Behavioral Learning Equilibria in the New Keynesian Model

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Abstract

We generalize the concept of behavioral learning equilibrium (BLE) to a general high dimensional linear system and apply it to the standard New Keynesian model. For each endogenous variable in the economy, boundedly rational agents learn to use a simple, but optimal AR(1) forecasting rule with parameters consistent with the observed sample mean and autocorrelation of past data. Agents do not fully recognize the complex structure of the economy, but learn to use an optimal parsimonious AR(1) rule, which satisfies the orthogonality condition for RE. We find that BLE exists, under general stationarity conditions, typically with near unit root autocorrelation parameters. BLE thus exhibits a novel feature, persistence amplification: the persistence in inflation and output gap is much higher than the persistence in exogenous fundamental driving factors. We provide a general framework to find and estimate BLE. We illustrate our approach on U.S. data and show that the standard New Keynesian model fits aggregate data reasonably well under a BLE. We analyze optimal monetary policy under BLE and, in contrast to the RE benchmark, we find that a finite optimal Taylor rule exits for a range of calibrations, and that the transmission channel of monetary policy is stronger under BLE at the estimated parameter values.

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1 Introduction

Rational Expectations Equilibrium (REE) requires that economic agents' subjective probability distributions coincide with the objective distribution that is determined, in part, by their subjective beliefs. There is a vast literature that studies the drawbacks of REE. Some of these drawbacks include the fact that REE requires an unrealistic degree of computational power and perfect information on the part of agents. Alternatively, the adaptive learning literature (see, e.g., Evans and Honkapohja (2001, 2013) and Bullard (2006) for extensive surveys and references) replaces Rational Expectations with beliefs that come from an econometric forecasting model with parameters updated using observed time series. A large part of this literature involves studying under which conditions learning will converge to the rational expectations equilibrium. When the perceived law of motion (PLM) of agents is correctly specified, convergence of adaptive learning to an REE can occur. However, in general the PLM will be misspecified. As shown in White (1994), an economic model or a probability model is only a more or less crude approximation to whatever might be the "true" relationships among the observed data and consequently it is necessary to view economic and/or probability models as misspecified to some greater or lesser degree. Whenever agents have *misspecified* PLMs a reasonable learning process may settle down to some sort of misspecification equilibrium. In the literature, different types of misspecification equilibria have been proposed, e.g. Restricted Perceptions Equilibrium (RPE) where the forecasting model is underparameterized (Sargent, 1991; Evans and Honkapohja, 2001; Adam, 2003; Branch and Evans, 2010) and Stochastic Consistent Expectations Equilibrium (SCEE) (Hommes and Sorger, 1998; Hommes et al., 2013), where agents learn the optimal parameters of a simple, parsimonious AR(1) rule.[1]

A SCEE is a very natural misspecification equilibrium, where agents in the economy do not know the actual law of motion or even recognize all relevant explanatory variables, but rather prefer a parsimonious forecasting model. The economy is too complex to fully understand and therefore, as a first-order approximation, agents forecast the state of the economy by simple autoregressive models (e.g. Fuster et al., 2010). In the simplest model applying this idea, agents run a univariate AR(1) regression to generate out-of-sample forecasts of the state of the economy. Hommes and Zhu (2014) provide the first-order SCEE with an *intuitive behavioral* interpretation and refer to them as a *Behavioral Learn-*

[1]Branch (2006) provides a stimulating survey discussing the connection between these types of misspecification equilibria.

ing Equilibrium (BLE). Although it is possible for some agents to use more sophisticated models, one may argue that these practices are neither straightforward nor widespread. A simple, parsimonious BLE seems a more plausible outcome of the coordination process of individual expectations in large complex socio-economic systems (Grandmont, 1998).

Hommes and Zhu (2014) formalize the concept of BLE in the simplest class of models one can think of: a one-dimensional linear stochastic model driven by an exogenous linear stochastic AR(1) process. Agents do not recognize, however, that the economy is driven by an exogenous AR(1) process y_t, but simply forecast the state of the economy x_t using a univariate AR(1) rule. The parameters of the AR(1) forecasting rule are not free, but fixed (or learned over time) according to the observed sample average and first-order sample autocorrelation. Within this simple, but general, class of models Hommes and Zhu (2014) fully characterize the existence and multiplicity of BLE and provide stability conditions under a simple adaptive learning scheme –Sample Autocorrelation Learning (SAC-learning). Although this class of models is simple, it contains two important standard applications: an asset pricing model driven by autocorrelated dividends and the New Keynesian Philips curve with inflation driven by autocorrelated output gap (or marginal costs).As shown in Fuhrer (2009), however, the skeleton model of the New Keynesian Philips curve with AR(1) driving variable leaves implicit the determination of real output and the role of monetary policy in influencing output and inflation.

In this paper we extend the BLE concept to a general n-dimensional linear stochastic framework and provide a method to estimate such models under BLE. As an application we consider the standard 3-equation dynamic stochastic general equilibrium (DSGE) model-the New Keynesian model-and study the empirical fit of the model and the role of monetary policy under BLE. Agents are boundedly rational and they do not know the exact form of the actual law of motion because of cognitive limitations, or they simply prefer a parsimonious prediction rule. Agents' perceived law of motion (PLM) is a simple univariate AR(1) process for each variable to be forecasted. The same consistency requirements are imposed upon BLE to pin down the parameters of the forecasting model: for each endogenous variable observed sample averages and first-order sample autocorrelations match the corresponding parameters of the forecasting rule. Agents thus learn the optimal AR(1) forecasting rule for each endogenous variable in the economy.

Numerous empirical studies show that overly parsimonious models with little parameter uncertainty can provide better forecasts than models consistent with the actual data-generating complex process (e.g. Nelson, 1972; Stock and Watson, 2007; Clark and West, 2007; Enders, 2010). In a similar vein (but without analytical results) Slobodyan and Wouters (2012) study a New Keynesian DSGE model with agents using an AR(2) forecasting rule. Chung and Xiao (2014) and Xiao and Xu (2014) study learning and predictions with an AR(1) or VAR(1) model in a two dimensional New Keynesian model with limited information and show, based on simulations, that the simple AR(1) model is more likely to prevail in reality when they make predictions. Laboratory experiments in the NK framework also show that simple forecasting rules such as AR(1) describe individual forecasting behavior surprisingly well (Assenza et al., 2014; Pfajfar and Zakelj, 2016).

Our paper bears clear resemblance to the seminal work of Krusell and Smith (1998), where the behavior of macroeconomic aggregates can be almost perfectly described by the mean of the wealth distribution and the aggregate productivity shock. In our BLE agents are boundedly rational and forecast macro variables by learning the mean and the first-order autocorrelations of all observable endogenous macro variables. Such "irrationality" is hard to detect however, as agents learn to forecast correctly the mean and the first-order persistence of each variable in the economy. In fact, for each endogenous variable the optimal univariate rule satisfies the orthogonality condition for rational expectations.

The main contributions of our paper are fourfold: (1) existence and stability conditions of BLE in a general linear framework, (2) a general framework to find and estimate a BLE, (3) empirical validation and comparison of BLE based on historical U.S. data, (4) optimal monetary policy analysis under BLE.

Many models of learning lead to excess volatility, that is, volatility under learning is typically much higher than under REE. Our BLE model exhibits another novel feature, *persistence amplification*: the persistence of inflation and output gap under BLE is significantly higher than under REE. In fact, even when autocorrelations of the exogenous shocks to fundamentals are small, inflation and output gap along BLE are typically near unit root processes. As a consequence, estimating the New Keynesian model under BLE leads to substantial differences in the model structure compared with the REE. We further analyze optimal monetary policy under BLE for calibrated and estimated parameter values. We find finite optimal Taylor rule coefficients under a wide range of calibrations, which is different from the REE case. Further, we find that transmission channel of monetary policy is stronger under BLE at the estimated parameters.

Related literature

The issue of persistence has been of great interest to macroeconomists and policy-makers. A number of models of frictions have been proposed to replicate persistence, such as habit formation in consumption, indexation to lagged inflation in price-setting, rule-of-thumb behavior, or various adjustment costs (Phelps, 1968; Taylor, 1980; Fuhrer and Moore, 1992, 1995; Christiano et al., 2005; Smets and Wouters, 2003, 2005; Boivin and Giannoni, 2006; Giannoni and Woodford, 2003). These papers essentially improve the empirical fit by adding lags in the model equations. Estimating these rich models with frictions under the assumption of rational expectations one typically finds that substantial degrees of habit persistence and inflation indexation are supported by the data. Those additional sources of persistence appear, therefore, necessary to match the inertia of macroeconomic variables. These estimations also typically involve highly persistent structural shocks. Our BLE model is applied to a New Keynesian framework without habit formation or indexation, but nevertheless exhibits strong persistence. Learning causes persistence amplification: small autocorrelations of exogenous shocks are strongly amplified as agents learn to coordinate on a simple AR(1) forecasting rule with near unit root parameters consistent with observed sample average and sample autocorrelations. The high persistence of inflation and output thus arises from a self-fulfilling mistake (Grandmont, 1998).

Our BLE concept fits with the literature employing adaptive learning to analyze the evolution of U.S. inflation and monetary policy. Adaptive learning can help in understanding some particular historical episodes, such as high inflation in the 1980s, which are often harder to explain under rational expectations. For example, Orphanides and Williams (2003) consider a form of imperfect knowledge in which economic agents rely on adaptive learning to form expectations. This form of learning represents a relatively modest deviation from rational expectations that nests it as a limiting case. They find that policies that would be efficient under rational expectations can perform poorly when knowledge is imperfect. Milani (2005, 2007) also assumes that agents form expectations through adaptive learning using correctly specified economic models and updating the parameters through constant-gain learning (CGL) based on historical data. He shows empirically that when learning replaces rational expectations, the estimated degrees of habits and indexation drop closer to zero, suggesting that persistence arises in the model economy mainly from expectations and learning. Eusepi and Preston (2011) study expectations-driven business cycles based on learning, and find that learning dynamics generate forecast errors similar to the Survey of Professional Forecasters. Estrella and Fuhrer (2002) study the shortcomings of REE models with a focus on inertia and shock propagation structure. Fuhrer (2009) provides a good survey on inflation persistence. He examines a number of empirical measures of reduced form persistence including the first-order autocorrelation and the autocorrelation function of the inflation series. He also investigates the sources of persistence, including learning of agents in a rational- expectation setting.

Our behavioral learning equilibrium concept is closely related to the Exuberance Equilibria (EE) in Bullard et al. (2008), where agents' perceived law of motion is misspecified. However, because of difficulty of computation, in Bullard et al. (2008) there are only numerical results on the exuberance equilibria, while here we analytically show the existence of BLE, its stability under learning and the persistence amplification in a general linear framework with application to the New Keynesian model. Another related misspecification equilibrium is Limited Information Learning Equilibrium (LILE) defined in Chung and Xiao (2014), which is defined by the least-squares projection of variables on the past information of the actual law of motion equal to that in the perceived law of motion. Different from the LILE, our general Behavioral Learning Equilibrium is defined by the conditions that sample means and first-order autocorrelations of each variable of the actual law of motion are consistent with those corresponding to the perceived law of motion. We further study the effects of monetary policy under the more plausible BLE. The concept of natural expectations in Fuster et al. (2010) and Fuster et al. (2011, 2012) is another misspecification concept, where agents use simple, misspecified models, e.g., linear autoregressive models. Natural expectations, however, do not pin down the parameters of the forecasting model through consistency requirements as for a restricted perceptions equilibrium nor do they allow the agents to learn an optimal misspecified model through empirical observations. Cho and Kasa (2015) study model validation in an environment where agents are aware of misspecification and try to detect it through adaptive learning. In our BLE misspecification is self-fulfilling and the outcome of the SAC-learning process.

The paper is organized as follows. Section 2 introduces the main concepts of BLE in a general n-dimensional setup, the results on existence and stability of BLE in a linear framework, and the estimation methodology. Section 3 applies BLE to the 3-equation New Keynesian model and studies the existence, stability and estimation results under BLE. Section 5 studies optimal monetary policy and how policy can mitigate persistence and volatility amplification under BLE. Finally, Section 5 concludes.

2 BLE in a Multivariate Framework

Hommes and Zhu (2014) introduced BLE in the simplest setting, a one-dimensional linear stochastic model driven by an exogenous linear stochastic AR(1) process. In this paper we generalize BLE to n-dimensional (linear) stochastic models driven by exogenous linear stochastic AR(1) processes of multiple shocks. To ease the exposition we initially follow the presentation in Hommes and Zhu (2014), but generalize their 1-dimensional model to a general n-dimensional framework. In addition, most macroeconomic models include features such as interest rate smoothing, habit formation in consumption or indexation in prices and wages, which introduces lagged state variables into the models. In this paper, we further extend our model with lagged state variables.

Let the law of motion of an economic system be given by the stochastic difference equation

$$\boldsymbol{x}_t \quad = \quad \boldsymbol{F}(\boldsymbol{x}^e_{t+1},\ \boldsymbol{x}_{t-1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t), \tag{2.1}$$

where $\boldsymbol{x}_t$ is an $n \times 1$ vector of endogenous variables denoted by $[x_{1t}, x_{2t}, \cdots, x_{nt}]'$ and $\boldsymbol{x}^e_{t+1}$ is the expected value of $\boldsymbol{x}$ at date $t+1$. This notation highlights that expectations may not be rational. Here $\boldsymbol{F}$ is a continuous n-dimensional vector function, $\boldsymbol{u}_t$ is a vector of exogenous stationary variables and $\boldsymbol{v}_t$ is a vector of white noise disturbances.

Agents are boundedly rational and do not know the exact form of the actual law of motion (2.1). They only use a simple, parsimonious forecasting model where agents' perceived law of motion (PLM) is a simple univariate AR(1) process for each variable to be forecasted. As shown in Enders (2010, p.84-85), coefficient uncertainty increases as the model becomes more complex, and hence it could be that an estimated AR(1) model forecasts a real ARMA(2,1) process better than an estimated ARMA(2,1) model. Numerous empirical studies also show that overly parsimonious models with little parameters uncertainty can provide better forecasts than models consistent with the actual data-generating complex process (e.g. Nelson, 1972; Stock and Watson, 2007; Clark and West, 2007). Thus agents' perceived law of motion (PLM) is assumed to be the simplest VAR model with minimum parameters, i.e. a restricted VAR(1) process

$$\boldsymbol{x}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}) + \boldsymbol{\delta}_t, \tag{2.2}$$

where $\boldsymbol{\alpha}$ is a vector denoted by $[\alpha_1, \alpha_2, \cdots, \alpha_n]'$, $\boldsymbol{\beta}$ is a diagonal matrix[2] denoted by $\begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & \beta_n \end{bmatrix}$ with $\beta_i \in (-1, 1)$ and $\{\boldsymbol{\delta}_t\}$ is a white noise process; $\boldsymbol{\alpha}$ is the unconditional mean of $\boldsymbol{x}_t$ and β_i is the first-order correlation coefficient of variable x_i. Given the perceived law of motion (2.2), the 2-period ahead forecasting rule for $\boldsymbol{x}_{t+1}$ that minimizes the mean-squared forecasting error is

$$\boldsymbol{x}_{t+1}^e = \boldsymbol{\alpha} + \boldsymbol{\beta}^{\mathbf{2}}(\boldsymbol{x}_{\boldsymbol{t-1}} - \boldsymbol{\alpha}). \tag{2.3}$$

Combining the expectations (2.3) and the law of motion of the economy (2.1), we obtain the implied actual law of motion (ALM)

$$\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}),\ \boldsymbol{x}_{t-1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t). \tag{2.4}$$

In the case that the ALM (2.4) is stationary, suppose the variance-covariance matrix $\boldsymbol{\Gamma}(0) := E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})']$ and the first order covariance matrix $\boldsymbol{\Gamma}(1) := E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{x}_{t+1} - \overline{\boldsymbol{x}})']$, where $\overline{\boldsymbol{x}}$ is the mean of $\boldsymbol{x}_t$. Let $\boldsymbol{\Omega}$ be the diagonal matrix in which the ith diagonal element is the variance of the ith process, that is $\boldsymbol{\Omega} = \mathrm{diag}[\gamma_{11}(0), \gamma_{22}(0), \cdots, \gamma_{nn}(0)]$, where $\gamma_{ii}(0)$ is the ith diagonal entry of $\boldsymbol{\Gamma}(0)$. Let $\boldsymbol{L}$ be the diagonal matrix in which the ith diagonal element is the first-order autocovariance of the ith process, that is $\boldsymbol{L} = \mathrm{diag}[\gamma_{11}(1), \gamma_{22}(1), \cdots, \gamma_{nn}(1)]$, where $\gamma_{ii}(1)$ is the ith diagonal entry of $\boldsymbol{\Gamma}(1)$. Let $\boldsymbol{G}$ denote the diagonal matrix in which the ith diagonal element is the first-order autocorrelation coefficient of the ith process $x_{i,t}$. Hence

$$\boldsymbol{G} = \boldsymbol{L}\boldsymbol{\Omega}^{-1}. \tag{2.5}$$

Behavioral Learning Equilibrium (BLE)

Extending Hommes and Zhu (2014), the concept of BLE is generalized as follows.

[2]Chung and Xiao (2014) also argue using simulations that the simple AR(1) model is more likely to prevail in reality because of limited information restrictions when they model predictions in a two dimensional New Keynesian model. In addition, as far as prediction is concerned, based on our numerous empirical analyses, the short-term forecasts based on AR(1) model are better than more general VAR models in most cases, because in more general VAR models too many parameters need to be estimated and hence coefficient uncertainty increases. Of course, general VAR models have other advantages, such as studying impulse response of other shocks.

Definition 2.1 *A vector* $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$*, where* μ *is a probability measure,* $\boldsymbol{\alpha}$ *is a vector and* $\boldsymbol{\beta}$ *is a diagonal matrix with* $\beta_i \in (-1, 1)$ $(i = 1, 2, \cdots, n)$*, is called a behavioral learning equilibrium (BLE) if the following three conditions are satisfied:*

S1 The probability measure μ *is a nondegenerate invariant measure for the stochastic difference equation (2.4);*

S2 The stationary stochastic process defined by (2.4) with the invariant measure μ *has unconditional mean* $\boldsymbol{\alpha}$*, that is, the unconditional mean of* x_i *is* α_i*,* $(i = 1, 2, \cdots, n)$*;*

S3 Each element x_i *for the stationary stochastic process of* $\boldsymbol{x}$ *defined by (2.4) with the invariant measure* μ *has unconditional first-order autocorrelation coefficient* β_i*,* $(i = 1, 2, \cdots, n)$*, that is,* $\boldsymbol{G} = \boldsymbol{\beta}$*.*

That is to say, a BLE is characterized by two natural observable consistency requirements: the unconditional means and the unconditional first-order autocorrelation coefficients generated by the actual (unknown) stochastic process (2.4) coincide with the corresponding statistics for the perceived linear VAR(1) process (2.2), as given by the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This means that in a BLE agents correctly perceive the two simplest and most important statistics: the mean and first-order autocorrelation (i.e., persistence) of each relevant variable of the economy, without fully understanding its structure and recognizing all explanatory variables and cross correlations. A BLE is parameter free, as along a BLE the two parameters of each linear forecasting rule are pinned down by simple and observable statistics. Hence, agents do not fully understand the linear structure of the stochastic economy, e.g. they do not take the cross-correlation of state variables into account, but rather use a parsimonious univariate AR(1) forecasting rule for each state variable. A simple BLE may be a plausible outcome of the coordination process of expectations of a large population. Laboratory experiments within the New Keynesian framework also provide empirical evidence of the use of simple univariate AR(1) forecasting rules to forecast inflation and output gap (Adam, 2007; Pfajfar and Zakelj, 2016; Assenza et al., 2014).

Furthermore, we note that along a BLE the orthogonality condition

$$
\begin{aligned}
&E[x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)] = 0, \\
&E\{[x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)]x_{i,t-1}\} = E\{[x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)](x_{i,t-1} - \alpha_i)\} = 0
\end{aligned}
$$

is satisfied. That is, the forecast $\alpha_i + \beta_i(x_{i,t-1} - \alpha_i)$ is the linear projection of $x_{i,t}$ on the vector $(1, x_{i,t-1})'$. For each variable, agents cannot detect the correlation between the forecasting error $x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)$ and the vector $(1, x_{i,t-1})'$ in the forecast model. The linear projection produces the smallest mean squared error among the class of linear forecasting rules (e.g., Hamilton (1994). Therefore, for each variable agents use the optimal forecast within their class of univariate AR(1) forecasting rules (Branch, 2006).

Sample autocorrelation learning

In the above definition of BLE, agents' beliefs are described by the linear forecasting rule (2.3) with fixed parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. However, the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are usually unknown to agents. In the adaptive learning literature, it is common to assume that agents behave like econometricians using time series observations to estimate the parameters as additional observations become available. Following Hommes and Sorger (1998), we assume that agents use sample autocorrelation learning (SAC-learning) to learn the parameters α_i and β_i, $i = 1, 2, \cdots, n$. That is, for any finite set of observations $\{x_{i,0}, x_{i,1}, \cdots, x_{i,t}\}$, the sample average is given by

$$\alpha_{i,t} = \frac{1}{t+1} \sum_{k=0}^{t} x_{i,k}, \tag{2.6}$$

and the first-order sample autocorrelation coefficient is given by

$$\beta_{i,t} = \frac{\sum_{k=0}^{t-1}(x_{i,k} - \alpha_{i,t})(x_{i,k+1} - \alpha_{i,t})}{\sum_{k=0}^{t}(x_{i,k} - \alpha_{i,t})^2}. \tag{2.7}$$

Hence $\alpha_{i,t}$ and $\beta_{i,t}$ are updated over time as new information arrives. It is easy to check that, independently of the choice of the initial values $(x_{i,0}, \alpha_{i,0}, \beta_{i,0})$, it always holds that $\beta_{i,1} = -\frac{1}{2}$, and that the first-order sample autocorrelation $\beta_{i,t} \in [-1, 1]$ for all $t \geq 1$.

As shown in Hommes and Zhu (2014), define

$$R_{i,t} = \frac{1}{t+1} \sum_{k=0}^{t} (x_{i,k} - \alpha_{i,t})^2,$$

then the SAC-learning is equivalent to the following recursive dynamical system[3]

$$\begin{cases} \alpha_{i,t} = \alpha_{i,t-1} + \dfrac{1}{t+1}(x_{i,t} - \alpha_{i,t-1}), \\ \beta_{i,t} = \beta_{i,t-1} + \dfrac{1}{t+1} R_{i,t}^{-1} \Big[(x_{i,t} - \alpha_{i,t-1}) \big(x_{i,t-1} + \dfrac{x_{i,0}}{t+1} - \dfrac{t^2+3t+1}{(t+1)^2}\alpha_{i,t-1} - \dfrac{1}{(t+1)^2} x_{i,t} \big) \\ \qquad - \dfrac{t}{t+1}\beta_{i,t-1}(x_{i,t} - \alpha_{i,t-1})^2 \Big], \\ R_{i,t} = R_{i,t-1} + \dfrac{1}{t+1}\Big[\dfrac{t}{t+1}(x_{i,t} - \alpha_{i,t-1})^2 - R_{i,t-1} \Big]. \end{cases} \tag{2.8}$$

The actual law of motion under SAC-learning is therefore given by

$$\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1}),\ \boldsymbol{x}_{t-1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t), \tag{2.9}$$

with $\alpha_{i,t}$, $\beta_{i,t}$ as in (2.8).

In Hommes and Zhu (2014), F is a one-dimensional linear function. In this paper $\boldsymbol{F}$ may be a general n-dimensional linear vector function and includes the lagged term $\boldsymbol{x}_{t-1}$.

2.1 Main results in a multivariate linear framework

Assume that a reduced form model is an n-dimensional linear stochastic process $\boldsymbol{x}_t$, driven by an exogenous VAR(1) process $\boldsymbol{u}_t$. More precisely, the actual law of motion of the economy is given by[4]

$$\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{x}_{t+1}^e,\ \boldsymbol{u}_t,\ \boldsymbol{v}_t) = \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{x}_{t+1}^e + \boldsymbol{b}_2\boldsymbol{x}_{t-1} + \boldsymbol{b}_3\boldsymbol{u}_t + \boldsymbol{b}_4\boldsymbol{v}_t, \tag{2.10}$$

$$\boldsymbol{u}_t = \boldsymbol{a} + \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{2.11}$$

where $\boldsymbol{x}_t$ is an $n \times 1$ vector of endogenous variables, $\boldsymbol{b}_0$ and $\boldsymbol{a}$ are vectors of constants, $\boldsymbol{b}_1, \boldsymbol{b}_2$ and $\boldsymbol{b}_4$ are $n \times n$ matrices of coefficients, $\boldsymbol{b}_3$ is an $n \times m$ matrix, $\boldsymbol{\rho}$ is an $m \times m$ matrix, $\boldsymbol{u}_t$ is an $m \times 1$ vector of exogenous variables which is assumed to follow a stationary VAR(1) as

[3] The system in (2.8) is a decreasing gain algorithm, where all observations receive equal weight and therefore the weight on the latest observation decreases as the sample size grows. There is also a constant gain correspondence of SAC-learning, where past observations are discounted at a geometric rate. This can be obtained the replacing the weights $\frac{1}{t+1}$ by some positive constant κ, see the online appendix to Hommes & Zhu (2014) for further details.

[4] As shown in Section 2.2 Our results on BLE still hold for the more general model including the term of lagged $\boldsymbol{x}_{t-1}$ in the RHS of Eq. (2.10).

shown in (2.11), and $\boldsymbol{v}_t$ is an $n \times 1$ vector of i.i.d. stochastic disturbance terms with mean zero and finite absolute moments, with variance-covariance matrix $\boldsymbol{\Sigma_v}$. That is, here all of the eigenvalues of $\boldsymbol{\rho}$ are assumed to be inside the unit circle. In addition, $\boldsymbol{\varepsilon}_t$ is assumed to be an $m \times 1$ vector of i.i.d. stochastic disturbance terms with mean zero and finite absolute moments, with variance-covariance matrix $\boldsymbol{\Sigma_\varepsilon}$ and is independent of $\boldsymbol{v}_t$.

Rational expectations equilibrium

Under the assumption that agents are rational, assume the perceived law of motion (PLM) corresponding to the minimum state variable REE of the model

$$\boldsymbol{x}_t^* = \boldsymbol{c}_0 + \boldsymbol{c}_1 \boldsymbol{x}_{t-1}^* + \boldsymbol{c}_2 \boldsymbol{u}_t + \boldsymbol{c}_3 \boldsymbol{v}_t. \tag{2.12}$$

Assuming that shocks $\boldsymbol{u}_t$ are observable when forecasting $\boldsymbol{x}_{t+1}$, the one-step ahead forecast is

$$E_t \boldsymbol{x}_{t+1}^* = \boldsymbol{c}_0 + \boldsymbol{c}_2 \boldsymbol{a} + \boldsymbol{c}_1 \boldsymbol{x}_t^* + \boldsymbol{c}_2 \boldsymbol{\rho} \boldsymbol{u}_t, \tag{2.13}$$

and the corresponding actual law of motion is

$$\boldsymbol{x}_t^* = \boldsymbol{b}_0 + \boldsymbol{b}_1(\boldsymbol{c}_0 + \boldsymbol{c}_2 \boldsymbol{a} + \boldsymbol{c}_1 \boldsymbol{x}_t^* + \boldsymbol{c}_2 \boldsymbol{\rho} \boldsymbol{u}_t) + \boldsymbol{b}_2 \boldsymbol{x}_{t-1} + \boldsymbol{b}_3 \boldsymbol{u}_t + \boldsymbol{b}_4 \boldsymbol{v}_t. \tag{2.14}$$

The rational expectations equilibrium (REE) is the fixed point of

$$\begin{aligned} \boldsymbol{c}_0 - \boldsymbol{b}_1 \boldsymbol{c}_1 \boldsymbol{c}_0 - \boldsymbol{b}_1 \boldsymbol{c}_0 &= \boldsymbol{b}_0 + \boldsymbol{b}_1 \boldsymbol{c}_2 \boldsymbol{a}, & (2.15)\\ \boldsymbol{c}_1 - \boldsymbol{b}_1 \boldsymbol{c}_1^2 &= \boldsymbol{b}_2, & (2.16)\\ \boldsymbol{c}_2 - \boldsymbol{b}_1 \boldsymbol{c}_1 \boldsymbol{c}_2 - \boldsymbol{b}_1 \boldsymbol{c}_2 \boldsymbol{\rho} &= \boldsymbol{b}_3, & (2.17)\\ \boldsymbol{c}_3 - \boldsymbol{b}_1 \boldsymbol{c}_1 \boldsymbol{c}_3 &= \boldsymbol{b}_4. & (2.18) \end{aligned}$$

A straightforward computation (see Appendix A) shows that the mean of the REE $\overline{\boldsymbol{x}^*}$ satisfies

$$\overline{\boldsymbol{x}^*} = (\boldsymbol{I} - \boldsymbol{b}_1 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_3 \boldsymbol{(I - \rho)^{-1}} \boldsymbol{a}], \tag{2.19}$$

where $\boldsymbol{I}$ denotes a comfortable identity matrix throughout the paper.

In the special case $\boldsymbol{\rho} = \rho \boldsymbol{I}$ [5] and $\boldsymbol{b}_2 = \boldsymbol{0}$, the rational expectation equilibrium $\boldsymbol{x}_t^*$ satisfies

$$\boldsymbol{x}_t^* = (\boldsymbol{I} - \boldsymbol{b}_1)^{-1} \boldsymbol{b}_0 + (\boldsymbol{I} - \boldsymbol{b}_1)^{-1} \boldsymbol{b}_1 (\boldsymbol{I} - \rho \boldsymbol{b}_1)^{-1} \boldsymbol{b}_3 \boldsymbol{a} + (\boldsymbol{I} - \rho \boldsymbol{b}_1)^{-1} \boldsymbol{b}_3 \boldsymbol{u}_t + \boldsymbol{b}_4 \boldsymbol{v}_t. \tag{2.20}$$

[5] Note that $\boldsymbol{\rho}$ is a matrix while ρ is a scalar number throughout the paper.

Thus its unconditional mean is

$$\overline{\boldsymbol{x}^*} = E(\boldsymbol{x}_t^*) = (1-\rho)^{-1}(\boldsymbol{I}-\boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0(1-\rho)+\boldsymbol{b}_3\boldsymbol{a}]. \tag{2.21}$$

Its variance-covariance matrix is

$$\boldsymbol{\Sigma}_{\boldsymbol{x}^*} = E[(\boldsymbol{x}_t^*-\overline{\boldsymbol{x}^*})(\boldsymbol{x}_t^*-\overline{\boldsymbol{x}^*})'] = (1-\rho^2)^{-1}(\boldsymbol{I}-\rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}[(\boldsymbol{I}-\rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3]' + \boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4' \tag{2.22}$$

Furthermore, the first-order autocovariance is,

$$\boldsymbol{\Sigma}_{\boldsymbol{x}^*\boldsymbol{x}^*_{-1}} = E[(\boldsymbol{x}_t^*-\overline{\boldsymbol{x}^*})(\boldsymbol{x}_{t-1}^*-\overline{\boldsymbol{x}^*})'] = \rho(1-\rho^2)^{-1}(\boldsymbol{I}-\rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}[(\boldsymbol{I}-\rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3]'. \tag{2.23}$$

The first-order autocorrelation of the i-element x_i^* of $\boldsymbol{x}^*$ is the i-th diagonal element of matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}^*\boldsymbol{x}^*_{-1}}$ divided by the corresponding i-th diagonal element of matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}^*}$. Furthermore, if $\boldsymbol{\Sigma}_{\boldsymbol{v}} = \mathbf{0}$, then the first-order autocorrelation of the i-element u_i of $\boldsymbol{u}$ is equal to ρ. In this case the persistence of the i-th variable x_i^* in the REE coincides exactly with the persistence of the exogenous driving force $u_{i,t}$. That is, in this case the persistence in the REE only inherits the persistence of the exogenous driving force.

Existence of BLE

Now assume that agents are boundedly rational and do not believe or do not recognize that the economy is driven by an exogenous VAR(1) process $\boldsymbol{u}_t$, but use a simple univariate linear rule to forecast the state $\boldsymbol{x}_t$ of the economy. Given that agents' perceived law of motion is a special VAR(1) process as shown in (2.2), the actual law of motion becomes

$$\boldsymbol{x}_t = \boldsymbol{b}_0 + \boldsymbol{b}_1[\boldsymbol{\alpha}+\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\boldsymbol{\alpha})] + \boldsymbol{b}_2\boldsymbol{x}_{t-1} + \boldsymbol{b}_3\boldsymbol{u}_t + \boldsymbol{b}_4\boldsymbol{v}_t, \tag{2.24}$$

with $\boldsymbol{u}_t$ given in (2.11). If all the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2$ for each $\beta_i \in [-1,1] (i = 1, 2, \cdots, n)$ lie inside the unit circle, then the system of $\boldsymbol{x}_t$ is stationary and hence its mean $\overline{\boldsymbol{x}}$ and first-order autocorrelation $\boldsymbol{G}$ exist.

The mean of $\boldsymbol{x}_t$ in (2.24) is computed as

$$\overline{\boldsymbol{x}} = (\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{\beta}^2-\boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0+\boldsymbol{b}_1\boldsymbol{\alpha}-\boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\alpha}+\boldsymbol{b}_3(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}]. \tag{2.25}$$

Imposing the first consistency requirement of a BLE on the mean, i.e. $\overline{\boldsymbol{x}} = \boldsymbol{\alpha}$, and solving for $\boldsymbol{\alpha}$ yields

$$\boldsymbol{\alpha}^* = (\boldsymbol{I}-\boldsymbol{b}_1-\boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0+\boldsymbol{b}_3(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}]. \tag{2.26}$$

Comparing with (2.19), we conclude that in a BLE the unconditional mean $\boldsymbol{\alpha}^*$ coincides with the REE mean. That is to say, in a BLE the state of the economy $\boldsymbol{x}_t$ fluctuates on average around its RE fundamental value $\boldsymbol{x}^*$.

Consider the second consistency requirement of a BLE on the first-order autocorrelation coefficient matrix $\boldsymbol{\beta}$ of the PLM. The second consistency requirement yields

$$\boldsymbol{G}(\boldsymbol{\beta}) = \boldsymbol{\beta}. \tag{2.27}$$

Recall from Section 2, both $\boldsymbol{G}$ and $\boldsymbol{\beta}$ are diagonal matrices. For convenience let G_i denote the i-th diagonal element of the matrix $\boldsymbol{G}$ in (2.5). Under the assumption that all of the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2$ for each $\beta_i \in [-1,1] (i = 1, 2, \cdots, n)$ lie inside the unit circle, from the theory of stationary linear time series, $G_i(\beta_1, \beta_2, \cdots, \beta_n) \in [-1, 1]$ and is a smooth function with respect to $(\beta_1, \beta_2, \cdots, \beta_n)$ and other model parameters, see Appendix B[6]. Based on Brouwer's fixed-point theorem for $(G_1, G_2, \cdots, G_n)$, there exists $\boldsymbol{\beta}^* = (\beta_1^*, \beta_2^2, \cdots, \beta_n^*)$ with each $\beta_i^* \in [-1, 1]$, such that $\boldsymbol{G}(\boldsymbol{\beta^*}) = \boldsymbol{\beta^*}$. We conclude:

Proposition 1 *If all the eigenvalues of* $\boldsymbol{\rho}$ *and* $\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2$ *for each* $\beta_i \in [-1, 1]$ *are inside the unit circle*[7]*, there exists at least one behavioral learning equilibrium* $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ *for the economic system (2.24) with* $\boldsymbol{\alpha}^* = (\boldsymbol{I} - \boldsymbol{b}_1 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_3(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}] = \overline{\boldsymbol{x}^*}$.

Stability under SAC-learning

In this subsection we study the stability of BLE under SAC-learning. The ALM of the economy under SAC-learning is given by

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{b}_0 + \boldsymbol{b}_1[\boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1})] + \boldsymbol{b}_2\boldsymbol{x}_{t-1} + \boldsymbol{b}_3\boldsymbol{u}_t + \boldsymbol{b}_4\boldsymbol{v}_t, \\ \boldsymbol{u}_t = \boldsymbol{a} + \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t. \end{cases} \tag{2.28}$$

with $\boldsymbol{\alpha}_t$, $\boldsymbol{\beta}_t$ updated based upon realized sample average and sample autocorrelation as in (2.8). Appendix C shows that the E-stability principle applies and that the stability under

[6]For example, refer to the expression (3.9) in Hommes and Zhu (2014) for the special 1-dimensional case $n = 1$ and $\boldsymbol{b}_2 = \boldsymbol{0}$. In Section 3 we consider the two-dimensional New Keynesian model and will compute the (complicated) expressions of $G_1(\beta_1, \beta_2)$ and $G_2(\beta_1, \beta_2)$ explicitly.

[7]The Schur-Cohn criterion theorem provides necessary and sufficient conditions for all eigenvalues to lie inside the unit circle, see Elaydi (1999). For specific models, one may find sufficient conditions to guarantee all eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2$ for each $\beta_i \in [-1, 1]$ are inside the unit circle, as shown below for the NK model.

SAC-learning is determined by the associated ordinary differential equation (ODE)[8]

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha},\boldsymbol{\beta}) - \boldsymbol{\alpha} = (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^2 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{\alpha} - \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\alpha} + \boldsymbol{b}_3(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}] - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{G}(\boldsymbol{\beta}) - \boldsymbol{\beta}, \end{cases} \tag{2.29}$$

where $\overline{\boldsymbol{x}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the mean given by (2.25) and $\boldsymbol{G}(\boldsymbol{\beta})$ is the diagonal first-order autocorrelation matrix. A BLE $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ corresponds to a fixed point of the ODE (2.29). Moreover, a BLE $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ is locally stable under SAC-learning, if it is a stable fixed point of the ODE (2.29). Therefore, we have the following property of SAC-learning stability.

Proposition 2 *A BLE $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ is locally stable (E-stable) under SAC-learning if*

(i) all the eigenvalues of $(\boldsymbol{I} - \boldsymbol{b}_1{\boldsymbol{\beta}^*}^2 - \boldsymbol{b}_2)^{-1}(\boldsymbol{b}_1 + \boldsymbol{b}_2 - \boldsymbol{I})$ have negative real parts[9]*, and*

(ii) all the eigenvalues of $\boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)$ have real parts less than 1, where $\boldsymbol{DG}_{\boldsymbol{\beta}}$ is the Jacobian matrix with the (i,j)-th entry $\frac{\partial G_i}{\partial \beta_j}$.

Proof. See Appendix C.

Recall from Subsection 2.1 that $G_i(\beta_1,\beta_2,\cdots,\beta_n) \in (-1,1)$ so that at least one BLE exists. The proposition above implies that the BLE may also be E-stable under SAC-learning. In the next section, we study BLE in a 3-equation New Keynesian model.

2.2 Estimating a BLE

As our application in the next section will illustrate, finding an analytical expression for a BLE is in general not possible. Therefore we provide a general method to approximate and estimate locally stable (E-stable) BLE under the system in (2.10) and (2.11). We first augment $\boldsymbol{x}_t$ with $\boldsymbol{u}_t$ to obtain

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{b}_3 \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{u}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_0 \\ \boldsymbol{a} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\rho} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{u}_{t-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^e_{t+1} \\ \boldsymbol{u}^e_{t+1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_4 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_t \\ \boldsymbol{\epsilon}_t \end{bmatrix} \tag{2.30}$$

If we define[10]

$$\begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{u}_t \end{bmatrix} = S_t, \quad \begin{bmatrix} \boldsymbol{v}_t \\ \boldsymbol{\epsilon}_t \end{bmatrix} = \eta_t, \quad \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{b}_3 \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} = \tilde{\gamma}, \tilde{\gamma}^{-1} \begin{bmatrix} \boldsymbol{b}_0 \\ \boldsymbol{a} \end{bmatrix} = \bar{\gamma}, \quad \tilde{\gamma}^{-1} \begin{bmatrix} \boldsymbol{b}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\rho} \end{bmatrix} = \gamma_1,$$

[8] See Evans and Honkapohja (2001) for discussion and a mathematical treatment of E-stability.

[9] The Routh-Hurwitz criterion theorem provides sufficient and necessary conditions for all the n eigenvalues having negative real parts, see Brock and Malliaris (1989).

[10] We assume the invertibility conditions of the corresponding matrices are satisfied throughout the paper.

$$\tilde{\gamma}^{-1}\begin{bmatrix}\boldsymbol{b}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \gamma_2,\ \ \tilde{\gamma}^{-1}\begin{bmatrix}\boldsymbol{b}_4 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}\end{bmatrix} = \gamma_3,$$

then we can re-write the law of motion as

$$S_t = \bar{\gamma} + \gamma_1 S_{t-1} + \gamma_2 S^e_{t+1} + \gamma_3 \eta_t, \tag{2.31}$$

Denote the learning vector for the mean parameters $\boldsymbol{\alpha}$, and the diagonal learning matrix for the first-order autocorrelation parameters $\boldsymbol{\beta}$, with the corresponding learning parameters by α_x and β_x for each endogenous variable x_t as before[11]. Then the agent's PLM, the corresponding expectations and the implied ALM are given as

$$\begin{cases} S_t = \boldsymbol{\alpha} + \boldsymbol{\beta}(S_{t-1} - \boldsymbol{\alpha}) + v_t \\ E_t S_{t+1} = \boldsymbol{\alpha} + \boldsymbol{\beta}^2(S_{t-1} - \boldsymbol{\alpha}) \\ S_t = (\bar{\gamma} + \boldsymbol{\alpha} - \boldsymbol{\beta}^2\boldsymbol{\alpha}) + \gamma_1 S_{t-1} + \gamma_2 \boldsymbol{\beta}^2 S_{t-1} + \gamma_3 \eta_t \end{cases} \tag{2.32}$$

Our main focus in this section is to estimate log-linearized DSGE models, where the mean $\boldsymbol{\alpha^*}$ is available based on (2.26). Without loss of generality, we focus on the special case where $\boldsymbol{\alpha^*} = 0$ for the remainder. Denoting by $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$ the variance-covariance and first-order covariance matrices as before, one can show that[12]

$$\begin{cases} Vec(\boldsymbol{\Gamma}(0)) = [I - M(\boldsymbol{\beta^*}) \otimes M(\boldsymbol{\beta^*})]^{-1}(\gamma_3 \otimes \gamma_3)Vec(\boldsymbol{\Sigma_\eta}) \\ Vec(\boldsymbol{\Gamma}(1)) = [I \otimes M(\boldsymbol{\beta^*})]Vec(\boldsymbol{\Gamma}(0)) \end{cases} \tag{2.33}$$

where $M(\boldsymbol{\beta}^*) = \gamma_1 + \gamma_2\boldsymbol{\beta}^{*2}$, and $\boldsymbol{\Sigma_\eta}$ is the variance-covariance matrix of i.i.d disturbances η_t. This implies, $\forall j = \{1, ..., N\}, \beta_j{}^* = \frac{Vec(\boldsymbol{\Gamma}(1))_{N(j-1)+j}}{Vec(\boldsymbol{\Gamma}(0))_{N(j-1)+j}} = G_j(\boldsymbol{\beta}^*, \theta)$, where θ represents the set of structural parameters in γ_1, γ_2 and γ_3. Then every BLE satisfies the following:

$$\begin{cases} S_t = \gamma_1 S_{t-1} + \gamma_2 \boldsymbol{\beta}^{*2} S_{t-1} + \gamma_3 \eta_t \\ \forall j = \{1, ..., N\}, \beta^*_j = \frac{Vec(\boldsymbol{\Gamma_1})_{N(j-1)+j}}{Vec(\boldsymbol{\Gamma_0})_{N(j-1)+j}} = G_j(\boldsymbol{\beta}^*, \theta) \end{cases} \tag{2.34}$$

The E-stability conditions of Proposition 2 are easily extended to this special case with zero mean. Accordingly, a BLE ($\mathbf{0}$, $\boldsymbol{\beta}^*$) is locally stable if all eigenvalues of $(I - \gamma_1 - \gamma_2\boldsymbol{\beta}^{*2})(\gamma_1 + \gamma_2 - I)$ have negative real parts, and all eigenvalues of $DG_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)$ have real parts less than one. Note that the first term governs the stability of mean coefficients, while the second one relates to stability of first-order autocorrelation coefficients.

[11] Without loss of generality, we assume the first N variables in S_t constitute the vector of forward-looking variables, and we introduce zeros for the remaining state variables and exogenous shocks.

[12] See Appendix A.

Iterative E-stability and Estimation of BLE

The first-order autocorrelation coefficients $\boldsymbol{\beta}^*$ in (2.34) are functions in terms of the structural parameters θ, which cannot be computed analytically. In order to find a BLE, we use a simple fixed-point iteration, which is formalized below in Algorithm I.

Algorithm I: Finding a BLE using Iterative E-stability

Denote by θ the set of structural parameters, and by $G(\boldsymbol{\beta^{(k)}}, \theta)$ the first-order autocorrelation function for a given θ.

- Step (0): Initialize the vector of learning parameters at $\boldsymbol{\beta^{(0)}}$.
- Step (I): At each iteration k, using the first-order autocorrelation function, update the vector of learning parameters as follows:

$$\boldsymbol{\beta^{(k)}} = G(\boldsymbol{\beta^{(k-1)}}) \tag{2.35}$$

  where $G(\boldsymbol{\beta^{(k-1)}})$ is known from step $k-1$.
- Step (II): Terminate if $||\boldsymbol{\beta^{(k)}} - (\boldsymbol{\beta^{(k-1)}})||_p < c$ for a small scalar $c > 0$[13] and a suitable norm distance $||.||_p$, otherwise repeat Step (I).

The iteration function plays an important role in Algorithm I, where our choice of the function $G(.)$ reduces it to the simplest fixed-point iteration. The convergence of this iteration is known as **Iterative E-stability** in the adaptive learning literature (Evans & Honkapohja, 2001). Defining $F(\boldsymbol{\beta}) = \boldsymbol{\beta} - G(\boldsymbol{\beta})$, one could also consider a Quasi-Newton iteration of the following type:

$$\boldsymbol{\beta^{(k)}} = \boldsymbol{\beta^{(k-1)}} + DF(\boldsymbol{\beta^{(k-1)}})^{-1} F(\boldsymbol{\beta^{(k-1)}}) \tag{2.36}$$

where $DF(\boldsymbol{\beta^{(k-1)}})$ denotes the Jacobian of $G(\boldsymbol{\beta})$ at iteration $k-1$[14]. This latter algorithm has been used in e.g. Farmer et. al. (2009) to compute MSV-solutions in Markov-switching models. We characterize the convergence properties of (2.35) and (2.36) below.

[13] Throughout the remainder of this paper, we use the common L^1-Norm as our norm distance, i.e. $||\boldsymbol{\beta^{(k)}} - \boldsymbol{\beta^{(k-1)}}||_p = \sum_{j=1}^{N} |\beta_j^{(k)} - \beta_j^{(k-1)}|$.

[14] We approximate the Jacobian using $\frac{\partial F_i(\boldsymbol{\beta^{(k)}})}{\partial \beta_j^{(k)}} \approx \frac{F_i(\boldsymbol{\beta^{(k)}} + h\vec{e_j}) - F_i(\boldsymbol{\beta^{(k)}})}{h}$, $\forall i, j \in \{1...N\}$, where $\vec{e_j}$ denotes a suitable unit vector.

Proposition 3 *If Algorithm I converges to a stationary law of motion for a given K, the corresponding $(\mathbf{0}, \boldsymbol{\beta}^{(\boldsymbol{K})}) = (\mathbf{0}, \boldsymbol{\beta}^{*})$ is a BLE. Furthermore, if all eigenvalues of $DG_{\boldsymbol{\beta}}(\boldsymbol{\beta}^{*})$ lie inside the unit circle, then $G(.)$ is a contraction on a neighbourhood $\hat{I}$ of $\boldsymbol{\beta}^{*}$ and for all $\boldsymbol{\beta}^{(\mathbf{0})} \in \hat{I}$, we have $lim_{k\to\infty} G^k(\boldsymbol{\beta}^{(\mathbf{0})}) = \boldsymbol{\beta}^{*}$. Then the resulting BLE is **iteratively E-stable**.*

Proof: See Appendix D.

Iterative E-stability replaces the ODE in (2.6) under E-stability with the difference equation in (2.35), and there is a simple connection between these two concepts: for E-stability, the real parts of all eigenvalues of $DG(\boldsymbol{\beta}^{*})$ must be less than one, while iterative E-stability requires the eigenvalues to lie inside the unit circle. This immediately implies that iterative E-stability is a stronger condition than E-stability. Therefore the fixed-point iteration (2.35) not only allows us to eliminate E-unstable BLE without additional steps, but it also serves as a potential equilibrium selection mechanism when multiple BLE exist. The next corollary formalizes this notion.

Corollary 1 *Let $\rho(G(\boldsymbol{\beta}^{*})) = max\{|\lambda| : \lambda \in \mathbb{C}$ an eigenvalue of $DG(\boldsymbol{\beta}^{*})\}$, and $\xi(G(\boldsymbol{\beta}^{*})) = max\{Re(\lambda) : \lambda \in \mathbb{C}$ an eigenvalue of $DG(\boldsymbol{\beta}^{*})\}$. Assume $\xi((I-(\gamma_1+\gamma_2\boldsymbol{\beta}^{*2}))(\gamma_1+\gamma_2-I)) < 0$ is satisfied*[15]*. Then the E-stability and iterative E-stability conditions jointly reduce to $\rho(DG_{\boldsymbol{\beta}^{*}}(\boldsymbol{\beta}^{*})) < 1$.*

It follows that:

(i) *E-stability is a necessary condition for iterative E-stability.* This implies (2.35) cannot converge to a locally unstable BLE. Note that this does not hold for the Quasi-Newton iteration in (2.36), which finds all fixed-points including E-unstable BLE. See Appendix D for more details. Since iterative E-stability eliminates locally unstable points and guarantees convergence under SAC-learning, we use this method instead of the Quasi-Newton iteration in our estimations.

(ii) *E-stability is not a sufficient condition for iterative E-stability.* This means, in general, there may exist other E-stable BLE that are not iteratively E-stable. Therefore in our estimations, we complement the fixed-point iteration in (2.35) with Monte Carlo simulations to check whether iterative E-stability captures all E-stable points.

[15] Recall that this condition determines the stability of mean dynamics, but does not relate to the stability of persistence parameters.

Our discussion so far is based on finding E-stable BLE for a given set of model parameters θ. In the following, we provide a straightforward extension of Algorithm I to accommodate the estimation of an (iteratively) E-stable BLE. In order to estimate the model, we add a set of measurement equations to the law of motion in (2.34) as follows:

$$y_t = \psi_0(\theta) + \psi_1(\theta) S_t + h_t \tag{2.37}$$

where y_t denotes a vector of observable variables, h_t is a vector of measurement errors, $\psi_0(\theta)$ and $\psi_1(\theta)$ are matrices of the deep parameters that relate the state variables S_t and the observable variables y_t. Together with (2.34), (2.37) yields the state-space representation of the DSGE model under BLE. The model is linear in the state variables S_t, but the equilibrium restrictions $\boldsymbol{\beta}^*$ are non-linear functions in terms of the deep parameters θ to be estimated. On the one hand, given the nonlinear function $\boldsymbol{\beta}^* = G(\boldsymbol{\beta}^*, \theta)$, we can find $\boldsymbol{\beta}^*$ using iterative E-stability for a given θ. On the other hand, whenever $\boldsymbol{\beta}^*$ is temporarily fixed at an arbitrary $\boldsymbol{\beta}^{(k)}$ at any stage of the iteration, the model reduces to a linear VAR model in state-space form that can be estimated using standard methods. Based on this idea, we consider an iterative routine where the structural parameters θ and belief parameters $\boldsymbol{\beta}$ are estimated sequentially, which is summarized in Algorithm II.

Proposition 4 *If Algorithm II converges for a given $\boldsymbol{\beta}^{(K)}$ and $\hat{\theta}^{(K)}$, the resulting $\boldsymbol{\beta}^{(K)}$ is a BLE. Furthermore, denote the set of learning coefficients and estimates at the final step as $(\boldsymbol{\beta}^*\ \theta^*)$, with $\boldsymbol{\beta}^* = G(\boldsymbol{\beta}^*, \theta^*)$. If $\rho(DG(\boldsymbol{\beta}^*\ \theta^*)) < 1$, then $G(.)$ is a contraction on a neighbourhood $\hat{I}$ of $\boldsymbol{\beta}^*$ and for all $\boldsymbol{\beta}^{(0)} \in \hat{I}$, we have $lim_{k\to\infty} G^k(\boldsymbol{\beta}^{(0)}, \theta^{(0)}) = (\boldsymbol{\beta}^*, \theta^*)$.*

Proof. See Appendix D.

Remark. In order to formally rule out explosive outcomes, one can augment the algorithm with a projection facility, where the next iteration is projected to a point inside the unit cube if the iteration $G(\boldsymbol{\beta}^{k-1})$ leads to $|\beta_i^{(k)}| > 1$ for some $i \in \{1, ..., N\}$. We do not observe such outcomes in the examples considered in this paper and therefore do not use a projection facility.

The estimation routine as described above corresponds to a straightforward extension of Algorithm I, where we allow the structural parameters θ (and consequently the matrices γ_1, γ_2 and γ_3) to be re-estimated at each step of the fixed-point iteration in (2.35). Our

[16] For a detailed textbook derivation of the likelihood function and the posterior distribution, see e.g. Greenberg (2012) or Herbst & Schorfheide (2015). In this paper, we make use of the routines available in Dynare to estimate the model at each step for a given set of fixed learning parameters.

Algorithm II: Bayesian Estimation of an (iteratively) E-stable BLE

Denote by $Y_{1:T} = \{Y_1 ... Y_T\}$ the matrix of the observable variables up to period T, and by $p(\theta)$ the prior distributions for the structural parameters θ that appear in matrices γ_1, γ_2 and γ_3. Consider the system characterized by (2.34) and (2.37):

$$\begin{cases} S_t = \gamma_1 S_{t-1} + \gamma_2 \boldsymbol{\beta}^{*2} S_{t-1} + \gamma_3 \eta_t \\ \beta_j^* = G_j(\boldsymbol{\beta}^*, \theta) \\ y_t = \psi_0(\theta) + \psi_1(\theta) S_t + h_t \\ \end{cases} \tag{2.38}$$

- **Step (0)** Initialize a set of learning parameters $\boldsymbol{\beta^{(0)}}$. At the (temporarily) fixed $\boldsymbol{\beta^{(0)}}$, the system (2.38) reduces to a standard state-space representation for the linearized DSGE model.

- **Step (I-a)** At each iteration k, one can obtain the likelihood function using the Kalman filter and the corresponding posterior distribution conditional on $\beta^{(k-1)}$ as follows[16]:

$$p(Y_{1:T}|\theta, \boldsymbol{\beta}^{(k-1)}) = \sum_{t=1}^{T} p(y_t|Y_{1:T-1}, \theta, \boldsymbol{\beta}^{(k-1)}); \quad p(\theta|Y_{1:T}, \boldsymbol{\beta}^{(k-1)}) = \frac{p(Y_{1:T}|\theta, \boldsymbol{\beta}^{(k-1)}) p(\theta)}{p(Y_{1:T}, \boldsymbol{\beta}^{(k-1)})} \tag{2.39}$$

  where $\boldsymbol{\beta^{(k-1)}}$ is obtained from step $k-1$, and $p(Y_{1:T}, \boldsymbol{\beta^{(k-1)}})$ denotes the marginal likelihood function. Denote by $\hat{\theta}^{(k)}$ the conditional posterior mode obtained from $\hat{\theta}^{(k)} = argmax_\theta p(\theta|Y_{1:T}, \boldsymbol{\beta^{(k-1)}})$.

- **Step (I-b)** Using $\hat{\theta}^{(k)}$, update the matrix of learning parameters:

$$\beta_j^{(k)} = G_j(\boldsymbol{\beta^{(k-1)}}, \hat{\theta}^{(k)}), \forall j = \{1 ... N\}, \tag{2.40}$$

- **Step(II)** Proceed to Step (III) if $||\boldsymbol{\beta^{(k)}} - \boldsymbol{\beta^{(k-1)}}|| < c$ and $||\hat{\theta}^{(k)} - \hat{\theta}^{(k-1)}|| < c$ for some scalar $c > 0$, otherwise repeat Step (I).

- **Step(III)** Use the Metropolis-Hastings algorithm to construct the posterior distribution *conditional on the BLE* at the posterior mode.

approach is especially similar to the computation of initial beliefs in Slobodyan & Wouters (2012), where the belief coefficients in $\boldsymbol{\beta}$ are treated as separate parameters and estimated along with θ. The main difference here is that we compute the beliefs consistent with the underlying BLE, such that the first-order autocorrelations in the PLM coincide with the ALM at the estimated posterior mode. In other words, the beliefs are consistent with the actual realizations. Our estimation approach is fast and easy to implement, because it allows us to approximate and estimate a BLE at the posterior mode as a sequence of linear models. Since the beliefs in $\boldsymbol{\beta}^{(k)}$ are updated at each step k based on the first-order autocorrelations of the state variables, the estimated parameters $\hat{\theta}^k$ tend to lead $\boldsymbol{\beta}^k$ towards the empirically relevant region. In turn, this allows the system to rapidly converge to the underlying BLE as we illustrate in the next section.

As an alternative to this algorithm that directly estimates a BLE, we also consider the estimation under SAC-learning in real time. Since iterative E-stability guarantees convergence under SAC-learning, letting the agents learn in real time along with the Kalman filter serves as an indirect approach to estimate a BLE, as well as a robustness check for the empirical fit of a BLE. The model under SAC-learning is conditionally linear for a given set of belief coefficients and therefore one can use the standard Kalman filter recursions to obtain the likelihood function, where the beliefs are updated in each step using the Kalman filter output. This approach has been used in estimating constant gain least squares and Kalman gain adaptive learning models in Milani (2005, 2007) and Slobodyan & Wouters (2012) respectively. In this paper we focus on the decreasing-gain SAC-learning algorithm since our primary interest is the estimation of the underlying fixed-point BLE, rather than the time-variation in beliefs.

3 Application: a New Keynesian model

3.1 A baseline model

Now we apply these results within the framework of a standard New Keynesian model along the lines of Woodford (2003) and Galí (2008). Consider a simple version, linearized around the zero inflation steady state, given by

$$\begin{cases} y_t = y^e_{t+1} - \varphi(i_t - \pi^e_{t+1}) + u_{y,t}, \\ \pi_t = \lambda \pi^e_{t+1} + \gamma y_t + u_{\pi,t}, \end{cases} \tag{3.1}$$

where y_t is the aggregate output gap, π_t is the inflation rate, y^e_{t+1} and π^e_{t+1} are expected output gap and expected inflation. Following Bullard and Mitra (2002) and Bullard et al. (2008) we study the NK-model (3.1) with adaptive learning. The terms $u_{y,t}, u_{\pi,t}$ are stochastic shocks and are assumed to follow AR(1) processes

$$u_{y,t} = \rho_y u_{y,t-1} + \varepsilon_{y,t}, \tag{3.2}$$

$$u_{\pi,t} = \rho_\pi u_{\pi,t-1} + \varepsilon_{\pi,t}, \tag{3.3}$$

where $\rho_i \in [0, 1)$ and $\{\varepsilon_{i,t}\}$ $(i = y, \pi)$ are two uncorrelated i.i.d. stochastic processes with zero mean and finite absolute moments with corresponding variances σ_i^2.

The first equation in (3.1) is an IS curve that describes the demand side of the economy. In an economy of rational or boundedly rational agents, it is a linear approximation to a representative agent's Euler equation. The parameter $\varphi > 0$ is related to the elasticity of intertemporal substitution in consumption of a representative household, and its inverse can be interpreted as a risk aversion coefficient. The second equation in (3.1) is the New Keynesian Phillips curve which describes the aggregate supply relation. This is obtained by averaging each firm's pricing decisions, while the parameter γ is related to the degree of price stickiness in the economy and the parameter $\lambda \in [0, 1)$ is the discount factor of a representative household.

We supplement the equations in (3.1) with standard Taylor-type policy rule, which represents the behavior of the monetary authority in setting the nominal interest rate:

$$i_t = \phi_\pi \pi_t + \phi_y y_t, \tag{3.4}$$

where i_t is the deviation of the nominal interest rate from the value that is consistent with inflation at target and output at potential and the parameters ϕ_π, ϕ_y, measuring the response of i_t to the deviation of inflation and output from long run steady states, are assumed to be non-negative[17].

Substituting the Taylor-type policy rule in equation (3.4) into the equations in (3.1) and writing the model in matrix form gives

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{x}^e_{t+1} + \boldsymbol{C}\boldsymbol{u}_t, \\ \boldsymbol{u}_t = \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, \end{cases} \tag{3.5}$$

[17] In our online appendix we also discuss *lagged* and *forward-looking* Taylor rules, responding to lagged and expected future values of y_t and π_t respectively.

where $\boldsymbol{x}_t = [y_t, \pi_t]'$, $\boldsymbol{u}_t = [u_{y,t}, u_{\pi,t}]'$, $\boldsymbol{\varepsilon}_t = [\varepsilon_{y,t}, \varepsilon_{\pi,t}]'$, $\boldsymbol{B} = \frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\begin{bmatrix} 1 & \varphi(1-\lambda\phi_\pi) \\ \gamma & \gamma\varphi+\lambda(1+\varphi\phi_y) \end{bmatrix}$,
$\boldsymbol{C} = \frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\begin{bmatrix} 1 & -\varphi\phi_\pi \\ \gamma & 1+\varphi\phi_y \end{bmatrix}$, $\boldsymbol{\rho} = \begin{bmatrix} \rho_y & 0 \\ 0 & \rho_\pi \end{bmatrix}$.

Before turning to BLE, we consider rational expectations equilibrium first.

3.2 Theoretical results

Comparing the NK model (3.5) with the general framework (2.10), we note that $\boldsymbol{a} = \mathbf{0}$ and $\boldsymbol{b}_0 = \mathbf{0}$. The rational expectation equilibrium (REE) fixed point in (2.15-2.18) then simplifies to

$$(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{\xi} \;=\; \mathbf{0} \tag{3.6}$$

$$\boldsymbol{\eta} \;=\; \mathbf{B}\boldsymbol{\eta}\boldsymbol{\rho} + \mathbf{C}. \tag{3.7}$$

Bullard and Mitra (2002) show that the REE is unique (determinate) if and only if $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$. The REE is then the stable stationary process with mean

$$\overline{\mathbf{x}^*} = \mathbf{0}. \tag{3.8}$$

In the symmetric case $\rho_i = \rho$ for $i = 1, 2, \cdots, n$, the REE $\mathbf{x}_t^*$ satisfies

$$\mathbf{x}_t^* = (\mathbf{I} - \rho\mathbf{B})^{\mathbf{-1}}\mathbf{C}\mathbf{u_t}. \tag{3.9}$$

Thus its covariance is

$$\boldsymbol{\Sigma}_{\mathbf{x}^*} = \mathbf{E}(\mathbf{x_t^*} - \overline{\mathbf{x}^*})(\mathbf{x_t^*} - \overline{\mathbf{x}^*})' \;=\; (1-\rho^2)^{-1}(\mathbf{I} - \rho\mathbf{B})^{-1}\mathbf{C}\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}[(\mathbf{I} - \rho\mathbf{B})^{-1}\mathbf{C}]'. \tag{3.10}$$

Furthermore, the first-order autocorrelation of the i-element x_i of $\mathbf{x}$ is equal to ρ. That is, in this case the persistence of the REE coincides exactly with the persistence of the exogenous driving force $\mathbf{u}_t$ and the first-order autocorrelations of output gap and inflation are the same, i.e. symmetric, equal to the autocorrelation in the driving force. Inflation and output gap only inherit the persistence of the shocks.

Behavioral learning equilibria

Bullard and Mitra (2002) study adaptive learning in this NK setting. They consider a PLM which coincides with the minimum state variable solution (MSV) of the form

$$\boldsymbol{x}_t = \widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{E}}\boldsymbol{x}_{t+1}^e + \widetilde{\boldsymbol{F}}\boldsymbol{u}_t, \tag{3.11}$$

where $\widetilde{\boldsymbol{D}}$, $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{F}}$ are conformable matrices. We will consider learning with misspecification. As in the general setup in Section 3, we assume that agents are boundedly rational and use simple univariate linear rules to forecast the output gap y_t and inflation π_t of the economy. We therefore deviate from Bullard and Mitra (2002) in two important ways: (i) our agents can not observe or do not use the exogenous shocks $\boldsymbol{u}_t$, and (ii) agents do not fully understand the linear stochastic structure and do not take into account the cross-correlation between inflation and output. Rather our agents learn a simple univariate AR(1) forecasting rule for each variable as shown in (2.2). However this AR(1) rule indirectly, in a boundedly rational way, takes exogenous shocks and cross-correlations of endogenous variables into account as agents learn the two parameters of each AR(1) rule consistent with the observable sample averages and first-order autocorrelations. The use of simple AR(1) rule is supported by evidence from the learning-to-forecast laboratory experiments in the NK framework in Adam (2007), Assenza et al. (2014) and Pfajfar and Zakelj (2016). The actual law of motion (3.5) becomes

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{B}[\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha})] + \boldsymbol{C}\boldsymbol{u}_t, \\ \boldsymbol{u}_t = \boldsymbol{\rho} u_{t-1} + \boldsymbol{\varepsilon}_t. \end{cases} \tag{3.12}$$

For the actual law of motion (ALM) (3.12), the REE determinacy condition $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$ implies the ALM is stationary, see Appendix E. Thus the means and first-order autocorrelations are

$$\begin{aligned} \overline{\boldsymbol{x}} &= (\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}\boldsymbol{\alpha} - \boldsymbol{B}\boldsymbol{\beta}^2\boldsymbol{\alpha}), \\ \boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \begin{bmatrix} G_1(\beta_1, \beta_2) & 0 \\ 0 & G_2(\beta_1, \beta_2) \end{bmatrix} = \begin{bmatrix} \mathrm{corr}(y_t, y_{t-1}) & 0 \\ 0 & \mathrm{corr}(\pi_t, \pi_{t-1})) \end{bmatrix}. \end{aligned}$$

In order to obtain analytical expressions for $G_1(\beta_1, \beta_2)$ and $G_2(\beta_1, \beta_2)$ we focus on the symmetric case with $\rho_y = \rho_\pi = \rho$. The first-order autocorrelations of output gap and inflation can be expressed in terms of the structural parameters through very complicated calculations (see Appendix F[18])

$$G_1(\beta_1,\beta_2) = \frac{\widetilde{f}_1}{\widetilde{g}_1} \tag{3.13}$$

$$G_2(\beta_1,\beta_2) = \frac{\widetilde{f}_2}{\widetilde{g}_2} \tag{3.14}$$

where

$$\begin{aligned}
\widetilde{f}_1 &= \sigma_1^2\Big\{(\rho+\lambda_1+\lambda_2-\lambda\beta_2^2)[1-\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)]+[\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\\
&\quad \rho\lambda_1\lambda_2][(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\lambda\beta_2^2\rho\lambda_1\lambda_2]\Big\}+\sigma_2^2\Big\{(\varphi\phi_\pi(\rho+\lambda_1+\lambda_2)-\varphi\beta_2^2))\\
&\quad [\varphi\phi_\pi-\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)]+[\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\varphi\phi_\pi\rho\lambda_1\lambda_2]\\
&\quad [\varphi\phi_\pi(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\varphi\beta_2^2\rho\lambda_1\lambda_2]\Big\},\\
\widetilde{g}_1 &= \sigma_1^2\Big\{[(1+\lambda^2\beta_2^4)-2\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)+(1+\lambda^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\\
&\quad -\rho\lambda_1\lambda_2[(1+\lambda^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+(1+\lambda^2\beta_2^4)\rho\lambda_1\lambda_2]\Big\}\\
&\quad +\sigma_2^2\Big\{[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)-2\varphi\phi_\pi\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)+((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\\
&\quad -\rho\lambda_1\lambda_2[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\varphi\phi_\pi\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)\\
&\quad +((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)\rho\lambda_1\lambda_2]\Big\},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\widetilde{f}_2 &= \sigma_1^2\Big\{\gamma^2[(\rho+\lambda_1+\lambda_2)-\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big\}+\sigma_2^2\Big\{[(1+\varphi\phi_y)(\rho+\lambda_1+\lambda_2)-\beta_1^2]\cdot\\
&\quad [(1+\varphi\phi_y)-\beta_1^2(\rho+\lambda_1+\lambda_2)]+[\beta_1^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-(1+\varphi\phi_y)\rho\lambda_1\lambda_2]\cdot\\
&\quad [(1+\varphi\phi_y)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\beta_1^2\rho\lambda_1\lambda_2]\Big\},\\
\widetilde{g}_2 &= \sigma_1^2\Big\{\gamma^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-(\rho\lambda_1\lambda_2)^2]\Big\}\\
&\quad +\sigma_2^2\Big\{[((1+\varphi\phi_y)^2+\beta_1^4)-2(1+\varphi\phi_y)\beta_1^2(\rho+\lambda_1+\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4)\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]-\rho\lambda_1\lambda_2[((1+\varphi\phi_y)^2+\beta_1^4)(\rho+\lambda_1+\lambda_2)-2(1+\varphi\phi_y)\beta_1^2\cdot\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4)\rho\lambda_1\lambda_2]\Big\},
\end{aligned} \tag{3.16}$$

$$\lambda_1+\lambda_2 = \frac{\beta_1^2+(\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}, \tag{3.17}$$

$$\lambda_1\lambda_2 = \frac{\lambda\beta_1^2\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}. \tag{3.18}$$

[18] Appendix E employs the VARMA$(1,\infty)$ representation of the model. Although it is possible to obtain the expressions of $\boldsymbol{G}(\boldsymbol{\alpha},\boldsymbol{\beta})$ using the direct method in Appendix B, the analytical expressions are much more complicated. Numerical computations based on the two methods are consistent and also coincide with the simple numerical simulation of the first-order autocorrelation coefficients of output gap and inflation obtained from simulated time series generated by the system (3.12), confirming the complicated expressions (3.13-3.14).

From these expressions, it is easy to see that $G_1(\beta_1, \beta_2)$ and $G_2(\beta_1, \beta_2)$ are analytic functions with respect to β_1 and β_2, independent of $\boldsymbol{\alpha}$.

The actual law of motion (3.5) depends on eight parameters φ, λ, γ, ϕ_y, ϕ_π, ρ, σ_1^2 and σ_2^2. Only the ratio σ_1^2/σ_2^2 of noise terms matters for the persistence $G_i(\beta_1, \beta_2)$ in (3.13) and (3.14). Hence, the existence of BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ depends on seven structural parameters φ, λ, γ, ρ, ϕ_y, ϕ_π and σ_1^2/σ_2^2 of the NK-model.

Using Proposition 1 and Proposition 2 we have the following properties for the New Keynesian model:

Corollary 2 *Under the contemporaneous Taylor rule, if $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, then there exists at least one BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$, where $\boldsymbol{\alpha}^* = \mathbf{0} = \overline{\boldsymbol{x}^*}$.*

Corollary 3 *Under the contemporaneous interest rate rule and the condition $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, a BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is locally stable under SAC-learning if all eigenvalues of $\boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*) = \left(\frac{\partial G_i}{\partial \beta_j}\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}$ have real parts less than 1.*

Proof. See Appendix G.

It is useful to discuss the special case in which shocks are not persistent, that is, $\rho = 0$ (no autocorrelation in the shocks). It is easy to see that

$$G_1(0,0)\big|_{\rho=0} = 0, \qquad G_2(0,0)\big|_{\rho=0} = 0.$$

That is to say $(\mathbf{0}, \mathbf{0})$ is a BLE for $\rho = 0$. Hence, when there is no persistence in the exogenous shocks, the BLE coincides with the rational expectation equilibrium.

It is also useful to briefly discuss the non-stationary case, that is, when the coefficient matrix $\boldsymbol{B}$ for expectations $\boldsymbol{x}_{t+1}^e$ in (3.5) has at least one eigenvalue outside the unit circle. In that case, SAC-learning of an AR(1) rule typically leads to explosive dynamics with $\alpha_t \to \pm\infty$ and $\beta_t \to 1$. In the non-stationary case, learning of BLE thus typically leads to explosive time paths of inflation and output.

Persistence amplification

We illustrate these results by some typical numerical calculations for empirically plausible parameter values. As in the Clarida et al. (1999) calibration we fix $\varphi = 1, \lambda = 0.99$. We fix $\gamma = 0.04$, which lies between the calibrations $\gamma = 0.3$ in Clarida et al. (1999) and $\gamma = 0.024$ in Woodford (2003). For the exogenous shocks, we set the ratio of shocks $\frac{\sigma_2}{\sigma_1} = 0.5$, which is within the possible range suggested in Fuhrer (2006). We consider the symmetric case $\rho_1 = \rho_2 = \rho = 0.5$, with weak persistence in the shocks. The baseline parameters on the policy response to inflation deviation and output gap follow a broad literature, $\phi_\pi = 1.5$, $\phi_y = 0.5$, see for example Fuhrer (2006, 2009). At these parameter values, the two eigenvalues of the Jacobian matrix $\boldsymbol{DG_\beta}(\boldsymbol{\beta}^*)$ are $0.5012 \pm 0.7348i$ (the real parts less than 1), which implies that the BLE is E-stable under SAC-learning based on our theoretical results. The numerical results shown below are robust across a range of plausible parameter values.

Figure 1 illustrates the existence of a unique E-stable BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$[19]. In order to obtain (β_1^*, β_2^*), we numerically compute the corresponding fixed point $\beta_2^*(\beta_1)$ satisfying $G_2(\beta_1, \beta_2^*) = \beta_2^*$ for each β_1 and the corresponding fixed point $\beta_1^*(\beta_2)$ satisfying $G_1(\beta_1^*, \beta_2) = \beta_1^*$ for each β_2 as illustrated in Figure 1. Hence their intersection point (β_1^*, β_2^*) satisfies $G_1(\beta_1^*, \beta_2^*) = \beta_1^*$ and $G_2(\beta_1^*, \beta_2^*) = \beta_2^*$.

A striking feature of the BLE is that the first-order autocorrelation coefficients of output gap and inflation $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$ are substantially higher than those at the REE, that is, persistence is much higher than the persistence $\rho (= 0.5)$ of the exogenous shocks. We refer to this phenomenon as *persistence amplification*. Agents fail to recognize the complete linear structure of the economy, but rather learn to coordinate on a simple AR(1) rule consistent with simple observable statistics, the mean and the first-order autocorrelation. As a result of this *self-fulfilling mistake*, shocks to the economy are strongly amplified.

Figure 2 illustrates how these results depend on the persistence ρ of the exogenous shocks. The figure shows the BLE, i.e. the first-order autocorrelations β_1^* of output gap and β_2^* of inflation, as a function of the parameter ρ. This figure clearly shows the *persistence amplification* along BLE, with much higher ACF than under RE, for all values of $0 < \rho < 1$. Especially for $\rho \geq 0.5$ we have $\beta_1^* \geq 0.9$ and $\beta_2^* > 0.95$, implying that output gap and inflation have significantly higher persistence than the exogenous driving forces. Figure 2 (right plot) also illustrates the *volatility amplification* under BLE compared to REE. For output gap the ratio of variances $\sigma_y^2/\sigma_{y^*}^2$ reaches a peak of about 2.5 for $\rho \approx 0.75$, while for inflation the ratio of variances $\sigma_\pi^2/\sigma_{\pi^*}^2$ reaches its peak of about 3.5 for $\rho \approx 0.65$. These results suggest that, given the same parameter values, the moments implied by BLE and REE are substantially different due to persistence and volatility amplification under BLE. Therefore if the model is estimated on the same dataset under BLE and

[19] Note that $(\alpha_1^*, \alpha_2^*) = (0, 0)$.

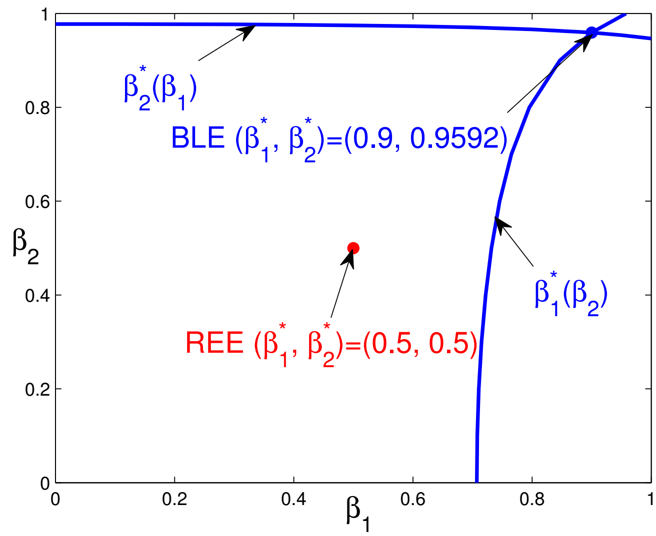


Figure 1: A unique BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$ obtained as the intersection point of the fixed point curves $\beta_2^*(\beta_1)$ and $\beta_1^*(\beta_2)$. The BLE exhibits strong persistence amplification compared to REE (red dot, with $\rho = 0.5$). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.

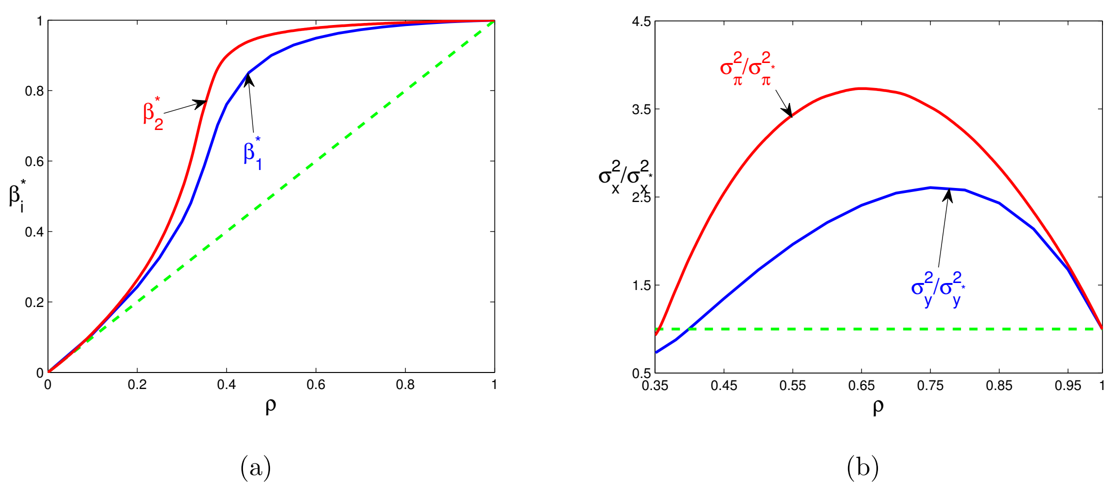


Figure 2: BLE (β_1^*, β_2^*) as a function of the persistence ρ of the exogenous shocks, for the contemporaneous Taylor rule. (a) $\beta_i^*(i = 1, 2)$ with respect to ρ; (b) the ratio of variances $(\sigma_y^2/\sigma_{y^*}^2,\ \sigma_\pi^2/\sigma_{\pi^*}^2)$ of the BLE (β_1^*, β_2^*) w.r.t. the REE. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.

REE, one might expect important differences in the resulting parameter values and the implied propagation mechanism of the model. We explore this idea in the next section by estimating the model under BLE and REE.

3.3 Estimation of The Baseline Model

Sample Period and Prior Distributions

In this section, we investigate the empirical fit of the 3-equation New Keynesian model under REE, BLE and SAC-learning. We augment the Taylor rule with an i.i.d monetary policy shock and an interest rate smoothing parameter to allow the model to match the inertia of the historical interest rate:

$$r_t = \rho_r r_{t-1} + (1-\rho_r)(\phi_\pi \pi_t + \phi_x y_t) + \epsilon_{r,t} \tag{3.19}$$

We estimate this small-scale system for the U.S economy over the period 1966:I-2016:IV using quarterly macroeconomic data. We also investigate whether our results are sensitive to structural breaks such as the large volatility reduction for most macroeconomic time series during the mid-80s, often referred to as the Great Moderation, or the near-zero level of nominal interest rates that followed the 2007-08 crisis period. We use the following simple measurement equations for output gap, inflation and interest rate[20] without measurement errors:

$$\begin{cases} log(y_t^{obs}) = \bar{\gamma} + y_t \\ log(\pi_t^{obs}) = \bar{\pi} + \pi_t \\ log(r_t^{obs}) = \bar{r} + r_t \end{cases} \tag{3.20}$$

where we use the cycle component of HP-filtered quarterly output as our measure of output gap y_t^{obs}[21], while π_t^{obs} and r_t^{obs} denote the quarterly historical inflation and interest rate series respectively. $\bar{\gamma}, \bar{\pi}$ and $\bar{r}$ correspond to the historical mean levels of output gap, inflation and interest rate. The model is estimated using the same prior distributions under all three specifications, which guarantees that any differences that arise between the estimations is due to the difference in the expectation formation rule. The prior distributions are kept close to those commonly assumed in the literature: the risk aversion

[20] See Appendix H for more details on the observable variables.

[21] As shown in the next section, our main results are not sensitive to the choice of output gap measure.

coefficient $\tau = \frac{1}{\varphi}$ is assigned a gamma distribution centered at 2 with a standard deviation of 0.5, covering values of approximately up to four. The slope of the Phillips curve γ is assigned a Beta distribution with mean 0.3 and standard deviation 0.15 which falls somewhere between the prior in An & Schorfheide (2007) and Smets & Wouters (2007), covering both flat and steep cases for the Phillips curve[22]. The policy response parameters for output gap and inflation are assigned beta distributions centered around 0.5 and 1.5, which are the typical values associated with the Taylor Rule in the literature. The autocorrelation coefficients have a Beta distribution centered at 0.5, and the standard deviations for the shock processes are assumed to follow an Inverted Gamma distribution with a mean of 0.1, same as Smets & Wouters (2007). The priors for the steady-state inflation rate, output growth and interest rate are normal distributions centered at their pre-sample means of 0.47, -0.2 and 0.72 respectively, where the pre-sample period covers from 1954:I to 1965:IV. Finally, we fix the HH discount rate λ at 0.99, which is a standard assumption in most empirical studies.

Convergence Diagnostics

Table 3.3 presents the posterior estimation results for the BLE, RE and SAC-learning models.[23] Before moving onto the estimation results, we first briefly discuss the convergence diagnostics of BLE and SAC-learning. Under BLE, initializing both β_y and β_π at fairly low values of 0.5 and using a convergence criterion $c = 10^{-5}$, the algorithm takes only 5 steps to converge[24]. The resulting BLE is $(\beta_y^*, \beta_\pi^*) = (0.88, 0.89)$ at the final step. This is fairly close to the sample-autocorrelation moments of the data over this period, which is $(0.87, 0.89)$. The left panel of Figure 3 shows the norm distances between two consecutive sets of $\boldsymbol{\beta}^{(k)}$ and $\theta^{(k)}$ at each step k, both of which rapidly converge towards 0. The largest eigenvalue of the Jacobian matrix $DG(\boldsymbol{\beta}^{(k)}, \theta^{(k)})$ remains strictly inside the

[22]In particular if we denote the nominal price stickiness as ω, its relation with γ is given as $\gamma = \frac{(1-\lambda\omega)(1-\omega)}{\omega}$ (Gali, 2008). Smets & Wouters (2007) assume a prior for ω with mean 0.5.

[23]BLE and REE models are estimated using the Dynare toolbox (Adjemian et. al, 2011), while our own toolbox is used for the SAC-learning estimations since it requires a slightly modified filter not available in Dynare. The posterior distributions are constructed using the Metropolis-Hastings algorithm with 250000 draws, using the first 50000 as the burn-in sample. The step size for the scale parameter of the jumping distribution's covariance matrix is adjusted in both models to obtain a rejection rate of 70% in both models, which is in the commonly assumed appropriate range for the MH algorithm.

[24]The results are robust to initial values of β_y and β_π.

unit circle during the estimation, and stabilizes after the second step. The right panel of the same figure shows the convergence of Algorithm I towards $\boldsymbol{\beta}^*$ at the estimated posterior mode with randomized initial values, suggesting that the estimated equilibrium is the unique iteratively E-stable BLE. A similar result also emerges when we examine the Monte Carlo simulations of the model under SAC-learning in Figure 4: The left panel shows the histograms of β_y and β_π over 1000 simulations under decreasing gain learning, while the right panel shows the constant gain equivalent with a small gain value of 0.001. It is readily seen none of the distributions show signs of multiplicity of equilibria. To formally check this, we provide the approximate distributions of β_y and β_π by smoothing the histograms and applying Hartigan's Dip Test of Unimodality[25]. The dip test does not reject the null hypothesis of unimodality, suggesting that there is no evidence of multiple BLE at the estimated parameter values based on the simulations. We also observe a small bias in the simulations for both cases, where the peak of the distributions slightly deviates from the underlying BLE denoted by the dotted line. These Monte Carlo simulations illustrate the advantage of SAC-learning over the standard least-squares learning approach: although the autocorrelation coefficients are fairly close to unity for both y_t and π_t, the time series never becomes explosive in our simulations. This is due to the natural projection facility of SAC-learning which makes explosive time paths less likely, as discussed in Section 2.

Figure 5 shows the mean and persistence coefficients along with the filtered variables of inflation and output gap under the SAC-learning estimation. For this specification, following the recommendation in Galimberti & Jacqueson (2017), we use a training-sample based initialization as follows: we use the unconditional moment of 0 for the intercept coefficients, and the diffuse moment 0 for the estimated variance of each variable R_t[26]. We use a period of five years over 1961:I-1965:IV as the transient period for the belief coefficients, and compute the likelihood from 1966:I onwards. It is readily seen that the mean coefficients do not substantially deviate from the unconditional mean of 0 and the persistence coefficients indeed converge over the estimation sample, with final values of

[25]Hartigan's Dip Test is based on checking for multimodality by using the maximum difference between the empirical distribution, and the theoretical unimodal distribution function that minimizes the maximum difference. The null hypothesis of the test is unimodality, see Hartigan & Hartigan (1985) for more details.

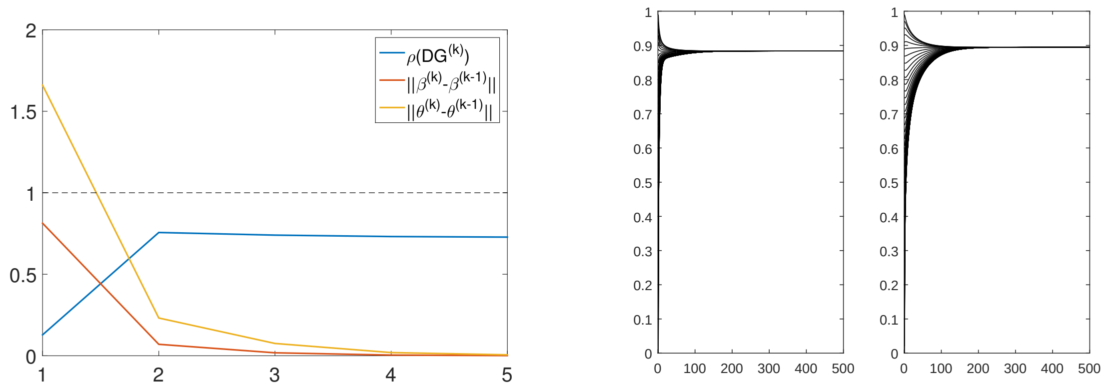
[26]The initial choice of first-order autocorrelations does not matter since $\beta_{1,t} = -\frac{1}{2}$ regardless of what $\beta_{0,t}$ is.

$\boldsymbol{\beta}^* = (0.86, 0.89)$: this is fairly close to the equilibrium resulting under the BLE estimation.

Posterior Results

Next we move onto the discussion of our structural parameter estimates. Starting with a comparison of the BLE and REE results, we observe several important differences: both persistence parameters for the inflation and output shocks, ρ_π and ρ_y, are substantially lower under BLE with 0.31 and 0.42 respectively, while they are 0.88 and 0.87 under REE. This is a direct consequence of the difference in the expectation formation rule, which shows the persistence amplification property of a BLE. The backward-looking expectation rule endogenously generates additional inertia for inflation and output gap, which in turn leads to much smaller persistence in the exogenous shocks to match the empirical autocorrelations of inflation and output gap. The low autocorrelation in u_π and u_y under BLE immediately imply higher estimated standard deviations for the i.i.d shocks of these AR(1) processes at 0.29 and 0.73, while these are 0.04 and 0.16 under REE[27]. Since interest-rate is not forward-looking, this result does not extend to the interest-rate smoothing ρ_r, which is estimated at 0.85 and 0.80, with similar levels of volatility at 0.29 and 0.3 under BLE and REE respectively. The steady-state parameters turn out fairly similar under both estimations, since they relate to sample means of the observable variables and are not affected by the expectation rule. The estimates of monetary policy parameters, ϕ_π and ϕ_y also turn out similar under both estimations, with 1.36 and 0.48 under BLE, and 1.39 and 0.46 under REE. Finally turning to the two structural parameters that determine the contemporaneous relation between the state variables, both the risk aversion coefficient $\frac{1}{\varphi}$ and the slope of the NKPC γ turn out fairly different: these are estimated at 3.02 and 0.035 under BLE, while they are 4.27 and 0.007 under REE respectively. These differences arise due to altered cross-restrictions under learning: the additional inertia that we introduce under learning comes at the cost of a weaker contemporaneous relation between the state variables and shocks. While the state variables are only related to the shocks through $\frac{1}{\varphi}$ and γ under BLE, they are also indirecly related through expectations under REE. As a result, the risk aversion coefficient turns out lower under learning, implying a larger direct impact from the ex-ante risk premium on output

[27]Note that for an AR(1) process $x_t = \rho_x x_{t-1} + u_t, u_t \sim iid(0, \sigma_u)$, the unconditional variance is given by $var(x) = \frac{\sigma_u}{1-\rho_x^2}$. This implies, as ρ_x increases, $var(x)$ also increases.



(a) Convergence towards $\boldsymbol{\beta}^{*}$ (b) Iterative E-stability of the unique $\boldsymbol{\beta}^{*}$

Figure 3: The estimated BLE is $(\beta_y, \beta_\pi) = (0.8835, 0.8944)$. The left panel shows the largest eigenvalue, and the norm distances between consecutive values of $\beta^{(k)}$, as well as $\theta^{(k)}$ across iterations. The second panel shows convergence towards the unique (iteratively) E-stable BLE with randomized initial values.

gap under BLE. Similarly, γ turns out higher under BLE, implying a stronger direct effect from output gap on inflation. It is also interesting to note that confidence intervals for γ are almost mutually exclusive under these two specifications, with a lower bound of 0.015 under BLE and an upper bound of 0.017 under REE[28].

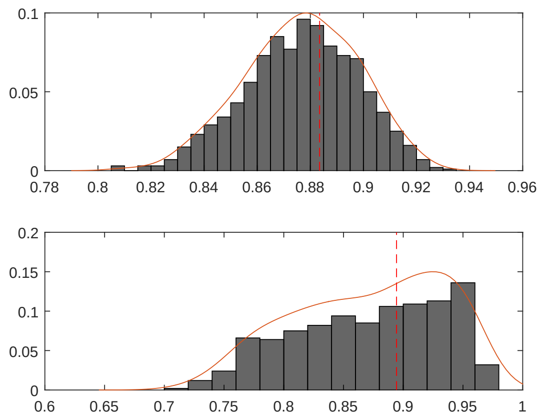
Overall, our results suggest important differences in the estimated parameters and the propagation of shocks under BLE. These changes lead to a substantial improvement in the empirical fit, evident from the (log marginal) likelihood of -337 under BLE compared with -348 under REE, which yields a Bayes' Factor 4.78 in favour of BLE[29]. Comparing the results to SAC-learning, it is readily seen that there are no substantial differences with the BLE specification. Relative to the REE estimations, the exogenous shocks have lower persistence and larger standard deviations, the risk aversion coefficient is lower, and Phillips curve slope is higher and monetary policy coefficients are similar. The only difference with the BLE model arises in the steady-state values of inflation and interest-rate, which turn out lower under the SAC-learning estimation. This difference arises since the

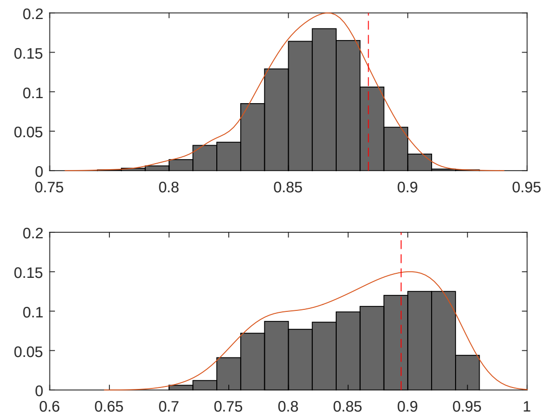
[28]As noted in the previous section, the first-order coefficients are computed based on the posterior mode values. Therefore, for completeness, the discussion and comparison of these results (both in this section and in the upcoming ones) is based on the posterior mode. It is worth noting that, given the small differences across the posterior means and modes, a similar discussion can be easily extended to the posterior mean.

[29]Based on Jeffrey's Guidelines (Greenberg, 2012), if the Bayes' Factor in favour of a model is larger than 2, then this provides *decisive support* for the model under consideration.

				REE				BLE				SAC			
			Laplace	-348				-337				-341			
			MHM	-348				-337				-341			
			Bayes Factor					4.78				3.04			
	Prior			Post.				Post.				Post.			
Para.	Dist.	Mean	St. Dev	Mode	Mean	5 %	95 %	Mode	Mean	5 %	95 %	Mode	Mean	5 %	95 %
η_y	Inv. Gamma	0.1	2	0.16	0.17	0.12	0.22	0.73	0.74	0.68	0.8	0.75	0.74	0.7	0.78
η_π	Inv. Gamma	0.1	2	0.04	0.04	0.03	0.05	0.29	0.3	0.27	0.32	0.3	0.31	0.29	0.32
η_r	Inv. Gamma	0.1	2	0.29	0.3	0.28	0.33	0.29	0.29	0.27	0.32	0.29	0.28	0.27	0.32
$\bar{y}$	Normal	-0.2	0.25	-0.15	-0.49	0.18	-0.12	-0.12	-0.4	0.17	-0.17	-0.21	-0.27	-0.5	-0.04
$\bar{\pi}$	Normal	0.47	0.25	0.7	0.71	0.53	0.9	0.79	0.79	0.67	0.91	0.41	0.5	0.34	0.65
$\bar{r}$	Normal	0.72	0.25	0.98	0.98	0.71	1.26	1.1	1.09	0.85	1.32	0.65	0.67	0.47	0.86
γ	Beta	0.3	0.15	0.007	0.01	0.002	0.017	0.035	0.037	0.015	0.065	0.048	0.039	0.023	0.056
$\frac{1}{\psi}$	Gamma	2	0.5	4.27	4.35	3.35	5.37	3.02	3.15	2.33	3.98	2.86	2.75	2.26	3.23
ϕ_π	Gamma	1.5	0.25	1.38	1.4	1.16	1.65	1.36	1.38	1.1	1.67	1.32	1.32	1.14	1.49
ϕ_y	Gamma	0.5	0.25	0.48	0.51	0.34	0.67	0.49	0.52	0.31	0.73	0.48	0.46	0.35	0.57
ρ_y	Beta	0.5	0.2	0.87	0.86	0.83	0.92	0.43	0.43	0.32	0.53	0.57	0.58	0.52	0.64
ρ_π	Beta	0.5	0.2	0.88	0.87	0.82	0.91	0.32	0.32	0.22	0.43	0.25	0.24	0.18	0.31
ρ_r	Beta	0.5	0.2	0.8	0.8	0.76	0.84	0.85	0.86	0.82	0.91	0.85	0.85	0.83	0.88

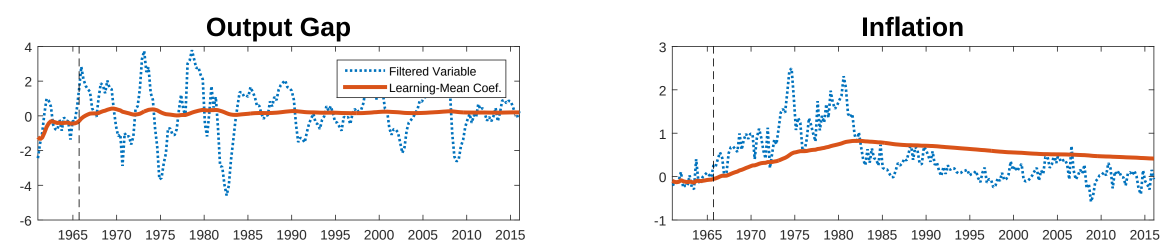
Table 1: Posterior Results under REE and BLE over the estimation period 1966:I-2016:IV. Lapl. refers to the Laplace likelihood approximation based on the posterior mode, while MHM refers to the Modified Harmonic Mean likelihood based on the posterior distribution. sacl caption: NKPC Estimation results under Decreasing Gain Learning with sample autocorrelation. Estimation Sample: 1966:I-2016:IV. The preceding 5 years 1961:I-1965:IV are used as a training sample for the learning coefficients (to avoid large fluctuations in the transitory dynamics.



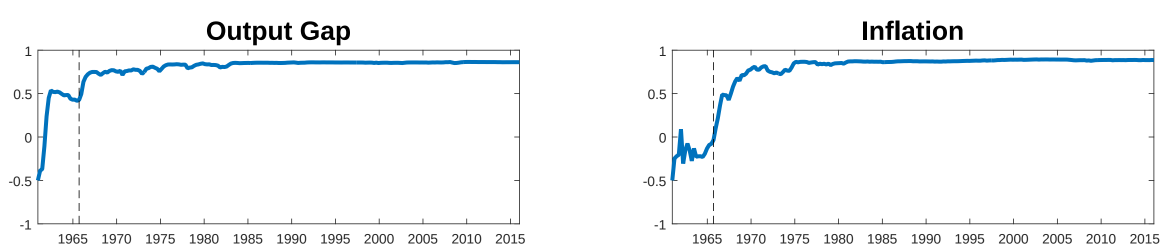


(a) β_y^* and β_π^* under Constant gain learning. (b) β_y^* and β_π^* under decreasing gain learning.

Figure 4: Monte Carlo Simulations: frequency distributions and unimodality test for β_y and β_π resulting from 1000 simulations. Hartigan's unimodality test p-values are 0.98 and 0.94 for β_y and β_π under constant gain simulations, and 0.99 and 0.99 for the decreasing gain simulations. Hence the null hypothesis of unimodality is not rejected for any of the distributions.



(a) Filtered Variables and mean parameters



(b) Persistence parameters

Figure 5: Filtered variables and learning parameters over the estimation sample under SAC-learning, where Kalman filter output is used to update the belief coefficients. Converged values of first-order autocorrelations are 0.86 and 0.89 for output gap and inflation respectively.

learning coefficients are time-varying under real-time SAC-learning and Figure 5 suggests that they are slightly above zero on average, which drives down the estimates of the steady-state parameters[30]. Other than these small differences, all parameter estimates are fairly close under BLE and SAC-learning, with implied HPD-intervals well within the range of each other. The likelihood turns out to be -341 under SAC-learning, which is still better than the fit of REE (the Bayes' Factor is 3.08 in favour of SAC-learning) but slightly worse than BLE. This result suggests that transitory dynamics and the resulting time-variation in the learning parameters do not improve the model fit in our decreasing-gain learning setup. Overall, these results also allow us to illustrate the advantage of using Algorithm II to estimate a BLE: The estimation under SAC-learning requires a relatively large burn-in sample for the convergence of first-order autocorrelation coefficients, which might become an issue since macroeconomic time series are typically not very long. Furthermore, as we have already seen in Figure 4, the simulations under learning have a relatively large Monte-Carlo variance. In other words, while the simulations converge to the underlying BLE on average, there might be relatively large deviations from the underlying fixed-point for any given simulation. Hence what comes out of the estimation in a real-time learning setup is similar to a single simulation of the model under learning, which in general may not accurately reflect the underlying fixed-point. In the following, we check whether these results are robust to different measures of output gap and different subsamples for the BLE and REE specifications.

Subsample Estimations

We first chek whether our main results hold across different sample periods. To this end, we consider three periods: 1966:I-1979:II, the period before Great Moderation; 1966:I-2008:IV, the period before Great Recession and the zero lower bound episode; and 1984:I-2008:IV, the Great Moderation period.

Table 2 reports the posterior mode and the corresponding Laplace approximation for all periods under BLE and REE. It is readily seen that the difference between parameter estimates are preserved across all three periods: BLE is characterized by lower persistence but larger standard deviation estimates in shocks, a steeper Phillips curve characterized by larger γ, and a smaller risk aversion coefficient. The estimation under BLE provides a

[30]Shutting off the learning dynamics about mean coefficients and letting the agents learn only about the first-order autocorrelations indeed yields steady-state values similar to BLE.

better model fit under all three subsamples, with Bayes' Factors of 1.67, 3.93 and 2.44 in favor of the BLE model.

Period	66:I-79:II		66:I-08:IV		84:I-08:IV	
	BLE	REE	BLE	REE	BLE	REE
Laplace	-118.63	-122.47	-313.16	-322.10	-34.32	-39.93
Bayes Factor	1.67		3.93		2.44	
Parameter	Mode		Mode		Mode	
η_y	0.94	0.24	0.76	0.17	0.53	0.09
η_π	0.38	0.06	0.3	0.04	0.18	0.08
η_r	0.20	0.20	0.31	0.32	0.15	0.15
$\bar{y}$	-0.22	-0.14	-0.09	-0.14	0.05	-0.14
$\bar{\pi}$	0.93	0.75	0.84	0.71	0.61	0.58
$\bar{r}$	1.03	0.86	1.25	1.08	1.14	0.97
γ	0.054	0.01	0.033	0.006	0.046	0.007
$\frac{1}{\psi}$	2.44	2.77	4.03	3.78	2.82	3.15
ϕ_π	1.03	1.07	1.29	1.31	1.55	1.57
ϕ_y	0.39	0.36	0.45	0.42	0.48	0.54
ρ_y	0.5	0.87	0.42	0.88	0.44	0.94
ρ_π	0.31	0.85	0.32	0.87	0.24	0.55
ρ_r	0.72	0.68	0.83	0.77	0.9	0.85

Table 2: Comparison of the sub-sample estimations under BLE and REE.

Alternative Definitions of Output Gap

Next we check whether our results are sensitive to which measure of output gap is considered by using two alternative definitions: output gap based on de-trended output, and output gap based on CBO's measure of potential output. The estimations over our main sample are reported in Table 3, which yield the same conclusions as before: BLE is characterized by lower persistence but larger standard deviation estimates in shocks, a steeper Phillips curve characterized by larger γ, and a smaller risk aversion coefficient. The likelihood is also better under BLE for both definitions, with Bayes' Factors of 11.49 and 10.41 respectively. As a final check, we compare the likelihoods across the three sub-samples with the alternative measures of output gap, which is reported in Table 4: it is readily seen that the likelihood under BLE is better for both measures under all sub-samples, although the difference for the Great Moderation period 1985-I:2008:IV is much smaller. These results are also consistent with our previous findings.

	BLE det.	REE det.	BLE CBO.	REE CBO.
Laplace	-360.96	-387.42	-342.9	-366.88
Bayes Factor	11.49		10.41	
	Post.		Post.	
Parameter	Mode	Mode	Mode	Mode
η_y	0.78	0.08	0.74	0.11
η_π	0.3	0.04	0.29	0.04
η_r	0.3	0.31	0.29	0.3
$\bar{y}$	-0.13	-0.19	-0.54	-0.42
$\bar{\pi}$	0.79	0.66	0.81	0.59
$\bar{r}$	1.09	0.92	1.15	1.05
γ	0.017	0.004	0.024	0.006
$\frac{1}{\psi}$	2.92	4.86	2.65	4.57
ϕ_π	1.44	1.45	1.41	1.43
ϕ_y	0.22	0.16	0.36	0.27
ρ_y	0.42	0.88	0.42	0.89
ρ_π	0.31	0.94	0.31	0.92
ρ_r	0.88	0.79	0.88	0.8

Table 3: Alternative estimations of the 3-equation NKPC model: We compare the results under BLE and REE with two alternative specifications of output gap. In the first case output gap is defined as the deviation of output from a quadratic trend, while in the latter we take the output gap based on CBO's measure of potential output.

	66:I-79:II		66:I-08:IV		84:I-08:IV	
	BLE	REE	BLE	REE	BLE	REE
CBO's estimate	-126.07	-127.6	-321.9	-344.4	-36.7	-46.9
Bayes Factor	0.66		9.77		4.43	
Detrended output	-129.9	-133.1	-333.1	-353.2	-48.4	-66.9
Bayes Factor	1.39		8.73		8.03	

Table 4: Sub-sample estimations with alternative definitions of output gap.

4 Optimal Monetary Policy

Our results so far illustrate that, when BLE and REE are examined under the same set of calibrated parameter values, BLE are characterized by persistence and volatility amplification, with much higher persistence and variance in output gap and inflation compared with REE. As a consequence, there are substantial differences in the estimated parameters and propagation structure when the 3-equation model is evaluated under BLE and REE. This leaves an important question for the optimal Taylor-rule parameters at the BLE. As shown in Boehm and House (2014), at the REE when the output gap and inflation are observed without error, it is typically optimal to respond infinitely strongly to observed deviation from the central bank's targets, while with measurement error the optimal Taylor rule coefficients are finite. How do the optimal values of Taylor rule parameters differ between the BLE and REE? In this section we try to answer this question by considering optimal monetary policy under both the calibrated and estimated parameter values.

Similar to Boehm and House (2014), Evans and Honkapohja (2003) and Woodford (2003), we assume that the central bank wishes to minimize an expected discounted sum of weighted squared inflation and output gap

$$(1-\vartheta)E\Big[\Sigma_{t=0}^{\infty}\vartheta^t[\omega\pi_t^2+(1-\omega)y_t^2]\Big]=\omega\sigma_\pi^2+(1-\omega)\sigma_y^2, \tag{4.1}$$

where ω is the relative weight that the central bank places on inflation. From the equations (F.12) and (F.13) in Appendix F,

$$\sigma_y^2 = \frac{\widetilde{g}_1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\rho\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}, \tag{4.2}$$

$$\sigma_\pi^2 = \frac{\widetilde{g}_2}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\rho\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}, \tag{4.3}$$

where $\widetilde{g}_1$, $\widetilde{g}_2$, λ_1 and λ_2 are given by the equations (3.15), (3.16), (3.17) and (3.18). In the following we study the optimal values (ϕ_y^*,ϕ_π^*) that minimize the central bank's loss function (4.1) at the BLE (β_1^*,β_2^*).

We first examine monetary policy under BLE and REE at calibrated parameter values: as before in our calibration exercise, we consider the parameters $\lambda=0.99,\varphi=1,\gamma=0.04,\rho=0.5,\frac{\sigma_2}{\sigma_1}=0.5$ for both BLE and REE. This ensures that the economic structure is the same under both specifications, but the implied moments of inflation and output gap are different under BLE and REE. We first consider the case $\omega=0.9$, that is, the central bank places relatively large weight on inflation. Interestingly, we find that the

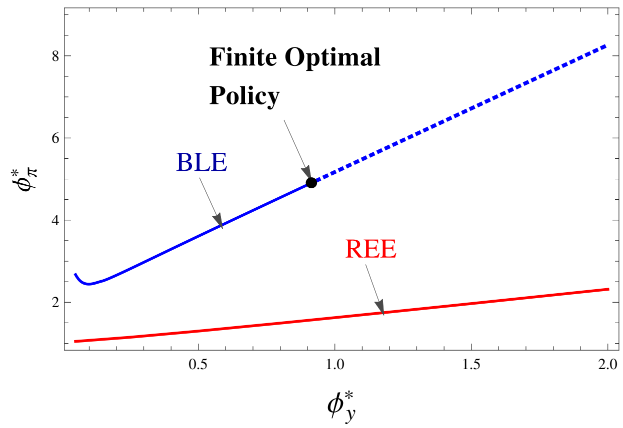
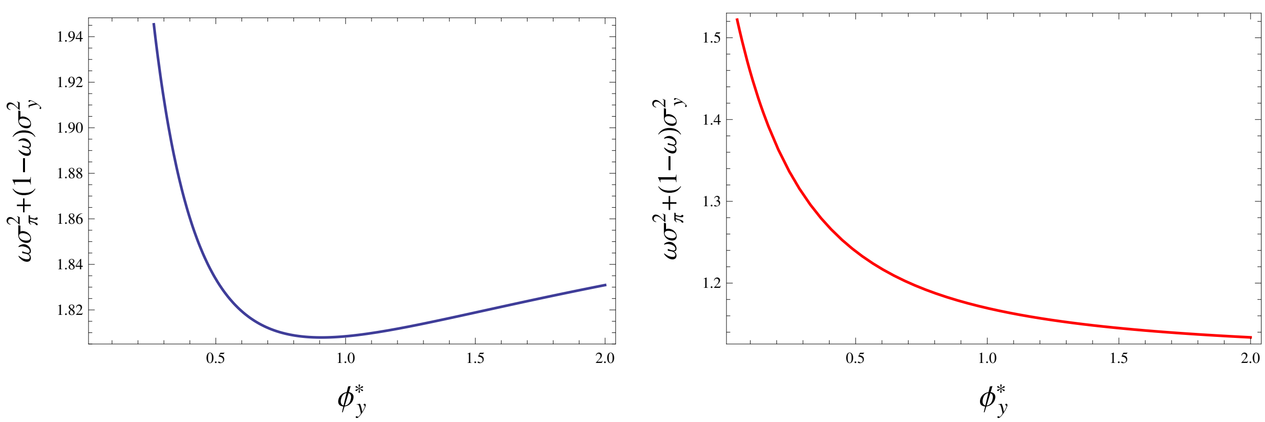


Figure 6: Optimal policies at the BLE and at the REE. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$ and $\omega = 0.9$.



(a) At the BLE

(b) At the REE

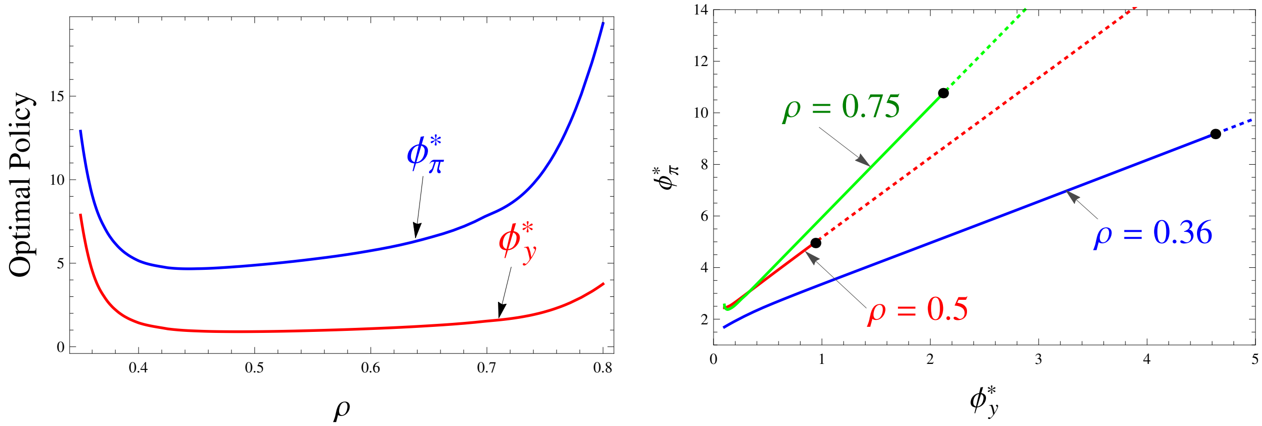
Figure 7: Loss function along the optimal paths (ϕ_y^*, ϕ_π^*) in Figure 6 at the BLE (a) and REE (b). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$ and $\omega = 0.9$.

optimal Taylor rule coefficients (ϕ_y^*, ϕ_π^*) are finite under BLE in this case[31]. As shown in Figure 6a, the corresponding optimal policy is $(\phi_y^*, \phi_\pi^*) = (0.9069, 4.8822)$.

This is different from REE, where there is no finite optimal policy except when measurement errors are considered, as shown in Boehm and House (2014). In fact, from Figure 6 it can be seen that in the case ϕ_y^* is small enough (i.e. < 0.9069) the coefficients ϕ_y^* and ϕ_π^* lie on a manifold and the loss function (4.1) decreases gradually along the manifold within the region, which is similar to REE but with higher ϕ_π^*. However, differently in the case $\phi_y^* > 0.9069$, the loss function (4.1) starts to increase, while in the REE the loss function (4.1) still decreases as shown in Figure 7. That is to say, there exist finite optimal Taylor rule coefficients at the BLE, but not at the REE. This is mainly because at the BLE the actual law of motion has higher volatility (especially for inflation) than at the REE in most cases and minimizing the loss function, i.e. minimizing the weighted variances of output gap and inflation, requires balancing the different responses in terms of policy parameters (ϕ_y, ϕ_π).

Next we investigate how optimal monetary policy changes as the persistence of the underlying shocks is varied. At the REE with measurement error the finite coefficients ϕ_y^* and ϕ_π^* increase as the persistence of shocks grows within some range, see Boehm and House (2014). At the BLE, in addition to this, we find that when the persistence of exogenous shocks becomes sufficiently small with $\rho < 0.4$, the finite coefficients ϕ_y^* and ϕ_π^* start increasing as shown in Figure 4a. Furthermore, Figure 4b suggests that the optimal manifold always moves up as the persistence of shocks ρ grows. The finite optimal policy lies at the point in the optimal manifold connecting the solid and dotted lines in Figure 4b. The location of the optimal point corresponding to finite optimal policies depends on the relative values of variances of output gap and inflation. In the case ρ is large enough, the loss function is mainly dominated by the variance of inflation and hence the optimal policy ϕ_π^* grows quickly converging to ∞ and the slope of $\frac{\phi_\pi^*}{\phi_y^*}$ converging to a relatively large constant. Given the parameters, for small enough ρ, the loss function is mainly

[31]We first select a policy parameter domain (e.g. $[0, 100] \times [1, 100]$) and define a lattice with some small step (*e.g.* 0.01). Then for each lattice point (ϕ_y, ϕ_π), we find the BLE $(\beta_1^*(\phi_y, \phi_\pi), \beta_2^*(\phi_y, \phi_\pi))$ and the corresponding central bank's loss function $\omega\sigma_\pi^2 + (1-\omega)\sigma_y^2$ at the BLE. Finally we interpolate the loss function with respect to (ϕ_y, ϕ_π) to find the finite optimal values. It is easy to get analytic expressions of REE and the corresponding variances. In contrast, it is impossible to obtain analytic expressions of the optimal policy parameters under BLE and therefore we have to rely on numerical approximations. We find consistent results using different ways to calculate the variances (i.e. based on (4.2) and (4.3) or computing the variances as in Appendix B).



(a) Optimal policy

(b) Optimal manifold

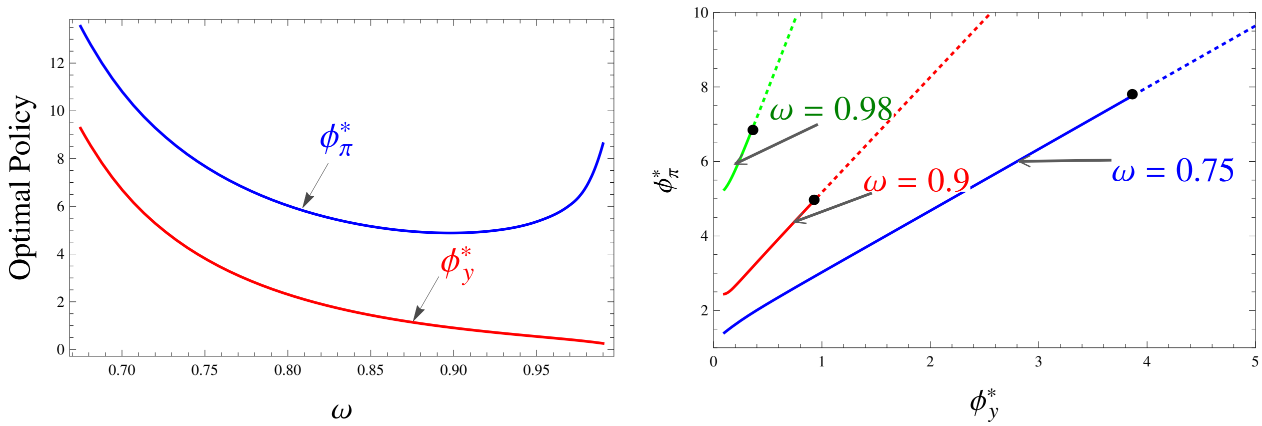
Figure 8: Optimal policies at the BLE with respect to ρ (a) and corresponding optimal manifolds for three different ρ (connection points of solid and dotted curves corresponding to finite optimal policies) (b). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \frac{\sigma_2}{\sigma_1} = 0.5$ and $\omega = 0.9$.

dominated by the variance of the output gap and also the finite optimal policy ϕ_y^* grows quickly converging to ∞ with a relatively small limit of $\frac{\phi_\pi^*}{\phi_y^*}$. Since both shock persistence parameters are fairly low in our estimation exercise, the optimal pair of coefficients for the U.S. data falls within this region as we will be discussed below.

In a similar vein, if the weight on inflation ω is large enough, the loss function is dominated by the variance of inflation. Figure 9a suggests that the optimal policy is $\phi_\pi^* \to \infty$ and $\phi_y^* \to 0$ for $\omega = 1$. Therefore, for large enough ω, finite optimal policy ϕ_π^* increases while ϕ_y^* decreases as ω grows, as shown in Figure 9. For small enough ω, the variance of output gap plays a dominant role and hence the optimal manifold increases as ω grows (see Figure 9b). For a range of ω values there exist finite optimal policies. As ω grows, the finite optimal policy ϕ_π^* first decreases and then increases, while ϕ_y^* decreases within the range of existence of finite optimal policy.

Next we investigate optimal monetary policy at our estimated parameter values. In this case the underlying economic structure is different under BLE and REE, while the moments of inflation and output gap are close under both specifications[32]. Similar to

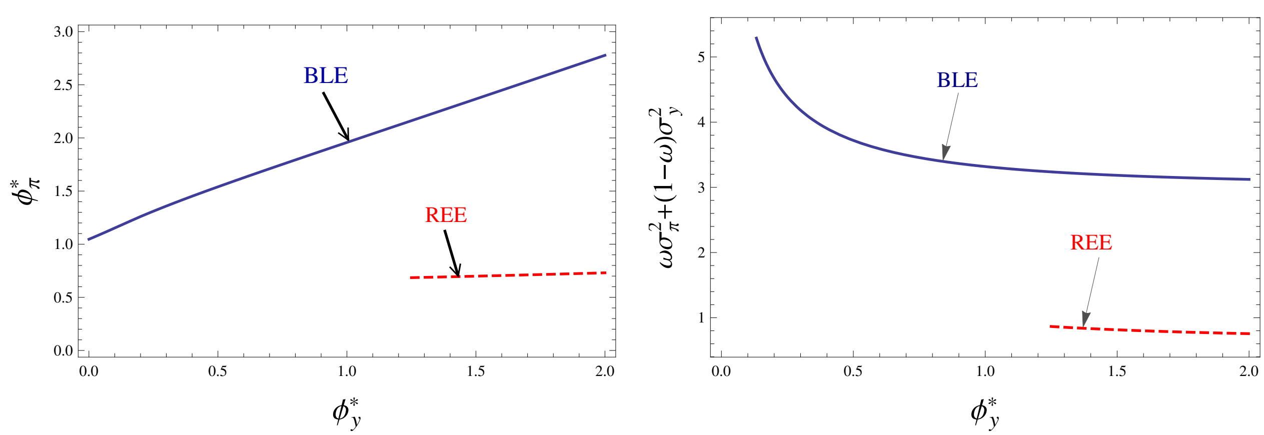
[32]Our analysis with the calibrated values is based on the expressions (3.15)-(3.18). This is no longer applicable since we add interest rate smoothing to our model and relax the assumption that exogenous shocks have the same persistence. Therefore we proceed by computing the fixed-points and the associated variances for each value of policy parameters using our Iterative E-stability algorithm.



(a) Optimal policy

(b) Optimal manifold

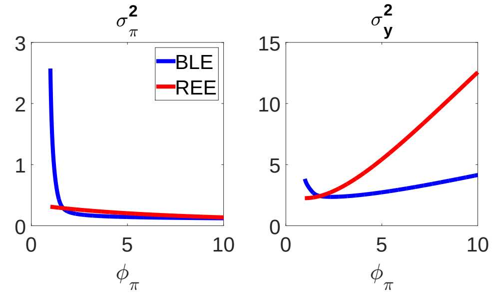
Figure 9: Optimal policies at the BLE with respect to ω (a) and corresponding optimal manifolds for three different ω (connection points of solid and dotted curves corresponding to finite optimal policies) (b) with the contemporaneous interest rate rule. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \frac{\sigma_2}{\sigma_1} = 0.5$ and $\rho = 0.5$.



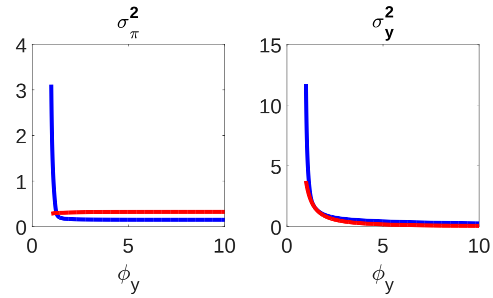
(a) Optimal manifold

(b) Loss function along optimal manifold

Figure 10: Optimal manifolds at the BLE and REE given each ϕ_y^* (a) and the corresponding loss function along the optimal paths (b) with the contemporaneous interest rate rule. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$ and $\omega = 0.5$.



(a) Variances w.r.t. ϕ_π.



(b) Variances w.r.t. ϕ_y.

Figure 11: Variances of inflation and output gap as the monetary policy parameters are varied over the empirically plausible range. The blue and red lines show the responses under BLE and REE respectively.

the calibration exercise, the finite pair of Taylor-coefficients does not exist under REE at the estimated parameter values. While the optimal Taylor-rule is still finite under BLE, the parameter values turn out to be arbitrarily large for both ϕ_y^* and ϕ_π^*. This result is already suggested in Figure 4, which shows that optimal coefficients start to rapidly increase when the persistence of exogenous shocks becomes too small. Recalling that our estimated shock persistence parameters are $\rho_y = 0.43$ and $\rho_\pi = 0.32$, the variance of output gap quickly dominates over this region, leading to a large pair of optimal values. The first row of Figure 11 shows how the variances of inflation and output gap change as we vary the monetary policy coefficients over a more empirically plausible range: we vary ϕ_π over $[1, 10]$ keeping ϕ_y fixed at its estimated value, and ϕ_y over $[0, 10]$ keeping ϕ_π at its estimated value[33]. It is readily seen that a relatively inactive monetary policy has a destabilizing effect under BLE: when ϕ_π falls below 1, or ϕ_y falls below 0.2, the underlying BLE is destabilized and the variances grow exponentially. This is similar to the standard indeterminacy result under REE, but the negative consequences of a passive monetary policy on the economy are larger under BLE. These two figures also show that there is a clear trade-off between inflation and output gap variances when ϕ_y is kept fixed, while the trade-off with higher values of ϕ_y is too small over this empirically plausible range. This also implies that the arbitrarily large pair of optimal coefficients is driven by the small trade-off as ϕ_y is varied.

[33]Since the estimated interest rate smoothing parameters are slightly different, we fix this parameter at $\rho_r = 0.85$ for both models to make the magnitudes of change in ϕ_π and ϕ_y equivalent under BLE and REE.

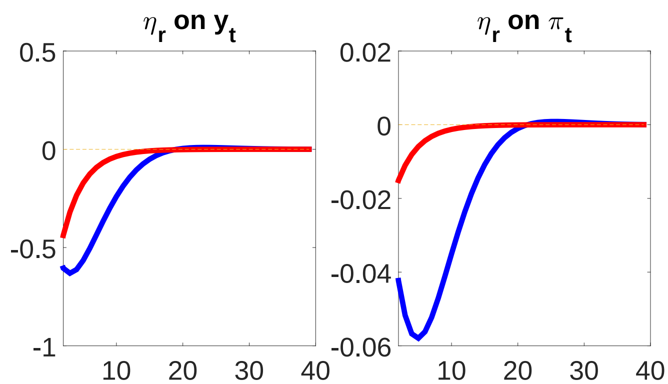


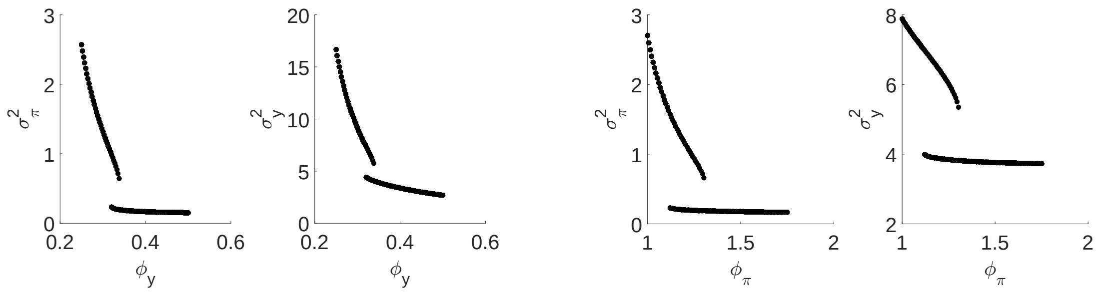
Figure 12: Impulse responses to a monetary policy shock of one standard deviation. The blue and red lines show the responses under BLE and REE respectively.

Figure 12 shows the impulse responses of inflation and output gap to a monetary policy shock of the same size[34]. Both the initial impact, as well as the cumulative impact of the shock are larger under BLE. This is because both the estimated slope of the Philips curve γ and the intertemporal elasticity of substitution φ are larger under BLE, leading to a stronger transmission channel of monetary policy shocks compared with REE. Furthermore the shock takes several quarters to reach its full impact under BLE, leading to hump-shaped responses for both inflation and output gap: this shows that the persistence amplification under BLE is also reflected in the system's response to an exogenous monetary policy shock. Together with Figure 11, this suggests that monetary policy has a stronger impact on the economy under BLE.

We close this section with an illustration of how multiplicity of equilibria can arise under BLE when monetary policy parameters are varied. In all calibration and estimation exercises that we discussed so far, the underlying BLE is unique. However, multiple stable BLE can arise for certain combinations of parameter values. One such case can be observed when we set the parameter values to their CBO-based output gap estimations as given in Table 3, and vary the values of monetary policy parameters.

In this case the estimated Philips curve slope γ is smaller at 0.024 compared with our benchmark estimation with $\gamma = 0.035$. Figure 11 illustrates how inflation and output gap variances change as we vary the monetary policy coefficients $\phi_y \in [0.25, 0.5]$ and $\phi_\pi \in [1, 1.75]$ in this case. While the change in variances follow the same overall pattern as in the benchmark case, we observe two co-existing E-stable BLE over a small range

[34]The shock size is one standard deviation at the estimated parameter values for each case, which is the same under BLE and REE with 0.29.



(a) Variances σ_y^2 and σ_π^2 w.r.t. ϕ_y

(b) Variances σ_y^2 and σ_π^2 w.r.t. ϕ_y

Figure 13: Optimal Policy under BLE and REE at the estimated parameter values.

of parameter values. The underlying BLE is unique at the estimated parameter values, but as the reaction coefficients become smaller, with $\phi_y < 0.33$ or $\phi_\pi < 1.3$, another BLE with higher variance and higher persistence for both inflation and output gap becomes stable and there is a range of parameter values where these two stable BLE co-exist. For smaller values of reaction coefficients, with $\phi_y < 0.31$ or $\phi_\pi < 1.1$, the low persistence/low variance equilibrium becomes unstable and only the high-variance equilibrium remains. These results suggest that, for certain parameter combinations, there is another important role for monetary policy in terms of ensuring that such high volatility equilibria are destabilized, which is only possible with a sufficiently active policy rule. In our example multiplicity of equilibria is driven by a flatter Philips curve compared to our benchmark case, but similar results can arise when other structural parameters are varied, such as the exogenous shock persistence parameters ρ_y and ρ_π, or the intertemporal elasticity of substitution φ.

5 Concluding Remarks

We have generalized the behavioral learning equilibrium concept to a general n-dimensional linear stochastic framework and provided a general method to find and estimate such BLE. We have applied our framework to the 3-equation New Keynesian model. Boundedly rational agents use univariate AR(1) forecasting rules for all endogenous variables. A BLE is parameter free, as along the BLE the two parameters of each rule are pinned down by two observable statistics: the unconditional mean and the first-order autocorrelation. Hence, to a first-order approximation the simple linear forecasting rule is consistent with observed market realizations. Agents gradually update the two coefficients –sample mean and first-order autocorrelation– of their linear rule through sample autocorrelation learning. In the long run, agents thus learn to coordinate on the best univariate linear forecasting rule for each endogenous state variable, without fully recognizing the more complex structure of the economy. In higher-dimensional systems, BLE exist under fairly general conditions and we provide simple stability conditions under learning. Coordination on a simple, parsimonious BLE is self-fulfilling and seems a plausible outcome of the coordination process of individual expectations in large complex socio-economic systems.

A striking feature of BLE is the strong *persistence amplification*: the persistence of output and inflation along a BLE is much higher, often near unit root, than the persistence in the exogenous shocks driving the economy. Due to these features, estimating the 3-equation model on historical data under BLE yields a substantially better model fit and a different economic structure compared with the REE model. This leaves an important role for monetary policy with the goal of stabilizing inflation and output. Different from REE, we find finite optimal Taylor rule coefficients at the BLE in our benchmark calibration. Furthermore, we observe a stronger transmission channel of monetary policy at the estimated parameter values under BLE. Future work should study BLE and corresponding optimal policies in more general New Keynesian models.

Appendix

A Mean of the rational expectations equilibrium

Using (2.10-2.11) and (2.15-2.18) the mean satisfies

$$\begin{aligned}
\overline{\mathbf{x}^*} &= (\boldsymbol{I}-\boldsymbol{c}_1)^{-1}(\boldsymbol{c}_0+\boldsymbol{c}_2\overline{\boldsymbol{u}}) \\
&= (\boldsymbol{I}-\boldsymbol{c}_1)^{-1}(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)^{-1}(\boldsymbol{b}_0+\boldsymbol{b}_1\boldsymbol{c}_2\boldsymbol{a})+(\boldsymbol{I}-\boldsymbol{c}_1)^{-1}\boldsymbol{c}_2(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a} \\
&= (\boldsymbol{I}-\boldsymbol{c}_1)^{-1}(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0+(\boldsymbol{b}_1\boldsymbol{c}_2(\boldsymbol{I}-\boldsymbol{\rho})+(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)\boldsymbol{c}_2)(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}] \\
&= [(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)(\boldsymbol{I}-\boldsymbol{c}_1)]^{-1}[\boldsymbol{b}_0+\boldsymbol{b}_3(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}] \\
&= (\boldsymbol{I}-\boldsymbol{b}_1-\boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0+\boldsymbol{b}_3(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}].
\end{aligned}$$

B Autocorrelation in the n-dimensional case

The purpose of this appendix is to show that the first-order autocorrelation coefficients of the stochastic stationary system (2.24) are continuous functions with respect to $(\beta_1,\beta_2,\cdots,\beta_n)$ and the other related parameters. Rewrite model (2.24) as

$$\begin{cases}\boldsymbol{x}_t-\overline{\boldsymbol{x}}=(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})+\boldsymbol{b}_3(\boldsymbol{u}_t-\overline{\boldsymbol{u}})+\boldsymbol{b}_4\boldsymbol{v}_t,\\ \boldsymbol{u}_t-\overline{\boldsymbol{u}}=\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})+\boldsymbol{\varepsilon}_t.\end{cases} \tag{B.1}$$

That is,

$$\begin{cases}\boldsymbol{x}_t-\overline{\boldsymbol{x}}=(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})+\boldsymbol{b}_3\boldsymbol{\varepsilon}_t+\boldsymbol{b}_4\boldsymbol{v}_t,\\ \boldsymbol{u}_t-\overline{\boldsymbol{u}}=\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})+\boldsymbol{\varepsilon}_t.\end{cases} \tag{B.2}$$

$$\begin{aligned}
\boldsymbol{\Gamma}(-1) &= E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'] \\
&= E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'+\boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})' \\
&\quad +\boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'\Big] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\boldsymbol{\Gamma}(0)+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\boldsymbol{\Gamma}(0)+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})']. \qquad \text{(B.3)}
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\Gamma}(0) &= E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] \\
&= E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})' + \boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})' + \boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t - \overline{\boldsymbol{x}})' + \boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'\Big] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)\boldsymbol{\Gamma}(1) + \boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] + \boldsymbol{b}_3E[\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] + \boldsymbol{b}_4E[\boldsymbol{v}_t(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)\boldsymbol{\Gamma}(1) + \boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] + \boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3' + \boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4'. \qquad \text{(B.4)}
\end{aligned}
$$

Note that $E[\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] = E\Big[\boldsymbol{\varepsilon}_t((\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}}))' + \boldsymbol{\varepsilon}_t(\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}}))' + \boldsymbol{\varepsilon}_t(\boldsymbol{b}_3\boldsymbol{\varepsilon}_t)' + \boldsymbol{\varepsilon}_t(\boldsymbol{b}_4\boldsymbol{v}_t)'\Big] = \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3'$ and $E[\boldsymbol{v}_t(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] = E\Big[\boldsymbol{v}_t((\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}}))' + \boldsymbol{v}_t(\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}}))' + \boldsymbol{v}_t(\boldsymbol{b}_3\boldsymbol{\varepsilon}_t)' + \boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{b}_4\boldsymbol{v}_t)'\Big] = \boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4'$.

Based on (B.3), (B.4) and $\boldsymbol{\Gamma}(-1) = \boldsymbol{\Gamma}(1)'$,

$$
\begin{aligned}
\boldsymbol{\Gamma}(0) &= (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)\boldsymbol{\Gamma}(0)(\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)' + (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'](\boldsymbol{b}_3\boldsymbol{\rho})' \\
&\quad + \boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] + \boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3' + \boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4'.
\end{aligned}
$$

In order to obtain the expression of $\boldsymbol{\Gamma}(0)$, we use column stacks of matrices. Suppose $vec(\boldsymbol{K})$ is the vectorization of a matrix $\boldsymbol{K}$ and $\otimes$ is the Kronecker product[35]. Under the assumption that all the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2$ are inside the unit circle, based on the property of Kronecker product[36], it is easy to see all the eigenvalues of $(\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)$ lie inside the unit circle and hence $[\boldsymbol{I} - (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)]^{-1}$ exist. Therefore,

$$
\begin{aligned}
vec(\boldsymbol{\Gamma}(0)) &= [\boldsymbol{I} - (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)]^{-1}[((\boldsymbol{b}_3\boldsymbol{\rho}) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2))vec(E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) \\
&\quad + (\boldsymbol{I} \otimes (\boldsymbol{b}_3\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})']) + vec(\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3' + \boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4')]. \qquad \text{(B.5)}
\end{aligned}
$$

Thus in order to obtain $\boldsymbol{\Gamma}(1)$ and $\boldsymbol{\Gamma}(0)$, we need calculate $E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']$ and $E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})']$.

$$
\begin{aligned}
& E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'] \\
={}& E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})' + \boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})' + \boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{u}_t - \overline{\boldsymbol{u}})' + \boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'\Big] \\
={}& E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t'] + \boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t'] \\
& + \boldsymbol{b}_3\boldsymbol{\varepsilon}_t[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t'] + \boldsymbol{v}_t[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t']\Big] \\
={}& (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)E[(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})']\boldsymbol{\rho}' + \boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']\boldsymbol{\rho}' + \boldsymbol{b}_3\boldsymbol{\Sigma}_{\varepsilon}.
\end{aligned}
$$

[35] One property of column stacks is that the column stack of a product of three matrices is $vec(ABC) = (C' \otimes A)vec(B)$. For more details on this and related properties, see Magnus and Neudecker(1988, Chapter 2) and Evans and Honkapohja (2001, Section 5.7).

[36] The eigenvalues of $\widehat{A} \otimes \widehat{B}$ are the mn numbers $\lambda_r\mu_s, r = 1, 2, \cdots, m, s = 1, 2, \cdots, n$ where $\lambda_1, \cdots, \lambda_m$ are the eigenvalues of $m \times m$ matrix $\widehat{A}$ and $\mu_1, \cdots, \mu_n$ are the eigenvalues of $n \times n$ matrix $\widehat{B}$; see Lancaster and Tismenetsky (1985).

Correspondingly,

$$\begin{aligned}
& vec(E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) \\
= & [\boldsymbol{I} - \boldsymbol{\rho} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)]^{-1}[vec(\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']\boldsymbol{\rho}') + vec(\boldsymbol{b}_3\boldsymbol{\Sigma}_\varepsilon)] \\
= & [\boldsymbol{I} - \boldsymbol{\rho} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)]^{-1}[(\boldsymbol{\rho} \otimes (\boldsymbol{b}_3\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) + (\boldsymbol{I} \otimes \boldsymbol{b}_3)vec\boldsymbol{\Sigma}_\varepsilon)] \\
= & [\boldsymbol{I} - \boldsymbol{\rho} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)]^{-1}[(\boldsymbol{\rho} \otimes (\boldsymbol{b}_3\boldsymbol{\rho}))[\boldsymbol{I} - \boldsymbol{\rho} \otimes \boldsymbol{\rho}]^{-1} + (\boldsymbol{I} \otimes \boldsymbol{b}_3)]vec(\boldsymbol{\Sigma}_\varepsilon). \qquad \text{(B.6)}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'] \\
= & E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})' + \boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})' + \boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})' + \boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\Big] \\
= & (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'] + \boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'].
\end{aligned}$$

Thus based on (B.6),

$$\begin{aligned}
& vec(E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})']) \\
= & (\boldsymbol{I} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2))vec(E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) + (\boldsymbol{I} \otimes (\boldsymbol{b}_3\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) \\
= & (\boldsymbol{I} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2))[\boldsymbol{I} - \boldsymbol{\rho} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2)]^{-1}[(\boldsymbol{\rho} \otimes (\boldsymbol{b}_3\boldsymbol{\rho}))[\boldsymbol{I} - \boldsymbol{\rho} \otimes \boldsymbol{\rho}]^{-1} + (\boldsymbol{I} \otimes \boldsymbol{b}_3)]vec(\boldsymbol{\Sigma}_\varepsilon) \\
& + (\boldsymbol{I} \otimes (\boldsymbol{b}_3\boldsymbol{\rho}))[\boldsymbol{I} - \boldsymbol{\rho} \otimes \boldsymbol{\rho}]^{-1}vec(\boldsymbol{\Sigma}_\varepsilon). \qquad \text{(B.7)}
\end{aligned}$$

Therefore based on (B.7), the expression of matrix $E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'$ can be obtained. Then by transposing the matrix $E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'$, we can obtain $vec(E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'])$. Furthermore, combining this with (B.6), we obtain the variance-covariance matrix $\boldsymbol{\Gamma}(0)$ from (B.5) and further $\boldsymbol{\Gamma}(1)$ from (B.3). Based on the properties of matrices operations, it is easy to see that the entries of matrices $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$ are smooth functions with respect to $(\beta_1, \beta_3, \cdots, \beta_n)$ and the other related parameters. Thus the first-order autocorrelation coefficients of the nontrivial stochastic stationary system (2.24) are continuous functions with respect to $(\beta_1, \beta_3, \cdots, \beta_n)$ and the other related parameters.

Moments for the zero-mean case

Taking (2.34) as the starting point and assuming $\bar{\gamma} = \mathbf{0}$, we have

$$S_t = \gamma_1 S_{t-1} + \gamma_2 \boldsymbol{\beta}^2 S_{t-1} + \gamma_3 \eta_t \qquad \text{(B.8)}$$

The first-order covariance matrix is given by:

$$\mathbb{E}[S_t S_{t-1}] = (\gamma_1 + \gamma_2 \boldsymbol{\beta}^2)\mathbb{E}[S_{t-1} S_{t-1}] + \gamma_3 \mathbb{E}[\eta_t S_{t-1}] \qquad \text{(B.9)}$$

We have $\mathbb{E}[\eta_t S_{t-1}] = 0$, while $\mathbb{E}[\eta_{t-1} S_{t-1}] = \mathbb{E}[\eta_{t-1}((\gamma_1 + \gamma_2 \boldsymbol{\beta}^2) S_{t-1} + \gamma_3 \eta_t] = \gamma_3 \boldsymbol{\Sigma_\eta}$. Further denoting $(\gamma_1 + \gamma_2 \boldsymbol{\beta}^2) = M(\boldsymbol{\beta})$, the first-order covariance matrix $\mathbb{E}[S_t S_{t-1}] = \boldsymbol{\Gamma}(-1)$ and the variance-covariance matrix $\mathbb{E}[S_t S_t] = \boldsymbol{\Gamma}(0)$, the expression in (B.9) reduces to:

$$\boldsymbol{\Gamma}(-1) = M(\boldsymbol{\beta})\boldsymbol{\Gamma}(0) \tag{B.10}$$

Taking the variance on both sides of (B.8) yields:

$$\boldsymbol{\Gamma}(0) = M(\boldsymbol{\beta})\boldsymbol{\Gamma}(0)M(\boldsymbol{\beta})' + \gamma_3 \boldsymbol{\Sigma_\eta} \gamma_3' \tag{B.11}$$

Vectorizing both sides implies:

$$Vec(\boldsymbol{\Gamma}(0)) = Vec(M(\boldsymbol{\beta})\boldsymbol{\Gamma}(0)M(\boldsymbol{\beta})' + \gamma_3 \boldsymbol{\Sigma_\eta} \gamma_3') \tag{B.12}$$

Using $Vec(ABC) = (C' \otimes A)Vec(B)$, and $Vec(A + B) = Vec(A) + Vec(B)$, the above expression reduces to:

$$Vec(\boldsymbol{\Gamma}(0)) = (M(\boldsymbol{\beta}) \otimes M(\boldsymbol{\beta}))Vec(\boldsymbol{\Gamma}(0)) + (\gamma_3 \otimes \gamma_3)Vec(\boldsymbol{\Sigma_\eta}) \tag{B.13}$$

Hence

$$Vec(\boldsymbol{\Gamma}(0)) = [I - M(\boldsymbol{\beta}) \otimes M(\boldsymbol{\beta})]^{-1}(\gamma_3 \otimes \gamma_3)Vec(\boldsymbol{\Sigma_\eta}) \tag{B.14}$$

which yields the expression in (2.33).

C Proof of Proposition 2 (stability under SAC-learning)

Set $\gamma_t = (1 + t)^{-1}$. For the state dynamics equations in (2.28) and (2.8)[37], since all functions are smooth, the SAC-learning rule satisfies the conditions (A.1-A.3) of Section 6.2.1 in Evans and Honkapohja (2001, p.124).

In order to check the conditions (B.1-B.2) of Section 6.2.1 in Evans and Honkapohja (2001, p.125), we rewrite the system in matrix form by

$$\boldsymbol{X}_t = \widetilde{\boldsymbol{A}}(\boldsymbol{\theta}_{t-1})\boldsymbol{X}_{t-1} + \widetilde{\boldsymbol{B}}(\boldsymbol{\theta}_{t-1})\boldsymbol{W}_t,$$

where $\boldsymbol{\theta}_t' = (\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t, \boldsymbol{R}_t), \boldsymbol{X}_t' = (1, \boldsymbol{x}_t', \boldsymbol{x}_{t-1}', \boldsymbol{u}_t')$ and $\boldsymbol{W}_t' = (1, \boldsymbol{v}_t', \boldsymbol{\varepsilon}_t')$,

$$\widetilde{\boldsymbol{A}}(\boldsymbol{\theta}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \boldsymbol{b_0} + \boldsymbol{b_1}(\boldsymbol{I} - \boldsymbol{\beta}^2)\boldsymbol{\alpha} + \boldsymbol{b_2 a} & \boldsymbol{b_1}\boldsymbol{\beta}^2 & \mathbf{0} & \boldsymbol{b_2 \rho} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{a} & \mathbf{0} & \mathbf{0} & \boldsymbol{\rho} \end{pmatrix},$$

[37]For convenience of theoretical analysis, one can set $\mathbf{S_{t-1}} = \mathbf{R_t}$.

$$\widetilde{\boldsymbol{B}}(\boldsymbol{\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & \mathbf{I} & \mathbf{b_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

Based on the properties of eigenvalues, see e.g. Evans and Honkapohja (2001, p.117), all the eigenvalues of $\widetilde{\boldsymbol{A}}(\boldsymbol{\theta})$ include 0 (multiple $n+1$), the eigenvalues of $\boldsymbol{\rho}$ and $\boldsymbol{b_1}\boldsymbol{\beta}^2$. Thus based on the assumptions, all the eigenvalues of $\widetilde{\boldsymbol{A}}(\boldsymbol{\theta})$ lie inside the unit circle. Moreover, it is easy to see all the other conditions for Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001) are also satisfied.

Since $\boldsymbol{x}_t$ is stationary, then the limits

$$\sigma_i^2 := \lim_{t\to\infty} E(x_{i,t} - \alpha_i)^2, \qquad \sigma^2_{x_i x_{i,-1}} := \lim_{t\to\infty} E(x_{i,t} - \alpha_i)(x_{i,t-1} - \alpha_i)$$

exist and are finite. Hence according to Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001, p.126), the associated ODE is

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{R}^{-1}[\boldsymbol{E} - \boldsymbol{\beta}\boldsymbol{\Omega}] = \boldsymbol{R}^{-1}\boldsymbol{\Omega}[\boldsymbol{E}\boldsymbol{\Omega}^{-1} - \boldsymbol{\beta}], \\ \dfrac{d\boldsymbol{R}}{d\tau} = \boldsymbol{\Omega} - \boldsymbol{R}, \end{cases} \tag{C.1}$$

where $\boldsymbol{R}$ is a diagonal matrix with the i-th diagonal entry R_i and $\boldsymbol{\Omega}$, $\boldsymbol{E}$ are also diagonal matrices as defined in Section 2. As shown in Evans and Honkapohja (2001), a BLE corresponds to a fixed point of the following ODE (C.2).

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{G} - \boldsymbol{\beta}. \end{cases} \tag{C.2}$$

Note that $\boldsymbol{\beta}$ and $\boldsymbol{G}$ are both diagonal matrices. The Jacobian matrix of C.2 is, in fact, equivalent to

$$\begin{pmatrix} (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^{*2})^{-1}(\boldsymbol{b}_1 - \boldsymbol{I}) & \boldsymbol{\varrho} \\ \mathbf{0} & \boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*) - \boldsymbol{I} \end{pmatrix},$$

where $\boldsymbol{DG}_{\boldsymbol{\beta}}$ is a Jacobian matrix with the (i,j)-th entry $\frac{\partial G_i}{\partial \beta_j}$ and the form of matrix $\boldsymbol{\varrho}$ is omitted since it is not needed in the proof. Therefore, if all the eigenvalues of $(\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^{*2})^{-1}(\boldsymbol{b}_1 - \boldsymbol{I})$ have negative real parts, and all the eigenvalues of $\boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)$ have real parts less than 1, the SAC-learning $(\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t)$ converges to the BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ as time t tends to ∞.

D Proof of Propositions 3 and 4

Proof of Proposition 3: The first part of the Proposition follows from Definition (2.1). The first consistency requirement is satisfied since $\boldsymbol{\alpha}^* = \mathbf{0}$ by assumption, and the second consistency requirement is satisfied since we have $G(\boldsymbol{\beta}^K) \approx \boldsymbol{\beta}^K$ after convergence, which implies $\boldsymbol{\beta}^K \approx \boldsymbol{\beta}^*$.

For the second part, define the spectral radius of a matrix A as $\rho(A) = max_i|\lambda_i(A)|$, where λ_i denotes an eigenvalue of A. Let $(\mathbf{0}, \boldsymbol{\beta}^*)$ be an iteratively E-stable fixed-point, such that $\rho(DG(\boldsymbol{\beta}^*)) < 1$. Using Lemma 5.6.10 of Horn & Johnson (1985), one can show that there exists a matrix norm $||.|| \in \mathbb{R}^n$ with

$$\rho(DG(\boldsymbol{\beta^*})) \leq ||DG(\boldsymbol{\beta^*})|| \leq \rho(DG(\boldsymbol{\beta^*})) + \epsilon$$

for some $\epsilon \in \mathbb{R}^+$. Defining $\epsilon < 1 - \rho((\boldsymbol{\beta^*}))$, it follows that $||DG(\boldsymbol{\beta^*})|| < 1$.
Let $\hat{D} \subset \mathbb{R}^N$ be a small open convex interval around $\boldsymbol{\beta^*}$, such that $\forall x \in D, \rho(x) < 1$ and $G(\boldsymbol{\beta}) : D \to \mathbb{R}^N$ has continuous partial derivatives. Letting $\boldsymbol{\beta_1}, \boldsymbol{\beta_2} \in D$, it follows that

$$||G(\boldsymbol{\beta_2}) - G(\boldsymbol{\beta_1})|| \leq \int_0^1 ||DG(\boldsymbol{\beta_1} + t(\boldsymbol{\beta_2} - \boldsymbol{\beta_1}))(\boldsymbol{\beta_2} - \boldsymbol{\beta_1})dt|| \leq$$

$$\int_0^1 ||DG(\boldsymbol{\beta_1} + t(\boldsymbol{\beta_2} - \boldsymbol{\beta_1}))|| \ ||(\boldsymbol{\beta_2} - \boldsymbol{\beta_1})||dt \leq q||\boldsymbol{\beta_2} - \boldsymbol{\beta_1}||$$

for some $q \in \mathbb{R}^+$ with $||DG(\boldsymbol{\beta^*})|| \leq q < 1$. Hence Banach fixed-point theorem implies that $G(\boldsymbol{\beta})$ is a contraction mapping on D with a Lipschitz constant $||DG(\boldsymbol{\beta^*})||$.

Next we shot that the condition $\rho(\boldsymbol{\beta^*}) < 1$ is not necessary for the Quasi-Newton iteration to be a contraction. The iteration is given as :

$$\boldsymbol{\beta^{(k+1)}} = \boldsymbol{\beta^{(k)}} - DF(\boldsymbol{\beta^{(k)}})^{-1}F(\boldsymbol{\beta^{(k)}})$$

with $F(\boldsymbol{\beta^{(k)}} = G(\boldsymbol{\beta^{(k)}}) - \boldsymbol{\beta^{(k)}}$. Defining $H(\boldsymbol{\beta}) = \boldsymbol{\beta} - DF(\boldsymbol{\beta})^{-1}F(\boldsymbol{\beta})$, we need to show that $H(\boldsymbol{\beta})$ is a contraction mapping. Note that

$$DH(\boldsymbol{\beta}) = DF(\boldsymbol{\beta})^{-2}D^2F(\boldsymbol{\beta})F(\boldsymbol{\beta})$$

with $DF(\boldsymbol{\beta}) = DG(\boldsymbol{\beta}) - I$ and $D^2F(\boldsymbol{\beta}) = D^2G(\boldsymbol{\beta})G(\boldsymbol{\beta})$, which implies

$$DH(\boldsymbol{\beta}) = D^2G(\boldsymbol{\beta})\boldsymbol{\beta} - D^2G(\boldsymbol{\beta})G(\boldsymbol{\beta})$$

Letting $\hat{D}$ be an open convex set around $\boldsymbol{\beta^*}$, and $\boldsymbol{\beta_1}, \boldsymbol{\beta_2}$ two points therein as before, it follows that

$$||H(\boldsymbol{\beta_2}) - H(\boldsymbol{\beta_1})|| \leq \int_0^1 ||DH(\boldsymbol{\beta_1} + t(\boldsymbol{\beta_2} - \boldsymbol{\beta_1}))(\boldsymbol{\beta_2} - \boldsymbol{\beta_1})dt|| \leq$$

$$\int_0^1 ||DH(\boldsymbol{\beta_1} + t(\boldsymbol{\beta_2} - \boldsymbol{\beta_1}))|| \ ||(\boldsymbol{\beta_2} - \boldsymbol{\beta_1})||dt \leq q||\boldsymbol{\beta_2} - \boldsymbol{\beta_1}||$$

for some $q \in \mathbb{R}^+$ with $||DH(\boldsymbol{\beta^*})|| \leq q < 1$. Note that $DH(\boldsymbol{\beta}) \to 0$ as $\boldsymbol{\beta} \to \boldsymbol{\beta^*}$. Accordingly, Banach fixed-point theorem implies that $H(\boldsymbol{\beta})$ is a contraction mapping on D with a Lipschitz constant $q \to 0$. Importantly, this result holds for *all* E-stable and E-unstable fixed-points satisfying $G(\boldsymbol{\beta^*}) = \boldsymbol{\beta^*}$, hence the Quasi-Newton iteration may also converge to E-unstable fixed-points.

Proof of Proposition 4: The first part of the proposition follows again from Definition (2.1). The first consistency requirement is satisfied by construction because the mean coefficients are assumed to be zero, and the second consistency requirement is satisfied as long as $\theta^{(k)} \approx \theta^{(k-1)}$, in which case we have $G(\boldsymbol{\beta^{(k-1)}}, \theta^{(k-1)}) = \boldsymbol{\beta^{(k)}} \approx \boldsymbol{\beta^{(k-1)}}$. Next we show that the contraction mapping argument extends to this case after re-writing the maximization problem. Note that at the last step, the estimation problem can be written as

$$\theta^* = argmax_\theta p(\theta|Y_{1:T}, \boldsymbol{\beta^*})$$

with $p(\theta|Y_{1:T}, \boldsymbol{\beta^*}) = \frac{p(Y_{1:T}|\theta, \boldsymbol{\beta^*})p(\theta)}{p(Y_{1:T})}$. At each step of the iteration, we have

$$\hat{\theta}^{(k)} = argmax_\theta p(\theta|Y_{1:T}, \boldsymbol{\beta^{(k)}})$$

Let $\eta(.)$ be some function such that $\hat{\theta^{(k)}} = \eta(Y_{1:T}, \boldsymbol{\beta^{(k)}})$. Since the observables $Y_{1:T}$ are the same at each iteration, we can further re-write this as $\hat{\theta^{(k)}} = \hat{\eta}(\boldsymbol{\beta^{(k)}})$ for some $\hat{\eta}$. Then the iteration reduces to:

$$\boldsymbol{\beta^{(k)}} = G(\boldsymbol{\beta^{(k-1)}}, \hat{\theta}^{(k-1)}) = G(\boldsymbol{\beta^{(k-1)}}, \hat{\eta}(\boldsymbol{\beta^{(k-1)}})) = \hat{G}(\boldsymbol{\beta^{(k-1)}})$$

for some $\hat{G(.)}$. The last expression has the same functional form as the fixed-point iteration of Proposition 3. Further note that, $\rho(DG(\boldsymbol{\beta^*})) < 1$ implies $\rho(D\hat{G}(\boldsymbol{\beta^*})) < 1$. Hence, assuming $\rho(DG(\boldsymbol{\beta^*})) < 1$, one can follow the same steps of the previous proof to show that $G(\boldsymbol{\beta}, \hat{\eta}(\boldsymbol{\beta}))$ is a contraction on a set $\hat{D} \subset \mathbb{R}^N$.

E Eigenvalues of matrix $\boldsymbol{B\beta}^2$

The characteristic polynomial of $\mathbf{B}\boldsymbol{\beta^2}$ is given by $h(\nu) = \nu^2 + c_1\nu + c_2$, where

$$c_1 = -\frac{\beta_1^2 + [\gamma\varphi + \lambda(1 + \varphi\phi_y)]\beta_2^2}{1 + \gamma\varphi\phi_\pi + \varphi\phi_y}, \quad c_2 = \frac{\lambda\beta_1^2\beta_2^2}{1 + \gamma\varphi\phi_\pi + \varphi\phi_y}.$$

Both of the eigenvalues of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ are inside the unit circle if and only if both of the following conditions hold (see Elaydi, 1999):

$$h(1) > 0, \quad h(-1) > 0, \quad |h(0)| < 1.$$

It is easy to see $h(-1) > 0, |h(0)| < 1$ for any $\beta_i \in [-1, 1]$. Note that

$$\begin{aligned} h(1) &= \frac{(1-\beta_1^2)(1-\lambda\beta_2^2) + \gamma\varphi\phi_\pi + \varphi\phi_y - (\gamma\varphi + \lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}, \\ &\geq \frac{\varphi[\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y]}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}. \end{aligned}$$

Thus if $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, then $h(1) > 0$. Therefore, both eigenvalues of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ lie inside the unit circle.

F First-order autocorrelation coefficients of output gap and inflation

Now we calculate $\boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Define $\boldsymbol{z}_t = \boldsymbol{x}_t - \bar{\mathbf{x}}$. Then in order to obtain $\boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, we first calculate $\boldsymbol{E}(z_t z_{t-1}')$ and $\boldsymbol{E}(z_t z_t')$. Rewrite model (3.12) into its VARMA$(1, \infty)$ representation

$$\boldsymbol{z}_t = \boldsymbol{B}\boldsymbol{\beta}^2 \mathbf{z}_{t-1} + \boldsymbol{C} \sum_{n=0}^{\infty} \rho^n \boldsymbol{\varepsilon}_{t-n}. \tag{F.1}$$

Since both eigenvalues of $\boldsymbol{B}\boldsymbol{\beta}^2$ lie inside the unit circle under the assumption $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$ (see Appendix E), then

$$\boldsymbol{z}_t = \mathbf{C}[\rho\mathbf{I} - \mathbf{C}^{-1}\boldsymbol{B}\boldsymbol{\beta}^2\mathbf{C}]^{-1} \sum_{n=0}^{\infty} [\rho^{n+1}\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\varepsilon}_{t-n}.$$

Note ρ is a scalar number and $\boldsymbol{I}$ is a 2×2 identity matrix. Based on i.i.d. assumption of $\boldsymbol{\varepsilon}_t$,

$$\begin{aligned} \boldsymbol{E} z_t \boldsymbol{z}_t' &= \mathbf{C}[\rho\mathbf{I} - \mathbf{C}^{-1}\boldsymbol{B}\boldsymbol{\beta}^2\mathbf{C}]^{-1} \sum_{n=0}^{\infty} [\rho^{n+1}\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\Sigma}[\rho^{n+1}\mathbf{I} - (\mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C})'] \cdot \\ & \quad [\rho\mathbf{I} - (\mathbf{C}^{-1}\boldsymbol{B}\boldsymbol{\beta}^2\mathbf{C})']^{-1}\mathbf{C}', \end{aligned} \tag{F.2}$$

where $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$.

In the following we try to obtain the expression of the matrix $\boldsymbol{E} z_t \boldsymbol{z}_t'$ and hence we first calculate the matrix $\sum_{n=0}^{\infty}[\rho^{n+1}\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\Sigma}[\rho^{n+1}\mathbf{I} - (\mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C})']$ and $\mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C}$. Note that

$$\boldsymbol{B}\boldsymbol{\beta}^2 = \frac{1}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}\left[\begin{array}{cc} \beta_1^2 & \varphi(1-\lambda\phi_\pi)\beta_2^2 \\ \gamma\beta_1^2 & (\gamma\varphi + \lambda(1+\varphi\phi_y))\beta_2^2 \end{array}\right].$$

$\boldsymbol{B}\boldsymbol{\beta}^2$ has two eigenvalues[38]

$$\begin{aligned}
\lambda_1 &= \frac{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2] + \sqrt{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]^2 - 4\lambda\beta_1^2\beta_2^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}}{2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}, \\
\lambda_2 &= \frac{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2] - \sqrt{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]^2 - 4\lambda\beta_1^2\beta_2^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}}{2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}.
\end{aligned}$$

Their corresponding eigenvectors are

$$\begin{aligned}
P_1 &= \Big[\frac{\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\ \lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big]', \\
P_2 &= \Big[\frac{\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\ \lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big]'.
\end{aligned}$$

Let $\boldsymbol{P} = [P_1, P_2]$. Then

$$\boldsymbol{C^{-1}B\beta^2C} = \boldsymbol{C^{-1}P}\left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right](C^{-1}\boldsymbol{P})^{-1},$$

where

$$\begin{aligned}
&\boldsymbol{C^{-1}P} \\
&= \left[\begin{array}{cc}
\frac{(1+\varphi\phi_y)\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \varphi\phi_\pi\Big(\lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big) & \frac{(1+\varphi\phi_y)\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \varphi\phi_\pi\Big(\lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big) \\
\frac{-\gamma\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \Big(\lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big) & \frac{-\gamma\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \Big(\lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big)
\end{array}\right] \\
&=: \left[\begin{array}{cc} d_1 & d_2 \\ d_3 & d_4 \end{array}\right].
\end{aligned}$$

Correspondingly

$$(C^{-1}\boldsymbol{P})^{-1} = \frac{1}{d_1 d_4 - d_2 d_3}\left[\begin{array}{cc} d_4 & -d_2 \\ -d_3 & d_1 \end{array}\right],$$

[38]In the special case $\lambda_1 = \lambda_2$, although $\boldsymbol{B}\boldsymbol{\beta}^2$ is not diagonalizable, the expressions of first-order auto-correlations (3.13) and (3.14) still hold based on the Jordan normal form of matrix $\boldsymbol{B}\boldsymbol{\beta}^2$. Without loss of generality, in the following we assume $\lambda_1 \neq \lambda_2$.

where

$$d_1 d_4 - d_2 d_3 = \det(\boldsymbol{C}^{-1} P) = \varphi(1-\lambda\phi_\pi)\beta_2^2(\lambda_2-\lambda_1).$$

Hence

$$\begin{aligned}
\boldsymbol{C^{-1}(B\beta^2)^{n+1}C} &= \boldsymbol{C^{-1}P}\begin{bmatrix} \lambda_1^{n+1} & 0 \\ 0 & \lambda_2^{n+1} \end{bmatrix}\boldsymbol{(C^{-1}P)^{-1}} \\
&= \frac{1}{d_1d_4-d_2d_3}\begin{bmatrix} d_1d_4\lambda_1^{n+1} - d_2d_3\lambda_2^{n+1} & d_1d_2(\lambda_2^{n+1}-\lambda_1^{n+1}) \\ d_3d_4(\lambda_1^{n+1}-\lambda_2^{n+1}) & d_1d_4\lambda_2^{n+1}-d_2d_3\lambda_1^{n+1} \end{bmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\boldsymbol{\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C} = \\
&\frac{1}{d_1d_4-d_2d_3}\begin{bmatrix} d_1d_4(\rho^{n+1}-\lambda_1^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_2^{n+1}) & -d_1d_2(\lambda_2^{n+1}-\lambda_1^{n+1}) \\ -d_3d_4(\lambda_1^{n+1}-\lambda_2^{n+1}) & d_1d_4(\rho^{n+1}-\lambda_2^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_1^{n+1}) \end{bmatrix}.
\end{aligned}$$

Therefore

$$\boldsymbol{[\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^{n+1}I - (C^{-1}(B\beta^2)^{n+1}C)']}$$

$$= \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} s_1(n+1) & s_2(n+1) \\ s_2(n+1) & s_3(n+1) \end{bmatrix},$$

where

$$\begin{aligned}
s_1(n+1) &= \sigma_1^2[d_1d_4(\rho^{n+1}-\lambda_1^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_2^{n+1})]^2 + \sigma_2^2[d_1d_2(\lambda_2^{n+1}-\lambda_1^{n+1})]^2, \\
s_2(n+1) &= \sigma_1^2 d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})[d_1d_4(\rho^{n+1}-\lambda_1^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_2^{n+1})] + \\
&\quad \sigma_2^2 d_1d_2(\lambda_1^{n+1}-\lambda_2^{n+1})[d_1d_4(\rho^{n+1}-\lambda_2^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_1^{n+1})], \\
s_3(n+1) &= \sigma_1^2[d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})]^2 + \sigma_2^2[d_1d_4(\rho^{n+1}-\lambda_2^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_1^{n+1})]^2.
\end{aligned}$$

Correspondingly it is natural to have

$$\begin{aligned}
&\boldsymbol{\sum_{n=0}^{\infty}[\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^{n+1}I - (C^{-1}(B\beta^2)^{n+1}C)']} \\
&= \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} \sum_{n=0}^{\infty} s_1(n+1) & \sum_{n=0}^{\infty} s_2(n+1) \\ \sum_{n=0}^{\infty} s_2(n+1) & \sum_{n=0}^{\infty} s_3(n+1) \end{bmatrix} \\
&= \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} s_1^* & s_2^* \\ s_2^* & s_3^* \end{bmatrix}, \qquad \text{(F.3)}
\end{aligned}$$

where

$$
\begin{aligned}
s_1^* &= \sigma_1^2\Big[(d_1d_4-d_2d_3)^2\frac{1}{1-\rho^2}-2d_1d_4(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_1}+(d_1d_4)^2\frac{1}{1-\lambda_1^2}\\
&\quad+2d_2d_3(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_2}-2d_1d_2d_3d_4\frac{1}{1-\lambda_1\lambda_2}+(d_2d_3)^2\frac{1}{1-\lambda_2^2}\Big]\\
&\quad+\sigma_2^2\Big[(d_1d_2)^2\big(\frac{1}{1-\lambda_2^2}-\frac{2}{1-\lambda_1\lambda_2}+\frac{1}{1-\lambda_1^2}\big)\Big], \qquad \text{(F.4)}\\
s_2^* &= \sigma_1^2\Big[d_3d_4\Big\{(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_2}-\frac{1}{1-\rho\lambda_1}\Big)+\frac{d_1d_4}{1-\lambda_1^2}-\frac{d_1d_4+d_2d_3}{1-\lambda_1\lambda_2}+\frac{d_2d_3}{1-\lambda_2^2}\Big\}\Big]+\sigma_2^2\cdot\\
&\quad\Big[d_1d_2\Big\{(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_1}-\frac{1}{1-\rho\lambda_2}\Big)+\frac{d_1d_4}{1-\lambda_2^2}-\frac{d_1d_4+d_2d_3}{1-\lambda_1\lambda_2}+\frac{d_2d_3}{1-\lambda_1^2}\Big\}\Big], \qquad \text{(F.5)}\\
s_3^* &= \sigma_1^2\Big[(d_3d_4)^2\big(\frac{1}{1-\lambda_2^2}-\frac{2}{1-\lambda_1\lambda_2}+\frac{1}{1-\lambda_1^2}\big)\Big]\\
&\quad+\sigma_2^2\Big[(d_1d_4-d_2d_3)^2\frac{1}{1-\rho^2}-2d_1d_4(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_2}+(d_1d_4)^2\frac{1}{1-\lambda_2^2}\\
&\quad+2d_2d_3(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_1}-2d_1d_2d_3d_4\frac{1}{1-\lambda_1\lambda_2}+(d_2d_3)^2\frac{1}{1-\lambda_1^2}\Big]. \qquad \text{(F.6)}
\end{aligned}
$$

Therefore based on (F.2) and (F.3), we can further obtain the expression of $\boldsymbol{E}z_t\boldsymbol{z}_t'$.

Note that

$$
[\boldsymbol{\rho I}-\boldsymbol{C}^{-1}(\boldsymbol{B\beta^2})\boldsymbol{C}]^{-1}=\frac{1}{\widetilde{m}}\begin{bmatrix} d_1d_4(\rho-\lambda_2)-d_2d_3(\rho-\lambda_1) & d_1d_2(\lambda_2-\lambda_1)\\ d_3d_4(\lambda_1-\lambda_2) & d_1d_4(\rho-\lambda_1)-d_2d_3(\rho-\lambda_2)\end{bmatrix},
$$

where $\widetilde{m}=(d_1d_4-d_2d_3)(\rho-\lambda_1)(\rho-\lambda_2)$, and

$$
\boldsymbol{C}[\boldsymbol{\rho I}-\boldsymbol{C}^{-1}(\boldsymbol{B\beta^2})\boldsymbol{C}]^{-1}=\frac{1}{\widetilde{m}(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}\begin{bmatrix} k_1 & k_2\\ k_3 & k_4\end{bmatrix},
$$

where

$$
\begin{aligned}
k_1 &= d_1d_4(\rho-\lambda_2)-d_2d_3(\rho-\lambda_1)-\varphi\phi_\pi d_3d_4(\lambda_1-\lambda_2), \qquad \text{(F.7)}\\
k_2 &= d_1d_2(\lambda_2-\lambda_1)-\varphi\phi_\pi[d_1d_4(\rho-\lambda_1)-d_2d_3(\rho-\lambda_2)], \qquad \text{(F.8)}\\
k_3 &= \gamma[d_1d_4(\rho-\lambda_2)-d_2d_3(\rho-\lambda_1)]+(1+\varphi\phi_y)d_3d_4(\lambda_1-\lambda_2), \qquad \text{(F.9)}\\
k_4 &= \gamma d_1d_2(\lambda_2-\lambda_1)+(1+\varphi\phi_y)[d_1d_4(\rho-\lambda_1)-d_2d_3(\rho-\lambda_2)]. \qquad \text{(F.10)}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\boldsymbol{E}z_t\boldsymbol{z}_t'=\widetilde{k}\cdot\\
&\begin{bmatrix} k_1^2s_1^*+2k_1k_2s_2^*+k_2^2s_3^* & k_1k_3s_1^*+(k_1k_4+k_2k_3)s_2^*+k_2k_4s_3^*\\ k_1k_3s_1^*+(k_1k_4+k_2k_3)s_2^*+k_2k_4s_3^* & k_3^2s_1^*+2k_3k_4s_2^*+k_4^2s_3^*\end{bmatrix} \text{(F.11)}
\end{aligned}
$$

where $\widetilde{k} = \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(d_1d_4-d_2d_3)^4(\rho-\lambda_1)^2(\rho-\lambda_2)^2}$, s_i^* is given in (F.4)-(F.6) and k_i is given in (F.7)-(F.10).

Through very technical and extremely complicated calculations[39], the variances of output gap and inflations can be further simplified as

$$\begin{aligned}
&E(y_t^2) = \frac{1}{\widetilde{k}}(k_1^2 s_1^* + 2k_1k_2s_2^* + k_2^2 s_3^*)\\
= \;& \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}\\
&\Big\{\sigma_1^2\Big[[(1+\lambda^2\beta_2^4) - 2\lambda\beta_2^2(\rho+\lambda_1+\lambda_2) + (1+\lambda^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\\
&-\rho\lambda_1\lambda_2[(1+\lambda^2\beta_2^4)(\rho+\lambda_1+\lambda_2) - 2\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) + (1+\lambda^2\beta_2^4)\rho\lambda_1\lambda_2]\Big]\\
&+\sigma_2^2\Big[[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4) - 2\varphi\phi_\pi\varphi\beta_2^2(\rho+\lambda_1+\lambda_2) + ((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\\
&-\rho\lambda_1\lambda_2[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho+\lambda_1+\lambda_2) - 2\varphi\phi_\pi\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)\\
&+((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)\rho\lambda_1\lambda_2]\Big]\Big\}, \qquad \text{(F.12)}\\
&E(\pi_t^2) = \frac{1}{\widetilde{k}}(k_3^2 s_1^* + 2k_3k_4s_2^* + k_4^2 s_3^*)\\
= \;& \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}\\
&\Big\{\sigma_1^2\Big[\gamma^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-(\rho\lambda_1\lambda_2)^2]\Big]\\
&+\sigma_2^2\Big[[((1+\varphi\phi_y)^2+\beta_1^4) - 2(1+\varphi\phi_y)\beta_1^2(\rho+\lambda_1+\lambda_2) + ((1+\varphi\phi_y)^2+\beta_1^4)\\
&(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)] - \rho\lambda_1\lambda_2[((1+\varphi\phi_y)^2+\beta_1^4)(\rho+\lambda_1+\lambda_2) - 2(1+\varphi\phi_y)\beta_1^2\,\cdot\\
&(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) + ((1+\varphi\phi_y)^2+\beta_1^4)\rho\lambda_1\lambda_2]\Big]\Big\}. \qquad \text{(F.13)}
\end{aligned}$$

Note that here $E(y_t^2)$ and $E(\pi_t^2)$ in fact depend on the trace $\lambda_1+\lambda_2$ and determinant $\lambda_1\lambda_2$.

With the expression of covariance matrix $\boldsymbol{E}z_t\boldsymbol{z}_t'$, in order to obtain the expressions of first-order autocorrelation coefficient of output gap and inflation, we need to further calculate the first-order autocovariance $\boldsymbol{E}z_t\boldsymbol{z}_{t-1}'$.

[39]Because of limit of pages, we first drop the calculations here. In case the readers need the details, we can provide separately.

Following the similar calculations to $\boldsymbol{E} z_t \boldsymbol{z}_t'$, we can obtain

$$
\begin{aligned}
\boldsymbol{E} z_t \boldsymbol{z}_{t-1}' &= \mathbf{C}[\rho\mathbf{I} - \mathbf{C}^{-1}\boldsymbol{B}\boldsymbol{\beta}^2\mathbf{C}]^{-1}\sum_{n=1}^{\infty}[\rho^{n+1}\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\Sigma}[\rho^n\mathbf{I} - (\mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^n\mathbf{C})'] \cdot \\
&\quad [\rho\mathbf{I} - (\mathbf{C}^{-1}\boldsymbol{B}\boldsymbol{\beta}^2\mathbf{C})']^{-1}\mathbf{C}' \\
&= \widetilde{k}\begin{bmatrix} k_1^2 w_1^* + k_1 k_2(w_2^* + w_3^*) + k_2^2 w_4^* & k_1 k_3 w_1^* + k_1 k_4 w_2^* + k_2 k_3 w_3^* + k_2 k_4 w_4^* \\ k_1 k_3 w_1^* + k_2 k_3 w_2^* + k_1 k_4 w_3^* + k_2 k_4 w_4^* & k_3^2 w_1^* + k_3 k_4 (w_2^* + w_3^*) + k_4^2 w_4^* \end{bmatrix},
\end{aligned}
$$

where $\widetilde{k},\ k_i$ are given in (F.11) and (F.7)-(F.10), and

$$
\begin{aligned}
w_1^* &= \sigma_1^2\Big\{(d_1d_4 - d_2d_3)^2\frac{\rho}{1-\rho^2} - d_1d_4(d_1d_4 - d_2d_3)\frac{\rho+\lambda_1}{1-\rho\lambda_1} + (d_1d_4)^2\frac{\lambda_1}{1-\lambda_1^2} \\
&\quad + d_2d_3(d_1d_4 - d_2d_3)\frac{\rho+\lambda_2}{1-\rho\lambda_2} - d_1d_2d_3d_4\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + (d_2d_3)^2\frac{\lambda_2}{1-\lambda_2^2}\Big\} \\
&\quad + \sigma_2^2(d_1d_2)^2\Big[\frac{\lambda_2}{1-\lambda_2^2} - \frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_1}{1-\lambda_1^2}\Big], \\
w_2^* &= \sigma_1^2 d_3d_4\Big\{(d_1d_4 - d_2d_3)\Big[\frac{\rho}{1-\rho\lambda_2} - \frac{\rho}{1-\rho\lambda_1}\Big] + \frac{d_1d_4\lambda_1}{1-\lambda_1^2} - \frac{d_1d_4\lambda_1 + d_2d_3\lambda_2}{1-\lambda_1\lambda_2} + \frac{d_2d_3\lambda_2}{1-\lambda_2^2}\Big\} \\
&\quad + \sigma_2^2 d_1d_2\Big\{(d_1d_4 - d_2d_3)\Big[\frac{\lambda_1}{1-\rho\lambda_1} - \frac{\lambda_2}{1-\rho\lambda_2}\Big] + \frac{d_2d_3\lambda_1}{1-\lambda_1^2} - \frac{d_1d_4\lambda_1 + d_2d_3\lambda_2}{1-\lambda_1\lambda_2} + \frac{d_1d_4\lambda_2}{1-\lambda_2^2}\Big\}, \\
w_3^* &= \sigma_1^2 d_3d_4\Big\{(d_1d_4 - d_2d_3)\Big[\frac{\lambda_2}{1-\rho\lambda_2} - \frac{\lambda_1}{1-\rho\lambda_1}\Big] + \frac{d_1d_4\lambda_1}{1-\lambda_1^2} - \frac{d_1d_4\lambda_2 + d_2d_3\lambda_1}{1-\lambda_1\lambda_2} + \frac{d_2d_3\lambda_2}{1-\lambda_2^2}\Big\} \\
&\quad + \sigma_2^2 d_1d_2\Big\{(d_1d_4 - d_2d_3)\Big[\frac{\rho}{1-\rho\lambda_1} - \frac{\rho}{1-\rho\lambda_2}\Big] + \frac{d_1d_4\lambda_2}{1-\lambda_2^2} - \frac{d_1d_4\lambda_2 + d_2d_3\lambda_1}{1-\lambda_1\lambda_2} + \frac{d_2d_3\lambda_1}{1-\lambda_1^2}\Big\}, \\
w_4^* &= \sigma_1^2(d_3d_4)^2\Big[\frac{\lambda_2}{1-\lambda_2^2} - \frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_1}{1-\lambda_1^2}\Big] + \sigma_2^2\Big\{(d_1d_4 - d_2d_3)^2\frac{\rho}{1-\rho^2} \\
&\quad - d_1d_4(d_1d_4 - d_2d_3)\frac{\rho+\lambda_2}{1-\rho\lambda_2} + (d_1d_4)^2\frac{\lambda_2}{1-\lambda_2^2} + d_2d_3(d_1d_4 - d_2d_3)\frac{\rho+\lambda_1}{1-\rho\lambda_1} \\
&\quad - d_1d_2d_3d_4\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + (d_2d_3)^2\frac{\lambda_1}{1-\lambda_1^2}\Big\}.
\end{aligned}
$$

Again through very technical calculations, the first-order auto-covariances of output gap and inflations are further simplified as

$$
\begin{aligned}
& E(y_t y_{t-1}) = \frac{1}{\tilde{\tilde{k}}}(k_1^2 w_1^* + k_1 k_2(w_2^* + w_3^*) + k_2^2 w_4^*) \\
= \;& \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \\
& \Big\{\sigma_1^2\Big[(\rho+\lambda_1+\lambda_2-\lambda\beta_2^2)[1-\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)]+[\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)- \\
& \rho\lambda_1\lambda_2][(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\lambda\beta_2^2\rho\lambda_1\lambda_2]\Big]+\sigma_2^2\Big[(\varphi\phi_\pi(\rho+\lambda_1+\lambda_2)-\varphi\beta_2^2)) \\
& [\varphi\phi_\pi-\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)]+[\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\varphi\phi_\pi\rho\lambda_1\lambda_2] \\
& [\varphi\phi_\pi(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\varphi\beta_2^2\rho\lambda_1\lambda_2]\Big]\Big\}, \qquad\qquad (F.14) \\
& E(\pi_t\pi_{t-1}) = \frac{1}{\tilde{\tilde{k}}}(k_3^2 w_1^* + k_3 k_4(w_2^* + w_3^*) + k_4^2 w_4^*) \\
= \;& \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \\
& \Big\{\sigma_1^2\Big[\gamma^2[(\rho+\lambda_1+\lambda_2)-\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big]+\sigma_2^2\Big[[(1+\varphi\phi_y)(\rho+\lambda_1+\lambda_2)-\beta_1^2]\cdot \\
& [(1+\varphi\phi_y)-\beta_1^2(\rho+\lambda_1+\lambda_2)]+[\beta_1^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-(1+\varphi\phi_y)\rho\lambda_1\lambda_2]\cdot \\
& [(1+\varphi\phi_y)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\beta_1^2\rho\lambda_1\lambda_2]\Big]\Big\}. \qquad\qquad (F.15)
\end{aligned}
$$

Therefore, the first-order autocorrelation coefficients of output gap and inflation

$$G_1(\beta_1,\beta_2)=\frac{E(y_t y_{t-1})}{E(y_t^2)},\quad G_2(\beta_1,\beta_2)=\frac{E(\pi_t\pi_{t-1})}{E(\pi_t^2)},$$

i.e. the equations (3.13)-(3.14).

G Stability for the Taylor rule

Based on Proposition 2, we only need to show that both of the eigenvalues of $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$ have negative real parts if $\gamma(\phi_\pi-1)+(1-\lambda)\phi_y>0$.

The characteristic polynomial of $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$ is given by $h(\nu)=\nu^2-c_1\nu+c_2$, where c_1 is the trace and c_2 is the determinant of matrix $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$. Direct calculation shows that

$$c_1 = \frac{-(1-\lambda)(1-\beta_1^2)-2\varphi(\gamma\phi_\pi+\phi_y)+\varphi(\gamma+\lambda\phi_y)(1+\beta_2^2)}{\triangle\ (1+\gamma\varphi\phi_\pi+\varphi\phi_y)}, \qquad (G.1)$$

$$c_2 = \frac{\varphi[\gamma(\phi_\pi-1)+(1-\lambda)\phi_y]}{\triangle\ (1+\gamma\varphi\phi_\pi+\varphi\phi_y)}, \qquad (G.2)$$

where $\triangle=\frac{(1-\beta_1^2)(1-\lambda\beta_2^2)+\gamma\varphi\phi_\pi+\varphi\phi_y-(\gamma\varphi+\lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$.

Both of the eigenvalues of $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$ have negative real parts if and only if $c_1 < 0$ and $c_2 > 0$ (these conditions are obtained by applying the *Routh-Hurwitz criterion theorem*; see Brock and Malliaris, 1989). If $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, from Appendix E it is easy to see $\triangle > 0$. Furthermore,

$$c_1 \leq \frac{-2\varphi[(\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y]}{\triangle\ (1+\gamma\varphi\phi_\pi + \varphi\phi_y)} < 0, \quad c_2 > 0.$$

H Data Appendix

The observable variables used in our estimations follow the definitions in Smets & Wouters (2007). Accordingly:

$$\begin{cases} y_t^{obs} = 100log(GDPC09_t/LNS_{index_t}) \\ \pi_t^{obs} = 100log(\frac{GDPDEF09_t}{GDPDEF09_{t-1}}) \\ r_t^{obs} = 100log(\frac{Funds_t}{4}) \end{cases} \tag{H.1}$$

where the time series are given as:

GDPC09: Real GDP, Billions of Chained 2009 Dollars, Seasonally Adjusted Annual Rate. Source: Federal Reserve Economic Data (FRED).

GDPDEF09: GDP-Implicit Price Deflator, 2009=100, Seasonally Adjusted. Source: FRED.

LNU00000000: Unadjusted civilian noninstitutional population, Thousands, 16 years over. Source: U.S. Bureau of Labor Statistics (BLS)

LNS10000000: Civilian noninstitutional populations, Thousands, 16 years & over, Seasonally Adjusted.

Source: BLS.

$LNS_{index} = \frac{LNS10000000}{LNS10000000(1992:03)}$

Source: FRED.

Funds:Federal Funds Rate, Daily Figure Averages in Percentages. Source: FRED.

The observable variable x_t^{obs} for the output gap in our main estimations is based on the HP-filtered series of y_t^{obs}, while the CBO-based output gap is defined as: $x_t = 100\frac{GDPC09-GDPPOT}{GDPPOT}$

with **GDPPOT**[40]: CBO's Estimate of the Potential Output, Billions of Chained 2009

[40]In order to calculate the potential output, CBO uses the theoretical framework in a standard Solow growth model setup, see CBO (2001) for more details.

Dollars, Not Seasonally Adjusted Quarterly Rate. Source: FRED.

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