Expectations and Learning: a Horse-race in a medium-scale DSGE model (Preliminary and incomplete, please do not cite or distribute)

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Abstract

We consider a horse-race between different expectation formation rules in the workhorse DSGE model Smets-Wouters (2007). We analyze the in- and out-of-sample fit of the model under a variety of specifications including Rational Expectations Equilibrium (REE), Behavioral Learning Equilibria (BLE), sample-autocorrelation learning (SAC) and recursive least squares learning (RLS) with AR(1) and AR(2) rules. Our contributions are threefold: (1) we show that small forecasting rules based on AR(1) and AR(2) outperform their REE counterpart both in-sample and out-of-sample, and this holds both in equilibrium with fixed-beliefs (BLE) and time-varying beliefs (SAC- and RLS-learning). (2) We disentangle the effects of three assumptions on agents' expectations, namely the information set, the timing and the time-variation of expectations. (3) We show that the SAC-learning algorithm based on sample moments is a plausible and computationally efficient alternative recursive least-squares learning.

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1 Introduction

The Rational Expectations Hypothesis, first introduced by Muth (1961), requires that economic agents' subjective probability distributions coincide with the objective distributions of models. This corresponds to a joint determination of beliefs and the state of the economy in self-referential frameworks, where the agents know the actual economic structure and form their expectations accordingly. There is a vast and growing literature that studies the drawbacks of Rational Expectations Equilibria (REE), which requires an unrealistic degree of computational power and knowledge on the agents' part. A strand of literature that emerged as an alternative to REE is adaptive learning, which replaces Rational Expectations (RE) with beliefs that come from an econometric forecasting model with parameters updated using observed time series from the past. Accordingly, agents are equipped with a Perceived Law of Motion (PLM) that affects the Actual Law of Motion (ALM), which then feeds back into the PLM in a self-referential manner. Much of the earlier literature on adaptive learning has focused on convergence to REE under adaptive learning. This is possible if the agents' information set actually nests the REE. But if we assume that agents use a misspecified information set to begin with, then convergence to a REE cannot occur. However, a reasonable learning process may still settle down on a misspecification equilibrium.

In recent literature, different types of misspecification equilibria have been proposed. A non-exhaustive list includes Restricted Perceptions Equilibria (RPE), which generally refers to underparameterized forecasting rules (see e.g. Sargent, 1991; Evans and Honkapohja, 2001; Adam, 2003; Branch and Evans, 2010, Lansing, 2009a, 2009b; Lansing & Ma, 2017); Natural Expectations (Fuster et. al., 2010, 2011, 2012), where agents use lower order autoregressive models than implied by the correct model, and Stochastic Consistent Expectations Equilibria SCEE (Hommes & Sorger, 1998), where agents use a simple linear AR(1) rule in a non-linear model.

In the simplest model applying the concept of SCEE, Hommes & Zhu (2014) introduce the univariate and parsimonious AR(1) rule in a linear framework, and define resulting equilibria as Behavioral Learning Equilibria (BLE), where the complete structure of the economy is too complex for the agents to fully understand. Therefore, as a first-order approximation, agents forecast the state of the economy by simple autoregressive models. A BLE is the simplest model applying this idea, where agents run a univariate AR(1) regression to generate out-of-sample forecasts of the relevant economic variables. In Hommes et. al. (2019), we extend the concept of BLE to general n-dimensional linear systems and provide a method for Bayesian estimation of linear systems under BLE. We show that estimating the baseline 3-equation New Keynesian model (Woodford, 2003) under BLE on U.S. data improves the model fit, and that monetary policy implications under BLE may be vastly different than the implications under REE.

In this paper, we extend our work in Hommes et. al. (2019) in several important dimensions. First, we refine our estimation algorithm to better handle medium- and large-scale DSGE models. Using this, we estimate a variant of the well-known Smets-Wouters (2007) model and show that the model fit under BLE improves relative to REE. We further provide an out-of-sample forecast exercise and show that the model delivers more accurate forecasts under BLE. This confirms our results in Hommes et. al. (2019) by showing that the model improvement continues to hold in a more realistic setup both in- and out-of-sample.

Second, on top of the BLE and REE models, we provide estimations and out-of-sample forecasts under three constant-gain learning models with different information sets: we consider AR(1) and MSV-learning models, where the information set corresponds to those used under BLE and REE respectively. We further consider an AR(2)-learning model, which can introduce an extrapolation on beliefs relative to the AR(1) model. Using this framework as a horse-race, we disentangle the effects of the information set from learning on parameter estimates and model fit. In particular, we find that the information set matters just as much as the learning dynamics.

Third, as a plausible alternative to constant-gain Recursive Least Squares (RLS) learning, we introduce the constant gain Sample Autocorrelation (SAC) learning for the special case of AR(1) learning. An advantage of this algorithm is over RLS is that it does not require the use of a projection facility: a well-known shortcoming of the constant-gain RLS learning is that, agents' PLM is not restricted to be stationary and as such, there are recurrent escapes from stable regions. Therefore RLS algorithms are typically complemented with a projection facility, which reinitializes the algorithm in the stable region if the realized data push the estimates into regions of instability. This is a computationally heavy step that requires ad-hoc assumptions, and the resulting estimates may be sensitive to how the projection facility is imposed. The SAC-learning algorithm, different from RLS, is a method of moments based estimator. Since the learning estimates depend on the realized sample moments, they are naturally restricted to be stationary. As such, SAC-learning comes already equipped with its own natural projection facility, and always generates stationary forecasts[1].

Fourth, we evaluate the timing assumption on expectations: standard REE models assume that agents' forecasts correctly take into account the current value of the relevant economic variables, which are in turn, partly determined by these forecasts. Therefore this assumption boils down to a joint determination of expectations and realizations, which

[1]In this paper, we limit ourselves to SAC-learning under AR(1) rule only. In ongoing research, Branch et. al. (2014) derive a more general constant-gain method of moments estimater that is applicale to other types of PLM. They define this generalization as Yule-Walker learning.

we will call the *period t assumption*. In adaptive learning, a common alternative is to assume a sequential timeline: expectations are formed using previous period's variables, then the variables are realized and expectations are updated. We will call this the *period t-1 assumption*. Empirical studies on adaptive learning made use of both assumptions, without making an explicit distinction between these two: e.g. Milani (2007, 2011) uses the perido t-1 assumption, whereas Slobodyan & Wouters (2011a, 2011b) and Ormeño & Molnár (2015) use the period t assumption, to name a few. In this paper, we test our models under both assumptions and show that some parameter estimates may be affected, with some differences in the model fit.

Finally, our work paves the way for a new initialization method in adaptive learning models with small forecasting rules. It is well known that empirical validation of adaptive learning models are sensitive to initial values, particularly in small samples (add references here). A growing body of literature recognizes that small forecasting rules provide a reasonable alternative to REE, see e.g. Slobodyan & Wouters (2011a, 2011b) and Ormeño & Molnár (2015). However, the standard approach in these papers is to initialize the beliefs from a distribution corresponding to REE, or to use other ad-hoc methods such as a training-sample. A conceptually better approach, however, is to initialize the beliefs at their model-consistent equilibrium, i.e. the corresponding Restricted Perceptions Equilibrium (RPE). In our framework, this corresponds to initializing the beliefs at the BLE when an AR(1) learning rule. In principle, our method is flexible enough to compute the corresponding RPE under different types of PLMs, such as an AR(2) rule.

Related Literature

Applications of adaptive learning in macroeconomic models has been of great interest to policymakers and academics alike. Our paper contributes to this growing line of literature in macroeconomics, see e.g. Evans and Honkapohja (2001), Branch (2006) and Bullard (2006) and Woodford (2013) for extensive reviews and surveys.

One shortcoming of REE models that receives attention in the literature is its failure to generate realistic expectation dynamics, and in particular being at odds with data coming from survey expectations. For example, Canova and Gambetti (2010) revisit the great moderation period and examine the role of expectations using reduced form methods. By using data from SPF, they find an important role for expectations that did not substantially change over time. Adam and Padulo (2011) estimate a forward-looking New Keynesian Phillips Curve (NKPC) using data from the Survey of Professional Forecasters SPF as a proxy for expected inflation and obtain reasonable estimates for the slope of the Phillips curve, which is an improvement over the REE model. Along similar lines, Del Negro and Eusepi (2011) use inflation expectations as an observed variable in their model estimations, and find evidence that the survey of expectations contains information not explained by other macroeconomic variables. Gennaioli et. al. (2016) show, by using survey expectations, that corporate investment plans depend on CFO's expectations of earnings growth. Forecast errors in CFO's expectations are predictable, which provides evidence in support of small extrapolative forecasting rules. Fuhrer (2017) shows that embedding survey data into DSGE models helps in several directions, such as reducing reliance on ad-hoc sources of persistence such as habit and indexation. A common feature in these studies is that they document the shortcomings of REE models along the expectations dimension and argue for the usefulness of incorporating data from survey expectations into these models.

A large part of the literature on adaptive learning focus on dynamics under MSV learning, i.e. the conditions under which the learning process converges to the underlying REE, e.g. Orphanides and Williams (2004) study monetary policy under MSV-learning and find that optimal policy is typically more aggressive to inflation under learning. Milani (2007) considers the estimation of the baseline New Keynesian model and finds that the model fit is improved under learning, while the dependence on some structural parameters such as habit and indexation is substantially reduced. Berardi and Galimberti (2017a) consider model specifications with time-varying gains under MSV-learning, and find higher estimates for the gain parameter on inflation.

More recently, a growing number of papers also considered small and/or misspecified forecasting rules as a convenient alternative to Rational Expectations and MSV learning, e.g. Lansing (200b) constructs a consistent expectations equilibrium (CEE), similar in spirit to our BLE concept, where agents use the optimal Kalman gain within their class of misspecified models. Along similar lines, Lansing and Ma (2017) use a CEE concept to study exchange rate dynamics. In a series of papers, Fuster et. al. (2010, 2011, 2012) study natural expectations characterized by an underestimation of the degree of mean reversion, which arises when agents use lower order autoregressive models than is warranted by the correct data generating process. As such, a BLE may be seen as an example of natural expectations when applied models of higher of autoregressive processes. Ormeño & Molnár (2015) investigate whether an adaptive learning model can fit the macroeconomic and survey data simultaneously, and find that this is true only when small forecasting rules are considered. The most relevant study for our paper is Slobodyan and Wouters (2012a), where the authors show that an AR(2) forecasting rule under Kalman gain learning substantially improves the model fit without a large effect on parameter estimates. As such, our paper can be seen as extending their work in several directions, where we disentangle the effects of initial beliefs, the timing of expectations and the learning algorithm on the model fit. Along similar lines, some studies investigate ARIMA type forecasting rules in an experimental setup, and find evidence in favor of small forecasting rules, see e.g. Beshears et. al. (2013) and Adam (2007).

A closely related body of work considers heterogeneous expectations and learning, where subsets of agents equipped with different types of forecasting rules or they are allowed to switch between different rules over time. For example, Cole and Milani (2016) show that incorporating heterogeneous expectations in a DSGE-VAR framework may help overcome some issues associated with New Keynesian models in terms of matching survey data. Audzeli and Slobodyan (2017) use a framework where agents are equipped with misspecified models and are allowed to evalute and change their forecasting models over time. They find that in some parameter regions, agents find it optimal to use their choice of a (misspecified) AR(1) rule. Gelain et. al. (2019) investigate hybrid expectations in Smets-Wouters (2007) model, where some agents use moving average rules. See Branch and McGough (2018) and Jump and Levine (2018) for an extensive overview of New Keynesian models with heterogeneous expectations.

A final line of research related to our paper is on the initialization of adaptive learning models: it has long been recognized that empirical studies on adaptive learning are sensitive to where the algorithms are initialization, which is particularly important when the sample size is small. Carceles-Poveda and Giannitsarou (2007) review several approaches to initialization in different learning algorithms, and conclude that the choice of initial conditions is very important if one is interested in using learning in an applied context. Berardi and Galimberti (2017b) review different initialization methods used in the literature, which are classified into REE-based, training-sample based and estimation-based methods. They find that equilibrium-related and training-sample based initials are less prone to distortions under small samples or model misspecification. In a further study, Berardi and Galimberti (2019) provide an alternative initialziation method, which is essentially a sequential procedure that estimates a learning sequence given the initials, and then estimates the initials given the estimates learning sequence until some convergence criterion is satisfied. Gibbs and Gaus (2018) document some of the stylized facts commonly observed in studies that estimate DSGE models under learning, which include an almost universal improvement in model fit and small estimated gain parameters. The latter result in particular implies that initial beliefs are an important determinant of fit in adaptive learning models. Our study can be seen as contributing to this literature by providing an alternative approach to initialization when small forecasting rules are considered. While our method and results are only illustrated using the AR(1) rule, it is easily extended to similar univariate forecasting rules.

2 Main Concepts

2.1 Behavioral Learning Equilibria

Hommes and Zhu (2014) introduce BLE in a simple setting with one-dimensional linear models, while Hommes et. al. (2019) extend the concept to general N-dimensional linear models and estimate the baseline New Keynesian DSGE model based on U.S. data. In this paper, we extend the empirical analysis of BLE to a more realistic setup by estimating the workhorse DSGE model Smets-Wouters (2007). To provide a brief overview of the main concepts, let the actual law of motion of the economy be given by

$$\begin{aligned} \boldsymbol{x}_t &= \boldsymbol{b}_0 + \boldsymbol{b}_1 \boldsymbol{x}^e_{t+1} + \boldsymbol{b}_2 \boldsymbol{x}_{t-1} + \boldsymbol{b}_3 \boldsymbol{u}_t + \boldsymbol{b}_4 \boldsymbol{v}_t, \qquad (2.1) \\ \boldsymbol{u}_t &= \boldsymbol{a} + \boldsymbol{\rho} \boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, \qquad (2.2) \end{aligned}$$

where $\boldsymbol{x}_t$ is a vector of endogenous variables and $\boldsymbol{u}_t$ is a vector of exogenous variables following a stationary VAR(1) process. Under a BLE, we assume that agents' perceived law of motion (PLM) is misspecified and follows a univariate AR(1) process:

$$\boldsymbol{x}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}) + \boldsymbol{\delta}_t, \qquad (2.3)$$

where $\boldsymbol{\alpha}$ denotes the vector of perceived means, and $\boldsymbol{\beta}$ denotes a diagonal matrix of perceived first-order autocorrelations. A BLE imposes two natural consistency requirements on the perceived coefficients: the unconditional means and unconditional first-order autocorrelation coefficients generated by the actual process 2.1 and 2.2, unknown to agents, coincide with the corresponding statistics for the perceived linear VAR(1) process 2.3, given by the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. More formally, a BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ satisfies

$$\begin{cases} \boldsymbol{\alpha}^* = E[\boldsymbol{x}_t] \\ \boldsymbol{\beta}^* = E[(\boldsymbol{x}_t - E[\boldsymbol{x}_t])(\boldsymbol{x}_{t-1} - E[\boldsymbol{x}_{t-1}])']. \end{cases} \qquad (2.4)$$

In Hommes et. al. (2019), we derive the existence and E-stability conditions for the general linear case. A BLE is characterized by the same unconditional mean as the corresponding REE. This means, on average, the endogenous variables fluctuate around the same values under a BLE and REE. In standard DSGE models linearized around a zero mean, this corresponds to $\boldsymbol{\alpha}^* = 0$. However, the matrix of first-order autocorrelation coefficients $\boldsymbol{\beta}^*$ is typically a highly non-linear function of the uderlying structural parameters, and there is no closed-form solution for it. Therefore the concept of BLE is accompanied by the notion of iterative E-stability (Evans, 1985), which is a simple fixed-point iteration that computes E-stable equilibria. Denoting the unconditional first-order autocorrelation coefficients as $G(\boldsymbol{\beta^*})$, iterative E-stability considers the following iteration:

$$\boldsymbol{\beta^{(k+1)}} = G(\boldsymbol{\beta^{(k)}}), 1 \leq k \leq N. \tag{2.5}$$

where $G(.)$ denotes the function of first-order autocorrelation coefficients. Given a reasonable initial value and a sufficiently large N, the above iteration converges to an E-stable BLE. Using this fixed-point iteration has the advantage of eliminating all E-unstable equilibria, since the algorithm cannot converge to such equilibria by construction[2]. In Hommes et. al. 2019, we further provide an estimation algorithm that uses the notion of iterative E-stability, which characterizes the underlying non-linear problem as a sequence linear problems via a *guess and verify* approach, where the guess values are updated across estimations using steps of the fixed-point iteration. The advantage of this approach is that it turns the estimation of BLE into a standard estimation problem, which can be implemented in standard toolboxes such as Dynare. In this paper, we implement (2.5) directly in our estimations by developing a toolkit that handles this problem. As such, estimation of a BLE works in the same manner as the standard estimation of Rational Expectations models. In a REE framework, for a given parameter draw, (i) the model is solved for the equilibrium, which yields a recursive backward-looking linear system, (ii) the likelihood is computed using the Kalman filter and prior distributions. For a BLE, only the first step differs from REE, while the second step works in the same manner. Further details of the estimation algorithms can be found in Appendix C.

2.2 AR(1) and AR(2) Learning

In a BLE, the learning coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are fixed at their equilibrium values, i.e. during the estimation, we assume that a sufficiently long learning process already took place in the past and the agents coordinated on the equilibrium. The importance of estimating such an equilibrium is that, it provides a one-to-one comparison with a REE, where the agents are assumed to have learned a REE in the past and have already coordinated on it. As such, the only difference between a BLE and REE is the information set that the agents use, i.e. a parsimonious AR(1) process that ignores cross-correlations, or the fully rational information set. In the literature, the information set associated with small forecasting rules have been compared with the REE models only in the context of learning. As such, our paper aims to distinguish the effects of learning and the information set from each other. To this end, we introduce learning dynamics on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, where we assume that agents do not know the exact equilibrium values, but instead act

[2]See Hommes et. al. (2019) for further details on the algorithm, and the accompanying online appendix for a comparison with the quasi-Newton algorithm.

like econometricians and update their beliefs every period as new observations become available. As such, agents' PLM and the corresponding expectations are replaced with the following[3]:

$$
\begin{cases} \boldsymbol{x}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}(\boldsymbol{x}_{t-1}) + \boldsymbol{\delta}_t, \\ E_t\boldsymbol{x}_{t+1} = (\boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}\boldsymbol{\alpha}_{t-1}) + \boldsymbol{\beta}_{t-1}^2\boldsymbol{x}_{t-1}, \end{cases} \tag{2.7}
$$

which assumes the following timeline: in every period t, agents form expectations using their information from the previous period $t-1$, after which the endogenous variables $\boldsymbol{x}_t$ are determined. Then the beliefs are updated using the new information from period t. The most common approach in the literature to update beliefs is constant-gain least squares. For the AR(1) model at hand, this is given as follows for each forward-looking variable x_t:

$$
\begin{cases} R_t = R_{t-1} + \gamma(Y_t^2 - R_{t-1}), \\ \theta_t = \theta_{t-1} + \gamma R_t^{-1} Y_{t-1}(S_t - \theta_{t-1}Y_{t-1}), \end{cases} \tag{2.8}
$$

with $\theta_t = [\alpha_t, \beta_t]$, $Y_t = [1, x_{t-1}]'$ and R_t the perceived variance of each variable x_t. γ denotes the gain coefficient and determines how much weight the agents place on the most recent data. In our empirical exercises, we treat this as a free parameter and estimate it along with the remaining structural parameters.

As an alternative to the least squares learning given above, we investigate the constant-gain sample-autocorrelation learning algorithm, which is given as follows for each variable x_t:

$$
\begin{cases} R_t = R_{t-1} + \gamma(x_t - \alpha_{t-1}), \\ \alpha_t = \alpha_{t-1} + \gamma(x_t - \alpha_{t-1}), \\ \beta_t = \beta_{t-1}\gamma R_t^{-1}[(x_t - \alpha_{t-1})(x_{t-1} - \alpha_{t-1}) - \beta_{t-1}(x_t - \alpha_{t-1})^2]. \end{cases} \tag{2.9}
$$

The difference between (2.8) and (2.9) is that, the first one makes use of the least squares estimator whereas the second one uses a method-of-moments estimator. The least squarest estimator has the well-known problem that the estimates on lagged variables are not restricted to be inside the unit circle, i.e the estimator may occasionally lead to explosive PLMs when sufficiently large shocks occur, which is known as escape dynamics in the adaptive learning literature. This is an issue in self-referential New Keynesian models, since an explosive PLM may push the implied ALM into explosive regions as well, in which case the model enters into a self-fulfilling explosive path. In order to deal

[3] For the special case of SAC-learning, we use a re-parameterized version of this:

$$
\begin{cases} \boldsymbol{x}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1}) + \boldsymbol{\delta}_t, \\ E_t\boldsymbol{x}_{t+1} = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1}). \end{cases} \tag{2.6}
$$

with this issue, constant-gain least squares models are equipped with a projection facility, which resets the learning algorithm in a stable region if the learning update leads to explosive updates. This typically introduces an additional step in the estimations and simulations, complicating the process and leading to an extra layer of computational burden. Further, the resulting time path of the systems may end up being sensitive to the way that the projection facility is implemented. SAC-learning offers a natural solution to this issue, because the estimates, based on the sample moments, are naturally restricted to stationary regions. In other words, SAC-learning is equipped with a natural projection facility that may reduce reliance on the projection facility[4]. Using both BLE and AR(1) learning estimates further allows us to disentangle the effects of learning from the information set, i.e. how much the learning dynamics improve the model fit on top of the improvement coming from BLE.

We further consider an AR(2) recursive least squares learning model for comparison, without computing the sample-autocorrelation and fixed-point equilibrium correspondences of this model[5]. In this case the PLM in 2.8 is replaced with:

$$\begin{cases} R_t = R_{t-1} + \gamma(Y_t^2 - R_{t-1}), \\ \theta_t = \theta_{t-1} + \gamma R_t^{-1} Y_t (S_t - \theta_{t-1} Y_t), \end{cases} \tag{2.10}$$

where $\theta_t = [\alpha_t, \beta_{1,t}, \beta_{2,t}]$ and $Y_t = [1, x_{t-1}, x_{t-2}]'$. This is the baseline PLM in Slobodyan & Wouters (2012a). A potential advantage of this PLM over the AR(1) rule is that it can generate an extrapolation bias in beliefs, where the most recent observation receives more weight relative to its AR(1) counterpart[6].

2.3 MSV Learning

The final model specification that we consider is MSV learning. This makes the assumption that agents use the correct functonal form associated with a REE, but do not know the exact equilibrium restrictions. The MSV solution of the ALM in (2.1) and (2.2) takes the following form:

$$\begin{cases} \boldsymbol{x}_t = \gamma_0 + \gamma_1 \boldsymbol{x}_{t-1} + \gamma_2 \boldsymbol{u}_t + \gamma_3 \boldsymbol{v}_t, \\ E_t \boldsymbol{x}_{t+1} = \gamma_0 + \gamma_1 \boldsymbol{x}_t + \gamma_2 \boldsymbol{\rho} \boldsymbol{u}_t, \end{cases} \tag{2.11}$$

[4]Note that, SAC-learning only ensures that the PLM is always stationary. This does not guarantee the stationarity of the implied ALM, which still needs to be checked separately.

[5]An extension of our equilibrium concept to AR(2) beliefs is under construction. A generalization of the SAC-learning algorithm to other types of PLMs is undertaken in Branch et. al. (2014).

[6]Empirical evidence in favour of the extrapolation bias has been found in e.g. Fuster et. al. (2010a, 2010b).

where we use the standard assumption that in a REE, 1-step ahead expectations and the endogenous variables are jointly determined. Plugging the implied 1-step ahead expectations into the ALM and solving for the underlying fixed-point yields the REE associated with the MSV solution[7]. Under learning, the PLM is again replaced with beliefs parameters that are updated every period, along with the 1-step ahead expectations:

$$\begin{cases} \boldsymbol{x}_t = \gamma_{0,t-1} + \gamma_{1,t-1}\boldsymbol{x}_{t-1} + \gamma_{2,t-1}\boldsymbol{u}_t + \gamma_{3,t-1}\boldsymbol{v}_t, \\ E_t\boldsymbol{x}_{t+1} = \gamma_{0,t-1} + \gamma_{1,t-1}\boldsymbol{x}_t + \gamma_{2,t-1}\boldsymbol{\rho}\boldsymbol{u}_t, \end{cases} \tag{2.12}$$

where we retain the REE assumption that expectations and endogenous variables are jointly determined every period. Accordingly, we use the following timing structure in a given period t: (I) agents' learning coefficients and PLM are based on information from period $t-1$ as denoted in (2.12), (II) 1-step ahead expectations $E_t\boldsymbol{x}_{t+1}$ and the contemporaneous endogenous variables $\boldsymbol{x}_t$ are jointly determined, (III) belief coefficients in agents' PLM are updated based on the new information $\boldsymbol{x}_t$. Notice that, this introduces an implicit difference in the timing assumption under BLE, AR(1)- and AR(2)-learning models which assumes expectations based on period $t-1$ information, and MSV-learning which assumes expectations based on period t information. Each case is consistent with the underlying timing structure used in the literature. A detailed treatment of the implications of the timing assumption is in Appendix E, which is also briefly discussed in the estimation section.

3 Estimation of Smets-Wouters Model

3.1 Measurement Equations and Prior Distributions

We consider a horse-race between the expectation formation rules discussed above in the Smets-Wouters (2007) model, henceforth SW, for the U.S historical quarterly macroeconomic data over the period 1966:I-2007:IV in order to avoid issues associated with the zero lower bound period after the crisis. The observable variables used in the estimation are the log difference of real GDP (y_t^{obs}), real consumption (c_t^{obs}), real investment (inv_t^{obs}), real wage (w_t^{obs}), log hours worked (l_t^{obs}), inflation (π_t^{obs}) and the federal funds rate (r_t^{obs})

[7]See Appendix C for further details.

for the U.S economy. Accordingly, the measurement equations are given as:

$$\begin{cases} d(log(y_t^{obs})) = \bar{\gamma} + (y_t - y_{t-1}), \\ d(log(c_t^{obs})) = \bar{\gamma} + (c_t - c_{t-1}), \\ d(log(inv_t^{obs})) = \bar{\gamma} + (inv_t - inv_{t-1}), \\ d(log(w_t^{obs})) = \bar{\gamma} + (w_t - w_{t-1}), \\ log(l_t^{obs}) = \bar{l} + l_t, \\ (log(\pi_t^{obs})) = \bar{\pi} + \pi_t, \\ (log(r_t^{obs})) = \bar{r} + r_t. \end{cases} \tag{3.1}$$

The model structure is the same as the original SW except for three minor deviations. The first is on the definition of output gap: in the original model, this is the deviation of output from its potential level, defined as output in the presence of flexible prices and wages. This definition requires the modelling of the flexible economy, which introduces additional forward-looking variables and further complicates the learning models. Instead we follow Slobodyan and Wouters (2012b) and define output gap as the deviation of output from its natural level based on the productivity process[8]. The second deviation is on the exogenous price and wage mark-up shocks, which follow ARMA(1,1) processes in the original model. However, as shown in Slobodyan and Wouters (2012b), mark-up shocks are typically reduced to near white noise processes once learning dynamics are introduced. In these cases the AR(1) and MA(1) parameters close being localy unidentified. Therefore we shut off the MA component of these shocks. The third difference is on the prior distribution of price stickiness ξ_p, which we tighten from $\xi_p \sim Beta(0.5, 0.2)$ to $\xi_p \sim Beta(0.75, 0.05)$. The reason for this tightening is based on our previous findings on the New Keynesian Phillips Curve (NKPC) in Hommes et. al. (2019), where we observe explosive outcomes and no BLE for large values of the slope of the Phillips curve (i.e. low values of price stickiness ξ_p). To set the ideas, consider a Phillips curve as follows:

$$\begin{cases} \pi_t = \beta E_t \pi_{t+1} + \gamma x_t + u_{\pi,t} \\ u_{\pi,t} = \rho_\pi u_{\pi,t-1} + \eta_t, \end{cases} \tag{3.2}$$

where x_t denotes the output. Figure (3.1) shows a projection of the regions with unique, multiple and explosive BLE over the parameter space $[\rho_\pi, \gamma]$, where the results come from the baseline 3-equation New Keynesian model thats nests the above equations. The red region with sufficiently large values γ correspond to parameter values where we encounter

[8]See the Appendix for further details.

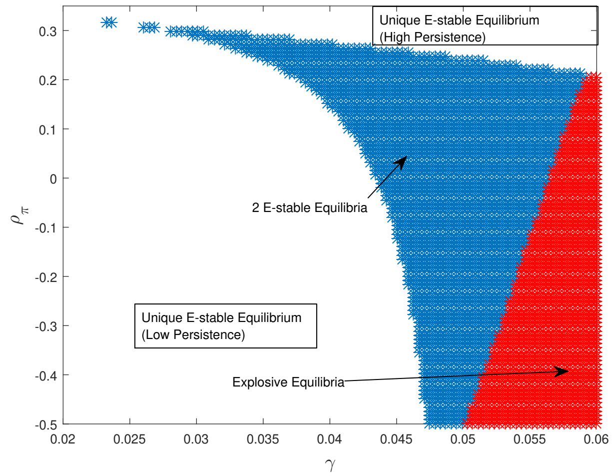


Figure 1: Unique, multiple and explosive equilibrium regions in the Phillips curve with over the parameters γ (slope of the Phillips curve) and ϕ_π (persistence of the cost-push shock.

exlosive BLE. We observe similar dynamics in the SW model, where small values of price stickiness ξ_p result in explosive BLE. SW model further incorporates a term with lagged inflation due to price indexation, which makes explosive outcomes more likely. We discuss this issue and how the existence of stationary BLE depend on different combinations of ξ_p, ρ_π and ι_p in further detail in latter sections. Here we content ourselves with using this preliminary example to justify the tigh prior on price stickiness. Later on, we further provide several alternative estimations with different restrictions on these parameters to see how the estimation results are affected by this issue.

The rest of the model, and the remaining prior distributions on the estimated parameter are left unchanged, see the model Appendix A for further details.

3.2 Estimation Methodology

3.2.1 Fixed-point Models

The estimations are performed in our stylized toolkit that performs the estimation of REE, BLE and learning models[9]. We make use of H. Uhlig's toolkit (1995) to compute the REE in our estimations[10]. For the BLE estimations, we use our iterative E-stability algorithm as outlined in the previous section, and explained in detail in Hommes et. al. (2019). Given a vector of initial values for the first-order autocorrelation coefficient of forward-looking variables, we use 200 iterations for each parameter draw and use the resulting fixed-point as an approximate BLE[11]. The initial values for the first-order au-

[9]The codes are freely available on the accompanying Github page of this paper.

[10]The estimations results for REE model do not substantially differ from the results in Dynare.

[11]The final estimates have a small sensitivity to the number of fixed-point iterations, but not enough to affect our results here qualitatively.

tocorrelation coefficients are determined pre-estimation : for forward-looking variables that are observable, i.e. inflation π_t and hours worked l_t, the initial value is set to the corresponding sample first-order autocorrelation. For the remaining forward-looking variables that are unobservable in the estimation, we take the unconditional moments implied by the estimated REE. We keep the initial values fixed at these values for all parameter draws, and assume the presence of a unique BLE during the estimation. Parameter draws where the fixed-point iteration fails to converge to a stationary equilibrium are discarded. We check the uniqueness or multiplicity of equilibria post-estimation, where we take the estimated parameter values at the posterior mode and repeat the fixed-point iteration for a grid of initial values over the unit circle[12].

The estimation methods for the REE and BLE models differ only in terms of equilibrium computation. Once the equilibrium is solve for, each model can be represented as a recursive linear system as:

$$S_t = \bar{S} + \Gamma_1 S_{t-1} + \Gamma_2 \eta_t, \tag{3.3}$$

with $S_t = [X_t, \epsilon_t]'$. As such, the models under BLE and REE differ differ in terms of Γ_1 and Γ_2[13]. Once the equilibrium is solved for and the recursive linear system is obtained for a given parameter draw, the likelihood is computed using the standard Kalman filter.

3.2.2 Learning Models

Similar to the fixed-point models, after plugging in the expectations every period, the learning models can be represented as a recursive system:

$$S_t = \bar{S} + \Gamma_{1,t} S_{t-1} + \Gamma_{2,t} \eta_t, \tag{3.4}$$

where the only difference between (3.3) comes from time-varying matrices $\Gamma_{1,t}$ and $\Gamma_{t,2}$. For these models, we take advantage of the sequential timing of events in a given period, which leads to a conditionally linear structure: at every period, the model is linear in state variables for a given set of belief coefficients and therefore a standard step of the Kalman filter recursion can be applied. In turn, for a given set of state variables using the Kalman filter output, the learning coefficients can be updated, which are then used in the next step of the recursion. This approach follows the description in Slobodyan and Wouters (2012a, 2012b).

For both fixed-point and learning models, we first obtain the posterior mode using C. Sim's *csminwel* optimization routine (1999). This is then used as a candidate density

[12]Alternatively, one could check for multiplicity of equilibria during the estimation by running the fixed-point iteration for a grid of initial values for every parameter draw. We do not follow this approach in our paper due to the computational burden it would impose during the estimations.

[13]Derivations of these matrices for each model can be found in the Appendix.

for the MCMC draws, where we use a random-walk Metropolis-Hastings algorithm[14]. For each model, we use two parallel Markov Chains where the scale coefficient of the covariance matrix is used to obtain an acceptance ratio between 30-35 %. Each Markov Chain contains 500000 draws, where the first half is discarded as a burn-in sample and convergence is checked via CUSUM plots[15].

Previous empirical studies on adaptive learning document that initial values play an important role in the resulting parameter estimates and model fit, see e.g. Slobodyan and Wouters (2012a), Berardi and Galimberti (2017b), and Gaus and Gibbs (2016). We therefore provide a variety of initial values for our learning models in Appendix D, where we mainly expose the estimated Rational and Behavioral Learning equilibria notions to generate different initial values.

3.3 Baseline Estimation Results

As our benchmark, Table 1 shows the estimation results under fixed-belief models BLE and REE, and learning models with PLMs under AR(1), AR(2) (with RLS-learning) and MSV information sets[16]. The learning coefficients under AR(1) and AR(2) models are initialized at the corresponding moments of the estimated BLE model, whereas the learning coefficients under MSV model are initialized using the moments from the estimated REE model. We denote these initial values as θ_{BLE^*} and θ_{REE^*} respectively[17].

Before analyzing the differences in estimated parameters, we first discuss the marginal density of each model under consideration baed on the Laplace approximation and the modified harmonic mean estimators[18]. Looking at the first two columns, we observe that the BLE model considerably improves upon the REE model, with a Bayes' Factor of 4.07 (Lapl.) and 6.23 (MHM) in favour of BLE[19]. Looking at the third column suggests that, introducing learning on top of BLE improves the model fit further: the Bayes' Factor

[14]For the BLE model, in order to minimize the computational costs, we compute the equilibrium values only for the posterior mode. The equilibrium values are then kept fixed throughout the MCMC exercise. As such, we compute the conditional posterior distributions for this model. The relatively small difference between Laplace and MHM densities suggests that this does not have a large impact on the distributions overall.

[15]We report only the parameter estimates in our paper, but further details and posterior distributions can be found in our Online Appendix.

[16]In our benchmark case, the AR(1) model is based on SAC-learning, whereas AR(2)- and MSV-learning models are based on RLS-learning.

[17]For the AR(1) and MSV-learning models, this method corresponds to a model-consistent method of initialization. For the AR(2) model, the corresponding equilibrium notion is in progress but not yet available. We therefore use the initial beliefs consistent with the closely related AR(1) model.

[18]Following our approach in Hommes et. al. (2019), we compare the Bayes' factors based on Jeffrey's Guidelines (Greenberg, 2012). Accordingly, a Bayes' Factor of larger than 2 implies decisive support for the model under consideration relative to REE model.

[19]Compared to other models, there is a larger difference between Laplace and MHM estimators in the BLE model. This is due to the uncertainty around inflation dynamics, which is discussed in further detail in latter sections.

relative to REE increases to 13.19 (Lapl.) and 12.13 (MHM) in favour of AR(1)-learning. Replacing the AR(1) learning model with an AR(2) process affects the model fit only marginally: the Laplace estimator favours the AR(1) model, whereas the MHM estimator favours the AR(2) model. The difference in Bayes' Factor between AR(1) and AR(2) models remains below 2 in either case.

The results on marginal densities have two important implications: first, the information set on expectations, in isolation from any learning effects, plays an important role in driving the model fit. Second, whether learning improves the model fit depends crucially on the information set: learning leads to substantial improvements in the model fit when expectations are based on AR(1) and AR(2) rules, whereas this effect disappears when the full REE information set is used. This is consistent with previous results in the literature, see e.g. Slobodyan and Wouters (2012a, 2012b) and Ormeño & Molnár (2015).

The MSV-learning model yields a slightly worse marginal density compared to REE model with a Bayes' Factor of -1.17 in favour of the REE model[20]. Earlier studies on MSV-learning in small-scale New Keynesian models, most notably Milani (2007), report substantial improvements under MSV-learning over the REE benchmark. Our results show that this result may no longer hold in a more realistic DSGE model.

Next we discuss the differences in parameter estimates across the models, where we start with a comparison between REE and BLE. The most important differences arise in friction parameters that introduce autocorrelation into the endogenous variables, i.e. habit λ and indexations ι_w , ι_p, and also in frictions parameters that determine the contemporaneous feedbacks between the endogenous variables, i.e. the Calvo probabilities ξ_w, ξ_p and capital adjustment cost ϕ. First looking at the Calvo probabilities, we observe a large difference in wage stickiness, which is estimated at 0.93 under REE, whereas this number decreases to 0.75 under BLE. A smaller difference is also observed for ξ_p, which is estimated at 0.72 under REE, whereas it decreases to 0.69 under BLE. Absent any other differences, these values suggest that the direct contemporaneous feedback from the real side of the economy to inflation and wages are larger under BLE. This is an intuitive result since under BLE, the contemporaneous channel through expectations is shut off. As a result, this channel is compensated by smaller Calvo probabilities. A similar result can be observed for capital adjustment cost ϕ, which determines the contemporaneous feedback from asset prices to investment (where a smaller adjustment cost translates into a stronger feedback). Not surprisingly, this value is smaller under BLE with $\phi = 1.04$, whereas it increases to $\phi = 5.36$ under REE. This is a direct consequence of the backward-looking

[20] The result on MSV-learning model is consistent with Slobodyan and Wouters (2012a), where they find that the data typically prefers VAR-type learning models over MSV-learning. The result between MSV-learning and REE models depend on the initial beliefs, and they report some initializations where the MSV-model performs better. We omit these cases here since our main focus is the small forecasting models.

investment expectations under BLE.

We next look at indexations ι_w, ι_p, and mark-up shock autocorrelations ρ_w, ρ_p. Both groups of parameters are designed to introduce more persistence into real wage and inflation variables. We observe that ι_w is substantially lower under BLE with 0.29 compared with 0.7 under REE, whereas the shock persistence ρ_w is at similar levels with 0.29 and 0.32 respectively. While the price indexation ι_p is moderately higher under BLE with 0.49 compared with 0.1 under REE, the picture is reversed for the shock persistence ρ_p with 0.05 under BLE and 0.69 under REE. These results are due to the backward-looking expectations under BLE, which introduces more persistence into the system, which in turn reduces some of the exogenous parameters that introduce more persistence into the system. A similar result is observed to a smaller degree for habit persistence λ, which introduces more persistence into consumption dynamics: this is smaller under BLE with 0.71, compared to 0.75 under REE.

In our previous paper Hommes et. al. (2019), we observe that, in the absence of any lagged endogenous variables (such as lagged consumption through habits, and lagged inflation through indexation in this model) the system under BLE is characterized by a persistence amplification compared to REE, where the exogenous shocks are amplified through expectations, which in turn leads to smaller persistence in the exogenous shocks under BLE. This picture becomes more complicated in the presence of lagged endogenous variables in this framework, since it is not only the exogenous shocks that introduce more persistence into the system. In this case, we observe that the shock persistence structure remains mostly similar under BLE and REE, with the exceptions of invesment and price mark-up shocks, both of which are smaller under BLE. Another important difference that arises in the exogenous shocks is in the standard deviations: since the feedback channel through expectations is cut-off under BLE, the model typically needs larger standard deviations to match the data, compared with REE. In the SW model, we observe that this occurs for investment and mark-up shocks.

A final parameter difference arises in the (inverse) elasticity of labor substitution σ_c: this is lower under BLE with 0.47 compared to REE with $\sigma_c = 1.38$. This parameter has a two-fold effect through the consumption Euler equation: first, it determines the feedback from ex-ante risk premium $(r_t - E_t\pi_{t+1})$ on consumption, where a smaller σ_c translates into a stronger feedback. In this respect, the effect is stronger under BLE, similar to the results with Calvo probabilities and capital adjustment cost. The second effect is through the relation between expected labor growth $l_t - E_t l_{t+1}$ and consumption: $\sigma_c > 1$ implies that labor growth and consumption are complementary, whereas $\sigma_c < 1$ implies that they are substitutes. As such, consumption and labor growth are complements under REE, whereas they become substitutes under BLE. The intuition for this result is as follows: under the simple mean-reverting AR(1) rule with $t-1$ timing of expectations, a positive labor growth today, i.e. $l_t - l_{t-1} > 0$, immediately implies a negative expected labor growth since $E_t[l_{t+1}] - l_t = \beta_l^2 l_{t-1} - l_t \approx l_{t-1} - l_t < 0$ (assuming a high autocorrelation). Hence, when agents observe a positive labor growth today,they expect it to turn negative in the next period, which leads them to lower their consumption today. We observe this result in not only BLE model but also AR(1) and AR(2) learning models, whereas the MSV-learning model yields a σ_c similar to REE. We will show in a latter section that this is in fact a direct result of the timing assumption of expectations, where the $t-1$ timing leads to $\sigma_c < 1$ across all models.

Next comparing the BLE result to AR(1) and AR(2) learning models, we see that the differences remain more modest: habit persistence, Calvo probabilities and price indexation become lower compared to BLE, while all of the remaining parameters remain fairly similar across all three models. The largest difference arises in the price Calvo probability ξ_p, which decreases to 0.59 and 0.6 under AR(1) and AR(2) models respectively, compared with 0.69 under BLE. This is a consequence of the result that was illustrated in Figure 3.1, where a BLE becomes less likely to exist with a steeper Phillips curve (i.e. lower price stickiness) in the baseline New Keynesian model. We will illustrate in latter sections that the same result also holds in the larger SW model, and no stationary equilibrium exists when small values of price stickiness is combined with a sufficiently large autocorrelation on the mark-up shock or a high price indexation. Finally considering the MSV-learning model, we observe that the parameter estimates remain largely similar to the REE model. Notable differences are the Calvo probabilities and indexations, in particular ξ_w and ι_w, both of which are lower under MSV-learning. The price stickiness ξ_p is also slightly lower under learning, whereas the price indexation and mark-up shock persistence switch places under REE and learning (i.e. price indexation is high under learning, whereas shock persistence is high under REE).

The estimated gain parameters turn out similar across all three learning models, with $0.009, 0.012$ and 0.01 under AR(1), AR(2) and MSV-learning respectively. It is worth noting that all of the estimated monetary policy parameters remain similar across all model specifications, with values well within the range of each other. This suggests that changing the information set on expectations or the introduction of learning do not change the estimated structure of monetary policy. However, this does not imply that the transmission channel of monetary policy remains the same, which we will illustrate in latter sections.

To provide a comparison between REE and BLE, table 2 provides the (unconditional) persistence and variances of all seven forward-looking variables under these two models. A notable result is that, the models are largely in agreement in terms of the persistence of these variables, with the exception of a small difference in asset prices. However, looking at the variances reveals a relatively large difference between the models: with the

Initial beliefs	REE		BLE		SAC θ_{BLE^*}		AR(2) θ_{BLE^*}		MSV θ_{REE^*}	
Timing	t		t-1		t-1		t-1		t	
	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.
ϕ	5,365	0,11	1,045	0,271	0,813	0,227	0.793	0.121	5,403	1,053
σ_c	1,386	0,051	0,471	0,059	0,436	0,056	0.472	0.059	1,246	0,234
λ	0,758	0,023	0,715	0,061	0,652	0,076	0.633	0.076	0,795	0,04
ξ_w	0,937	0,015	0,752	0,035	0,72	0,045	0.707	0.039	0,713	0,044
σ_l	1,591	0,696	1,858	0,713	1,913	0,706	2.051	0.707	1,679	0,702
ξ_p	0,727	0,034	0,694	0,015	0,586	0,04	0.596	0.037	0,689	0,044
ι_w	0,702	0,103	0,293	0,114	0,324	0,116	0.315	0.117	0,397	0,099
ι_p	0,105	0,051	0,489	0,062	0,321	0,163	0.325	0.142	0,648	0,092
ψ	0,511	0,135	0,435	0,139	0,487	0,144	0.516	0.138	0,517	0,119
ϕ_p	1,597	0,069	1,392	0,07	1,419	0,076	1.445	0.074	1,5	0,072
r_π	1,737	0,17	1,66	0,18	1,655	0,186	1.678	0.183	1,705	0,166
ρ	0,864	0,018	0,887	0,019	0,887	0,019	0.888	0.019	0,865	0,02
r_y	0,117	0,028	0,133	0,035	0,145	0,035	0.144	0.032	0,162	0,031
$r_{\delta y}$	0,161	0,019	0,144	0,019	0,142	0,02	0.14	0.019	0,16	0,02
$\bar{\pi}$	0,744	0,091	0,766	0,061	0,657	0,1	0.6839	0.097	0,689	0,1
$\bar{\beta}$	0,166	0,077	0,292	0,099	0,241	0,094	0.266	0.0938	0,193	0,062
$\bar{l}$	0,563	0,641	-0,275	0,748	-0,461	0,692	-0.2316	0.930	0,861	0,854
$\bar{\gamma}$	0,408	0,01	0,404	0,012	0,409	0,013	0.40	0.014	0,411	0,01
α	0,185	0,017	0,145	0,017	0,145	0,017	0.149	0.017	0,186	0,017
ρ_a	0,93	0,023	0,921	0,024	0,926	0,025	0.923	0.021	0,928	0,022
ρ_b	0,246	0,056	0,325	0,018	0,546	0,074	0.508	0.071	0,37	0,089
ρ_g	0,988	0,005	0,991	0,002	0,976	0,012	0.988	0.011	0,986	0,012
ρ_i	0,758	0,046	0,423	0,065	0,49	0,067	0.439	0.069	0,797	0,041
ρ_r	0,059	0,049	0,083	0,051	0,083	0,051	0.081	0.050	0,057	0,037
ρ_p	0,692	0,065	0,05	0,012	0,029	0,022	0.028	0.020	0,078	0,056
ρ_w	0,323	0,074	0,293	0,034	0,255	0,072	0.217	0.067	0,898	0,029
ρ_{ga}	0,534	0,076	0,544	0,075	0,528	0,075	0.527	0.077	0,544	0,078
η_a	0,446	0,028	0,479	0,029	0,476	0,031	0.469	0.028	0,458	0,026
η_b	2,51	0,206	2,48	0,578	1,901	0,464	1.894	0.433	2,565	0,611
η_g	0,507	0,026	0,491	0,026	0,491	0,026	0.496	0.026	0,501	0,027
η_i	0,419	0,042	1,465	0,074	1,467	0,075	1.465	0.074	0,404	0,027
η_r	0,216	0,011	0,211	0,011	0,211	0,011	0.211	0.010	0,214	0,011
η_p	0,093	0,016	0,303	0,014	0,296	0,017	0.300	0.016	0,174	0,013
η_w	0,203	0,022	0,533	0,026	0,54	0,029	0.533	0.028	0,1	0,006
gain					0.009	0,003	0.012	0.006	0,01	0,004
Laplace	1039.892		1030.530		1009.649		1012.49		1042.887	
Bayes F.	1		4.07		13.19		11.88		-1.15	
MHM	1039.631		1025.54		1011,946		1009.59		1042.169	
Bayes F.	1		6.23		12.13		13.16		-1.17	

Table 1: Main estimations over sample period 1966:I-2008:IV, where the period 1965:I-1965:IV is treated as pre-sample data to initialize the Kalman filter. The rows Laplace and MHM refer to Laplace approximation and modified harmonic mean estimators of the marginal likelihood. The model SAC uses the constant gain AR(1) sample autocorrelation learning, whereas AR(2) and MSV models are evaluated with constant gain recursive least squares. Bayes Factors are calculated taking the REE model as baseline specification.

exception of inflation that asset prices, all forward-looking variables are characterized by more volatility under BLE. This is an interesting result since, among the forward-looking variables, only inflation and labor (hours worked) are used as observables[21], while the remaining variables are treated as unobservables in the estimation. This result suggests that the unobserved component of the model is characterized by more volatility under BLE. The small difference between the first-order autocorrelations suggests that using either BLE or REE model to initialize the AR(1) and AR(2) learning models would yield similar results, whereas the difference between variances suggests that the speed of adjustment (i.e. gain) may be affected by this choice. We investigate this in further detail in latter sections.

Next, we investigate the impulse responses under BLE, REE and AR(2) models for some key variables and selected shocks, which is illustrated in Figure 2. Specifically, we focus on investment, output, inflation and real wage responses to productivity, government spending and monetary policy shocks. A comparison between these shocks across the models is particularly straightforward since their estimated moments remain almost identical across all model specifications.

The purpose of this exercise is to see the effects of the information set on the impulse responses, in isolation from any learning effects. Note that due to time-varying beliefs under the AR(2)-learning model, the model-implied impulse responses are different at each period[22]. To facilitate a comparison between the fixed-belief BLE and REE models, we take the average beliefs over the last 10 years of the estimation sample for the AR(2)-learning model. Specifically, we first compute the average beliefs over the period 1999:I-2008:IV, and then compute the corresponding impulse responses[23].

Based on the impulse responses in Figure 2, several observations stand out: first, it is readily seen that the contemporaneous effect of a given shock is typically larger under REE compared to BLE and AR(2) models. This is due to the assumption of unobserved shocks, as well as the t-1 timing of expectations, both of which delay the effect of a given shock. In turn, the shocks typically take longer to die out under BLE and AR(2) models. This suggests that expectations based on small forecasting rules typically reduce the initial impact of a shock, while increasing the duration of the effect. This is another form of persistence amplification under AR(1) and AR(2) rules that was discussed above.

A notable exception to the above observation is the govenment spending shock: while the smaller but more gradual effect relative to REE still holds for investment, we observe

[21]For inflation, both the unconditional variance implied by both models is close to the sample variance of 0.35, whereas for labor, the moment under BLE is closer to the sample variance of 7.62.

[22]The implications of the time-variation in beliefs are discussed in great detail in Slobodyan & Wouters (2012a, 2012b). In this paper, we focus our attention on the implications of the information set instead.

[23]The resulting IRFs are not sensitive to the specific period of averaging, as long as a sufficiently long transient period is taken into account. Similar results are also obtained if we average over the period-specific impulse responses.

that the impact on output is actually larger under BLE and AR(2) model. This suggests that the government spending multipliers are larger under small forecasting rules, which is consistent with previous findings in the literature (see e.g. Quaghebeur, 2018). The intuition is as follows: a positive government spending shock crowds out private investment, and it is readily seen that investment indeed decreases in all model specifications. Under REE, this effect is further propagated through the expectations channel, whereas introducing AR(1) and AR(2) beliefs slows down the channel through expectations. As a result, the initial effects on investment are smaller and the effects on output larger. Another interesting result arises in inflation, where a positive government spending shock is deflationary under REE but inflationary under BLE and AR(2) models. A final observation is on the responses of inflation under BLE and AR(2) models: the responses of inflation are notably smaller under BLE compared to AR(2). This result stems from the fact that the estimated price stickiness ξ_p is larger under BLE compared to AR(2), which leads to smaller responses of inflation.

	BLE		REE	
Variable	Persistence	Variance	Persistence	Variance
Rental Rate of Capital	0.998	41.49	0.990	11.24
Asset Price	0.61	10.71	0.504	16.68
Consumption	0.996	58.47	0.986	19.56
Investment	0.993	312.9	0.980	151.04
Labor	0.971	7.71	0.96	5.76
Inflation	0.835	0.3	0.813	0.29
Wage	0.995	31.8	0.966	4.72

Table 2: First-order autocorrelations and variance of seven forward-looking variables under BLE and REE models.

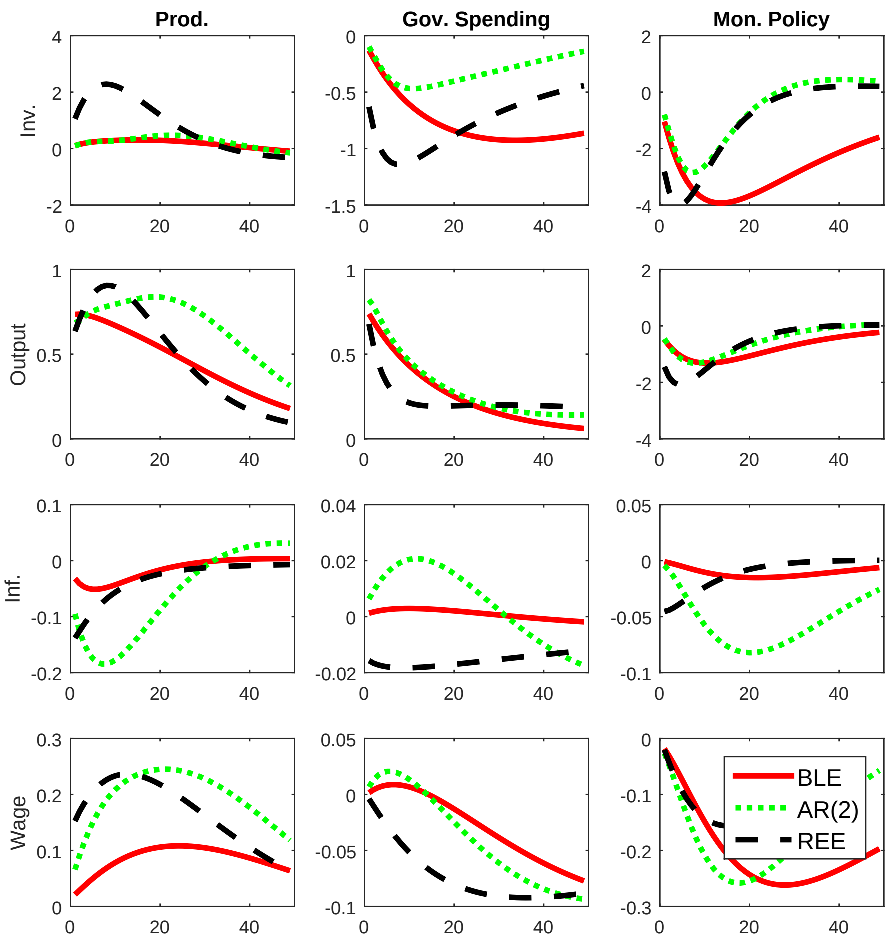


Figure 2: Impulse responses of key variables to selected shocks (productivity, government spending and monetary policy) under REE, BLE and AR(2)-learning models. Since the beliefs under AR(2) are time-varying, the implied impulse response are also different every period. Therefore in order to facilitate a comparison between the fixed-belief REE and BLE models, we plot the impulse responses under average beliefs β_1 and β_2 over the last ten years of the estimation sample, i.e. 1999:I-2008:IV.

3.4 Projection Facilities and RLS- vs. SAC-learning

In this section we provide a detailed comparison of the standard RLS-learning algorithm with the SAC-learning algorithm. As we mentioned in the introduction section, one of the well-known problems with RLS-learning is the potential instability of the resulting parameter estimates: since the parameters of the estimated PLM are not restricted, an update on the learning coefficients may well result in an explosive PLM at a given period.

In order to avoid this issue, the standard practice in the learning literature is to impose a projection facility on the PLM. If the learning update leads to an explosive model at a given period, the projection facility maps the explosive model into a stable model. The easiest and most standard approach in practice is to simply ignore the updates that lead to instabilities. More precisely, if a learning update implies a root outside the unit circle in (3.4), then the update in that period is discarded. This has the interpretation that people do not use explosive models in reality.

While projection facilities perform well in terms of dealing with the potential instability of RLS algorithms, there are two issues with the approach: first, the resulting patterns on the learning coefficients depend on the choice of how a projection facility is implemented, which has to depend on ad-hoc decisions by the researcher. Second, projection facilities are computationally costly, which complicates the estimation of learning models in practice. In particular, projection facilities require checking the eigenvalues of the system in (3.4) for every parameter draw and at every period.

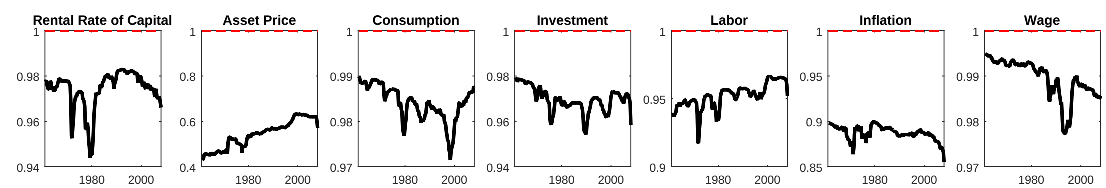
SAC-learning provides a potential alternative to deal with both of these issues: since the learning process depends on the sample moments, agents' PLM is equipped with a natural projection facility. This ensures that the learning process remains non-explosive throughout the estimations and simulations[24].

Figures 3 and 4 show the filtered paths of the learning coefficients for SAC-learning and RLS models (both AR(1) and AR(2)) respectively. It is readily seen that, for the AR(1) models, the filtered patterns of the perceived mean coefficients are similar. For the persistence coefficients, there are some differences in the patterns since the models have different parameterizations[25]. for the AR(2) model, recall that the initialization of the beliefs are based on the BLE model, hence β_2 starts from 0. We observe that the β_2 coefficients decisively move either in the positive or negative direction, suggesting that the second lag is supported by the data. In particular, we observe that for asset prices

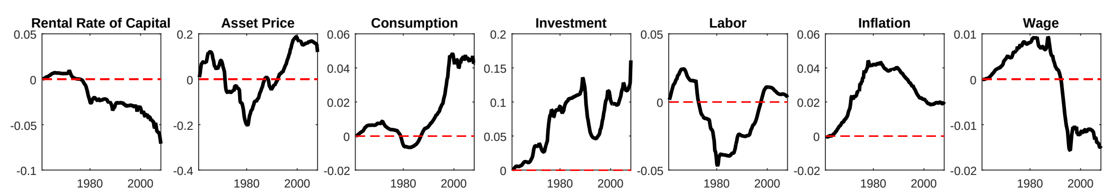
[24]Note that this only guarantess the stability of the PLM. The implied ALM can still become explosive even if the PLM is stationary. However, one can easily check potential regions of this prior to the estimation. In our example of AR(1) beliefs, if the implied ALM remains non-explosive under random walk beliefs, then it would also remain non-explosive under all classes of AR(1) rules. After confirming this, the model can be estimated without using the projection facility.

[25]Recall that for the RLS model, we use $X_t = \alpha + \beta X_t$ for the PLM, whereas for SAC-learning we have $X_t = \alpha + \beta(X_t - \alpha)$. The observed differences in the patterns for β are due to this difference in parameterization.

and inflation, the second lag moves in the positive direction, while the first lag moves in the negative (leaving the persistence mostly unchanged). For the remaining variables, β_2 moves in the negative direction while β_1 leaves the unit circle (although the magnitudes are smaller compared to inflation and asset prices). For the variables with $\beta_1 > 1$, this suggests an extrapolative behaviour relative to the BLE model. Evidence in favour of this has been suggested in some recent literature, see e.g. Fuster et. al. (2010). Further, since the belief coefficients show transitory behaviour during the estimation sample, a theoretical AR(2) model with $\beta_1 > 1$ and $\beta_2 < 0$ may further improve the model, which is left to future research. Figure 4e shows the frequency of the projection facility activity for the AR(1)- and AR(2)-RLS models: it is readily seen that projectiction facility is occasionally activated throughout the sample, particularly over early 80s and 90s. The projection facility is active for 9 % of the sample in the AR(1) model, whereas this number increases to 14 % for the AR(2) model. The figure is ommitted for the SAC-learning model, where the projection facility is never activated: this demonstrates the advantage of SAC-learning over recursive least squares, and suggests that generalizing this algorithm to all types of PLM (e.g. through Yule-Walker learning as in Branch et. al., 2014) may be a fruitful avenue of research.

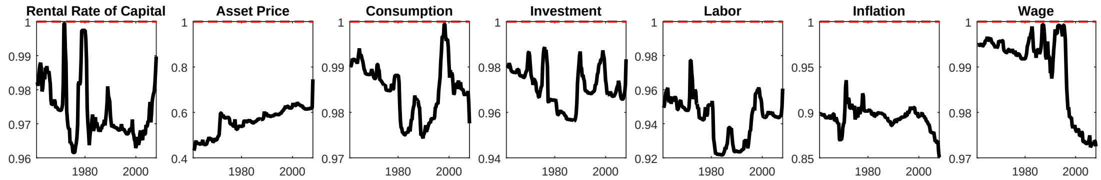


(a) SAC-learning, β coefficients.

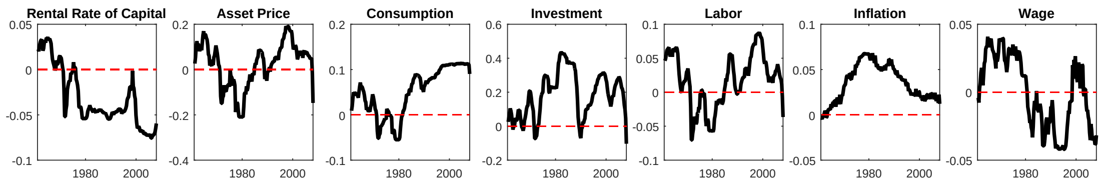


(b) SAC-learning, α coefficients.

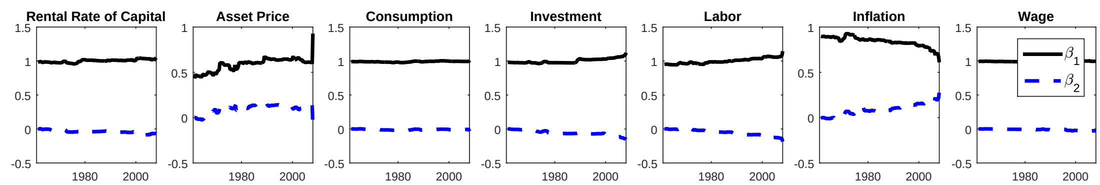
Figure 3: Filtered mean and persistence coefficients for SAC learning over the estimation sample 1966:I-2008:IV.



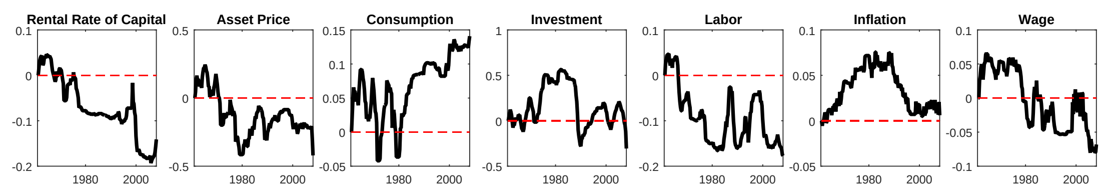
(a) AR(1)-RLS, β coefficients.



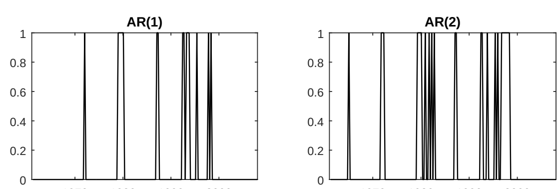
(b) AR(1)-RLS, α coefficients.



(c) AR(2)-RLS, β_1 and β_2 coefficients.



(d) AR(2)-RLS, α coefficients.



(e) Projection facilities in AR(1)- and AR(2)-RLS.

Figure 4: Filtered mean and persistence coefficients for AR(1) and AR(2) RLS models over the estimation sample 1966:I-2008:IV. The last panel shows the periods where the projection facility is binding for the RLS models. This is omitted for the SAC-learning model since the projection facility is never binding.

In the next section, we consider the learning models under a decreasin gain specification as a robustness check. Two more alternative model specifications and robustness checks are left to the Appendix. The effect of initial values in the learning models is undertaken in Appendix D, and the effects of the timing assumption in all model specifications is considered in Appendix E. The marginal densities reported in the following sections and Appendices are based on the Laplace approximation.

3.5 Estimations with Decreasing Gains

In this section we consider AR(1)- and AR(2)-learning models under a decreasing gain specification. As our results so far make it clear, using BLE with every parameter draw may be useful in terms of the model fit, but it comes at a large computational cost. Further, if a researcher wants to deviate from the AR(1) rule, the theoretical equilibrium concepts and related approximation algorithms are not yet available. Therefore a decreasing gain setup may be useful in mitigating these issues: in this case, unlike a constant gain learning algorithm, agents use the full sample available to them at all times, which means the weight on the final observation keeps decreasing as the sample size grows. Therefore learning coefficients converge to fixed-points as the sample size grows, which act as a reasonable approximation for the underlying equilibrium.

Table 3 shows the estimation results under decreasing gains for AR(1) and AR(2) models, whereas Table 4 shows the converged moments at the end of the estimation sample. First comparing the moments in Table 4 with the BLE moments in Table 2, it is readily seen that the equilibrium persistence and the converged persistence under decreasing gains remain fairly similar, although there is some uncertainty around inflation and asset price persistence. For the AR(2) model, we again observe that the same five forward-looking variables converge to $\beta_1 > 1$ and $\beta_2 < 0$ (the exceptions being inflation and asset prices), suggesting varying degrees of extrapolation in these variables. The main difference between the BLE estimations and decreasing gains again arises in the estimate of ξ_p, which is lower than the BLE estimate, similar to the constant gain estimations. This suggests that a learning process with decreasing gains indeed provides a good proxy for the equilibrium moments, but it may lead to differences in some estimated parameters that are sensitive to the learning process.

	SAC		RLS-AR(1)		RLS-AR(2)	
	Post.		Post.		Post.	
Parameter	Mode	Std.	Mode	Std.	Mode	Std.
ξ_w	0,718	0,05	0,697	0,042	0,72	0,038
ξ_p	0,501	0,035	0,552	0,036	0,518	0,038
ι_w	0,346	0,117	0,331	0,117	0,364	0,118
ι_p	0,49	0,117	0,371	0,118	0,298	0,129
r_π	1,704	0,184	1,729	0,18	1,698	0,18
ρ	0,887	0,019	0,89	0,019	0,886	0,019
r_y	0,141	0,033	0,133	0,033	0,142	0,035
$r_{\delta y}$	0,133	0,02	0,137	0,019	0,134	0,019
Laplace	1037,134		1019,481		1045,267	

Table 3: Estimations of AR(1) (both RLS and SAC-learning) and AR(2) models under decreasing gains, as an approximation of the underlying Behavioral Learning Equilibria.

	SAC	RLS-AR(1)	RLS-AR(2)	
Variable	β_1	β_1	β_1	β_2
Rental Rate of Cap.	0,96	0,98	1,28	-0,31
Asset Price	0,82	0,89	0,33	0,21
Consumption	0,98	0,98	1,29	-0,3
Investment	0,93	0,96	1,57	-0,62
Labor	0,94	0,95	1,39	-0,44
Inflation	0,83	0,92	0,58	0,3
Wage	0,98	0,98	1,16	-0,19

Table 4: Final values of β_1 and β_2 coefficients in learning models with decreasing gains.

3.6 Rolling Window Estimations

As a final comparison between the in-sample fit between the models, we compute a rolling-window sample estimation of each model under comparison. This exercise proceeds as follows: we consider a length of 80 quarters (20 years) for each estimation with a presample of 4 quarters, starting from our original first period of 1966:I (with 1965:I-1965:IV as the presample period). The models are then re-estimated using a rolling window of 20 years until we reach the final period of 2008:IV. This results in 96 distinct estimations for each model under consideration. For all models, we use the initial values and timing assumptions as reported in our main estimation table. Figure 5 shows the resulting rolling window marginal densities (final period of each model in the x-axis) and the corresponding Bayes' Factors relative to the REE model. A Bayes' Factor above zero provides evidence against REE relative to the model under consideration, while the opposite holds when the Bayes' Factor is below zero[26].

Several observations stand out from the table. First, for the MSV-learning model, the Bayes' Factor typically remains within the $[-2, 2]$ band. A notable exception is around early 2000 where the fit of the REE model substantially worsens for a few periods, which leads a positive jump in all model comparisons. The BLE, SAC-learning and AR(1) models perform worse than the REE model over the initial model until early to mid 1990s. Particularly the SAC-learning model provides a bad fit until 1993, after which the model fit improves and Bayes' Factor typically remains above 2 for the rest of the sample. These results suggest that the model improvements in BLE, SAC-learning and AR(2)-learning models relative to REE are robust across smaller subsamples of the datasets. This strenghtens our conclusion that the assumption on the information set (i.e. AR(1) or AR(2) small forecasting rules against MSV forecasting rule) matters as much as the time-variation in beliefs due to learning.

Using these rolling-window estimations, we next move on to an out-of-sample forecasting exercise.

[26]Again using the Jeffrey's Guidelines for the model comparison here, we consider a Bayes' Factor above two to be decisive support relative to REE for a model under consideration.

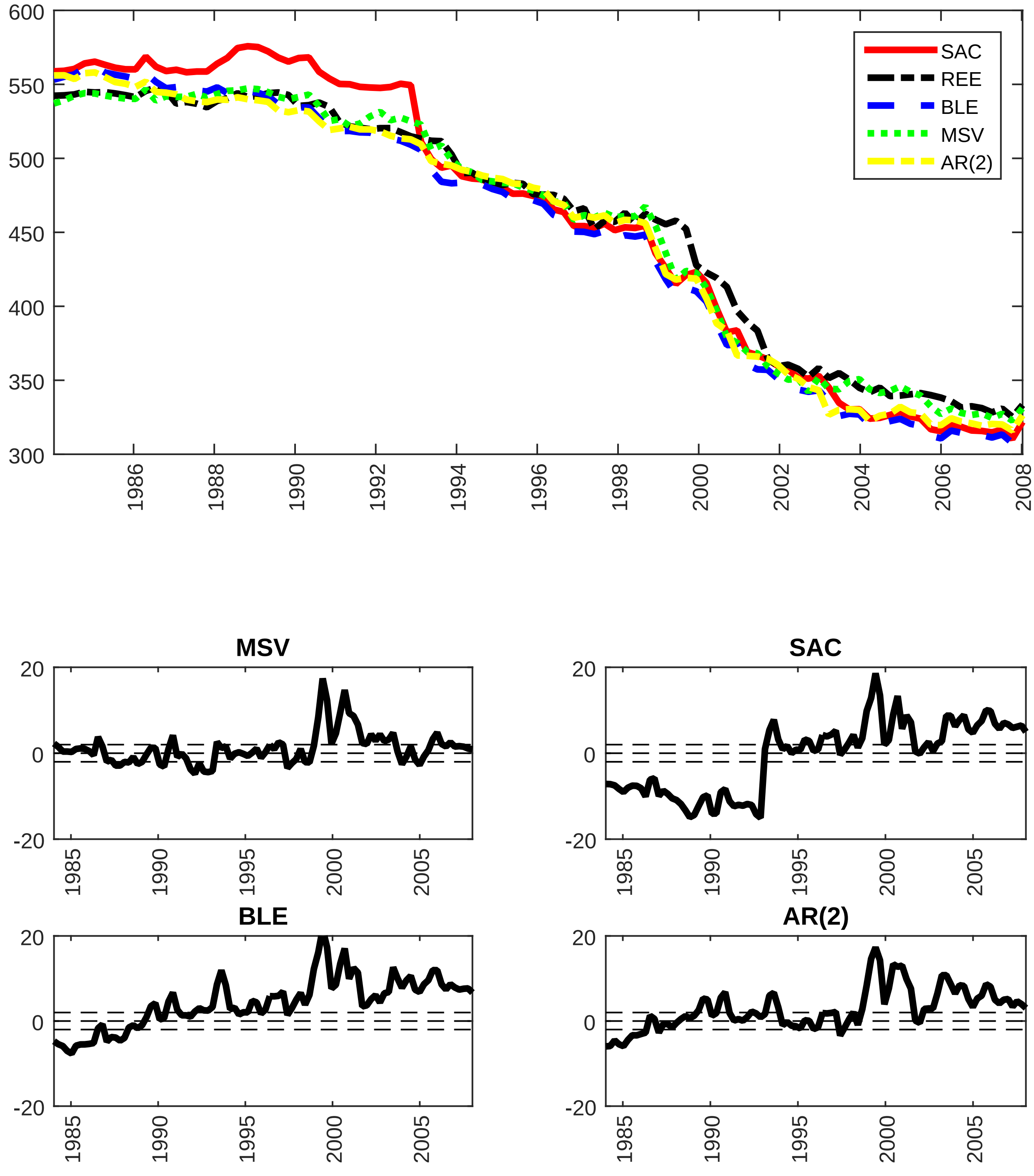


Figure 5: Rolling window marginal densities (based on Laplace approximation) and Bayes Factors (with REE as the base model). All models are re-estimated every period on a 20-year rolling window basis, starting with the period 1965:I-1984:IV. The x-axis in the figures correspond to the final period in a given estimation.

4 Out-of-sample Forecasts

Next we consider out-of-sample forecasting performance of each of the 5 models using the 96 auxiliary models considered above. For each estimated model, we forecast up to 12 periods (i.e. 3 years) ahead and compute the corresponding root mean squared errors (RMSE)[27]. We focus on Table 5, which shows the relative performance in terms of percentage gains/losses of all remaining models relative the RE benchmark. First looking at the learning models, MSV-learning, SAC-learning and AR(2)-learning models perform better than REE over shorter horizons, while their performance deteriorates over longer forecasting horizons with 8 and 12 quarters. The performance of SAC-learning model is better than MSV-learning, but they both show the same U-shaped pattern where the relative performance starts to decline after 2- and 4-step ahead forecasts. This result is consistent with those previously reported in Slobodyan & Wouters (2012b), suggesting that the REE cross-restrictions are more useful over longer horizons, while learning models provide more accurate forecasts over shorter horizons. Among the three learning models, the largest percentage gains are obtained under AR(2)-learning, which increases up to a 7.29 % improvement over REE at the 4 quarter horizon. Interestingly, this model also leads to the biggest deterioration over longer horizons, with a performance worse by 3.13 % with 3-year ahead forecasts.

Next looking at the BLE model, we see the largest improvements in the overall measure with up to 9.8%. The BLE model also yields the U-shaped pattern of increasing gains up to 4 quarters ahead, while its performance relative to RE declines over 8 and 12 quarter forecasts. At 12 quarters ahead, the performance of BLE and REE models is almost identical, with a difference of -0.28% in favour of the REE model. This shows that the equilibrium restrictions under BLE are particularly useful for forecasting: these restrictions resolve the trade-off between improved short-term and poor long-term forecasts of the learning models.

Finally we also consider a forecast combination exercise including both learning and BLE models (kitchen sink forecat combination with equal weight on all models), which yields positive improvements at all horizons relative to the REE model and a peak gain of 7.56% at 4 quarters ahead. These results again suggest that imposing correct cross-restrictions (i.e. AR(1) vs. MSV) is more important than the additional flexibility introduced by the learning dynamics.

[27]The results are robust across other similar measures such as mean squared errors or mean absolute errors.

perc gains-MSV								
horizon	Δy_t	Δc_t	Δinv_t	Δw_t	π_t	r_t	l_t	overall
1	-2,71	7,04	-5,54	-0,64	-1,82	6,02	-3,45	1,3
2	1,84	12,03	-0,34	0,69	-3,84	9,73	0,7	2,86
4	-0,19	6,43	4,47	0,89	-18,57	5,72	4,3	2,58
8	-2,37	0,45	-1,03	-2	-37,33	-8,4	5,11	-1,26
12	-1,53	-2,5	-5,49	-0,41	-41,22	-18,88	3,91	-6,33
perc gains-BLE								
horizon	dy	dc	dinve	dw	infl	r	lab	overall
1	6,85	5,72	5,47	1	15,79	21,83	7,91	6,57
2	4,35	10,08	-2,51	-0,96	30,57	23,3	7,24	8,06
4	-2,08	8,25	2,45	-1,93	41,42	16,27	0,22	9,79
8	2,86	10,23	2,69	-2,5	39,74	4,08	-5,13	4,5
12	2,71	7,86	-3,29	-6,1	38,99	1,93	-5,61	-0,28
perc gains-SAC								
horizon	dy	dc	dinve	dw	infl	r	lab	overall
1	8,04	6,21	6,26	-3,8	8,9	19	8,04	5,76
2	4,5	7,94	-3,18	-6,17	16,96	17,01	6,66	5,21
4	-2,47	4,62	1,58	-5,72	22,47	4,97	-1,38	5,81
8	3,06	8,4	2,01	-3,46	29,58	-12,07	-4,57	2,98
12	4,86	8,29	-3,49	-4,46	29,78	-14,85	-0,41	-2,51
perc gains-AR(2)								
horizon	dy	dc	dinve	dw	infl	r	lab	overall
1	8.61	9.10	7.89	-3.87	8.512	26.69	9.46	6.89
2	5.78	12.20	-0.38	-7.68	18.04	26.09	6.37	6.67
4	-1.74	6.75	2.51	-6.73	21.43	13.06	-5.12	7.29
8	4.10	10.13	2.87	-4.18	16.12	-13.66	-13.34	2.99
12	5.98	9.85	-2.99	-5.88	17.13	-27.61	-10.69	-3.13
perc gains-COMB								
horizon	dy	dc	dinve	dw	infl	r	lab	overall
1	6,03	6,54	3,8	0,2	9,93	16,26	5,94	5,39
2	5,18	9,26	1,22	-0,6	19,84	18,08	7,19	6,91
4	0,42	6,53	3,98	-1,02	26,86	14,11	4,59	7,56
8	1,92	5,92	2,04	-1,49	27,18	6,62	1,17	5,12
12	2,6	4,26	-1,81	-2,33	22,34	3,06	0,94	0,35

Table 5: Out-of-sample forecasting performance relative to the REE benchmark: each panel reports the percentage gains and losses of the model relative to REE based on root mean squared errors. Each model is re-estimated every period over a rolling-window 20 year (80 quarters) period, with an additional year to initialize the Kalman filter. The first model starts with an estimation period of 1965:I-1984:IV, whereas the final model is estimated over the period 1988:IV-2008:III. This results in 96 models with a forecasting period of 1985:I-2008:IV.

5 Fixed-point iterations and Simulations

As briefly discussed in the main estimation section, the existence of a BLE crucially depends on a subset of parameters. In particular, the Phillips curve and the BLE with respect to inflation dynamics crucially depends on price stickiness ξ_p, as well the other parameters directly affecting the persistence of inflation, namely ρ_p and ι_p: a steeper Phillips curve (i.e. lower ξ_p), combined with a sufficiently high price indexation ι_p and mark-up shock persistence ρ_p can easily push the underlying equilibrium into non-stationary regions. We explore the regions of existence and stability in further detail in Section 5. In this section, we first explore the regions of existence and stability with respect to a few selected parameters.

First, in order to show the effects of the equilibrium uncertainty on the marginal density of a BLE, Table 6 reports estimations of BLE with three different restrictions on the estimated parameter values along with the benchmark estimation with no restrictions on the first column. In the second column, it is readily seen that shutting off the indexation ι_p is costly in terms of the model fit: this results in a higher ξ_p and ρ_p, but these increases are not sufficient to make up for the loss in the model fit. The third column shows the results when ρ_p is shut off, in which case ξ_p and ι_p estimates increase to make up for the loss in persistence. In this case we observe that the model fit actually improves. Similarly on the fourth column, we consider the same scenario where ξ_p is also fixed at 0.9, on top of a fixed $\rho_p = 0$. This specification yields the best model fit. The result on the last two column, where the model fit is considerably improved with the parameter restrictions, stems from the uncertainty due to existence of BLE: in the baseline estimation, the data pushes the estimates of these parameters into the border of the stability region, which introduces more uncertainty in the densities and results in a relatively poor model fit. Shutting of some of the parameters allows the model to get away from the non-existence boundary, which leads to a considerable improvement in the marginal density in some cases. Our baseline results are nevertheless based on the full mode specification, since the values of these parameters have important implications for inflation dynamics and in particular the impulse responses of inflation to a wide variety of shocks.

Next we demonstrate regions of existence for several parameter combinations in Figure 6. Figure 6a shows fixed-point iterations of BLE at the estimated parameter values using randomized initial values over the unit cube[28]. Recall that in our estimation exercise, the initial point for the fixed-point iteration is kept fixed throughout the estimation (i.e. for all parameter draws). We observe convergence towards the same values for all initial values, suggesting that the equilibrium is indeed unique at the estimated parameter values. However, the equilibrium is sensitive to small changes with respect to parameters affecting

[28]We use 100 randomized initial points to generate this figure.

the Phillips curve. Figure 6b shows the projection of the fixed-point iterations over the parameter space $[\rho_p, \xi_p]$[29]: the red region corresponds to parameter combinations where the fixed-point iteration does not converge to a stationary equilibrium. It is readily seen that small values of ξ_p combined with sufficiently large values of ρ_p result in no equilibrium (or convergence to an explosive equilibrium). The estimated posterior mode is close to the border of the stable region, which explains the uncertainty around inflation dynamics that we observe in our estimations. A similar figure is obtained if we repeat the exercise with respect to $[\iota_p, \xi_p]$, which is omitted here for brevity.

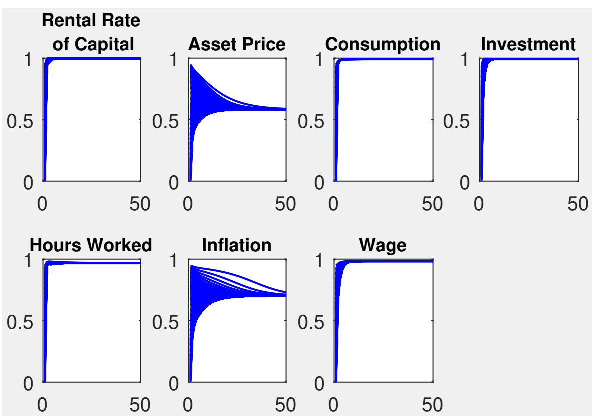
Figure 6c shows a special case with multiple equilibria in order to demonstrate how this may arise in the SW model: we repeat the fixed-point iteration with randomized initial values, where we change the price stickiness to $\xi_p = 0.71$, and monetary policy inflation reaction to $\phi_\pi = 1.05$, while the rest of the parameters are left at the posterior mode. We observe that with this parameterization, a second, high persistence equilibrium is created, which can be clearly seen from asset price and inflation persistence that converge to different values depending on the initial values. The impact on the persistence of the remaining forward-looking variables is negligible since all of them are already near unit root processes. Figure 6d shows another projection of fixed-point iterations with respect to parameters $[\xi_p, \phi_\pi]$. The region with two stationary equilibria are marked blue: it is readily seen that there is a small region where we get multiple stable equilibria, with $\xi_p \in [0.7, 0.75]$ and $\phi_\pi \in [1, 1.2]$. Smaller values of ξ_p and ϕ_π result in a region where only the high-persistence equilibrium exists, whereas sufficiently large values of $\xi_p > 0.75$ ensure that there is always a unique low-persistence equilibrium. These results are similar to what we obtained in Hommes et. al. (2019)[30].

[29]This figure is generated using a grid of length 25 for each parameter, where the fixed-point iteration is repeated 10 times with different initial values over the unit cube.

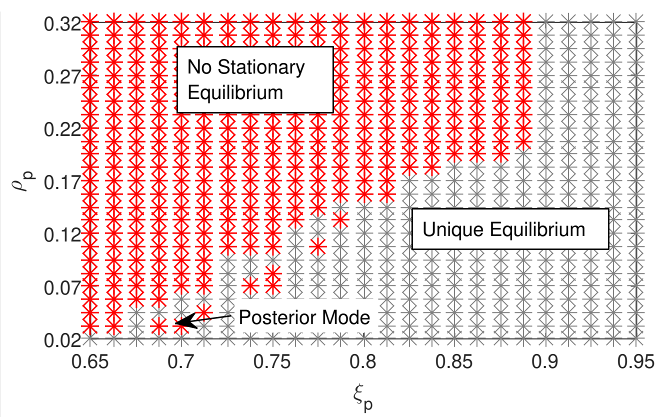
[30]The main difference between NKPC model in Hommes et. al. (2019) and SW model here is that, we do find evidence of any Cusp bifurcations, which characterizes the starting point of the multiple equilibria region. This is due to the large size of SW model, which makes exploration of larger parameter spaces computationally challenging.

Restrictions	None		$\iota_p = 0$		$\rho_p = 0$		$\xi_p = 0.9$ $\rho_p = 0$	
	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.
ξ_p	0,694	0,015	0,812	0,037	0,779	0,026	0,9	
ι_p	0,489	0,062	0		0,736	0,041	0,843	0,036
ρ_p	0,05	0,012	0.31	0.005	0		0	
r_π	1,66	0,18	1,676	0,183	1,632	0,182	1,654	0,179
ρ	0,887	0,019	0,889	0,019	0,886	0,02	0,886	0,019
r_y	0,133	0,035	0,14	0,034	0,136	0,035	0,135	0,035
$r_{\delta y}$	0,144	0,019	0,137	0,019	0,139	0,019	0,142	0,02
Laplace	1030,530		1041,761		1020,953		1018,85	
Bayes F.	1		-4.77		4.34		5.21	

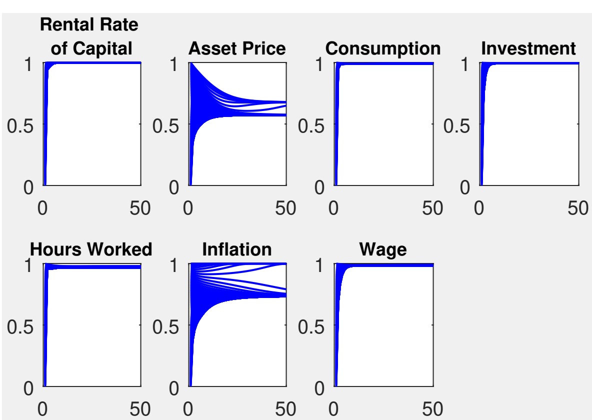
Table 6: Alternative estimations of BLE model with some parameter restrictions on inflation dynamics.



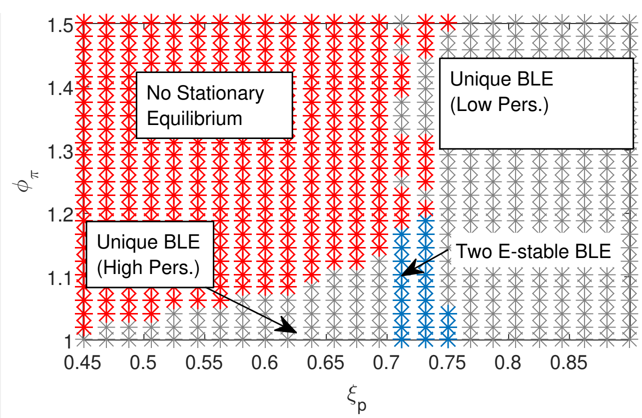
(a) Fixed-point iterations at the posterior mode.



(b) Stability regions over ξ_p and ρ_p space.



(c) Fixed-point iterations at $\xi_p = 0.71, \phi_\pi = 1.05$



(d) Stability and multiplicity regions over ξ_p and ϕ_π space.

Figure 6: Fixed point iterations stability regions in SW model.

5.1 Simulations under Learning

Finally, we consider several Monte Carlo simulation exercises of the SW model under AR(1)-learning. First, this allows us to confirm the stability results in Figure 6, and second, it allows us to investigate the model's behaviour under learning. For the Monte Carlo simulations, we focus mainly on the estimated parameters under BLE since our main purpose is to investigate the stability of the estimated equilibrium. We provide a brief comparison between SAC- and RLS-learning at the end of this section.

Recall that the iterative E-stability (i.e. convergence of the fixed-point iterations) theoretically guarantess that the equilibrium under consideration is also E-stable, i.e. stable under learning[31]. Therefore one would expect the model simulations to remain stable under learning. Figure 7 shows the distributions of inflation persistence from six different Monte Carlo experiments[32]. In each experiment, we simulate the model under a given parameterization 100 different times. Each simulation starts from a randomized initial point and is of length 10000. We discard the first 1000 periods of each simulation as burn-in period, and collect the remaining 9000 periods from each simulation, which results in a total of 900000 periods to generate the distributions reported in the figure.

Figure 7a shows three experiments at the estimated posterior mode, where we use gain values of 0.01 (estimated gain value), 0.025 and 0.05. It is readily seen that the experiments result in a unimodal distribution in each case, confirming the results from fixed-point iterations that the equilibrium is unique and E-stable at the estimated posterior mode. The dispersion of inflation persistence increases with larger gain values, which also results in a downward bias that increases as the gain value grows. Nevertheless, it is readily seen distributions remain strictly within the unit circle, which illustrates the natural projection facility of SAC-learning algorithm.

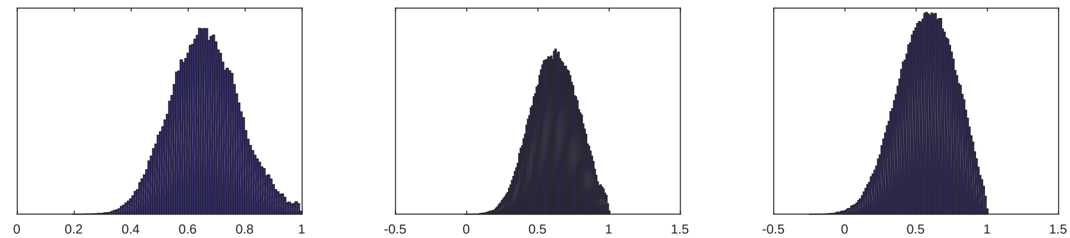
Figure 7b shows the results of the same experiments, where we now set $\xi_p = 0.5$. At these values, the fixed-point iterations fail to converge to a stationary equilibrium, i.e. there is no iteratively E-stable and stationary equilibria. Under learning, this translates into tight distributions centered around near unit-root values. This result shows another advantage of our equilibrium approach and fixed-point iterations: purely based on MC experiments in the second panel, it is not possible to distinguish whether an equilibrium exists or not. This distinction would be even more vague if we repeat the exercise under RLS-learning, where a binding projection facility further complicates the interpretation of the underlying distributions. Therefore taking advantage of the underlying equilibrium concept and the accompanying fixed-point iterations seems crucial in this context.

We provide one final MC experiment in Figure 8 with values $\xi_p = 0.71$, $\phi_\pi = 1.05$,

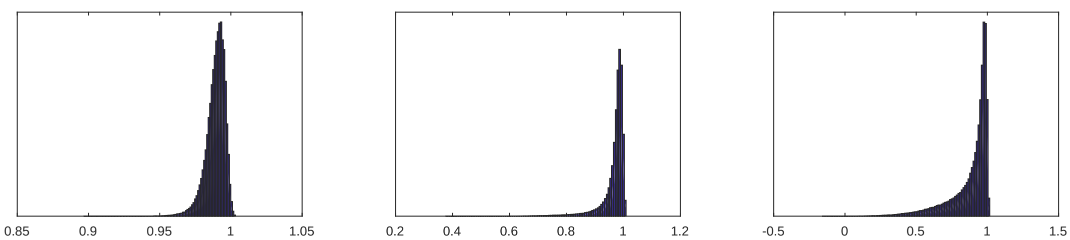
[31] See Hommes et. al. (2019) for a formal treatment of this result.

[32] We report the results only for the persistence of inflation for brevity, but similar results hold for all forward-looking variables.

where we observe two different iteratively E-table equilibria. The corresponding distribution with a gain of 0.01 under SAC-learning turns out bimodal as one might expect, where a second mode emerges close to 1. This second high-persistence equilibrium turns out to have a small basin of attraction, and we observe convergence towards these values in only a small number of simulations. Increasing the gain value reduces its basin of attraction further, in which case we no longer observe the bimodality.



(a) Estimated parameters under BLE specification. Gains are 0.01, 0.025 and 0.05 respectively.



(b) Estimated parameters under BLE with $\xi_p = 0.5$. Gains are 0.01, 0.025 and 0.05 respectively.

Figure 7: Distributions of β_π from simulations with constant gain SAC-learning.

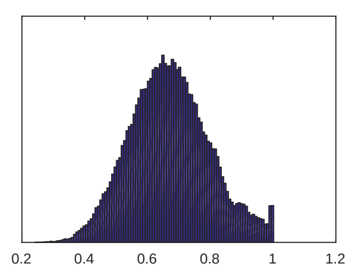


Figure 8: Distribution of β_π from Monte Carlo experiment with two (iteratively) E-stable equilibra: we use the estimated parameter values under BLE, with $\xi_p = 0.71$, $\phi_\pi = 1.05$ and gain 0.01.

We close the section with a comparison between SAC-learning and RLS under the AR(1) rule. Figure 10 shows the persistence of inflation from a one-time simulation under

SAC- and RLS-learning with gain values of $0.01, 0.02$ and 0.05. It is readily seen that the resulting patterns of the persistence coefficient is fairly similar under both algorithms. The projection facility is activated 0.3%, 3% and 4% of the time under RLS-learning if we impose the projection facility only on the implied ALM. If the projection facility is imposed both on the ALM and PLM, these frequencies increase to 5%, 9% and 20% respectively. This illustrates the advantage of SAC-learning over RLS, where the projection facility is never activated regardless of how it is imposed (i.e. only on ALM or both ALM and PLM). As such, SAC-learning can be used in model estimations without the need for a projection facility, which results in a considerable computational advantage over RLS-learning or other similar learning methods.

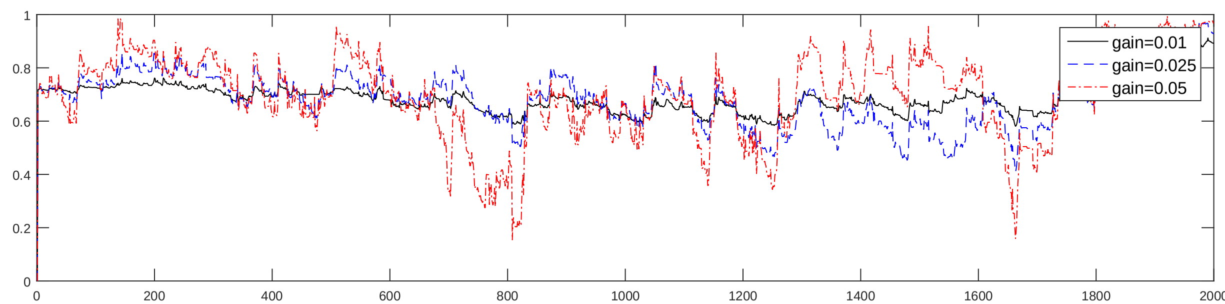
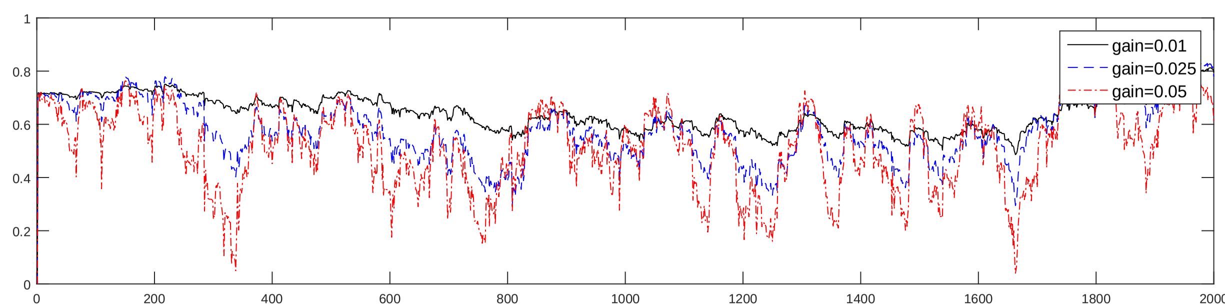


Figure 9: Persistence of inflation under AR(1)-RLS learning. Frequency of projection facilities: 0.3%, 3% and 4% respectively if the projection facility is imposed on the ALM only, whereas they increase to 5%, 9% and 20% if the projection facility is imposed on both ALM and PLM.



(a) Persistence of inflation under SAC-learning.

Figure 10: Persistence of inflation in a simulation with different gain values under AR(1)-RLS and SAC-learning. The frequency of projection facilities is 0 in all cases regardless of how it is imposed.

6 Conclusions

In this paper, we introduced the concept of Behavioral Learning Equilibria into the workhorse Smets-Wouters (2007) DSGE model. Our main estimation results complement our findings in Hommes et. al. (2019), where we find that the empirical fit of the baseline 3-equation New Keynesian model is better than the Rational Expectations counterpart. By further estimating the model under small forecasting rules with AR(1)- and AR(2)-learning, as well as MSV-learning that accounts for a fully rational information set, we disentangled the effects of the information set and learning on the model fit. In particular, we showed assumptions on these two concepts play almost equally important roles in driving both the in- and out-of-sample fit of the model. The key result in our paper emerges as the usefulness of the equilibrium concept that we introduce. While an AR(1) or AR(2) learning models with time-varying beliefs may outperform a Behavioral Learning Equilibrium with fixed-beliefs in-sample, the equilibrium restrictions turn out to be more useful in out-of-sample exercises, particularly over longer horizons. Further, we showed that estimating a model under learning without accounting for the underlying equilibrium may lead to large differences between estimated models and simulations of the same model. As such, using the appropriate equilibrium restrictions help reduce the disparity between estimations and simulations under learning. Finally, within the context of a parsimonious AR(1) model, we showed that Sample Autocorrelation Learning is a plausible alternative o Recursive Least-squares Learning that works well without requiring a projection facility.

Our work opens up two avenues of future research. First, our results call attention to the general class of Restricted Perceptions Equilibria that consider different dimensions of misspefications, and accompanying solution algoritms to make these equilibria operational. Second, Sample Autocorrelation Learning, which is based on a method-of-moments estimator for the AR(1) rule, should be generalized to account for any class of PLM in order to complement the corresponding Restricted Perceptions Equilibrium concepts.

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Appendix

A Supplementary Model Description and Priors

The model consists of 13 equations linearized around the steady-state growth path, supplemented with seven exogenous structural shocks. We deviate from the benchmark model by slightly restricting the parameter space of the model, where we assume all shocks follow an AR(1) process[33]. In this section we briefly outline the resulting linearized model economy that is used in our estimation. To start with the demand side of the economy, the aggregate resource constraint is given by:

$$\begin{cases} y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g \\ \epsilon_t^g = \rho_g \epsilon_{t-1}^g + \eta_t^g \end{cases} \tag{A.1}$$

where y_t, c_t, i_t and z_t are the output, consumption, investment and capital utilization rate respectively, while c_y, i_y and z_y are the steady-state shares in output of the respective variables. The second equation in (A.1) defines the exogenous spending shock ϵ_t^g where η_t^g is an i.i.d-normal disturbance for spending. The consumption Euler equation is given by:

$$\begin{cases} c_t = c_1 c_{t-1} + (1-c_1)\mathbb{E}_t c_{t+1} + c_2(l_t - \mathbb{E}_t l_{t+1}) - c_3(r_t - \mathbb{E}_t \pi_{t+1}) + \epsilon_t^b \\ \epsilon_t^b = \rho_b \epsilon_{t-1}^b + \eta_t^b \end{cases} \tag{A.2}$$

with $c_1 = \frac{\lambda}{\gamma}/(1+\frac{\lambda}{\gamma})$,$c_2 = (\sigma_c - 1)(w_{ss}l_{ss}/c_{ss})/(\sigma_c(1+\frac{\lambda}{\gamma})$,$c_3 = (1-\frac{\lambda}{\gamma})/((1+\frac{\lambda}{\gamma})\sigma_c)$, where λ, γ and σ_c denote the habit formation in consumption, steady state-growth rate and the elasticity of intertemporal substitution respectively, while x_{ss} corresponds to the steady-state level of a given variable x. The equation implies that current consumption is a weighted average of the past and expected future consumption, expected growth in hours worked and the ex-ante real interest rate. ϵ_t^b corresponds to the risk premium shock modeled as an AR(1) process, where η_t^b is an i.i.d-normal disturbance. Next, the investment Euler equation is defined as:

$$\begin{cases} i_t = i_1 i_{t-1} + (1-i_1)\mathbb{E}_t i_{t+1} + i_2 q_t + \epsilon_t^i \\ \epsilon_t^i = \rho_i \epsilon_{t-1}^i + \eta_t^i \end{cases} \tag{A.3}$$

[33] In particular, the benchmark model has more structure on the exogenous shocks, where the mark-up shocks each follow an ARMA(1,1) process, and the technology and government spending shocks follow a VAR(1) process. We refer the reader to SW for more details about the microfoundations of the benchmark model.

with $i_1 = \frac{1}{1+\bar{\beta}\gamma}$, $i_2 = \frac{1}{(1+\bar{\beta}\gamma)(\gamma^2\phi)}$, where $\bar{\beta} = \beta\gamma^{-\sigma_c}$, ϕ is the steady-state elasticity of capital adjustment cost and β is the HH discount factor. q_t denotes the real value of existing capital stock. Similar to the consumption Euler, the equation implies that investment is a weighted average of past and expected future consumption, as well as the real value of existing capital stock. ϵ_t^i represents the AR(1) investment shock , where η_t^i is an i.i.d-normal disturbance. The value of capital-arbitrage equation is given by:

$$q_t = q_1\mathbb{E}_t q_{t+1} + (1-q_1)\mathbb{E}_t r_{t+1}^k - (r_t + \mathbb{E}_t\pi_{t+1}) + \tfrac{1}{c_3}\epsilon_t^b \tag{A.4}$$

with $q_1 = \bar{\beta}(1-\delta)$, implying the real value of capital stock is a weighted average of its expected future value and expected real rental rate on capital, net of ex-ante real interest rate and the risk premium shock. The production function is characterized as:

$$\begin{cases} y_t = \phi_p(\alpha k_t^s + (1-\alpha)l_t + \epsilon_t^a) \\ \epsilon_t^a = \rho_a\epsilon_{t-1}^a + \eta_t^a \end{cases} \tag{A.5}$$

Where k_t^s denotes the capital services used in production, α is the share of capital in production and ϕ_p is (one plus) the share of fixed costs in production. ϵ_t^a denotes the AR(1) total factor productivity shock. Capital is assumed to be the sum of the previous amount of capital services used and the degree of capital utilization, hence:

$$k_t^s = k_{t-1} + z_t \tag{A.6}$$

Moreover, the degree of capital utilization is a positive function of the degree of rental rate, $z_t = z_1 r_t^k$, with $z_1 = \frac{1-\psi}{\psi}$, ψ being the elasticity of the capital utilization adjustment cost. Next the equation for installed capital is given by:

$$k_t = k_1 k_{t-1} + (1-k_1)i_t + k_2\epsilon_t^i \tag{A.7}$$

with $k_1 = \frac{1-\delta}{\gamma}$, $k_2 = (1 - \frac{1-\delta}{\gamma})(1+\bar{\beta}\gamma)\gamma^2\phi$. The price mark-up equation is given by:

$$\mu_t^p = \alpha(k_t^s - l_t) + \epsilon_t^a - w_t \tag{A.8}$$

Which means the price mark-up μ_t^p is the marginal product of labor net of the current wage. The NKPC is characterized as:

$$\begin{cases} \pi_t = \pi_1\mathbb{E}_t\pi_{t+1} - \pi_2\mu_t^p + \epsilon_t^p \\ \epsilon_t^p = \rho_p\epsilon_{t-1}^p + \eta_t^p \end{cases} \tag{A.9}$$

with $\pi_1 = \bar{\beta}\gamma$,$\pi_2 = (1-\beta\gamma\xi_p)(1-\xi_p)/[\xi_p((\phi_p-1)\epsilon_p+1)]$, where ξ_p corresponds to the degree of price stickiness, while ϵ_p denotes the Kimball goods market aggregator. The equation implies that current inflation is determined by the expected future inflation, the price mark-up and the AR(1) price mark-up shock ϵ_t^p, where η_t^p is an i.i.d-normal disturbance. The rental rate of capital is given by:

$$r_t^k = -(k_t - l_t) + w_t \tag{A.10}$$

Which implies the rental rate of capital is decreasing in the capital-labor ratio and increasing in the real wages. The wage mark-up is given as the real wages net of marginal rate of substitution between working and consuming, hence:

$$\mu_t^w = w_t - (\sigma_l l_t + \frac{1}{1-\lambda/\gamma}(c_t - \frac{\lambda}{\gamma}c_{t-1}) \tag{A.11}$$

Where σ_l denotes the elasticity of labor supply. The real wage equation is given by:

$$\begin{cases} w_t = w_1 w_{t-1} + (1-w_1)(\mathbb{E}_t w_{t+1} + \mathbb{E}_t \pi_{t+1}) - w_2 \mu_t^w + \epsilon_t^w \\ \epsilon_t^w = \rho_w \epsilon_{t-1}^w + \eta_t^w \end{cases} \tag{A.12}$$

with $w_1 = 1/(1+\bar{\beta}\gamma)$, and $w_2 = ((1-\bar{\beta}\gamma\xi_w)(1-\xi_w)/(\xi_w(\phi_w-1)\epsilon_w+1))$. Hence the real wage is a weighted average of the past and expected wage, expected inflation, the wage mark-up and the wage mark-up shock ϵ_t^w , where η_t^w is an i.i.d-normal disturbance. Finally, monetary policy is assumed to follow a standard generalized Taylor rule:

$$\begin{cases} r_t = \rho r_{t-1} + (1-\rho)(r_\pi \pi_t + r_y x_y) + r_{dy}(\Delta x_t) + \epsilon_t^r \\ \epsilon_t^r = \rho_r \epsilon_{t-1}^r + \eta_t^r \end{cases} \tag{A.13}$$

Where x_t denotes the output gap, and ϵ_t^r is the AR(1) monetary policy shock, with η_t^r the i.i.d-normal disturbance. Hence the monetary policy responds output gap growth on top of inflation and the output gap. In this paper, following the approach in Slobodyan & Wouters (2012), we deviate from SW (2007) and model the output gap as the deviation of output from the underlying productivity process, i.e. $x_t = y_t - \epsilon_t^a$ [34]. For convenience, Table 7 provides a summary of all structural (deep) parameters of the system.

[34] In the benchmark model, the output gap is defined as the deviation of output from the potential output, which requires modeling the flexible economy. We adapt the definition considered in this paper particularly because it does not require the flexible economy. This substantially reduces the number of forward-looking variables, and accordingly the number of first-order autocorrelations that are numerically approximated.

Table 7: **Smets-Wouters (2007): Structural Parameters of the System.**

γ : Steady-state growth rate
g: Steady-state share of government spending
$\bar{l}$ ($\bar{\pi}$) Steady-state level of labour (inflation)
δ : Depreciation rate of capital
ϵ_t^g : Exogenous spending shock
ϵ_t^b : Risk premium shock
ϵ_t^i : Investment-specific technology shock
ϵ_t^a : Productivity shock
ϵ_t^w (ϵ_t^p) : Wage (Price) mark-up shock
ϵ_t^r : Monetary policy shock
$\iota_p(\iota_w)$: Price (Wage) indexation
λ : Habit formation in consumption
ϕ : Capital adjustment cost
ϕ_p : (One plus) the share of fixed costs in production
ψ : Capital utilization adjustment cost
$\xi_p(\xi_w)$: Calvo price (wage) stickiness
σ_c : (Inverse of) the elasticity of intertemporal substitution for labor.
σ_l : Elasticity of labor supply with respect to the real wage
β : Household discount factor
α : Share of capital in production
$\epsilon_p(\epsilon_w)$: Kimball Goods (Labor) market aggregator
ϕ_w : (One plus) steady-state labor mark-up
ρ : Interest rate smoothing
$r_y(r_{dy})$: Policy reaction to output gap (output gap growth)
r_π Policy reaction to inflation

Table 8: Fixed parameters and the prior distributions of the estimated parameters for the Smets-Wouters (2007) model.

Fixed Parameters			
δ	0.025		
ϕ_w	1.5		
g	0.18		
ϵ_p	10		
ϵ_w	10		
Prior	Distribution	Mean	Var.
Parameters related to nominal and real frictions			
ϕ	Normal	4	1.5
σ_c	Normal	1.5	0.375
λ	Beta	0.7	0.1
ξ_w	Beta	0.5	0.1
σ_l	Normal	2	0.75
ξ_p	Beta	0.5 (0.75)	0.1 (0.05)
ψ	Beta	0.5	0.15
ϕ_p	Normal	1.25	0.125
Policy related parameters			
r_π	Normal	1.5	0.25
ρ	Beta	0.75	0.1
r_y	Normal	0.125	0.05
r_{dy}	Normal	0.125	0.05
Steady-state related parameters			
$\bar{\pi}$	Gamma	0.625	0.1
$\bar{\beta}$	Gamma	0.25	0.1
$\bar{l}$	Normal	0	2
$\bar{\gamma}$	Normal	0.4	0.1
α	Normal	0.3	0.05
Parameters related to shock persistence			
ρ_a	Beta	0.5	0.2
ρ_b	Beta	0.5	0.2
ρ_g	Beta	0.5	0.2
ρ_i	Beta	0.5	0.2
ρ_r	Beta	0.5	0.2
ρ_p	Beta	0.5	0.2
ρ_w	Beta	0.5	0.2
Shock variance parameters			
η_a	Inv. Gamma	0.1	2
η_b	Inv. Gamma	0.1	2
η_g	Inv. Gamma	0.1	2
η_i	Inv. Gamma	0.1	2
η_r	Inv. Gamma	0.1	2
η_p	Inv. Gamma	0.1	2
η_w	Inv. Gamma	0.1	2

B Kalman Filter with adaptive learning

This section describes the Kalman filter used in the estimations of learning models. The main filter block follows standard steps, and we use the filter output at the end of each iteration to update the belief parameters.

Denote by $S_{0|0}, P_{0|0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, \boldsymbol{R}_0$ the initial state vector, state covariance matrix, initial belief parameters and their initial covariance matrix respectively. Denoting by L, N and T the number of shocks, forward-looking variables and length of the dataset respectively, the recursion is given as follows:

$$
\begin{cases}
\quad \textbf{For } \mathbf{t = 1 : T} \\
S_{t|t-1} = \tilde{\gamma}_{t-1} + (\gamma_1 + \gamma_2 \beta_{t-1}^2) S_{t-1|t-1}, \\
P_{t|t-1} = (\gamma_1 + \gamma_2 \beta_{t-1}^2) P_{t-1|t-1} (\gamma_1 + \gamma_2 \beta_{t-1}^2)' + \gamma_3 \Sigma_\eta \gamma_3', \\
v_t = Y_t - \phi_0 - \phi_1 S_{t|t-1}, \\
\Sigma_t = \phi_1 P_{t|t-1} \phi_1', \\
S_{t|t} = S_{t|t-1} + P_{t|t-1} \phi_1' \Sigma_t^{-1} v_t, \\
P_{t|t} = P_{t|t-1} \phi_1' \Sigma_t^{-1} \phi_1 P_{t|t-1}, \\
p(y_t | \alpha_{t-1}, \beta_{t-1}) = -\frac{L}{2} ln(2\pi) - \frac{1}{2} ln|\Sigma_t| - \frac{1}{2}(v_t' \Sigma_t^{-1} v_t) \\
\qquad \textbf{For } \mathbf{i = 1 : N} \\
\qquad \alpha_{i,t} = \alpha_{i,t-1} + \frac{1}{t+1}(x_{i,t|t} - \alpha_{i,t-1}), \\
\qquad \beta_{i,t} = \beta_{i,t-1} + \frac{1}{t+1} R_{i,t}^{-1} \Big[(x_{i,t|t} - \alpha_{i,t-1}) \big(x_{i,t-1|t-1} + \frac{x_{i,0|0}}{t+1} - \frac{t^2+3t+1}{(t+1)^2} \alpha_{i,t-1} - \frac{1}{(t+1)^2} x_{i,t|t} \big) \\
- \frac{t}{t+1} \beta_{i,t-1} (x_{i,t|t} - \alpha_{i,t-1})^2 \Big], \\
\qquad R_{i,t} = R_{i,t-1} + \frac{1}{t+1} \Big[\frac{t}{t+1} (x_{i,t|t} - \alpha_{i,t-1})^2 - R_{i,t-1} \Big]. \\
\qquad \textbf{End} \\
\quad \textbf{End}
\end{cases}
\tag{B.1}
$$

C Details on Equilibrium Algorithms

Iterative E-stability for BLE

The iterative E-stability algoritm outlined in Hommes et. al. (2019) is given below.

In this paper during the estimations, rather than checking for convergence at every step, we use a fixed number of 200 iterations for every parameter draw, which is typically long enough to ensure convergence. The convergence criterion is checked in Section 5 for

Approximation of a BLE using Iterative E-stability

Denote by θ the set of structural parameters, and by $G(\boldsymbol{\beta^{(k)}}, \theta)$ the first-order autocorrelation function for a given θ.

- Step (0): Initialize the vector of learning parameters at $\boldsymbol{\beta^{(0)}}$.
- Step (I): At each iteration k, using the first-order autocorrelation functions, update the vector of learning parameters as

$$\boldsymbol{\beta^{(k)}} = G(\boldsymbol{\beta^{(k-1)}}, \theta), \tag{C.1}$$

  where $G(\boldsymbol{\beta^{(k-1)}}, \theta)$ is known from iteration $k-1$.
- Step (II): Terminate if $||\boldsymbol{\beta^{(k)}} - \boldsymbol{\beta^{(k-1)}}||_p < \epsilon$, for a small scalar $\epsilon > 0$. and a suitable norm distance $||.||_p$, otherwise repeat Step (I).

the fixed-point iterations with over randomized initial values.

Minimum State Variable REE Solution

The solution takes the following, along with the 1-step ahead expectations:

$$\begin{cases} X_t = \alpha + \beta X_{t-1} + d\epsilon_t + e\epsilon_{t-1}, \\ E_t X_{t+1} = \alpha + \beta X_t + d\rho\epsilon_t + e\epsilon_t. \end{cases} \tag{C.2}$$

We first solve for the underlying Rational Expectations Equilibrium (REE) for matrices β, d and e. Since we focus on zero-mean systems in this paper, $\alpha = 0$ will always hold in a REE.

The fixed-point condition for β is given as follows:

$$\beta = (A - C\beta)^{-1}B \Rightarrow C\beta^2 - A\beta + B = 0.$$

This yields a quadratic expression in terms β, which is solved numerically using Uhlig's toolkit (1995). Given β, we have the followuing expressions for d:

$$d = (A - C\beta)^{-1}(c(d\rho + e) + D),$$

which yields

$$vec(d) = (I \otimes (A - C\beta) - (\rho' \otimes C))^{-1}(vec(Ce + D)).$$

And for e we have:

$$(A - C\beta)^{-1}E,$$

which yields

$$e = (A - C\beta)^{-1}E.$$

Hence, a solution for β pins down e, which in turns yields d.

D Initial values in learning models

In this section, we consider estimation results under alternative initial values for the learning models, which are reported in Tables 9, 10 and 11.

We start with an evaluation of whether initializing at BLE or REE makes a difference for the learning models with a small forecasting rule, i.e. SAC-learning and RLS-AR(1) and -AR(2), in Table 9. Recall from the previous section that the persistence of forward-looking variables generated under the estimated BLE and REE are fairly similar, whereas their volatilities are substantially larger under BLE. This may lead to differences in the properties of learning models depending on which model is used to initialize the learning process. To see the effect of this, we consider two initializations for the learning models with small forecasting rules: θ^*_{REE} which fixes the initial beliefs at moments generated from the estimated REE model, and θ^*_{REE} which fixes the initial beliefs at moments generated from the estimated BLE model[35]. We observe that the model fit is better with the BLE-based initialization for SAC- and AR(2) RLS-learning models, whereas the fit remains fairly similar for the AR(1) RLS model. Further, for the AR(1) models, we observe that the estimated Calvo probabilities are lower when the model is initialization at the BLE-based moments, while the monetary policy and gain parameters remain mostly unchanged. For the AR(2) model, we observe that all estimated parameters remain mostly unchanged, while the estimated gain parameter is substantially smaller under REE-based initialization. These results suggest that in all model specifications, at least one important difference is observed in an estimated parameter and the model fit.

Next we consider MSV-learning model, for which we consider three different initializations reported in Table 10: (i) θ_{REE} that re-computes the the initial beliefs by deriving the underlying REE for every new parameter draw, (ii) θ^*_{REE} as the benchmark case that fixes the initial beliefs at the estimated moments, (iii) $\theta_{diffuse}$ that initializes the belief coefficients at diffuse values[36]. The third model provides a substantially worse model fit compared to the two REE-based initializations, whereas the difference between the REE-based initializations is negligible. The first model imposes the REE-restrictions on the model for all initial values, whereas the second model imposes the restrictions at the best-fitting REE-model only. We observe that, under θ_{REE}, the estimated parameters are closer to the original REE model, wheras in the second case the parameter estimates go in the direction of the other learning models, e.g. the estimated Calvo probabilities are lower in the second case.

Finally, for the SAC-learning model, we also consider three additional initializations: on top of θ^*_{REE} and θ^*_{BLE} that were already discussed, we consider θ_{REE} and θ_{BLE} that re-compute the initial beliefs for every parameter draw, and $\theta_{diffuse}$ as discussed above for MSV-learning. We observe that diffuse initials again yield the worst model fit[37]. In this case we do not see a noteworthy difference between θ_{REE} and θ^*_{REE} in terms of both estimated parameters and the model fit. There are however some differences between θ_{BLE} and θ^*_{BLE}: the fit under θ_{BLE} is substantially is worse, and it is closer to the estimations

[35]This corresponds to a 2-step approach, where we first take the estimated REE and BLE models and simulate a large dataset from each model. Then the initial beliefs are generated by estimating an AR(1) or AR(2) model based on the simulated data.

[36]We use initial values of 0 for all moments in this exercise, as suggested in Berardi and Galimberti (2017b).

[37]The difference is much smaller compared to MSV-learning, which is due to faster convergence under AR(1) beliefs given the smaller size of the learning models.

under BLE model without any learning. This difference again arises due to uncertainty around inflation dynamics under BLE. As one might expect, the differences between θ^*_{BLE} and θ_{BLE} are close to the difference between the original SAC-learning and BLE models as reported in the main estimation section. This is because the first model imposes the BLE restrictions at every parameter draw, whereas the second model only uses the model for the initialization and does not impose the cross restrictions that would be consistent with BLE for every parameter draw.

Our main takeaway from this section is that, imposing the equilibrium restrictions for every parameter draw does not typically improve the model fit. Using the fixed-point equilibrium model (i.e. BLE and REE) only to initialize the beliefs yields the best model fit in both MSV-learning and AR(1)-learning cases. Hence we discover a trade-off here: imposing the equilibrium conditions at every parameter draw (i.e. θ_{BLE} and θ_{REE}) comes at a computational cost since the equilibria have to be re-computed for every parameter draw. They further come at a loss in the model fit, since the resulting equilibrium conditions are more restrictive than using only the estimated model. However, not imposing the equilibrium restrictions means the underlying initial conditions are consistent only with the estimated parameters, not the whole parameter space. Therefore in empirical studies, this trade-off must be carefully weighted before deciding on the final model specification. If the model fit is the primary interest (e.g. when the model is to be used for forecasting), then using the fixed-belief models to only for initialization may be the better approach, whereas if the parameter estimations are also important for the researcher, then using θ_{BLE} and θ_{REE} may be more relevant. In this paper, for our out-of-sample forecasting exercise, we proceed with our results in the main estimation table.

For the AR(1) model, we consider 5 different initializations: -θ_{BLE}: re-computes the BLE for every parameter draw. This is the most computationally expensive model, since it computes the BLE for every draw and adds learning on top of it. The resulting likelihood is almost the worst. Not surprising since the learning dynamics push the estinmate ξ_p down, whereas the BLE restriction leads to non-existence over smaller values. So there is a tension between those two dynamics here. -θ^*_{BLE}: yields the best fit. Computationally more efficient than the above. However this method, and the remaining ones, have one important disadvantage: imposing the BLE restriction for every parameter draw ensures that there will be an underlying equilibrium at the final parameter estimates. Therefore when we take these estimates and simulate the model, the resulting moments will be reasonably close to the estimated moments. In the other cases, there is no underlying BLE. Hence when we take the estimates and simulate the model, some variables, e.g. inflation, may display very different dynamics than the estimated ones, i.e. not accounting for the existence of BLE results in a disconnection between the estimations and simulations of the same model. Put differently, using the BLE restriction ensures that these two exercises

Learning	SAC				RLS-AR(1)				RLS-AR(2)			
Initial	θ^*_{BLE}		θ^*_{REE}		θ^*_{BLE}		θ^*_{REE}		θ^*_{BLE}		θ^*_{REE}	
Parameter	Posterior		Posterior		Posterior		Posterior		Posterior			
	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.
ξ_w	0,72	0,045	0,787	0,031	0,698	0,041	0,692	0,041	0.707	0.039	0,73	0,041
ξ_p	0,586	0,04	0,69	0,02	0,602	0,04	0,689	0,042	0.596	0.037	0,583	0,036
ι_w	0,324	0,116	0,42	0,097	0,318	0,126	0,303	0,126	0.315	0.117	0,346	0,117
ι_p	0,321	0,163	0,291	0,063	0,323	0,159	0,31	0,158	0.325	0.142	0,257	0,126
r_π	1,655	0,186	1,619	0,186	1,641	0,187	1,642	0,187	1.678	0.183	1,658	0,182
ρ	0,887	0,019	0,89	0,019	0,888	0,019	0,888	0,019	0.888	0.019	0,888	0,019
r_y	0,145	0,035	0,143	0,034	0,141	0,035	0,144	0,035	0.144	0.032	0,142	0,035
$r_{\delta y}$	0,142	0,02	0,13	0,02	0,142	0,02	0,148	0,02	0.14	0.019	0,134	0,02
gain	0,009	0,003	0,008	0,003	0,014	0,004	0,015	0,004	0.012	0.006	0,005	0,002
Laplace	1009,659		1020,12		1008,25		1008,63		1012,16		1017,544	

Table 9: A comparison of RLS- and SAC-learning algorithms.

Initial	θ_{REE}		θ^*_{REE}		$\theta_{diffuse}$	
	Post.		Post.		Post.	
	Mode	Std.	Mode	Std.	Mode	Std.
ξ_w	0,937	0,015	0,713	0,044	0,746	0,026
ξ_p	0,729	0,035	0,689	0,044	0,763	0,033
ι_w	0,699	0,092	0,397	0,099	0,272	0,057
ι_p	0,108	0,048	0,648	0,092	0,84	0,063
r_π	1,713	0,159	1,705	0,166	1,357	0,186
ρ	0,867	0,017	0,865	0,02	0,875	0,019
r_y	0,122	0,033	0,162	0,031	0,161	0,031
$r_{\delta y}$	0,158	0,019	0,16	0,02	0,125	0,018
gain	0,011	0,006	0,01	0,004	0,018	0,001
Laplace	1043,147		1042,888		1195,678	

Table 10: Alternative estimations of the MSV-learning model with different initial beliefs.

yield similar results. -θ_{REE}: yields a surprisingly good fit, but still worse than the initial based on estimated equilibria. -θ^*_{REE}: yields the second best fit. Not surprising since the moments come from the best fitting REE. The moments under AR(1) vs. REE may not be aligned for each parameter draw, but the best-fitting REE and the best-fitting AR(1) model still generate similar moments in terms of first-order autocorrellations as we already saw. $\theta_{diffuse}$: yields the worst fit, same as MSV-learning.

Initial	θ_{BLE}		θ^*_{BLE}		θ_{REE}		θ^*_{REE}		$\theta_{diffuse}$	
	Posterior									
	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.	Mode	Std.
ξ_w	0,714	0,038	0,72	0,045	0,733	0,041	0,72	0,041	0,747	0,045
ξ_p	0,793	0,014	0,586	0,04	0,533	0,044	0,586	0,04	0,629	0,039
ι_w	0,292	0,114	0,324	0,116	0,372	0,118	0,324	0,118	0,311	0,116
ι_p	0,554	0,074	0,321	0,163	0,504	0,143	0,321	0,149	0,715	0,107
r_π	1,681	0,193	1,655	0,186	1,648	0,188	1,655	0,187	1,586	0,191
ρ	0,887	0,019	0,887	0,019	0,888	0,02	0,887	0,019	0,89	0,02
r_y	0,14	0,035	0,145	0,035	0,141	0,035	0,145	0,035	0,117	0,034
$r_{\delta y}$	0,141	0,019	0,142	0,02	0,14	0,021	0,142	0,02	0,127	0,019
gain	0,006	0,003	0,009	0,003	0,012	0,004	0,009	0,003	0,004	0,001
Laplace	1027,34		1009,65		1023,84		1020,12		1042,68	

Table 11: Alternative estimations of the SAC-learning model with different initial beliefs.

E Timing of expectations

In this section, we investigate the implications of the timinig assumption of expectations for all models under consideration. Recall from Section 1 that under BLE and SAC-learning[38], we assume that expectations and the endogenous variables are sequen-

[38]Similarly, AR(1) and AR(2)-RLS models are evaluated under the $t-1$ timing assumption, and re-estimating with t timing yields results similar to BLE and SAC-learning, therefore they are omitted

tially determined: in a given period t, first expectations are formed with the information previous period $t-1$, after which the endogenous variables are realized. The standard assumption in a REE model is different than the above, where expectations and endogenous variables are jointly determined, which is the assumption that we also kept in the MSV-learning model. In this section we provide re-estimations of our all 4 specifications with both t and $t-1$ assumptions.

The main impact of the timing assumption is observed on the parameters that relate to contemporaneous cross-restrictions of endogenous variables, namely σ_c and ϕ. With $t-1$ timing, expectations enter the model with a one-period lag and therefore some parameters increase to strenghten the contemporaneous cross-restrictions. In particular, σ_c becomes lower with $t-1$ timing, which increases the direct impact of ex-ante risk premium on consumption; and ϕ becomes lower, which increases the direct impact of asset prices on real investment. These effects are observed in all models with the exception of REE. Further, again in all models with the exception of REE, habit persistence and Calvo probabilities become larger with t timing, while the differences are either negligible or go in the opposite direction under REE. The effect on the indexations are ambigious, which sometimes increase and sometimes decrease, while the estimated gain and monetary poliyc parameters remain similar across all specifications.

Overall, BLE, SAC and MSV-learning models all prefer the $t-1$ timing assumption, while the REE model prefers the t timing based on the Laplace approximation.

here.

	BLE				SAC				MSV				REE			
Timing	t-1		t		t-1		t		t-1		t		t-1		t	
Initial					θ^*_{BLE}		θ^*_{BLE}		θ^*_{REE}		θ^*_{REE}					
Parameter	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.	Post. Mode	Std.
ϕ	1,045	0,271	3,685	0,357	0,813	0,227	3,558	0,465	2,847	0,758	5,403	1,053	6,445	0,89	5,365	0,11
σ_c	0,471	0,059	1,111	0,127	0,436	0,056	1,164	0,24	0,678	0,136	1,246	0,234	1,334	0,117	1,386	0,051
λ	0,715	0,061	0,738	0,017	0,652	0,076	0,731	0,029	0,707	0,068	0,795	0,04	0,808	0,033	0,758	0,023
ξ_w	0,752	0,035	0,79	0,018	0,72	0,045	0,787	0,032	0,665	0,044	0,713	0,044	0,931	0,016	0,937	0,015
ξ_p	0,694	0,015	0,706	0,028	0,586	0,04	0,696	0,02	0,643	0,053	0,689	0,044	0,724	0,031	0,727	0,034
ι_w	0,293	0,114	0,452	0,098	0,324	0,116	0,423	0,098	0,504	0,125	0,397	0,099	0,687	0,089	0,702	0,103
ι_p	0,489	0,062	0,19	0,056	0,321	0,163	0,291	0,063	0,628	0,115	0,648	0,092	0,099	0,048	0,105	0,051
r_π	1,66	0,18	1,617	0,162	1,655	0,186	1,62	0,186	1,62	0,172	1,705	0,166	1,708	0,158	1,737	0,17
ρ	0,887	0,019	0,894	0,019	0,887	0,019	0,892	0,019	0,886	0,018	0,865	0,02	0,879	0,016	0,864	0,018
r_y	0,133	0,035	0,134	0,033	0,145	0,035	0,143	0,034	0,148	0,032	0,162	0,031	0,092	0,039	0,117	0,028
$r_{\delta y}$	0,144	0,019	0,131	0,018	0,142	0,02	0,13	0,019	0,129	0,017	0,16	0,02	0,152	0,017	0,161	0,019
gain					0,009	0,003	0,008	0,003	0,009	0,002	0,01	0,004				
Laplace	1030,05		1037,6		1009,65		1032,83		1038,67		1042,9		1044,17		1039,89	

Table 12: Effects of the timing assumption of expectations on the model fit and parameter estimates.

F HPD intervals and Posterior Distributions of Estimated Parameters

In this section, we provide the 95% HPD intervals of all models estimated and reported in Table 1, as well as the posterior distributions of the estimated parameters. Overall, the posterior distributions are typically symmetric and the posterior mean remains fairly close to the posterior mode (the vertical red line), suggesting that the Laplace approximations used throughout the paper provide a sufficiently good approximation to the true posterior distributions.

F.1 REE

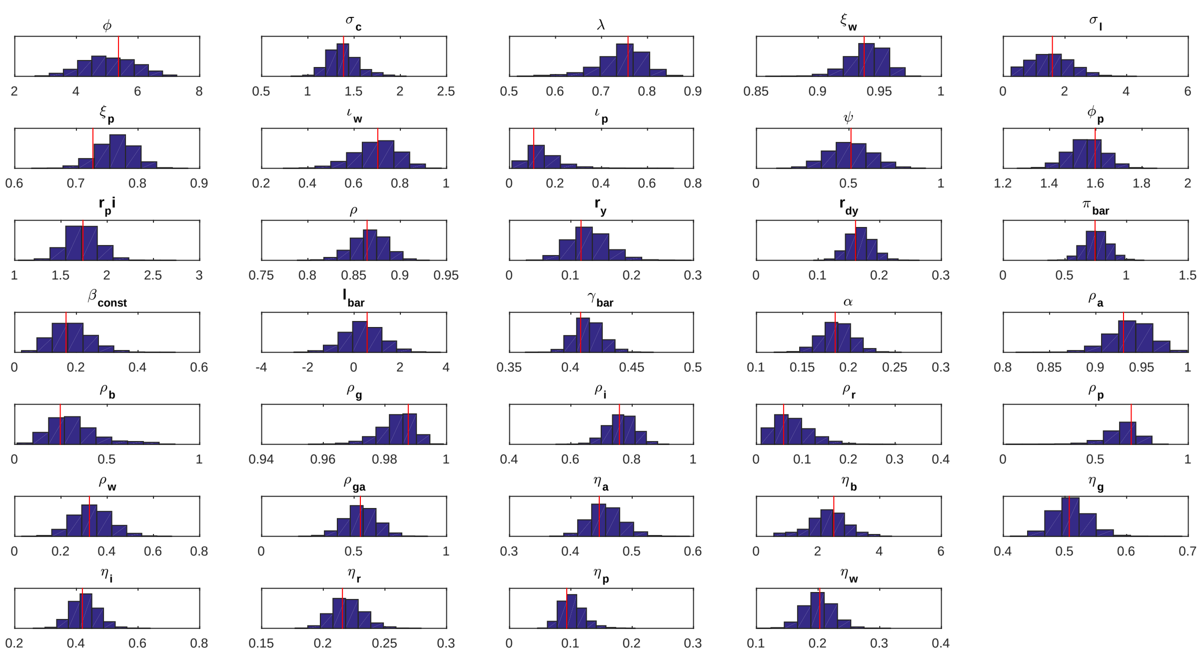

Figure 11: Posterior Distributions of estimated parameters under REE.

Initial Beliefs	REE		BLE		SAC θ^*_{BLE}		AR(2) θ^*_{REE}		MSV θ^*_{BLE}	
Timing	t		t-1		t-1		t-1		t	
	Post.		Post.		Post.		Post.		Post.	
	5 %	95 %	5 %	95 %	5 %	95 %	5 %	95 %	5 %	95 %
ϕ	3,54	6,74	0,5	1,32	0,5	1,65	0,51	1,47	3,54	7,66
σ_c	1,08	1,74	0,27	0,47	0,33	0,56	0,37	0,63	0,83	1,65
λ	0,63	0,84	0,53	0,81	0,53	0,83	0,52	0,79	0,71	0,86
ξ_w	0,91	0,97	0,69	0,86	0,64	0,81	0,62	0,79	0,61	0,78
σ_l	0,25	2,69	0,41	3,1	0,55	3,24	0,67	3,35	0,34	2,88
ξ_p	0,7	0,82	0,62	0,72	0,59	0,77	0,52	0,69	0,68	0,84
ι_w	0,51	0,87	0,2	0,71	0,12	0,57	0,12	0,57	0,2	0,64
ι_p	0,03	0,3	0,26	0,65	0,1	0,65	0,11	0,66	0,41	0,79
ψ	0,29	0,73	0,14	0,65	0,2	0,73	0,24	0,76	0,25	0,74
ϕ_p	1,41	1,71	1	1,29	1,21	1,53	1,31	1,62	1,35	1,63
r_π	1,4	2,06	1,25	2,01	1,26	2,01	1,23	2	1,34	2,04
ρ	0,83	0,9	0,85	0,93	0,85	0,93	0,85	0,93	0,82	0,91
r_y	0,07	0,19	0,08	0,23	0,07	0,22	0,06	0,2	0,09	0,23
$r_{\delta y}$	0,13	0,21	0,12	0,22	0,1	0,19	0,1	0,18	0,12	0,2
$\bar{\pi}$	0,57	0,95	0,66	0,99	0,48	0,87	0,51	0,92	0,51	0,91
$\bar{\beta}$	0,07	0,31	0,13	0,5	0,11	0,47	0,1	0,45	0,07	0,33
$\bar{l}$	-1,25	1,98	-2,49	0,54	-1,8	1,05	-2,71	0,93	-1,01	1,96
$\bar{\gamma}$	0,39	0,44	0,39	0,44	0,39	0,44	0,38	0,43	0,4	0,44
α	0,15	0,22	0,08	0,15	0,11	0,18	0,12	0,19	0,15	0,22
ρ_a	0,89	0,98	0,87	0,96	0,88	0,98	0,88	0,98	0,89	0,97
ρ_b	0,1	0,66	0,37	0,65	0,39	0,68	0,35	0,65	0,2	0,55
ρ_g	0,97	0,99	0,94	0,98	0,95	0,99	0,95	1	0,95	1
ρ_i	0,67	0,85	0,31	0,56	0,36	0,63	0,31	0,57	0,69	0,88
ρ_r	0,01	0,15	0,02	0,22	0,02	0,21	0,02	0,21	0,01	0,15
ρ_p	0,45	0,79	0,01	0,09	0,01	0,1	0,01	0,1	0,01	0,24
ρ_w	0,18	0,49	0,19	0,41	0,12	0,4	0,09	0,36	0,85	0,97
ρ_{ga}	0,37	0,7	0,39	0,64	0,38	0,68	0,37	0,68	0,39	0,71
η_a	0,41	0,51	0,48	0,65	0,43	0,56	0,42	0,53	0,41	0,52
η_b	0,77	3,4	1,18	2,95	1,24	3,67	1,33	3,23	1,46	3,89
η_g	0,45	0,56	0,43	0,53	0,44	0,55	0,45	0,56	0,45	0,57
η_i	0,35	0,51	1,32	1,63	1,33	1,66	1,33	1,65	0,36	0,48
η_r	0,2	0,25	0,19	0,24	0,19	0,24	0,19	0,24	0,19	0,24
η_p	0,07	0,14	0,29	0,36	0,27	0,33	0,27	0,34	0,14	0,2
η_w	0,16	0,25	0,49	0,61	0,49	0,61	0,48	0,6	0,09	0,12
gain	0,01	0,06	0,01	0,06	0.003	0.015	0.0041	0.019	0.003	0.019
MHM	-1039,51		-1025,23		-1012,21		-1009,63		-1042,22	
Acc. Rate	0,323		0,343		0,343		0,335		0,328	

Table 13: This table reports the 95 % HPD intervals under all five expectation formation rules corresponding to the main estimation results reported in the paper.

F.2 BLE

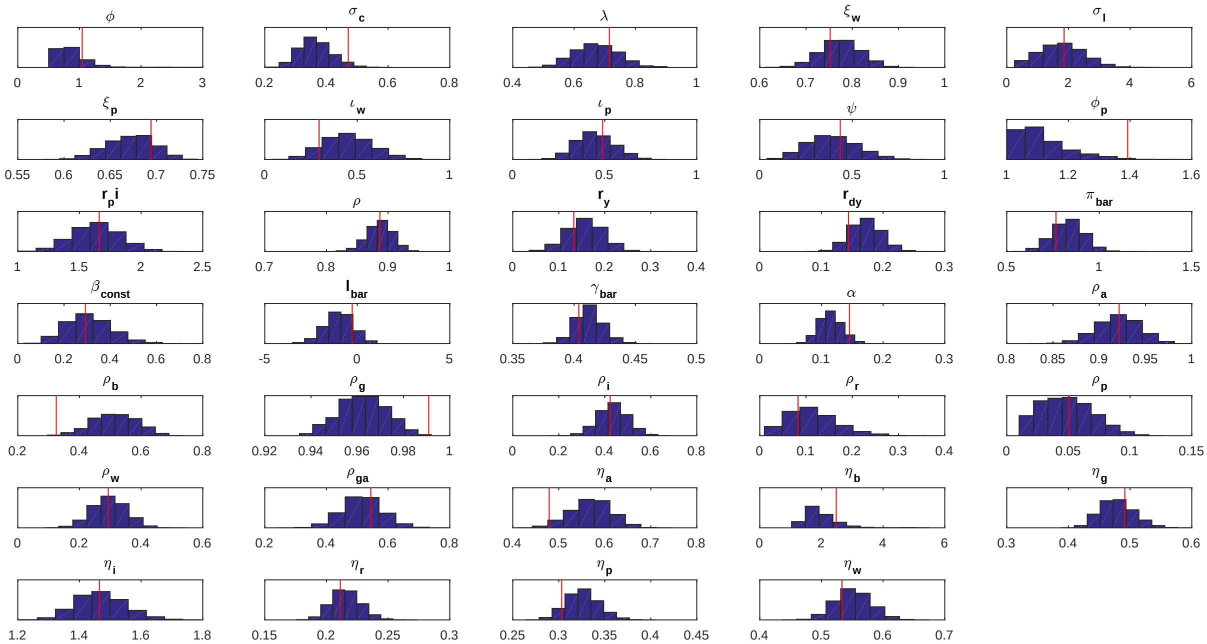

Figure 12: Posterior Distributions of estimated parameters under BLE.

F.3 SAC-learning

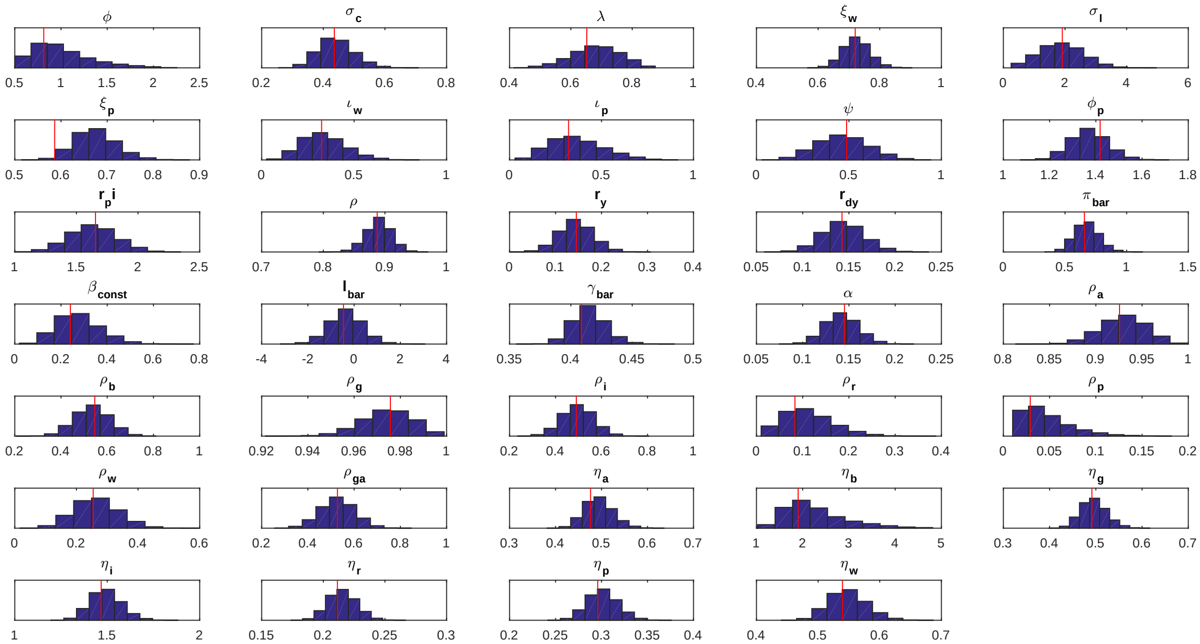

Figure 13: Posterior Distributions of estimated parameters under SAC-learning.

F.4 MSV-learning

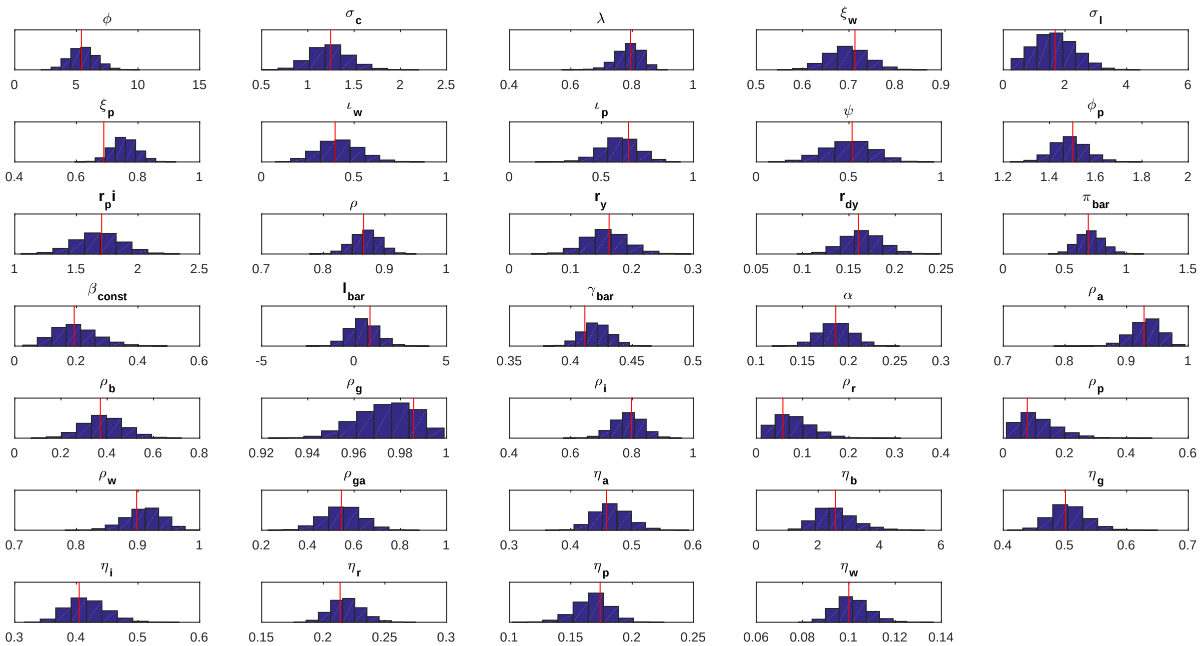

Figure 14: Posterior Distributions of estimated parameters under MSV-learning.

F.5 AR(2)-learning

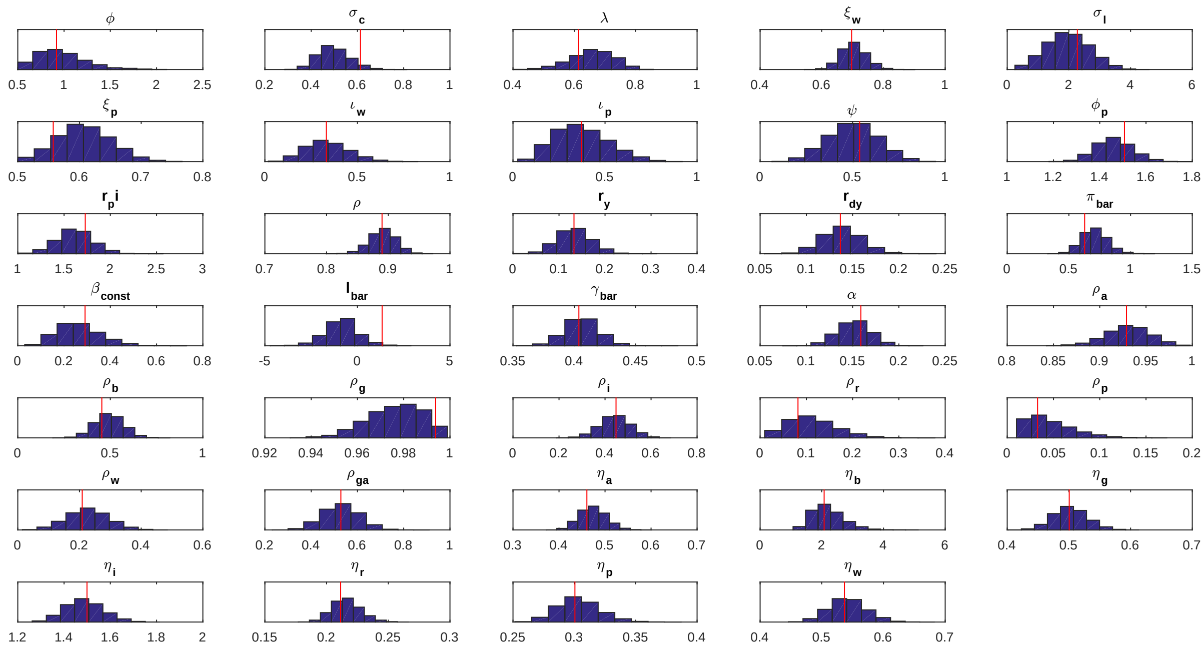

Figure 15: Posterior Distributions of estimated parameters under AR2-learning.