Restricted Perceptions and Regime Switches *

(Preliminary and incomplete, do not cite or distribute)

Tolga Ozden[†], Rafael Wouters[‡]

May 5, 2019

Abstract

We consider the estimation of Markov-switching DSGE models under adaptive learning (AL) in order to study the interaction between expectations and the business cycle over the Zero Lower Bound period. We assume that regime shifts by monetary policy are not directly observed by agents, instead they indirectly infer about the regimes to the extent that it feeds back into their information set. This setup results in so-called Restricted Perceptions Equilibria (RPE) consistent with a given information set, and standard E-stability conditions are applicable to these equilibria. We illustrate these conditions in two environments: a basic Fisherian setup, and the benchmark 3-equation New Keynesian model. We then use a variant of the Kim & Nelson (1999) filter to estimate MS-DSGE models under constant gain adaptive learning. Based on estimations of the 3-equation NKPC and workhorse Smets-Wouters models, our results can be summarized as follows: adaptive learning models outperform the REE benchmark in all cases and the Regime-switching REE model in most cases, suggesting that Markov-switching and Adaptive Learning approaches can be complementary in terms of model fit. Furthermore, we observe that the impulse responses and shock propagation differ under AL and REE setups: both supply and demand shocks can be subject to different patterns under AL over the transition period to the ZLB episode, depending on the information set. Focusing particularly on government spending shocks, we find that the proportional change in fiscal multipliers over the ZLB period is smaller under AL compared to the REE benchmark. This suggests that standard models may severely overestimate the impact of a fiscal expansion over this period following the 2007-08 financial crisis.

JEL Classification: E37; E65; C11; C32 .

Keywords: Adaptive Learning; Markov-Switching; Bayesian Estimation of DSGE Models; Zero Lower Bound.

---

*The research project is funded by the National Bank of Belgium under *the Research Program for Young Researchers*. The views expressed in this paper are those of the author and do not necessarily reflect the views of the National Bank of Belgium or any other institution to which the author is affiliated.

[†]T.ozden@uva.nl, University of Amsterdam

[‡]Rafael.wouters@nbb.be, National Bank of Belgium

1 Introduction

With the onset of the Global Financial Crisis in 2007-08 and the subsequent drop of interest rates to near-zero levels among the leading central banks, there has been increased interest among policymakers and central bankers alike about the ZLB constraint on nominal interest rates. There is still ongoing debate about the precise impact of the zero lower bound constraint on the economy as a whole and in particular about its macroeconomic cost in terms of aggregate GDP levels. Monetary and fiscal policy recommendations of standard macroeconomic models are mixed: for instance there is no consensus on the propagation of a government spending shock and the size of a government spending multiplier during this period. As a consequence, while some researchers recommend a fiscal austerity program to jump-start the economy during a ZLB episode, others think it is best to adapt a fiscal consolidation strategy. A common approach in most macroeconomic models examining the ZLB episode is the assumption of Rational Expectations Equilibria (REE): agents are assumed to have perfect information about the underlying regime along with all other cross-correlations of the relevant macroeconomic variables and form their expectations accordingly. In this paper, we contribute to the growing literature on the ZLB episode by relaxing the perfect information assumption, and instead estimating DSGE models under adaptive learning subject to the ZLB constraint.

In standard REE-DSGE models, the perfect foresight assumption about regime switches typically leads to short periods of anticipated ZLB episodes: the expected duration of this period is typically between three to five quarters in most DSGE models estimated on the U.S. economy, see e.g. Lindé et al. (2017) and Ji & Xiao (2016), while the ZLB episode between 2009 and 2016 lasted for around twenty eight quarters. Another shortcoming of the standard REE models is the overestimation of the impact of forward guidance on the macroeconomy, a phenomenon known as the "forward guidance puzzle" Del Negro et al. (2012). These shortcomings call for a relaxation of REE-restrictions and introduce expectational frictions into the models. A plausible and popular method to introduce such frictions is adaptive learning, which relaxes the assumption that agents have perfect knowledge about the underlying economic relations and the corresponding cross-correlations. Instead, they have their own sub-models, possibly under parameterized, that may not coincide with the correct economic structure. Agents act as econometricians and update their models each period as new observations become available. There is a vast and growing literature on the empirical validation of adaptive learning in DSGE models as well as monetary and fiscal policy implications of adaptive learning, see Evans & Honkapohja (2012) for a textbook treatment and Woodford (2013) for a comprehensive review of the more recent work. Much of the earlier literature on adaptive learning focused on the learnability of Rational Expectations Equilibria and MSV-learning, focusing on small and temporary deviations from perfect foresight models. Milani (2007) and Eusepi & Preston (2011) are earlier examples of expectations-driven business cycles and how MSV-learning can improve the empirical properties of small-scale DSGE models, while Bullard & Mitra (2002) and Bullard & Eusepi (2014) examine monetary policy implications of this type of learning. In more recent work, Slobodyan & Wouters (2012a) and Slobodyan & Wouters (2012b) show that further deviations from perfect foresight models with the use of small forecasting rules can lead to further improvements in the fit of a medium-scale DSGE model. On a similar vein, Quaghebeur (2018) examines fiscal policy implications of a VAR-type adaptive learning rule and finds that government spending multipliers are larger under adaptive learning. Evans et al. (2008) and Evans & Honkapohja (2010) examine the implications of adaptive learning for fiscal policy.

Branch et al. (2013) analyzes the theoretical properties of MSV-learning in Markov-switching models where agents are informed about the regime switches; and Lansing (2018) that analyzes the ZLB episode in a calibrated setup under adaptive learning where regime switches are unobserved. The closest study to our framework is Gust et al. (2018), which examines the ZLB episode in a Markov-switching setup under Bayesian learning. Our key difference from these papers and one of our main contributions is to extend their framework to non-MSV and non-rational beliefs, and to estimate the resulting DSGE models during the ZLB episode. We then examine the consequences of deviating from the REE during this period, particularly how it might contribute to a prolonging of the crisis and how it might change implications of standard DSGE models about the potential impact of a government spending shock during this episode.

There are various different approaches to modeling the ZLB constraint: Some researchers use a perfect foresight & endogenous duration approach, which allows for a joint determination of expectations and regime switches; see e.g. Maih (2015) or Lindé et al. (2017). Another approach which is more common in VAR-literature is to use a threshold-switching method, where the economy is assumed to be in the ZLB regime if interest rates fall below some pre-specified level, see e.g. Bonam et al. (2017). A final approach is to use a Markov-switching framework, where the presence of the ZLB regime is determined by its predictive density, see e.g. Binning & Maih (2016). Lindé et al. (2017) show that Markov-switching and endogenous duration approaches typically lead to similar results as long as the ZLB constraint is accounted for. In this paper, we use the Markov-switching (MS) approach to take into account the constraint. Aside from the ZLB episode, MS approach recently gained popularity in DSGE literature to model structural changes such as monetary policy switches or volatility breaks, see e.g. Sims & Zha (2006), Davig & Leeper (2007), Sims et al. (2008), Liu et al. (2011), Liu & Mumtaz (2011), Bianchi (2016) and Bianchi & Ilut (2017) for some of the recent work. While Markov-switching and adaptive learning have both been increasingly popular classes of time-varying DSGE models in recent years, there is surprisingly little work on DSGE models that combine both approaches. Therefore, our paper is also the first one to explicitly unify Markov-switching and adaptive learning methods in an estimation context.

Our key assumption is that the underlying regime changes are unobserved to economic agents: agents only indirectly become aware of regime changes to the extent that these switches have an observable and strong enough impact on their information set. To set the ideas, consider the following example: A central bank follows a simple Taylor rule that reacts to inflation in setting interest rates. This will only be known to economic agents to the extent that the central bank discloses its goal of inflation targeting, but the agents never know the exact reaction coefficient. Accordingly, the agents will not find out if the central bank suddenly and discreetly decides to change its reaction coefficient. Instead, the agents will slowly find out about this regime shift as long as it leads to observable consequences in the interest rate and the resulting inflation levels.

Farmer et al. (2009) and Farmer et al. (2011) explore the class of REE in Markov-switching models. Since we assume that regimes are never observed, an equilibrium concept in our framework can never coincide with a Rational Expectations Equilibrium. Instead, in this limited information environment, there are so-called Restricted Perceptions Equilibria (RPE) where the agents' misspecification of the economy becomes self-fulfilling and the system settles on a non-rational equilibrium. To start with, we compute these equilibria in a 1-dimensional setup with the Fisherian equation, where the agents' perceived law of motion (PLM) has the form of a Minimum State Variables (MSV) solution, except that the PLM does not take into account the possibility of regime-switches. We show that standard E-stability conditions apply to these equilibria, and therefore the systems will converge to the underlying equilibria under standard recursive algorithms such as constant-gain least-squares. Furthermore, the E-stability and convergence results continue to hold even if one of the underlying regimes is E-unstable as long as the remaining regimes are sufficiently E-stable. This is a simple extension of the long-run determinacy result of Davig & Leeper (2007), which they call the long-run Taylor principle. We therefore denote our result as the long-run *E-stability principle*. We then extend this idea further to higher dimensional systems, where the PLM can also deviate from the MSV-solution in the form of small VAR-type forecasting rules: this allows the information set of the agents to be smaller than the MSV-solution due to, for example, unobserved shocks or unaccounted cross-correlations. The underlying RPE are too complicated to compute either analytically or numerically in this more general setup, although the underlying systems can always be simulated to observe the system behaviour and E-stability.

Next we consider a variant of the Kim & Nelson (1999) filter to estimate our class of MS-DSGE models under adaptive learning, and we apply the filter to the Bayesian likelihood estimation of two standard DSGE models: The first one is the 3-equation NKPC model along the lines of Woodford (2013), which provides a good starting point to expose our main results. The second one is the more complex and empirically relevant Smets & Wouters (2007) model, which is popular among central bankers and policy makers as a benchmark for policy analysis. Our estimation results can be summarized as follows: The MS-AL models outperform the standard REE benchmark in all cases, and the also the regime-switching REE models in a majority of cases. Furthermore, we observe important differences in the impulse response and shock propagation structure of the models under consideration. The models have important implications for the impact of government spending shocks in particular: we find that, during normal times, government spending multipliers tend to be larger under adaptive learning compared to REE. Over the ZLB period with inactive monetary policy, the multipliers become larger for both REE and AL models compared with normal times. However, the proportional change for the REE model is substantially larger compared to all AL models. This suggests that the benchmark REE model may severely overestimate the effect of a fiscal expansion following the 2007-08 crisis.

The paper is organized as follows: Section 2 illustrates the main concepts in a simple framework with one-forward looking variable. Section 3 shows the computation and E-stability results of the two classes of Restricted Perceptions Equilibria in DSGE models. Section 4 provides the filter used in our estimations, while sections 5 and 6 discuss the estimations results in the 3-equation NKPC and SW models respectively. Finally Section 7 concludes.

2 Preliminaries: Fisherian Model of Inflation Determination

Consider a simple model of Fisherian inflation determination without regime switching:

$$\begin{cases} i_t = E_t \pi_{t+1} + r_t \\ r_t = \rho r_{t-1} + v_t \\ i_t = \alpha \pi_t - u_t \end{cases}$$

where r_t is the exogenous AR(1) ex-ante real interest rate, i_t is the nominal interest rate, π_t is inflation, and $\{u_t, v_t\}$ are IID shock processes. We assume that monetary policy follows a simple rule by adjusting nominal interest rate to inflation, denoted by α[1]. We can re-write the model in terms of inflation as follows:

$$\begin{cases} \pi_t = \frac{1}{\alpha}(E_t\pi_{t+1} + r_t) + u_t, \\ r_t = \rho r_{t-1} + v_t. \end{cases}$$

We start with this simplified setup since it has been analyzed in Davig & Leeper (2007), which is one of the first studies on expectations in a regime switching setup. With the standard MSV solution, the agents' perceived law of motion (PLM) takes the form of $\pi_t = ar_t$. The REE solution for a is then pinned down by iterating the PLM forward to obtain the one-step ahead expectations, plugging the expectations back into the implied actual law of motion (ALM) and computing the associated fixed point where the agents' PLM exactly coincides with the implied ALM. This yields $a = \frac{1}{\alpha-\rho}$. Hence the law of motion evolves according to $\pi_t = \frac{1}{\alpha-\rho}r_t$ at the REE. In this benchmark case, the equilibrium is determinate if $\alpha > 1$, i.e. monetary policy is sufficiently aggressive.

Davig & Leeper (2007) consider scenarios where the interest rate coefficient α is subject to regime switches. Focusing on a two regime environment, assume that α changes stochastically between two regimes, $s_t = \{1, 2\}$ subject to the transition matrix:

$$Q = \begin{pmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{pmatrix}$$

Then inflation dynamics are given as:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}(E_t\pi_{t+1} + r_t) + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

Denoting $\pi_{i,t} = \pi_t(s_t = i)$, we can re-cast the model into a multivariate form:

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \pi_{1,t} \\ \pi_{2,t} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} E_t\pi_{1,t+1} \\ E_t\pi_{2,t+1} \end{bmatrix} + \begin{bmatrix} r_t \\ r_t \end{bmatrix} + \begin{bmatrix} u_t \\ u_t \end{bmatrix}$$

Since the ALM is regime-dependent, the corresponding PLM is also regime-dependent if agents are rational. Denoting by a_i the regime-specific coefficients, the PLM is given by:

$$\pi_{i,t} = a_i r_t$$

In other words, agents hold a seperate PLM for each regime in the economy. Then the corresponding regime-dependent 1-step ahead expectations are given by:

$$\begin{cases} E_t[\pi_{t+1}|s_t = 1] = (p_{11}a_1 + p_{12}a_2)\rho r_t \\ E_t[\pi_{t+1}|s_t = 2] = (p_{21}a_1 + p_{22}a_2)\rho r_t \end{cases}$$

[1] For the remainder, we assume that $Var(r_t) = 1$ to simplify the exposure.

Davig & Leeper (2007) show that, in this case, the equilibrium is determinate as long as the so-called *long-run Taylor principle (LRTP)* is satisfied:

$$\alpha_1\alpha_2 > 1 - ((1-\alpha_2)p_{11} + (1-\alpha_1)p_{22})$$

A key insight of this principle is that, the long-run dynamics of the model can be determinate even if one of the underlying regimes is indeterminate, provided there is at least one regime that is sufficiently determinate or the probability of entering into the indeterminate regime is sufficiently small. In what follows, we first extend the long-run determinacy insight into the concept of learnability, i.e. *E-stability* of equilibria. Our key assumption is that agents do not directly observe or take into account the regime shifts that occur in the economy when they form their expectations. Accordingly, their PLMs and the implied 1-step ahead expectations are given as follows, which are not regime dependent:

$$\pi_t = ar_t \Rightarrow E_t\pi_{t+1} = aE_tr_{t+1} = a\rho r_t$$

The implied ALM is then given by:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}(a\rho+1)r_t + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

The assumed form of PLM here does not nest the regime-dependent MSV solution. Therefore any resulting notion of equilibrium under this scenario cannot coincide with the full-information Rational Expectations Equilibrium. However, one can still consider a non-rational & limited-information equilibrium associated with the above PLM. This type of equilibrium is commonly referred to as a Restricted Perceptions equilibrium in the adaptive learning literature (Evans & Honkapohja, 2012): the agents use a restricted (and misspecified) information set, which becomes self-fulfilling at the underlying equilibrium.

Unlike a REE, one cannot use the method of undetermined coefficients as above to pin down the value of a associated with the RPE. Instead, following Hommes & Zhu (2014), we impose a moment consistency requirement on the model to pin down the value of a: the coefficient a determines the *perceived correlation* between inflation and real rate of interest in the PLM, i.e. $a = \frac{E[\pi_t r_t]}{E[r_t r_t]}$. In an equilibrium, the unconditional correlation $\frac{E[\pi_t r_t]}{E[r_t r_t]}$, implied by the ALM should be equal to a. In other words, the information used by the agent should be consistent with the information that arises in the actual outcomes. In an equilibrium as such, agents make systematic mistakes to the extent that they do not use the full-information forecasting rule. Instead, they use the best forecasting rule within the class of their information set. Computing the associated moment in our example, the ALM yields:

$$\frac{E[\pi_t r_t]}{E[r_t r_t]} = E[\frac{1}{\alpha(s_t)}\beta\rho + \frac{1}{\alpha(s_t)}]$$

This expression above involves the ergodic distribution of the Markov chain, which we denote by P. This is given by $P = [\frac{1-p_{22}}{2-p_{11}-p_{22}}, \frac{1-p_{11}}{2-p_{11}-p_{22}}]$[2]. Using this, the equilibrium coefficient, which we denote by a^{RPE} is given by[3]:

$$a^{RPE} = \frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}-p_{22}) - \rho\alpha_1(1-p_{22}) - \rho\alpha_2(1-p_{11})}$$

Further note that, the regime-specific MSV solutions are given by $a^{R_i} = \frac{1}{\alpha_i - \rho}$, $i \in \{1,2\}$. Given these expressions, the underlying RPE boils down to a weighted average of the regime-specific equilibria. Instead of the standard determinacy of Rational Expectations models, our main concept of interest in this case is E-stability: Bullard & Eusepi (2014) shows that there is tight link between determinacy and E-stability of REE and in some special cases these conditions may even coincide. E-stability governs whether the agents can learn the above fixed-point by starting from an arbitrary point a_0, and updating their beliefs about the coefficient each period as new observations become available. As shown in Evans & Honkapohja (2012), E-stability is governed by the mapping from agents' PLM to the implied ALM, defined as the T-map. In our example, the T-map is given by:

$$T : a \to T(a) = \frac{E[\pi_t r_t]}{E[r_t r_t]} = (a\rho + 1)\frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}p_{22})}$$

The T-map is locally stable if the Jacobian matrix has eigenvalues with real parts less than one. When this local stability condition is satisfied, the equilibrium is said to be E-stable. In our example, the eigenvalue and the associated E-stability condition are given by:

$$\frac{DT(a)}{D(a)} = \frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}-p_{22})} < 1$$

.

Re-arranging the above expression yields:

$$\alpha_1\alpha_2 > \frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{2-p_{11}-p_{22}}$$

With the above expression, the E-stability criterion reduces to a condition similar to *LRTP*. In order to satisfy E-stability, we need a more aggressive monetary policy rule α_1 whenever: (i) the average time spent in regime 1 (P_1) decreases, (ii) the average time spent in regime 2 (P_2) increases, or (iii) the monetary policy rule in regime 2 (α_2) becomes less aggressive. This suggests that it is possible to have E-stability despite having an E-unstable regime, as long as there is a sufficiently E-stable regime and the model does not spend too much time in the unstable regime on average. This is an intuitive extension of Davig & Leeper's insight on long-run determinacy to the learnability of equilibria, therefore we denote this as *the principle of long-run E-stability.*

[2]Note that ergodic distribution is obtained by solving $P'Q = P$.
[3]See Appendix A for details.

For the remainder of our analysis, we add an indexation term to the inflation equation with ι_p and consider the system:

$$\begin{cases} \pi_t = \frac{1}{\alpha}(E_t\pi_{t+1} + \iota_p \pi_{t-1} r_t) + u_t, \\ r_t = \rho r_{t-1} + v_t. \end{cases}$$

This small extension allows us to also study the autocorrelation structure. In this case the regime-specific solution takes the form of:

$$\pi_t = a r_t + b\pi_{t-1}.$$

Using the moment consistency requirements for both a and b, the T-map in this case is given by:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} E[(\pi_t - b\pi_{t-1})r_t] \\ \frac{E[(\pi_t - a r_t)\pi_{t-1}]}{E[\pi_t^2]} \end{pmatrix}$$

With the addition of lagged inflation, the moments appearing in the above expression already become analytically intractable, therefore the values a^{RPE} and b^{RPE} are obtained numerically in the examples below[4]. Agents' forecasts along such an RPE imply systematic forecast errors: since they do not know the underlying regimes and only use a univariate rule, they essentially have a weighted average based on the ergodic distribution at the RPE, where they minimize the unconditional forecast errors. In general, we assume that agents do not simply remain at the RPE-consistent values but instead update their beliefs each period as new observations become available, using a constant-gain least squares method à la Evans & Honkapohja (2012). Accordingly in our example, using the notation $\theta = [a, b]'$ and $y_t = [r_t, \pi_{t-1}]'$, the vector of coefficients are updated as follows:

$$\begin{cases} R_t = R_{t-1} + \gamma(y_t^2 - R_{t-1}) \\ \theta_t = \theta_{t-1} + \gamma R_t^{-1} y_t(\pi_t - \theta_{t-1} y_t) \end{cases}$$

where γ denotes the gain value, i.e. the weight that agents put into the most recent observation. The learning algorithm allows agents to put more weight into recent observations, thereby giving them more flexibility. We illustrate the model dynamics for a number of scenarios. Figure 1 shows four examples where both regime-specific MSV solutions, as well as the underlying RPE are E-stable. Panel (a) shows a decreasing gain learning case (DGL) with $\gamma = \frac{1}{t}$, where the learning coefficients converge towards their RPE-consistent values. In this case the beliefs are asymptotically locked in at the RPE as each new observation receives less weight. Panel (b) shows a constant gain learning (CGL) case with $\gamma = 0.01$, in which case the learning coefficients oscillate around their RPE-consistent values.

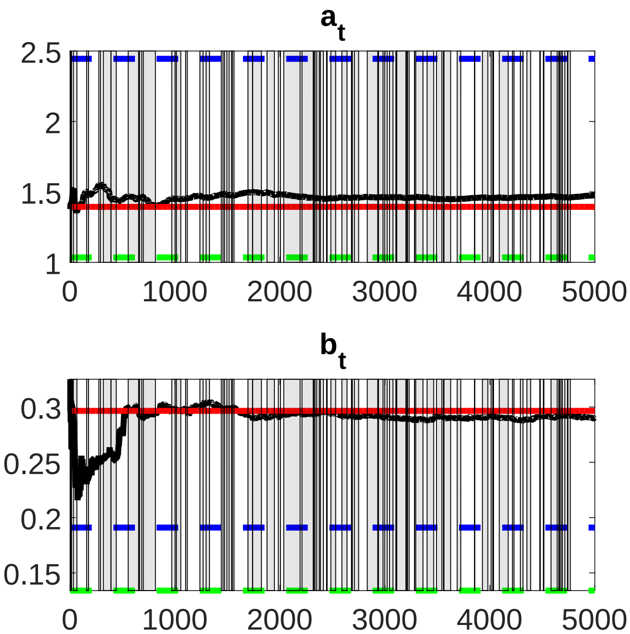
An interesting feature of our setup is that agents may forget past regime shifts under the right conditions: as the gain coefficient becomes larger relative to the underlying regime persistence parameters p_{11} and p_{22}, agents become more myopic and put less weight into past observations. In these scenarios, as past regime shifts are forgotten, the learning coefficents become more likely to escape from the RPE towards regime-specific equilibria. Panels (c) and

[4]See Appendix B for details on the implied moments.

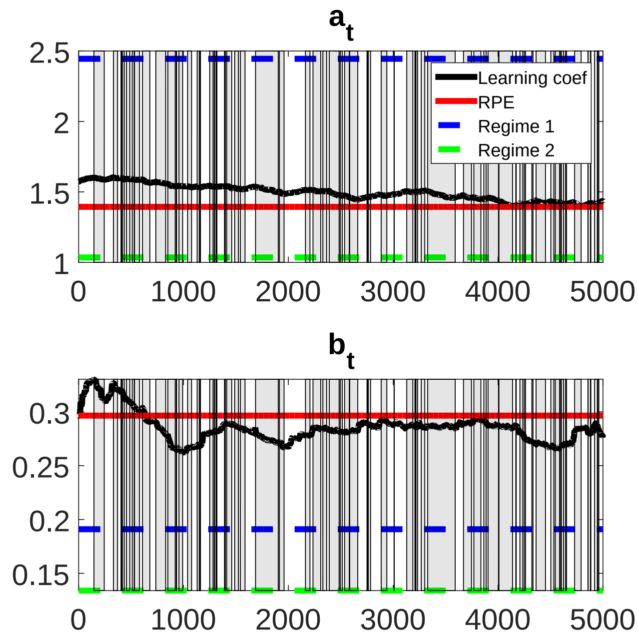
(d) of Figure 1 illustrate two such cases, where we increase the persistence of the underlying regimes. In these cases, once a regime shift occurs, the learning coefficients first jump towards their RPE-consistent values. As the system spends enough time in the new regime, the regime shift is forgotten and the learning coefficients jump towards their regime specific values. This occurrence is particularly important from an empirical point of view as it characterizes how the system behaves when exiting a very persistent regime, or entering into a new regime that has not been observed before, such as the recent ZLB episode.

Figure 2 illustrates two examples where one equilibrium is stable, while the second equilibrium is borderline E-unstable. Given the parameter values, the underlying RPE is E-stable. Panel (a) shows convergence towards the underlying RPE under DGL with $\gamma = \frac{1}{t}$, while Panel (b) shows a CGL case with $\gamma = 0.01$, and the learning coefficients oscillate around the RPE-consistent values. In this case with one E-unstable regime, we observe the same phenomenon of jumps as the system switches from the persistent E-stable regime, to the less persistent E-unstable regime. This feature is also relevant for the ZLB episode that will be studied in the upcoming sections, since this regime has been found to be E-unstable under a wide variety of models and specifications. In the next section, we extend our analysis to a general multivariate setup in order to study New Keynesian models.

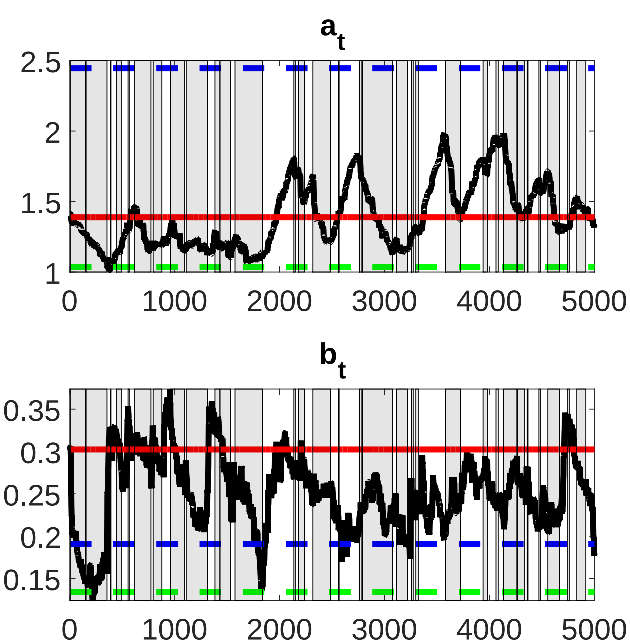
Figure 1: Learning coefficients along with the RPE-consistent and regime-specific values. The parameters $\iota_p = 0.25, \rho = 0.9, \alpha_1 = 1.5, \alpha_2 = 2, \sigma_u^2 = 0.1$ are fixed for all simulations, while p_{11}, p_{22} and γ are varied. Given the values of α_1 and α_2, both regime-specific equilibria, as well as the RPE are E-stable.



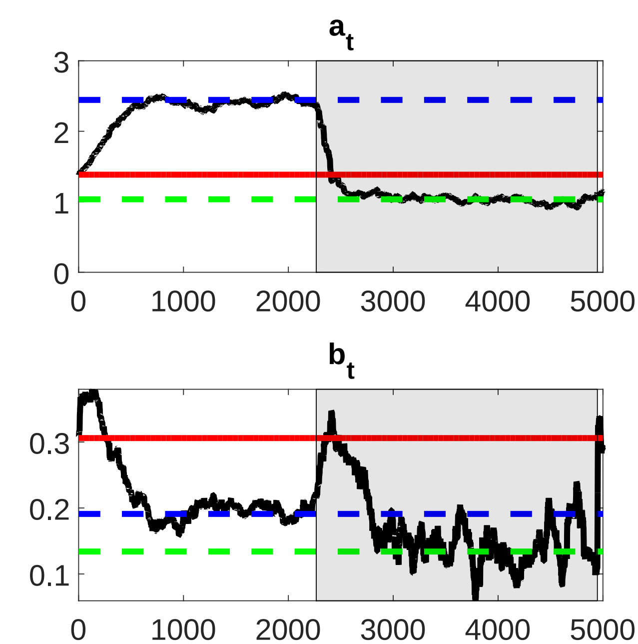
(a) $\gamma = \frac{1}{t}, p_{11} = p_{22} = 0.975$.



(b) $\gamma = 0.001, p_{11} = p_{22} = 0.975$.

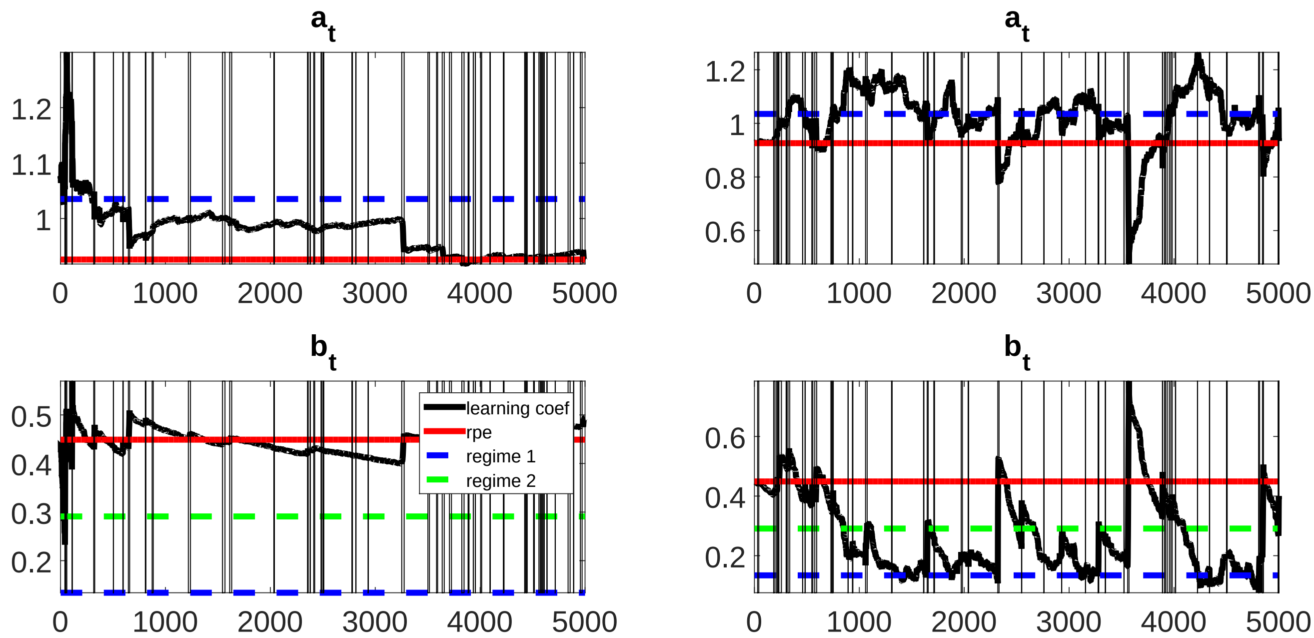


(c) $\gamma = 0.01, p_{11} = p_{22} = 0.99$.



(d) $\gamma = 0.01, p_{11} = p_{22} = 0.999$.

Figure 2: Learning coefficients along with the RPE-consistent and regime-specific values. The parameters $\iota_p = 0.25, \rho = 0.9, \alpha_1 = 2, \alpha_2 = 1.15, \sigma_u^2 = 0.1$ are fixed for all simulations, while p_{11}, p_{22} and γ are varied. The first regime is E-stable, while the second regime is E-unstable. At the given values for p_{11} and p_{22} below, the RPE is E-stable.



(a) $\gamma = \frac{1}{t}, p_{11} = 0.99, p_{22} = 0.9.$

(b) $\gamma = 0.01, p_{11} = 0.99, p_{22} = 0.9..$

3 Restricted Perceptions Equilibria in MS-DSGE Models: General Case

Our simple example in the previous section illustrates the idea of restricted perceptions in Markov-switching DSGE models, where the underlying Restricted Perceptions Equilibrium is still reasonably easy to compute. In this section, we generalize our results to cases with multiple forward-looking variables, where the underlying equilibrium quickly becomes intractable. Accordingly, first consider the data generating process:

$$\begin{cases} X_t = A(s_t) + B(s_t)X_{t-1} + C(s_t)E_tX_{t+1} + D(s_t)\epsilon_t \\ \epsilon_t = \rho\epsilon_{t-1} + \eta_t \end{cases}$$

where X_t denotes the state-variables that depend on their lags, 1-step ahead expectations and the structural shocks ϵ_t, which itself follow a VAR(1) process. We assume that the corresponding matrices A, B, C and D contain the structural parameters of the model, some of which are subject to regime switches captured by s_t. Using this, we extend our analysis to the more general multivariate case. We further allow the vector of intercept coefficients $A(s_t)$ to be non-zero in this case, which is particularly important for analyzing the steady-state stability of the ZLB episode. Consider again a PLM of the general form:

$$\begin{cases} X_t = a + bX_{t-1} + d\epsilon_t \\ E_t X_{t+1} = a + bX_t + d\rho\epsilon_t \end{cases}$$

where we assume that structural shocks are contemporaneously observed while the state variables are not, which is a common assumption in the adaptive learning literature. Further note that the above specification nests many benchmark PLMs as a special case: when a, b and d are all non-zero, the PLM takes the form of a regime-specific solution and the only source of misspecification is the unobserved regimes. With $d = 0$, the PLM admits a variety of VAR(1)-type learning rules, which assumes unobserved shocks and allows some or all cross-correlations to be misspecified. A diagonal matrix b further reduces the PLM to univariate autoregressive rules. These types of AR/VAR rules have been successfully applied in recent past to improve the empirical fit of otherwise standar DSGE models, see e.g. Slobodyan & Wouters (2012b) and Gaus & Gibbs (2018). Plugging the expectations back into the first expression yields the implied ALM:

$$X_t = A(s_t) + B(s_t)X_{t-1} + C(s_t)a + C(s_t)bX_t + (C(s_t)d\rho + D(s_t))\epsilon_t$$

which can be re-written as

$$X_t = a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t$$

where $a(s_t) = (I - C(s_t)b)^{-1}(A(s_t) + C(s_t)a), b(s_t) = (I - C(s_t)b)^{-1}B(s_t)$ and $d(s_t) = (I - C(s_t)b)^{-1}(C(s_t)d\rho + D(s_t))$. In this case the T-map is given as:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} E[X_t - b(s_t)X_{t-1} - d(s_t)\epsilon_t] \\ E[(X_t - a(s_t) - d(s_t)\epsilon_t)X'_{t-1})E[X_t X'_t]^{-1}] \\ E[(X_t - a(s_t) - b(s_t)X_{t-1})\epsilon'_t]E[\epsilon_t\epsilon'_t]^{-1} \end{pmatrix}$$

Appendix C provides the first and second moments that appear here in a general setup with m regimes. While the vector a^{RPE} and d^{RPE} are easily computed for a given matrix b^{RPE}, the computation of b^{RPE} involves a 2^{nd} order matrix polynomial, for which we do yet have a straightforward and reliable method. Therefore for the models considered in the remainder of this paper, we rely on simulations whenever lagged variables are involved. In special cases without lagged variables, i.e. $b = 0$, a^{RPE} and d^{RPE} are easily computed. Using our expressions in Appendix C for a 2-regime setup without lagged variables, we obtain the T-map:

$$\begin{pmatrix} a \\ d \end{pmatrix} \rightarrow \begin{pmatrix} A_i + C_i a(s_t) \\ P_1(C_1 d(s_t)\rho + D_1)\Sigma_\epsilon(p_{11} + p_{21}\frac{P_2}{P_1}) + P_2(C_2 d(s_t)\rho + D_2)\Sigma_\epsilon(p_{22} + p_{12}\frac{P_1}{P_2}) \end{pmatrix}$$

Solving for the implied fixed-point yields the following equilibrium:

$$a^{RPE} = (I - P_1(p_{11}+1-p_{22})C_1 - P_2(p_{22}+1-p_{11})C_2)^{-1}(P_1(p_{11}+1-p_{22})A_1 + P_2(p_{22}+1-p_{11})A_2)$$

$$vec(d) = [I - (\rho\Sigma_\epsilon(p_{11} + p_{21}\frac{P_2}{P_1})' \otimes (P_1C_1)) - (\rho\Sigma_\epsilon(p_{22} + p_{12}\frac{P_1}{P_2})' \otimes (P_2C_2))]^{-1}$$
$$[vec(P_1D_1\Sigma_\epsilon(p_{11} + p_{21}\frac{P_2}{P_1})) + vec(P_2D_2\Sigma_\epsilon(p_{22} + p_{12}\frac{P_1}{P_2}))]$$

The associated Jacobian is given by:

$$DT_{(a,d)} = \begin{pmatrix} P_1(p_{11} + (1-p_{22}))C_1 + P_2(p_{22} + (1-p_{11}))C_2 & 0 \\ 0 & \begin{array}{c} vec^{-1}_{n,n}(\Sigma_\epsilon(p_{22} + p_{12}\frac{P_2}{P_1})' \otimes P_1C_1 + \\ \Sigma_\epsilon(p_{11} + p_{21}\frac{P_1}{P_2})' \otimes P_2C_2) \end{array} \end{pmatrix}$$

Denoting by $r_\sigma(DT)$ the spectral radius of the Jacobian matrix above, the long-run E-stability principle is satisfied for the RPE if the real part of $r_\sigma(DT)$ is inside the unit circle. Further note that the stability of learning coefficients for the intercept and shock terms are independent of each other, implying the first and second diagonal terms of $DT_{(a,d)}$ govern the stability of intercept and shock terms respectively.

The RPE-consistent values derived above are not available in the general setup with $b \neq 0$. In this case there is no straightforward way of checking the E-stability condition, since the associated Jacobian matrix is also not available. Therefore for the remainder of our paper, we rely on simulations to check for E-stability and assume that the stability conditions are satisfied if the simulations show no sign of explosive dynamics. This is illustrated below for the 3-equation hybrid New Keynesian model, before we move onto the estimations.

3.1 Constant-gain Least Squares in the 3-eqution NKPC

In this section, in order to illustrate our results, we consider the baseline 3-equation NKPC model given as:

$$\begin{cases} x_t = \iota_y x_{t-1} + (1-\iota_y)E_t x_{t+1} - \frac{1}{\tau}(r_t - E_t\pi_{t+1}) + \epsilon_{x,t} \\ \pi_t = \iota_p\beta(1-\iota_p)E_t\pi_{t+1} + \kappa x_t + \epsilon_{\pi,t} \\ r_t = max\{0, \rho r_t + (1-\rho)(\phi_x x_t + \phi_\pi \pi_t) + \eta_{r,t}\} \\ \epsilon_{y,t} = \rho_y \epsilon_{y,t-1} + \eta_{y,t} \\ \epsilon_{\pi,t} = \rho_\pi \epsilon_{\pi,t-1} + \eta_{\pi,t} \end{cases}$$

Following Linde et. al. (2018), we can re-cast the interest rate rule above as a Markov switching process with two regimes, where:

$$\begin{cases} r_t(s_t = 1) = \rho r_{t-1} + (1-\rho)(\phi_x x_t + \phi_\pi \pi_t) + \eta^1_{r,t}\} \\ r_t(s_t = 2) = \eta^2_{r,t}\} \end{cases}$$

subject to the same transition matrix same as in the first example. The presence of noise in the second regime is meant to capture the fact that, although interest rates are very close to zero in empirical data, they are never exactly equal to zero in the post-2007 period. This setup provides a convenient approximation to model the zero lower bound episode by making the assumption that the regime switches are exogenous. We first consider a special case without lagged variables, i.e. $\iota_p = 0, \iota_y = 0$ and $\rho_r = 0$, and an otherwise standard calibration with parameter values $\phi_y = 0.125, \phi_\pi = 1.5, \rho = 0, \kappa = 0.02, \tau = 1, \beta = 0.99, \sigma_y = 0.5, \sigma_\pi = 0.25, \sigma_r^I = 0.1, \sigma_r^{II} = 0.01, \rho_y = 0.25, \rho_\pi = 0.25, \iota_p = 0, \iota_y = 0, p_{11} = 0.99, p_{22} = 0.95$.

Given this calibration without lagged variables, using our expressions from above, the E-stability conditions for mean and shock dynamics are satisfied. For the shock dynamics, the E-stability condition is satisfied with both regime-specific equilibria[5]. For the mean dynamics, the E-stability principle is satisfied in the normal regime, while it is violated in the ZLB regime: this is similar to the key result from Evans et al. (2008) that studies steady-state learning in a non-linear setup and finds that the liquidity trap equilibrium is E-unstable. Our result here shows that, the E-unstability result still applies even if the model does not switch to the liquidity-trap equilibrium. Furthermore, as long as the expected duration of the ZLB regime is sufficiently short, the long-run E-stability principle is satisfied; this is indeed the case with the above calibration of $p_{11} = 0.99$ and $p_{22} = 0.95$. Accordingly, the underlying RPE is E-stable under a constant gain least squares learning, provided that the gain parameter γ is sufficiently small.

We next set $\iota_y = \iota_p = \rho = 0.1$ and examine the dynamic behaviour of the model via simulations. Given the small values on the lagged variables, we expect the E-stability conditions above to hold in this case as well. In this case the learning model is given as:

$$\begin{cases} R_t = R_{t-1} + \gamma(S_{t-1}^2 - R_{t-1}) \\ \theta_t = \theta_{t-1} + \gamma R_t^{-1} S_{t-1}(S_t - \theta_{t-1} S_{t-1}) \end{cases}$$

where θ_t denotes the information set that the agent uses when forming his expectations, given as:

$$\theta_t = [1, \epsilon_{y,t}, \epsilon_{\pi t}, y_{t-1}, \pi_{t-1}, r_{t-1}]'$$

In order to assess the overall stability of model dynamics, we first use a Monte Carlo simulation with a constant gain value of $\gamma = 0.01$. Figure 3 shows frequency distributions of all six learning ceofficients on output gap, inflation and interest rate respectively. It is readily seen that the resulting distributions are unimodal, suggesting oscillations around a unique E-stable RPE.

Next we examine the time paths for the learning coefficients in a given simulation. For this exercise, we provide an additional deviation from our current setup and also consider a simulation with endogenous regime switching. While the exogenous switching setup provides a convenient way to model and estimate the ZLB episode, a potential shortcoming of this approach is that, in reality, the switches to the ZLB episode are not exogenous events. Rather, the ZLB regime only becomes relevant if the economic conditions drive the interest rates to near

[5] See Appendix C for details on the regime-specific equilibria and the associated E-stability conditions.

zero levels. Therefore we also consider the case with endogenous transition probalities as follows:

$$p_{11}(t) = \frac{\theta}{\theta + exp(-\phi(r_t^* - \underline{r}))}, p_{22}(t) = \frac{\theta}{\theta + exp(\phi(r_t^* - \underline{r}))}$$

where r_t^* denotes the shadow interest rate, i.e. the interest rate that would prevail in the absence of ZLB constraint; $\underline{r}$ denotes the value of the lower bound for interest rates; and θ and ϕ are hyperparameters to be calibrated. Following Binning & Maih (2016), we set $\theta = 1$ and $\phi = 1000$, which lead to a very sharp transition to the ZLB regime once r_t^* falls even slightly below $\underline{r}$. For our exercise, we set $\underline{r} = -1$, which is equivalent to assuming a steady-state interest rate level of 1 with a lower bound of 0.

Figure 4 shows two simulations with a gain of $\gamma = 0.01$ with exogenous (left panel) and endogenous (right panel) transition probabilities. In each figure, the dotted lines provide the regime-specific values for the underlying REE. Although we do not have the RPE-consistent values in this case, the behaviour of the learning coefficients in the exogenous switching case suggest that the results are similar to the 1-dimensional case: the shock coefficients typically show a gradual movement towards the regime-specific values, while the lagged coefficients display a sharp jump immediately following the switch, before starting to move in the direction of regime-specific values. Given our calibration, the ZLB episodes are very short-lived with durations of 1 or 2 periods only[6], but similar results can be observed for the lagged coefficients. In particular, the jumps in the lagged coefficients is evident immediately following a binding ZLB period. The shock coefficients remain close to the regime-specific values of the normal regime, since the ZLB episodes are not sufficiently long-lived to induce a movement in these. These results indicate that the propagation of regime switches on expectations under adaptive learning works in a similar fashion regardless of the nature of regime switches, i.e. endogenous or exogenous. This is a natural result since regime switches are unobserved by the agents in either case. Taking this into account, we present our estimation framework and result by assuming exogenous probabilities, but complement our results with simulations under endogenous switches whenever appropriate.

A robust phenomenon in our simulation exercises up to this point is the observed jumps in the learning coeffients during the ZLB regimes. These jumps are more pronounced particularly when the system exits persistent and long-lived regimes; enters into new regimes that have not been previously observed; or when the gain coefficient is sufficiently high. Note that these jumps are similar to what one would observe in a Rational Expectations framework, where the PLMs immediately switch following a regime change, since such changes are assumed to be observed by agents. However different than a REE in our framework, the direction of the jumps are either towards the RPE or regime-specific values, which do not coincide with REE-consistent values in general. Therefore a natural question that arises is whether we can observe similar jumps in PLMs when we take our learning approach to the data, and how the direction and magnitude of such jumps under adaptive learning differ from REE. To address this question, we next move onto the discussion of our estimation framework of Markov-Switching DSGE models under adaptive learning.

[6] While these episodes can be made more persistent with additional assumptions, our primary interest here is how the learning coefficients react immediately following the binding constraint.

Figure 3: Frequency distributions of learning coefficients from 500 simulations of length 5000 for the two-regime NKPC. The columns shows the learning coefficients on output gap, inflation and interet rate; while the rows are the steady-state, lagged inflation, output gap, interest rate, and the two shock coefficients respectively. A casual inspection suggests the distributions are unimodal, suggesting oscillations around a unique E-stable RPE.

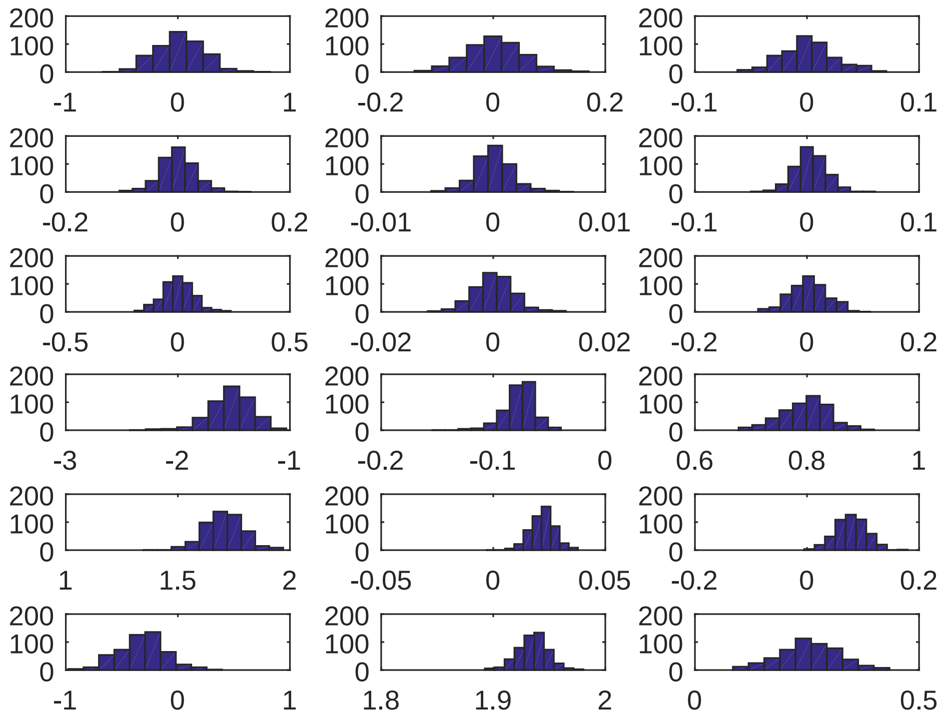
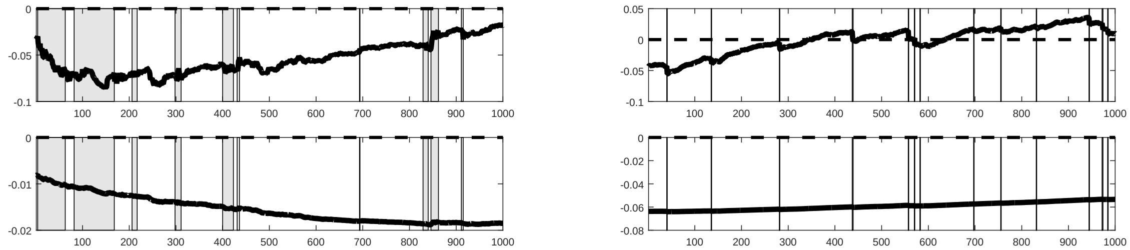
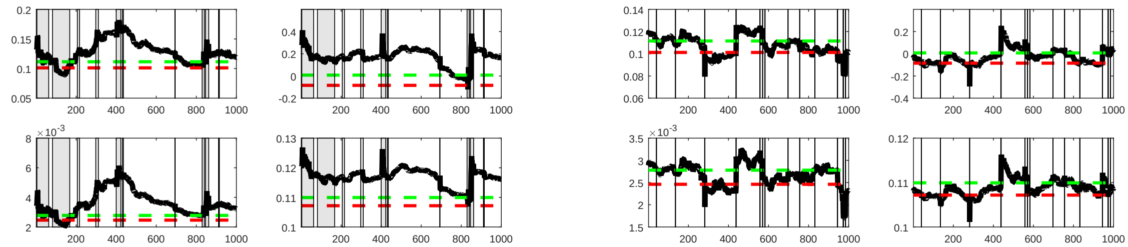


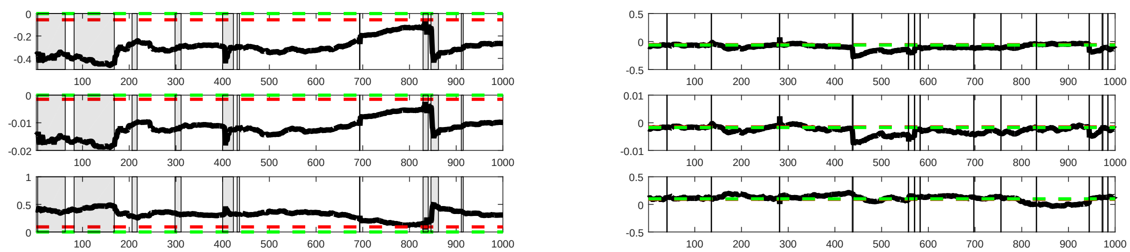
Figure 4: Simulations of length 5000, with a burn-in period of 4000, for the two-regime NKPC with exogenous and endogenous regime switching. The left panel shows the results with exogenous regime switching, while the right panel shows the results with endogenous switching.



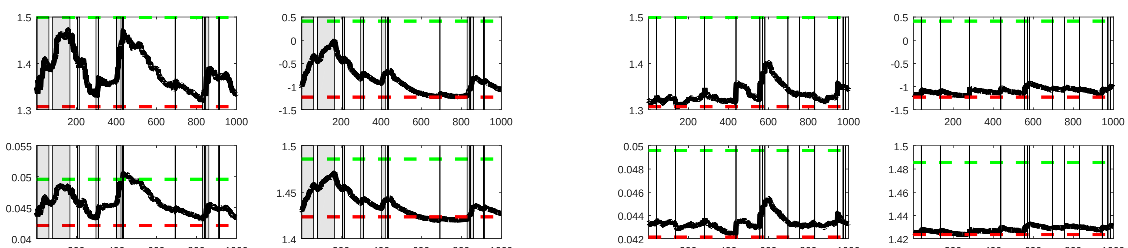
(a) Mean coefficients learning



(b) Lagged interest rate learning



(c) Lagged inflation and output gap learning



(d) Shocks coefficients learning

4 Bayesian Estimation of Markov-Switching DSGE Models under Adaptive Learning: Filtering Algorithm

The benchmark algorithm for Markov-switching state-space models is the modified Kalman filter by Kim & Nelson (henceforth KN-filter): in a Markov-switching model with m distinct regimes, a dataset of size T leads to m^T distinct timelines, which quickly makes the standard Kalman filter intractable. The main idea in the KN-filter is to introduce a so-called *collapsing* technique to deal with this issue, which amounts to taking weighted averages of the state vector and covariance matrix at each iteration of the filter. This effectively reduces the number of timelines at each iteration by an order of m, thereby making the filter tractable again. The standard recommendation is to carry as many lags of the states as appears in the transition equation. Since we only consider we consider DSGE models that have a reduced-form VAR(1) representation in this paper, only a version of the filter with the single lag is presented here, although the same framework can be easily extended to any VAR(p) framework. Accordingly, if there are m different regimes in the model, we carry m different timelines in each period. Therefore there are m^2 different sets of variables in the forecasting and updating steps of each iteration. These are then collapsed at the end of each iteration to reduce to m sets of variables. An important question is how to introduce adaptive learning into this framework. We use an approach that is consistent with the theoretical framework of the previous section: the agents have a unique PLM based on observables, independent of the regime switches. We model this formally by collapsing the m different states further at each iteration to obtain the filtered states, which are then used for the adaptive learning step. The unique learning coefficients are then used in each distinct timeline of the next period's iteration[7]. We denote the state-space representation of our model as follows:

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)} S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)} \epsilon_t, & , \epsilon_t \sim N(0, \Sigma) \\ y_t = E + F S_t \end{cases}$$

where S_t and y_t denote the (unobserved) state variables and observables respectively, while E and F collect the parameters that relate the state variables to the observables. The matrices of structural parameters are time-varying with respect to both the Markov state s_t and the belief coefficients Φ_t. Given this representation, Table 1 the filter for the general case, while Figure 5 illustrates the special case of two regimes, which is our main focus in this paper.

[7] A natural alternative here is to apply the adaptive learning step distinctly to each collapsed state; one can then take a weighted average of these expectations to obtain the filtered expectations. Our results in the upcoming sections are not sensitive to such an alternative, but we only present the results under the first approach since it is more in the spirit of our theoretical framework.

Table 1: **KM-filter for Markov-Switching DSGE Models under Adaptive Learning**

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)} S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)} \epsilon_t, \quad , \epsilon_t \sim N(0, \Sigma) \\ y_t = E + F S_t \end{cases}$$

0) Initial States :

$\tilde{S}^i_{0|0}, \tilde{P}^i_{0|0}, Pr[S_0 = i|\Phi_0], \Phi_0$ given.

1) Kalman Filter Block with the standard measurement and transition equations:

For $t = 1 : N$

For $\{S_{t-1} = i, S_t = j\}$

$$\begin{cases} S^{(i,j)}_{t|t-1} = \gamma_1^{(j)} S^{(i)}_{t-1|t-1} + \gamma_2^{(j)} \\ P^{(i,j)}_{t|t-1} = \gamma_1^{(j)} P^{(i)}_{t-1|t-1} \gamma_1^{(j)} + \gamma_3^{(j)} \Sigma^{(j)} (\gamma_3^{(j)})' \\ v^{(i,j)}_{t|t-1} = (y_t - F^{(j)} S^{(i,j)}_{t|t-1}) \\ Fe^{(i,j)} = F^{(j)} P^{(i,j)}_{t|t-1} F^{(j)} \\ S^{(i,j)}_{t|t} = S^{(i,j)}_{t|t-1} + P^{(i,j)}_{t|t-1} (F^{(j)})' (Fe^{(i,j)})^{-1} v^{(i,j)} \\ P^{(i,j)}_{t|t} = P^{(i,j)}_{t|t-1} (F^{(j)})' (Fe^{(i,j)})^{-1} F^{(j)} P^{(i,j)}_{t|t-1} \end{cases}$$

2) Hamilton Block for transition probabilities:

Denote: $Pr[S_{t-1} = i, S_t = j|\Phi_{t-1}] = pp^{i,j}_{t|t-1}$,$f(y_t|\Phi_{t-1})$ the marginal likelihood,
$Pr[S_{t-1} = i, S_t = j|\Phi_t] = pp^{i,j}_{t|t}$ and $Pr[S_t = j|\Phi_t] = \tilde{pp}^{j}_{t|t}$.

$$\begin{cases} pp^{(i,j)}_{t|t-1} = Q(i,j) pp^{(i)}_{t-1|t-1} \\ f(y_t|\Phi_{t-1}) = \sum_{j=1}^M \sum_{i=1}^M f(y_t|S_{t-1} = i, S_t = j, \Phi_{t-1}) pp^{(i,j)}_{t|t-1} \\ pp^{(i,j)}_{t|t} = \frac{f(y_t|S_{t-1}=i, S_t=j, \Phi_{t-1}) pp^{(i,j)}_{t|t-1}}{f(y_t|\Phi_{t-1})} \\ p^{j}_{t|t} = \sum_i^M pp^{(i,j)}_{t|t-1} \end{cases}$$

3) Collapsing to reduce the number of states from m^2 to m:

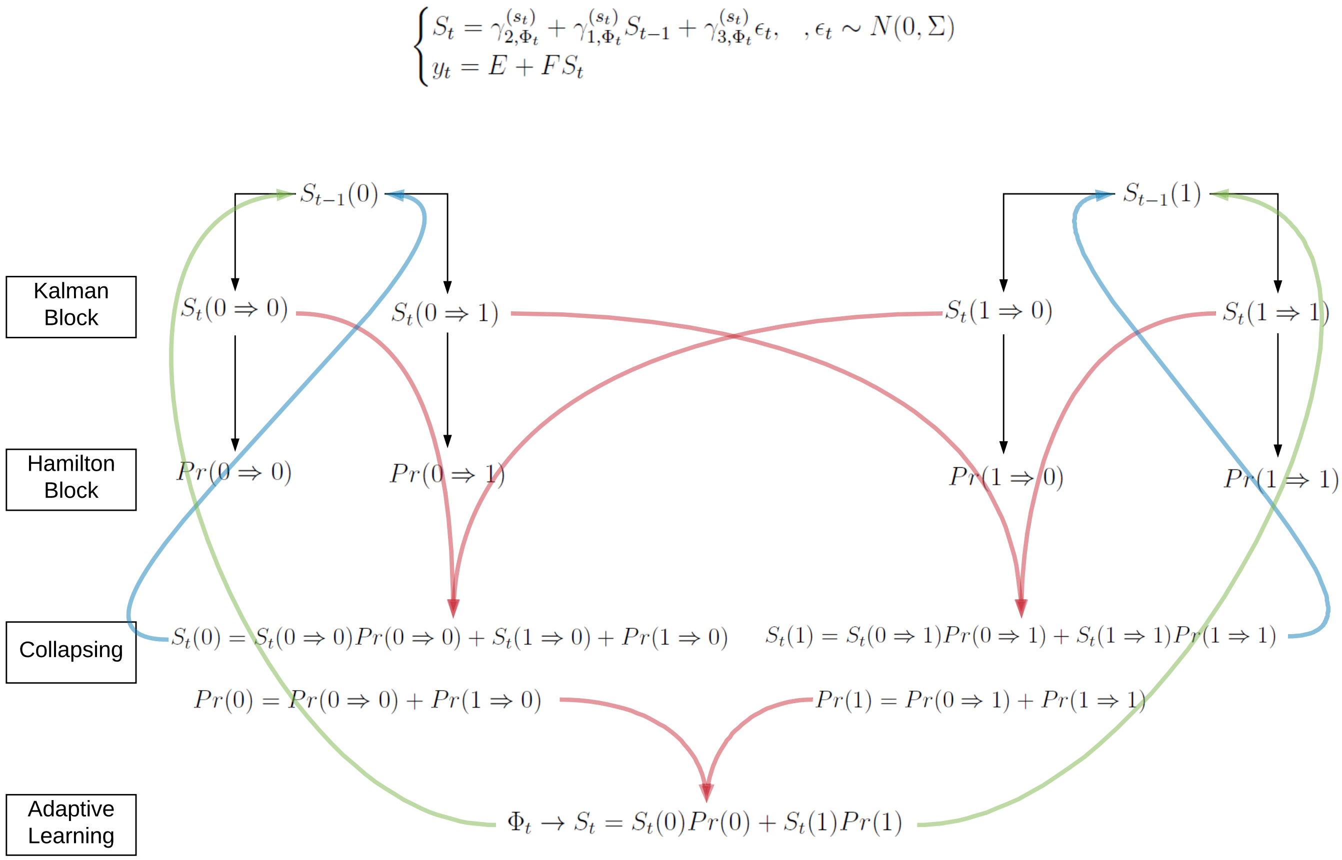
$$\begin{cases} S^{(i)}_{t|t} = \frac{\sum_{i=1}^M pp^{(i,j)}_{t|t} S^{(i,j)}_{t|t}}{p^{(j)}_{t|t}} \\ P^{(i)}_{t|t} = \frac{\sum_{i=1}^M pp^{(i,j)}_{tt} (P^{(i,j)}_{t|t} + (S^{(j)}_{t|t} - S^{(i,j)}_{t|t})(S^{(j)}_{t|t} - S^{(i,j)}_{t|t})')}{p^{(j)}_{t|t}} \end{cases}$$

4) Update expectations based on filtered states:

Updating Expectations based on Filtered States:

$$\begin{cases} \tilde{S}_{t|t} = \sum_{j=1}^M p^{(j)}_{t|t} S^{(j)}_{t|t} \\ \Phi_t = \Phi_{t-1} + \gamma R_t^{-1} \tilde{S}_{t-1|t-1} (\tilde{S}_{t|t} - \Phi^T_{t-1} \tilde{S}_{t-1|t-1})^T \\ R_t^{-1} = R_{t-1} + \gamma (\tilde{S}_{t-1|t-1} \tilde{S}^T_{t-1|t-1} - R_{t-1}) \end{cases}$$

Figure 5: Illustration of the filter in a 2-regime model.



4.1 Initial Beliefs

The first practical issue with the above filter is Step (0), i.e. where to initialize the beliefs. This has been shown to play a key role in driving the estimation results in previous studies, and various different approaches have been considered: Milani (2007) uses an estimation-based approach, where the initial beliefs are treated as structural parameters and estimated jointly along with the other parameters of interest; Slobodyan & Wouters (2012b,2012a) consider REE-based and training-sample based approaches along with the estimation-based approach; while Berardi & Galimberti (2017c) proposes a smoothing-based approach. A common result in these studies is that the results are generally sensitive to initial beliefs, and the best-fitting approach depends on the specific model under consideration; see Berardi & Galimberti (2017a, 2017b) for a detailed overview.

In this paper, following the approach in Slobodyan & Wouters (2012b), we present our main results under the REE-based initial beliefs and check the sensitivity of these results to alternative specifications. Accordingly, as our main approach, we estimate the benchmark (non-switching) REE, and use the relevant moments implied by the matrices γ_1 and γ_3 from this estimation in the initialization step of the learning models[8]. As our alternatives, we consider three approaches: (i) training-sample approach, where we initialize the filter at diffuse moments and estimate it on a training sample[9], then use the final values of the learning coefficients to initialize the main estimation; (ii) estimation-based approach, where we first estimate all belief coefficients jointly with the structural parameters in a VAR model, then use the resulting estimates to initialize the main estimation; (iii) a filter-based approach, where, at each step of the estimation, the beliefs are initialized at diffuse points and ran once, then the converged values of the belief coefficients are used as initial values to run the filter for a second time. The results of the alternative estimations are provided in the Appendix, and while the relative fit of each model is sensitive to the initial beliefs, our main conclusions continue to hold in all specifications.

4.2 Projection Facilities

The second practical issue with the above filter is how to retain the stability of the underlying model with adaptive learning. In most operational macro models, the MSV-solution includes lagged state variables due to properties such as habit formation, indexation in prices and wages, interest rate smoothing, etc. When these parameters in PLM are updated each period in a constant gain setup, they may easily end up in an unstable region, which then feeds back into the implied ALM and leads to explosive dynamics. Models subject to the zero lower bound constraint are more prone to encounter this problem, since typically an inactive monetary policy rule implies indeterminacy and E-unstability for the regime-specific dynamics. A common way in the adaptive learning literature to deal with these potential instabilities is to impose a so-called projection facility on the model, which projects the learning parameters into a point in the stable region when the instability is encountered. The simplest approach to do this is to leave the parameters at their previous value if the update leads to an instability, which is the method adopted in Slobodyan & Wouters (2012a). In this paper we use a variant of this simple idea: we stop updating the learning parameters each period, if the update pushes the largest eigenvalue of the system outside the unit circle. We base our notion of stability on the ergodic distribution of the Markov chain: the model is said to be explosive at any set of parameter values, if the largest eigenvalue of the ergodic distribution is outside the unit circle. Accordingly, we allow the regime-specific models to be temporarily explosive, as long as the underlying distribution is still stable. Importantly, this approach also allows the agents' PLM to become temporarily explosive as long as the underlying ergodic distribution is stable. These choices do not have an impact on our estimation of the small-scale NKPC since we do not encounter unstable regions in this case; but they turn out to play an important role in the estimation of medium-scale SW model as we show in the upcoming sections.

5 Estimation of the 3-equation NKPC

In this section, as a first step, we estimate the small-scale 3-equation NKPC model presented in the previous section. In this framework there are 18 parameters to be estimated, to which we

[8]The REE-implied intercepts are always zero, therefore the vector γ_2 is always initialized at the vector of zeros

[9]For both models considered in the latter sections, our training sample is based on pre-1966 period of aggregate U.S. variables.

assign prior distributions consistent with the previous literature: the risk aversion coefficient has a gamma distribution with a mean 2 and standard deviation 0.5 as in An & Schorfheide (2007). The monetary policy reaction coefficients are given Gamma distributions centered at 1.5 and 0.5 respectively, with a standard deviation of 0.25; these are the standard values associated with the Taylor rule. Interest rate smoothing, shock persistence and shock standard deviation coefficients are consistent with Smets & Wouters (2007), the first two having a Beta distribution with mean 0.5 and st. dev 0.2, while the latter is assigned a Gamma distribution with mean 0.1 and st. dev 2. The indexation parameters ι_y and ι_p are assigned Beta distributions with mean 0.25 and standard deviation 0.1[10]. The regime probabilities are taken from Lindé et al. (2017): the exit probability of the normal regime is a Beta distribution with mean 0.1 and standard deviation 0.05, while the exit probability of ZLB regime is a Beta distribution with a mean 0.3 and st. dev 0.01. The gain coefficient follows the same distribution as in Slobodyan & Wouters (2012b) with a gamma distribution and a mean of 0.035, but we assume a tighter distribution with a standard deviation of 0.015. The slope of Phillips curve is assigned a Beta distribution centered at 0.3 with a standard deviation of 0.15, which has a slightly larger mean and variance compared toAn & Schorfheide (2007).

We use quarterly U.S. data over the period 1966:I-2016:IV on interest rates, inflation and output gap with the following simple measurement equations:

$$\begin{cases} y_t^{obs} = \bar{y} + y_t \\ \pi_t^{obs} = \bar{\pi} + \pi_t \\ r_t^{obs} = \bar{r} + r_t \end{cases}$$

where the mean parameters $\bar{y}, \bar{\pi}$ and $\bar{r}$ are assigned normal distributions based on the pre-1966 period, and output gap is based on the potential output gap measure of CBO. Table 2 shows the point-estimates for five model specifications: for the constant-gain learning cases, we consider a univariate AR(1) rule, a VAR(1) rule including output gap, inflation and interest rates, and the MSV-type rule that assumes regime switches are unobserved as discussed above. The last two columns show the two benchmark Rational Expectations cases with and without regime switching[11]. First looking at the resulting likelihoods based on Laplace approximation, we can already see a pattern: the MS-REE model leads to a substantial improvement over the benchmark REE model, implying that the regime shift on interest rates plays an important role in driving the model fit. Adding adaptive learning on top of Markov-switching improves the likelihood further: all three adaptive learning specifications outperform the REE-MS model. Both of these results, individually, are consistent with the previous results found in the literature, i.e, it is well known that both Markov-switching and adaptive learning typically improve the model fit compared with the benchmark case. Our results here show that these two results are also complementary, i.e. putting the two together improves the results further compared

[10]This prior has a lower mean and tighter standard deviation than usual; e.g. the priors for indexation parameters in Smets-Wouters model in a beta distribution with a mean of 0.5 and st. dev. of 0.2. In this setup, the MS-REE model leads explosive outcomes at the ZLB regime when the indexation parameters are sufficiently large. Therefore we choose to restrict these parameters to a lower regime by assuming a tight prior.

[11]The REE-MS specification is estimated using J. Maih's RISE toolbox (Maih (2015)), while the standard REE case is obtained from the Dynare toolbox (Adjemian et al. (2011)). For the adaptive learning specifications, we use our filter as presented in the previous section. Note that RISE toolbox uses a variant of the same KN filter, hence the estimations are based on the same filter except for the adaptive learning component.

with the individual cases.

Next we turn to a discussion of our parameter estimates: the results under REE-MS are generally similar to the REE model with the exception of the risk aversion parameter τ, which is fairly high under REE-MS with 5.08 compared with 3.74 under REE. Comparing the MSV-learning case to REE-MS, the differences are minimal: the only differences arise in the interest rate smoothing ρ_r, output gap reaction ϕ_y and the risk aversion coefficient τ, where the first two are higher and the latter lower under MSV-learning. Another difference is the exit probability from ZLB regime, which is estimated to be lower under MSV-learning. At the estimated coefficients, this implies the ZLB regime has an expected duration of 4.2 quarters under MS-REE, while this is 7.7 under MSV-learning. In other words, the estimated persistence of the ZLB regime is higher under the learning specification. These differences become more pronounced as we move onto the AR(1) and VAR(1) cases: the exit probability from ZLB regime under AR(1)-learning is the same as in MSV case with an expected duration of 7.7, while the ZLB regime is even more persistent under VAR(1)-learning with an expected duration of 9.1 quarters. These results show that, in general, the expected duration of the ZLB regime is much larger under any type of learning compared with the MS-REE case. Further, in both AR(1) and VAR(1) cases, the NKPC slope κ is substantially larger at 0.013 and 0.011 respectively, compared with 0.006, 0.006 and 0.001 under MSV-learning, REE-MS and REE cases: this is a direct consequence of assuming unobserved shocks, which shifts the transmission channel from expectations to the parameter κ. Another important difference between AR(1) and VAR(1) compared with the remaining specifications is the shock persistence terms: in both cases, the persistence of shocks are much smaller compared with the remaining cases. This results from the backward-looking nature of AR(1) and VAR(1) rules, which shift some of the exogenous persistence in the shocks onto expectations, thereby resulting in lower estimates for the shock persistence parameters. Furthermore, the gain coefficient increases as we reduce the information set: it is lowest for the MSV-learning case and highest for the univariate AR(1) rule. Among the three learning rules, the most parsimonious AR(1) rule yields the best fit with a Laplace approximation of -281, while the VAR(1) rule yields the worst one with -292. This is confirmed by the Bayes' Factors, which yields decisive support in favor of the learning models for all specifications compared with the REE or MS-REE benchmarks.

Figure 6 shows the filtered regime probabilities for the AR(1)-learning case along with the historical interest rates over the estimation sample. It is readily seen that the estimated regime probability sharply rises in both cases in 2009 as the interest rates are lowered to near-zero levels. The same pattern can be observed in other learning cases of VAR(1)- and MSV-learning model, as well as REE-MS, which we omit here. This figure suggests that there are no differences in the estimated pattern of entering and exiting the ZLB regime, which is intuitive since the learning process should not have an impact on this. We next turn to the time variation in the belief coefficients over the estimation sample, which is shown in the second panel of Figure 6 for the AR(1) model[12]. The most important difference arises in the pattern of intercept coefficients for MSV-learning with observed shocks, and AR(1) & VAR(1) learning with unobserved shocks: while there is considerable time-variation in the intercept parameters in the latter cases, these

[12] We omit the plots of time-varying beliefs for VAR- and MSV-learning models for brevity, although the differences are briefly discussed

is much less variation for MSV-learning. This suggests that when the exogenous shocks are assumed to be observed, the changes in the endogenous variables are attributed to these shocks and to changes in the feedback coefficients from exogenous shocks to endogenous variables. When the shocks do not enter into agents' information set, they attribute more changes to the time-variation in the intercept terms. In the AR(1) case, there is a sudden drop in the intercept terms during the 2007-08 crisis period, and these coefficients remain there during the remainder of the sample. The same observation also applies to the VAR(1) case, where the intercept of interest rates is also characterized with a sudden drop along with inflation and output gap. This suggests that there was a level shift in the agents beliefs for the endogenous variables during this period. This level shift does not arise in the MSV-learning case since the changes are attributed to the exogenous shocks. Accordingly, while all learning specifications provide a plausible characterization of the economy since they provide a better fit than the REE models, there is considerable difference in the implied evolution of beliefs during the sample period. This in turn suggests there might be important differences in how the economy responds to the same shock under different specifications. Figure 7 shows the impulse responses of output to supply and demand shocks for all estimated models: we consider the responses to one unit supply and demand shocks η_y and η_π respectively. The black and red lines at the left and right sides of each panel show the IRFs from MS-REE model. It is readily seen that there is considerable time-variation in the IRFs for both shocks under all learning models. A noticeable jump takes place during the switch to the ZLB episode under all learning models, although the patterns are different. In the AR(1) model, there is a quick shift after the switch, and the time-variation is otherwise relatively small. In the VAR(1) model, there is a very large jump at the beginning of the switch, followed by a gradual shift downwards. It is interesting to see that in this case the period-specific IRFs gradually move towards the IRF of MS-REE model. The pattern in the MSV model is more gradual, where there is a small jump with the switch, after which the IRF gradually becomes larger.

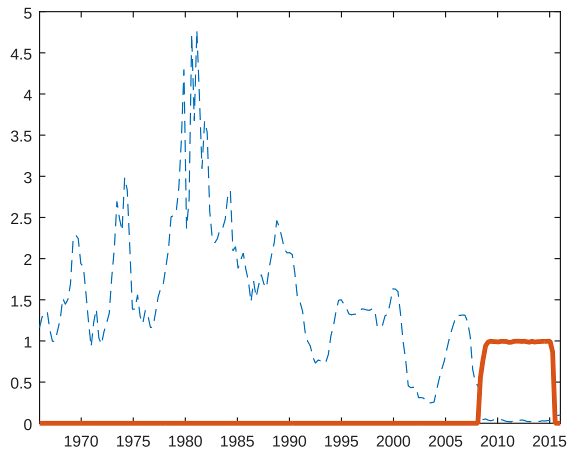
The pattern in impulse responses make it clear that there are sizeable differences in the response of the model to the same shock, depending on the assumption of the information set and the perceived law of motion: this suggests that implied stochastic structure in the economy is different across models even in this small-scale setup. This raises the question of whether the differences will carry over to a more realistic model setup, which we analyze in the next section with the Smets & Wouters (2007) model.

Table 2: Estimation Sample: 1966:I-2016:IV based on the U.S data, where the observables are the output gap (based on CBO's historical estimates), inflation and interest rate.The Markov-switching REE model is obtained from RISE toolbox, while the standard REE case is provided by Dynare. The AR(1), VAR(1) and MSV-learning cases are based on our algorithm above.

Param.	Prior			REE Post.		MS-REE Post.		AR(1) Post.		VAR(1) Post.		MSV Post.	
	Dist	Mean	St. Dev	Mode	St. Dev	Mode	St. Dev	Mode	St. Dev	Mode	St. Dev	Mode	St. Dev
$\bar{y}$	Normal	0	0.25	-0.33	0.235	-0.2	0.238	-0.14	0.21	-0.13	0.207	-0.31	0.2
$\bar{\pi}$	Gamma	0.62	0.25	0.56	0.149	0.66	0.181	0.49	0.074	0.43	0.128	0.73	0.086
$\bar{r_1}$	Gamma	1	0.25	1.04	0.17	1.24	0.252	1.08	0.098	1.09	0.147	1.28	0.129
κ	Beta	0.3	0.15	0.001	0.0008	0.005	0.002	0.013	0.004	0.011	0.005	0.006	0.003
τ	Gamma	2	0.5	3.74	0.573	5.08	0.64	3.59	0.529	3.45	0.465	2.97	0.425
ϕ_π	Gamma	1.5	0.25	1.44	0.175	1.46	0.167	1.34	0.136	1.42	0.176	1.42	0.182
ϕ_y	Gamma	0.5	0.25	0.36	0.091	0.32	0.087	0.32	0.055	0.5	0.098	0.41	0.073
ρ_y	Beta	0.5	0.2	0.03	0.022	0.85	0.043	0.3	0.065	0.22	0.065	0.79	0.028
ρ_π	Beta	0.5	0.2	0.34	0.068	0.89	0.032	0.04	0.028	0.03	0.023	0.04	0.028
ρ_r	Beta	0.5	0.2	0.87	0.026	0.81	0.03	0.85	0.021	0.88	0.019	0.87	0.017
η_y	Inv. G.	0.1	2	0.48	0.054	0.14	0.028	0.38	0.019	0.49	0.025	0.2	0.01
η_π	Inv. G.	0.1	2	0.15	0.008	0.03	0.009	0.13	0.008	0.17	0.009	0.2	0.01
η_{r_1}	Inv. G.	0.1	2	0.29	0.015	0.32	0.018	0.31	0.015	0.31	0.017	0.31	0.017
ι_y	Beta	0.25	0.1	0.71	0.049	0.28	0.057	0.44	0.027	0.61	0.032	0.36	0.029
ι_π	Beta	0.25	0.1	0.53	0.027	0.05	0.024	0.35	0.029	0.5	0.039	0.63	0.035
$\bar{r_2}$	Normal	0.1	0.25			0.03	0.002	0.03	0.002	0.03	0.002	0.03	0.002
η_{r_2}	Uniform	0.005	0.05			0.01	0.001	0.01	0.002	0.01	0.001	0.01	0.001
$1-p_{11}$	Beta	0.1	0.05			0.01	0.007	0.02	0.01	0.02	0.009	0.02	0.009
$1-p_{22}$	Beta	0.3	0.1			0.24	0.051	0.13	0.047	0.13	0.049	0.13	0.049
gain	Gamma	0.035	0.015					0.04	0.009	0.05	0.001	0.01	0.004
Lapl					-360.16		-304.95		-281.78		-292.39		-289.5
Bayes					1		23.98		34.04		29.43		30.68

Figure 6: Top panel shows the historical interest rates along with the estimated regime probabilities in AR(1)-learning model. The estimated regime probabilities are similar across all regime-switching models. The bottom two panels show the perceived mean and persistence coefficients in the AR(1)-learning for output gap and inflation. The coeffients in PLMs for MSV- and VAR-learning rules are omitted here but can be found in the appendix.

Historical interest rates along with the estimated regime probability:



AR(1) intercept (left) and persistence (right) coefficients in the PLM, with output gap and inflation on the first and second row respectively:

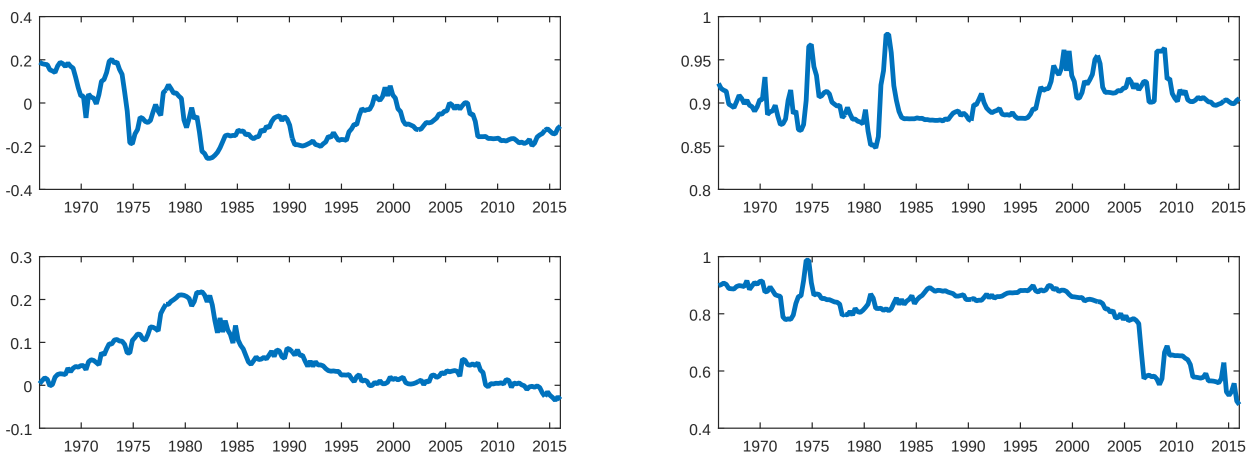
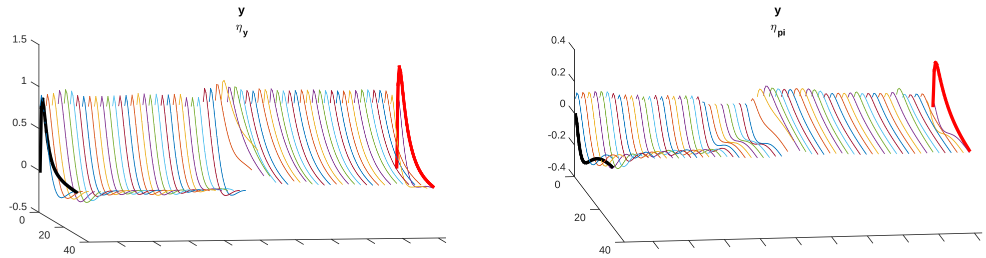
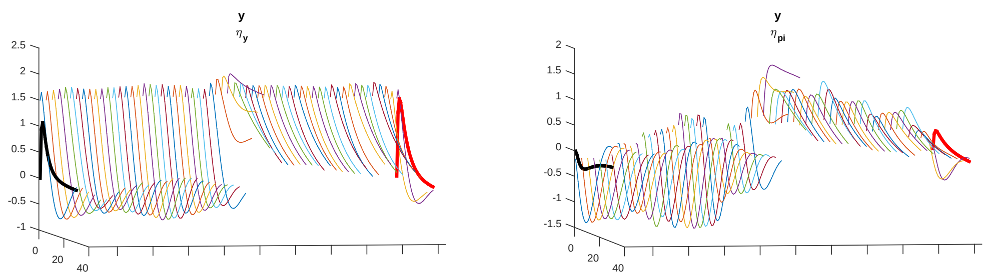


Figure 7: Implied time variation in the IRFs for learning models: responses of output gap to supply and demand shocks during the last 15 years of the estimation sample, 2002:I-2016:IV. The solid black (normal regime) and red (ZLB regime) lines on each side are the regime-specific IRFs of the MSV-REE model.

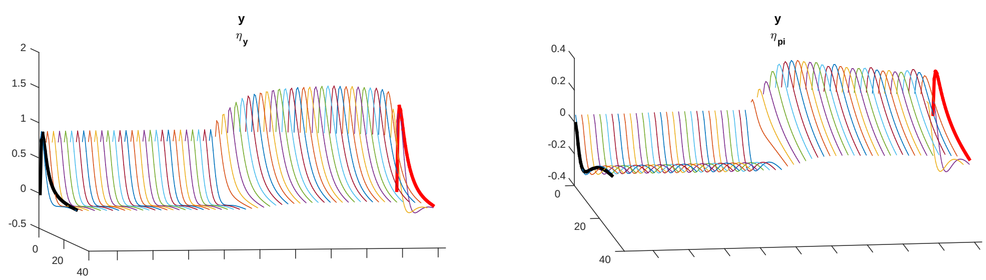
AR(1):



VAR(1):



MSV:



6 Estimation of Smets-Wouters

In this section we consider the estimation of Smets-Wouters (2007). The details of the model are omitted here for brevity; the readers are referred to Smets & Wouters for a close-up of the model. The prior distributions of the benchmark parameters are identical to Smets-Wouters, while the ZLB and learning related parameters are the same as in the 3-equation model above. The model features seven structural shocks: productivity, risk-premium, government spending, investment, monetary policy, and price & wage mark-up shocks, denoted by a, b, g, i, r, p and w, see Smets & Wouters (2003,2007) for more details. Our only deviation from the benchmark model is to assume that the mark-up shocks follow an AR(1) process, instead of the original

ARMA(1,1) assumption: as shown in Smets & Wouters (2003,2007), these shock processes are typically close to being white noise when expectations are assumed to be backward-looking, in which case the AR(1) and MA(1) terms are close to being locally unidentified. Therefore we assume away the MA(1) terms and model these shocks as AR(1) processes. The rest of the model is left unchanged and our measurement equations also follow the benchmark case with seven observables as follows:

$$\begin{cases} d(log(y_t^{obs})) = \bar{\gamma} + (y_t - y_{t-1}) \\ d(log(c_t^{obs})) = \bar{\gamma} + (c_t - c_{t-1}) \\ d(log(inv_t^{obs})) = \bar{\gamma} + (inv_t - inv_{t-1}) \\ d(log(w_t^{obs})) = \bar{\gamma} + (w_t - w_{t-1}) \\ log(l_t^{obs}) = \bar{l} + l_t \\ (log(\pi_t^{obs})) = \bar{\pi} + \pi_t \\ (log(r_t^{obs})) = \bar{r} + r_t \end{cases}$$

where $d(log(y_t^{obs}))$, $d(log(c_t^{obs}))$, $d(log(inv_t^{obs}))$ and $d(log(w_t^{obs}))$ denote real output, consumption, investment and wage growths with the common growth rate $\bar{\gamma}$ respectively, while $log(l_t^{obs})$, $(log(\pi_t^{obs}))$ and $(log(r_t^{obs}))$ denote (normalized) hours worked, inflation rate and federal funds rate respectively. We use the same estimation sample as in the 3-equation NKPC with quarterly U.S. data covering the period from 1966:I to 2016:IV. Tables 3 and 4 show the results for the learning and benchmark REE and MS-REE cases. We observe that our main conclusions from the NKPC-estimation carry over to this more realistic setup: the MS-REE model considerably improves upon the REE benchmark with likelihoods -1140 and -1194 respectively, which shows the empirical importance of explicitly modeling the ZLB episode. The MSV-learning setup yields a likelihood of -1169, which, unlike the 3-equation setup, is worse than the REE-MS case. This indicates the time-variation under MSV-learning does not lead to any meaningful improvements over the MS-REE model, and in fact the time-variation in the belief parameters comes at a cost in terms of model fit. Next looking at the parsimonious AR(1) learning case, we have a likelihood of -1125, which is another substantial improvement over both the MSV-learning and MS-REE cases. Similarly, the VAR(1) learning case yields a likelihood of -1136, which is worse than the AR(1) model but better than MS-REE and MSV-learning cases. The model with the univariate, backward-looking AR(1) expectation rule is thus our preferred specification based on these results. Taken together with the NKPC estimations, all of our results are consistent with the previous literature on learning: Milani (2007) shows that MSV-learning performs better that REE in the small 3-equation setup, while Slobodyan & Wouters (2012b) find that this result does not extend to the medium-scale setup of SW model, where MSV-learning does not lead to improvements over the REE model. Instead using a univariate backward-looking rule in Slobodyan & Wouters (2012a), they find that the model fit improves considerably. Our results in this section and the previous one confirm these results, and show that they continue to hold in a Markov-switching setup where the ZLB episode is considered.

Next we turn to a discussion of our parameter estimates and we focus particularly on the learning, ZLB and nominal friction parameters. Starting with the estimated exit probabilities from the normal and ZLB regimes, we can see the same result as in the 3-equation model, where the ZLB regime becomes more persistent under the learning cases. In particular, the estimated

exit probability from the ZLB regime is 0.27 under the MS-REE model, indicating an expected ZLB duration of of 3.7 quarters. This probability is substantially larger under the learning specifications with a value of 0.13 under both AR(1) and MSV-learning cases, which suggests an expected ZLB duration of 7.7 quarters. Especially the expected duration under the MS-REE case is lower compared to the 3-equation model, suggesting there is downward pressure for the persistence of this regime. It is important to note that the expected ZLB durations both under the MS-REE and learning cases are substantially lower than the empirical length of the ZLB episode, which lasted around 7 years or 28 quarters for the U.S. economy. However, our results suggest that adaptive learning is indeed one way of increasing the expected duration of this episode, since the expected duration in the Smets-Wouters model is estimated more than twice under adaptive learning compared with the MS-REE case.
The estimated gain values for learning models are lower compared with the 3-equation model, with values of 0.02, 0.01 and 0.006 for AR(1), VAR(1) and MSV-learning cases respectively. Branch & Evans (2006) find that values over the range of $[0.005, 0.05]$ provide a good fit for the Survey of Professional Forecasters (SPF) dataset. Accordingly, the resulting gain value under AR(1)- and VAR(1)-learning is in the empirically relevant range, while under MSV-learning it is substantially closer to zero. This suggests either that the MSV information set is not supported by the data, or that the initial beliefs provide a better fit than the time-variation in beliefs under the MSV-setup. At the estimated value under MSV-learning, the initial beliefs receive a weight of $(1 - 0.006)^{200} \approx 0.3$ at the end of the sample; i.e. initial beliefs receive a considerable weight even towards the end of the estimation sample. The resulting likelihoods suggest that this does not lead to substantial differences in the model fit, which is also evident from the remaining parameter estimates. The only exceptions are the price and wage Calvo probabilities, which are slightly lower under the MSV-learning case compared with the MS-REE. The difference is more pronounced under AR(1)-learning, where the Calvo probabilities keep decreasing further. This suggests nominal persistence parameters become less pronounced as expectations become more sluggish, which is also consistent with the findings in previous literature. Another important difference that arises under AR(1)-learning is the shock persistence terms: the price and wage mark-up shocks become near white noise processes with persistence near zero, while the persistence of investment shock also becomes substantially smaller. Overall, our results suggest that the univariate AR(1)-learning case provides the best model fit and leads to some important differences in the model structure as a whole, while the MSV-learning remains very close to the MS-REE model. In both cases, however, there is a sizable increase in expected duration of the ZLB episode, which brings the persistence of this regime closer to the empirically relevant region.

We next examine the estimated regime probabilities and the time-variation in learning parameters for the best performing AR(1) model. We start with the estimated regime probabilities over the estimation sample for the AR(1) learning mode, which is shown in Figure 8: similar to the 3-equation model, we observe a sharp switch to the ZLB regime in the beginning of 2009 where the interest rates drop to near-zero levels, while the economy exits from the ZLB regime towards the end of 2016. The same pattern is also observed in the other learning models and MS-REE model, which is omitted here. An important difference between the SW model here and the 3-equation model is the activity of projection facility, which was irrelevant in the small model because it remains stable throughout the estimation sample and as a consequence, the projection facility is never imposed. In the larger SW-model, however, the model is occasionally driven into unstable regions, thereby activating the projection facility. While we omit the

Table 3: Estimation period: 1966:I-2016:IV

			REE		REE-MS		AR(1)		VAR(1)		MSV	
Prior			Post.		Post.		Post.		Post.		Post.	
Para.	Dist	Mean	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.
ϕ	Normal	4	5.43	1.021	6.3	0.98	4.52	0.526	5.59	0.622	6.03	0.856
σ_c	Normal	1.5	1.3	0.101	1.2	0.081	1.17	0.129	1.01	0.03	1.22	0.244
λ	Beta	0.7	0.77	0.041	0.82	0.029	0.79	0.021	0.87	0.015	0.84	0.033
ξ_w	Beta	0.5	0.93	0.016	0.95	0.012	0.74	0.03	0.76	0.029	0.6	0.035
σ_l	Normal	2	2.09	0.688	1.83	0.705	2.26	0.677	1.61	0.626	2.23	0.568
ξ_p	Beta	0.5	0.81	0.03	0.82	0.022	0.76	0.015	0.77	0.037	0.77	0.04
ι_w	Beta	0.5	0.84	0.069	0.81	0.072	0.56	0.127	0.64	0.123	0.71	0.116
ι_p	Beta	0.5	0.08	0.038	0.09	0.043	0.18	0.017	0.6	0.063	0.72	0.076
ψ	Beta	0.5	0.79	0.079	0.77	0.08	0.7	0.1	0.62	0.107	0.65	0.093
ϕ_p	Normal	1.25	1.55	0.072	1.66	0.079	1.56	0.053	1.55	0.058	1.54	0.062
r_π	Normal	1.25	1.45	0.155	1.49	0.168	1.62	0.169	1.63	0.174	1.68	0.178
ρ	Beta	0.75	0.85	0.019	0.88	0.02	0.88	0.02	0.88	0.02	0.87	0.019
r_y	Normal	0.125	0.05	0.02	0.08	0.021	0.13	0.028	0.12	0.027	0.13	0.027
r_{dy}	Normal	0.125	0.17	0.019	0.17	0.018	0.14	0.014	0.12	0.015	0.13	0.017
$\bar{\pi}$	Gamma	0.625	0.72	0.093	0.79	0.111	0.57	0.087	0.83	0.062	0.65	0.077
$\bar{\beta}$	Gamma	0.25	0.13	0.052	0.15	0.058	0.18	0.067	0.35	0.068	0.24	0.082
$\bar{l}$	Normal	0	-0.14	1.056	0.54	0.997	2.03	0.588	2.54	0.438	2.84	0.601
$\bar{\gamma}$	Normal	0.4	0.38	0.015	0.4	0.012	0.4	0.008	0.42	0.006	0.43	0.01
α	Normal	0.3	0.19	0.015	0.19	0.016	0.17	0.015	0.18	0.016	0.18	0.013
Laplace			-1194.87		-1140.24		-1125.17		-1136.4		-1169.49	
Bayes			1		23.45		29.97		25.19		10.85	

Table 4: Estimation period: 1966:I-2016:IV

Prior			REE		REE-MS		AR(1)		VAR(1)		MSV	
			Post.		Post.		Post.		Post.		Post.	
Para.	Dist	Mean	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.	Mode	St. Dev.
ρ_a	Beta	0.5	0.96	0.01	0.95	0.011	0.95	0.017	0.96	0.014	0.96	0.014
ρ_b	Beta	0.5	0.36	0.091	0.31	0.067	0.28	0.066	0.35	0.065	0.38	0.062
ρ_g	Beta	0.5	0.99	0.005	0.99	0.006	0.99	0.001	0.99	0.003	0.99	0.004
ρ_i	Beta	0.5	0.81	0.04	0.74	0.041	0.48	0.06	0.56	0.056	0.83	0.038
ρ_r	Beta	0.5	0.09	0.05	0.08	0.052	0.17	0.059	0.09	0.055	0.06	0.044
ρ_p	Beta	0.5	0.8	0.05	0.77	0.049	0.05	0.027	0.05	0.028	0.07	0.047
ρ_w	Beta	0.5	0.06	0.038	0.05	0.036	0.07	0.046	0.09	0.05	0.87	0.032
ρ_{ga}	Beta	0.5	0.51	0.076	0.5	0.081	0.51	0.079	0.54	0.075	0.53	0.077
η_a	Inv. G.	0.1	0.44	0.024	0.44	0.024	0.44	0.023	0.44	0.023	0.45	0.023
η_b	Inv. G.	0.1	0.21	0.025	0.23	0.022	0.29	0.016	0.33	0.017	0.24	0.016
η_g	Inv. G.	0.1	0.48	0.024	0.49	0.025	0.48	0.025	0.48	0.025	0.48	0.023
η_i	Inv. G.	0.1	0.35	0.031	0.36	0.032	0.78	0.039	0.8	0.04	0.37	0.022
$\eta_{r,N}$	Inv. G.	0.1			0.22	0.012	0.21	0.011	0.21	0.012	0.21	0.011
$\eta_{r,ZLB}$	Unif.	0.05	0.21	0.011	0.01	0.001	0.01	0.001	0.01	0.001	0.01	0.001
η_p	Inv. G.	0.1	0.06	0.011	0.06	0.012	0.06	0.003	0.15	0.008	0.16	0.011
η_w	Inv. G.	0.1	0.37	0.023	0.37	0.022	0.38	0.019	0.38	0.018	0.19	0.012
$gain$	Gamma	0.035					0.02	0.003	0.01	0.003	0.006	0.001
$1-p_{11}$	Beta	0.1			0.02	0.006	0.02	0	0.02	0.008	0.02	0.009
$1-p_{22}$	Beta	0.1			0.27	0.033	0.13	0.05	0.13	0.046	0.13	0.051
$\bar{r_{zlb}}$	Normal	0.05			0.03	0.002	0.03	0.002	0.03	0.002	0.03	0.002
Laplace			-1194.87		-1140.24		-1125.17		-1136.4		-1169.49	
Bayes			1		23.45		29.97		25.19		10.85	

corresponding Figures for the projection facility, it is active for 10 to 20 data points over the estimation sample for the learning models, corresponding to between 5 to 10 % of the entire sample. For this reason, *how* the projection facility is imposed plays a key role in the system dynamics: recall from the previous section that we only impose the projection facility when the underlying ergodic distribution becomes unstable, while the regime-specific models are allowed to be unstable. In this case, although the ZLB-regime is unstable for long periods, the normal regime is sufficiently stable most of the time such that the projection facility is only activated a couple of times. This way of imposing the projection facility is justified on the grounds of our RPE concept, where the regime-specific models can become temporarily explosive as long as the underlying ergodic distribution is stable; we are able to keep the activity of projection facility to a minimum based on this notion.

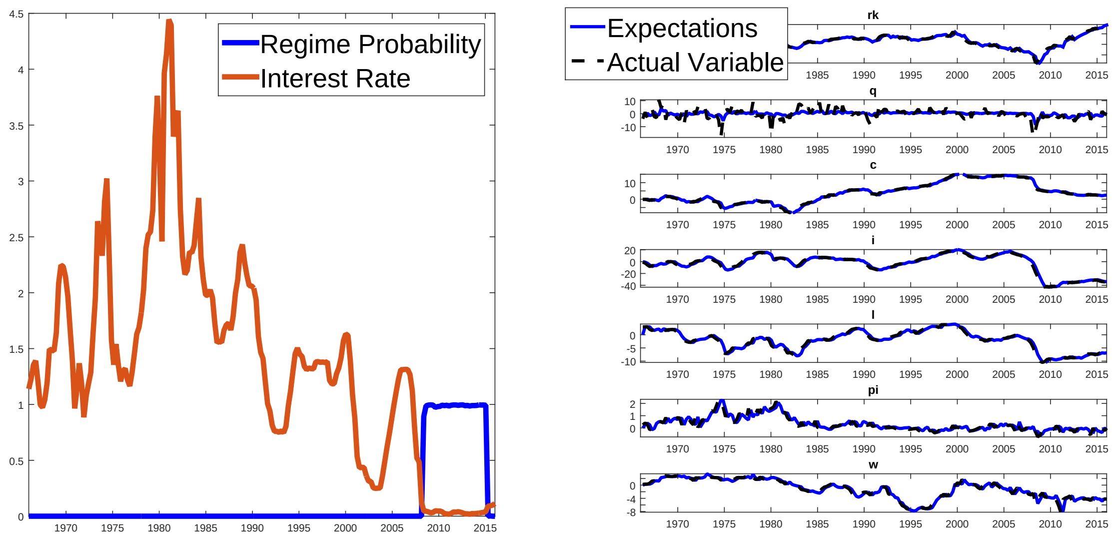
Next we examine how the learning parameters for mean and first-order autocorrelation evolve over the estimation sample for the AR(1) model, which are shown in the last two panels of 8. In this setup we can observe more interesting dynamics in the implied time variation: since shocks are unobserved in this case, the intercept terms play an important role in agents' expectations. A particularly interesting case is the financial crisis period, where we observe an almost uniform reduction in all coefficients. Particularly the coefficients on investment and real value of capital (denoted by i and q) respectively) decrease by a relatively large amount, while remaining coefficients decrease moderately. This implies that, under AR(1)-learning with unobserved shocks, the crisis and subsequent *Great Recession* period is interpreted as a level shift in the variables by agents, as opposed to a series of adverse shocks. We can observe similar jumps in the first-order autocorrelation parameters, where some of the parameters temporarily go above one. In other words, when the crisis hits and the agents observe the large downward movement in the endogenous variables, they compensate for this by temporarily switching from a trend-following rule to an extrapolative one. This can be interpreted as a temporary wave of pessimism in the economy, where the agents expect more downward movement following the adverse shock after the crisis.
A well-known problem in the VAR literature is that the standard models tend to over-predict the growth rates in the post-crisis period of 2007-08: after the crisis period, there has been a reduction in the growth rates of real in the U.S, particularly for output, consumption and investment. As a consequence, the standard models tend to over-predict these variables if no additional structural break is introduced into the model. This applies to models that both take into account the zero lower bound episode and those that do not. Figure 9 shows the forecast errors of both MS-REE and AR(1) learning models. It is readily seen that in the MS-REE case, the forecasts of the growth rates are indeed consistently larger than the actual values. Lindé et al. (2017) propose to deal with this issue in the SW model by introducing a structural break in the risk-premium shock. Accordingly, it is assumed that there is a permanent increase in the risk premium after the crisis (i.e. a level shift in the b-shock), which pushes down the forecasts of the real growth rates. We plot the forecasts under adaptive learning in the second column of Figure 9. It is readily seen that the issue of over-predicting does not arise under adaptive learning. This suggests that time-variation under adaptive learning causes as strong enough downward shift in the forecasts. Accordingly, time-varying beliefs and the potential pessimism of agents following the crisis arise as an alternative way to deal with the over-prediction problem

over this period[13].

[13] This result raises the question of whether the level shift expectations would still be prevalent if we also add a structural break in the b-shock, i.e. is it the shift in the risk premium or agents' beliefs that dominates. We leave this question to future work.

Figure 8: Top panel: estimated regime probabilities and projection facility in AR(1)- and MSV-learning cases. Bottom two panels: learning coefficients in the AR(1)-learning model.

Filtered ZLB regime probability:
Historical interest rates along with the estimated regime probability (left), expectations along with the forward-looking variables (right):



AR(1) intercept (left) and persistence (right) coefficients in the PLM, with output gap and inflation on the first and second row respectively:

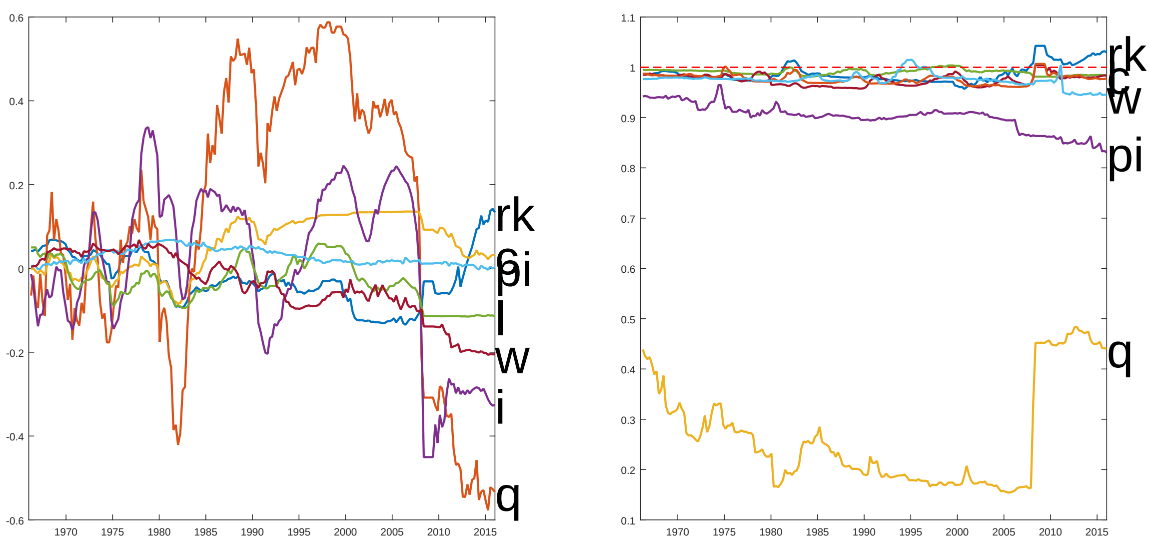
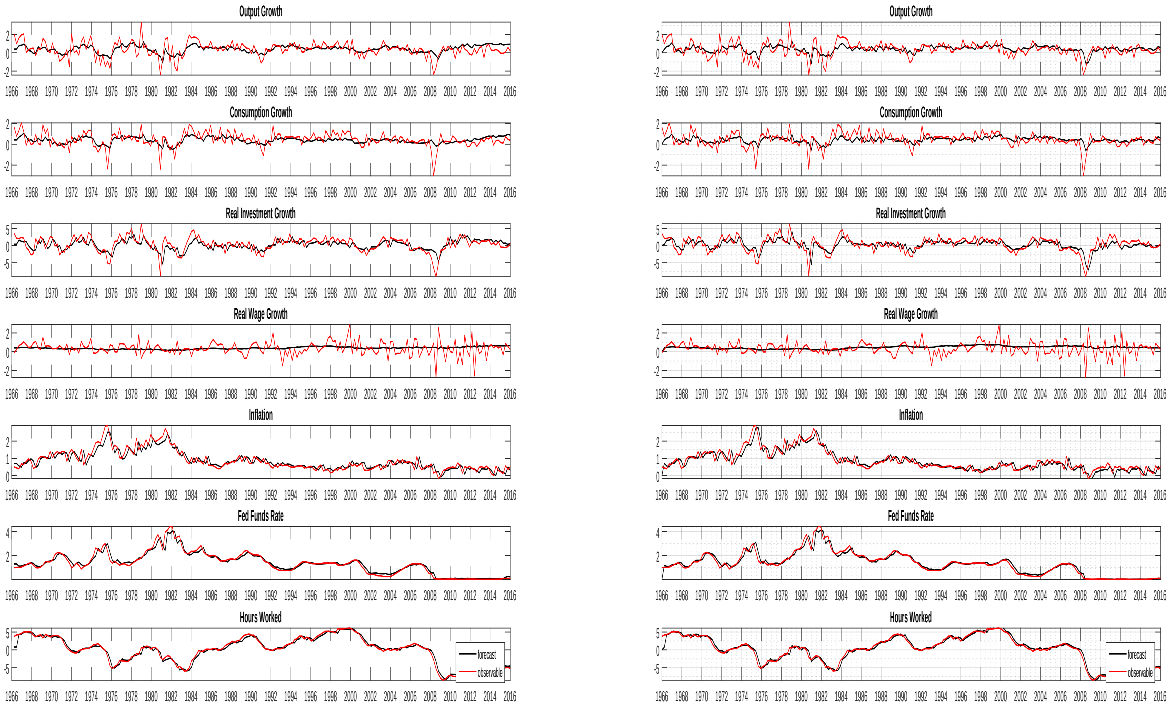


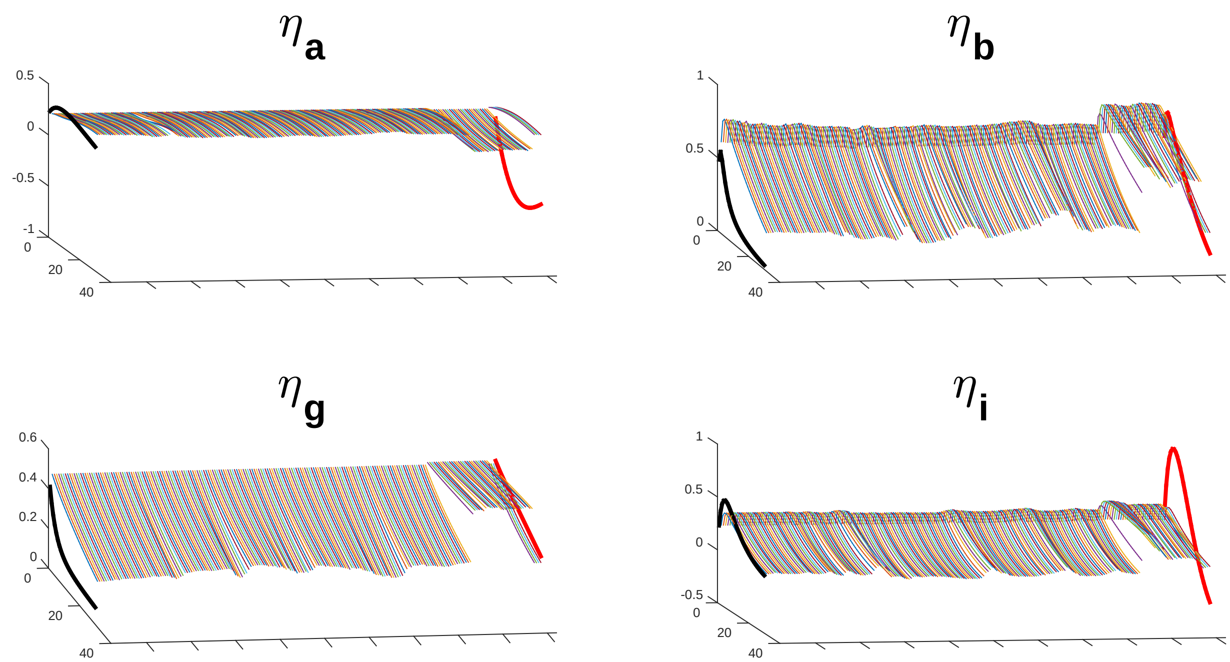
Figure 9: In-sample forecasts (i.e. the forecasting step of the filter) for all observables: MS-REE and AR(1) models respectively.



Impulse Responses

Figure 10: Comparison of AR(1) learning IRFs with REE IRFs. Each IRF shows a one standard deviation shock of η_a,η_b,η_g,η_i respectively.

Output:



Inflation:

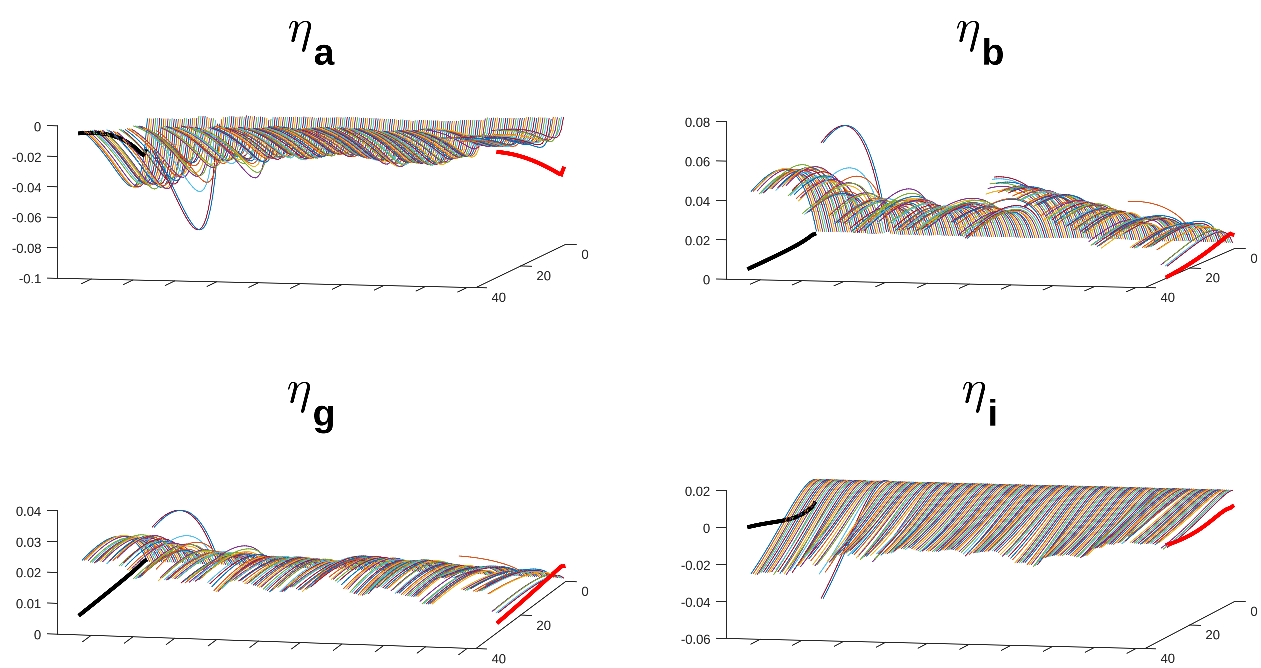
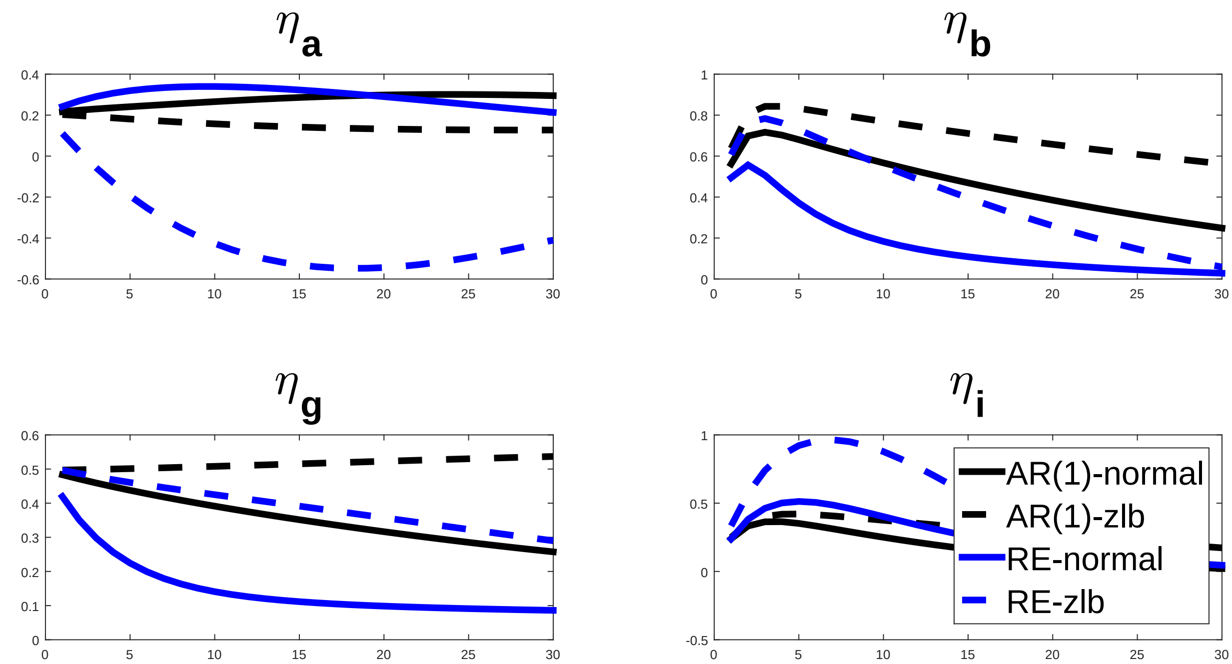
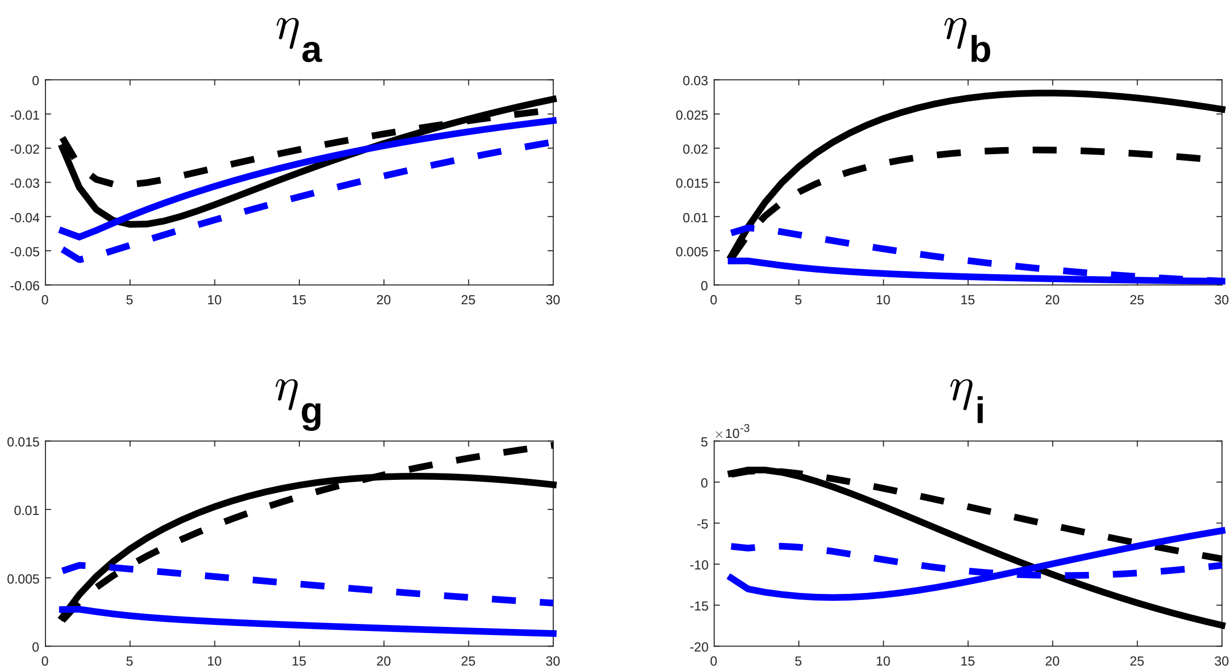


Figure 11: Comparison of AR(1) learning IRFs with REE IRFs. Each IRF shows a one standard deviation shock of η_a,η_b,η_g,η_i respectively.

Output:



Inflation:



We next compare the impulse responses under both AR(1)-learning MS-REE case. As we mention in the previous section, the impulse responses under adaptive learning are different each period since belief coefficient are updated each period, which feeds back into the autocovariance structure. We focus particularly on two key variables: output and inflation.

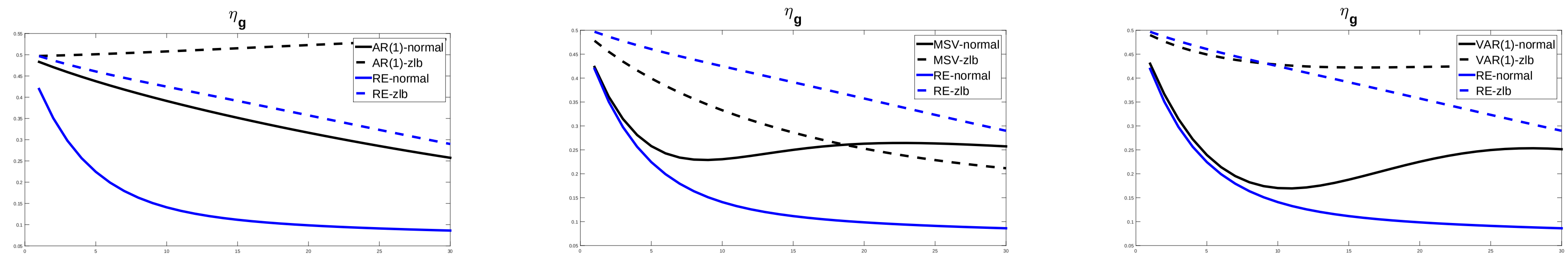
The first observation is that in general, regardless of the model specification, a shock of the same size tends to have a larger impact under the ZLB regime compared with the normal one. This is an intuitive results since nominal interest rates do not adjust at the ZLB regime to mitigate the effects of a given shock, which leads to larger effects. This applies in particular to two important shocks, i.e. risk premium b and government spending g which proxy for the impacts of a financial crisis and expansionary monetary policy respectively. When we examine the impulse responses under AR(1)-learning, we observe that they typically take much longer to reach their peak values, and also take substantially longer to die out: this result is due to the additional inertia and persistence introduced into the system by the backward-looking nature of the PLM. Accordingly, the shocks take longer to reach their maturity, as well as longer to die out. One common result that emerges the learning approach (both AR(1) that is plotted here, and the MSV- and VAR-learning cases that are omitted here), however, is that an adverse b-shock during the ZLB episode is going to have a larger and longer-lasting deflationary effect on the economy, suggesting the REE model underestimates this channel.

Next we focus on the effects of a government spending g-shock. It is readily seen from Figure 11 that, during normal times, a government spending shock has a largar impact under learning for both output and inflation. The larger effect on output in this case is driven by the fact that, when shocks are not observed, the fiscal expansion starts to *crowd in* government spending instead of crowding out. Accordingly, a fiscal expansion leads to larger effects under the normal regime with learning. This result is consistent with previous studies that examined government spending multipliers under adaptive learning, see e.g. Quaghebeur (2018). A second observation is that, when the economy switches to the ZLB regime, the impact of the shock becomes larger under both learning and MS-REE setups. This is a natural result since in a ZLB regime, interest rates do not rise to mitigate the expansionary effect of the government spending shock, leading to a larger overall impact. An interesting observation, however, is that the proportional change in the IRFs is smaller under learning, i.e. the difference induced by the switch to ZLB is larger under the REE model. This is examined in more detail in the next Figure 6.

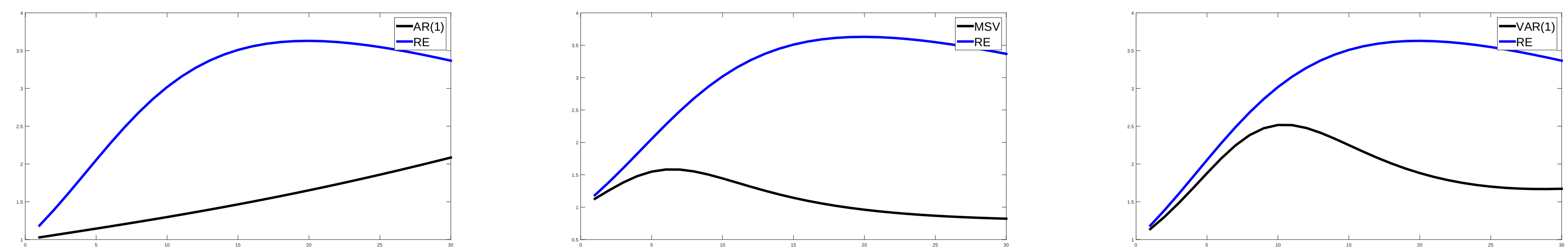
Figure 6 shows the fiscal multipliers for all three learning models and compares them with the REE case. We define the fiscal multiplier in this context as $FM = \frac{\partial y}{\partial \eta_g} / \frac{\partial g}{\partial \eta_g}$, i.e. the proportional change in output y relative to the AR(1) government spending process g, in response to a one standard deviation increase in the government spending shock η_g. This is shown in the first panel of Figure 6 for all learning models. Consistently across all three learning models, we find that the multiplier is larger than the REE model during the normal regime. The results for the ZLB regime is mixed: while the multiplier remains larger than REE for the AR(1) and VAR(1) models during the ZLB regime, it is smaller under MSV model. However, if look at the proportional change in the multipliers, i.e. $\frac{FM^{ZLB}}{FM^{normal}}$, then we observe that this is smaller than REE across all three learning models. The third panel shows the same responses for inflation, which shows a similar picture with smaller proportional changes under all learning models. This suggests that the REE model overestimates the effect of a fiscal expansion during the

Figure 12: Government spending shocks and a comparison of fiscal multipliers in all learning models with the REE benchmark. Fiscal multipliers are defined as the ratio of the increase in output and the increase in the AR(1) government spending process, i.e. $FM = \frac{\partial y}{\partial \eta_g} / \frac{\partial g}{\partial \eta_g}$. The second and third panels then compare the response the proportional change in the normal and ZLB regimes in each case, i.e. $\frac{FM^{ZLB}}{FM^{normal}}$. The third panel follows in a similar fashion, where output is replaced with inflation.

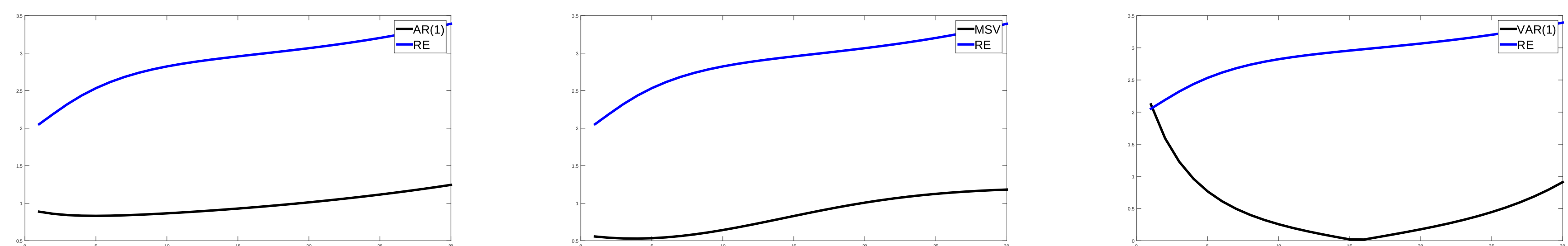
Government spending shocks (one standard deviation):



Fiscal multipliers: each curve compares the ratio of the output response under normal regime with the zlb regime.



Effects of government spending shocks on inflation: each curve compares the ratio of the inflation response under normal regime with the zlb regime.



ZLB period. In other words, the learning models suggest that the effects of a fiscal expansion will not be that much different than a fiscal expansion during normal times, compared to the REE model. Given that two of the three learning models outperform the REE model in terms of model fit, this suggests that one should be cautious in using the REE model for policy recommendations.

7 Conclusions

The literature on macroeconomics made great strides in the estimation of both Markov-switching and adaptive learning models in recent years, which are two alternative ways of introducing time-variation into DSGE models. Although there are numerous examples of both classes of models that have been successfully taken to the data, there is surprisingly little work on Markov-switching DSGE models under adaptive learning. In this paper, we provided a first attempt to estimate this class of DSGE models by combining these two approaches under the same roof. The resulting framework has a nice intuitive interpretation where the agents' do not know the details of a complex non-linear economy, but use simple linear rules to form their expectations about the future state of the economy. Our simulations show that, using simple adaptive rules allows the agents' to take into account the structural changes in the economy, albeit in an indirect manner. More importantly, our estimations indicate that the two approaches can be complementary rather than substitutes: it both the 3-equation NKPC and Smets-Wouters models we consider, imposing the Markov-switching structure on standard models improves the empirical fit, while imposing adaptive learning leads to further improvements. Furthermore, our results on the ZLB episode suggest that a wave of pessimism, modeled as a downward shift of expectations, may well have contributed to the Great Recession period characterized by low growth rates, which is a well-established idea in literature as an expectations-driven business cycle. Our paper can be extended in many directions: the key role played by expectations in our paper is an inevitable result since it is the only possible factor that can drive the low growth rates. Therefore a particularly important question is whether expectations can still play a key role in driving the Great Recession, when there are other types of structural breaks during the ZLB episode, such a break in the risk premium shock as in Lindé et al. (2017). Another important issue is the macroeconomic cost of the ZLB episode under adaptive learning: since the estimation results indicate different structural parameters and impulse responses in general, it is also plausible that the model-implied cost of ZLB is different under learning compared with REE. A final interesting extension is to study the interaction between expectations and monetary policy when there are more subtle regime switches in the monetary policy regime, such as the Great Moderation period of 1980s which is typically associated with a switch to more aggressive monetary policy. We these issues to future work.

Appendix

A Special case with 2 regimes, no lagged variables

Note that in the special case with $\iota_p = 0$, the model can be written as a generic 1-dimensional Markov-switching model of the form:

$$\begin{cases} \pi_t = a(s_t) r_t + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

where $a(s_t) = \frac{1}{\alpha(s_t)}(a\rho + 1)$. The moments necessary for the T-map are given as follows:

$$E[\pi_t r_t] = P_1 E[\pi_t r_t | S_t = 1] + P_2 E[\pi_t r_t | S_t = 2]$$

$$E[\pi_t r_t | S_t = 1] = E[a(s_t) r_t^2 | S_t = 1, S_{t-1} = 1] p_{11} + E[a(s_t) r_t^2 | S_t = 1, S_{t-1} = 2](1 - p_{22}) \frac{P_2}{P_1}$$

$$= a_1 p_{11} + a_1 (1 - p_{22}) \frac{P_2}{P_1}$$

Similarly, we have

$$E[\pi_t r_t | S_t = 1] = a_2 p_{22} + a_2 (1 - p_{11}) \frac{P_1}{P_2}$$

which yields

$$E[\pi_t r_t] = P_1 (a_1 p_{11} + a_1 (1 - p_{22}) \frac{P_2}{P_1}) + P_2 (a_2 p_{22} + a_2 (1 - p_{11}) \frac{P_1}{P_2})$$

Plugging in the steady-state probabilities P_1 and P_2, the T-map is given as follows:

$$a \to T(a) = \frac{\alpha_1 (1 - p_{22}) + \alpha_2 (1 - p_{11})}{\alpha_1 \alpha_2 (2 - p_{11} - p_{22})} (a\rho + 1)$$

with the E-stability condition

$$DT_a = \frac{\alpha_1 (1 - p_{22}) + \alpha_2 (1 - p_{11})}{\alpha_1 \alpha_2 (2 - p_{11} - p_{22})} \rho < 1$$

Re-arranging the expression above yields the LRES condition presented in Section 1. Further note that the regime-specific T-maps, and the associated regime-specific E-stability conditions are given by:

$$a \to \frac{a\rho + 1}{\alpha_i}$$

$$DT_a = \frac{\rho}{\alpha_i} < 1$$

which implies that *regime specific E-stability is a sufficient, but not necessary condition* for LRES.

B 1-dimensional case with m regimes

Note that the Fisherian model considered in Section 2 can be written as a generic 1-dimensional Markov-switching model of the form:

$$\begin{cases} \pi_t = a(s_t)r_t + b(s_t)\pi_{t-1} + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

where $a(s_t) = \frac{1}{\alpha(s_t)}(a\rho + ba + 1)$ and $b(s_t) = \frac{b^2+\iota_p}{\alpha(s_t)}$. In this Appendix we consider the general case with m regimes, with transition matrix given by:

$$Q = \begin{bmatrix} p_{11} & \dots & p_{1m} \\ \vdots & \dots & \vdots \\ p_{m1} & \dots & p_{mm} \end{bmatrix}$$

The 2-regime setup of Section is the special case with $m = 2$. We omit the first moment $E[\pi_t]$, which is trivially given as zero. Using this, we compute the second moments starting with the conditional variance. We have:

$$E[\pi_t^2] = \sum_{i=1}^{m} P_i E[\pi_t^2 | S_t = i]$$

$$E[\pi_t^2 | S_t = i] \sum_{i=1}^{m} E[\pi_t^2 | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

where P_i denotes the i^{th} element of the steady-state vector of the Markov chain.

$$= \sum_{j=1}^{m} E[a(s_t)^2 r_t^2 + b(s_t)^2 \pi_{t-1}^2 + u_t^2 + 2b(s_t)a(s_t)r_t\pi_{t-1} | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} E[a_i^2 \sigma_r^2 + b_i^2 \pi_{t-1}^2 + \sigma_r^2 + 2b_1 a_i r_t \pi_{t-1} | S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

Note that this last expression implies m equations in m unknowns for the conditional variances, *given* the conditional covariances $E[\pi_t r_t | S_t = j]$. Using this, the unconditional variance is given by:

$$E[\pi_t^2] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (a_i^2 \sigma_r^2 + b_i^2 E[\pi_{t-1}^2 | S_t = j] + \sigma_r^2 + 2b_1 a_i r_t E[\pi_{t-1} | S_{t-1} = j]) p_{ji} \frac{P_j}{P_i}$$

Next we move onto the covariance term $E[\pi_t r_t]$:

$$E[\pi_t r_t] = \sum_{i=1}^{m} P_i E[\pi_t r_t | S_t = i]$$

$$E[\pi_t r_t | S_t = i] = \sum_{j=1}^{m} E[\pi_t r_t | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} E[b(s_t)\pi_{t-1} r_t + a(s_t) r_t^2 | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$\sum_{i=1}^{m} (b_i \rho E[\pi_t r_t | S_t = j] + a_i \sigma_r^2) p_{ji} \frac{P_j}{P_i}$$

Note that again, the last expression implies m equations in m unknowns for the conditional covariances. Using this, the unconditional covariance is given by:

$$E[\pi_t r_t] = \sum_{i=1} m P_i \sum_{j=1}^{m} (b_i \rho E[\pi_t r_t | S_t = j] + a_i \sigma_r^2) p_{ji} \frac{P_j}{P_i}$$

Next we compute the first-order autocovariance:

$$E[\pi_t \pi_{t-1}] = \sum_{i=1}^{m} P_i E[\pi_t \pi_{t-1} | S_t = i]$$

$$E[\pi_t \pi_{t-1} | S_t = i] = \sum_{j=1}^{m} E[b(s_t)\pi_{t-1}^2 + a(s_t)\pi_{t-1} r_t | S_t = 1, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} (b_i E[\pi_t^2 | S_t = j] + a_i \rho E[\pi_t r_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Given the conditional covariance and conditional variance terms, the above expression yields the conditional autocovariances. Hence the unconditional autocovariance is given as:

$$E[\pi_t \pi_{t-1}] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (b_i E[\pi_t^2 | S_t = j] + a_i \rho E[\pi_t r_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Finally note that:

$$E[a(s_t)\pi_{t-1} r_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} a_i \rho E[\pi_t r_t | S_t = j] p_{ji} \frac{P_j}{P_i}$$

and

$$E[b(s_t)\pi_{t-1}r_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} b_i\rho E[\pi_t r_t|S_t = j]p_{ji}\frac{P_j}{P_i}$$

Recalling the T-map $\begin{pmatrix} a \\ b \end{pmatrix} \to T(a,b) = \begin{pmatrix} E[(\pi_t - b\pi_{t-1})r_t] \\ \frac{E[(\pi_t - ar_t)\pi_{t-1}]}{E[\pi_t^2]} \end{pmatrix}$, the above conditions fully pin down $T(a,b)$. It is generally not possible to obtain analytical expressions for this mapping, and therefore the RPE values a^{RPE} amd b^{RPE}. Therefore our results in Section 1 are computed numerically for given values of parameters.

C N dimensional case with m regimes

Note that, after plugging in the PLM into ALM, the model considered in Section can be re-written as a generic Markov-switching model of the form:

$$\begin{cases} X_t = a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t \\ \epsilon_t = \rho\epsilon_{t-1} + \eta_t \end{cases}$$

where $a(s_t) = A(s_t) + C(s_t)(a + ba)$, $b(s_t) = B(s_t) + C(s_t)b^2$ and $d(s_t) = C(s_t)(bd + d\rho) + D(s_t)$. We need the first and second moments of this system in order to compute the the resulting T-map for the RPE. Starting with the first moment, we have:

$$E[X_t] = \sum_{i=1}^{m} P_i E[X_t|S_t = i]$$

$$E[X_t|S_t = i] = \sum_{j=1}^{m} [a_i + b_i X_{t-1} + d_i\epsilon_t|S_{t-1} = j]p_{ji}\frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} (a_i + b_i E[X_t|S_t = j])p_{ji}\frac{P_j}{P_i}$$

The expression above implies m equations in m unknowns for the conditions means. Using this yields:

$$E[X_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (a_i + b_i E[X_t|S_t = j])p_{ji}\frac{P_j}{P_i}$$

Moving onto the second moments and starting with the covariance term, we have:

$$E[X_t\epsilon_t'] = \sum_{i=1}^{m} P_i E[X_t\epsilon_t'|S_t = i]$$

$$E[X_t\epsilon_t'|S_t=i]E[)a(s_t)+b(s_t)X_{t-1}+d(s_t)\epsilon_t)\epsilon_t'|S_t=i]$$

$$=\sum_{j=1}^{m}E[(a_i+b_iX_{t-1}+d_i\epsilon_t)\epsilon_t'|S_{t-1}=j]p_{ji}\frac{P_j}{P_i}$$

$$=\sum_{j=1}^{m}(b_i\rho E[X_t\epsilon_t'|S_t=j]+d_i\Sigma_\epsilon)p_{ji}\frac{P_j}{P_i}$$

The last expression again implies m equations in m unknowns for the conditional covariances. The unconditional covariance is then given by:

$$E[X_t\epsilon_t']=\sum_{i=1}^{m}P_i\sum_{j=1}^{m}(b_i\rho E[X_t\epsilon_t'|S_t=j]+d_i\Sigma_\epsilon)p_{ji}\frac{P_j}{P_i}$$

Next we compute:

$$E[X_tX_t']=\sum_{i=1}^{m}P_iE[X_tX_t'|S_t=i]$$

$$E[X_tX_t'|S_t=i]=E[a(s_t)a(s_t)'+2a(s_t)X_{t-1}'b(s_t)'+2a(s_t)\epsilon_td(s_t)'+$$
$$b(s_t)X_{t-1}X_{t-1}'b(s_t)'+2b(s_t)X_{t-1}\epsilon_td(s_t)'+d(s_t)\epsilon_t\epsilon_t'd(s_t)'|S_t=i]$$

$$=\sum_{j=1}^{m}E[a_ia_i'2a_iX_{t-1}'b_i'+2a_i\epsilon_t'd_i'+b_iX_{t-1}X_{t-1}'b_i'+2b_iX_{t-1}\epsilon_t'd_i'+d_i\epsilon_t\epsilon_t'd_i'|S_t=j]p_{ji}\frac{P_j}{P_i}$$

Given the conditional means and covariances, the last expressions implies m equations in m unknowns for the conditional moments $E[X_tX_t'|S_t=i]$. The unconditional moment is then given by:

$$E[X_tX_t']=\sum_{i=1}^{m}P_i\sum_{j=1}^{m}(a_ia_i'+2a_iE[X_t'|S_t=j]b_i'+b_iE[X_tX_t'|S_t=j]b_i'+2b_iE[X_t\epsilon_t'|S_t=j]\rho'd_i'+d_i\Sigma_\epsilon d_i')p_{ji}\frac{P_j}{P_i}$$

Finally we compute the autocovariance term:

$$E[X_tX_{t-1}']=\sum_{i=1}^{m}P_iE[X_tX_{t-1}'|S_t=i]$$

$$E[X_tX_{t-1}'|S_t=i]=E[a_iX_{t-1}'+b_iX_{t-1}X_{t-1}'+d_i\rho\epsilon_{t-1}X_{t-1}'|S_t=i]=$$

$$\sum_{j=1}^{m}(a_i E[X_t|S_t=j]+b_i E[X_t X_t'|S_t=j]+d_i\rho E[\epsilon_t X_t'|S_t=j])p_{ji}\frac{P_j}{P_i}$$

The last expression is pinned by the conditional first and second moments computed above. The unconditional autocovariance is then given as:

$$E[X_t X_t']=\sum_{i=1}^{m}\sum_{j=1}^{m}(a_i E[X_t|S_t=j]+b_i E[X_t X_t'|S_t=j]+d_i\rho E[\epsilon_t X_t'|S_t=j])p_{ji}\frac{P_j}{P_i}$$

Recall that the T-map is given by:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} E[X_t - bX_{t-1} - d\epsilon_t] \\ E[(X_t - a - d\epsilon_t)X_{t-1}')E[X_t X_t']^{-1}] \\ E[(X_t - a - bX_{t-1})\epsilon_t']E[\epsilon_t\epsilon_t']^{-1} \end{pmatrix}$$

Hence, given the first and second moments computed above, the T-maps for a, b and c are pinned down. Similar to 1-dimensional case, it is generally not possible to find analytical expressions for these matrices. Further note that, the T-map for $b \rightarrow T(a,b,c)$ involves a 4^{th} order matrix polynomial of dimension N. This means there can be up to $\binom{4N}{N}$ for b. To our knowledge, there is no straightforward and general method to compute the full set of solutions to this problem. In this paper, we do not compute these fixed-points and rely on Monte Carlo simulations when necessary.

Further note that the regime-specific T-maps are given by:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} A_i + C_i(a+ba) \\ B_i + C_i b^2 \\ C_i(bd + d\rho) + D_i \end{pmatrix}$$

These simply correspond to the standard MSV solutions given the regime-specific matrices. Computing the fixed-points yield the regime-specific equilibria as follows:

$$\begin{cases} a^{R_i} = (I - C_i - C_i b^{R_i})^{-1} A_i \\ vec(D^{R_i}) = (I - (I \otimes (C_i b^{R_i})))vec(d) + (\rho \otimes C_i)vec(d) + vec(D_i) \end{cases}$$

which yields the regime-specific values for a^{R_i} and d^{R_i} respectively, for a given matrix b^{R_i}. The second-order polynomial for b^{R_i} can be solved using standard toolboxes such as Adjemian et al. (2011) and Uhlig et al. (1995), which then completely pins down the regime-specific MSV. Denoting $\theta = (a,b,d)'$, the associated Jacobian for E-stability condition is given by:

$$\frac{DT}{D\theta} = \begin{bmatrix} C_i + C_i b & vec^{-1}_{n,n}(a' \otimes C_i) & 0 \\ 0 & 2C_i b & 0 \\ 0 & vec^{-1}_{n,n}(d' \otimes C_i) & C_i b + vec^{-1}_{n,n}(\rho' \otimes C_i) \end{bmatrix}$$

where $vec^{-1}_{n,n}$ denotes the matricization of a vector to an (n, n) matrix.

References

Adjemian, S., Bastani, H., Juillard, M., Mihoubi, F., Perendia, G., Ratto, M., & Villemot, S. (2011). Dynare: Reference manual, version 4.

An, S. & Schorfheide, F. (2007). Bayesian analysis of dsge models. *Econometric reviews*, 26(2-4), 113–172.

Berardi, M. & Galimberti, J. K. (2017a). Empirical calibration of adaptive learning. *Journal of Economic Behavior & Organization*, 144, 219–237.

Berardi, M. & Galimberti, J. K. (2017b). On the initialization of adaptive learning in macroeconomic models. *Journal of Economic Dynamics and Control*, 78, 26–53.

Berardi, M. & Galimberti, J. K. (2017c). Smoothing-based initialization for learning-to-forecast algorithms. *Macroeconomic Dynamics*, (pp. 1–16).

Bianchi, F. (2016). Methods for measuring expectations and uncertainty in markov-switching models. *Journal of Econometrics*, 190(1), 79–99.

Bianchi, F. & Ilut, C. (2017). Monetary/fiscal policy mix and agents' beliefs. *Review of economic Dynamics*, 26, 113–139.

Binning, A. & Maih, J. (2016). Implementing the zero lower bound in an estimated regime-switching dsge model.

Bonam, D., de Haan, J., Soederhuizen, B., et al. (2017). *The effects of fiscal policy at the effective lower bound*. Technical report, Netherlands Central Bank, Research Department.

Branch, W. A., Davig, T., & McGough, B. (2013). Adaptive learning in regime-switching models. *Macroeconomic Dynamics*, 17(5), 998–1022.

Branch, W. A. & Evans, G. W. (2006). A simple recursive forecasting model. *Economics Letters*, 91(2), 158–166.

Bullard, J. & Eusepi, S. (2014). When does determinacy imply expectational stability? *International Economic Review*, 55(1), 1–22.

Bullard, J. & Mitra, K. (2002). Learning about monetary policy rules. *Journal of monetary economics*, 49(6), 1105–1129.

Davig, T. & Leeper, E. M. (2007). Generalizing the taylor principle. *American Economic Review*, 97(3), 607–635.

Del Negro, M., Giannoni, M., & Patterson, C. (2012). The forward guidance puzzle. *Center for Economic Policy Research (CEPR).*

Eusepi, S. & Preston, B. (2011). Expectations, learning, and business cycle fluctuations. *American Economic Review*, 101(6), 2844–72.

Evans, G. W., Guse, E., & Honkapohja, S. (2008). Liquidity traps, learning and stagnation. *European Economic Review*, 52(8), 1438–1463.

Evans, G. W. & Honkapohja, S. (2010). 11 expectations, deflation traps and macroeconomic policy. *Twenty Years of Inflation Targeting: Lessons Learned and Future Prospects*, (pp. 232).

Evans, G. W. & Honkapohja, S. (2012). *Learning and expectations in macroeconomics*. Princeton University Press.

Farmer, R. E., Waggoner, D. F., & Zha, T. (2009). Understanding markov-switching rational expectations models. *Journal of Economic theory*, 144(5), 1849–1867.

Farmer, R. E., Waggoner, D. F., & Zha, T. (2011). Minimal state variable solutions to markov-switching rational expectations models. *Journal of Economic Dynamics and Control*, 35(12), 2150–2166.

Gaus, E. & Gibbs, G. C. (2018). *Expectations and the empirical fit of DSGE models*. Technical report, Mimeo.

Gust, C., Herbst, E., & Lopez-Salido, J. D. (2018). Forward guidance with bayesian learning and estimation.

Hommes, C. & Zhu, M. (2014). Behavioral learning equilibria. *Journal of Economic Theory*, 150, 778–814.

Ji, Y. & Xiao, W. (2016). Government spending multipliers and the zero lower bound. *Journal of Macroeconomics*, 48, 87–100.

Lansing, K. J. (2018). Endogenous regime switching near the zero lower bound.

Lindé, J., Maih, J., & Wouters, R. (2017). *Estimation of Operational Macromodels at the Zero Lower Bound*. Technical report, Mimeo.

Liu, P. & Mumtaz, H. (2011). Evolving macroeconomic dynamics in a small open economy: An estimated markov switching dsge model for the uk. *Journal of Money, Credit and Banking*, 43(7), 1443–1474.

Liu, Z., Waggoner, D. F., & Zha, T. (2011). Sources of macroeconomic fluctuations: A regime-switching dsge approach. *Quantitative Economics*, 2(2), 251–301.

Maih, J. (2015). Efficient perturbation methods for solving regime-switching dsge models.

Milani, F. (2007). Expectations, learning and macroeconomic persistence. *Journal of monetary Economics*, 54(7), 2065–2082.

Quaghebeur, E. (2018). Learning and the size of the government spending multiplier. *Macroeconomic Dynamics*, (pp. 1–36).

Sims, C. A., Waggoner, D. F., & Zha, T. (2008). Methods for inference in large multiple-equation markov-switching models. *Journal of Econometrics*, 146(2), 255–274.

Sims, C. A. & Zha, T. (2006). Were there regime switches in us monetary policy? *American Economic Review*, 96(1), 54–81.

Slobodyan, S. & Wouters, R. (2012a). Learning in a medium-scale dsge model with expectations based on small forecasting models. *American Economic Journal: Macroeconomics*, 4(2), 65–101.

Slobodyan, S. & Wouters, R. (2012b). Learning in an estimated medium-scale dsge model. *Journal of Economic Dynamics and control*, 36(1), 26–46.

Smets, F. & Wouters, R. (2003). An estimated dynamic stochastic general equilibrium model of the euro area. *Journal of the European economic association*, 1(5), 1123–1175.

Smets, F. & Wouters, R. (2007). Shocks and frictions in us business cycles: A bayesian dsge approach. *American economic review*, 97(3), 586–606.

Uhlig, H. F. et al. (1995). A toolkit for analyzing nonlinear dynamic stochastic models easily.

Woodford, M. (2013). Macroeconomic analysis without the rational expectations hypothesis. *Annu. Rev. Econ.*, 5(1), 303–346.