Restricted Perceptions and the Zero Lower Bound Episode *

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Abstract

We consider the estimation of Markov-switching DSGE models under adaptive learning (AL) in order to study the interaction between expectations and the business cycle over the Zero Lower Bound period. We assume that regime shifts by monetary policy are not directly observed by agents, instead they indirectly infer about the regimes to the extent that it feeds back into their information set. This setup results in so-called Restricted Perceptions Equilibria (RPE) consistent with a given information set, and standard E-stability conditions are applicable to these equilibria. We illustrate these conditions in two environments: a basic Fisherian setup, and the benchmark 3-equation New Keynesian model. We then use a variant of the Kim & Nelson (1999) filter to estimate MS-DSGE models under constant gain adaptive learning. Based on estimations of the 3-equation NKPC and workhorse Smets-Wouters models, our results can be summarized as follows: adaptive learning models outperform the REE benchmark in all cases and the Regime-switching REE model in most cases, suggesting that Markov-switching and Adaptive Learning approaches can be complementary. Furthermore, we observe that the impulse responses and shock propagation differ under AL and REE setups. Particularly, a financial shock and a government spending shock of the same size typically has a longer-lasting impact under adaptive learning, suggesting that the Rational Expectations models may severely underestimate both the impact of 2007-08 financial crisis, as well as the impact of a fiscal stimulus during the zero lower bound episode that followed the crisis.

JEL Classification: E37; E65; C11; C32 .

Keywords: Adaptive Learning; Markov-Switching; Bayesian Estimation of DSGE Models; Zero Lower Bound.

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1 Introduction

With the onset of the Global Financial Crisis in 2007-08 and the subsequent drop of interest rates to near-zero levels among the leading central banks, there has been increased interest among policymakers and central bankers about the ZLB constraint on nominal interest rates. There is still ongoing debate about the precise impact of the zero lower bound constraint on the economy as a whole and in particular about its macroeconomic cost in terms of aggregate GDP levels. Monetary and fiscal policy recommendations of standard macroeconomic models are mixed: for instance there is no consensus on the propagation of a government spending shock and the size of a government spending multiplier during this period. As a consequence, while some researchers recommend a fiscal austerity program to jump-start the economy during a ZLB episode, others think it is best to adapt a fiscal consolidation strategy. A common approach in most macroeconomic models examining the ZLB episode is the assumption of Rational Expectations Equilibria (REE): agents are assumed to have perfect information about the underlying regime along with all other cross-correlations of the relevant macroeconomic variables and form their expectations accordingly. In this paper, we contribute to the growing literature on the ZLB episode by relaxing the perfect information assumption, and instead estimating DSGE models under adaptive learning subject to the ZLB constraint.

In standard REE-DSGE models, the perfect foresight assumption about regime switches typically leads to short periods of anticipated ZLB episodes: the expected duration of this period is typically between three to five quarters in most DSGE models estimated on the U.S. economy, see e.g. Lindé et al. (2017) and Ji & Xiao (2016), while the ZLB episode between 2009 and 2016 lasted for around twenty eight quarters. Another shortcoming of the standard REE models is the overestimation of the impact of forward guidance on the macroeconomy, a phenomenon known as the "forward guidance puzzle" Del Negro et al. (2012). These shortcomings call for a relaxation of REE-restrictions and introduce expectational frictions into the models. A plausible and popular method to introduce such frictions is adaptive learning, which relaxes the assumption that agents have perfect knowledge about the underlying economic relations and the corresponding cross-correlations. Instead, they have their own sub-models, possibly under parameterized, that may not coincide with the correct economic structure. Agents act as econometricians and update their models each period as new observations become available. There is a vast and growing literature on the empirical validation of adaptive learning in DSGE models as well as monetary and fiscal policy implications of adaptive learning, see Evans & Honkapohja (2012) for a textbook treatment and Woodford (2013) for a comprehensive review of the more recent work. Much of the earlier literature on adaptive learning focused on the learnability of Rational Expectations Equilibria and MSV-learning, focusing on small and temporary deviations from perfect foresight models. Milani (2007) and Eusepi & Preston (2011) are earlier examples of expectations-driven business cycles and how MSV-learning can improve the empirical properties of small-scale DSGE models, while Bullard & Mitra (2002) and Bullard & Eusepi (2014) examine monetary policy implications of this type of learning. In more recent work, Slobodyan & Wouters (2012a) and Slobodyan & Wouters (2012b) show that further deviations from perfect foresight models with the use of small forecasting rules can lead to further improvements in the fit of a medium-scale DSGE model. On a similar vein, Quaghebeur (2018) examines fiscal policy implications of a VAR-type adaptive learning rule and finds that government spending multipliers are larger under adaptive learning. Evans et al. (2008) and Evans & Honkapohja (2010) examine the implications of adaptive learning for fiscal policy. Branch et al. (2013) analyzes the theoretical properties of MSV-learning in Markov-switching models where agents are informed about the regime switches; and Lansing (2018) that analyzes the ZLB episode in a calibrated setup under adaptive learning where regime switches are unobserved. The closest study to our framework is Gust et al. (2018), which examines the ZLB episode in a Markov-switching setup under Bayesian learning. Our key difference from these papers and one of our main contributions is to extend their framework to non-MSV and non-rational beliefs, and to estimate the resulting DSGE models during the ZLB episode. We then examine the consequences of deviating from the REE during this period, particularly how it might contribute to a prolonging of the crisis and how it might change implications of standard

DSGE models about the potential impact of a government spending shock during this episode.

There are various different approaches to modeling the ZLB constraint: Some researchers use a perfect foresight & endogenous duration approach, which allows for a joint determination of expectations and regime switches; see e.g. Maih (2015) or Lindé et al. (2017). Another approach which is more common in VAR-literature is to use a threshold-switching method, where the economy is assumed to be in the ZLB regime if interest rates fall below some pre-specified level, see e.g. Bonam et al. (2017). A final approach is to use a Markov-switching framework, where the presence of the ZLB regime is determined by its predictive density, see e.g. Binning & Maih (2016). Lindé et al. (2017) show that Markov-switching and endogenous duration approaches typically lead to similar results as long as the ZLB constraint is accounted for. In this paper, we use the Markov-switching (MS) approach to take into account the constraint. Aside from the ZLB episode, MS approach recently gained popularity in DSGE literature to model structural changes such as monetary policy switches or volatility breaks, see e.g. Sims & Zha (2006), Davig & Leeper (2007), Sims et al. (2008), Liu et al. (2011), Liu & Mumtaz (2011), Bianchi (2016) and Bianchi & Ilut (2017) for some of the recent work. While Markov-switching and adaptive learning have both been increasingly popular classes of time-varying DSGE models in recent years, there is surprisingly little work on DSGE models that combine both approaches. Therefore, our paper is also the first one to explicitly unify Markov-switching and adaptive learning methods in an estimation context.

Our key assumption is that the underlying regime changes are unobserved to economic agents: agents only indirectly become aware of regime changes to the extent that these switches have an observable and strong enough impact on their information set. To set the ideas, consider the following example: A central bank follows a simple Taylor rule that reacts to inflation in setting interest rates. This will only be known to economic agents to the extent that the central bank discloses its goal of inflation targeting, but the agents never know the exact reaction coefficient. Accordingly, the agents will not find out if the central bank suddenly and discreetly decides to change its reaction coefficient. Instead, the agents will slowly find out about this regime shift as long as it leads to observable consequences in the interest rate and the resulting inflation levels.

Farmer et al. (2009) and Farmer et al. (2011) explore the class of REE in Markov-switching models. Since we assume regimes are never observed, an equilibrium concept in our framework can never coincide with a Rational Expectations Equilibrium. Instead, in this limited information environment, there are so-called Restricted Perceptions Equilibria (RPE) where the agents' misspecification of the economy becomes self-fulfilling and the system settles on a non-rational equilibrium. We compute these equilibria for two classes of Perceived Law of Motion (PLM): In the first case we assume that agents' PLM has the form of a Minimum State Variable (MSV) solution, except that the PLM does not take into account the possibility of regime-switches. In the second case, we take this idea further and consider PLMs based on small forecasting rules of VAR-type, where the information set of the agents is allowed to be smaller than the MSV-solution due to, for example, unobserved shocks or unaccounted cross-correlations. We show that standard E-stability conditions apply to these equilibria, and therefore the systems will converge to the underlying equilibria under standard recursive algorithms such as constant-gain least-squares. Furthermore, the E-stability and convergence results continue to hold even if one of the underlying regimes is E-unstable as long as the remaining regimes are sufficiently E-stable. This is a simple extension of the long-run determinacy result of Davig & Leeper (2007), which they call the long-run Taylor principle. We therefore denote our result as the long-run E-stability principle.

Next we build a variant of the Kim & Nelson (1999) filter to estimate our class of MS-DSGE models under adaptive learning, and we apply the filter to the Bayesian estimation of two standard DSGE models: The first one is the 3-equation NKPC model along the lines of Woodford (2013), which provides a good starting point to expose our main results. The second one is the more complex and empirically relevant Smets & Wouters (2007) model, which is popular among central bankers and policy makers as a benchmark for policy analysis. Our estimation results can be summarized as follows: The MS-AL models outperform the standard REE benchmark in all cases, and the also the regime-switching REE models in a majority of cases. Furthermore, we observe important differences in the impulse response and shock propagation structure of the models under consideration. For instance, a financial shock of the same size typically has a longer-lasting impact under adaptive learning, while a government spending shock may have a larger impact. These results suggest that benchmark REE models may severely underestimate the crisis duration, as well as underestimate the impact of a fiscal stimulus package.

The paper is organized as follows: Section 2 illustrates the main concepts in a simple framework with one-forward looking variable. Section 3 shows the computation and E-stability results of the two classes of Restricted Perceptions Equilibria in DSGE models. Section 4 provides the filter used in our estimations, while sections 5 and 6 discuss the estimations results in the 3-equation NKPC and SW models respectively. Finally Section 7 concludes.

2 Preliminaries: Fisherian Model of Inflation Determination

Consider a simple model of Fisherian inflation determination without regime switching:

$$\begin{cases} i_t = E_t\pi_{t+1} + \iota_p\pi_{t-1} + r_t \\ r_t = \rho r_{t-1} + v_t \\ i_t = \alpha\pi_t - u_t \end{cases}$$

where r_t is the exogenous AR(1) ex-ante real interest rate, i_t is the nominal interest rate, π_t is inflation, and $\{u_t, v_t\}$ are IID shock processes. We assume that monetary policy follows a simple rule by adjusting nominal interest rate to inflation, denoted by α[1]. We can re-write the model in terms of inflation as follows:

$$\begin{cases} \pi_t = \frac{1}{\alpha}(E_t\pi_{t+1} + \iota_p\pi_{t-1} + r_t) + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

To set the ideas, we first focus on the special case where $\iota_p = 0$, which has been analyzed in Davig & Leeper (2007). With the standard MSV solution, the agents' perceived law of motion (PLM) takes the form of $\pi_t = ar_t$. The REE solution for a is then pinned down by iterating the PLM forward to obtain the one-step ahead expectations, plugging the expectations back into the implied actual law of motion (ALM) and computing the associated fixed point where the agents' PLM exactly coincides with the implied ALM. This yields $a = \frac{1}{\alpha-\rho}$. Hence the law of motion evolves according to $\pi_t = \frac{1}{\alpha-\rho}r_t$ at the REE. In this benchmark case, the equilibrium is determinate if $\alpha > 1$, i.e. monetary policy is sufficiently aggressive.
Davig & Leeper (2007) consider scenarios where the interest rate coefficient α is subject to regime switches. Focusing on a two regime environment, assume that α changes stochastically between two regimes, $s_t = \{1, 2\}$ subject to the transition matrix:

$$Q = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{pmatrix}$$

Then inflation dynamics are given as:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}(E_t\pi_{t+1} + r_t) + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

Denoting $\pi_{i,t} = \pi_t(s_t = i)$, we can re-cast the model into a multivariate form:

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \pi_{1,t} \\ \pi_{2,t} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} E_t\pi_{1,t+1} \\ E_t\pi_{2,t+1} \end{bmatrix} + \begin{bmatrix} r_t \\ r_t \end{bmatrix} + \begin{bmatrix} u_t \\ u_t \end{bmatrix}$$

Since the ALM is regime-dependent, the corresponding PLM is also regime-dependent if agents are rational. Denoting by a_i the regime-specific coefficients, the PLM is given by:

$$\pi_{i,t} = a_i r_t$$

[1]For the remainder, we assume that $Var(r_t) = 1$ to simplify the exposure.

In other words, agents hold a seperate PLM for each regime in the economy. Then the corresponding regime-dependent 1-step ahead expectations are given by:

$$\begin{cases} E_t[\pi_{t+1}|s_t = 1] = (p_{11}a_1 + p_{12}a_2)\rho r_t \\ E_t[\pi_{t+1}|s_t = 2] = (p_{21}a_1 + p_{22}a_2)\rho r_t \end{cases}$$

Davig & Leeper (2007) show that, in this case, the equilibrium is determinate as long as the so-called *long-run Taylor principle (LRTP)* is satisfied:

$$\alpha_1\alpha_2 > 1 - ((1-\alpha_2)p_{11} + (1-\alpha_1)p_{22})$$

A key insight of this principle is that, the long-run dynamics of the model can be determinate even if one of the underlying regimes is indeterminate, provided there is at least one regime that is sufficiently determinate or the probability of entering into the indeterminate regime is sufficiently small. In what follows, we first extend the long-run determinacy insight into the concept of learnability, i.e. *E-stability* of equilibria. Our key assumption is that agents do not directly observe or take into account the regime shifts that occur in the economy when they form their expectations. Accordingly, their PLMs and the implied 1-step ahead expectations are given as follows, which are not regime dependent:

$$\pi_t = ar_t \Rightarrow E_t\pi_{t+1} = aE_t r_{t+1} = a\rho r_t$$

The implied ALM is then given by:

$$\begin{cases} \pi_t = \frac{1}{\alpha(s_t)}(a\rho + 1)r_t + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

The assumed form of PLM here does not nest the regime-dependent MSV solution. Therefore any resulting notion of equilibrium under this scenario cannot coincide with the full-information Rational Expectations Equilibrium. However, one can still consider a non-rational & limited-information equilibrium associated with the above PLM. This type of equilibrium is commonly referred to as a Restricted Perceptions equilibrium in the adaptive learning literature (Evans & Honkapohja, 2012): the agents use a restricted (and misspecified) information set, which becomes self-fulfilling at the underlying equilibrium.

Unlike a REE, one cannot use the method of undetermined coefficients as above to pin down the value of a associated with the RPE. Instead, following Hommes & Zhu (2014), we impose a moment consistency requirement on the model to pin down the value of a: the coefficient a determines the *perceived correlation* between inflation and real rate of interest in the PLM, i.e. $a = \frac{E[\pi_t r_t]}{E[r_t r_t]}$. In an equilibrium, the unconditional correlation $\frac{E[\pi_t r_t]}{E[r_t r_t]}$, implied by the ALM should be equal to a. In other words, the information used by the agent should be consistent with the information that arises in the actual outcomes. In an equilibrium as such, agents make systematic mistakes to the extent that they do not use the full-information forecasting rule. Instead, they use the best forecasting rule within the class of their information set. Computing the associated moment in our example, the ALM yields:

$$\frac{E[\pi_t r_t]}{E[r_t r_t]} = E[\frac{1}{\alpha(s_t)}\beta\rho + \frac{1}{\alpha(s_t)}]$$

This expression above involves the ergodic distribution of the Markov chain, which we denote by P. This is given by $P = [\frac{1-p_{22}}{2-p_{11}-p_{22}}, \frac{1-p_{11}}{2-p_{11}-p_{22}}]$[2]. Using this, the equilibrium coefficient, which we denote by a^{RPE} is given by[3]:

$$a^{RPE} = \frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}-p_{22}) - \rho\alpha_1(1-p_{22}) - \rho\alpha_2(1-p_{11})}$$

[2] Note that ergodic distribution is obtained by solving $P'Q = P$.

[3] See Appendix A for details.

Further note that, the regime-specific MSV solutions are given by $a^{R_i} = \frac{1}{\alpha_i - \rho}$, $i \in \{1, 2\}$. Given these expressions, the underlying RPE boils down to a weighted average of the regime-specific equilibria. Instead of the standard determinacy of Rational Expectations models, our main concept of interest in this case is E-stability: Bullard & Eusepi (2014) shows that there is tight link between determinacy and E-stability of REE and in some special cases these conditions may even coincide. E-stability governs whether the agents can learn the above fixed-point by starting from an arbitrary point a_0, and updating their beliefs about the coefficient each period as new observations become available. As shown in Evans & Honkapohja (2012), E-stability is governed by the mapping from agents' PLM to the implied ALM, defined as the T-map. In our example, the T-map is given by:

$$T : a \to T(a) = \frac{E[\pi_t r_t]}{E[r_t r_t]} = (a\rho + 1)\frac{\alpha_1(1 - p_{22}) + \alpha_2(1 - p_{11})}{\alpha_1\alpha_2(2 - p_{11}p_{22})}$$

The T-map is locally stable if the Jacobian matrix has eigenvalues with real parts less than one. When this local stability condition is satisfied, the equilibrium is said to be E-stable. In our example, the eigenvalue and the associated E-stability condition are given by:

$$\frac{DT(a)}{D(a)} = \frac{\alpha_1(1 - p_{22}) + \alpha_2(1 - p_{11})}{\alpha_1\alpha_2(2 - p_{11} - p_{22})} < 1$$

.

Re-arranging the above expression yields:

$$\alpha_1\alpha_2 > \frac{\alpha_1(1 - p_{22}) + \alpha_2(1 - p_{11})}{2 - p_{11} - p_{22}}$$

With the above expression, the E-stability criterion reduces to a condition similar to *LRTP*. In order to satisfy E-stability, we need a more aggressive monetary policy rule α_1 whenever: (i) the average time spent in regime 1 (P_1) decreases, (ii) the average time spent in regime 2 (P_2) increases, or (iii) the monetary policy rule in regime 2 (α_2) becomes less aggressive. This suggests that it is possible to have E-stability despite having an E-unstable regime, as long as there is a sufficiently E-stable regime and the model does not spend too much time in the unstable regime on average. This is an intuitive extension of Davig & Leeper's insight on long-run determinacy to the learnability of equilibria, therefore we denote this as *the principle of long-run E-stability.*

For the remainder of our analysis, we focus on cases with $\iota_p \neq 0$ where the PLM takes the following form:

$$\pi_t = ar_t + b\pi_{t-1}$$

Using the moment consistency requirements for both a and b, the T-map in this case is given by:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} E[(\pi_t - b\pi_{t-1})r_t] \\ \frac{E[(\pi_t - ar_t)\pi_{t-1}]}{E[\pi_t^2]} \end{pmatrix}$$

With the addition of lagged inflation, the moments appearing in the above expression already become analytically intractable, therefore the values a^{RPE} and b^{RPE} are obtained numerically in the examples below[4]. Agents' forecasts along such an RPE imply systematic forecast errors: since they do not know the underlying regimes and only use a univariate rule, they essentially have a weighted average based on the ergodic distribution at the RPE, where they minimize the unconditional forecast errors. In general, we assume that agents do not simply remain at the RPE-consistent values but instead update their beliefs each period as new observations become available, using a constant-gain least squares method à la Evans & Honkapohja (2012). Accordingly in our example, using the notation $\theta = [a, b]'$ and $y_t = [r_t, \pi_{t-1}]'$, the vector of

[4]See Appendix B for details on the implied moments.

coefficients are updated as follows:

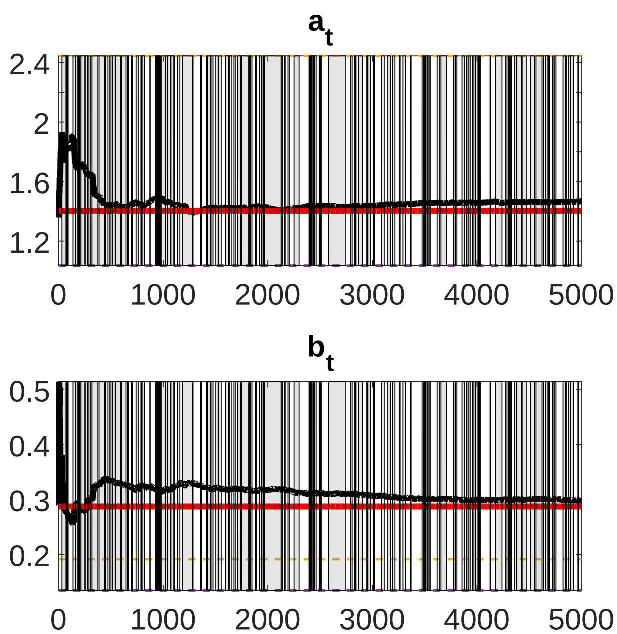
$$\begin{cases} R_t = R_{t-1} + \gamma(y_t^2 - R_{t-1}) \\ \theta_t = \theta_{t-1} + \gamma R_t^{-1} y_t(\pi_t - \theta_{t-1} y_t) \end{cases}$$

where γ denotes the gain value, i.e. the weight that agents put into the most recent observation. The learning algorithm allows agents to put more weight into recent observations, thereby giving them more flexibility. We illustrate the model dynamics for a number of scenarios. Figure 1 shows four examples where both regime-specific MSV solutions, as well as the underlying RPE are E-stable. Panel (a) shows a decreasing gain learning case (DGL) with $\gamma = \frac{1}{t}$, where the learning coefficients converge towards their RPE-consistent values. In this case the beliefs are asymptotically locked in at the RPE as each new observation receives less weight. Panel (b) shows a constant gain learning (CGL) case with $\gamma = 0.01$, in which case the learning coefficients oscillate around their RPE-consistent values.

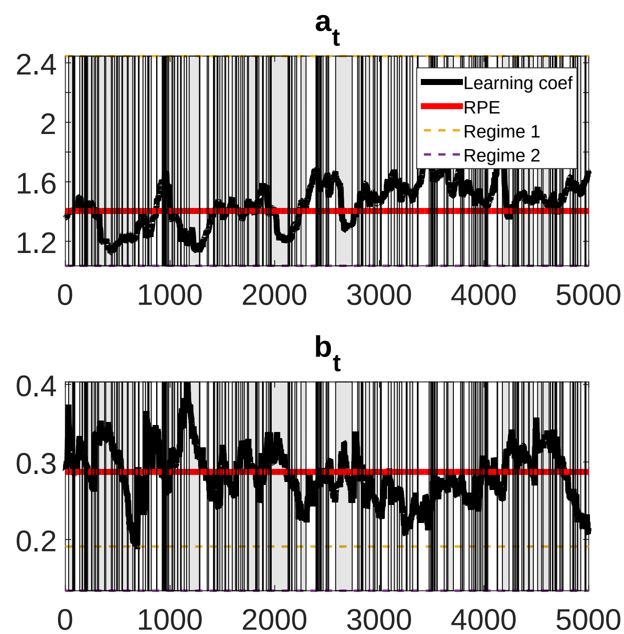
An interesting feature of our setup is that agents may forget past regime shifts under the right conditions: as the gain coefficient becomes larger relative to the underlying regime persistence parameters p_{11} and p_{22}, agents become more myopic and put less weight into past observations. In these scenarios, as past regime shifts are forgotten, the learning coeffecents become more likely to escape from the RPE towards regime-specific equilibria. Panels (c) and (d) of Figure 1 illustrate two such cases, where we increase the persistence of the underlying regimes. In these cases, once a regime shift occurs, the learning coefficients first jump towards their RPE-consistent values. As the system spends enough time in the new regime, the regime shift is forgotten and the learning coefficients jump towards their regime specific values. This occurrence is particularly important from an empirical point of view as it characterizes how the system behaves when exiting a very persistent regime, or entering into a new regime that has not been observed before, such as the recent ZLB episode.

Figure 2 illustrates two examples where one equilibrium is stable, while the second equilibrium is borderline E-unstable. Given the parameter values, the underlying RPE is E-stable. Panel (a) shows convergence towards the underlying RPE under DGL with $\gamma = \frac{1}{t}$, while Panel (b) shows a CGL case with $\gamma = 0.01$, and the learning coefficients oscillate around the RPE-consistent values. In this case with one E-unstable regime, we observe the same phenomenon of jumps as the system switches from the persistent E-stable regime, to the less persistent E-unstable regime. This feature is also relevant for the ZLB episode that will be studied in the upcoming sections, since this regime has been found to be E-unstable under a wide variety of models and specifications. In the next section, we extend our analysis to a general multivariate setup in order to study New Keynesian models.

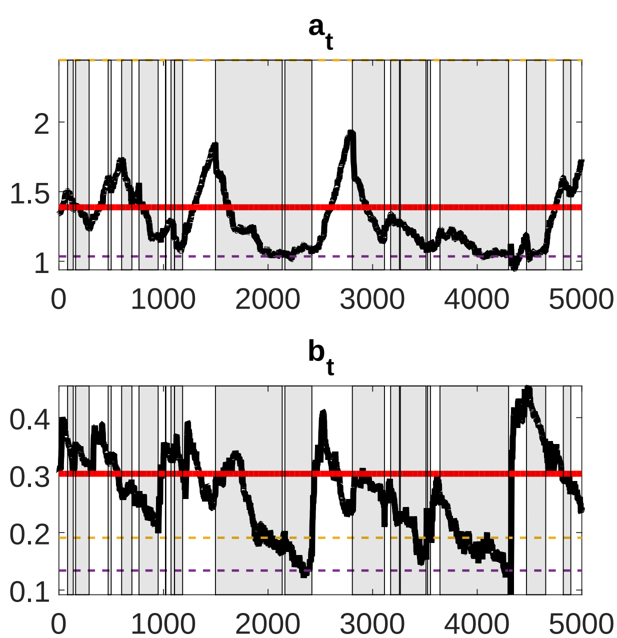
Figure 1: Learning coefficients along with the RPE-consistent and regime-specific values. The parameters $\iota_p = 0.25, \rho = 0.9, \alpha_1 = 1.5, \alpha_2 = 2, \sigma_u^2 = 0.1$ are fixed for all simulations, while p_{11}, p_{22} and γ are varied. Given the values of α_1 and α_2, both regime-specific equilibria, as well as the RPE are E-stable.



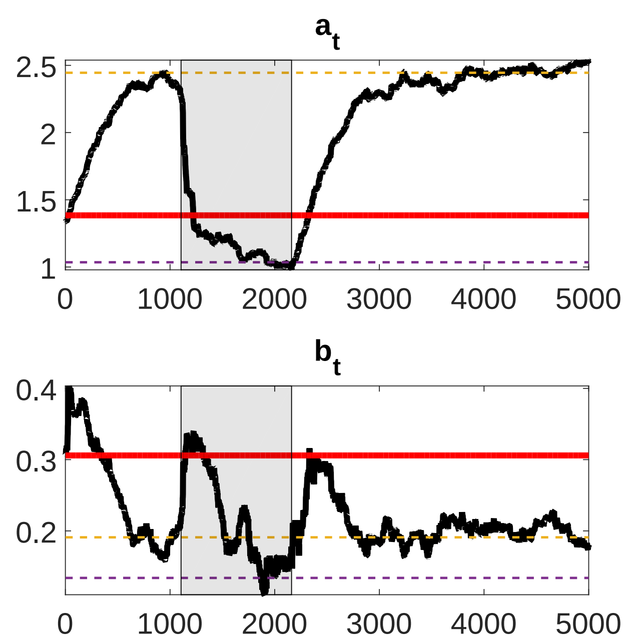
(a) $\gamma = \frac{1}{t}, p_{11} = p_{22} = 0.95$.



(b) $\gamma = 0.01, p_{11} = p_{22} = 0.95$.

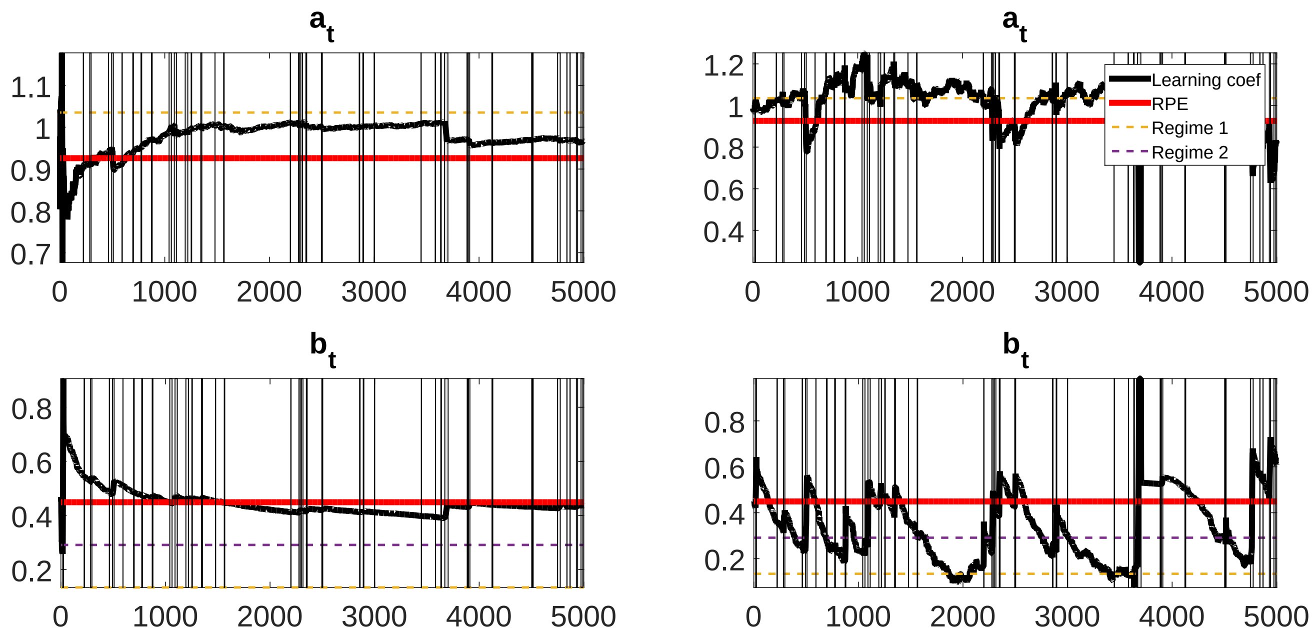


(c) $\gamma = 0.01, p_{11} = p_{22} = 0.99$.



(d) $\gamma = 0.01, p_{11} = p_{22} = 0.999$.

Figure 2: Learning coefficients along with the RPE-consistent and regime-specific values. The parameters $\iota_p = 0.25, \rho = 0.9, \alpha_1 = 2, \alpha_2 = 1.15, \sigma_u^2 = 0.1$ are fixed for all simulations, while p_{11}, p_{22} and γ are varied. The first regime is E-stable, while the second regime is E-unstable. At the given values for p_{11} and p_{22} below, the RPE is E-stable.



(a) $\gamma = \frac{1}{t}, p_{11} = 0.99, p_{22} = 0.9$.

(b) $\gamma = 0.01, p_{11} = 0.99, p_{22} = 0.9$..

3 Restricted Perceptions Equilibria in MS-DSGE Models: General Case

Our simple example in the previous section illustrates our idea of restricted perceptions in Markov-switching DSGE models, where we can analytically compute the Restricted Perceptions Equilibrium. In this section, we generalize our results to cases with multiple forward-looking variables, where the underlying equilibrium quickly becomes intractable. Accordingly, first consider the data generating process:

$$\begin{cases} X_t = A(s_t) + B(s_t)X_{t-1} + C(s_t)E_tX_{t+1} + D(s_t)\epsilon_t \\ \epsilon_t = \rho\epsilon_{t-1} + \eta_t \end{cases}$$

where X_t denotes the state-variables that depend on their lags, 1-step ahead expectations and the structural shocks ϵ_t, which itself follow a VAR(1) process. We assume that the corresponding matrices A, B, C and D contain the structural parameters of the model, some of which are subject to regime switches captured by s_t. Using this, we extend our analysis to the more general multivariate case. We further allow the vector of intercept coefficients $A(s_t)$ to be non-zero in this case, which is particularly important for analyzing the steady-state stability of the ZLB episode. Consider again a PLM of the general form:

$$\begin{cases} X_t = a + bX_{t-1} + d\epsilon_t \\ E_tX_{t+1} = (a + ba) + b^2X_{t-1} + (bd + d\rho)\epsilon_t \end{cases}$$

where we assume that structural shocks are contemporaneously observed while the state variables are not, which is a common assumption in the adaptive learning literature. Further note that the above specification nests many benchmark PLMs as a special case: when a, b and d are all non-zero, the PLM takes the form of a regime-specific solution and the only source of misspecification is the unobserved regimes. With $d = 0$, the PLM admits a variety of VAR(1)-type learning rules, which assumes unobserved shocks and allows some or all cross-correlations to be misspecified. A diagonal matrix b further reduces the PLM to univariate autoregressive rules. These types of AR/VAR rules have been successfully applied in recent past to improve the empirical fit of otherwise standar DSGE models, see e.g. Slobodyan & Wouters (2012b) and Gaus & Gibbs (2018). Plugging the expectations back into the first expression yields the implied ALM:

$$X_t = (A(s_t) + C(s_t)(a + ba)) + (C(s_t)b^2 + B(s_t))X_{t-1} + (C(s_t)(bd + d\rho) + D(s_t))\epsilon_t$$

which can be re-written as

$$X_t = a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t$$

where $a(s_t) = A(s_t)+C(s_t)(a+ba)$, $b(s_t) = B(s_t)+C(s_t)b^2$ and $d(s_t) = C(s_t)(bd+d\rho)+D(s_t)$. In this case the T-map is given as:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} E[X_t - bX_{t-1} - d\epsilon_t] \\ E[(X_t - a - d\epsilon_t)X'_{t-1})E[X_tX'_t]^{-1}] \\ E[(X_t - a - bX_{t-1})\epsilon'_t]E[\epsilon_t\epsilon'_t]^{-1} \end{pmatrix}$$

Appendix C provides the first and second moments that appear here in a general setup with m regimes. While the vector a^{RPE} and d^{RPE} are easily computed for a given matrix b^{RPE}, the computation of b^{RPE} involves a 4^{th} order matrix polynomial, for which we do yet have a straightforward and reliable method. Therefore for the models considered in the remainder of this paper, we rely on simulations whenever lagged variables are involved. However in the special case without lagged variables, a^{RPE} and d^{RPE} are easily computed. Using our expressions in Appendix C for a 2-regime setup without lagged variables, we obtain the T-map:

$$\begin{pmatrix} a \\ d \end{pmatrix} \rightarrow \begin{pmatrix} A_i + C_i a \\ P_1(C_1 d\rho + D_1)\Sigma_\epsilon(p_{11} + p_{21}\frac{P_2}{P_1}) + P_2(C_2 d\rho + D_2)\Sigma_\epsilon(p_{22} + p_{12}\frac{P_1}{P_2}) \end{pmatrix}$$

Solving for the implied fixed-point yields the following equilibrium:

$$a^{RPE} = (I - P_1(p_{11}+1-p_{22})C_1 - P_2(p_{22}+1-p_{11})C_2)^{-1}(P_1(p_{11}+1-p_{22})A_1 + P_2(p_{22}+1-p_{11})A_2)$$

$$vec(d) = [I - (\rho\Sigma_\epsilon(p_{11} + p_{21}\frac{P_2}{P_1})' \otimes (P_1C_1)) - (\rho\Sigma_\epsilon(p_{22} + p_{12}\frac{P_1}{P_2})' \otimes (P_2C_2))]^{-1}$$
$$[vec(P_1D_1\Sigma_\epsilon(p_{11} + p_{21}\frac{P_2}{P_1})) + vec(P_2D_2\Sigma_\epsilon(p_{22} + p_{12}\frac{P_1}{P_2}))]$$

The associated Jacobian is given by:

$$DT_{(a,d)} = \begin{pmatrix} P_1(p_{11} + (1-p_{22}))C_1 + P_2(p_{22} + (1-p_{11}))C_2 & 0 \\ 0 & \begin{matrix} vec^{-1}_{n,n}(\Sigma_\epsilon(p_{22} + p_{12}\frac{P_2}{P_1})' \otimes P_1C_1 + \\ \Sigma_\epsilon(p_{11} + p_{21}\frac{P_1}{P_2})' \otimes P_2C_2) \end{matrix} \end{pmatrix}$$

Denoting by $r_\sigma(DT)$ the spectral radius of the Jacobian matrix above, the long-run E-stability principle is satisfied for the RPE if the real part of $r_\sigma(DT)$ is inside the unit circle. Further note that the stability of learning coefficients for the intercept and shock terms are independent of each other, implying the first and second diagonal terms of $DT_{(a,d)}$ govern the stability of intercept and shock terms respectively. This special case provides an important benchmark for the simple 3-equation New Keynesian model that we study next.

3.1 Constant-gain Least Squares in the 3-eqution NKPC

In this section, in order to illustrate our results, we consider the baseline 3-equation NKPC model given as:

$$\begin{cases} x_t = E_t x_{t+1} - \frac{1}{\tau}(r_t - E_t\pi_{t+1}) + \epsilon_{x,t} \\ \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \epsilon_{\pi,t} \\ r_t = max\{0, \rho r_t + (1-\rho)(\phi_x x_t + \phi_\pi \pi_t) + \eta_{r,t}\} \\ \epsilon_{y,t} = \rho_y \epsilon_{y,t-1} + \eta_{y,t} \\ \epsilon_{\pi,t} = \rho_\pi \epsilon_{\pi,t-1} + \eta_{\pi,t} \end{cases}$$

We can re-cast the interest rate rule above as a Markov process with two regimes, where:

$$\begin{cases} r_t(s_t = 1) = \rho r_{t-1} + (1-\rho)(\phi_x x_t + \phi_\pi \pi_t) + \eta^1_{r,t}\} \\ r_t(s_t = 2) = \eta^2_{r,t}\} \end{cases}$$

subject to the same transition matrix same as in the first example. The presence of noise in the second regime is meant to capture the fact that, although interest rates are very close to zero in empirical data, they are never exactly equal to zero in the post-2007 period (Lindé et al. (2017)). We consider a standard calibration with parameter values $\phi_y = 0.5, \phi_\pi = 1.5, \rho = 0, \kappa = 0.01, \beta = 0.99, \sigma_y = 0.7, \sigma_\pi = 0.3, \sigma_r^I = 0.3, \sigma_r^{II} = 0.01, \rho_y = 0.5, \rho_\pi = 0.5, p_{11} = 0.99, p_{22} = 0.9$. In the absence of interest rate smoothing, i.e. $\rho_r = 0$, the model reduces to the special case provided in the previous section, for which we can analytically check the E-stability conditions of the underlying RPE. Given this calibration, we have the following results: for the shock dynamics, the E-stability condition is satisfied with both regime-specific equilibria, as well as the RPE[5]. For the mean dynamics, the E-stability principle is satisfied in the normal regime, while it is violated in the ZLB regime: this is the key result from Evans et al. (2008) and many other papers that study steady-state learning at the ZLB. However, as long as the expected duration of the ZLB episode is not too long, the long-run E-stability principle is satisfied; this is indeed the case with our calibration of $p_{11} = 0.99$ and $p_{22} = 0.9$. Accordingly, the underlying RPE is E-stable under a constant gain least squares learning, provided the gain parameter γ is sufficiently small:

$$\begin{cases} R_t = R_{t-1} + \gamma(S^2_{t-1} - R_{t-1}) \\ \theta_t = \theta_{t-1} + \gamma R_t^{-1} S_{t-1}(S_t - \theta_{t-1} S_{t-1}) \end{cases}$$

where θ_t denotes the information set that the agent uses when forming his expectations. In order to check how the learning dynamics behave when lagged variables are also included, we consider the following overparameterized information set:

$$\theta_t = [1, \epsilon_{y,t}, \epsilon_{\pi_t}, y_{t-1}, \pi_{t-1}, r_{t-1}]'$$

We first simulate the model under recurrent regime switching with many ZLB episodes in order to study overall stability dynamics, and then focus on a one-time switch to and from the ZLB episode which is more relevant from an empirical point of view.

3.1.1 Recurrent Switching

In order to assess the overall stability of model dynamics, we first use a Monte Carlo simulation with a constant gain value of $\gamma = 0.01$. As we already know, the RPE is E-stable in terms of steady-state and shock coefficient dynamics when lagged variables are absent from the model. In this case, the regime-specific values involving b_i are zero matrices for both regimes, but it is unclear where the RPE-consistent matrix is located and whether it is stable. Figure 3 shows frequency distributions of all six learning ceofficients on output gap, inflation and interest rate respectively. It is readily seen that the resulting distributions are unimodal, suggesting oscillations around a unique E-stable RPE. Figure 4 shows the convergence and oscillation paths

[5]See Appendix C for details on the regime-specific equilibria and the associated E-stability conditions.

from two simulations with a decreasing and constant gain. Under DGL, we observe convergence towards a single point since the gain on new variables continues to decrease as t grows. Under CGL with $\gamma = 0.01$, we observe oscillations around a neighbourhood for each coefficient with no sign of divergence, which support our hypothesis that the underlying RPE is E-stable. A more important observation is that the jumps in learning coefficients during regime switches, which is similar to what we observed with the Fisherian equation. We next provide a close-up on this behaviour.

Figure 3: Frequency distributions of learning coefficients from 500 simulations of length 5000 for the two-regime NKPC. The columns shows the learning coefficients on output gap, inflation and interet rate; while the rows are the steady-state, lagged inflation, output gap, interest rate, and the two shock coefficients respectively. A casual inspection suggests the distributions are unimodal, suggesting oscillations around a unique E-stable RPE.

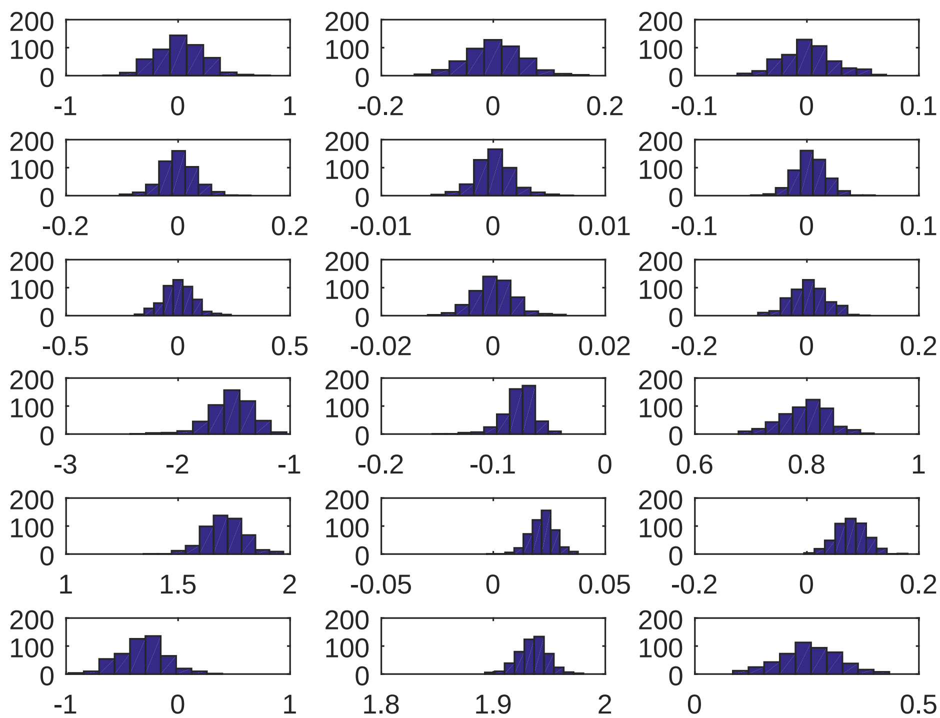
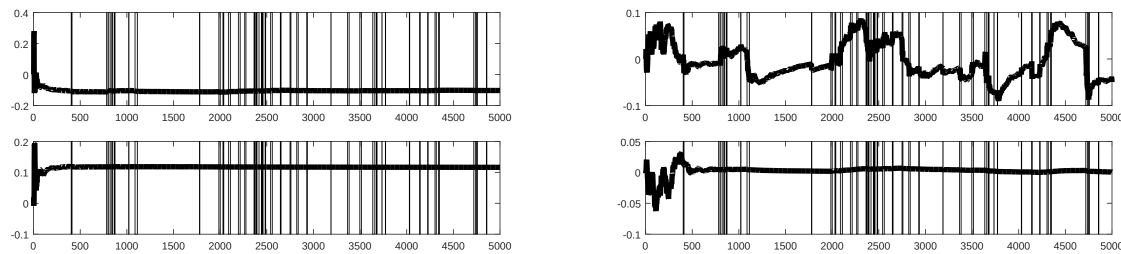
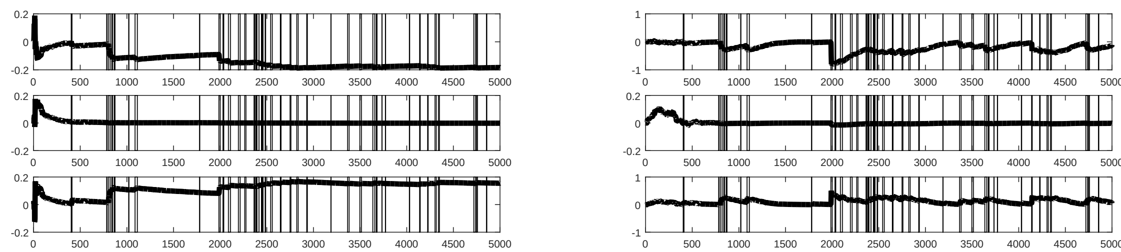


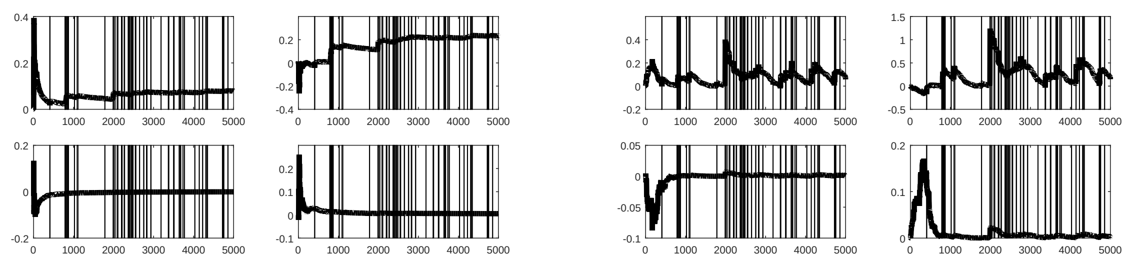
Figure 4: Simulations of length 5000 for the two-regime NKPC with recurrent regime switching. The left panel shows the results from a DGL simulation, while the right panel shows a CGL simulation with $\gamma = 0.01$.



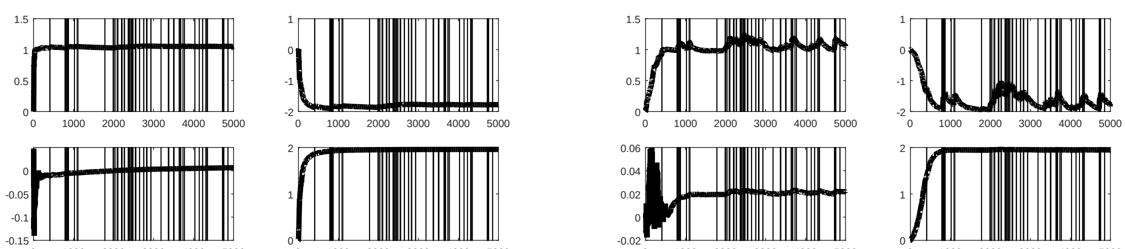
(a) Steady-state learning



(b) Lagged interest rate learning



(c) Lagged inflation and output gap learning



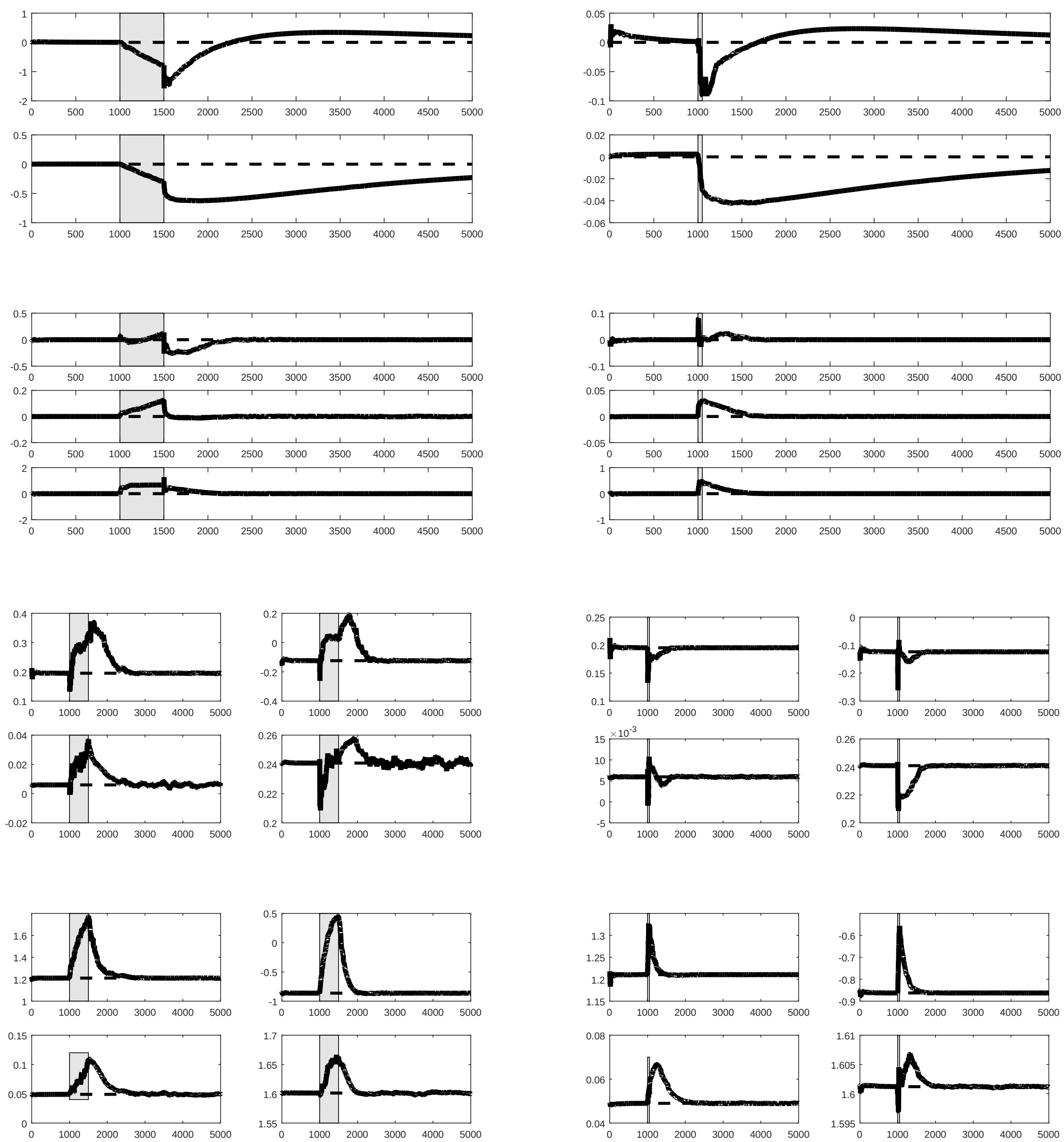
(d) Shocks coefficients learning

3.1.2 One-time Switch to ZLB regime

Figure 5 shows two simulations under CLG with $\gamma = 0.01$, with a single entry to and exit from the ZLB regime. The left column provides a case with an excessively long ZLB episode of length 2000, which allows us to examine how each learning coefficient behaves during the ZLB regime. The right column provides a case with a short-lived ZLB episode of length 50, which is closer to the type of ZLB regime that we observe in practice. First focusing on the long-lived regime, we observe that all lagged variable and shock coefficients show stable behaviour during the ZLB regime. The shock coefficients gradually converge towards new equilibrium values during the ZLB episode, while gradually reverting back to their previous values after the exit. On the contrary, the lagged variables are characterized by jumps immediately after the regime change both during the entry and exit, while gradually reverting back to their previous values after the switch. These jumps are in line with what we observed in our 1-dimensional example, despite the fact that we cannot compute the underlying RPE. On the contrary, the steady-state coefficients seem to be on a divergent path after entry to the ZLB regime, while this divergence is halted after exiting the ZLB regime. We can make similar observations with the short-lived ZLB episode with respect to the jumps in the learning coefficients. In this case the episode is too short for the divergence of steady-state parameters to kick in, therefore the mean coefficients seem more stable overall. This example illustrates how bursts of E-unstable regimes do not pose a problem for the overall stability of the model dynamics, as long as the unstable regimes are sufficiently short-lived. However, the E-unstability of steady-state dynamics indicate that one should be careful the inclusion of these coefficients in the PLM. This is discussed in more detail in the upcoming estimation sections.

A robust phenomenon in our simulation exercises up to this point is the observed jumps in the learning coeffients during the ZLB regimes. These jumps are more pronounced particularly when the system exits persistent and long-lived regimes; enters into new regimes that have not been previously observed; or when the gain coefficient is sufficiently high. Notice that these jumps are similar to what one would observe in a Rational Expectations framework, where the PLMs immediately switch following a regime change, since such changes are assumed to be observed by agents. However different than a REE in our framework, the direction of the jumps are either towards the RPE or regime-specific values, which do not coincide with REE-consistent values in general. Therefore a natural question that arises is whether we can observe similar jumps in PLMs when we take our learning approach to the data, and how the direction and magnitude of such jumps under adaptive learning differ from REE. To address this question, we next move onto the discussion of a filter that can handle the estimation of Markov-Switching DSGE models under adaptive learning.

Figure 5: Simulations of length 5000 for the two-regime NKPC with a one-time entry to and exit from the ZLB regime, both under CGL with $\gamma = 0.01$. The left panel shows a long-lived ZLB episode of length 500 periods, which illustrates which parameters converge and which ones diverge over the ZLB episode. The right panel shows a short-lived ZLB-episode of length 50 periods, which is more relevant from an empirical viewpoint.



4 Bayesian Estimation of Markov-Switching DSGE Models under Adaptive Learning: Filtering Algorithm

The benchmark algorithm for Markov-switching state-space models is the modified Kalman filter by Kim & Nelson (henceforth KN-filter): in a Markov-switching model with m distinct regimes, a dataset of size T leads to m^T distinct timelines, which quickly makes the standard Kalman filter intractable. The main idea in the KN-filter is to introduce a so-called *collapsing* technique to deal with this issue, which amounts to taking weighted averages of the state vector and covariance matrix at each iteration of the filter. This effectively reduces the number of timelines at each iteration by an order of m, thereby making the filter tractable again. The standard recommendation is to carry as many lags of the states as appears in the transition equation. Since we only consider we consider DSGE models that have a reduced-form VAR(1) representation in this paper, only a version of the filter with the single lag is presented here, although the same framework can be easily extended to any VAR(p) framework. Accordingly, if there are m different regimes in the model, we carry m different timelines in each period. Therefore there are m^2 different sets of variables in the forecasting and updating steps of each iteration. These are then collapsed at the end of each iteration to reduce to m sets of variables. An important question is how to introduce adaptive learning into this framework. We use an approach that is consistent with the theoretical framework of the previous section: the agents have a unique PLM based on observables, independent of the regime switches. We model this formally by collapsing the m different states further at each iteration to obtain the filtered states, which are then used for the adaptive learning step. The unique learning coefficients are then used in each distinct timeline of the next period's iteration[6]. We denote the state-space representation of our model as follows:

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)} S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)} \epsilon_t, & , \epsilon_t \sim N(0, \Sigma) \\ y_t = E + F S_t \end{cases}$$

where S_t and y_t denote the (unobserved) state variables and observables respectively, while E and F collect the parameters that relate the state variables to the observables. The matrices of structural parameters are time-varying with respect to both the Markov state s_t and the belief coefficients Φ_t. Given this representation, Table 1 the filter for the general case, while Figure 6 illustrates the special case of two regimes, which is our main focus in this paper.

[6] A natural alternative here is to apply the adaptive learning step distinctly to each collapsed state; one can then take a weighted average of these expectations to obtain the filtered expectations. Our results in the upcoming sections are not sensitive to such an alternative, but we only present the results under the first approach since it is more in the spirit of our theoretical framework.

Table 1: **KM-filter for Markov-Switching DSGE Models under Adaptive Learning**

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)} S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)} \epsilon_t, \quad , \epsilon_t \sim N(0, \Sigma) \\ y_t = E + F S_t \end{cases}$$

0) Initial States :

$\tilde{S}_{0|0}^{i}, \tilde{P}_{0|0}^{i}, Pr[S_0 = i|\Phi_0], \Phi_0$ given.

1) Kalman Filter Block with the standard measurement and transition equations:

For $t = 1 : N$

For $\{S_{t-1} = i, S_t = j\}$

$$\begin{cases} S_{t|t-1}^{(i,j)} = \gamma_1^{(j)} S_{t-1|t-1}^{(i)} + \gamma_2^{(j)} \\ P_{t|t-1}^{(i,j)} = \gamma_1^{(j)} P_{t-1|t-1}^{(i)} \gamma_1^{(j)} + \gamma_3^{(j)} \Sigma^{(j)} (\gamma_3^{(j)})' \\ v_{t|t-1}^{(i,j)} = (y_t - F^{(j)} S_{t|t-1}^{(i,j)}) \\ Fe^{(i,j)} = F^{(j)} P_{t|t-1}^{(i,j)} F^{(j)} \\ S_{t|t}^{(i,j)} = S_{t|t-1}^{(i,j)} + P_{t|t-1}^{(i,j)} (F^{(j)})' (Fe^{(i,j)})^{-1} v^{(i,j)} \\ P_{t|t}^{(i,j)} = P_{t|t-1}^{(i,j)} (F^{(j)})' (Fe^{(i,j)})^{-1} F^{(j)} P_{t|t-1}^{(i,j)} \end{cases}$$

2) Hamilton Block for transition probabilities:

Denote: $Pr[S_{t-1} = i, S_t = j|\Phi_{t-1}] = pp_{t|t-1}^{i,j}, f(y_t|\Phi_{t-1})$ the marginal likelihood,

$Pr[S_{t-1} = i, S_t = j|\Phi_t] = pp_{t|t}^{i,j}$ and $Pr[S_t = j|\Phi_t] = \tilde{pp}_{t|t}^{j}$.

$$\begin{cases} pp_{t|t-1}^{(i,j)} = Q(i,j) pp_{t-1|t-1}^{(i)} \\ f(y_t|\Phi_{t-1}) = \sum_{j=1}^{M} \sum_{i=1}^{M} f(y_t|S_{t-1} = i, S_t = j, \Phi_{t-1}) pp_{t|t-1}^{(i,j)} \\ pp_{t|t}^{(i,j)} = \frac{f(y_t|S_{t-1}=i, S_t=j, \Phi_{t-1}) pp_{t|t-1}^{(i,j)}}{f(y_t|\Phi_{t-1})} \\ p_{t|t}^{j} = \sum_{i}^{M} pp_{t|t-1}^{(i,j)} \end{cases}$$

3) Collapsing to reduce the number of states from m^2 to m:

$$\begin{cases} S_{t|t}^{(i)} = \frac{\sum_{i=1}^{M} pp_{t|t}^{(i,j)} S_{t|t}^{(i,j)}}{p_{t|t}^{(j)}} \\ P_{t|t}^{(i)} = \frac{\sum_{i=1}^{M} pp_{tt}^{(i,j)} (P_{t|t}^{(i,j)} + (S_{t|t}^{(j)} - S_{t|t}^{(i,j)})(S_{t|t}^{(j)} - S_{t|t}^{(i,j)})')}{p_{t|t}^{(j)}} \end{cases}$$

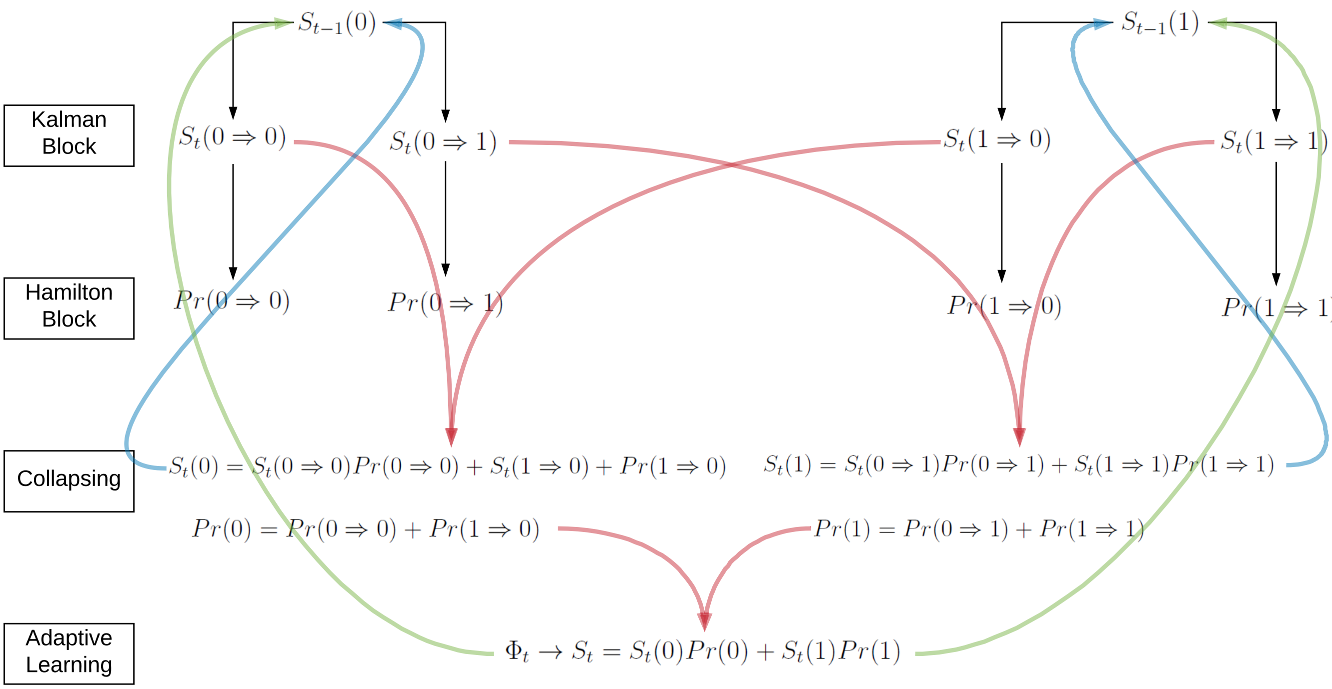
4) Update expectations based on filtered states:

Updating Expectations based on Filtered States:

$$\begin{cases} \tilde{S}_{t|t} = \sum_{j=1}^{M} p_{t|t}^{(j)} S_{t|t}^{(j)} \\ \Phi_t = \Phi_{t-1} + \gamma R_t^{-1} \tilde{S}_{t-1|t-1} (\tilde{S}_{t|t} - \Phi_{t-1}^{T} \tilde{S}_{t-1|t-1})^{T} \\ R_t^{-1} = R_{t-1} + \gamma (\tilde{S}_{t-1|t-1} \tilde{S}_{t-1|t-1}^{T} - R_{t-1}) \end{cases}$$

Figure 6: Illustration of the filter in a 2-regime model.

$$\begin{cases} S_t = \gamma_{2,\Phi_t}^{(s_t)} + \gamma_{1,\Phi_t}^{(s_t)} S_{t-1} + \gamma_{3,\Phi_t}^{(s_t)} \epsilon_t, & , \epsilon_t \sim N(0, \Sigma) \\ y_t = E + F S_t \end{cases}$$



4.1 Initial Beliefs

The first practical issue with the above filter is Step (0), i.e. where to initialize the beliefs. This has been shown to play a key role in driving the estimation results in previous studies, and various different approaches have been considered: Milani (2007) uses an estimation-based approach, where the initial beliefs are treated as structural parameters and estimated jointly along with the other parameters of interest; Slobodyan & Wouters (2012b,2012a) consider REE-based and training-sample based approaches along with the estimation-based approach; while Berardi & Galimberti (2017c) proposes a smoothing-based approach. A common result in these studies is that the results are generally sensitive to initial beliefs, and the best-fitting approach depends on the specific model under consideration; see Berardi & Galimberti (2017a, 2017b) for a detailed overview.

In this paper, following the approach in Slobodyan & Wouters (2012b), we present our main results under the REE-based initial beliefs and check the sensitivity of these results to alternative specifications. Accordingly, as our main approach, we estimate the benchmark (non-switching) REE, and use the relevant moments implied by the matrices γ_1 and γ_3 from this estimation in the initialization step of the learning models[7]. As our alternatives, we consider three approaches: (i) training-sample approach, where we initialize the filter at diffuse moments and estimate it

[7] The REE-implied intercepts are always zero, therefore the vector γ_2 is always initialized at the vector of zeros

on a training sample[8], then use the final values of the learning coefficients to initialize the main estimation; (ii) estimation-based approach, where we first estimate all belief coefficients jointly with the structural parameters in a VAR model, then use the resulting estimates to initialize the main estimation; (iii) a filter-based approach, where, at each step of the estimation, the beliefs are initialized at diffuse points and ran once, then the converged values of the belief coefficients are used as initial values to run the filter for a second time. The results of the alternative estimations are provided in the Appendix, and while the relative fit of each model is sensitive to the initial beliefs, our main conclusions continue to hold in all specifications.

4.2 Projection Facilities

The second practical issue with the above filter is how to retain the stability of the underlying model with adaptive learning. In most operational macro models, the MSV-solution includes lagged state variables due to properties such as habit formation, indexation in prices and wages, interest rate smoothing, etc. When these parameters in PLM are updated each period in a constant gain setup, they may easily end up in an unstable region, which then feeds back into the implied ALM and leads to explosive dynamics. Models subject to the zero lower bound constraint are more prone to encounter this problem, since typically an inactive monetary policy rule implies indeterminacy and E-unstability for the regime-specific dynamics. A common way in the adaptive learning literature to deal with these potential instabilities is to impose a so-called projection facility on the model, which projects the learning parameters into a point in the stable region when the instability is encountered. The simplest approach to do this is to leave the parameters at their previous value if the update leads to an instability, which is the method adopted in Slobodyan & Wouters (2012a). In this paper we use a variant of this simple idea: we stop updating the learning parameters each period, if the update pushes the largest eigenvalue of the system outside the unit circle. We base our notion of stability on the ergodic distribution of the Markov chain: the model is said to be explosive at any set of parameter values, if the largest eigenvalue of the ergodic distribution is outside the unit circle. Accordingly, we allow the regime-specific models to be temporarily explosive, as long as the underlying distribution is still stable. Importantly, this approach also allows the agents' PLM to become temporarily explosive as long as the underlying ergodic distribution is stable. These choices do not have an impact on our estimation of the small-scale NKPC since we do not encounter unstable regions in this case; but they turn out to play an important role in the estimation of medium-scale SW model as we show in the upcoming sections.

5 Estimation of the 3-equation NKPC

In this section, as a first step, we estimate the small-scale 3-equation NKPC model presented in the previous section. In this framework there are 18 parameters to be estimated, to which we assign prior distributions consistent with the previous literature: the risk aversion coefficient has a gamma distribution with a mean 2 and standard deviation 0.5 as in An & Schorfheide (2007). The monetary policy reaction coefficients are given Gamma distributions centered at 1.5 and 0.5 respectively, with a standard deviation of 0.25; these are the standard values associated with the Taylor rule. Interest rate smoothing, shock persistence and shock standard deviation coefficients are consistent with Smets & Wouters (2007), the first two having a Beta distribution with mean 0.5 and st. dev 0.2, while the latter is assigned a Gamma distribution with mean 0.1 and st. dev 2. The regime probabilities are taken from Lindé et al. (2017): the exit probability of the normal regime is a Beta distribution with mean 0.1 and standard deviation 0.05, while the exit probability of ZLB regime is a Beta distribution with a mean 0.3 and st. dev 0.01. The gain coefficient follows the same distribution as in Slobodyan & Wouters (2012b) with a gamma distribution and a mean of 0.035, but we assume a tighter distribution with a standard deviation of 0.015. The slope of Phillips curve is assigned a Beta distribution

[8] For both models considered in the latter sections, our training sample is based on pre-1966 period of aggregate U.S. variables.

centered at 0.3 with a standard deviation of 0.15, which has a slightly larger mean and variance compared toAn & Schorfheide (2007).

We use quarterly U.S. data over the period 1966:I-2016:IV on interest rates, inflation and output gap with the following simple measurement equations:

$$\begin{cases} y_t^{obs} = \bar{y} + y_t \\ \pi_t^{obs} = \bar{\pi} + \pi_t \\ r_t^{obs} = \bar{r} + r_t \end{cases}$$

where the mean parameters $\bar{y}, \bar{\pi}$ and $\bar{r}$ are assigned normal distributions based on the pre-1966 period, and output gap is based on the potential output gap measure of CBO. Table 2 shows the point-estimates for five model specifications: for the constant-gain learning cases, we consider a univariate AR(1) rule, a VAR(1) rule including output gap, inflation and interest rates, and the MSV-type rule that assumes regime switches are unobserved as discussed above. The last two columns show the two benchmark Rational Expectations cases with and without regime switching[9]. First looking at the resulting likelihoods based on Laplace approximation, we can already see a pattern: the MS-REE model leads to a substantial improvement over the benchmark REE model, implying that the regime shift on interest rates plays an important role in driving the model fit. Adding adaptive learning on top of Markov-switching improves the likelihood further: all three adaptive learning specifications outperform the REE-MS model. Both of these results, individually, are consistent with the previous results found in the literature, i.e, it is well known that both Markov-switching and adaptive learning typically improve the model fit compared with the benchmark case. Our results here show that these two results are also complementary, i.e. putting the two together improves the results further compared with the individual cases.

Next we turn to a discussion of our parameter estimates: the results under REE-MS are generally similar to the REE model with the exception of the risk aversion parameter τ, which is lower under REE-MS with 2.75 compared with 4.57 under REE. Comparing the MSV-learning case to REE-MS, the differences are minimal: the only differences arise in the interest rate smoothing ρ_r, output gap reaction ϕ_y and the risk aversion coefficient τ, which turn out slightly higher under MSV-learning. Another difference is the exit probability from ZLB regime, which is estimated to be lower under MSV-learning. At the estimated coefficients, this implies the ZLB regime has an expected duration of 5.9 quarters under MS-REE, while this is 7.69 under MSV-learning. In other words, the estimated persistence of the ZLB regime is higher under the learning specification. These differences become more pronounced as we move onto the AR(1) and VAR(1) cases: the exit probability from ZLB regime under AR(1)-learning is the same as in MSV case with an expected duration of 7.69, while the ZLB regime is even more persistent under VAR(1)-learning with an expected duration of 9.1 quarters. These results show that, in general, the expected duration of the ZLB regime is much larger under any type of learning compared with the MS-REE case. Further, in both AR(1) and VAR(1) cases, the NKPC slope κ is substantially larger at 0.036 and 0.029 respectively, compared with 0.005, 0.004 and 0.006 under MSV-learning, REE-MS and REE cases: this is a direct consequence of assuming unobserved shocks, which shifts the transmission channel from expectations to the parameter κ. Another important difference between AR(1) and VAR(1) compared with the remaining specifications is the shock persistence terms: in both cases, the persistence of shocks are much smaller compared with the remaining cases. This results from the backward-looking nature of AR(1) and VAR(1) rules, which shift some of the exogenous persistence in the shocks onto expectations, thereby resulting in lower estimates for the shock persistence parameters. Furthermore, the gain coefficient increases as we reduce the information set: it is lowest for the MSV-learning case and highest for the univariate AR(1) rule. Among the three learning rules,

[9]The REE-MS specification is estimated using J. Maih's RISE toolbox (Maih (2015)), while the standard REE case is obtained from the Dynare toolbox (Adjemian et al. (2011)). For the adaptive learning specifications, we use our filter as presented in the previous section. Note that RISE toolbox uses a variant of the same KN filter, hence the estimations are based on the same filter except for the adaptive learning component.

the most parsimonious AR(1) rule yields the best fit with a Laplace approximation of -273, while the VAR(1) rule yields the worst one with -303. This is confirmed by the Bayes' Factors, which yields decisive support in favor of the learning models for all specifications compared with the REE or MS-REE benchmarks.

Figure 7 shows the filtered regime probabilities for the AR(1)-learning and MS-REE cases along with the historical interest rates over the estimation sample. It is readily seen that the estimated regime probability sharply rises in both cases in 2009 as the interest rates are lowered to near-zero levels. Both models indicate that towards the end of 2016, the economy is out of the ZLB regime as the interest rates slowly start to increase. The same pattern can be observed in other learning cases of VAR(1)- and MSV-learning, which we omit here. This figure suggests that there are no differences in the estimated pattern of entering and exiting the ZLB regime, which is intuitive since the learning process should not have an impact on this. We next turn to the time variation in the belief coefficients over the estimation sample, which is shown in Figure 8 for all three learning cases. The most important difference arises in the pattern of intercept coefficients for MSV-learning with observed shocks, and AR(1) & VAR(1) learning with unobserved shocks: while there is considerable time-variation in the intercept parameters in the latter cases, these is much less variation for MSV-learning. This suggests that when the exogenous shocks are assumed to be observed, the changes in the endogenous variables are attributed to these shocks and to changes in the feedback coefficients from exogenous shocks to endogenous variables. When the shocks do not enter into agents' information set, they attribute more changes to the time-variation in the intercept terms. In the AR(1) case, there is a sudden drop in the intercept terms during the 2007-08 crisis period, and these coefficients remain there during the remainder of the sample. The same observation also applies to the VAR(1) case, where the intercept of interest rates is also characterized with a sudden drop along with inflation and output gap. This suggests that there was a level shift in the agents beliefs for the endogenous variables during this period. This level shift does not arise in the MSV-learning case since the changes are attributed to the exogenous shocks. Accordingly, while all learning specifications provide a plausible characterization of the economy since they provide a better fit than the REE models, there is considerable difference in the implied evolution of beliefs during the sample period. This in turn suggests there might be important differences in how the economy responds to the same shock under different specifications. Figure 9 shows the impulse responses for all estimated models[10]: we consider the responses to one unit supply and demand shocks η_y and η_π respectively. As one might expect, the AR(1) and VAR(1) cases are very similar, while there is virtually no difference between MSV-learning and MSV-REE cases. A positive supply shock is expansionary and inflationary under all specifications, and the impact is larger under the ZLB regime since monetary policy does not mitigate the impact of this shock. An important difference between the observed shock and unobserved shocks cases is how long the shock takes to "kick in": under MSV-learning and MS-REE cases, the effect is maximal on impact, and the difference between the normal and ZLB regimes is therefore also maximal on impact. On the contrary, under the AR(1) and VAR(1) cases the effect is more gradual and takes a few periods to reach its maximum impact, leading to a hump-shaped response. Therefore the difference between normal and ZLB regimes also takes several periods to become visible. The impact of a demand shock is inflationary in all regimes and specifications, hence there is no difference with regards to the impact of this shock on inflation. However, there are important differences with regards to output: in the ZLB regime, a demand shock is always expansionary but the impact is again more gradual in the unobserved shock cases, leading to a hump-shaped response. In the normal regime, a demand shock is always contractionary in the observed shock cases, while it is initially expansionary and only becomes contractionary after 8 quarters in the unobserved shock cases.

As the above discussion makes it clear, we can already see some important differences in the

[10]Note that, due to time-variation in beliefs, the impulse responses are different each period under the adaptive learning specifications. Therefore for Figure 9, we consider two *representative* periods based on 2006Q1 and 2011Q1 for the impulses under normal and ZLB regimes respectively. We leave the analysis of the time-variation in the impulse responses for the Smets-Wouters model.

estimations and the implied stochastic structure of the economy even in small-scale setup. This raises the question of whether the differences will carry over to a more realistic model setup, which we analyze in the next section with the Smets & Wouters (2007) model.

Table 2: Estimation Sample: 1966:I-2016:IV based on the U.S data, where the observables are the output gap (based on CBO's historical estimates), inflation and interest rate.The Markov-switching REE model is obtained from RISE toolbox, while the standard REE case is provided by Dynare. The AR(1), VAR(1) and MSV-learning cases are based on our algorithm above.

Parameter	Prior			Posterior AR(1)	VAR(1)	MSV	REE-MS	REE
	Dist	Mean	St. Dev	Mode	Mode	Mode	Mode	Mode
$\bar{y}$	Normal	0	0.25	-0.26	0.06	-0.32	-0.17	0.24
$\bar{\pi}$	Gamma	0.62	0.41	0.38	0.78	0.7	0.39	0.17
$\bar{r_1}$	Gamma	1	0.25	0.84	1.42	0.84	0.68	1.11
κ	Beta	0.3	0.15	0.036	0.029	0.005	0.004	0.006
τ	Gamma	2	0.5	2.41	2.7	3.08	2.75	4.57
ϕ_π	Gamma	1.5	0.25	1.48	1.52	1.56	1.56	1.42
ϕ_y	Gamma	0.5	0.25	0.4	0.4	0.45	0.27	0.27
ρ_r	Beta	0.5	0.2	0.87	0.93	0.9	0.8	0.8
ρ_y	Beta	0.5	0.2	0.30	0.44	0.89	0.92	0.93
ρ_π	Beta	0.5	0.2	0.05	0.06	0.85	0.92	0.89
η_y	Inv. Gamma	0.1	2	0.76	0.75	0.1	0.1	0.1
η_π	Inv. Gamma	0.1	2	0.28	0.28	0.03	0.03	0.04
η_{r_1}	Inv. Gamma	0.1	2	0.32	0.33	0.32	0.32	0.3
$\bar{r_2}$	Normal	0.1	0.25	0.03	0.04	0.03	0.03	-
η_{r_2}	Uniform	0.005	0.05	0.02	0.02	0.01	0.01	-
$1-p_{11}$	Beta	0.1	0.05	0.02	0.02	0.02	0.02	-
$1-p_{22}$	Beta	0.3	0.1	0.13	0.11	0.13	0.17	-
$gain$	Gamma	0.035	0.015	0.039	0.027	0.0246	-	-
Laplace				-273.6	-303.6	-289.07	-317.02	-368.49
Bayes' Factor				41.2	28.4	34.5	22.4	1

Figure 7: Historical interest rates along with the estimated regime probabilities in AR(1)-learning and benchmark MS-REE cases.

AR(1) learning and the benchmark MS-REE:

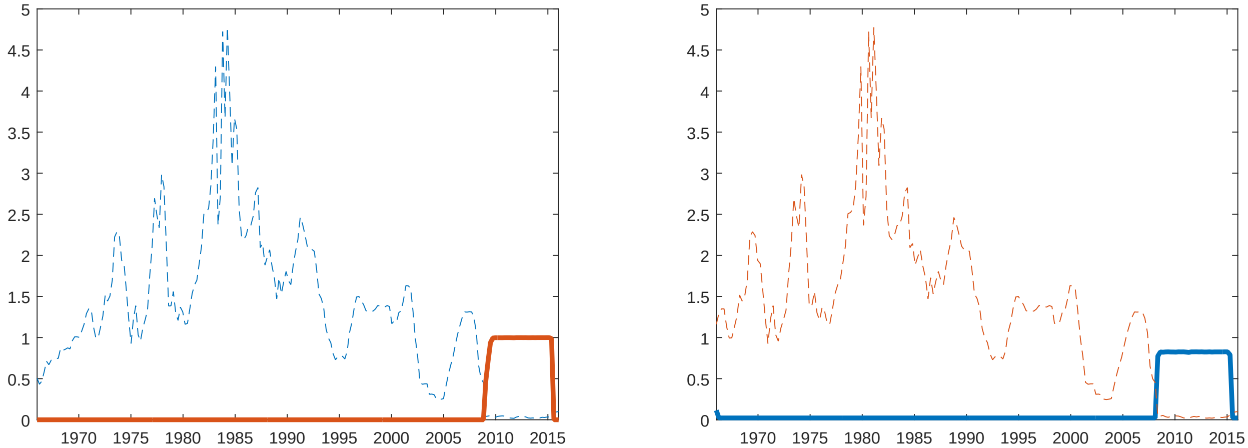
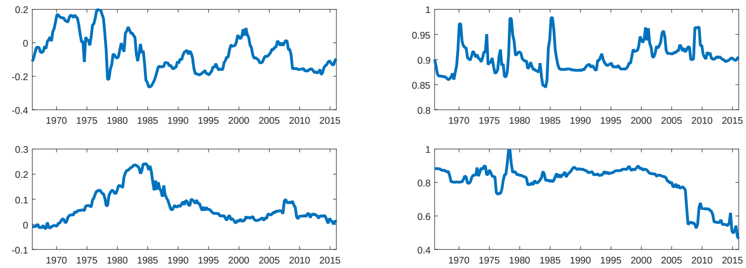
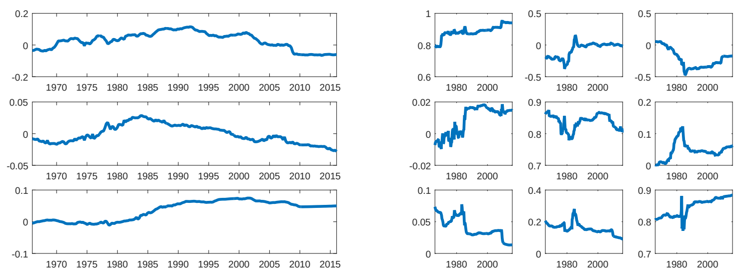


Figure 8: Expectation coefficients for AR(1), VAR(1) and MSV-learning cases.

AR(1) intercept and persistence coefficients, with output gap and inflation on the first and second row respectively.



VAR(1) intercept and cross-correlation coefficients. The left panel shows the perceived intercepts of output gap, inflation and interest rates respectively. The right panel shows the impact of lagged output gap, inflation and interest rates on the first, second and third rows respectively.



MSV learning coefficients. The right panel shows the perceived intercepts of output gap, inflation and interest rates; the middle panel shows the impact of lagged interest rates on output gap, inflation and interest rates; and the right panel shows feedback coefficients from supply and demand shocks on output gap and inflation respectively.

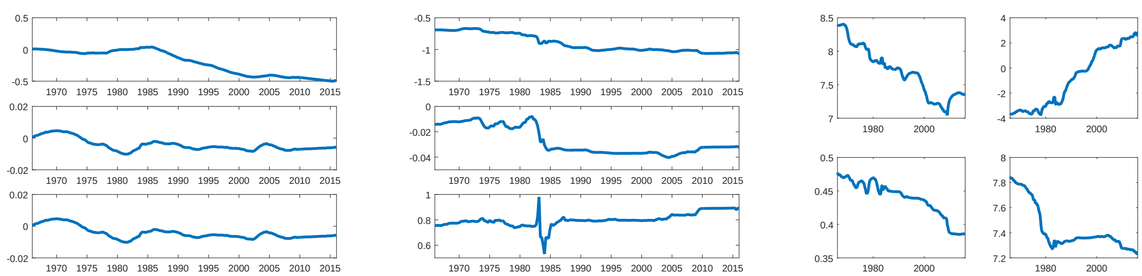
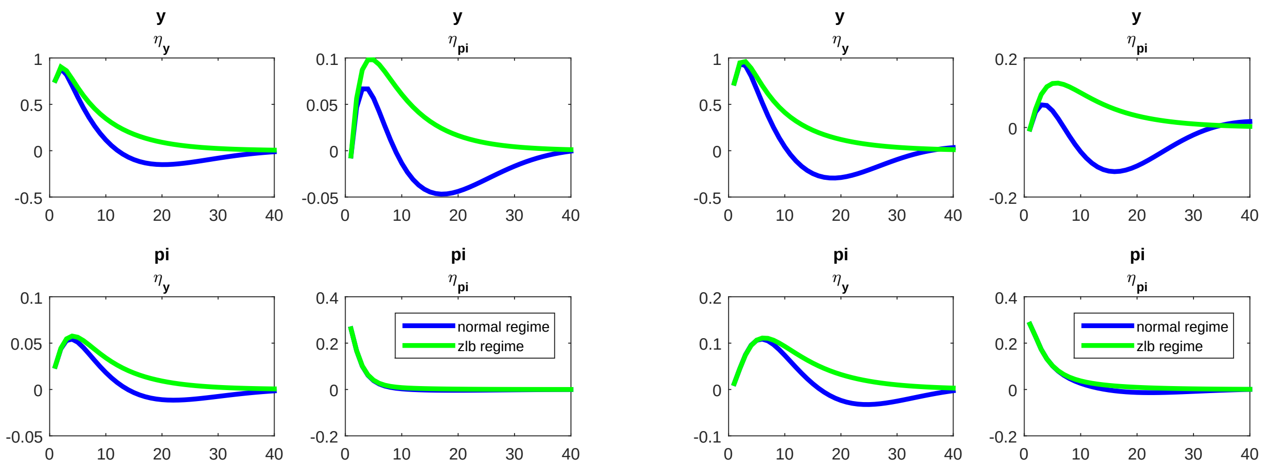
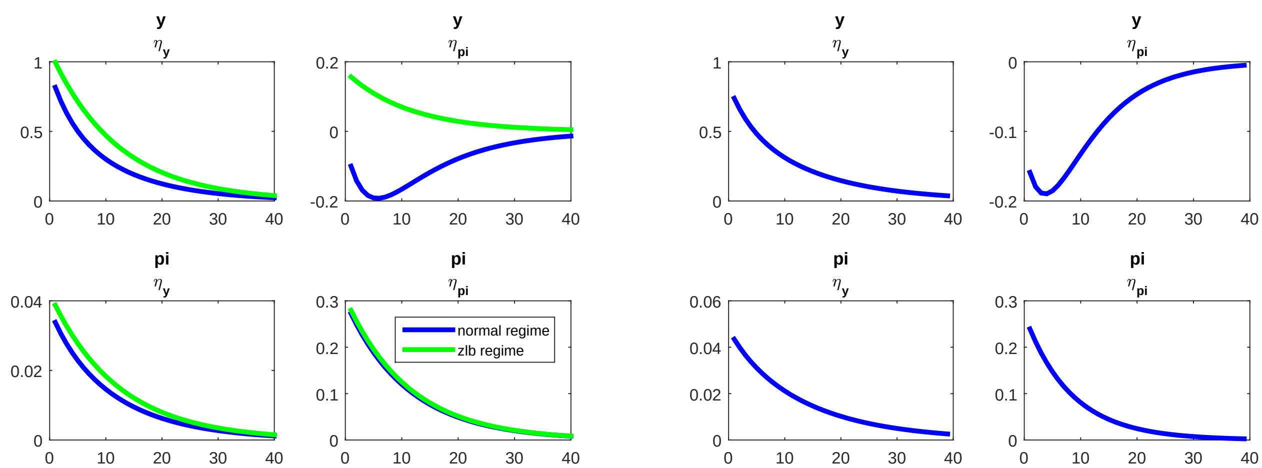


Figure 9: Impulse responses of time-varying PLMs are based on an arbitrary period.

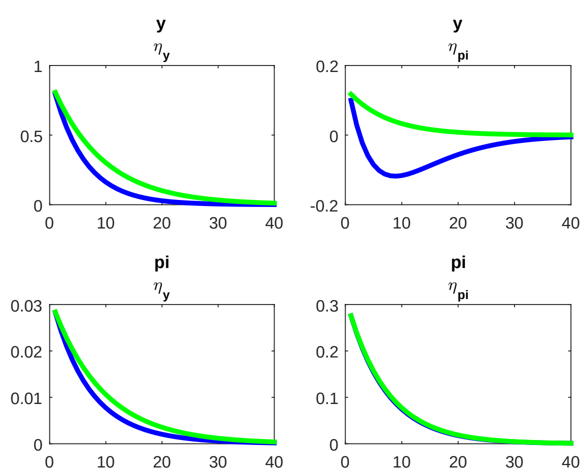
AR(1) and VAR(1) beliefs:



MS and standard REE benchmarks:



MSV beliefs:



6 Estimation of Smets-Wouters

In this section we consider the estimation of Smets-Wouters (2007). The prior distributions of the benchmark parameters are identical to Smets-Wouters, while the ZLB and learning related parameters are the same as in the 3-equation model above. The model features seven structural shocks: productivity, risk-premium, government spending, investment, monetary policy, and price & wage mark-up shocks, denoted by a, b, g, i, r, p and w, see Smets & Wouters (2003,2007) for more details. Our only deviation from the benchmark model is to assume that the mark-up shocks follow an AR(1) process, instead of the original ARMA(1,1) assumption: as shown in Smets & Wouters (2003,2007), these shock processes are typically close to being white noise when expectations are assumed to be backward-looking, in which case the AR(1) and MA(1) terms are close to being locally unidentified. Therefore we assume away the MA(1) terms and model these shocks as AR(1) processes. The rest of the model is left unchanged and our measurement equations also follow the benchmark case with seven observables as follows:

$$\begin{cases} d(log(y_t^{obs})) = \bar{\gamma} + (y_t - y_{t-1}) \\ d(log(c_t^{obs})) = \bar{\gamma} + (c_t - c_{t-1}) \\ d(log(inv_t^{obs})) = \bar{\gamma} + (inv_t - inv_{t-1}) \\ d(log(w_t^{obs})) = \bar{\gamma} + (w_t - w_{t-1}) \\ log(l_t^{obs}) = \bar{l} + l_t \\ (log(\pi_t^{obs})) = \bar{\pi} + \pi_t \\ (log(r_t^{obs})) = \bar{r} + r_t \end{cases}$$

where $d(log(y_t^{obs}))$, $d(log(c_t^{obs}))$, $d(log(inv_t^{obs}))$ and $d(log(w_t^{obs}))$ denote real output, consumption, investment and wage growths with the common growth rate $\bar{\gamma}$ respectively, while $log(l_t^{obs})$, $(log(\pi_t^{obs}))$ and $(log(r_t^{obs}))$ denote (normalized) hours worked, inflation rate and federal funds rate respectively. We use the same estimation sample as in the 3-equation NKPC with quarterly U.S. data covering the period from 1966:I to 2016:IV. Table 3 shows the full sample results for the learning and benchmark REE and MS-REE cases. We observe that our main conclusions from the NKPC-estimation carry over to this more realistic setup: the MS-REE model considerably improves upon the REE benchmark with likelihoods -1171 and -1213 respectively, which shows the empirical importance of explicitly modeling the ZLB episode. The MSV-learning setup yields a likelihood of -1168, which is very close to the REE-MS case. This indicates the time-variation under MSV-learning does not lead to any meaningful improvements over the MS-REE model. Next looking at the parsimonious AR(1) learning case, we have a likelihood of -1135, which is another substantial improvement over both the MSV-learning and MS-REE cases. The model with the univariate, backward-looking expectation rule is thus our preferred specification based on this results. Taken together with the NKPC estimations, all of our results are consistent with the previous literature on learning: Milani (2007) shows that MSV-learning performs better that REE in the small 3-equation setup, while Slobodyan & Wouters (2012b) find that this result does not extend to the medium-scale setup of SW model, where MSV-learning does not lead to improvements over the REE model. Instead using a univariate backward-looking rule in Slobodyan & Wouters (2012a), they find that the model fit improves considerably. Our results in this section and the previous one confirm these results, and show that they continue to hold in a Markov-switching setup where the ZLB episode is considered.

Next we turn to a discussion of our parameter estimates and we focus particularly on the learning, ZLB and nominal friction parameters. Starting with the estimated exit probabilities from the normal and ZLB regimes, we can see the same result as in the 3-equation model, where the ZLB regime becomes more persistent under the learning cases. In particular, the estimated exit probability from the ZLB regime is 0.29 under the MS-REE model, indicating an expected ZLB duration of of 3.5 quarters. This probability is substantially larger under the learning specifications with a value of 0.13 under both AR(1) and MSV-learning cases, which suggests an expected ZLB duration of 7.7 quarters. Especially the expected duration under the MS-REE case is lower compared to the 3-equation model, suggesting there is downward pressure for the persistence of this regime. It is important to note that the expected ZLB durations both under the MS-REE and learning cases are substantially lower than the empirical length of the ZLB episode, which lasted around 7 years or 28 quarters for the U.S. economy. However, our results suggest that adaptive learning is indeed one way of increasing the expected duration of this episode, since the expected duration in the Smets-Wouters model is estimated more than twice under adaptive learning compared with the MS-REE case.

The estimated gain values for both AR(1)- and MSV-learning cases are substantially lower compared with the 3-equation model, with values of 0.012 and 0.001 respectively. Branch & Evans (2006) find that values over the range of $[0.005, 0.05]$ provide a good fit for the Survey of Professional Forecasters (SPF) dataset. Accordingly, the resulting gain value under AR(1)-learning is in the empirically relevant range, while under MSV-learning it is substantially closer to zero. This suggests either that the MSV information set is not supported by the data, or that the initial beliefs provide a better fit than the time-variation in beliefs under the MSV-setup. At the estimated value under MSV-learning, the initial beliefs receive a weight of $(1-0.001)^200 \approx 0.82$ at the end of the sample; i.e. beliefs remain very close to the initial beliefs during the entire estimation sample. The resulting likelihoods suggest that this does not lead to substantial differences in the model fit, which is also evident from the remaining parameter estimates. The only exceptions are the price and wage Calvo probabilities, which are slightly lower under the MSV-learning case compared with the MS-REE. The difference is more pronounced under AR(1)-learning, where the Calvo probabilities keep decreasing further. This suggests nominal persistence parameters become less pronounced as expectations become more sluggish, which is also consistent with the findings in previous literature. Another important difference that arises under AR(1)-learning is the shock persistence terms: the price and wage mark-up shocks become near white noise processes with persistence near zero, while the persistence of investment shock also becomes substantially smaller. Finally, the implied consumption dynamics are very different under AR(1) learning: both the capital adjustment cost ϕ and habit formation λ parameters are lower compared to the remaining cases, while the inverse intertemporal elasticity of substitution decreases to below one. An interesting implication of this is that, under AR(1) learning, consumption and labor become substitutes rather than complements.

Overall, our results suggest that the univariate AR(1)-learning case provides the best model fit and leads to some important differences in the model structure as a whole, while the MSV-learning remains very close to the MS-REE model. In both cases, however, there is a sizable increase in expected duration of the ZLB episode, which brings the persistence of this regime closer to the empirically relevant region.

Table 3: Estimation period: 1966:I-2016:IV

Prior			AR(1)	MSV	VAR(1)	REE	MS-REE
Param	Dist	Mean	Mode	Mode	Mode	Mode	Mode
ϕ	Normal	4	1.02	3.75	4.88	4.87	6.4
σ_c	Normal	1.5	0.88	1.48	0.82	1.36	1.14
λ	Beta	0.7	0.55	0.73	0.88	0.75	0.83
ξ_w	Beta	0.5	0.7	0.75	0.71	0.93	0.95
σ_l	Normal	2	2.41	2.18	2.09	1.98	1.74
ξ_p	Beta	0.5	0.62	0.67	0.64	0.8	0.83
ι_w	Beta	0.5	0.43	0.63	0.48	0.84	0.81
ι_p	Beta	0.5	0.40	0.23	0.42	0.07	0.08
ψ	Beta	0.5	0.49	0.69	0.72	0.83	0.69
ϕ_p	Normal	1.25	1.49	1.57	1.52	1.59	1.56
r_π	Normal	1.25	1.64	1.62	1.65	1.5	1.35
ρ	Beta	0.75	0.88	0.88	0.88	0.85	0.86
r_y	Normal	0.125	0.13	0.12	0.12	0.05	0.06
r_{dy}	Normal	0.125	0.14	0.14	0.13	0.17	0.19
$\bar{\pi}$	Gamma	0.625	0.71	0.86	0.88	0.75	0.76
$\bar{\beta}$	Gamma	0.25	0.14	0.19	0.37	0.21	0.25
$\bar{l}$	Normal	0	0.73	1.59	2.39	-1.2	0.15
$\bar{\gamma}$	Normal	0.4	0.4	0.43	0.39	0.4	0.41
α	Normal	0.3	0.15	0.16	0.17	0.17	0.18
ρ_a	Beta	0.5	0.98	0.95	0.99	0.96	0.95
ρ_b	Beta	0.5	0.35	0.42	0.21	0.36	0.29
ρ_g	Beta	0.5	0.99	0.99	0.99	0.98	0.98
ρ_i	Beta	0.5	0.48	0.83	0.52	0.83	0.76
ρ_r	Beta	0.5	0.15	0.07	0.09	0.08	0.16
ρ_p	Beta	0.5	0.03	0.64	0.03	0.81	0.78
ρ_w	Beta	0.5	0.04	0.19	0.06	0.06	0.05
ρ_{ga}	Beta	0.5	0.52	0.5	0.49	0.53	0.51
η_a	Inv. Gamma	0.1	0.46	0.43	0.44	0.44	0.45
η_b	Inv. Gamma	0.1	0.68	0.21	0.63	0.21	0.23
η_g	Inv. Gamma	0.1	0.48	0.48	0.48	0.49	0.48
η_i	Inv. Gamma	0.1	1.36	0.36	1.53	0.36	0.34
$\eta_{r,N}$	Inv. Gamma	0.1	0.21	0.21	0.21	0.21	0.23
$\eta_{r,ZLB}$	Gamma	0.03	0.01	0.01	0.01	-	0.01
η_p	Inv. Gamma	0.1	0.27	0.07	0.26	0.05	0.06
η_w	Inv. Gamma	0.1	0.73	0.35	0.74	0.37	0.37
$gain$	Gamma	0.035	0.012	0.001	0.016	-	-
$1-p_{11}$	Beta	0.1	0.02	0.02	0.2	-	0.01
$1-p_{22}$	Beta	0.1	0.13	0.13	0.13	-	0.29
$\bar{r_{zlb}}$	Normal	0.05	0.03	0.03	0.03	-	0.03
Laplace			-1135	-1168	-1141	-1213	-1171
Bayes' Factor			33.88	19.54	31.27	1	18.24

We next turn to the estimated regime probabilities and activity of projection facilities over the estimation sample for the learning models, which is shown in Figure 10: similar to the 3-equation model, in both AR(1)- and MSV-learning cases, we observe a sharp switch to the ZLB regime in the beginning of 2009 where the interest rates drop to near-zero levels, while the economy exits from the ZLB regime towards the end of 2016. The same pattern is also observed in the MS-REE model, which is omitted here. An important difference between the SW model here and the 3-equation model is the activity of projection facility, which was irrelevant in the small model because it remains stable throughout the estimation sample and as a consequence, the projection facility is never imposed. In the larger SW-model, however, the model is occasionally driven into unstable regions, thereby activating the projection facility. This is shown in the second row of the same figure. The right panel shows the regime-specific and weighted-average (based on the ergodic distribution) eigenvalues along with the activity of projection facility for the MSV-learning case. It is readily seen that, although the ZLB regime is closer to the instability region compared to the normal regime, neither regime becomes unstable in this case and the projection facility is not activated. This is partially due to the fact that the information set under MSV-learning does not affect the autocorrelation structure as much; and partially due to the small gain coefficient under this specification. The left panel shows the same figures for the AR(1)-learning case: in this case, we observe that projection facility is activated twice: the first time is during the beginning of 2000s, and the second time is after the crisis period in 2009. An important observation is that, the ZLB-regime is unstable for extended periods of time, especially during the period from 1990 to 2010. For this case, *how* the projection facility is imposed plays a key role in the system dynamics: recall from the previous section that we only impose the projection facility when the underlying ergodic distribution becomes unstable, while the regime-specific models are allowed to be unstable. In this case, although the ZLB-regime is unstable for long periods, the normal regime is sufficiently stable most of the time such that the projection facility is only activated a couple of times. This way of imposing the projection facility is justified on the grounds of our RPE concept, where the regime-specific models can become temporarily explosive as long as the underlying ergodic distribution is stable; we are able to keep the activity of projection facility to a minimum based on this notion.

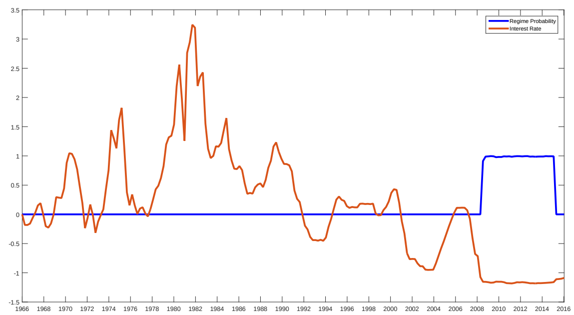
Next we examine how the learning coefficients evolve over the estimation sample, which are shown in Figures 11 and 12. In the MSV-learning case, there are 15 variables in the information set of the agent, with the intercept, 7 exogenous shocks and 7 endogenous backward-looking variables. Therefore we only provide the plots and discussion of two of these variables for exposition, but similar arguments can be extended to all variables in the information set. Figure 11 shows the intercept coefficients under both learning cases: the first panel shows the AR(1) while the second one is MSV-learning. As one might expect, the time-variation under MSV-learning is relatively small, where all coefficients remain close to zero and there are no observable jumps. This confirms our conclusion from the previous section that, when shocks are assumed to be observable, the intercept terms receive much less weight in forming expectations. On the contrary with the AR(1) case, we observe much more interesting dynamics: since shocks are unobserved in this case, the intercept terms play an important role in agents' expectations. A particularly interesting case is the financial crisis period, where we observe an almost uniform reduction in all coefficients. Particularly the coefficients on investment and real value of capital (denoted by i and q) respectively) decrease by a relatively large amount, while remaining coefficients decrease moderately. This implies that, under AR(1)-learning with unobserved shocks, the crisis and subsequent *Great Recession* period is interpreted as a level shift in the variables by agents, as opposed to a series of adverse shocks. Figure 12 shows the learning coefficients on persistence for the AR(1) case, and lagged inflation for the MSV-learning case. The same results applies to the MSV-learning again, where we do not observe any meaningful time-variation in the coefficients, especially during the crisis period. While we omit the remaining coefficients for the MSV case on lagged variables and exogenous shocks, similar patterns can be observed for these cases as well. Next looking at the AR(1) case which is shown in the first panel, we can again observe a jump in the coefficients during the crisis period, where some of the persistence parameters temporarily go above one. In other words, when the crisis hits and the agents observe the large downward movement in the endogenous variables, they compensate for this by temporarily switching from a trend-following rule to an extrapolative one. This can be interpreted as a temporary wave of pessimism in the economy, where the agents expect more downward movement following the adverse shock after the crisis.

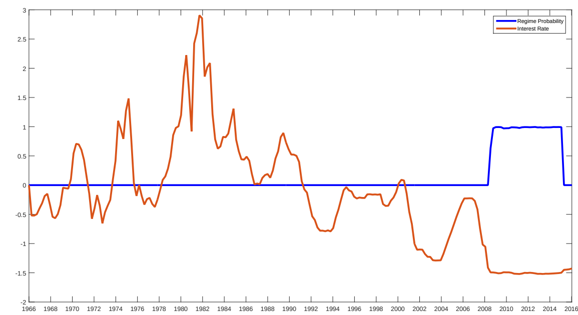
A well-known problem in the VAR literature is that the standard models tend to over-predict the growth rates in the post-crisis period of 2007-08: after the crisis period, there has been a reduction in the growth rates of real in the U.S, particularly for output, consumption and investment. As a consequence, the standard models tend to over-predict these variables if no additional structural break is introduced into the model. This applies to models that both take into account the zero lower bound episode and those that do not. The first row of Figure 13 shows the forecast errors of both REE and MS-REE models. It is readily seen that in both cases, the forecasts of the growth rates are indeed consistently larger than the actual values. Lindé et al. (2017) propose to deal with this issue in the SW model by introducing a structural break in the risk-premium shock. Accordingly, it is assumed that there is a permanent increase in the risk premium after the crisis (i.e. a level shift in the b-shock), which pushes down the forecasts of the real growth rates. We plot the forecasts under adaptive learning in the second row of Figure 13. It is readily seen that the issue of over-predicting does not arise under adaptive learning. This suggests that time-variation under adaptive learning causes as strong enough downward shift in the forecasts. Accordingly, time-varying beliefs and the potential pessimism of agents following the crisis arise as an alternative way to deal with the over-prediction problem over this period[11].

[11] This result raises the question of whether the level shift expectations would still be prevalent if we also add a structural break in the b-shock, i.e. is it the shift in the risk premium or agents' beliefs that dominates. We leave this question to future work.

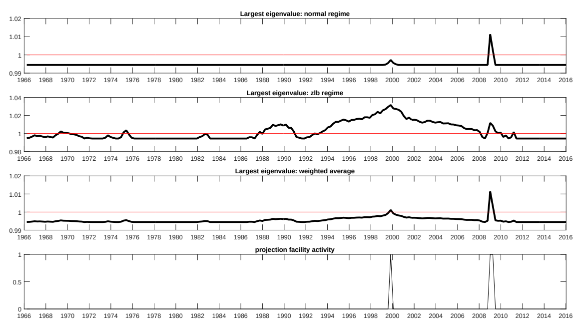
Figure 10: Estimated regime probabilities and projection facility in AR(1)- and MSV-learning cases.

Filtered ZLB regime probability:





Eigenvalues and Projection Facilities:



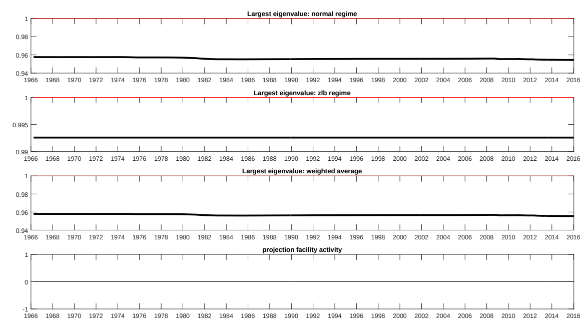


Figure 11: Learning coefficients in the AR(1)- and MSV-learning cases: Intercepts.

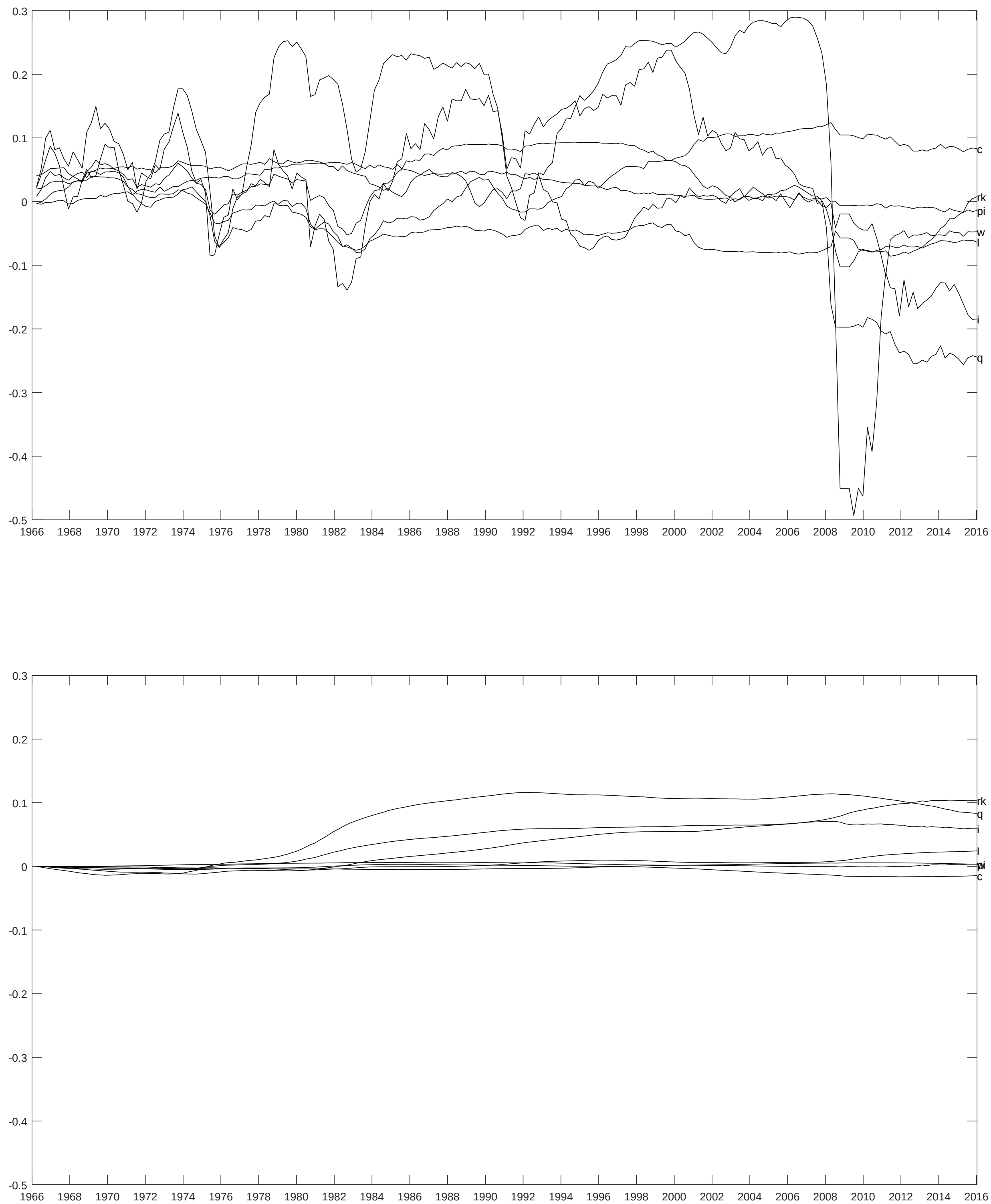
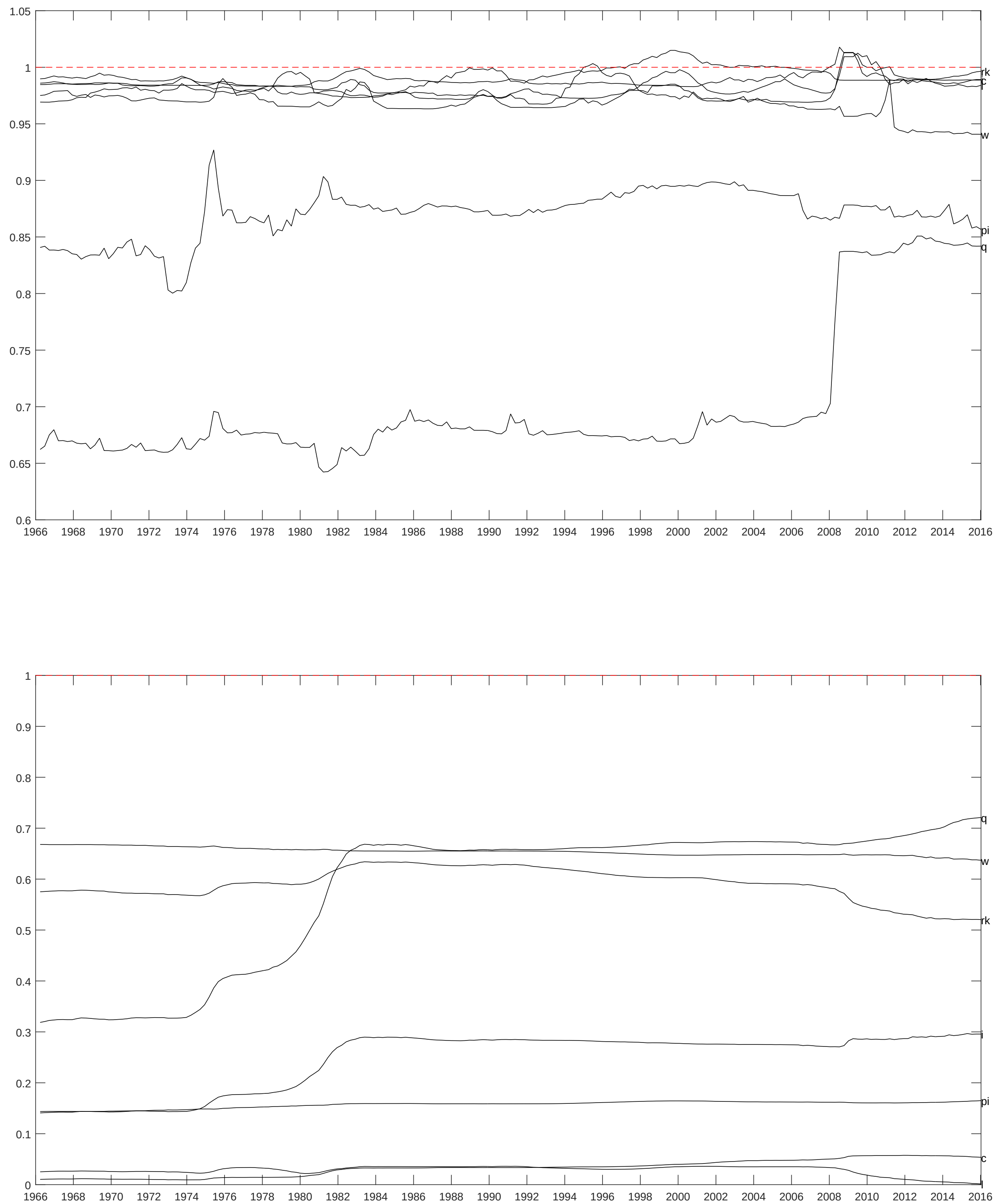


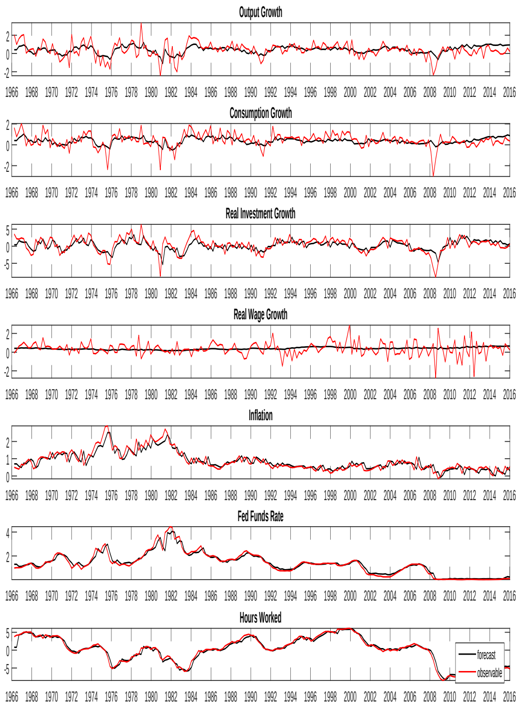
Figure 12: Learning coefficients in the AR(1)- and MSV-learning cases: Persistence coefficients under AR(1)-learning, and lagged inflation under MSV-learning.

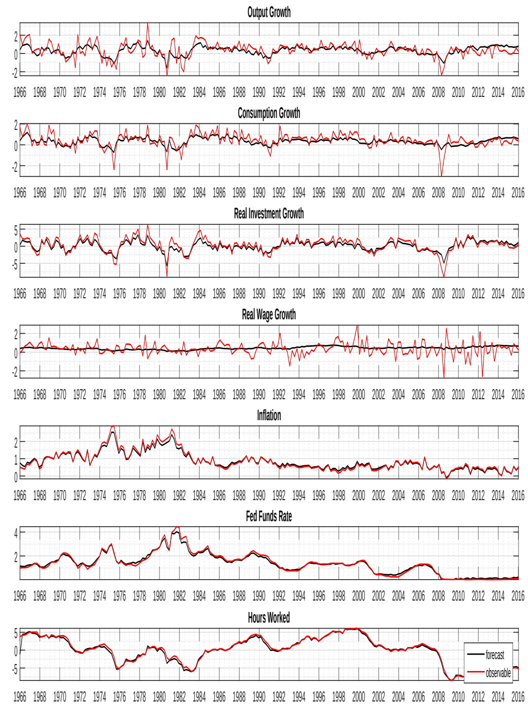


6.1 Forecast Errors

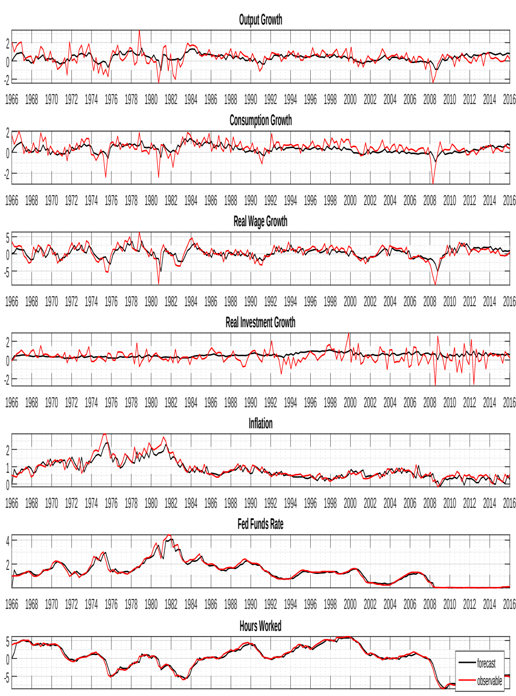
Figure 13: In-sample forecasts (i.e. the forecasting step of the filter) for all observables.

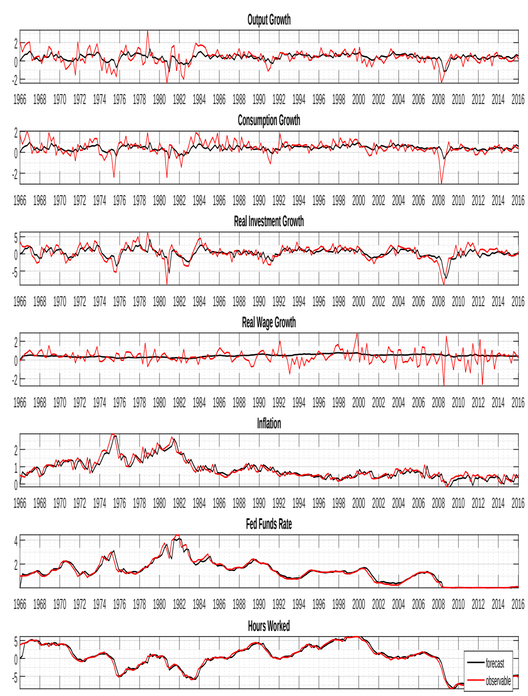
MS-REE and REE:





MSV- and AR(1)-learning:





We next compare the impulse responses under both AR(1)- and MSV-learning to MS-REE case. As we mention in the previous section, the impulse responses under adaptive learning are different each period since belief coefficient are updated each period, which feeds back into the autocovariance structure. We therefore start by comparing the impulse responses based on two representative periods, 2006Q1 for the normal regime before the crisis, and 2011Q1 for the ZLB regime after the crisis. We focus particularly on two key variables: output and inflation.

The first observation is that in general, regardless of the model specification, a shock of the same size tends to have a larger impact under the ZLB regime compared with the normal one. This is an intuitive results since nominal interest rates do not adjust at the ZLB regime to mitigate the effects of a given shock, which leads to larger effects. This applies in particular to two important shocks, i.e. risk premium b and government spending g which proxy for the impacts of a financial crisis and expansionary monetary policy respectively. First examining the b-shock on output under MSV-learning, we observe that the impact is slightly larger under the normal regime with learning compared to the REE model. This picture is reversed when we look at the ZLB regime and the impact becomes smaller under learning. Given that the only difference between MSV-learning and REE model in this case is the unobserved regime shift, the feedback of the regime shift through expectations is smaller under learning. As a consequence, we observe a smaller jump in the impulse responses under learning compared with the REE. Interestingly, however, the impact of the same shock on inflation is substantially larger under learning. This result arises due to the smaller price stickiness under learning, which leads to a steeper NKPC and a larger feedback from the real side of the economy on inflation. When we examine the same impulse responses under AR(1)-learning, we observe that they take much longer to reach their peak values, and also take substantially longer to die out: this result is due to the additional inertia and persistence introduced into the system by the backward-looking nature of the PLM. Accordingly, the shocks take longer to reach their maturity, as well as longer to die out. One common result that emerges under both learning approaches, however, is that an adverse b-shock during the ZLB episode is going to have a larger and longer-lasting deflationary effect on the economy, suggesting the REE model underestimates this channel.

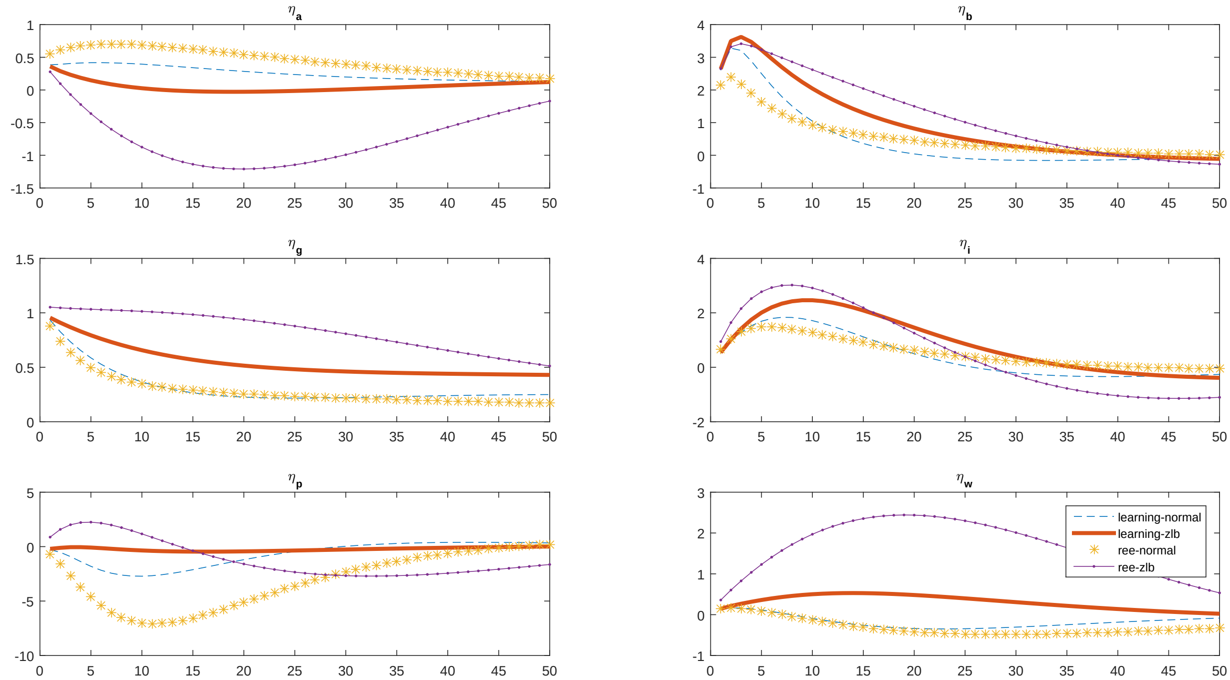
Next we focus on the effects of a government spending g-shock. Again first looking at the case of MSV-learning, we observe similar effects under the normal regime on output compared with REE. While the impact is larger under the ZLB regime for both cases, the jump is much smaller under learning. Similar to the b-shock, this smaller difference under learning is explained by the fact that inactive monetary policy under learning does not feedback through expectations. Since the regime shift is observed at the REE model, the shift is taken into account by expectations, leading to a larger effect. Despite the smaller jump under learning on output, we can again observe a larger inflationary effect under learning due to lower price stickiness. This indicates that the lower price stickiness under learning dominates the effect of unobserved regime shifts, thereby generating a larger inflationary impact. When we consider the same impulse responses under AR(1)-learning, the impacts are substantially larger on both output and inflation. The larger effect on output in this case is driven by the fact that, when shocks are not observed, the fiscal expansion starts to *crowd in* government spending instead of crowding out. Accordingly, a fiscal expansion leads to larger effects under both normal and ZLB regimes with AR(1)-learning. This result is consistent with previous studies that examined government spending multipliers under adaptive learning, see e.g. Quaghebeur (2018). It is also worth noting that with AR(1)-learning, the g-shock takes more than 50 quarters to reach its maturity under the ZLB regime, which almost resembles a permanent shock at the usual business cycle frequency; this shortcoming is typically observed in a variety of backward-looking expectation rules as pointed out in Gaus & Gibbs (2018).

Aside from the risk premium and government spending shocks, another interesting result under learning is that the price and and mark-up shocks play a much smaller role for both output and inflation dynamics. Particularly under AR(1)-learning, the impacts of both mark-up shocks are much smaller in both the normal and ZLB regimes. This essentially implies the role played by the ad-hoc and exogenous mark-up shocks can be shifted towards expectations as already observed in Slobodyan & Wouters (2012b); in this paper we simply show that this result also extends to the ZLB episode. Finally for the technology shock a, there is a sign reversal with the regime switch, where a positive a-shock is expansionary and contractionary in the normal and ZLB regimes respectively, while remaining deflationary in both regimes. Similar to the government spending shock, we observe a smaller jump under the MSV-learning specification, while the AR(1)-learning delivers a larger jump between the regimes.

As a final note on the impulse responses under learning, we examine that the model-implied time variation in the responses over the estimation period, which is provided in Figures 16 and 17 for MSV- and AR(1)-cases respectively. The general pattern in MSV-learning shows very small variations as one might expect, since the estimated gain coefficient is very close to zero. Therefore in this case, all time variation is essentially dwarfed by the jump after the switch to the ZLB episode. AR(1)-learning case clearly shows more variation over the estimation sample. Focusing on the ZLB period, the overall picture indicates that the biggest jump in the impulse responses happens at the initial periods of the ZLB episode, after which the impulses gradually become smaller until the economy switches back to the normal regime, which leads to another jump towards the end of the sample. Importantly however, we do not see any sign reversals in either learning specification conditional on either regime, which confirms that our discussion above with two representative periods before and after the crisis is indeed applicable to other periods in general.

Figure 14: Comparison of MSV learning IRFs with MSV-RE IRFs. One unit shocks of η_a,η_b,η_g,η_i,η_p,η_w respectively.

Output:



Inflation:

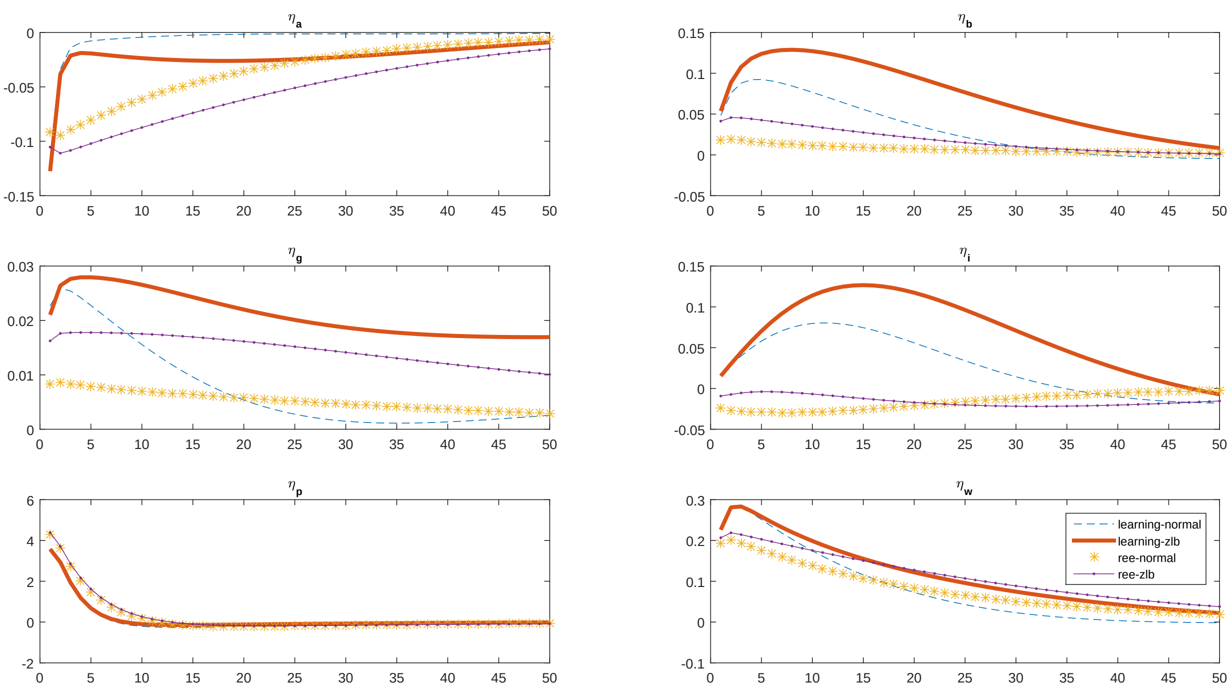
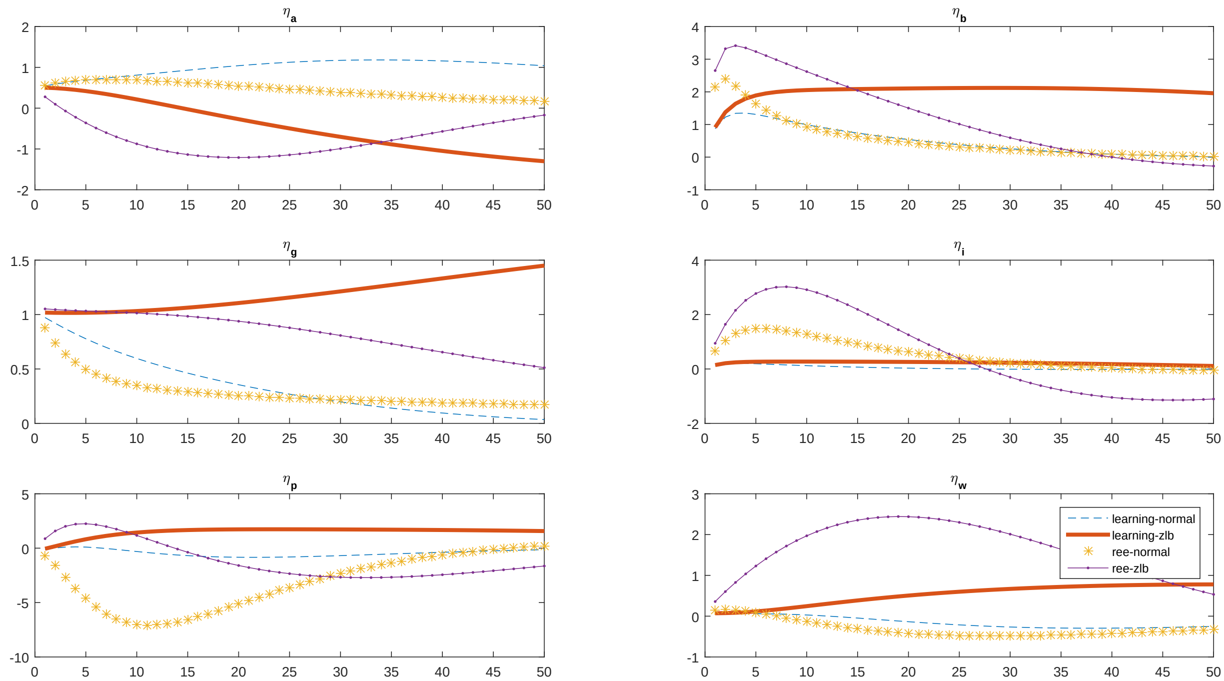


Figure 15: Comparison of AR(1) learning IRFs with MSV-RE IRFs. One unit shocks of η_a,η_b,η_g,η_i,η_p,η_w respectively.

Output:



Inflation:

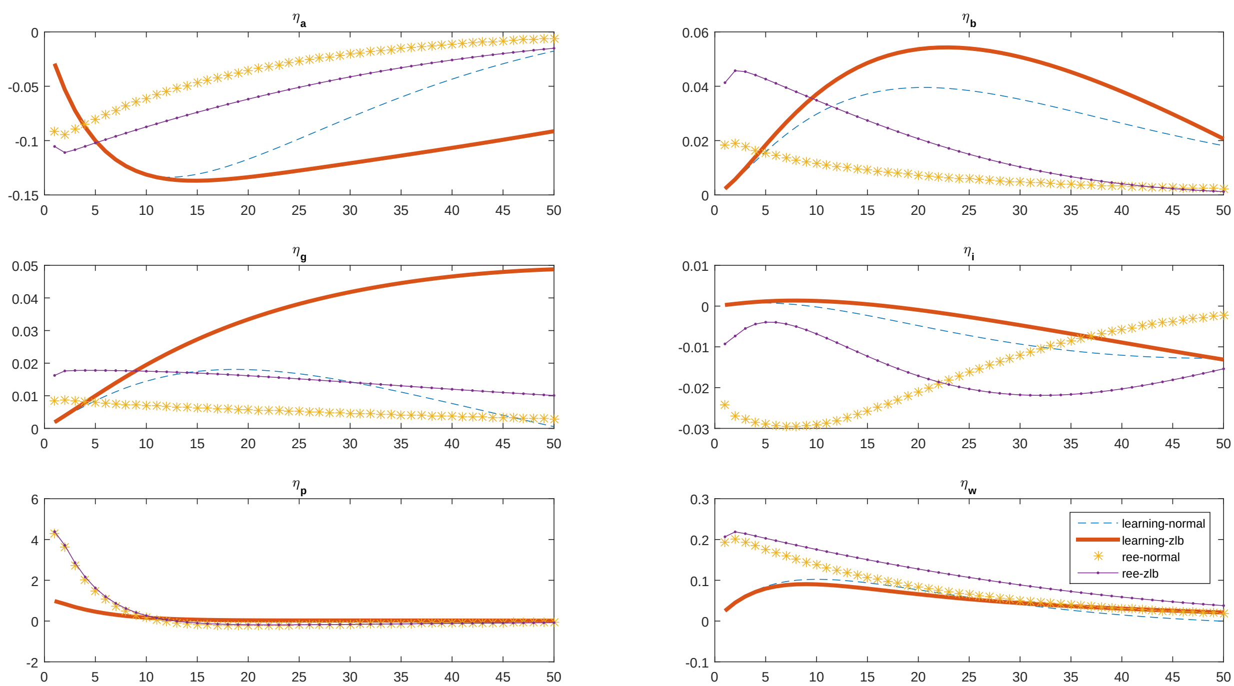
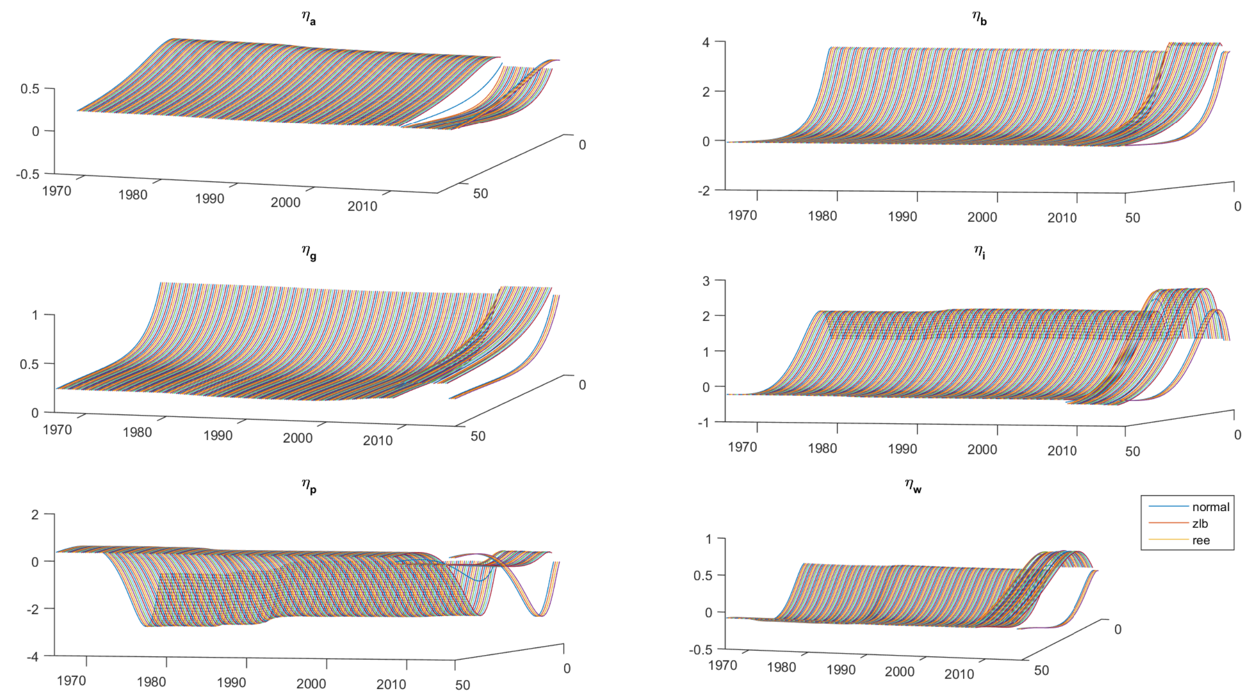


Figure 16: Time variation over the estimation period in the impulse responses in the MSV-learning case.

Output:



Inflation:

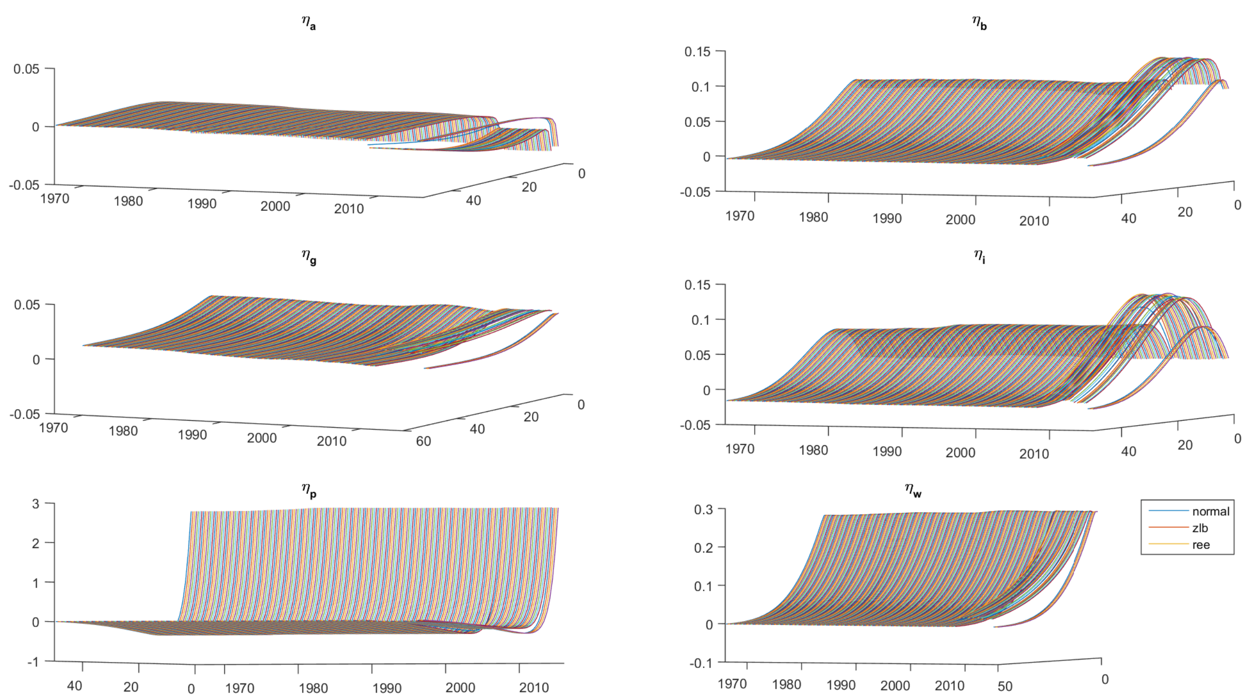
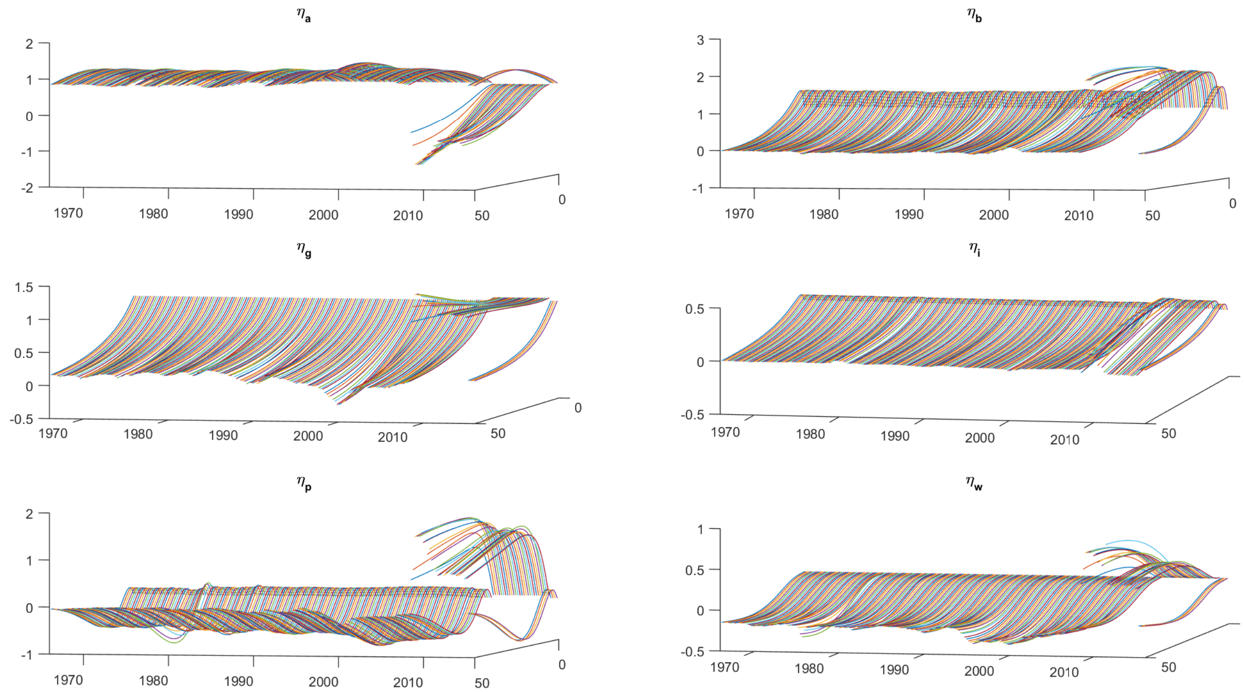
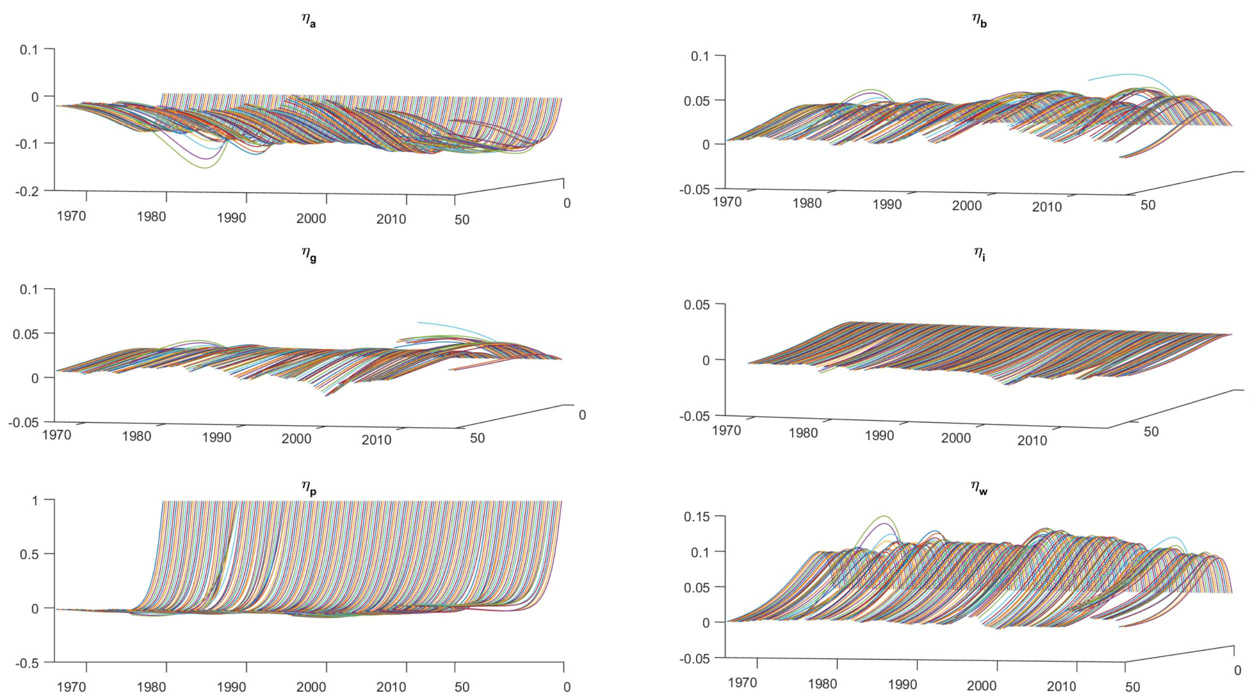


Figure 17: Time variation over the estimation period in the impulse responses in the AR(1)-learning case.

Output:



Inflation:



7 Conclusions

The literature on macroeconomics made great strides in the estimation of both Markov-switching and adaptive learning models in recent years, which are two alternative ways of introducing time-variation into DSGE models. Although there are numerous examples of both classes of models that have been successfully taken to the data, there is surprisingly little work on Markov-switching DSGE models under adaptive learning. In this paper, we provided a first attempt to estimate this class of DSGE models by combining these two approaches under the same roof. The resulting framework has a nice intuitive interpretation where the agents' do not know the details of a complex non-linear economy, but use simple linear rules to form their expectations about the future state of the economy. Our simulations show that, using simple adaptive rules allows the agents' to take into account the structural changes in the economy, albeit in an indirect manner. More importantly, our estimations indicate that the two approaches can be complementary rather than substitutes: it both the 3-equation NKPC and Smets-Wouters models we consider, imposing the Markov-switching structure on standard models improves the empirical fit, while imposing adaptive learning leads to further improvements. Furthermore, our results on the ZLB episode suggest that a wave of pessimism, modeled as a downward shift of expectations, may well have contributed to the Great Recession period characterized by low growth rates, which is a well-established idea in literature as an expectations-driven business cycle. Our paper can be extended in many directions: the key role played by expectations in our paper is an inevitable result since it is the only possible factor that can drive the low growth rates. Therefore a particularly important question is whether expectations can still play a key role in driving the Great Recession, when there are other types of structural breaks during the ZLB episode, such a break in the risk premium shock as in Lindé et al. (2017). Another important issue is the macroeconomic cost of the ZLB episode under adaptive learning: since the estimation results indicate different structural parameters and impulse responses in general, it is also plausible that the model-implied cost of ZLB is different under learning compared with REE. A final interesting extension is to study the interaction between expectations and monetary policy when there are more subtle regime switches in the monetary policy regime, such as the Great Moderation period of 1980s which is typically associated with a switch to more aggressive monetary policy. We these issues to future work.

Appendix

A Special case with 2 regimes, no lagged variables

Note that in the special case with $\iota_p = 0$, the model can be written as a generic 1-dimensional Markov-switching model of the form:

$$\begin{cases} \pi_t = a(s_t) r_t + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

where $a(s_t) = \frac{1}{\alpha(s_t)}(a\rho + 1)$. The moments necessary for the T-map are given as follows:

$$E[\pi_t r_t] = P_1 E[\pi_t r_t | S_t = 1] + P_2 E[\pi_t r_t | S_t = 2]$$

$$E[\pi_t r_t | S_t = 1] = E[a(s_t) r_t^2 | S_t = 1, S_{t-1} = 1] p_{11} + E[a(s_t) r_t^2 | S_t = 1, S_{t-1} = 2](1 - p_{22}) \frac{P_2}{P_1}$$

$$= a_1 p_{11} + a_1 (1 - p_{22}) \frac{P_2}{P_1}$$

Similarly, we have

$$E[\pi_t r_t | S_t = 1] = a_2 p_{22} + a_2 (1 - p_{11}) \frac{P_1}{P_2}$$

which yields

$$E[\pi_t r_t] = P_1(a_1 p_{11} + a_1(1-p_{22})\frac{P_2}{P_1}) + P_2(a_2 p_{22} + a_2(1-p_{11})\frac{P_1}{P_2})$$

Plugging in the steady-state probabilities P_1 and P_2, the T-map is given as follows:

$$a \to T(a) = \frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}-p_{22})}(a\rho+1)$$

with the E-stability condition

$$DT_a = \frac{\alpha_1(1-p_{22}) + \alpha_2(1-p_{11})}{\alpha_1\alpha_2(2-p_{11}-p_{22})}\rho < 1$$

Re-arranging the expression above yields the LRES condition presented in Section 1. Further note that the regime-specific T-maps, and the associated regime-specific E-stability conditions are given by:

$$a \to \frac{a\rho+1}{\alpha_i}$$

$$DT_a = \frac{\rho}{\alpha_i} < 1$$

which implies that *regime specific E-stability is a sufficient, but not necessary condition* for LRES.

B 1-dimensional case with m regimes

Note that the Fisherian model considered in Section 2 can be written as a generic 1-dimensional Markov-switching model of the form:

$$\begin{cases} \pi_t = a(s_t)r_t + b(s_t)\pi_{t-1} + u_t \\ r_t = \rho r_{t-1} + v_t \end{cases}$$

where $a(s_t) = \frac{1}{\alpha(s_t)}(a\rho + ba + 1)$ and $b(s_t) = \frac{b^2+\iota_p}{\alpha(s_t)}$. In this Appendix we consider the general case with m regimes, with transition matrix given by:

$$Q = \begin{bmatrix} p_{11} & \dots & p_{1m} \\ \vdots & \dots & \vdots \\ p_{m1} & \dots & p_{mm} \end{bmatrix}$$

The 2-regime setup of Section is the special case with $m = 2$. We omit the first moment $E[\pi_t]$, which is trivially given as zero. Using this, we compute the second moments starting with the conditional variance. We have:

$$E[\pi_t^2] = \sum_{i=1}^{m} P_i E[\pi_t^2 | S_t = i]$$

$$E[\pi_t^2 | S_t = i] \sum_{i=1}^{m} E[\pi_t^2 | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

where P_i denotes the i^{th} element of the steady-state vector of the Markov chain.

$$= \sum_{j=1}^{m} E[a(s_t)^2 r_t^2 + b(s_t)^2 \pi_{t-1}^2 + u_t^2 + 2b(s_t)a(s_t) r_t \pi_{t-1} | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} E[a_i^2\sigma_r^2 + b_i^2\pi_{t-1}^2 + \sigma_r^2 + 2b_1 a_i r_t \pi_{t-1} | S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

Note that this last expression implies m equations in m unknowns for the conditional variances, *given* the conditional covariances $E[\pi_t r_t | S_t = j]$. Using this, the unconditional variance is given by:

$$E[\pi_t^2] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (a_i^2\sigma_r^2 + b_i^2 E[\pi_{t-1}^2 | S_t = j] + \sigma_r^2 + 2b_1 a_i r_t E[\pi_{t-1} | S_{t-1} = j]) p_{ji} \frac{P_j}{P_i}$$

Next we move onto the covariance term $E[\pi_t r_t]$:

$$E[\pi_t r_t] = \sum_{i=1}^{m} P_i E[\pi_t r_t | S_t = i]$$

$$E[\pi_t r_t | S_t = i] = \sum_{j=1}^{m} E[\pi_t r_t | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} E[b(s_t)\pi_{t-1} r_t + a(s_t) r_t^2 | S_t = i, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$\sum_{i=1}^{m} (b_i \rho E[\pi_t r_t | S_t = j] + a_i \sigma_r^2) p_{ji} \frac{P_j}{P_i}$$

Note that again, the last expression implies m equations in m unknowns for the conditional covariances. Using this, the unconditional covariance is given by:

$$E[\pi_t r_t] = \sum_{i=1} m P_i \sum_{j=1}^{m} (b_i \rho E[\pi_t r_t | S_t = j] + a_i \sigma_r^2) p_{ji} \frac{P_j}{P_i}$$

Next we compute the first-order autocovariance:

$$E[\pi_t \pi_{t-1}] = \sum_{i=1}^{m} P_i E[\pi_t \pi_{t-1} | S_t = i]$$

$$E[\pi_t \pi_{t-1} | S_t = i] = \sum_{j=1}^{m} E[b(s_t)\pi_{t-1}^2 + a(s_t)\pi_{t-1} r_t | S_t = 1, S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} (b_i E[\pi_t^2 | S_t = j] + a_i \rho E[\pi_t r_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Given the conditional covariance and conditional variance terms, the above expression yields the conditional autocovariances. Hence the unconditional autocovariance is given as:

$$E[\pi_t \pi_{t-1}] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (b_i E[\pi_t^2 | S_t = j] + a_i \rho E[\pi_t r_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Finally note that:

$$E[a(s_t)\pi_{t-1}r_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} a_i \rho E[\pi_t r_t | S_t = j] p_{ji} \frac{P_j}{P_i}$$

and

$$E[b(s_t)\pi_{t-1}r_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} b_i \rho E[\pi_t r_t | S_t = j] p_{ji} \frac{P_j}{P_i}$$

Recalling the T-map $\binom{a}{b} \to T(a,b) = \begin{pmatrix} E[(\pi_t - b\pi_{t-1})r_t] \\ \frac{E[(\pi_t - ar_t)\pi_{t-1}]}{E[\pi_t^2]} \end{pmatrix}$, the above conditions fully pin down $T(a,b)$. It is generally not possible to obtain analytical expressions for this mapping, and therefore the RPE values a^{RPE} amd b^{RPE}. Therefore our results in Section 1 are computed numerically for given values of parameters.

C N dimensional case with m regimes

Note that, after plugging in the PLM into ALM, the model considered in Section can be re-written as a generic Markov-switching model of the form:

$$\begin{cases} X_t = a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t \\ \epsilon_t = \rho\epsilon_{t-1} + \eta_t \end{cases}$$

where $a(s_t) = A(s_t) + C(s_t)(a + ba)$, $b(s_t) = B(s_t) + C(s_t)b^2$ and $d(s_t) = C(s_t)(bd + d\rho) + D(s_t)$. We need the first and second moments of this system in order to compute the the resulting T-map for the RPE. Starting with the first moment, we have:

$$E[X_t] = \sum_{i=1}^{m} P_i E[X_t | S_t = i]$$

$$E[X_t | S_t = i] = \sum_{j=1}^{m} [a_i + b_i X_{t-1} + d_i \epsilon_t | S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} (a_i + b_i E[X_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

The expression above implies m equations in m unknowns for the conditions means. Using this yields:

$$E[X_t] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (a_i + b_i E[X_t | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Moving onto the second moments and starting with the covariance term, we have:

$$E[X_t \epsilon_t'] = \sum_{i=1}^{m} P_i E[X_t \epsilon_t' | S_t = i]$$

$$E[X_t \epsilon_t' | S_t = i] E[)a(s_t) + b(s_t)X_{t-1} + d(s_t)\epsilon_t)\epsilon_t' | S_t = i]$$

$$= \sum_{j=1}^{m} E[(a_i + b_i X_{t-1} + d_i \epsilon_t)\epsilon_t' | S_{t-1} = j] p_{ji} \frac{P_j}{P_i}$$

$$= \sum_{j=1}^{m} (b_i \rho E[X_t \epsilon_t' | S_t = j] + d_i \Sigma_\epsilon) p_{ji} \frac{P_j}{P_i}$$

The last expression again implies m equations in m unknowns for the conditional covariances. The unconditional covariance is then given by:

$$E[X_t \epsilon_t'] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (b_i \rho E[X_t \epsilon_t' | S_t = j] + d_i \Sigma_\epsilon) p_{ji} \frac{P_j}{P_i}$$

Next we compute:

$$E[X_t X_t'] = \sum_{i=1}^{m} P_i E[X_t X_t' | S_t = i]$$

$$E[X_t X_t' | S_t = i] = E[a(s_t)a(s_t)' + 2a(s_t)X_{t-1}'b(s_t)' + 2a(s_t)\epsilon_t d(s_t)' +$$
$$b(s_t)X_{t-1}X_{t-1}'b(s_t)' + 2b(s_t)X_{t-1}\epsilon_t d(s_t)' + d(s_t)\epsilon_t \epsilon_t' d(s_t)' | S_t = i]$$

$$= \sum_{j=1}^{m} E[a_i a_i' 2a_i X_{t-1}' b_i' + 2a_i \epsilon_t' d_i' + b_i X_{t-1} X_{t-1}' b_i' + 2b_i X_{t-1} \epsilon_t' d_i' + d_i \epsilon_t \epsilon_t' d_i' | S_t = j] p_{ji} \frac{P_j}{P_i}$$

Given the conditional means and covariances, the last expressions implies m equations in m unknowns for the conditional moments $E[X_t X_t' | S_t = i]$. The unconditional moment is then given by:

$$E[X_t X_t'] = \sum_{i=1}^{m} P_i \sum_{j=1}^{m} (a_i a_i' + 2a_i E[X_t' | S_t = j] b_i' + b_i E[X_t X_t' | S_t = j] b_i' + 2b_i E[X_t \epsilon_t' | S_t = j] \rho' d_i' + d_i \Sigma_\epsilon d_i') p_{ji} \frac{P_j}{P_i}$$

Finally we compute the autocovariance term:

$$E[X_t X_{t-1}'] = \sum_{i=1}^{m} P_i E[X_t X_{t-1}' | S_t = i]$$

$$E[X_t X_{t-1}' | S_t = i] = E[a_i X_{t-1}' + b_i X_{t-1} X_{t-1}' + d_i \rho \epsilon_{t-1} X_{t-1}' | S_t = i] =$$
$$\sum_{j=1}^{m} (a_i E[X_t | S_t = j] + b_i E[X_t X_t' | S_t = j] + d_i \rho E[\epsilon_t X_t' | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

The last expression is pinned by the conditional first and second moments computed above. The unconditional autocovariance is then given as:

$$E[X_t X_t'] = \sum_{i=1}^{m} \sum_{j=1}^{m} (a_i E[X_t | S_t = j] + b_i E[X_t X_t' | S_t = j] + d_i \rho E[\epsilon_t X_t' | S_t = j]) p_{ji} \frac{P_j}{P_i}$$

Recall that the T-map is given by:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} E[X_t - bX_{t-1} - d\epsilon_t] \\ E[(X_t - a - d\epsilon_t)X_{t-1}')E[X_t X_t']^{-1}] \\ E[(X_t - a - bX_{t-1})\epsilon_t'] E[\epsilon_t \epsilon_t']^{-1} \end{pmatrix}$$

Hence, given the first and second moments computed above, the T-maps for a, b and c

are pinned down. Similar to 1-dimensional case, it is generally not possible to find analytical expressions for these matrices. Further note that, the T-map for $b \to T(a,b,c)$ involves a 4^{th} order matrix polynomial of dimension N. This means there can be up to $\binom{4N}{N}$ for b. To our knowledge, there is no straightforward and general method to compute the full set of solutions to this problem. In this paper, we do not compute these fixed-points and rely on Monte Carlo simulations when necessary.

Further note that the regime-specific T-maps are given by:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} A_i + C_i(a+ba) \\ B_i + C_i b^2 \\ C_i(bd + d\rho) + D_i \end{pmatrix}$$

These simply correspond to the standard MSV solutions given the regime-specific matrices. Computing the fixed-points yield the regime-specific equilibria as follows:

$$\begin{cases} a^{R_i} = (I - C_i - C_i b^{R_i})^{-1} A_i \\ vec(D^{R_i}) = (I - (I \otimes (C_i b^{R_i})))vec(d) + (\rho \otimes C_i)vec(d) + vec(D_i) \end{cases}$$

which yields the regime-specific values for a^{R_i} and d^{R_i} respectively, for a given matrix b^{R_i}. The second-order polynomial for b^{R_i} can be solved using standard toolboxes such as Adjemian et al. (2011) and Uhlig et al. (1995), which then completely pins down the regime-specific MSV. Denoting $\theta = (a,b,d)'$, the associated Jacobian for E-stability condition is given by:

$$\frac{DT}{D\theta} = \begin{bmatrix} C_i + C_i b & vec^{-1}_{n,n}(a' \otimes C_i) & 0 \\ 0 & 2C_i b & 0 \\ 0 & vec^{-1}_{n,n}(d' \otimes C_i) & C_i b + vec^{-1}_{n,n}(\rho' \otimes C_i) \end{bmatrix}$$

where $vec^{-1}_{n,n}$ denotes the matricization of a vector to an (n,n) matrix.

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