the authors are affiliated.

[†]T.ozden@uva.nl, University of Amsterdam

[‡]Rafael.wouters@nbb.be, National Bank of Belgium

to do list:

-any differences between Dynare and the toolkit estimations? Based on the estimations of REE on pre-crisis data, there does not seem to any (they return identical estimates).
-initial beliefs matter quite a lot for the learning models, in particular for MSV-learning and the estimates of Calvo probabilities. This is not surprising but should take some space in the discussion.
-does the prior on gain make a difference? (current version vs. Slobodyan Wouters version)
-3-equation model and the surrounding discussion comes out of the paper (to be extended with forward guidance, (possibly) long-term interest rates and SPF data, including the dispersion of beliefs.)
-update the estimates with REE-consistent initial beliefs (at every parameter draw)
-endogenous switching model: there are alternative forms for the transition function which I might try, but the last version with one parameter fixed and parameter with tighter prior seems to do the trick so I am leaving it as it is.
-definition of shadow rate stays as it is (shadow rate lagged on previous shadow rate–> this generates persistent deviations from actual funds rate and the deviations will not correspond to monetary policy shocks).
-initial R matrix for AR(1) and VAR(1) matrix–check again that this setup is consistent with Slob. Wouters version.
-Simulations additional exercises: is the projection facility still binding frequently if the gain is very small? If so, there is another potential problem there that makes the facility bind.
-Exogenous switching simulations do not really go well with the flow of the paper, put them in the appendix (add one more exercise where the gain is much smaller, see what happens to the projection facility. In other words, is there a small enough gain such that the facility never binds?–as an approximation for the underlying RPE)
-MCMC–does it work when the projection facility is not imposed?
-Is there a clean version for the SPF data first two moments (mean and dispersion)–and when does it start?
-MCMC rise code–compare the current version with Raf's version
-fiscal multiplier definition–depends on the horizon, check what horizon other people used (in particular the Schorfheide student JME paper)
-for robustness checks of the estimated models: 1) estimations without mean, 2) fixed R matrix consistent with REE (generalized stochastic gradient learning), 3) fixed initial beliefs–i.e. current version, 4) maybe different sample periods. Repeat all of these only at the posterior mode, and most of them will probably go into appendix/online appendix.
-Include posterior mode values for the likelihood, Laplace can be potentially misleading sometimes.
-b shock: why is it not persistent in all of these models? (different than operationalizing REE paper). Potential candidates for this: reformulation of b-shock (i.e. its different than the original model, check this), or the definition of output gap (no flexible economy.)
-scales in figure 10: make them more consistent, makes it more confusing. This figure might also need to just come out. VAR1 and MSV forecasts in fact do not look that nice.
-VAR(1) model needs a different name, because its not a VAR.
-counterfactual simulations (code mostly ready, needs the latest estimations though). Also need to think about what to include. Distributions of inflation/output growth/interest rate/duration

of ZLB regime/frequency of ZLB regime, etc.
-In Section 2: first introduce the mean dynamics before putting in the lagged variables. Is the solution analytically tractable in this case? Is it possible to illustrate all points in one big figure? Switch out sections 2.2 and 2.3 in that case.
-impulse responses: emphasize that they move in the same direction, maybe don't focus as much on the fiscal multiplier.
-pages 17-23 on NKPC come out, discussed above.
-jumps in the learning parameters in Section 2–see how much this extends to the SW model estimates. Try to connect sections 2 and SW section as much as possible.

1 Introduction

With the onset of the Global Financial Crisis in 2007-08 and the subsequent drop of interest rates to near-zero levels among leading central banks, there has been increased interest among policymakers and central bankers alike about the ZLB constraint on nominal interest rates. There is still ongoing debate about the precise impact of the zero lower bound constraint on the economy as a whole and in particular about its macroeconomic cost in terms of aggregate GDP levels. Monetary and fiscal policy recommendations of standard macroeconomic models are mixed: for instance there is no consensus on the propagation of a government spending shock and the size of a government spending multiplier during this period. As a consequence, while some researchers recommend a fiscal austerity program to jump-start the economy during the ZLB episode, others think it is best to adapt a fiscal consolidation strategy. A common approach in most macroeconomic models examining the ZLB episode is the assumption of Rational Expectations Equilibria (REE): agents are assumed to have perfect information about the underlying economic conditions along with all other cross-correlations of the relevant macroeconomic variables and form their expectations accordingly. In this paper, we contribute to the growing literature on the ZLB episode by relaxing the perfect information assumption, and instead estimating DSGE models under adaptive learning subject to the ZLB constraint.

In standard REE-DSGE models, the perfect foresight assumption about regime switches leads to short periods of anticipated ZLB episodes: the expected duration of this period is typically between three to five quarters in most DSGE models estimated on the U.S. economy, see e.g. Lindé et al. (2017) and Ji & Xiao (2016), while the ZLB episode between 2009 and 2016 lasted for twenty eight quarters. Another shortcoming of the standard REE models is the overestimation of the impact of forward guidance on the macroeconomy, usually known as the forward guidance puzzle Del Negro et al. (2012). These shortcomings call for a relaxation of REE-restrictions and introduce informational frictions into the models. A plausible and popular method to introduce such frictions is adaptive learning, which relaxes the assumption that agents have perfect knowledge about the underlying economic conditions. Instead, they have their own sub-models, possibly under- or over-parameterized, that may not coincide with the correct economic structure. Agents act as econometricians and update their models each period as new observations become available. There is a vast and growing literature on the empirical validation of adaptive learning in DSGE models as well as monetary and fiscal policy implications of adaptive learning, see Evans & Honkapohja (2012) for a textbook treatment and Woodford (2013) for a comprehensive review of the more recent work. Much of the earlier literature on adaptive learning focused on the learnability of Rational Expectations Equilibria and MSV-learning, focusing on small and temporary deviations from perfect foresight models. Milani (2007) and Eusepi & Preston (2011) are earlier examples of expectations-driven business cycles and how MSV-learning can improve the empirical properties of small-scale DSGE models, while Bullard & Mitra (2002) and Bullard & Eusepi (2014) examine monetary policy implications of this type of learning. In more recent work, Slobodyan & Wouters (2012a) and Slobodyan & Wouters (2012b) show that further deviations from perfect foresight models with the use of small forecasting rules can lead to further improvements in the fit of a medium-scale DSGE model. On a similar vein, Quaghebeur (2018) examines fiscal policy implications of a VAR-type adaptive learning rule and finds that government spending multipliers are larger under adaptive learning. Evans et al. (2008) and Evans & Honkapohja (2010) examine the implications of adaptive learning for fiscal policy.

There are various different approaches to modeling the ZLB constraint: Some researchers use a perfect foresight & endogenous duration approach, which allows for a joint determination of expectations and regime switches; see e.g. Maih (2015) or Lindé et al. (2017). Another approach which is more common in VAR-literature is to use a threshold-switching method, where the economy is assumed to be in the ZLB regime if interest rates fall below some pre-specified level, see e.g. Bonam et al. (2017). A final approach is to use a Markov-switching framework, where the presence of the ZLB regime is determined by its predictive density, see e.g. Binning & Maih (2016). Lindé et al. (2017) show that Markov-switching and endogenous duration approaches typically lead to similar results as long as the ZLB constraint is accounted for. In this paper, we use the Markov-switching (MS) approach to take into account the constraint. Aside from the ZLB episode, MS approach recently gained popularity in DSGE literature to model structural changes such as monetary policy switches or volatility breaks, see e.g. Sims & Zha (2006), Davig & Leeper (2007), Sims et al. (2008), Liu et al. (2011), Liu & Mumtaz (2011), Bianchi (2016), Bianchi & Ilut (2017) and Bianchi & Melosi (2017) for some of the recent work. Other related work includes Bullard & Duffy (2004) that studies learning about unanticipated structural change in productivity in an RBC framework, and Hollmayr & Matthes (2015) that studies consequences of fiscal policy shifts when agents have uncertainty about the switch.

While Markov-switching and adaptive learning have both been increasingly popular classes of time-varying DSGE models in recent years, there is surprisingly little work on DSGE models that combine both approaches. Closely related theoretical work includes Branch et al. (2013) that studies the properties of MSV-learning in Markov-switching models where agents are informed about regime switches but learn the remaining economic parameters; and Airaudo & Hajdini (2019) that studies equilibria in a Markov-switching framework where agents use an optimal AR(1) rule without accounting for regime switches. Empirical studies closely related to our work include Gust et al. (2018) that examines the ZLB episode and forward guidance in a Markov-switching setup under Bayesian learning, where agents are aware of regime switches but have to infer about the underlying regime of the current economy; and Lansing (2018) that analyzes the ZLB episode in a calibrated setup under adaptive learning where regime switches are unobserved. Our key difference from these empirical papers and one of our main contributions is to extend their framework to non-MSV and non-rational beliefs, and to estimate the resulting DSGE models during the ZLB episode. We then examine the consequences of deviating from the REE during this period, particularly how it might contribute to a prolonging of the crisis and how it might change implications of standard DSGE models about the potential impact of a government spending shock during this episode.

Our key assumption is that the underlying regime changes are unobserved to economic agents. Instead they use a constant gain econometric model, where they only indirectly become aware of regime changes if these switches have an observable and strong enough impact on their information set. To set the ideas, consider the following example: A central bank follows a simple Taylor rule that reacts to inflation in setting interest rates. This will only be known to economic agents to the extent that the central bank discloses its goal of inflation targeting, but the agents never know the exact reaction coefficient. Accordingly, the agents will not find out if the central bank suddenly and discreetly decides to change its reaction coefficient. Instead, the agents will slowly find out about this regime shift as long as it leads to observable consequences in the interest rate and the resulting inflation levels.

Farmer et al. (2009) and Farmer et al. (2011) explore the class of REE in Markov-switching models. Since we assume that regimes are never observed, an equilibrium concept in our framework can never coincide with a Rational Expectations Equilibrium. Instead, in this limited information environment, there are so-called Restricted Perceptions Equilibria (RPE) where the agents' misspecification of the economy becomes self-fulfilling and the system settles on a non-rational equilibrium. To start with, we compute these equilibria in a 1-dimensional setup with the Fisherian equation, where the agents' perceived law of motion (PLM) has the form of a Minimum State Variables (MSV) solution, except that the PLM does not take into account the possibility of regime-switches. We show that standard E-stability conditions apply to these equilibria, and therefore the systems will converge to the underlying equilibria under standard recursive algorithms such as constant-gain least-squares. Furthermore, the E-stability and convergence results continue to hold even if one of the underlying regimes is E-unstable as long as the remaining regimes are sufficiently E-stable. This is a simple extension of the long-run determinacy result of Davig & Leeper (2007), which they call the long-run Taylor principle. We therefore denote our result as the long-run *E-stability principle*. We then extend this idea further to higher dimensional systems, where the PLM can also deviate from the MSV-solution in the form of small VAR-type forecasting rules: this allows the information set of the agents to be smaller than the MSV-solution due to, for example, unobserved shocks or unaccounted cross-correlations. The underlying RPE are too complicated to compute either analytically or numerically in this more general setup, although the underlying systems can always be simulated to observe the system behaviour and E-stability.

Next we consider a variant of the Kim & Nelson (1999) filter to estimate our class of MS-DSGE models under adaptive learning, and we apply the filter to the Bayesian likelihood estimation of two standard DSGE models: The first one is the 3-equation NKPC model along the lines of Woodford (2013), which provides a good starting point to expose our main results. The second one is the more complex and empirically relevant Smets & Wouters (2007) model, which is popular among central bankers and policy makers as a benchmark for policy analysis. Our estimation results can be summarized as follows: The MS-AL models outperform the standard REE benchmark in all cases, and the also the regime-switching REE models in a majority of cases. Furthermore, we observe important differences in the impulse response and shock propagation structure of the models under consideration. The models have important implications for the impact of government spending shocks in particular: we find that, during normal times, government spending multipliers tend to be larger under adaptive learning compared to REE. Over the ZLB period with inactive monetary policy, the multipliers become larger for both REE and AL models compared with normal times. However, the proportional change for the REE model is substantially larger compared to all AL models. This suggests that the benchmark REE model may severely overestimate the effect of a fiscal expansion following the 2007-08 crisis.

The paper is organized as follows: Section 2 illustrates the main concepts in a simple framework with one-forward looking variable. Section 3 shows the computation and E-stability results of the two classes of Restricted Perceptions Equilibria in DSGE models. Section 4 provides the filter used in our estimations, while sections 5 and 6 discuss the estimations results in the 3-equation NKPC and SW models respectively. Finally Section 7 concludes.

2 Preliminaries

2.1 Fisher Equation and Long-run E-stability

Consider first a simple model of Fisherian inflation determination without regime switching:

$$\begin{cases} i_t = E_t\pi_{t+1} + r_t \\ r_t = \rho r_{t-1} + v_t \\ i_t = \alpha\pi_t, \end{cases} \tag{2.1}$$

where r_t is the exogenous AR(1) ex-ante real interest rate, i_t is the nominal interest rate, π_t is inflation, and v_t is an IID shock process. We assume that monetary policy follows a simple rule by adjusting nominal interest rate to inflation, denoted by α[1]. After eliminating nominal interest rate i_t, the system can be re-written as follows:

$$\begin{cases} \pi_t = \frac{1}{\alpha}(E_t\pi_{t+1} + r_t), \\ r_t = \rho r_{t-1} + v_t. \\ \end{cases} \tag{2.2}$$

We use this small setup as our starting because, since it has been analyzed in Davig & Leeper (2007), which is one of the first studies on expectations in a regime switching setup; as well as in Airaudo & Hajdini (2019), which is the first study on small forecasting rules in a regime switching setup. The standard Minimum State Variable (MSV) solution takes the form of

$$\pi_t = dr_t. \tag{2.3}$$

In terms of adaptive learning terminology, (2.3) is known as the the *Perceived Law of Motion* (PLM). The Rational Expectations Equilibrium (REE) value of a is then pinned down by iterating the PLM forward to obtain the one-step ahead expectations, plugging the expectations back into the actual law of motion (2.2) and computing the associated fixed point, which yields $a = \frac{1}{\alpha-\rho}$. Hence the law of motion under REE is given by $\pi_t = \frac{1}{\alpha-\rho}r_t$. In this benchmark case, the equilibrium is said to be determinate if $\alpha > 1$, i.e. monetary policy is sufficiently aggressive.

Davig & Leeper (2007) consider scenarios where the interest rate coefficient α is subject to regime switches. Focusing on a two regime environment, assume that α changes stochastically between two regimes, $s_t = \{1, 2\}$ subject to the transition matrix:

[1]For the remainder, we assume that $Var(r_t) = 1$ to simplify the exposure.

$$Q = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{pmatrix}$$

Then inflation dynamics are given as:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}(E_t \pi_{t+1} + r_t) \\ r_t = \rho r_{t-1} + v_t. \end{cases} \tag{2.4}$$

Denoting by $\pi_{i,t} = \pi_t(s_t = i)$, we can rewrite the model in a multivariate form:

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \pi_{1,t} \\ \pi_{2,t} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} E_t \pi_{1,t+1} \\ E_t \pi_{2,t+1} \end{bmatrix} + \begin{bmatrix} r_t \\ r_t \end{bmatrix}. \tag{2.5}$$

Since the ALM is regime-dependent, expectations are also regime-dependent in a REE framework. Denoting by a_i the regime-specific solutions, the corresponding regime-dependent 1-step ahead expectations are given by:

$$\begin{cases} E_t[\pi_{t+1}|s_t = 1] = (p_{11}d_1 + p_{12}d_2)\rho r_t, \\ E_t[\pi_{t+1}|s_t = 2] = (p_{21}d_1 + p_{22}d_2)\rho r_t. \end{cases}$$

In other words, agents hold two distinct laws of motion, know the transition matrix Q and form their expectations correctly after observing the current regime s_t. Davig & Leeper (2007) show that, in this setup, the equilibrium is determinate as long as the *long-run Taylor principle (LRTP)* is satisfied:

$$\alpha_1 \alpha_2 > 1 - ((1-\alpha_2)p_{11} + (1-\alpha_1)p_{22}) \tag{2.6}$$

A key insight of this principle is that, the long-run dynamics of the model will be determinate even if one of the underlying regimes is indeterminate, provided there is at least one regime that is sufficiently determinate or the probability of entering into the indeterminate regime is sufficiently small. In what follows, we first relax the assumption of full information to replace it with that of learning, and then we extend the long-run determinacy insight into the concept of learnability, i.e. *E-stability* of equilibria.

Our main assumption in this paper is that agents do not directly observe or take into the regime shifts that occur in the economy when forming their expectations. Instead, they hold period-specific expectations that are updated each period as new observations become available. Therefore in our setup, regime switches are unknown to agents ex-ante, but only affect agents' expectations ex-post depending on their observable consequences. Before turning to learning, it is useful to first study the equilibrium properties of this setup. Hence assume that the economy evolves according to (2.4) with two regimes, where agents hold a regime-independent PLM and expectations as follows:

$$\pi_t = d r_t \Rightarrow E_t \pi_{t+1} = d E_t r_{t+1} = d \rho r_t. \tag{2.7}$$

The implied Actual Law of Motion (ALM) is then given by:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}(d\rho + 1) r_t \\ r_t = \rho r_{t-1} + v_t. \end{cases} \tag{2.8}$$

The assumed form of PLM here does not nest the regime-dependent MSV solution. Therefore, any resulting notion of equilibrium under this scenario cannot coincide with the full-information REE. Instead, we consider a limited-information equilibrium associated with the above PLM. This type of equilibrium is commonly referred to as a Restricted Perceptions Equilibrium (RPE) in the adaptive learning literature (see e.g. (Evans & Honkapohja, 2012)), where the agents use a restricted (and therefore misspecified) information set, which becomes self-fulfilling at the underlying equilibrium[2].

Unlike a REE, one cannot use the method of undetermined coefficients as above to pin down the value of d associated with the RPE. Instead, following Hommes & Zhu (2014), we impose a moment consistency requirement on the model to pin down the value of d: the coefficient d determines the *perceived correlation* between inflation and real rate of interest in the PLM, i.e. $d = \frac{E[\pi_t r_t]}{E[r_t r_t]}$. In an RPE, the unconditional correlation $\frac{E[\pi_t r_t]}{E[r_t r_t]}$, implied by the ALM is equal to d. In other words, the agents' forecasting rule is consistent with the actual outcomes on average but it is misspecified along each regime. The associated unconditional moment in our example is given as:

$$\frac{E[\pi_t r_t]}{E[r_t r_t]} = E[\frac{1}{\alpha(s_t)} d\rho + \frac{1}{\alpha(s_t)}], \tag{2.9}$$

which involves involves the long-run distribution (i.e. ergodic distribution) of the Markov chain denoted by P. Given the transition matrix Q, this follows $P = [\frac{1-p_{22}}{2-p_{11}-p_{22}}, \frac{1-p_{11}}{2-p_{11}-p_{22}}]$[3]. Then the underlying RPE coefficient, which we denote as d^{RPE}, is given by[4]:

$$d^{RPE} = \frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}-p_{22}) - \rho\alpha_1(1-p_{22}) - \rho\alpha_2(1-p_{11})}. \tag{2.10}$$

Further note that, the regime-specific MSV solutions (i.e. the solution when the economy is always in regime i) are given by $d^i = \frac{1}{\alpha_i - \rho}$, $i \in \{1, 2\}$. Given these expressions, the underlying RPE boils down to a weighted average of the regime-specific equilibria, where the weights are determined by the long-run distribution of the regimes. Instead of the standard determinacy of Rational Expectations models, our main concept of interest in this case is E-stability[5]. E-stability governs whether the agents can learn the above fixed-point by starting from an arbitrary point a_0, and updating their beliefs about the coefficient each period using a recursive system as new observations become available. As shown in Evans & Honkapohja (2012), E-stability is governed by the mapping from agents' PLM to the implied ALM, defined as the T-map. In our example, the T-map is given by:

$$T : d \rightarrow T(d) = \frac{E[\pi_t r_t]}{E[r_t r_t]} = (d\rho + 1)\frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}p_{22})} \tag{2.11}$$

The T-map is locally stable if the Jacobian matrix has roots with real parts less than one. When the local stability condition is satisfied, the equilibrium is said to be E-stable. In our

[2] Airaudo & Hajdini (2019) study RPE resulting from an AR(1) rule in this setup, i.e. not only the regimes are unobserved but the functional form of PLM is also misspecified.

[3] Note that ergodic distribution is obtained by solving $P'Q = P$.

[4] See Appendix A for details.

[5] Bullard & Eusepi (2014) shows that there is tight link between determinacy and E-stability of REE and in some special cases these conditions may even coincide.

case, the root and the associated E-stability condition are given as:

$$\frac{DT(d)}{D(d)} = \frac{\alpha_1(1-p_{22})+\alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}-p_{22})}\rho < 1, \tag{2.12}$$

which, after re-arranging, yields:

$$\alpha_1\alpha_2 > \frac{\alpha_1(1-p_{22})+\alpha_2(1-p_{11})}{2-p_{11}-p_{22}}. \tag{2.13}$$

This results in an E-stability criterion similar to that of *LRTP*. In order to obtain E-stability, a more aggressive monetary policy rule α_1 is needed whenever: (i) the average time spent in regime 1 (P_1) decreases, (ii) the average time spent in regime 2 (P_2) increases, or (iii) the monetary policy rule in regime 2 (α_2) becomes less aggressive. This suggests that it is possible to have E-stability despite having an E-unstable regime, as long as there is a sufficiently E-stable regime and the model does not spend too much time in the unstable regime on average. This is an intuitive extension of Davig & Leeper's insight on long-run determinacy to the learnability of equilibria, therefore we denote this as *the principle of long-run E-stability.*

2.2 Regime Switches and Constant Gain Learning

The RPE in (2.10) serves as a preliminary starting point to study the overall stability dynamics. However, our main point of interest in this paper is to study the transitory dynamics under adaptive learning when there is a monetary policy regime switch. In order to also consider learning dynamics about persistence, we first extend the model with a fraction ι_p of agents that have backward-looking expectations based on the previous period, while the remaining fraction $1-\iota_p$ form their expectations rationally as before. Accordingly, consider the following extension of the model:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}(\tilde{E}_t\pi_{t+1} + \iota_p\pi_{t-1} + r_t), \\ \tilde{E}_t\pi_{t+1} = \iota_p\pi_{t-1} + (1-\iota_p)E_t\pi_{t+1}, \\ r_t = \rho r_{t-1} + v_t, \end{cases} \tag{2.14}$$

where $\tilde{E}_t$ denotes aggregate expectations operator and E_t refers to the Rational Expectations as before. Assuming again that agents' do not observe the regime switches, the associated PLM of the rational agents is given as[6]:

$$\pi_t = dr_t + b\pi_{t-1}, \tag{2.15}$$

where the T-map is given by:

$$\begin{pmatrix} d \\ b \end{pmatrix} \rightarrow \begin{pmatrix} E[(\pi_t - b(s_t)\pi_{t-1})r_t] \\ \frac{E[(\pi_t - d(s_t)r_t)\pi_{t-1}]}{E[\pi_t^2]}, \end{pmatrix} \tag{2.16}$$

[6]We assume that rational agents take into account the presence of backward-looking agents when forming their expectations.

with $b(s_t) = \frac{\iota_p}{\alpha(s_t)-(1-\iota_p)b}$ and $d(s_t) = \frac{(1-\iota_p)d\rho+1}{\alpha(s_t)-(1-\iota_p)b}$. With the addition of lagged inflation, the moments appearing in the above expression become analytically intractable, therefore the values a^{RPE} and b^{RPE} and the associated E-stability conditions are obtained numerically in the examples below[7].

Next we introduce adaptive learning into the system, where beliefs about a and b are updated each period as new observations become available, using a constant-gain least squares method à la Evans & Honkapohja (2012). Denoting by $\theta = [d, b]'$ and $y_t = [r_t, \pi_{t-1}]'$, the coefficients are updated using:

$$\begin{cases} R_t = R_{t-1} + \gamma(y_t^2 - R_{t-1}), \\ \theta_t = \theta_{t-1} + \gamma R_t^{-1} y_t(\pi_t - \theta_{t-1} y_t), \end{cases} \tag{2.17}$$

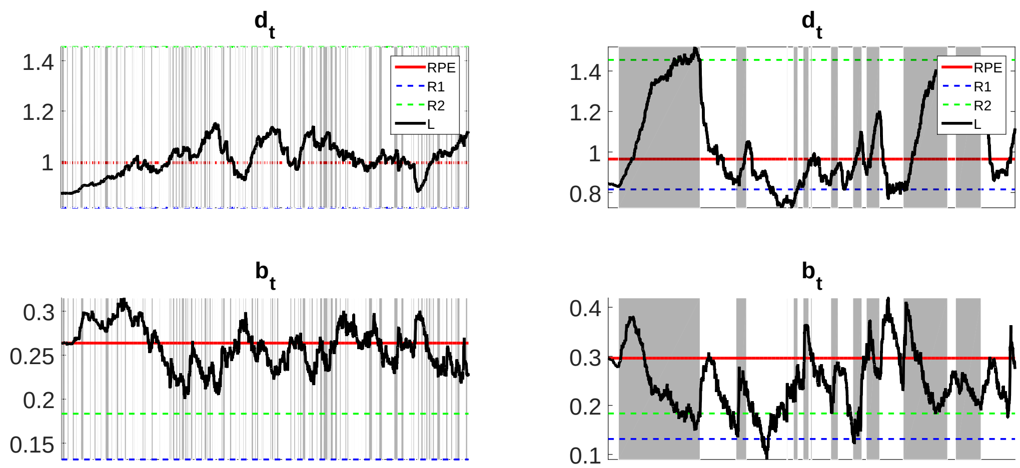
where γ denotes the gain value, i.e. the weight that agents put into the most recent observation. A costant gain implies geometric discounting of the past and allows agents to put more weight into recent observations, thereby allowing them to potentially detect the consequences of regime switches. We first illustrate the model dynamics for a parameterization where both regime-specific MSV-solutions, as well as the underlying RPE are E-stable. Figure 1 shows two simulations with different gain values and transition probabilities. Panel (a) shows an example with frequent regime switches, $p_{11} = p_{22} = 0.9$, and a small gain value of 0.005. In this case the learning coefficients oscillate around the RPE-consistent values, illustrating the stability of the system. An interesting feature of the RPE is that, while d^{RPE} is a weighted-average of the regime-specific equilibrium values, b^{RPE} is larger than the regime-specific values. This suggests that RPE is not necessarily a simple weighted average of the underlying regime-specific equilibria, and that regime-switching may induce persistence amplification in the system[8]. Panel (b) shows instead an example with more persistent regimes, $p_{11} = p_{22} = 0.99$, and a larger gain value of 0.01. It is readily seen that when the gain value is sufficiently large and the regime durations are long, the system converges towards the regime-specific values, i.e. agents forget about the past regime switches. When the regime shift occurs, there are two possible outcomes: if the RPE and the new regime specific value are in the same direction, such as for d_t, then the learning process gradually moves towards the new direction. If the RPE and the new regime specific value are in different directions, such as for b_t in this example, then the learning process first jumps towards the RPE, then starts to gradually move towards the regime specific value. This figure illustrates that, under the right circumstances, regime switches may be characterized by periods of temporarily amplified persistence. More importantly, learning of the new regime can be very quick, especially when exiting a very long regime or entering into a new regime that has not been observed before. These results are in line with Hollmayr & Matthes (2015), where unacticipated structural change leads to a temporary period of fast learning and amplified volatility. In our framework, this phenomenon occurs as a temporary shift towards the RPE. These characteristics are particularly important from an empirical point of view, since the recent ZLB episode is similar to such a switch from a persistent regime to a new regime that was not experienced in the recent past. To examine further where the jumps in the learning coefficients are coming from, Figure 2 shows the perceived covariance matrix

[7] See Appendix B for details on the implied moments.

[8] We do not plot the RPE as a function of the structural parameters for brevity, but our simulations show that persistence amplification arises whenever the persistence of the exogenous driving process ρ is sufficiently large.

along with the agents' forecast errors for the second simulation with $p_{11} = p_{22} = 0.99$ and $\gamma = 0.01$. It is readily seen that regime switches are accompanied both by jumps in the forecast errors, as well as jumps in the perceived perceived variance of inflation and perceived covariance between inflation and real interest rates. As such, the accelerated learning and jumps in the coefficients over regime switches are driven not only by the increased forecast errors, but also due to perceived changes in the covariance structure of the system.

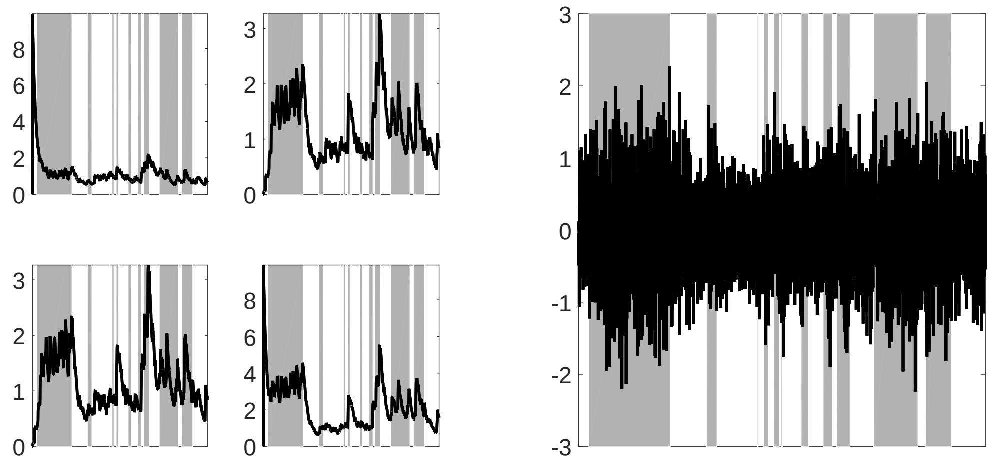
Figure 1: Learning coefficients along with the RPE-consistent and regime-specific values. The parameters $\iota_p = 0.25, \rho = 0.9, \alpha_1 = 1.5, \alpha_2 = 2$, are fixed in both simulations. Given the values of α_1 and α_2, both regime-specific equilibria and the RPE are E-stable.



(a) $\gamma = 0.005, p_{11} = p_{22} = 0.9$.

(b) $\gamma = 0.01, p_{11} = p_{22} = 0.995$.

Figure 2: Perceived covariance matrix and forecast errors from the simulation on the right in Figure 1 with $\gamma = 0.01, p_{11} = p_{22} = 0.995$. The left panel shows the perceived covariance matrix, i.e. $E[y_t y_t']$, while the right panel shows the forecast errors, i.e. $(\pi_t - \theta_{t-1} y_t)$.



(a) $\gamma = 0.005, p_{11} = p_{22} = 0.9$.

(b) $\gamma = 0.01, p_{11} = p_{22} = 0.995$.

2.3 Mean Dynamics

As a final step in this section, to motivate learning dynamics about the mean, we assume that nominal interest rates react to deviations of inflation from its non-zero target rate $\bar{\pi}$, i.e. $i_t - \bar{\pi} = \alpha(s_t)(\pi_t - \bar{\pi})$. After re-arranging, this can be re-written as:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}((\alpha(s_t) - 1)\bar{\pi} + \tilde{E}_t \pi_{t+1} + r_t), \\ \tilde{E}_t \pi_{t+1} = \iota_p \pi_{t-1} + (1 - \iota_p) E_t \pi_{t+1}, \\ r_t = \rho r_{t-1} + v_t, \end{cases} \tag{2.18}$$

where the rational agents' PLM is given by:

$$\pi_t = a + b\pi_{t-1} + dr_t, \tag{2.19}$$

and the T-map:

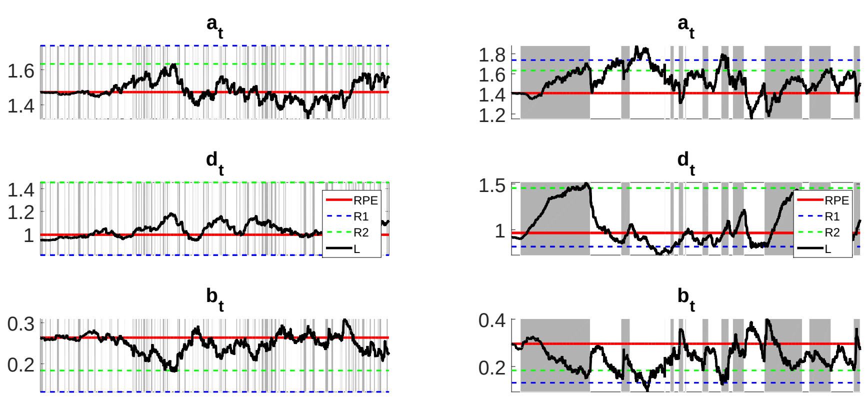
$$\begin{pmatrix} a \\ b \\ d \end{pmatrix} \rightarrow \begin{pmatrix} E[(\pi_t - b(s_t) X_{t-1} - d(s_t)\epsilon_t)] \\ E[\frac{(\pi_t - a(s_t) - b(s_t)\pi_{t-1}) r_t}{E[r_t^2]}] \\ \frac{E[(\pi_t - a(s_t) d(s_t) r_t)\pi_{t-1}]}{E[\pi_t^2]} \end{pmatrix}. \tag{2.20}$$

with $a(s_t) = \frac{(\alpha(s_t)-1)\bar{\pi} + (1-\iota_p)a}{\alpha(s_t) - (1-\iota_p)b}$, $b(s_t) = \frac{\iota_p}{\alpha(s_t) - (1-\iota_p)b}$ and $d(s_t) = \frac{(1-\iota_p)d\rho + 1}{\alpha(s_t) - (1-\iota_p)b}$. Figure 3 illustrates two simulations with the same parameterization from before and $\bar{\pi} = 2$, where both regime-specific equilibria and the RPE are E-stable. Panel (a) again shows frequent regime switches with a small gain value, while Panel (b) shows infrequent switches with a larger gain. We observe that in this case, the RPE value for a is lower than both regime-specific values, which confirms our result from the previous section that the RPE is not necessarily given as a simple weighted average of the regimes. In this case, the lower value of a^{RPE} suggests that the perceived inflation target is lower under RPE than both regime specific values. While we observe oscillations near the RPE-value in the first simulation, the second one shows jumps towards the RPE value along with regime switches, followed by a gradual movement towards the regime-specific values as the regime persists.

A standard result in New Keynesian models is that the mean dynamics are not stable under learning with a passive interest rate rule. This has received considerable attention in the recent macro literature since the monetary policy is restricted to be passive during the ZLB episode, which leads to E-unstability for mean dynamics. Within our framework, using the notion of long-run E-stability such a regime leads to temporary episodes of instability and a return to stable dynamics once the regime switches back. Figure 4 illustrates a parameterization with such dynamics: monetary policy switches between $\alpha_1 = 2$ and $\alpha_2 = 0.75$[9] with transition probabilities $p_{11} = 0.99$ and $p_{22} = 0.95$. At these values, the first regime is E-unstable and the RPE is still stable. We shut off $\iota_p = 0$ and $\rho_r = 0$ such that the dynamics around b^{RPE} and d^{RPE} are still stable. It is readily seen that a_t still fluctuates around the RPE value, but now with short bursts of unstable periods. Further, one can still observe the same phenomenon of jumps not only in b_t, but also in a_t, which is visible in both simulations. It is important to note that the overall stability of the model is driven purely by the exogeneity of regime transition probabilities, and we consider extensions with endogenous probabilities in latter sections that can give rise to unstable dynamics.

In the next section, we extend our analysis to a general multivariate setup in order to approximate the ZLB episode as a regime-switch in New Keynesian models.

Figure 3: Learning coefficients along with the RPE-consistent and regime-specific values. The parameters $\bar{\pi} = 2, \iota_p = 0.25, \rho = 0.9, \alpha_1 = 1.5, \alpha_2 = 2$, are fixed in both simulations. Given the values of α_1 and α_2, both regime-specific equilibria and the RPE are stable.

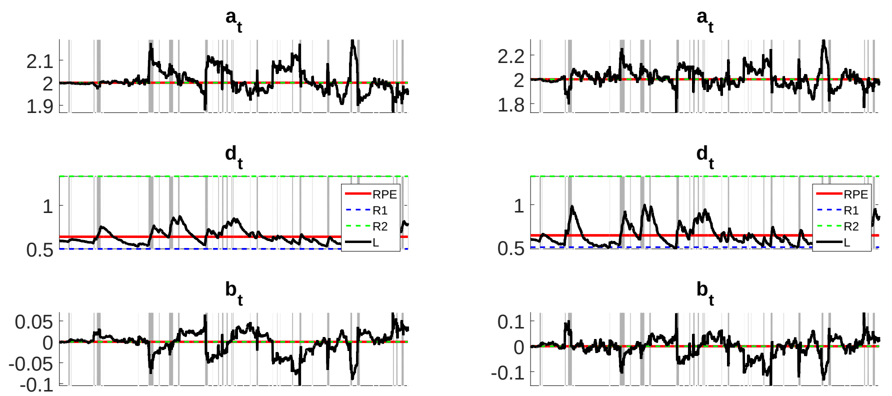


(a) $\gamma = 0.001, p_{11} = p_{22} = 0.99$.

(b) $\gamma = 0.005, p_{11} = p_{22} = 0.999$.

[9]The reaction coefficient of 0.9 is close to the stability region and still far from a passive policy rule; we use this value since even lower values have a large impact in this 2-equation setup. Simulations with a passive rule are provided in latter sections in more realistic model setups.

Figure 4: Learning coefficients along with the RPE-consistent and regime-specific values. The parameters $\mu = 2, \iota_p = 0, \rho = 0, \alpha_1 = 0.75, \alpha_2 = 2$, are fixed in both simulations. Given the values of α_1 and α_2, the mean dynamics in regime 1 are not E-stable but the RPE is E-stable.



(a) $\gamma = 0.005, p_{11} = 0.9, p_{22} = 0.95$. (b) $\gamma = 0.01, p_{11} = 0.99, p_{22} = 0.95$.

3 General Setup and Estimation

Our simple example in the previous section with only one forward-looking variable serves as an introduction to the main concepts that we consider in this maper. In this section, we extend our notion of restricted perceptions to a general class of linear multivariate models. Consider the following data generating process:

$$\begin{cases} X_t = A(s_t) + B(s_t)X_{t-1} + C(s_t)E_t X_{t+1} + D(s_t)\epsilon_t \\ \epsilon_t = \rho\epsilon_{t-1} + \eta_t \end{cases} \tag{3.1}$$

where X_t denotes the state-variables that may depend on their lags, 1-step ahead expectations and the structural shocks ϵ_t, which itself follow a VAR(1) process. We assume that the corresponding matrices A, B, C, D and E contain the structural parameters of the model, some of which are subject to regime switches captured by s_t[10]. The corresponding regime-independent PLM of agents is given by:

$$\begin{cases} X_t = a + bX_{t-1} + d\epsilon_t, \\ E_t X_{t+1} = a + bX_t + (d\rho)\epsilon_t, \end{cases} \tag{3.2}$$

where we use a *period t dating* assumption, i.e. structural shocks and contemporaneous variables are observed at the time of forming expectations[11]. Further note that the above specification nests many benchmark PLMs as a special case, which will be discussed below. Plugging the expectations in (3.2) back into (3.1) yields the implied ALM:

$$X_t = A(s_t) + B(s_t)X_{t-1} + C(s_t)a + C(s_t)bX_t + (C(s_t)(d\rho) + D(s_t))\epsilon_t, \tag{3.3}$$

[10] One may also consider regime switches in the shock processes, we omit these cases here.

[11] The alternative is to assume period t-1 dating, which uses information from the previous period, which effectively leads to 2-step ahead forecasts in expectations.

which can be re-written as

$$X_t = a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t, \tag{3.4}$$

where $a(s_t) = (I - C(s_t)b)^{-1}(A(s_t) + C(s_t)a), b(s_t) = (I - C(s_t)b)^{-1}B(s_t)$ and $d(s_t) = (I - C(s_t)b)^{-1}(C(s_t)(d\rho) + D(s_t))$. In this case the T-map is given as:

$$\begin{pmatrix} a \\ b \\ d \end{pmatrix} \rightarrow \begin{pmatrix} E[X_t - b(s_t)X_{t-1} - d(s_t)\epsilon_t] \\ E[(X_t - a(s_t) - d(s_t)\epsilon_t)E[X_tX_t']^{-1}] \\ E[(X_t - a(s_t) - b(s_t)X_{t-1}]E[\epsilon_t\epsilon_t']^{-1} \end{pmatrix} \tag{3.5}$$

Appendix C provides the first and second moments that appear here for a general setup with m regimes. While equilibrium values for a^{RPE} and d^{RPE} are easily computed for a given matrix b^{RPE}, the matrix b^{RPE} is intractable, which means the corresponding E-stability conditions are also intractable. Therefore for the multivariate models considered in the remainder of this paper, we rely on simulations to check for the stability of the systems. Finally, to introduce adaptive learning, denote by $\Phi_t = [a, d, b]'$ and $Y_t = [X_{t-1}, \epsilon_t]'$, the coefficients are updated using least squares:

$$\begin{cases} R_t = R_{t-1} + \gamma(Y_t^2 - R_{t-1}), \\ \Phi_t = \Phi_{t-1} + \gamma R_t^{-1}y_t(X_t - \Phi_{t-1}Y_t). \end{cases} \tag{3.6}$$

With the introduction of adaptive learning into the the Markov-switching framework, we have a system characterized by two types of time variation, which can be written in the following compact state-space form:

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)}S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)}\eta_t, \quad , \eta_t \sim N(0, \Sigma) \end{cases} \tag{3.7}$$

with $S_t = [X_t', \epsilon_t']'$ and $\gamma_{1,\Phi_t}^{(s_t)}, \gamma_{2,\Phi_t}^{(s_t)}$ and $\gamma_{3,\Phi_t}^{(s_t)}$ conformable matrices in terms of structural parameters, which depend on the assumption of the PLM. We next discuss the estimation of the general model in (3.7).

3.1 Estimation

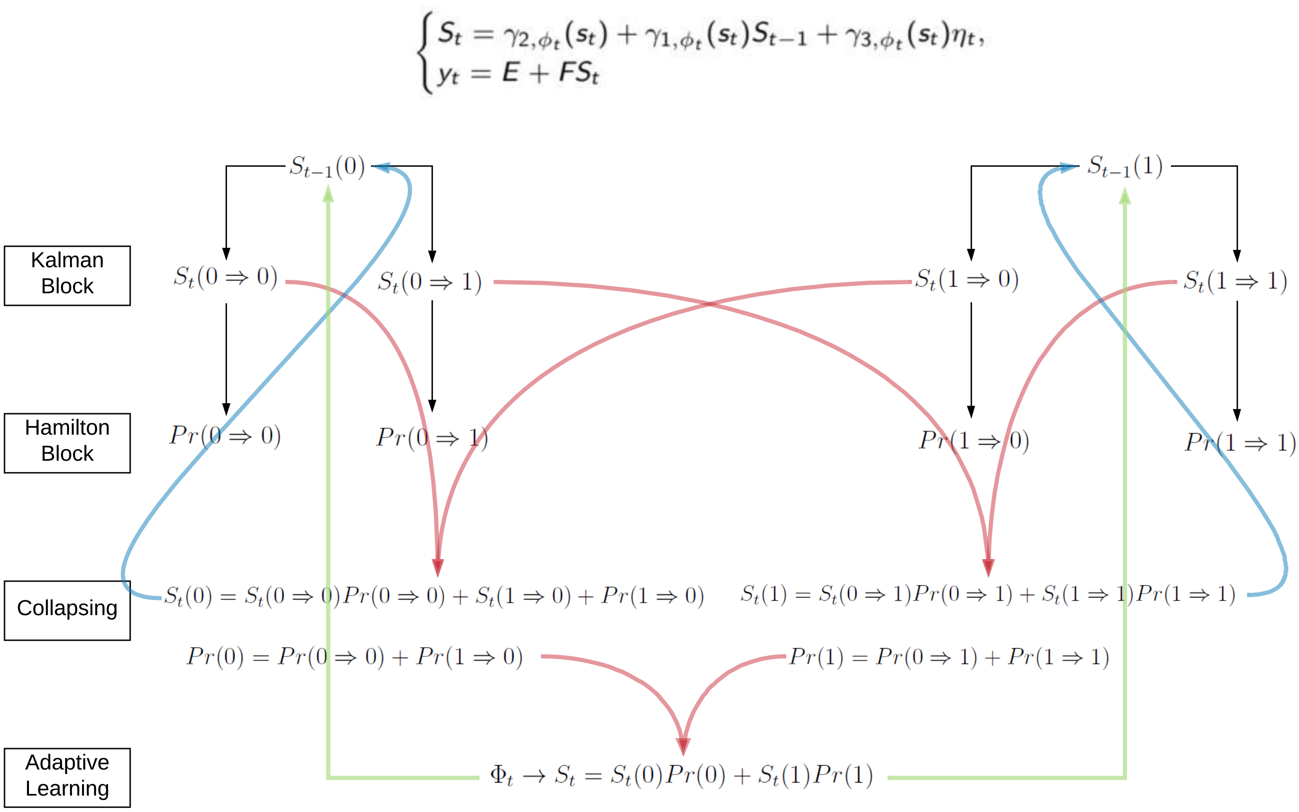
The benchmark algorithm for Markov-switching state-space models is the modified Kalman filter by Kim & Nelson (henceforth KN-filter): in a Markov-switching model with m regimes, a dataset of size T leads to m^T possible timelines, which quickly makes the standard Kalman filter intractable. The main idea in the KN-filter is to introduce a so-called *collapsing* technique to deal with this issue, which amounts to taking weighted averages of the state vector and covariance matrix at each iteration of the filter. This effectively reduces the number of timelines at each iteration by an order of m, thereby making the filter tractable again. The standard recommendation is to carry as many lags of the states as appears in the transition equation. Since we only consider we consider DSGE models that have a reduced-form VAR(1) representation in this paper, only a version of the filter with a single lag is presented here, although the same framework can be easily extended to any VAR(p) framework. Accordingly, if there are m different regimes in the model, we carry m different timelines in each period. Therefore there are m^2 different sets of variables in the forecasting and updating steps of each iteration. These are then collapsed at the end of each iteration to reduce to m sets of variables. An important question is how to introduce adaptive learning into this framework. We use an approach that is consistent with the theoretical framework of the previous section: the agents have a unique PLM based on observables, independent of the regime switches. We model this formally by collapsing the m different states further at each iteration to obtain the final states estimated by the filter, which are then used for the adaptive learning step. The unique learning coefficients are then used in each Kalman filter timeline of the next period's iteration[12]. Extending (3.7) with a set of measurement equations, the state-space representation of the model is given as follows:

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)} S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)} \eta_t, & ,\eta_t \sim N(0,\Sigma) \\ y_t = E + FS_t, \end{cases} \tag{3.8}$$

Given this representation, Figure 5 illustrates the KM-filter for the special case of two regimes, see Appendix B for the details. The filter as illustrated below yields the likelihood function, which is then combined with a set of prior distributions for Bayesian inference.

[12] A natural alternative here is to apply the adaptive learning step distinctly to each collapsed state; one can then take a weighted average of these expectations to obtain the filtered expectations. Our results in the upcoming sections are not sensitive to such an alternative, but we only present the results under the first approach since it is more in the spirit of our theoretical framework.

Figure 5: Illustration of the filter in a 2-regime model.



Initial Beliefs

A first practical issue in empirical studies on learning is where to initialize the beliefs. This has been shown to play a key role in driving the estimation results and model fit in previous studies, and various different approaches have been considered: Milani (2007) uses an estimation-based approach, where the initial beliefs are treated as free parameters and estimated jointly along with the other structural parameters of the model; Slobodyan & Wouters (2012b,2012a) consider REE-based and training-sample based approaches along with the estimation-based approach; while Berardi & Galimberti (2017c) proposes a smoothing-based approach. A common result in these studies is that the results are generally sensitive to initial beliefs, and the best-fitting approach depends on the specific model under consideration; see Berardi & Galimberti (2017a, 2017b) for a detailed overview on initial beliefs.
In this paper we follow the approach in Slobodyan & Wouters (2012b) with REE-based initial beliefs[13]. Specifically, we first estimate the benchmark REE model without regime switching over the pre-ZLB period. The relevant moments implied by the matrices γ_1 and γ_3 from this estimation are used as initial beliefs for the learning models[14]. This method has the advantage of not being computationally demanding since the initial beliefs are only computed once and remain fixed throughout the rest of the estimation.

Projection Facilities

A second issue with the estimation of adaptive learning models relates to retaining the stationarity of the model. A well-known issue with adaptive learning, particularly with constant gain recursive least squares learning, is that the stationarity of the underlying models are not guaranteed. Particularly when the PLM involves lagged state variables, the learning process may occasionally push the system into non-stationary and explosive regions, even if the underlying equilibrium is stable. Models subject to the zero lower bound constraint may be even more prone to encounter this problem, since typically an inactive monetary policy implies indeterminacy and E-unstability for the regime-specific dynamics.
A common method in the adaptive learning literature to deal with these potential instabilities is to impose a projection facility on the model, which forces the model dynamics to be stationary by projecting the learning parameters into the stable region whenever instability is encountered. The simples approach to do this is to leave the parameters at their previous value if the learning update leads to non-stationarity, which is the method adopted in Slobodyan & Wouters (2012a). Specifically, we set up the projection facility as follows in our estimations: we stop updating the learning parameters each period if the update pushes the largest eigenvalue of the ergodic distribution of the model (3.7) outside the unit circle. In other words, we allow the regime-

[13] Alternative specifications with different initial beliefs methods can be found in the online appendix to this paper. Specifically, we consider: (i) a training-sample based approach where the filter at diffuse moments and the first 5-year period is discarded; (ii) an estimation-based approach where beliefs are estimated as free parameters in an auxiliary model, which are then used in the learning estimations; (iii) a filter-based approach, where, for each parameter draw, the beliefs are initialized at diffuse points and the filter is ran once, then the converged values of the beliefs coefficients are used as initial values for the second run of the filter. The results of the alternative estimations are provided in the Appendix, and while the relative fit of each model is sensitive to the initial beliefs, our main conclusions continue to hold in all specifications.

[14] The REE-implied intercepts are always zero, therefore the vector γ_2 is always initialized at the vector of zeros

specific dynamics to be temporarily non-stationary as long as the underlying implied ergodic distribution remains stable. Importantly, this approach also allows the agents' PLM (3.2) to become temporarily explosive as long as the underlying ergodic distribution is stable. These choices reflect our desire to keep the projection facility as inactive as possible; and while they do not have an impact on our estimation results in general, but they indeed affect the frequency of periods with an active projection facility.

It is important to note that, since the projection facility imposes stability on the model, it essentially overrides the E-stability of the underlying system. In other words, the estimated models are stable regardless of whether E-stability holds. This is a necessary restriction since the T-map (3.5) for the general system is not tractable, i.e. we do not have a simple expression that verifies whether the model is E-stable or not for a given parameter draw. Therefore, to check the E-stability of the models, we resort to Monte Carlo (MC) simulations at the posterior mode, i.e. stability in the estimations is assumed ex-ante via projection facility, and it is verified ex-post at the point estimates via MC simulations.

Learning Rules

Our discussion up to this point is based on the information set consistent with the MSV solution, where the only source of model misspecification arises from unobserved regimes. However, in principle, any information set may be considered in agents' PLM. In this paper, we will focus our attention on three types of learning rules: (i) MSV-consistent rule as discussed before, (ii) a VAR-based rule, which assumes unobserved shocks[15], (iii) an AR(1)-based rule, which assumes a univariate process for each forward-looking variable[16]. These types of forecasting rules have been successfully applied in recent past to improve the empirical fit or match stylized facts of otherwise standard DSGE models, see e.g. Slobodyan & Wouters (2012b) and Gaus & Gibbs (2018) and Di Pace et al. (2016). In the following sections, this framework is applied both to a baseline 3-equation New Keynesian model, and to the Smets-Wouters (2007) model to investigate the ZLB episode for the U.S. economy.

4 Baseline Hybrid New Keynesian Model

In this section, as a first step, we estimate a variant of the baseline hybrid NKPC model. Accordingly, consider the following system:

$$\begin{cases} x_t = \iota_y x_{t-1} + (1-\iota_y) E_t x_{t+1} - \frac{1}{\tau}(r_t - E_t \pi_{t+1}) + \epsilon_{x,t}, \\ \pi_t = \iota_p \pi_{t-1} + \beta(1-\iota_p) E_t \pi_{t+1} + \kappa x_t + \epsilon_{\pi,t}, \\ r_t = max\{0, \rho r_{t-1} + (1-\rho)(\phi_x x_t + \phi_\pi \pi_t) + \eta_{r,t}\}, \\ \epsilon_{x,t} = \rho_y \epsilon_{x,t-1} + \eta_{x,t}, \\ \epsilon_{\pi,t} = \rho_\pi \epsilon_{\pi,t-1} + \eta_{\pi,t}, \end{cases}$$

where x_t, π_t and r_t denote the output gap, inflation and interest rates respectively, while $\epsilon_{x,t}$ and $\epsilon_{\pi,t}$ are exogenous AR(1) processes. Following Lindé et al. (2017), we re-cast the interest rate rule above as a Markov switching process with two regimes:

[15] In terms of (3.2), this assumes d is the zero matrix

[16] In terms of (3.2), this assumes d is the zero matrix and b is diagonal.

$$\begin{cases} r_t(s_t = 1) = \rho r_{t-1} + (1-\rho)(\phi_x x_t + \phi_\pi \pi_t) + \eta^1_{r,t} \\ r_t(s_t = 2) = \eta^2_{r,t} \end{cases}$$

where monetary policy follows a standard Taylor rule in the normal regime (regime 1), and a white noise process in the ZLB regime (regime 2). In this framework there are 18 parameters to be estimated, to which we assign prior distributions consistent with the previous literature: the risk aversion coefficient τ has a gamma distribution with a mean 2 and standard deviation 0.5 as in An & Schorfheide (2007). The monetary policy reaction coefficients ϕ_π and ϕ_x are given Gamma distributions centered at 1.5 and 0.5 respectively, with a standard deviation of 0.25; these are the standard values associated with the Taylor rule. Interest rate smoothing ρ_r, shock persistence ρ_x, ρ_π and shock standard deviation coefficients η_x, η_π and η_r are consistent with Smets & Wouters (2007), the first two having a Beta distribution with mean 0.5 and st. dev 0.2, while the latter is assigned a Gamma distribution with mean 0.1 and st. dev 2. The indexation parameters ι_y and ι_p are assigned Beta distributions with mean 0.25 and standard deviation 0.1[17]. We assign uninformative Uniform priors for the exit probabilities $1 - p_{11}$ and $1 - p_{22}$ over the interval $[0, 1]$. This deviates from *Lindet al.* (2017), where the exit probabilities are assigned more informative Beta distributions. The gain coefficient is assigned a prior close to Slobodyan & Wouters (2012b) with a gamma distribution and a mean of 0.035, but we assume a tighter distribution with a standard deviation of 0.015. The slope of Phillips curve is assigned a Beta distribution centered at 0.3 with a standard deviation of 0.15, which has a slightly larger mean and variance compared toAn & Schorfheide (2007).

We use quarterly U.S. data over the period 1966:I-2016:IV on interest rates, inflation and output gap with the following simple measurement equations:

$$\begin{cases} y^{obs}_t = \bar{y} + y_t \\ \pi^{obs}_t = \bar{\pi} + \pi_t \\ r^{obs}_t = \bar{r} + r_t \end{cases}$$

where the mean parameters $\bar{y}, \bar{\pi}$ and $\bar{r}$ are assigned normal distributions based on the pre-1966 period, and output gap is based on the potential output gap measure of CBO. Table 1 shows the estimates at the posterior mode for five model specifications: the first two columns show the benchmark Rational Expectations cases with (REE-MS) and without (REE) regime switching[18]. First looking at the resulting model fit based on Laplace approximation, one can see that the MS-REE model leads to a substantial improvement over the benchmark REE model, implying that the regime shift on interest rates plays an important role in driving the model fit. Adding adaptive learning on top of Markov-switching improves the likelihood further:

[17] This prior has a lower mean and tighter standard deviation than usual; e.g. the priors for indexation parameters in Smets-Wouters model in a beta distribution with a mean of 0.5 and st. dev. of 0.2. In this setup, the MS-REE model leads explosive outcomes at the ZLB regime when the indexation parameters are sufficiently large. Therefore we choose to restrict these parameters to a lower regime by assuming a tight prior.

[18] The REE-MS specification is estimated using J. Maih's RISE toolbox (Maih (2015)), while the standard REE case is obtained from the Dynare toolbox (Adjemian et al. (2011)). For the constant-gain learning cases, we consider the three learning rules as discussed above. For the adaptive learning specifications, we use our filter as presented in the previous section. Note that RISE toolbox uses a variant of the same KN filter, hence the estimations are based on the same filter except for the adaptive learning component.

all three adaptive learning specifications outperform the REE-MS model, where the AR(1) rule yields the best model fit and the model fit under VAR(1) and MSV rules are fairly close to each other. Both of these results, individually, are consistent with the previous results found in the literature, i.e, it is well known that both Markov-switching and adaptive learning typically improve the model fit compared with the benchmark case. Our results here show that these two results are also complementary, i.e. putting the two together improves the results further compared with the individual cases.

Next we turn to a discussion of our parameter estimates: the results under REE-MS are generally similar to the REE model with the exception of the risk aversion parameter τ, which is fairly high under REE-MS with 5.08 compared with 3.74 under REE. These values are closer to REE under all learning cases, with 2.97, 3.47 and 3.57 under MSV, VAR(1) and AR(1) rules respectively. The slope of the Phillips curve is estimated at a small value of 0.001 under REE, while in ranges between $[0.004, 0.007]$ under REE-MS and learning models, with values close to each other under learning models and larger than MS-REE. This suggests the difference at the estimated slope comes from both the regime switch on interest rates, as well as learning. The estimated historical mean interest rates is somewhat lower under REE model compared to the remaining models since this parameter also accounts for the ZLB regime in the REE model, which introduces a downward bias in this parameter. We do not observe any important differences in the estimated Taylor rule coefficients, which are estimated at similar values over the range $[1.42, 1.46]$ for inflation reaction and $[0.32, 0.48]$ for output gap reaction. A similar argument applies to interest rate smoothing, which ranges over $[0.81, 0.88]$. While there are some differences in the estimated indexation and shock persistence terms, these two sets of parameters should be interpreted jointly: for some specifications the indexation term is low but the shock persistence term is high, such as inflation in REE-MS model, while in other cases indexation term is high but the shock persistence term is low, such as output gap in the VAR(1)-learning model. This suggests either of these terms can act as the source of inertia in the models, and we do not observe important differences overall when these two parameter groups are considered together. Similarly, there are some differences in the estimated standard deviations of the i.i.d shocks η_y and η_π, but the implied variances of the AR(1) processes $\epsilon_{x,t}$ and $\epsilon_{\pi,t}$ are fairly similar across all models once the shock persistence terms are taken into account.

The gain coefficient is estimated at 0.01 under AR(1)- and MSV-learning model, while it is higher at 0.03 under VAR(1) model. These estimates are within the range typically found in previous studies, see e.g. Slobodyan & Wouters (2012b) and Milani (2007). Further, Branch & Evans (2006) find that values over the range of $[0.005, 0.05]$ provide a good fit for the Survey of Professional Forecasters (SPF) dataset, which is consistent with these results. The projection facility, which is imposed based on the implied ergodic distribution of each model as described above, is never activated in all learning models for this small model[19]. The corresponding Monte Carlo simulations of each model can be found in latter sections.

Another important difference between REE-MS and learning models is the expected exit proba-

[19] However, as discussed above, this result is somewhat sensitive to how the projection facility is defined. In particular, the active eigenvalue of the system typically exceeds one temporarily once the economy switches to the ZLB episode. Hence if the projection facility is activated based on this, then it is occasionally binding for some periods especially around the crisis period. Nevertheless, the overall activity of the projection facility never exceeds 5% and the parameter estimates are not substantially affected by how this is set up.

bility from the ZLB regime, which is lower under all learning models: at the estimated parameter values, the probability is 0.21 under REE-MS which implies an expected duration of 4.76 quarters, whereas in the learning models this probability is 0.035, implying an expected duration of 28.57 quarters. This result suggests that under Rational Expectations, agents' unconditional expectations about the duration of the ZLB episode needs to be very short to make the model consistent with the data. Under adaptive learning models, the exit probability does not interact with the agents' expectations since their PLM is independent of the regime switches. Therefore the resulting probability simply reflects the unconditional expected duration over the sample period, which is 28 quarters.

Figure 6 shows the filtered regime probabilities for the AR(1)-learning case along with the historical interest rates over the estimation sample. It is readily seen that the estimated regime probability sharply rises in both cases in 2009 as the interest rates are lowered to near-zero levels[20] Next Figure 6 shows the filtered learning parameters for the AR(1) model over the estimation sample[21]. Focusing on the crisis and subsequent ZLB period, it is readily seen that there is a drop in the perceived intercept terms for both inflation and output gap, although the drop in output gap is sharper. For the persistence terms, there is a sharp increase in the output gap term immediately following the crisis, which gradually decreases; this pattern is similar to our simulations with the Fisher equation, where the regime switch leads to a temporary period of amplified persistence that gradually subsides. We do not observe a sharp change of the same magnitude in inflation persistence and in fact it slightly drops following the crisis. This is not surprising since the change in inflation during this period is smaller compared to output gap, and the short deflationary period is reflected in the intercept term rather than the first-order autocorrelation.

Figure 7 shows the impulse responses of output gap to supply and demand shocks for all estimated models: we consider the responses to one unit supply and demand shocks η_y and η_π respectively. The black and red lines at the left and right sides of each panel show the IRFs from MS-REE model. It is readily seen that there is considerable time-variation in the IRFs for both shocks under all learning models. A common result in all three learning models is the jump in the IRFs during the switch to the ZLB episode, but there are some differences in the overall pattern. The AR(1) model is characterized by a relative quick shift of only a few periods, and the time-variation is otherwise relatively small afterwards. In the VAR(1) model, there is a sizeable jump at the beginning of the switch, followed by a gradual shift downward.s It is interesting to see that in this case the period-specific IRFs gradually move towards the IRF of MS-REE model. The pattern in the MSV model is more gradual, where there is a small jump with the switch, after which the IRFs gradually become smaller for a while before stabilizing. The pattern in impulse responses make it clear that there are sizeable differences in the response of the model to the same shock, depending on the assumption of the information set and the perceived law of motion. This suggests that implied stochastic structure in the economy is different across models even in this small-scale setup. An important question raised by this result is whether the differences across regimes are smaller or larger under learning models compared with the REE-MS. Before discussing these differences, in the next section we

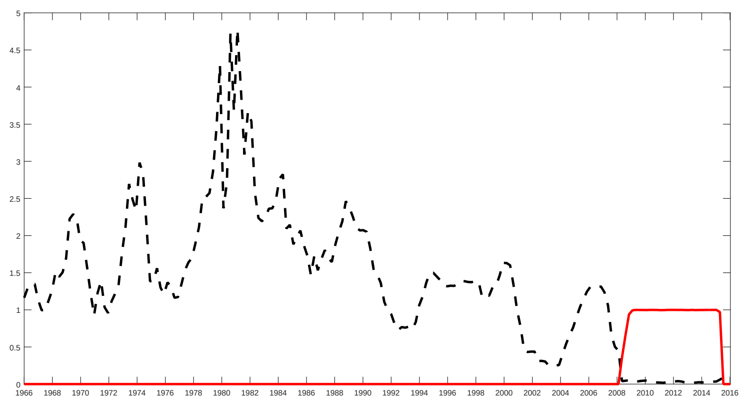
[20] An almost identical pattern is readily observed in other learning cases of VAR(1)- and MSV-learning model, as well as REE-MS, which we omit here.

[21] We show the learning parameters only for the AR(1) model since it is the best-fitting and also the most parsimonious one among the learning models, while the parameter space for the VAR(1)- and MSV-learning models are much larger. A brief discussion on these two cases can be found in the appendix.

first move onto a more realistic model setup and estimate the Smets & Wouters (2007) model.

Figure 6: Top panel shows the historical interest rates along with the estimated regime probabilities in AR(1)-learning model. The estimated regime probabilities are similar across all regime-switching models. The bottom two panels show the perceived mean and persistence coefficients in the AR(1)-learning for output gap and inflation. The coeffients in PLMs for MSV- and VAR-learning rules are omitted here but can be found in the appendix.

Historical interest rates along with the estimated regime probability:



AR(1) intercept (left) and persistence (right) coefficients in the PLM, with output gap and inflation on the first and second row respectively:

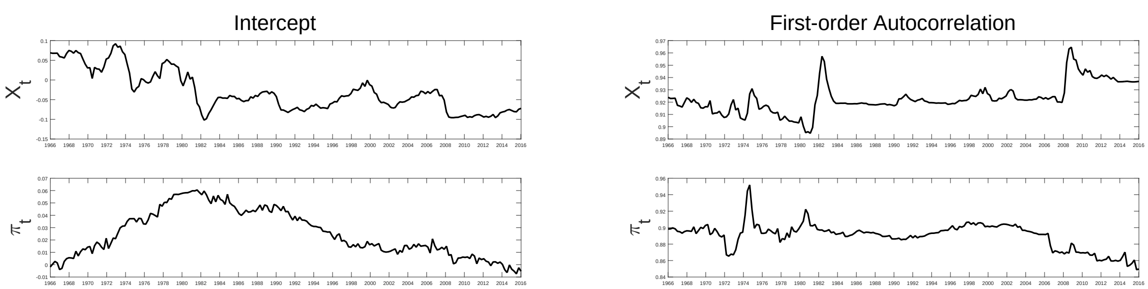
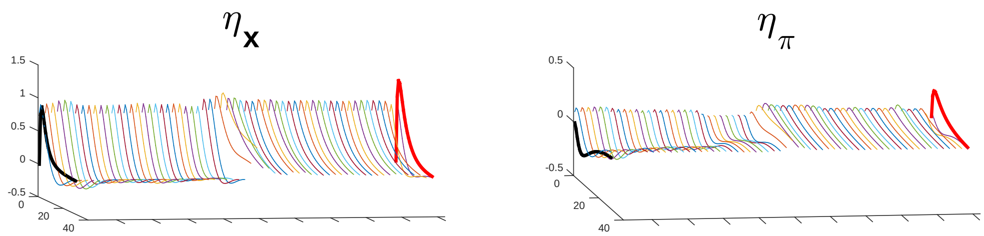


Table 1: Estimation Sample: 1966:I-2016:IV based on the U.S data, where the observables are the output gap (based on CBO's historical estimates), inflation and interest rate.The Markov-switching REE model is obtained from RISE toolbox, while the standard REE case is provided by Dynare. The AR(1), VAR(1) and MSV-learning cases are based on our algorithm above.

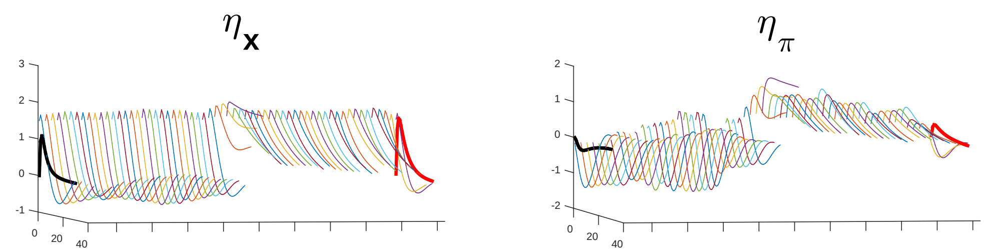
Parameter	Prior			REE		MS-REE		MSV		VAR(1)		AR(1)	
				Posterior		Posterior		Posterior		Posterior		Posterior	
	Dist	Mean	St. Dev	Mode	St. Dev	Mode	St. Dev	Mode	St. Dev	Mode	St. Dev	Mode	St. Dev
$\bar{y}$	Normal	0	0.25	-0.33	0.235	-0.22	0.24	-0.31	0.2	-0.18	0.214	-0.29	0.205
$\bar{\pi}$	Gamma	0.62	0.25	0.56	0.149	0.64	0.16	0.73	0.086	0.6	0.14	0.63	0.081
$\bar{r_1}$	Gamma	1	0.25	1.04	0.17	1.21	0.23	1.28	0.129	1.25	0.162	1.15	0.112
κ	Beta	0.3	0.15	0.001	0.0008	0.004	0.0026	0.006	0.003	0.007	0.004	0.007	0.004
τ	Gamma	2	0.5	3.74	0.573	5.08	0.66	2.97	0.425	3.47	0.52	3.57	0.509
ϕ_π	Gamma	1.5	0.25	1.44	0.175	1.45	0.167	1.42	0.182	1.42	0.193	1.46	0.151
ϕ_y	Gamma	0.5	0.25	0.36	0.091	0.32	0.087	0.41	0.073	0.48	0.131	0.44	0.117
ρ_y	Beta	0.5	0.2	0.03	0.022	0.85	0.03	0.79	0.028	0.27	0.069	0.31	0.065
ρ_π	Beta	0.5	0.2	0.34	0.068	0.89	0.032	0.04	0.028	0.03	0.026	0.03	0.026
ρ_r	Beta	0.5	0.2	0.87	0.026	0.8	0.03	0.87	0.017	0.88	0.03	0.88	0.03
η_y	Inv. Gamma	0.1	2	0.48	0.054	0.14	0.028	0.2	0.01	0.53	0.04	0.38	0.018
η_π	Inv. Gamma	0.1	2	0.15	0.008	0.03	0.009	0.2	0.01	0.16	0.02	0.13	0.006
η_{r1}	Inv. Gamma	0.1	2	0.29	0.015	0.32	0.018	0.31	0.017	0.31	0.017	0.31	0.014
ι_y	Beta	0.25	0.1	0.71	0.049	0.28	0.057	0.37	0.029	0.65	0.051	0.45	0.026
ι_π	Beta	0.25	0.1	0.53	0.027	0.05	0.024	0.63	0.035	0.49	0.07	0.42	0.025
$\bar{r_2}$	Normal	0.1	0.25			0.03	0.002	0.03	0.002	0.03	0.002	0.03	0.002
η_{r_2}	Uniform	0.005	0.05			0.01	0.001	0.01	0.001	0.01	0.001	0.01	0.001
$1-p_{11}$	Unif.	0.5	0.29			0.005	0.005	0.005	0.009	0.006	0.009	0.006	0.01
$1-p_{22}$	Unif.	0.5	0.29			0.21	0.06	0.1035	0.049	0.035	0.049	0.035	0.047
gain	Gamma	0.035	0.015					0.014	0.004	0.033	0.008	0.013	0.007
				Laplace	-360.16		-305.11		287.26		288.27		286.81
				Bayes F.	1		23.9		31.66		31.22		31.86

Figure 7: Implied time variation in the IRFs for learning models: responses of output gap to supply and demand shocks during the last 15 years of the estimation sample, 2002:I-2016:IV. The solid black (normal regime) and red (ZLB regime) lines on each side are the regime-specific IRFs of the MSV-REE model.

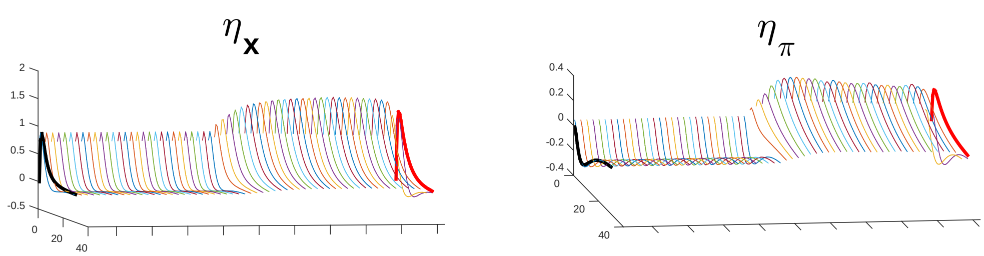
AR(1):



VAR(1):



MSV:



5 Smets-Wouters Model

In this section we consider the estimation of Smets-Wouters (2007). The details of the model are omitted here for brevity; the readers are referred to Smets & Wouters for a close-up of the model. We have two small deviations from the benchmark model: first, we assume the price and wage mark-up shocks follow exogenous AR(1) processes, instead of the original ARMA(1,1) assumption[22]. Second, we shut off the flexible economy side of the model, which is used to obtain the potential output and the associated level of output gap. Instead, we follow Slobodyan &

[22] This is due to the fact that, as shown in Slobodyan & Wouters (2012a), these shock processes are typically close to being white noise when expectations are based on small learning rules, in which case the AR(1) and MA(1) terms are close to being locally unidentified. Therefore we assume away the MA(1) terms in these shocks.

Wouters (2012a) and derive the output gap from the natural level of output, based on the underlying productivity process. This has the advantage of reducing the size of the model, thereby making its estimation computationally less demanding. The rest of the model, along with the prior distributions and measurement equations remain unchanged. Accordingly, the estimation is based on seven observables on the U.S. data over the period 1966:I-2016:IV as follows:

$$\begin{cases} d(log(y_t^{obs})) = \bar{\gamma} + (y_t - y_{t-1}) \\ d(log(c_t^{obs})) = \bar{\gamma} + (c_t - c_{t-1}) \\ d(log(inv_t^{obs})) = \bar{\gamma} + (inv_t - inv_{t-1}) \\ d(log(w_t^{obs})) = \bar{\gamma} + (w_t - w_{t-1}) \\ log(l_t^{obs}) = \bar{l} + l_t \\ (log(\pi_t^{obs})) = \bar{\pi} + \pi_t \\ (log(r_t^{obs})) = \bar{r} + r_t \end{cases}$$

where $d(log(y_t^{obs}))$, $d(log(c_t^{obs}))$, $d(log(inv_t^{obs}))$ and $d(log(w_t^{obs}))$ denote real output, consumption, investment and wage growths with the common growth rate $\bar{\gamma}$ respectively, while $log(l_t^{obs})$, $(log(\pi_t^{obs}))$ and $(log(r_t^{obs}))$ denote (normalized) hours worked, inflation rate and federal funds rate respectively. We use the same estimation sample as in the 3-equation NKPC with quarterly U.S. data covering the period from 1966:I to 2016:IV. Tables 3 and **??** show the results for the learning and benchmark REE and MS-REE cases. We observe that our main conclusions from the NKPC-estimation carry over to this medium-scale setup: the MS-REE model considerably improves upon the REE benchmark with likelihoods -1140 and -1194 respectively, which shows the empirical importance of explicitly modeling the ZLB episode. The MSV-learning setup yields a likelihood of -1159, which, unlike the 3-equation setup, is worse than the REE-MS case. This indicates the time-variation under MSV-learning does not lead to any meaningful improvements over the MS-REE model, and in fact the time-variation in the belief parameters comes at a cost in terms of model fit. Next looking at the parsimonious AR(1) learning case, we observe a likelihood of -1118, which is another substantial improvement over both the MSV-learning and MS-REE cases. Similarly, the VAR(1) learning case yields a likelihood of -1120, which is slightly worse than the AR(1) model but still better than MS-REE and MSV-learning cases. The model with small AR(1) forecasting rule is thus the preferred specification based on these results. Next briefly looking at the estimated structural parameter values, we see some differences in the learning and REE models: the estimated wage stickiness, and to some extend also price stickiness, are lower under learning models. Shock persistence and indexation terms, when considered together, are at similar levels across all models specifications (with some models having high indexation and low persistence and vice versa for others). Capital cost adjustment elasticity ϕ and intertemporal elasticity of substitution σ_c are also slightly lower under learning models. The remaining estimated parameters, in particualr the monetary policy, steady-state related and shock standard deviation parameters are fairly similar across all models and well within the range of each other. Taken together with the NKPC estimations, all of our results are consistent with the previous literature on learning: Milani (2007) shows that MSV-learning performs better that REE in the small 3-equation setup, while Slobodyan & Wouters (2012b) find that this result does not extend to the medium-scale setup of SW model, where MSV-learning does not lead to improvements over the REE model. Instead using a univariate backward-looking rule in Slobodyan & Wouters (2012a), they find that the model fit improves considerably. Our results in this section and the previous one confirm these results, and show that they continue to hold in a Markov-switching setup where the ZLB episode is considered.

In terms of the estimated exit probabilities from the ZLB regime, we observe the same pattern as in the baseline model: the exit probability under REE-MS moedl is 0.27 with an expected duration of 3.7 quarters, which is close to but somewhat lower than the estimate in the baseline model estimation. The estimated exit probability in the learning models is identical to that of baseline model with an expected duration of 7.7 quarters. This confirms that this estimate only reflects the underlying empirical duration of the ZLB and the prior distribution, regardless of the model specification and the assumed PLM.

The estimated gain values for learning models in this case turn out to be 0.017, 0.007 and 0.01 for AR(1), VAR(1) and MSV-learning models respectively. The implied speed of learning is highest for the AR(1) model in this case, different than the 3-equation model where largest gain is attained by the VAR(1) model. Nevertheless, all gain coefficients are consistent with ranges reported in previous empirical studies in the literature, similar to the 3-equation model.

The estimated regime probabilities for both REE and learning models also look fairly similar to that of NKPC model as already shown in Figure 6, therefore they are omitted here. Figure 8 shows the filtered learning parameters on intercept and first-order autocorrelation for the AR(1) learning model[23]. While not all variables are characterized by a jump in the learning parameters during the crisis period, the pattern is still visible for some variables: the real value of capital q_t, labor l_t and investment i_t variables show a downward break in the mean coefficient, and an upward jump in the perceived persistence term. Further, the patterns for both perceived mean and persistence in inflation follow the same pattern as in the 3-equation model: the perceived mean shows a downward trend after 1980s. It is characterized by only a small break downward break during the crisis period, while the persistence term has a slightly larger downward break. Furthermore, for the case of AR(1) model, the projection facility is activated for 3 subsequent quarters during this period, indicating that some of these jumps combined with an inactive monetary policy rule lead to a period of temporarily explosive dynamics, and the jumps would be potentially large in the absence of the projection facility[24].

A similar break in the perceived intercept terms can be seen in the MSV-learning model, and to a smaller extent also in the VAR(1)-learning model[25]. To see the effect of these changes in the perceived mean parameters on the model fit, Figure 9 plots the in-sample forecasts for the growth rates of output, consumption, investment and wages for the MS-REE and all three learning model. A well-known issue with standard REE models over the post-crisis period is the over-prediction of these growth rates: the sudden downward shift in the interest rates implies an increase in the growth rates in the model variables, whereas in fact the growth rates have been

[23]Similar to the small scale model, we only show the learning parameters for the most parsimonious and best fitting AR(1) here, while omitting the figures for VAR(1) and MSV learning cases since the parameter space for these cases becomes too large. More details on the corresponding figures can be found in the appendix.

[24]Overall, the projection facility is activated for 8 quarters in total in the AR(1) model, 3 of which following the crisis and the remaining around late 1990s and early 2000s. This correspond 3.9% of the total estimation sample. The projection facility never binds for the MSV and VAR(1) models. Hence if we also take into account the 3-equation specifications, the AR(1) model in SW is the only specification where the projection facility is activated.

[25]This break is smaller in the VAR(1) model due to the small estimated gain coefficient.

Table 2: Estimation period: 1966:I-2016:IV

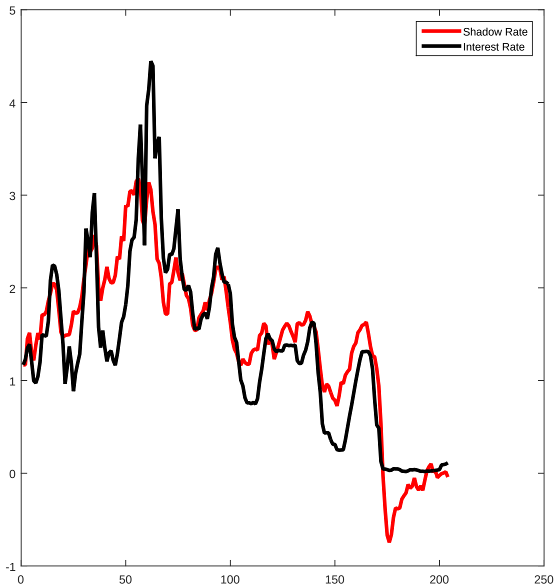
	Prior		REE Post.		REE-MS Post.		MSV Post.		VAR(1) Post.		AR(1) Post.		AR(1)-endo Post.	
Para	Dist	Mean	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.
ϕ	Normal	4	5.43	1.021	6.3	0.98	5.86	0.937	5.06	0.923	4.21	0.97	4.37	0.992
σ_c	Normal	1.5	1.3	0.101	1.2	0.081	0.84	0.086	0.98	0.034	1.14	0.284	1.09	0.257
λ	Beta	0.7	0.77	0.041	0.82	0.029	0.86	0.026	0.86	0.028	0.78	0.061	0.8	0.054
ξ_w	Beta	0.5	0.93	0.016	0.95	0.012	0.6	0.035	0.78	0.034	0.75	0.031	0.75	0.031
σ_l	Normal	2	2.09	0.688	1.83	0.705	1.89	0.651	1.41	0.679	2.07	0.646	1.99	0.648
ξ_p	Beta	0.5	0.81	0.03	0.82	0.022	0.8	0.021	0.76	0.041	0.73	0.026	0.73	0.025
ι_w	Beta	0.5	0.84	0.069	0.81	0.072	0.66	0.114	0.64	0.12	0.56	0.121	0.56	0.122
ι_p	Beta	0.5	0.08	0.038	0.09	0.043	0.71	0.081	0.57	0.099	0.25	0.062	0.25	0.055
ψ	Beta	0.5	0.79	0.079	0.77	0.08	0.56	0.105	0.66	0.106	0.71	0.107	0.7	0.107
ϕ_p	Normal	1.25	1.55	0.072	1.66	0.079	1.46	0.063	1.51	0.071	1.54	0.068	1.53	0.069
r_π	Normal	1.25	1.45	0.155	1.49	0.168	1.71	0.176	1.6	0.172	1.6	0.168	1.51	0.163
ρ	Beta	0.75	0.85	0.019	0.88	0.02	0.86	0.019	0.88	0.022	0.88	0.022	0.88	0.02
r_y	Normal	0.125	0.05	0.02	0.08	0.021	0.13	0.029	0.13	0.026	0.12	0.026	0.08	0.012
r_{dy}	Normal	0.125	0.17	0.019	0.17	0.018	0.17	0.017	0.17	0.017	0.14	0.017	0.14	0.014
$\bar{\pi}$	Gamma	0.625	0.72	0.093	0.79	0.111	0.55	0.082	0.74	0.093	0.6	0.083	0.58	0.077
$\bar{\beta}$	Gamma	0.25	0.13	0.052	0.15	0.058	0.18	0.068	0.2	0.071	0.18	0.072	0.17	0.072
$\bar{l}$	Normal	0	-0.14	1.056	0.54	0.997	1.85	0.851	1.01	0.905	1.68	0.964	1.26	0.964
$\bar{\gamma}$	Normal	0.4	0.38	0.015	0.4	0.012	0.42	0.007	0.4	0.011	0.39	0.013	0.39	0.014
α	Normal	0.3	0.19	0.015	0.19	0.016	0.17	0.015	0.18	0.017	0.17	0.017	0.16	0.016
r_{zlb}^{-}	Normal	0.05			0.03	0.002	0.03	0.002	0.03	0.002	0.03	0.002	0.03	0.002
Laplace			-1194.87		-1140.24		-1160.68		-1117.161		-1120.105		-1128.21	
Bayes F.			1		23.73		14.85		33.75		32.47		28.95	

Table 3: Estimation period: 1966:I-2016:IV

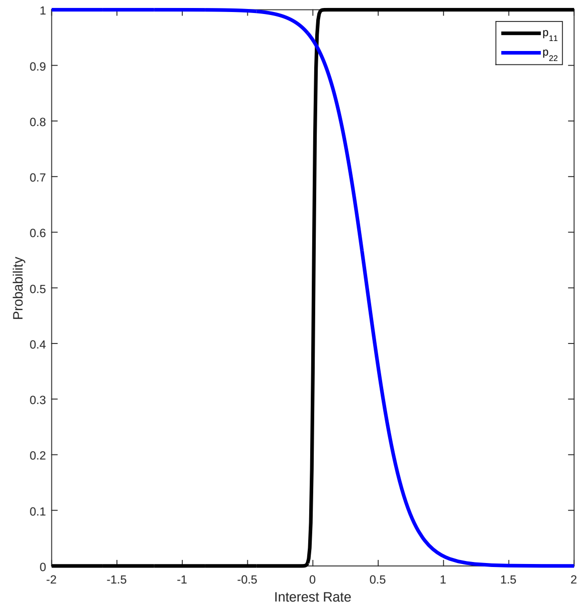
	Prior		REE Post.		REE-MS Post.		MSV Post.		VAR(1) Post.		AR(1) Post.		AR(1)-endo Post.	
Para	Dist	Mean	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.
α	Normal	0.3	0.19	0.015	0.19	0.016	0.17	0.015	0.18	0.017	0.17	0.017	0.16	0.016
ρ_a	Beta	0.5	0.96	0.01	0.95	0.011	0.95	0.013	0.97	0.013	0.96	0.015	0.96	0.015
ρ_b	Beta	0.5	0.36	0.091	0.31	0.067	0.34	0.059	0.27	0.063	0.29	0.061	0.29	0.06
ρ_g	Beta	0.5	0.99	0.005	0.99	0.006	0.99	0.007	0.99	0.006	0.99	0.005	0.99	0.006
ρ_i	Beta	0.5	0.81	0.04	0.74	0.041	0.78	0.046	0.57	0.055	0.49	0.061	0.5	0.061
ρ_r	Beta	0.5	0.09	0.05	0.08	0.052	0.09	0.042	0.13	0.053	0.17	0.057	0.05	0.017
ρ_p	Beta	0.5	0.8	0.05	0.77	0.049	0.08	0.04	0.05	0.036	0.04	0.026	0.04	0.025
ρ_w	Beta	0.5	0.06	0.038	0.05	0.036	0.89	0.032	0.08	0.05	0.08	0.047	0.08	0.046
ρ_{ga}	Beta	0.5	0.51	0.076	0.5	0.081	0.53	0.078	0.54	0.078	0.51	0.079	0.51	0.078
η_a	Inv. Gamma	0.1	0.44	0.024	0.44	0.024	0.46	0.024	0.44	0.024	0.44	0.024	0.44	0.024
η_b	Inv. Gamma	0.1	0.21	0.025	0.23	0.022	0.24	0.017	0.31	0.015	0.29	0.019	0.29	0.018
η_g	Inv. Gamma	0.1	0.48	0.024	0.49	0.025	0.48	0.023	0.48	0.024	0.48	0.024	0.48	0.024
η_i	Inv. Gamma	0.1	0.35	0.031	0.36	0.032	0.39	0.027	0.8	0.037	0.77	0.036	0.77	0.037
η_{r_N}	Inv. Gamma	0.1	0.22	0.01	0.22	0.011	0.22	0.011	0.21	0.011	0.22	0.011	0.21	0.011
$\eta_{r_{ZLB}}$	Gamma	0.03			0.01	0.001	0.01	0.001	0.01	0.001	0.01	0.001	0.01	0.001
η_p	Inv. Gamma	0.1	0.06	0.011	0.06	0.012	0.15	0.009	0.15	0.009	0.07	0.011	0.07	0.009
η_w	Inv. Gamma	0.1	0.37	0.023	0.37	0.022	0.18	0.011	0.38	0.019	0.38	0.019	0.38	0.019
$gain$	Gamma	0.035					0.01	0.003	0.01	0.002	0.02	0.004	0.02	0.001
$1-p_{11}$	Unif.	0.5			0.02	0.006	0.01	0.005	0.01	0.004	0.01	0.004		
$1-p_{22}$	Unif.	0.5			0.27	0.033	0.04	0.033	0.04	0.022	0.03	0.021		
$\theta_N/100$	Unif.	0.5											0.01	0.021
$\theta_{zlb}/100$	Unif.	0.5											0.17	0.071
$\phi_N/1000$	Unif.	0.5											0.12	0.052
$\phi_{zlb}/1000$	Unif.	0.5											0.007	0.004
Laplace			-1194.87		-1140.24		-1160.68		-1117.161		-1120.105		-1128.21	
Bayes F.			1		23.73		14.85		33.75		32.47		28.95	

slightly lower than the pre-crisis historical averages, particularly for output, consumption and investment. As a consequence, the models tend to over-predict these variables if no additional structural break is introduced into the model. It is readily seen in Figure 9b that this is indeed the case under MS-REE for the growth rates of output, consumption and investment. As opposed to this, Figure 9a shows that this over-prediction issue does not arise in the AR(1) model: we interpret this downward shift as a consequence of the time-variation in the perceived mean parameters. Accordingly, the lower growth rates over the post-crisis period emerge as a simple consequence of a pessimistic wave reflected in the perceived mean parameters. A similar figure can be seen for the MSV-learning model, where output and investment growth rate forecasts are more in line with the observables compared to the MS-REE model, while the consumption growth is actually under-predicted in this model. The VAR(1)-learning model still over-predicts the output and investment growth rates similar to MS-REE, which might be a consequence of the low estimated gain parameter.

As a final consideration before moving onto the impulse responses, we show the agents' forecast errors over the etimation period for all learning models in Figure 10. For the AR(1) model, we observe only a sharp increase during the crisis period, after which forecast errors quickly revert back to their pre-crisis levels for most variables. This suggests a quick adoption of the new environment by the agents: the initial large adverse shock comes as a surprise to the agents, leading to the large forecast error over that period. The learning process temporarily speeds up after this, supported by the jump in perceived persistence for some of these variables, and the forecast errors quickly revert back to their pre-crisis levels. The same phenomenon can be seen in the VAR(1)-learning model where the initial shock leads to a sharp increase in the forecast errors. In this case the reduction in forecast errors is more gradual as a consequence of the larger model that agents use, combined with the smaller estimated gain coefficient. For the MSV-learning model, we observe quite the opposite pattern: the forecast errors gradually increase over the ZLB period, implying agents' forecasting performance is declining. This suggests that the MSV information set, combined with the constant gain approach, does not capture well the structural change over this period and results in increasingly large forecast errors for the agents. This might also explain why the MSV-learning model does not perform well in terms of the model fit compared to other model specifications.



Shadow Rate
Interest Rate



p_{11}
p_{22}
Probability
Interest Rate

Figure 8: learning coefficients for intercept and first-order autocorrelation in the AR(1)-learning model.

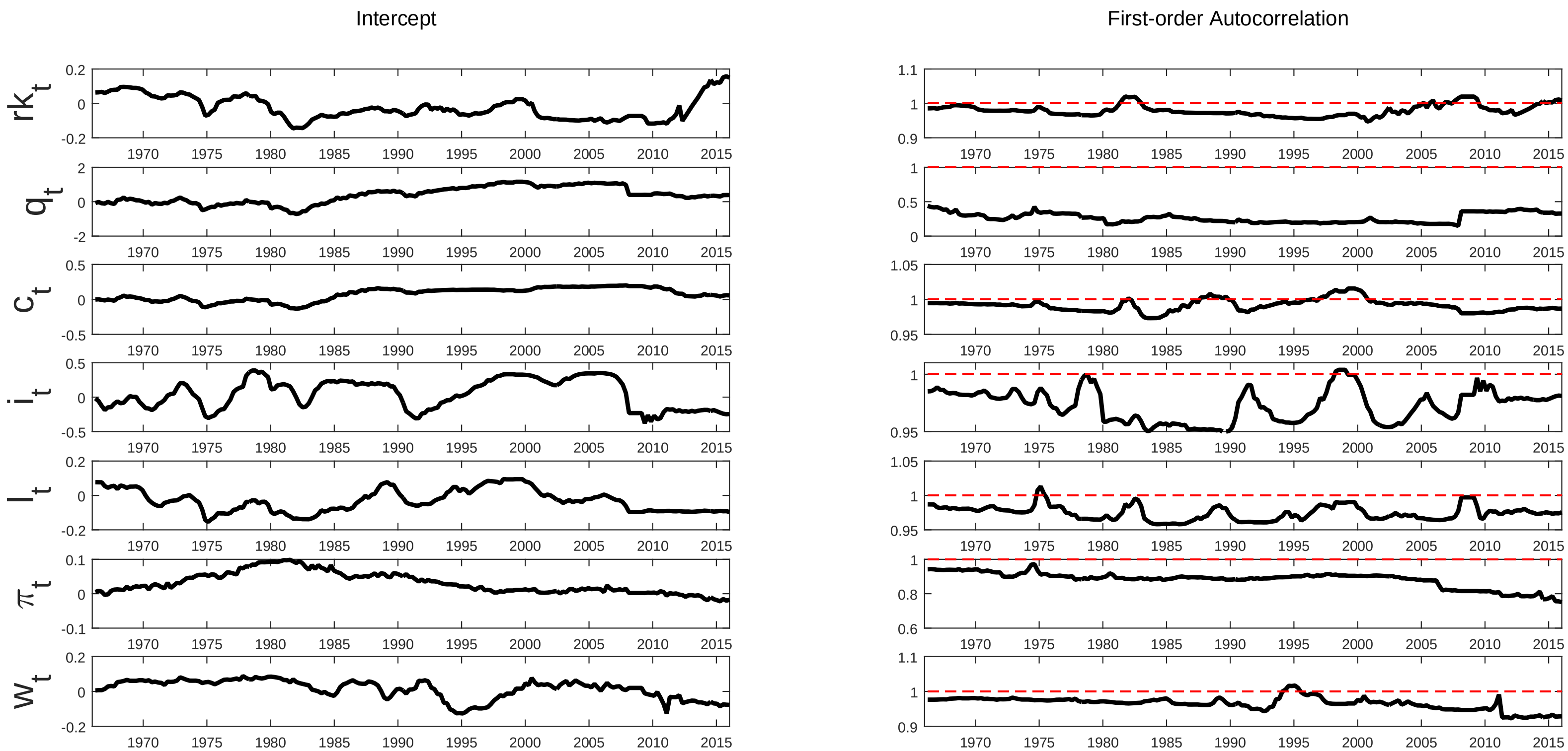
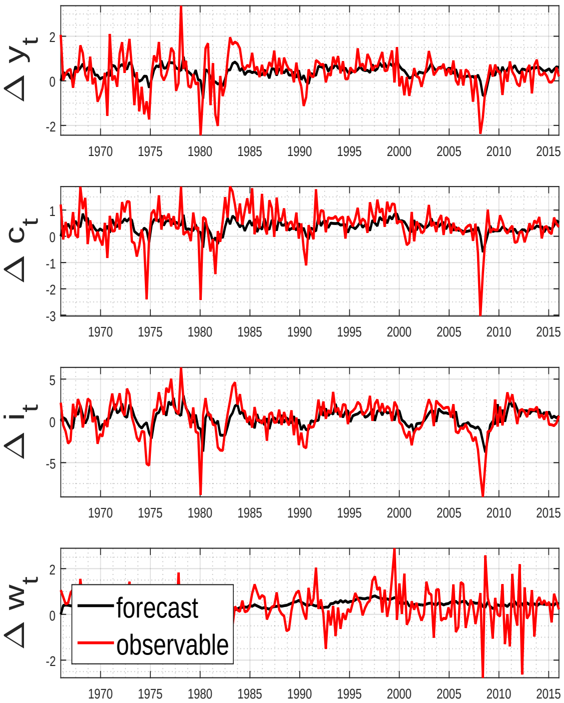
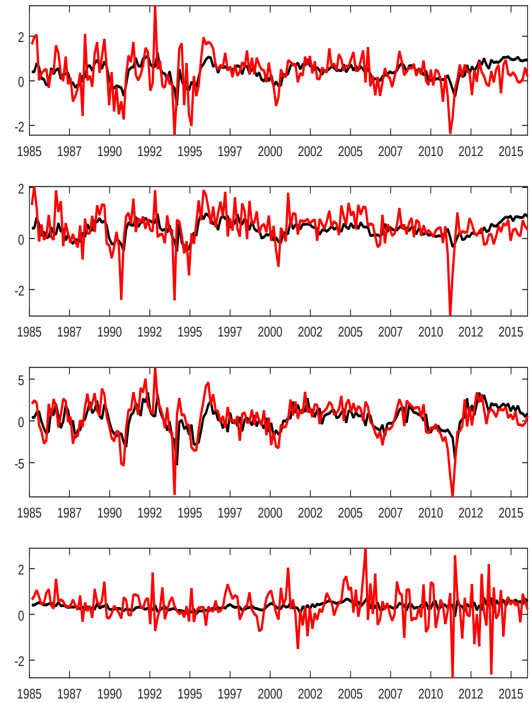


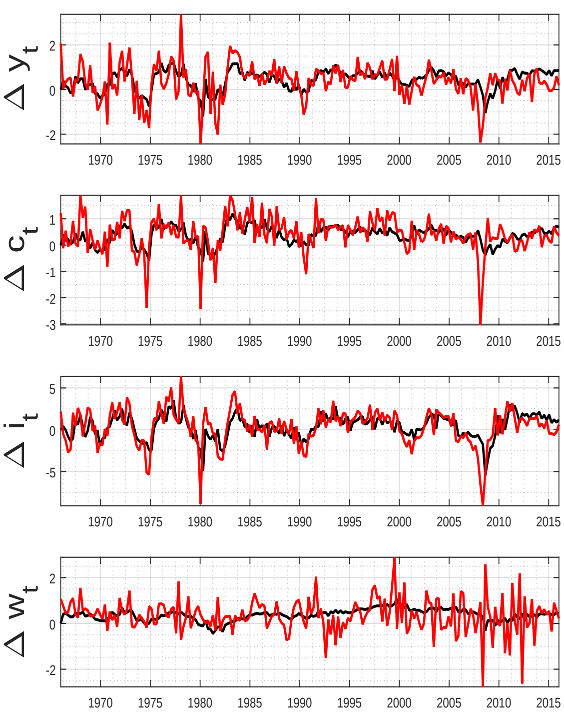
Figure 9: In-sample forecasts (i.e. the forecasting step of the filter) for all observables: AR(1) and MS-REE models respectively.



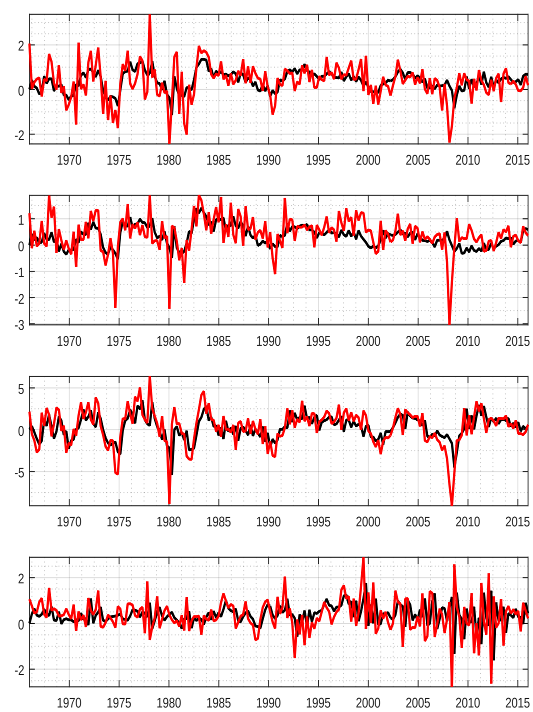
(a) AR(1)



(b) REE

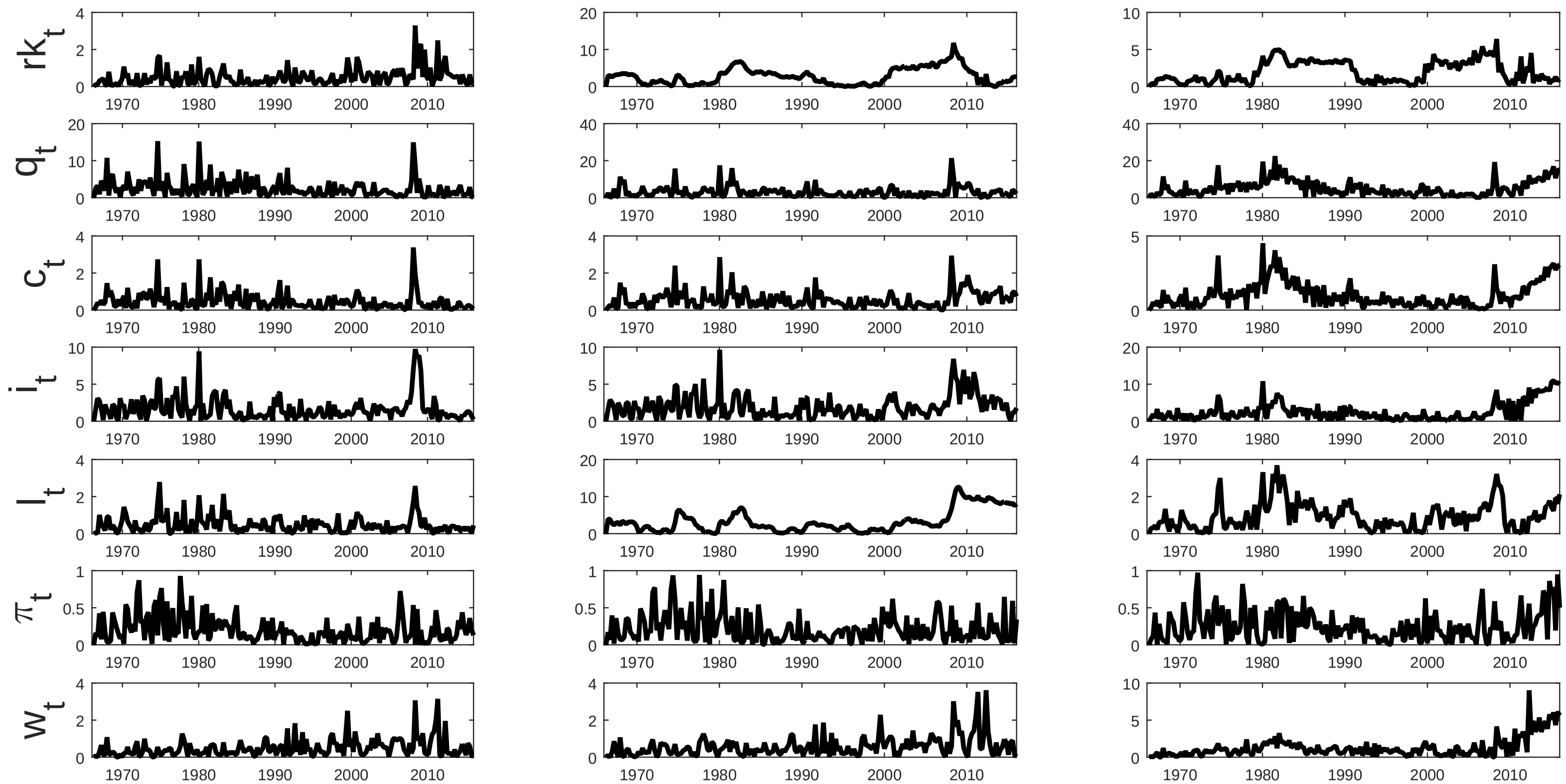


(c) VAR(1)



(d) MSV

Figure 10: Agents' forecast errors in learning models



(a) AR(1)

(b) VAR(1)

(c) MSV

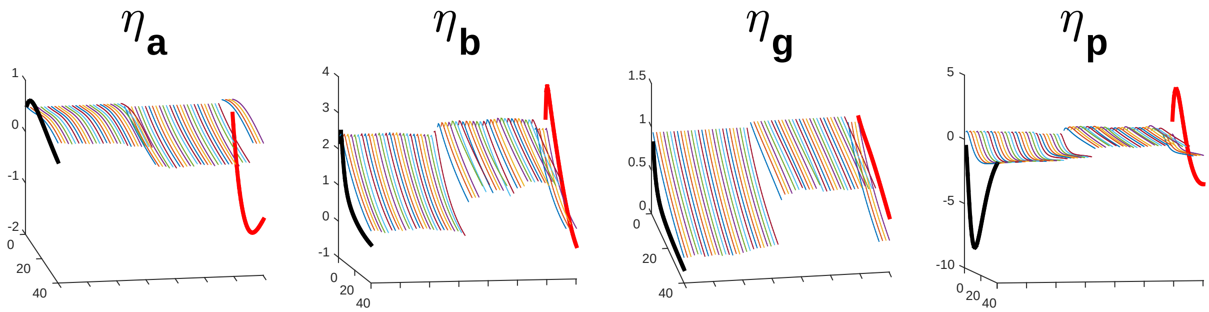
Impulse Responses

We next compare the impulse responses on output under learning models with the REE-MS model, which is presented in Figure 11. Similar to small-scale model, we present only the last 15 years over the estimation sample for the learning models, while the black and red lines on the left and right-hand sides correspond to the normal- and ZLB-regime IRFs under the REE-MS model. In this case we present four shocks, where the risk-premium shock η_b and price mark-up shock η_p are similar to the supply and demand shocks used in the small-scale model in functional form, while the remaining two shocks η_a and η_g correspond to productivity and government spending shocks respectively.

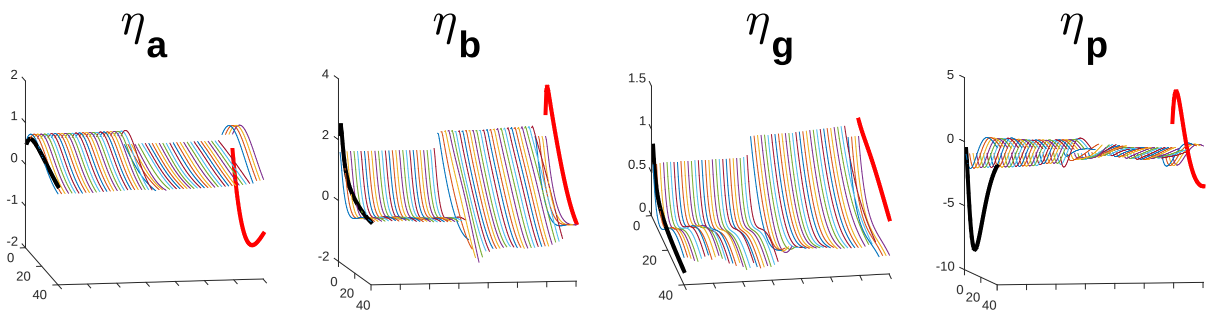
We observe that the overall time-variation in the learning models in small compared to the jump during the switch to the ZLB episode in the AR(1)- and VAR(1)-learning models, which is in line with the fast adoption of the new regime as discussed in the previous section. This small relative time-variation in the AR(1) model is similar to what we observed in the small-scale model, while for the VAR(1) model, the small gain parameter explains the smaller relative time-variation compared to the previous case. As opposed to these two, MSV-learning case is characterized by a more gradual shift after the switch to the ZLB episode, where the jump in the IRFs is smaller and there is more time variation in the IRFs during the ZLB period. An important result that becomes evident from this figure is that, the direction of changes in the IRFs with the switch to the ZLB episode is always the same under REE and learning models. However, there are clearly differences in the magnitudes of these changes.
To investigate further the magnitude in the changes in impulse responses, we consider the following exercise: we designate 5-year periods during the normal regime before the crisis (2002:I-2006:IV) and during the ZLB regime after the crisis (2011:I-2015:IV). For the learning models, we then compute the average differences in the impulse responses between the two regimes. We further compute the minimum and maximum differences in the IRFs to serve as a pseudo confidence interval for these differences. Figure 12 plots these IRF differentials, along with the corresponding difference under REE-MS case. What becomes quickly evident is that, the differences in learning models are smaller than the differences in the REE model in a vast majority of cases: with the exception of the b-shock in VAR(1) and MSV models, the black line (learning model) and the associated pseudo confidence interval remains below the blue line (REE model). This result suggests that the standard REE models may overestimate the impact of the ZLB regime and its impact on the propagation of shocks on the underlying model. As an application of this, Figure 13 shows the differences in the fiscal multipliers for the REE-MS and learning models[26]. In the case of REE-MS model, the difference increases up to 0.9 after a period of 12 quarters, which suggests that the impact of a fiscal stimulus over the ZLB regime will be nearly twice as much as a fiscal stimulus over the normal regime. For the learning models, this difference remains below the REE-MS line for up to 40 quarters, which suggests that the REE framework overestimates the impact of a fiscal stimulus over the ZLB regime.

[26]The fiscal multiplier is computed as $FM = \frac{\partial y}{\partial \eta_g} / \frac{\partial g}{\partial \eta_g}$ in both regimes. The figure shows the difference between these two multipliers for all models under consideration.

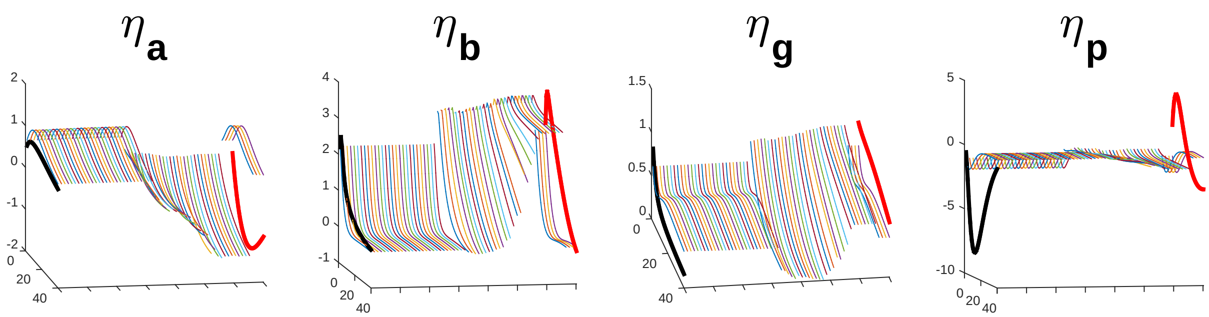
Figure 11: Comparison of AR(1) learning IRFs with REE IRFs. Each IRF shows a one standard deviation shock of η_a,η_b,η_g,η_p respectively.



(a) AR(1)

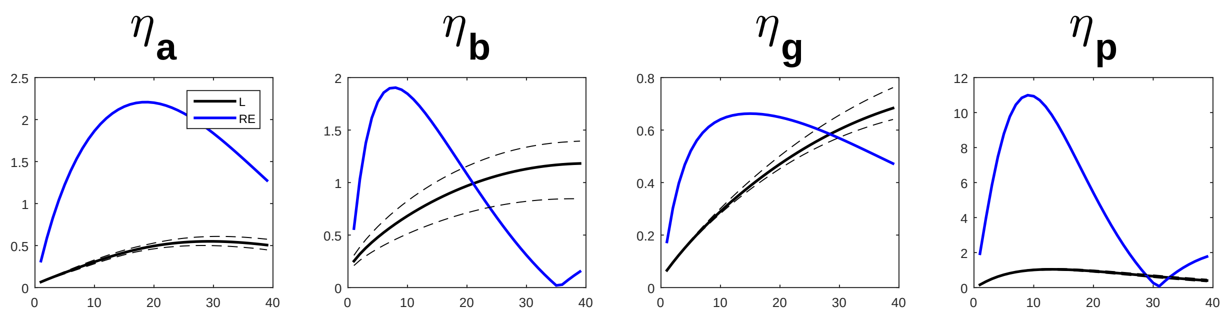


(b) VAR(1)

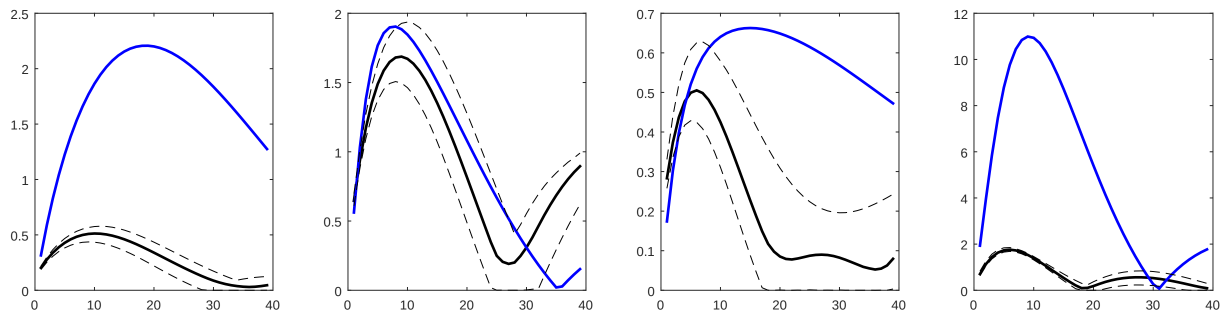


(c) MSV

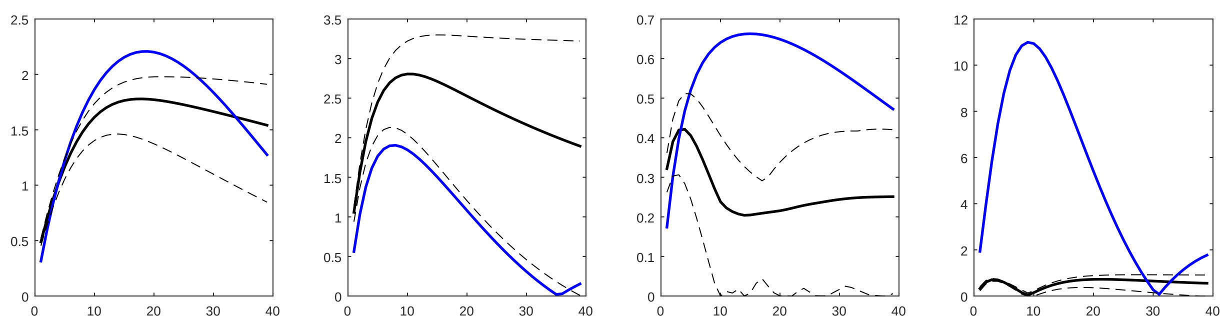
Figure 12: Comparison of AR(1) learning IRFs with REE IRFs. Each IRF shows a one standard deviation shock of η_a,η_b,η_g,η_p respectively.



(a) AR(1)

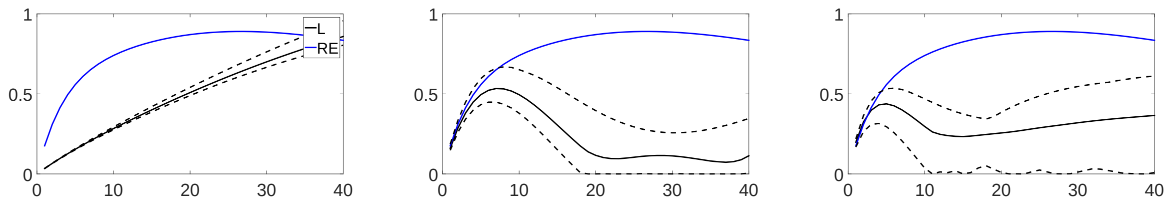


(b) VAR(1)



(c) MSV

Figure 13: Comparison of fiscal multiplier differences between REE and learning models over the normal and ZLB regimes.



(a) AR(1) (b) VAR(1) (c) MSV

6 E-stability and Simulations

This section investigates the E-stability of the 3-equation and Smets-Wouters models that have been estimated in the previous sections. Recall from Section 3 that the underlying E-stability conditions for a general system of the form (3.4) is not available in closed form. Therefore, while the projection facility ensures that the models remain stable throughout the estimation, this does not guarantee the stability of the resulting systems under learning.

A well known result in the literature is that, when monetary policy is inactive, New Keynesian models are generally indeterminate and the mean dynamics are E-unstable under standard parameterizations. This result may also apply to the model specifications considered in this paper. As a first step, we illustrate this for the 3-equation model in a simplified framework to build intuition. Accordingly, consider the 3-equation model under MSV-learning, where we shut off all three parameters related to lagged endogenous variables, i.e. $\rho_r = 0, \iota_x = 0, \iota_\pi = 0$ to reduce the T-map in (3.5) to an analytically tractable form. We use the following parameterization for the remaining parameters: $\kappa = 0.02, \tau = 1, \beta = 0.99, \rho_y = 0.25, \rho_\pi = 0.25$. First consider the regime-specific T-maps and the associated Jacobian matrix:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to T(a,d) = \begin{pmatrix} A_i + C_i a \\ C_i d\rho + D_i \end{pmatrix}, \frac{\partial T(a,d)}{\partial [a,d]} = \begin{pmatrix} C_i & 0 \\ 0 & (\rho C_i)' \end{pmatrix}, \tag{6.1}$$

where the underlying Rational Expectations Equilibrium is given by $a = (I - C_i)^{-1} A_i$ and $vec(d) = (I - \rho' \otimes C_i)^{-1} vec(D_i)$. At the given parameterization with an inactive Taylor rule, the largest root relating to the mean dynamics (i.e. the largest eigenvalue of C_i) is larger than 1, hence the model is E-unstable[27]. If we instead consider the Long-run E-stability with an active and a passive monetary policy rule, then we get the T-map:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to T(a,d) = \begin{pmatrix} P_1(A_1 + C_1 a) + P_2(A_2 + C_2 a) \\ P_1(C_2 d\rho + D_2) + P_2(C_2 d\rho + D_2) \end{pmatrix}, \frac{\partial T(a,d)}{\partial [a,d]} = \begin{pmatrix} P_1 C_1 + P_2 C_2 & 0 \\ 0 & P_1(\rho C_1)' + P_2(\rho C_2)' \end{pmatrix}, \tag{6.2}$$

[27] In this case the eigenvalues of C_i are given as $\lambda_{1,2} = \frac{\tau}{\tau + \phi_x + \kappa\phi_\pi}(\frac{-p}{2} \pm \sqrt{\frac{p^2}{4} - q})$, with $p = -(1 + \frac{\kappa + \beta\tau + \beta\phi_x}{\tau})$ and $q = \frac{\beta\tau + \beta\phi_x + \kappa\beta\phi_\pi}{\tau}$.

where the RPE is given by

$$\begin{cases} a = (I - P_1C_1 - P_2C_2)^{-1}(P_1A_1 + P_2A_2), \\ d = (I - \rho' \otimes (P_1C_1) - \rho' \otimes (P_2C_2))^{-1}(vec(P_1D_1) + vec(P_2D_2)). \end{cases} \tag{6.3}$$

Hence, in the absence of lagged variables, the equilibrium and the associated stability conditions are simply weighted averages of the underlying regime-specific equilibria. Considering a standard Taylor rule with $\phi_\pi = 1.5$ and $\phi_x = 0.125$, the long-run E-stability is satisfied as long as $P_2 < \frac{1}{3}$, i.e. as long as the system does not spend an excessive amount of time in the passive regime. We conjecture that this result also applies to the estimated model specifications with lagged variables, i.e. the model dynamics are E-unstable during the ZLB regime, but the system as a whole may still be stable as long as it the duration of the passive regime is sufficiently short.

To investigate the stability of the estimated model specifications under learning, we use the following Monte Carlo experiment: we simulate all models at the estimated posterior mode 100 times, each of length 2000, where each simulation is initialized at the REE-based initial values (i.e. the same beliefs used in the estimation) and discard the first half of each simulation as the burn-in period. We collect the distributions of the learning parmaeters and the frequency of the projection facility activity over the remaining periods, which we use as an indicator of E-stability for the underlying system. Generally, the projection facility will be binding either when the underlying equilibrium dynamics are close to the non-stationary region, or when the equilibrium dynamics are E-unstable; a Monte Carlo experiment is unable to distinguish between these two. Nevertheless, it provides us with an overall picture of how much the system depends on the projection facility to remain in the stationary region.
Table 4 shows the results on the projection facility for all models under consideration. For a majority of cases, the frequency ranges between $[1.3\%, 6.3\%]$, i.e. the facility does not bind very often. This suggests that these models remain within the stationary region for over 90 % of the periods, including the ZLB regimes with inactive monetary policy. There are two exceptions with a higher frequency: SW model with AR(1), where the projection facility binds 14.7% of the time, and the NKPC model with the VAR(1) rule, where the facility binds 32.1% of the time. Particularly in the latter case, this high frequency is caused by the high estimated parameter value of 0.03. We provide one more experiment with this model with a lower gain value of 0.01, where the projection facility frequency indeed decreases to 19.2%. Overall, we conclude that all model specifications are able to generate stable dynamics most of the time, where the best performing model relies on the projection facility only 1.3 % of the time, whereas the worst one relies on the projection facility 32.1 % of the time). The underlying distributions of the learning parameters for all models specifications can be found in the Appendix. With the exception of a few parameters[28], the distributions are typically unimodal, suggesting stationary dynamics around a unique equilibrium.

[28]E.g. inflation persistence has a seemingly bimodal distribution both in the NKPC and SW models, with one mode around 0.6 and another one close to unit root. This could be an indicator or two stable equilibria, or simply a result of non-stationary dynamics that cluster around unit root due to the projection facility. We do not investigate this further since the simulations are only meant to investigate the overall stability of the models.

Table 4: Frequency of projection facility activity in all learning model. The results are based in 100 simulations each of length 2000, where the first half is discarded as burn-in sample in each case.

	NKPC	SW
AR(1)	1.5 %	14.7 %
VAR(1)	32.1 %	6.3 %
VAR(1) (gain= 0.01)	19.2 %	
MSV	5.1 %	1.3 %

Endogenous Regime Probabilities

As a final exercise, we consider a simulation with endogenous regime probabilities, where the exogenous regime probabilities are substituted with endogenous ones depending on how close interest rates are to the ZLB. The goal of this exercise is to illustrate that the learning dynamics will show similar patterns regardless of the nature of the regime switch: in both cases, the regime switch will be an exogenous event to the agents since their PLM only accounts for the structural break ex-post. Accordingly, consider the time-varying endogenous transition probabilities as follows:

$$p_{11}(t) = \frac{\theta}{\theta + exp(-\phi(r_t^* - \underline{r}))}, p_{22}(t) = \frac{\theta}{\theta + exp(\phi(r_t^* - \underline{r}))}$$

where r_t^* denotes the shadow interest rate, i.e. the interest rate that would prevail in the absence of ZLB constraint; $\underline{r}$ denotes the value of the lower bound for interest rates; and θ and ϕ are hyperparameters to be calibrated. Following Binning & Maih (2016), we set $\theta = 1$ and $\phi = 1000$, which lead to a very sharp transition to the ZLB regime once r_t^* falls even slightly below $\underline{r}$. For our exercise, we set $\underline{r} = -1$, which is equivalent to assuming a steady-state interest rate level of 1 with a lower bound of 0. For this exercise, we again use the 3-equation model under MSV-learning with the same parameterization from before, i.e. $\kappa = 0.02, \tau = 1, \beta = 0.99, \rho_y = 0.25, \rho_\pi = 0.25$. We set $\iota_p = \iota_x = 0.1$ to also include learning parameters on lagged variables[29].

Figure 14 shows two simulations with a gain of $\gamma = 0.01$ with exogenous (left panel) and endogenous (right panel) transition probabilities. In each figure, the dotted lines provide the regime-specific values for the underlying REE. Although we do not have the RPE-consistent values in this case, the behaviour of the learning coefficients in the exogenous switching case suggest that the results are similar to the 1-dimensional case: the shock coefficients typically show a gradual movement towards the regime-specific values, while the lagged coefficients display a sharp jump immediately following the switch, before starting to move in the direction of regime-specific values. Given our calibration, the ZLB episodes are very short-lived with durations of 1 or 2 periods only[30], but similar results can be observed for the lagged coefficients.

[29]We want to ensure the projection facility never binds for this exercise, therefore these two parameters are set to low values.

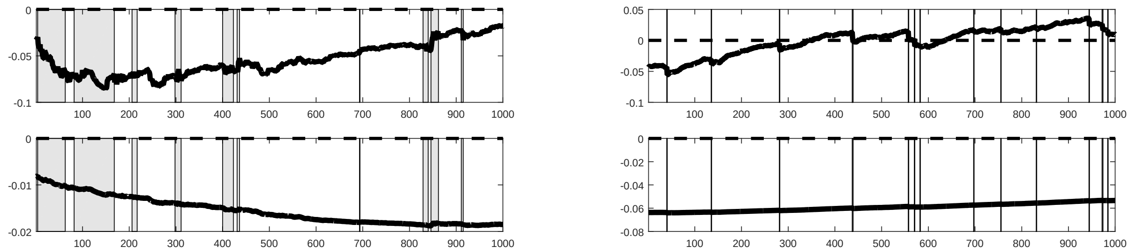
[30]While these episodes can be made more persistent with additional assumptions, our primary interest here is how the learning coefficients react immediately following the binding constraint.

In particular, the jumps in the lagged coefficients is evident immediately following a binding ZLB period. The shock coefficients remain close to the regime-specific values of the normal regime, since the ZLB episodes are not sufficiently long-lived to induce a movement in these. These results indicate that the propagation of regime switches on expectations under adaptive learning works in a similar fashion regardless of the nature of regime switches, i.e. endogenous or exogenous. This is a natural result since regime switches are unobserved by the agents in either case. As a final note we note that, the jumps in the learning parameters (on laggged variables) during regime switches, which were observed in the introductory Fisher model as well as some of the estimated models, is also evident in these simulations. This suggests that the temporarily amplified persistence and fast learning during regime switches is a rather robust phenomenon.

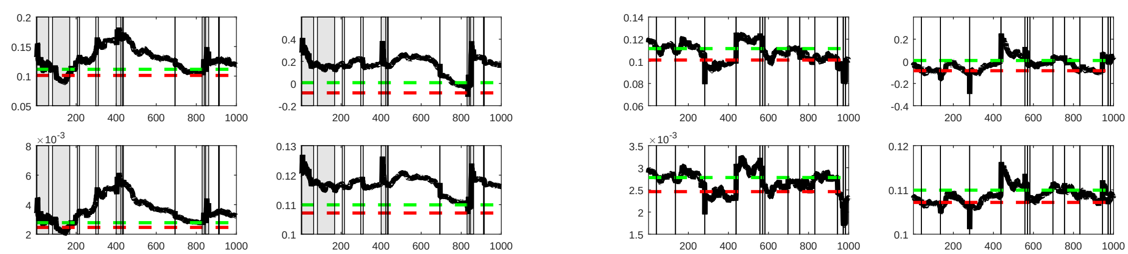
7 Conclusions

The literature on macroeconomics made great strides in the estimation of both Markov-switching and adaptive learning models in recent years, which are two alternative ways of introducing time-variation into DSGE models. Although there are numerous examples of both classes of models that have been successfully taken to the data, there is surprisingly little work on Markov-switching DSGE models under adaptive learning. In this paper, we provided a first attempt to estimate this class of DSGE models by combining these two approaches under the same roof. The resulting framework has an intuitive interpretation where the agents' do not know the details of a complex non-linear economy, but use simple linear rules to form their expectations about the future state of the economy. Our simulations show that, using simple adaptive rules allows the agents' to take into account the structural changes in the economy, albeit in an indirect manner. More importantly, our estimations indicate that the two approaches can be complementary: it both the 3-equation NKPC and Smets-Wouters models we consider, imposing the Markov-switching structure on standard models improves the empirical fit, while imposing adaptive learning leads to further improvements. Furthermore, our results on the ZLB episode suggest that a wave of pessimism, modeled as a downward shift of expectations, may well have contributed to the Great Recession period characterized by low growth rates, which is a well-established idea in literature as an expectations-driven business cycle. Comparing the impulse responses under AL and RE models, we find that the IRFs typically move in the same direction with the switch to the ZLB regime under both classes of models. However, there may be differences in the magnitudes of these changes. Focusing particularly on government spending shocks and fiscal multipliers, we find that the proportional change in fiscal multipliers over the ZLB period is smaller under AL compared to the REE benchmark. This suggests that standard RE models may overestimate the impact of a fiscal expansion over the ZLB period following the 2007-08 financial crisis.

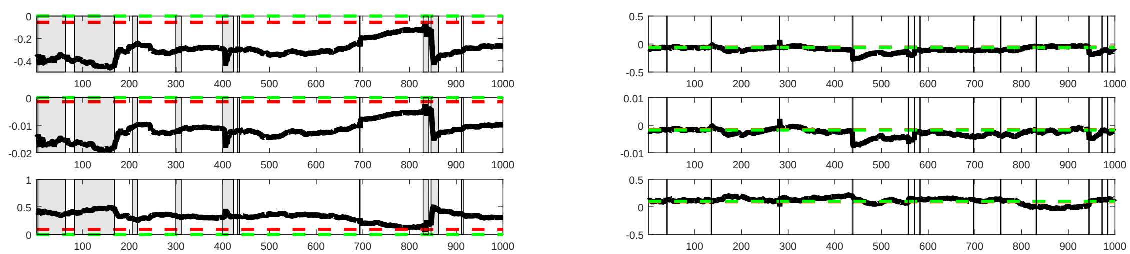
Figure 14: Simulations of length 5000, with a burn-in period of 4000, for the two-regime NKPC with exogenous and endogenous regime switching. The left panel shows the results with exogenous regime switching, while the right panel shows the results with endogenous switching.



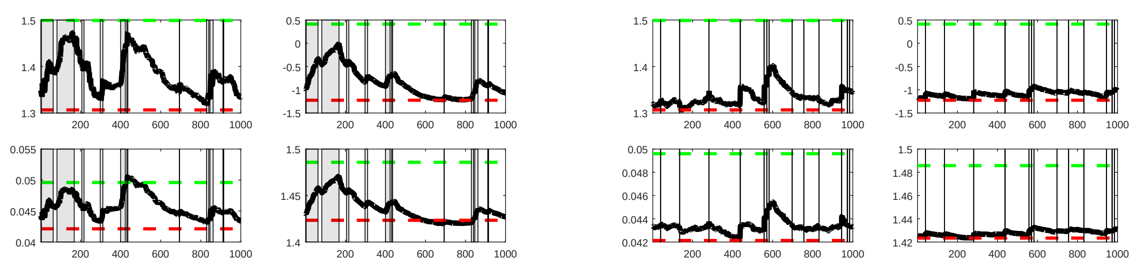
(a) Mean coefficients learning



(b) Lagged interest rate learning



(c) Lagged inflation and output gap learning



(d) Shocks coefficients learning

Appendix

A RPE and T-map

Special case with 2 regimes, no lagged variables

Note that in the special case with $\iota_p = 0$, the model can be written as a generic 1-dimensional Markov-switching model of the form:

$$\begin{cases} \pi_t = d(s_t) r_t + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

where $d(s_t) = \frac{d\rho - 1}{\alpha(s_t)}$. The moments necessary for the T-map are given as follows:

$$E[\pi_t r_t] = P_1 E[\pi_t r_t | S_t = 1] + P_2 E[\pi_t r_t | S_t = 2]$$

$$E[\pi_t r_t | S_t = 1] = E[d(s_t) r_t^2 | S_t = 1, S_{t-1} = 1] p_{11} + E[d(s_t) r_t^2 | S_t = 1, S_{t-1} = 2](1 - p_{22}) \frac{P_2}{P_1}$$

$$= d_1 p_{11} + d_1 (1 - p_{22}) \frac{P_2}{P_1}$$

Similarly, we have

$$E[\pi_t r_t | S_t = 1] = d_2 p_{22} + d_2 (1 - p_{11}) \frac{P_1}{P_2}$$

which yields

$$E[\pi_t r_t] = P_1 (d_1 p_{11} + d_1 (1 - p_{22}) \frac{P_2}{P_1}) + P_2 (d_2 p_{22} + d_2 (1 - p_{11}) \frac{P_1}{P_2})$$

Plugging in the steady-state probabilities P_1 and P_2, the T-map is given as follows:

$$d \to T(d) = \frac{\alpha_1 (1 - p_{22}) + \alpha_2 (1 - p_{11})}{\alpha_1 \alpha_2 (2 - p_{11} - p_{22})} (d\rho + 1)$$

with the E-stability condition

$$DT_a = \frac{\alpha_1 (1 - p_{22}) + \alpha_2 (1 - p_{11})}{\alpha_1 \alpha_2 (2 - p_{11} - p_{22})} \rho < 1$$

Re-arranging the expression above yields the LRES condition presented in Section 1. Further note that the regime-specific T-maps, and the associated regime-specific E-stability conditions are given by:

$$d \to \frac{d\rho + 1}{\alpha_i}$$

$$DT_d = \frac{\rho}{\alpha_i} < 1$$

which implies that regime specific E-stability of all regime-specific equilibria is a sufficient, but not necessary condition for LRES.

1-dimensional case with m regimes

Note that the Fisherian model considered in Section 2 can be written as a generic 1-dimensional Markov-switching model of the form:

$$\begin{cases} \pi_t = d(s_t)r_t + b(s_t)\pi_{t-1} \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

where $b(s_t) = \frac{\iota_p}{\alpha(s_t)-(1-\iota_p)b}$ and $d(s_t) = \frac{(1-\iota_p)d\rho+1}{\alpha(s_t)-(1-\iota_p)b}$. In this Appendix we consider the general case with m regimes, with transition matrix given by:

$$Q = \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \cdots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{bmatrix}$$

The 2-regime setup of Section is the special case with $m = 2$. We omit the first moment $E[\pi_t]$, which is trivially given as zero. Using this, we compute the second moments starting with the conditional variance. We have:

$$E[\pi_t^2] = \sum_{i=1}^{m} P_i E[\pi_t^2 | S_t = i]$$

$$E[\pi_t^2 | S_t = i] \sum_{i=1}^{m} E[\pi_t^2 | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

where P_i denotes the i^{th} element of the steady-state vector of the Markov chain.

$$= \sum_{j=1}^{m} E[d(s_t)^2 r_t^2 + b(s_t)^2 \pi_{t-1}^2 + u_t^2 + 2b(s_t)d(s_t)r_t\pi_{t-1} | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} E[d_i^2 \sigma_r^2 + b_i^2 \pi_{t-1}^2 + \sigma_r^2 + 2b_1 d_i r_t \pi_{t-1} | S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

Note that this last expression implies m equations in m unknowns for the conditional variances, *given* the conditional covariances $E[\pi_t r_t | S_t = j]$. Using this, the unconditional variance is given by:

$$E[\pi_t^2] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (d_i^2 \sigma_r^2 + b_i^2 E[\pi_{t-1}^2 | S_t = j] + \sigma_r^2 + 2b_1 d_i r_t E[\pi_{t-1} | S_{t-1} = j]) p_{ji} \frac{P_j}{P_i}$$

Next we move onto the covariance term $E[\pi_t r_t]$:

$$E[\pi_t r_t] = \sum_{i=1}^{m} P_i E[\pi_t r_t | S_t = i]$$

$$E[\pi_t r_t | S_t = i] = \sum_{j=1}^{m} E[\pi_t r_t | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} E[b(s_t)\pi_{t-1} r_t + d(s_t) r_t^2 | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$\sum_{i=1}^{m} (b_i \rho E[\pi_t r_t | S_t = j] + d_i \sigma_r^2) p_{ji} \frac{P_j}{P_i}$$

Note that again, the last expression implies m equations in m unknowns for the conditional covariances. Using this, the unconditional covariance is given by:

$$E[\pi_t r_t] = \sum_{i=1} m P_i \sum_{j=1}^{m} (b_i \rho E[\pi_t r_t | S_t = j] + d_i \sigma_r^2) p_{ji} \frac{P_j}{P_i}$$

Next we compute the first-order autocovariance:

$$E[\pi_t \pi_{t-1}] = \sum_{i=1}^{m} P_i E[\pi_t \pi_{t-1} | S_t = i]$$

$$E[\pi_t \pi_{t-1} | S_t = i] = \sum_{j=1}^{m} E[b(s_t)\pi_{t-1}^2 + d(s_t)\pi_{t-1} r_t | S_t = 1, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} (b_i E[\pi_t^2 | S_t = j] + d_i \rho E[\pi_t r_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Given the conditional covariance and conditional variance terms, the above expression yields the conditional autocovariances. Hence the unconditional autocovariance is given as:

$$E[\pi_t \pi_{t-1}] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (b_i E[\pi_t^2 | S_t = j] + d_i \rho E[\pi_t r_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Finally note that:

$$E[d(s_t)\pi_{t-1}r_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} d_i \rho E[\pi_t r_t | S_t = j] p_{ji} \frac{P_j}{P_i}$$

and

$$E[b(s_t)\pi_{t-1}r_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} b_i \rho E[\pi_t r_t | S_t = j] p_{ji} \frac{P_j}{P_i}$$

Recalling the T-map $\begin{pmatrix} d \\ b \end{pmatrix} \rightarrow T(d,b) = \begin{pmatrix} E[(\pi_t - b\pi_{t-1})r_t] \\ \frac{E[(\pi_t - dr_t)\pi_{t-1}]}{E[\pi_t^2]} \end{pmatrix}$, the above conditions fully pin down $T(d,b)$. It is generally not possible to obtain analytical expressions for this mapping, and therefore the RPE values d^{RPE} amd b^{RPE}. Therefore our results in Section 1 are computed numerically for given values of parameters.

N dimensional case with m regimes

Note that, after plugging in the PLM into ALM, the model considered in Section can be re-written as a generic Markov-switching model of the form:

$$\begin{cases} X_t = a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t \\ \epsilon_t = \rho\epsilon_{t-1} + \eta_t \end{cases}$$

where $a(s_t) = A(s_t) + C(s_t)(a + ba)$, $b(s_t) = B(s_t) + C(s_t)b^2$ and $d(s_t) = C(s_t)(bd + d\rho) + D(s_t)$. We need the first and second moments of this system in order to compute the the resulting T-map for the RPE. Starting with the first moment, we have:

$$E[X_t] = \sum_{i=1}^{m} P_i E[X_t | S_t = i]$$

$$E[X_t | S_t = i] = \sum_{j=1}^{m} [a_i + b_i X_{t-1} + d_i \epsilon_t | S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} (a_i + b_i E[X_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

The expression above implies m equations in m unknowns for the conditions means. Using this yields:

$$E[X_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (a_i + b_i E[X_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Moving onto the second moments and starting with the covariance term, we have:

$$E[X_t\epsilon_t'] = \sum_{i=1}^{m} P_i E[X_t\epsilon_t'|S_t = i]$$

$$E[X_t\epsilon_t'|S_t = i]E[)a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t)\epsilon_t'|S_t = i]$$

$$= \sum_{j=1}^{m} E[(a_i + b_i X_{t-1} + d_i\epsilon_t)\epsilon_t'|S_{t-1} = j]p_{ji}\frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m}(b_i\rho E[X_t\epsilon_t'|S_t = j] + d_i\Sigma_\epsilon)p_{ji}\frac{P_j}{P_i}$$

The last expression again implies m equations in m unknowns for the conditional covariances. The unconditional covariance is then given by:

$$E[X_t\epsilon_t'] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m}(b_i\rho E[X_t\epsilon_t'|S_t = j] + d_i\Sigma_\epsilon)p_{ji}\frac{P_j}{P_i}$$

Next we compute:

$$E[X_tX_t'] = \sum_{i=1}^{m} P_i E[X_tX_t'|S_t = i]$$

$$E[X_tX_t'|S_t = i] = E[a(s_t)a(s_t)' + 2a(s_t)X_{t-1}'b(s_t)' + 2a(s_t)\epsilon_t d(s_t)' +$$
$$b(s_t)X_{t-1}X_{t-1}'b(s_t)' + 2b(s_t)X_{t-1}\epsilon_t d(s_t)' + d(s_t)\epsilon_t\epsilon_t' d(s_t)'|S_t = i]$$

$$= \sum_{j=1}^{m} E[a_i a_i' 2a_i X_{t-1}' b_i' + 2a_i\epsilon_t' d_i' + b_i X_{t-1}X_{t-1}' b_i' + 2b_i X_{t-1}\epsilon_t' d_i' + d_i\epsilon_t\epsilon_t' d_i'|S_t = j]p_{ji}\frac{P_j}{P_i}$$

Given the conditional means and covariances, the last expressions implies m equations in m unknowns for the conditional moments $E[X_tX_t'|S_t = i]$. The unconditional moment is then given by:

$$E[X_tX_t'] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m}(a_i a_i' + 2a_i E[X_t'|S_t = j]b_i' + b_i E[X_tX_t'|S_t = j]b_i' + 2b_i E[X_t\epsilon_t'|S_t = j]\rho' d_i' + d_i\Sigma_\epsilon d_i')p_{ji}\frac{P_j}{P_i}$$

Finally we compute the autocovariance term:

$$E[X_tX_{t-1}'] = \sum_{i=1}^{m} P_i E[X_tX_{t-1}'|S_t = i]$$

$$E[X_tX'_{t-1}|S_t=i] = E[a_iX'_{t-1} + b_iX_{t-1}X'_{t-1} + d_i\rho\epsilon_{t-1}X'_{t-1}|S_t=i] =$$

$$\sum_{j=1}^{m}(a_iE[X_t|S_t=j] + b_iE[X_tX'_t|S_t=j] + d_i\rho E[\epsilon_tX'_t|S_t=j])p_{ji}\frac{P_j}{P_i}$$

The last expression is pinned by the conditional first and second moments computed above. The unconditional autocovariance is then given as:

$$E[X_tX'_t] = \sum_{i=1}^{m}\sum_{j=1}^{m}(a_iE[X_t|S_t=j] + b_iE[X_tX'_t|S_t=j] + d_i\rho E[\epsilon_tX'_t|S_t=j])p_{ji}\frac{P_j}{P_i}$$

Recall that the T-map is given by:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} E[X_t - bX_{t-1} - d\epsilon_t] \\ E[(X_t - a - d\epsilon_t)X'_{t-1})E[X_tX'_t]^{-1}] \\ E[(X_t - a - bX_{t-1})\epsilon'_t]E[\epsilon_t\epsilon'_t]^{-1} \end{pmatrix}$$

Hence, given the first and second moments computed above, the T-maps for a, b and c are pinned down. Similar to 1-dimensional case, it is generally not possible to find analytical expressions for these matrices. Further note that, the T-map for $b \rightarrow T(a,b,c)$ involves a 4^{th} order matrix polynomial of dimension N. This means there can be up to $\binom{4N}{N}$ for b. To our knowledge, there is no straightforward and general method to compute the full set of solutions to this problem. In this paper, we do not compute these fixed-points and rely on Monte Carlo simulations when necessary.

Further note that the regime-specific T-maps are given by:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} A_i + C_i(a + ba) \\ B_i + C_ib^2 \\ C_i(bd + d\rho) + D_i \end{pmatrix}$$

These simply correspond to the standard MSV solutions given the regime-specific matrices. Computing the fixed-points yield the regime-specific equilibria as follows:

$$\begin{cases} a^{R_i} = (I - C_i - C_ib^{R_i})^{-1}A_i \\ vec(D^{R_i}) = (I - (I \otimes (C_ib^{R_i})))vec(d) + (\rho \otimes C_i)vec(d) + vec(D_i) \end{cases}$$

which yields the regime-specific values for a^{R_i} and d^{R_i} respectively, for a given matrix b^{R_i}. The second-order polynomial for b^{R_i} can be solved using standard toolboxes such as Adjemian et al. (2011) and Uhlig et al. (1995), which then completely pins down the regime-specific MSV. Denoting $\theta = (a,b,d)'$, the associated Jacobian for E-stability condition is given by:

$$\frac{DT}{D\theta} = \begin{bmatrix} C_i + C_i b & vec_{n,n}^{-1}(a' \otimes C_i) & 0 \\ 0 & 2C_i b & 0 \\ 0 & vec_{n,n}^{-1}(d' \otimes C_i) & C_i b + vec_{n,n}^{-1}(\rho' \otimes C_i) \end{bmatrix}$$

where $vec_{n,n}^{-1}$ denotes the matricization of a vector to an (n, n) matrix.

B Kim-Nelson Filter with Adaptive Learning

Table 5: **KM-filter for Markov-Switching DSGE Models under Adaptive Learning**

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)} S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)} \epsilon_t, \quad , \epsilon_t \sim N(0, \Sigma) \\ y_t = E + F S_t \end{cases}$$

0) Initial States :

$\tilde{S}_{0|0}^i, \tilde{P}_{0|0}^i, Pr[S_0 = i|\Phi_0], \Phi_0$ given.

1) Kalman Filter Block with the standard measurement and transition equations:

For $t = 1 : N$

For $\{S_{t-1} = i, S_t = j\}$

$$\begin{cases} S_{t|t-1}^{(i,j)} = \gamma_1^{(j)} S_{t-1|t-1}^{(i)} + \gamma_2^{(j)} \\ P_{t|t-1}^{(i,j)} = \gamma_1^{(j)} P_{t-1|t-1}^{(i)} \gamma_1^{(j)} + \gamma_3^{(j)} \Sigma^{(j)} (\gamma_3^{(j)})' \\ v_{t|t-1}^{(i,j)} = (y_t - F^{(j)} S_{t|t-1}^{(i,j)}) \\ Fe^{(i,j)} = F^{(j)} P_{t|t-1}^{(i,j)} F^{(j)} \\ S_{t|t}^{(i,j)} = S_{t|t-1}^{(i,j)} + P_{t|t-1}^{(i,j)} (F^{(j)})' (Fe^{(i,j)})^{-1} v^{(i,j)} \\ P_{t|t}^{(i,j)} = P_{t|t-1}^{(i,j)} (F^{(j)})' (Fe^{(i,j)})^{-1} F^{(j)} P_{t|t-1}^{(i,j)} \end{cases}$$

2) Hamilton Block for transition probabilities:

Denote: $Pr[S_{t-1} = i, S_t = j|\Phi_{t-1}] = pp_{t|t-1}^{i,j}$,$f(y_t|\Phi_{t-1})$ the marginal likelihood,

$Pr[S_{t-1} = i, S_t = j|\Phi_t] = pp_{t|t}^{i,j}$ and $Pr[S_t = j|\Phi_t] = pp_{t|t}^{\tilde{j}}$.

$$\begin{cases} pp_{t|t-1}^{(i,j)} = Q(i,j) pp_{t-1|t-1}^{(i)} \\ f(y_t|\Phi_{t-1}) = \sum_{j=1}^M \sum_{i=1}^M f(y_t|S_{t-1} = i, S_t = j, \Phi_{t-1}) pp_{t|t-1}^{(i,j)} \\ pp_{t|t}^{(i,j)} = \frac{f(y_t|S_{t-1}=i, S_t=j, \Phi_{t-1}) pp_{t|t-1}^{(i,j)}}{f(y_t|\Phi_{t-1})} \\ p_{t|t}^j = \sum_i^M pp_{t|t-1}^{(i,j)} \end{cases}$$

3) Collapsing to reduce the number of states from m^2 to m:

$$\begin{cases} S_{t|t}^{(i)} = \frac{\sum_{i=1}^M pp_{t|t}^{(i,j)} S_{t|t}^{(i,j)}}{p_{t|t}^{(j)}} \\ P_{t|t}^{(i)} = \frac{\sum_{i=1}^M pp_{tt}^{(i,j)} (P_{t|t}^{(i,j)} + (S_{t|t}^{(j)} - S_{t|t}^{(i,j)})(S_{t|t}^{(j)} - S_{t|t}^{(i,j)})')}{p_{t|t}^{(j)}} \end{cases}$$

4) Update expectations based on filtered states:

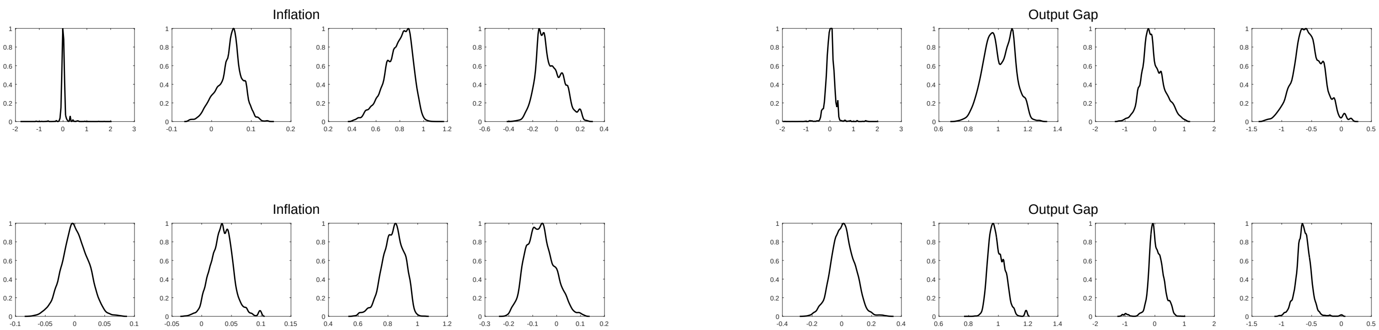
Updating Expectations based on Filtered States:

$$\begin{cases} \tilde{S}_{t|t} = \sum_{j=1}^M p_{t|t}^{(j)} S_{t|t}^{(j)} \\ \Phi_t = \Phi_{t-1} + \gamma R_t^{-1} \tilde{S}_{t-1|t-1} (\tilde{S}_{t|t} - \Phi_{t-1}^T \tilde{S}_{t-1|t-1})^T \\ R_t^{-1} = R_{t-1} + \gamma (\tilde{S}_{t-1|t-1} \tilde{S}_{t-1|t-1}^T - R_{t-1}) \end{cases}$$

C Distributions of Learning Parameters from Monte Carlo Simulations

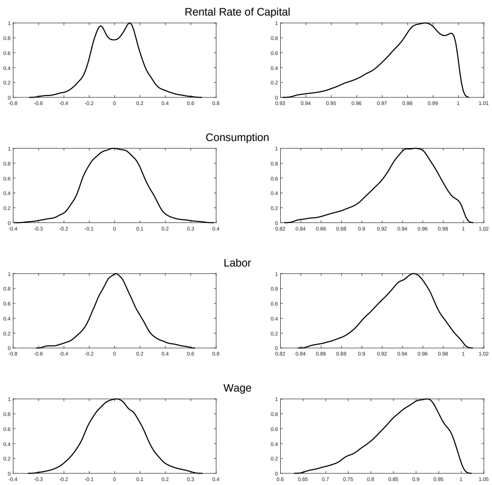
NKPC, AR(1)

NKPC, VAR(1) second row with $gain = 0.01$

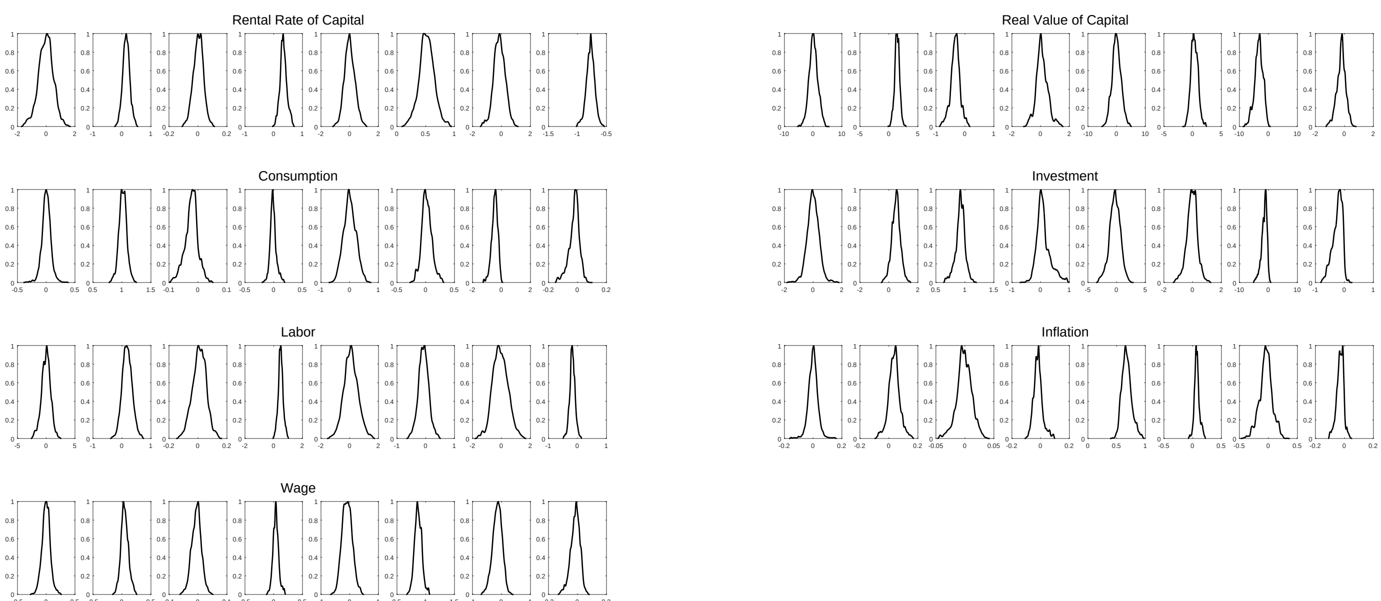


NKPC, MSV

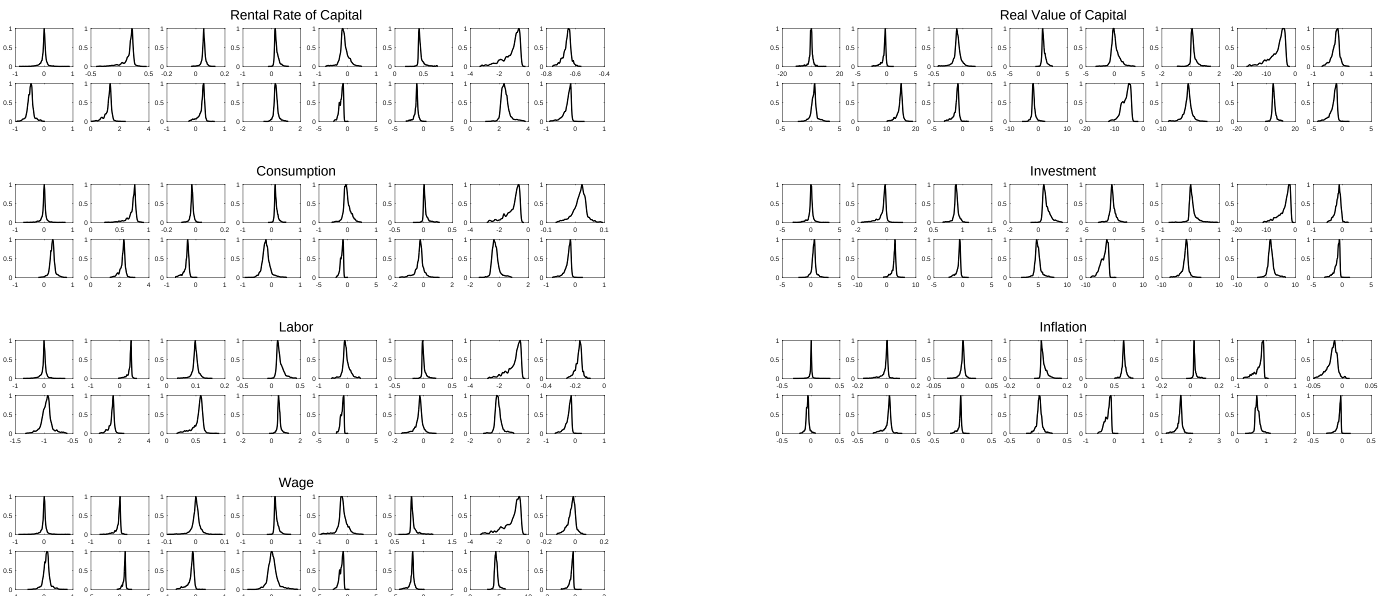
SW, AR(1)



SW, VAR(1)



SW, MSV



Rental Rate of Capital
Real Value of Capital
Consumption
Investment
Labor
Inflation
Wage

References

Adjemian, S., Bastani, H., Juillard, M., Mihoubi, F., Perendia, G., Ratto, M., & Villemot, S. (2011). Dynare: Reference manual, version 4.

Airaudo, M. & Hajdini, I. (2019). Regime switching consistent expectations equilibria: General results with application to monetary policy.

An, S. & Schorfheide, F. (2007). Bayesian analysis of dsge models. *Econometric reviews*, 26(2-4), 113–172.

Berardi, M. & Galimberti, J. K. (2017a). Empirical calibration of adaptive learning. *Journal of Economic Behavior & Organization*, 144, 219–237.

Berardi, M. & Galimberti, J. K. (2017b). On the initialization of adaptive learning in macroeconomic models. *Journal of Economic Dynamics and Control*, 78, 26–53.

Berardi, M. & Galimberti, J. K. (2017c). Smoothing-based initialization for learning-to-forecast algorithms. *Macroeconomic Dynamics*, (pp. 1–16).

Bianchi, F. (2016). Methods for measuring expectations and uncertainty in markov-switching models. *Journal of Econometrics*, 190(1), 79–99.

Bianchi, F. & Ilut, C. (2017). Monetary/fiscal policy mix and agents' beliefs. *Review of economic Dynamics*, 26, 113–139.

Bianchi, F. & Melosi, L. (2017). Escaping the great recession. *American Economic Review*, 107(4), 1030–58.

Binning, A. & Maih, J. (2016). Implementing the zero lower bound in an estimated regime-switching dsge model.

Bonam, D., de Haan, J., Soederhuizen, B., et al. (2017). *The effects of fiscal policy at the effective lower bound*. Technical report, Netherlands Central Bank, Research Department.

Branch, W. A., Davig, T., & McGough, B. (2013). Adaptive learning in regime-switching models. *Macroeconomic Dynamics*, 17(5), 998–1022.

Branch, W. A. & Evans, G. W. (2006). A simple recursive forecasting model. *Economics Letters*, 91(2), 158–166.

Bullard, J. & Duffy, J. (2004). Learning and structural change in macroeconomic data. *FRB of St. Louis Working Paper No.*

Bullard, J. & Eusepi, S. (2014). When does determinacy imply expectational stability? *International Economic Review*, 55(1), 1–22.

Bullard, J. & Mitra, K. (2002). Learning about monetary policy rules. *Journal of monetary economics*, 49(6), 1105–1129.

Davig, T. & Leeper, E. M. (2007). Generalizing the taylor principle. *American Economic Review*, 97(3), 607–635.

Del Negro, M., Giannoni, M., & Patterson, C. (2012). The forward guidance puzzle. *Center for Economic Policy Research (CEPR).*

Di Pace, F., Mitra, K., & Zhang, S. (2016). Adaptive learning and labour market dynamics.

Eusepi, S. & Preston, B. (2011). Expectations, learning, and business cycle fluctuations. *American Economic Review*, 101(6), 2844–72.

Evans, G. W., Guse, E., & Honkapohja, S. (2008). Liquidity traps, learning and stagnation. *European Economic Review*, 52(8), 1438–1463.

Evans, G. W. & Honkapohja, S. (2010). 11 expectations, deflation traps and macroeconomic policy. *Twenty Years of Inflation Targeting: Lessons Learned and Future Prospects*, (pp. 232).

Evans, G. W. & Honkapohja, S. (2012). *Learning and expectations in macroeconomics.* Princeton University Press.

Farmer, R. E., Waggoner, D. F., & Zha, T. (2009). Understanding markov-switching rational expectations models. *Journal of Economic theory*, 144(5), 1849–1867.

Farmer, R. E., Waggoner, D. F., & Zha, T. (2011). Minimal state variable solutions to markov-switching rational expectations models. *Journal of Economic Dynamics and Control*, 35(12), 2150–2166.

Gaus, E. & Gibbs, G. C. (2018). *Expectations and the empirical fit of DSGE models.* Technical report, Mimeo.

Gust, C., Herbst, E., & Lopez-Salido, J. D. (2018). Forward guidance with bayesian learning and estimation.

Hollmayr, J. & Matthes, C. (2015). Learning about fiscal policy and the effects of policy uncertainty. *Journal of Economic Dynamics and Control*, 59, 142–162.

Hommes, C. & Zhu, M. (2014). Behavioral learning equilibria. *Journal of Economic Theory*, 150, 778–814.

Ji, Y. & Xiao, W. (2016). Government spending multipliers and the zero lower bound. *Journal of Macroeconomics*, 48, 87–100.

Lansing, K. J. (2018). Endogenous regime switching near the zero lower bound.

Lindé, J., Maih, J., & Wouters, R. (2017). *Estimation of Operational Macromodels at the Zero Lower Bound.* Technical report, Mimeo.

Liu, P. & Mumtaz, H. (2011). Evolving macroeconomic dynamics in a small open economy: An estimated markov switching dsge model for the uk. *Journal of Money, Credit and Banking*, 43(7), 1443–1474.

Liu, Z., Waggoner, D. F., & Zha, T. (2011). Sources of macroeconomic fluctuations: A regime-switching dsge approach. *Quantitative Economics*, 2(2), 251–301.

Maih, J. (2015). Efficient perturbation methods for solving regime-switching dsge models.

Milani, F. (2007). Expectations, learning and macroeconomic persistence. *Journal of monetary Economics*, 54(7), 2065–2082.

Quaghebeur, E. (2018). Learning and the size of the government spending multiplier. *Macroeconomic Dynamics*, (pp. 1–36).

Sims, C. A., Waggoner, D. F., & Zha, T. (2008). Methods for inference in large multiple-equation markov-switching models. *Journal of Econometrics*, 146(2), 255–274.

Sims, C. A. & Zha, T. (2006). Were there regime switches in us monetary policy? *American Economic Review*, 96(1), 54–81.

Slobodyan, S. & Wouters, R. (2012a). Learning in a medium-scale dsge model with expectations based on small forecasting models. *American Economic Journal: Macroeconomics*, 4(2), 65–101.

Slobodyan, S. & Wouters, R. (2012b). Learning in an estimated medium-scale dsge model. *Journal of Economic Dynamics and control*, 36(1), 26–46.

Smets, F. & Wouters, R. (2007). Shocks and frictions in us business cycles: A bayesian dsge approach. *American economic review*, 97(3), 586–606.

Uhlig, H. F. et al. (1995). A toolkit for analyzing nonlinear dynamic stochastic models easily.

Woodford, M. (2013). Macroeconomic analysis without the rational expectations hypothesis. *Annu. Rev. Econ.*, 5(1), 303–346.