2 Representation of Integer

2.0 Why binary?

Scales: **Binary**（二进制）, **Octal**（八进制）, **Decimal**（十进制）, **Hexadecimal**（十六进制）.

Given a number N, we can use R-scale to represent it, that will cost $\log_R N$ digits. The basic number of R-scale $(0, 1, \dots, R-1)$ should be represent in different ways, so each digit should cost R space, then the total space cost D can be represented as:

$$D = R \log_R N = \ln N \frac{R}{\ln R}$$
$$\frac{\partial D}{\partial R} = \ln N \cdot \frac{\ln R - 1}{(\ln R)^2}$$

When $R = e = 2.71828...$, $\frac{\partial D}{\partial R} = 0$, which means D reaches its minimum.

Since $R \in \mathbb{N}$, we can get the optimal solution: $R = 2$ or $R = 3$.

Binary is more convenient for circuit design than ternary, so we choose binary to represent data in computer.

2.1 Encoding of Integer

$$m = X_0 X_1 X_2 \dots X_n$$

- Signed magnitude（原码）
  - X_0 represent the sign of the integer m (0: *non-negative*, 1: *non-positive*).
  - $X_1 X_2 \dots X_n$ is the binary form of $|m|$.
  - As a result, the number 0 has two codes $100...00$ or $000...00$.
  - Not convenient for calculation.
- One's complement（反码）
  - X_0 represent the sign of the integer m (0: *non-negative*, 1: *non-positive*).
  - $X_1 X_2 \dots X_n$ is the binary form of the number m if $X_0 = 0$;
  - $X_1 X_2 \dots X_n$ is the bitwise NOT of the binary form of the number $-m$ if $X_0 = 1$.
  - As a result, the number 0 has two codes $000...00$ or $111...11$.
  - Not convenient enough for calculation, either.
- Two's complement （补码）
  - X_0 represent the sign of the integer m (0: *non-negative*, 1: *non-positive*).
  - $X_1 X_2 \dots X_n$ is the binary form of the number m if $X_0 = 0$;
  - $X_1 X_2 \dots X_n$ is 1 plus the bitwise NOT of the binary form of the number $-m$ if $X_0 = 1$.
  - Actually, $X_1 X_2 \dots X_n$ is the binary form of the number $(2^n + m)$ if $X_0 = 1$, so $X_0 X_1 X_2 \dots X_n$ is actually $(2^{n+1} + m)$. That is,

    $$[m]_{TC} = [m]_2 \quad (0 \le m < 2^n)$$
    $$[m]_{TC} = [2^{n+1} + m]_2 \quad (-2^n \le m < 0)$$

    The two's complement has the same effect as *mod* 2^{n+1}.
  - As a result, the number 0 only has one code $000...00$.
  - How to write a negative number in two's complement?

        - From the lowest digit of its absolute number's binary code, when we encounter `0` and the first `1`, we do not change their digits; then we change the digits after the first `1` to their opposite numbers (`0` to `1`, `1` to `0`). （从其绝对值的二进制编码的最低位开始，遇到的0和第一个1不变，之后的所有数取反。）
    - **CAN NOT** compare two numbers in two's complement directly.
    - $[X]_{TC} + [-X]_{TC} = 2^{n+1} = [0]_{TC}$
    - $[X+Y]_{TC} = [X]_{TC} + [Y]_{TC}$

      $[X-Y]_{TC} = [X]_{TC} + [-Y]_{TC}$

> [*Example*] (different encoding methods in computer)
>
> $$[-102]_{10} = [11100110]_S = [10011001]_{OC} = [10011010]_{TC}$$
>
> where S stands for *Signed magnitude*, OC stands for *One's complement* and TC stands for *Two's complement*.
>
> So -102 has the code of 10011010 in two's complement, as a result, it is stored as $[9A]_H = [9A]_{16}$ in computer.

> [*Example*] The encoding method of *int* in the computer is *two's complement*.
>
> $[100...00]_{TC} = -2147483648 = -2^{31}$, $[011...11]_{TC} = 2147483647$, so the range of *int* in computer is $[-2147483648, 2147483647]$.

2.2 Encoding of Unsigned Integer

$$m = X_0 X_1 X_2 \ldots X_n$$

The range is $[0, 2^{n+1} - 1]$, it is also same as *mod* 2^{n+1}.

The two's complement of a signed number can also be regard as a unsigned number, which means the code does not change! But the meaning of the code changes, a negative number becomes a quite big positive number.

> [*Example*] **Warning** In C compiler, if the two operands are a signed number and an unsigned number, the signed number will be implicitly transformed to unsigned number.
>
> ```
> unsigned int length = 0;
> for (int i = 0; i <= length - 1; ++ i)
>     // do something ...
> ```
>
> In the program above, `length` is an unsigned number while `i` and `1` are a signed number, the compiler will automatically transform `1` to unsigned number, and calculate `length - 1` which is `0-1` in unsigned number, so the result will be $2^{32} - 1$, which is the maximum number of *unsigned int*. The comparison between `i` and `length - 1` will also be treated as unsigned number comparison, so the loop will never end because all the *unsigned int* number is not greater than $2^{32} - 1$.

How to use *unsigned* in programming?

- Use *unsigned int* to represent set (subset).
- Use *unsigned int* as a modulo system.

2.3 Bitwise Operators

Operator `&`, `|`, `^`, `~`, `<<`, `>>` in C language are bitwise operator. They can fully use the feature of the binary number. Here are some useful methods to use the bitwise operator:

- Masking（掩码）：Use operator `&` to extract some certain digit of an number:

  > [*Example*] `0x8C & 0x0F = 0x0C` extract the digit `C`; `0x238C & 0x0FF0 = 0x0380` extract the digits `38`.

- Set or check in certain digits: Use operator `&` or `|` to set 1 or 0 in some certain digit, or check if certain digit is 0 or 1.

  > [*Example*] `0x0C | 0xF0 = 0xFC`; `0x238C & 0x0FFF = 0x038C`;
  >
  > Check if the last binary digit of `x` is 1: `x & 1`.

- Represent a set: The i-th binary digit represent whether p_j is in the set. The `&` operator can be used to get the *intersection* of two sets; the `|` operator can be used to get the *union* of two set; the `^` operator can be used to get the *symmetric difference* of two sets.

  > [*Example*] $\{1,2,4,6\}$ can be represent as `01010110`, $\{1,3,5\}$ can be represent as `00101010`; `01010110 & 00101010 = 01111110` represent $\{1,2,3,4,5,6\}=\{1,2,4,6\}\cup\{1,3,5\}$.

- Digit-extension（位扩展）: the C programming language will do it automatically, after the extension, the number **WILL NOT** change.
  - 0-extension（0扩展）: the transformation of *unsigned* numbers, all the extension digit will be filled with 0.
  - signed-extension（带符号扩展）: the transformation of *signed* numbers, all the extension digit will be filled with the sign digit of the original number.

  > [*Example*]
  >
  > `(unsigned short)111...11 = (int)000..00111...11`
  >
  > `(short)011...11 = (int)000...00011...11`
  >
  > `(short)111...11 = (int)111...11111...11`

- Digit-truncation（位截断）: the C programming language will do it automatically. Force to truncate, so the meanings may be different.

  > [*Example*]
  >
  > ```c
  > int i = 32768;
  > short j = (int) i;
  > int k = (short) j;
  > ```
  >
  > Both `i` and `j` have code of `0x8000` in the two's complement, which represent the number of `-32768` in *signed short*. When cast to *int* again, the number is still `-32768` according to the digit-extension rules, that is, `k` have a code of `0xFFFF8000` in the two's complement, which is different from original number `32768`.

- Shift-truncation（移位）: the C programming language has the operator `<<` and `>>`.
  - Left-shift（左移）: throw away the high digits, and filled the low digits with 0. Usually, `x << 1` has the same effect as `x * 2`. Left-shift may cause overflow and get the wrong result.

- Right-shift（右移）: throw away the low digits, and filled the high digits with 0 (logic right-shift（逻辑右移）); or throw away the low digits, and filled the high digits with the sign digit of original number (arithmetic right-shift（算术右移）). Just like digit-extension, compiler will automatically choose one of the methods. Similarly, `x >> 1` has the same effect as `x / 2` usually.

  > [*Example*]
  >
  > `x >> y` get the result $\lfloor x/2^y \rfloor$, so if `x` is negative, we may get unexpected result (because the result is not zero-correction（向零取整）), that is, the result is different from `x / (1 << y)`.
  >
  > Let's say `y = 2`, when `x` is negative, the result of `x >> 2` may be different from `x / 4`.

- **Warning**: In the shift operator `x << y` or `x >> y`, if `y` is greater than the digit-length of `x`, then the C compiler / MIPS will automatically do the modulo operation in `y`.

  > [*Example*] If the `x` has 32 digits, the result will be `x >> (y%32)`.

2.4 Logic Operators

Logic Operators `&&`, `||`, `!` only get the result *true* (not 0) or *false* (0).

> [*Example*] Short-circuit evaluation in logic operator: `true || p == 1`, we don't need to check if `p==1`, we can get the result is `true`.

The Comparison between Logic Operators and Bitwise Operators:

- Logic Operators: only has *true* or *false*, don't care about the actual numbers.
- Bitwise Operators: are operations between actual numbers.

2.5 The +/- Operators

Suppose the digit numbers of the operands are w.

Unsigned Addition Operation

Unsigned addition operation is same as addition operation under modulo 2^w, that is,

$$UAdd_w(u,v) = (u+v)\ mod\ 2^w$$

Signed Addition Operation (TC)

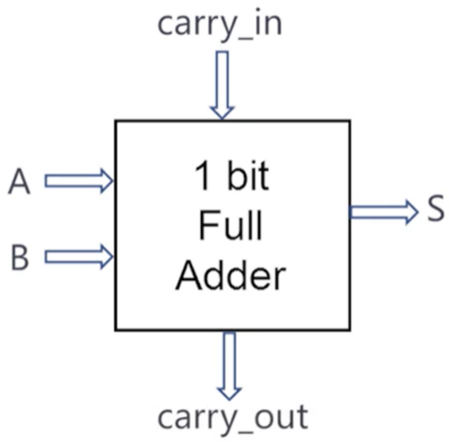
According to the addition formula in two's complement, we still have:

$$TAdd_w(u,v) = (u+v)\ mod\ 2^w$$

> That means, in the following program, `sum1` is equal to `sum2`.
>
> ```
> int u, v;
> int sum1 = (int)((unsigned int)u + (unsigned int v);
> int sum2 = u + v;
> ```
>
> **Explanation** *The numbers are the same, only the ways we look at the numbers change.*

Serial Carry Adder

So the *addition operation* can be implemented with many **1-bit Full Adder**s（一位全加器）.



where, *A* and *B* are operands, *carry_in* is the carry of lower digits, *carry_out* is the carry of higher digits, *S* is the result of this digit.

We can have:

```
S = A ^ B ^ carry_in
carry_out = (A & B) | (A & carry_in) | (B & carry_in)
```

With many *1-bit full adders* connected together, we get a **serial carry adder** (simple but slow).

Unsigned Subtraction Operation

Unsigned subtraction operation is same as substract operation under modulo 2^w, that is,

$$USub_w(u, v) = u - v = u - v + 2^w = u + (2^w - v) = u + \bar{v} + 1$$

Signed Subtraction Operation (TC)

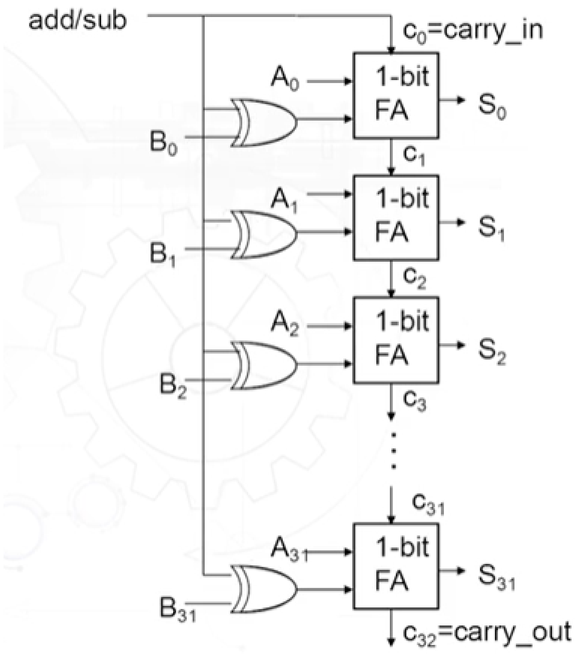
According to the addition formula in two's complement, $[A - B]_{TC} = [A + (-B)]_{TC} = [A]_{TC} + [-B]_{TC}$. And we know that $[B]_{TC} + [-B]_{TC} = 0$, so we have $[-B]_{TC} = 0 - [B]_{TC} = \overline{[B]_{TC}} + 1$.

$$TSub_w(u, v) = u - v = u + \bar{v} + 1$$

Arithmetic/Logic Unit

Thus, we can design a *serial carry adder-subtracter* （串行加减法器） in the following structure.

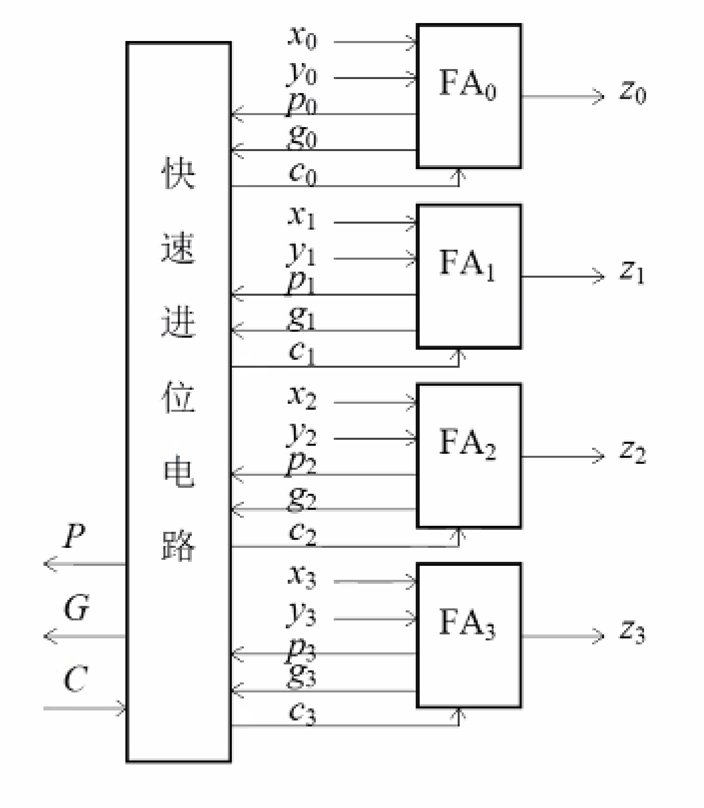
When doing addition operation, the signal `add/sub` is `0`; when doing subtraction operation, the signal is `1`.



We use a *xor-gate* to implement the bitwise NOT operation in the subtraction.

This unit can also do other logic/arithmetic things, so we call it **ALU** (**Arithmetic/Logic Unit**).

> **Carry Lookahead Adder**
>
> We summarize the carry signals, then we get:
>
> ```
> c1 = (x0 & c0) | (y0 & c0) | (x0 & y0);
> // c2 = (x1 & c1) | (y1 & c1) | (x1 & y1);
> c2 = (x1 & x0 & y0) | (x1 & x0 & c0) | (x1 & y0 & c0) |
>      (y1 & x0 & y0) | (y1 & y0 & c0) | (y1 & x0 & c0) | (x1 & y1);
> // ...
> ```
>
> We can use some notations to make the formulas simpler.
>
> $$g_i \triangleq x_i \text{ and } y_i$$
> $$p_i \triangleq x_i \text{ or } y_i$$
>
> then we have:
>
> ```
> c1 = g0 | (p0 & c0);
> c2 = g1 | (p1 & g0) | (p1 & p0 & c0);
> c3 = g2 | (p2 & g1) | (p2 & p1 & g0) | (p2 & p1 & p0 & c0);
> c4 = g3 | (p3 & g2) | (p3 & p2 & g1) | (p3 & p2 & p1 & g0) |
>      (p3 & p2 & p1 & p0 & c0);
> ```
>
> With the formulas above, we can calculate carry of 4 digits in one special unit, which speed up the process of calculation. We call it **Carry Lookahead Adder** (**CLA**).



2.6 Overflow

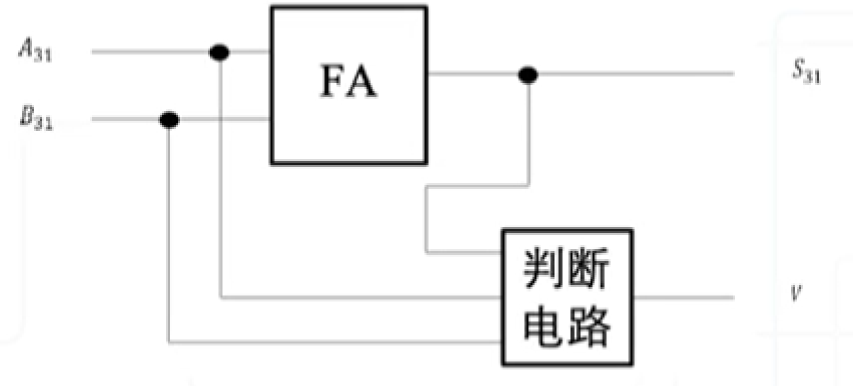
Overflow: the result of operation is out of the range of number.

Two number with the opposite sign in addition operation or two number with the same sign in subtraction operation **CAN NOT** cause *overflow*. Only two number with the same sign in addition operation or two number with the opposite sign in subtraction operation **MAY** cause *overflow*.

Operation	Operand A	Operand B	Overflow Result
C = A + B	$A \geq 0$	$B \geq 0$	$C < 0$
C = A + B	$A < 0$	$B < 0$	$C \geq 0$
C = A - B	$A \geq 0$	$B < 0$	$C < 0$
C = A - B	$A < 0$	$B \geq 0$	$C \geq 0$

How to check if overflow happens?

- Check the sign digit: suppose the highest digit number is 31 ($n = 32$).



```
Overflow_Sign = (A31 & B31 & (~ S31)) | ((~ A31) & (~ B31) & S31);
Overflow_Sign = (A(n-1) & B(n-1) & ~ (S(n-1))) | (~ (A(n-1)) & ~ (B(n-1)) &
C(n-1));
```

- Check the carry of the highest digit c_{n-1} and the carry of the second-highest digit c_n.
  - If $c_{n-1} = c_n$, then **NO** overflow.
  - If $c_{n-1} \neq c_n$, then **OVERFLOW**.

  c_{n-1} is the carry_in of $(n-1)$ digit and c_n is the carry_out of $(n-1)$ digit.

  P.S: We count the digit from 0 to $(n-1)$.

  So we can add a *xor-gate* between carry_in and carry_out of the digit $(n-1)$, that is,

```
Overflow_Sign = carry_in(n-1) ^ carry_out(n-1);
```

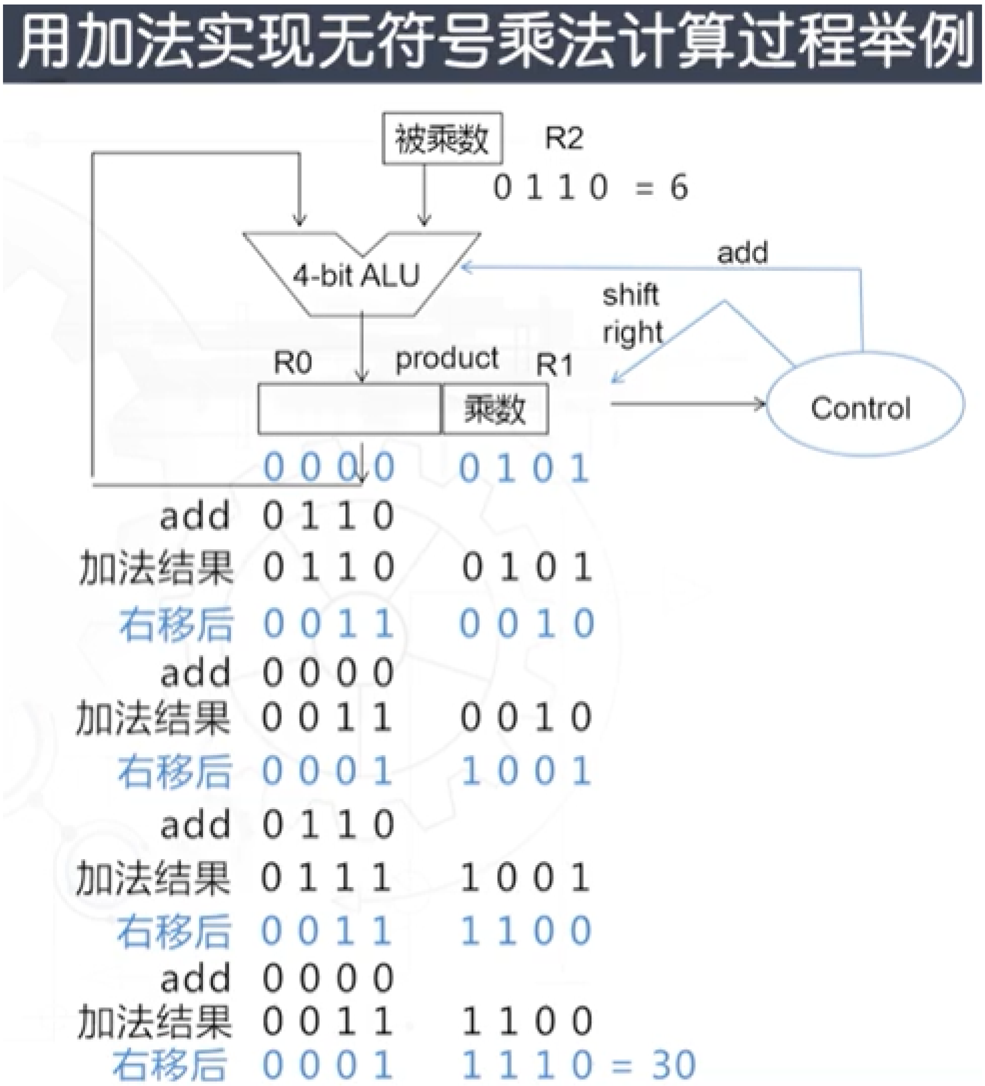
- Double sign-digit. Extend the sign digit to 2 digits, and use `00` to represent positive and `11` to represent negative. Then if the result's sign digits are `01` or `10`, overflow happens.

2.7 The * Operator

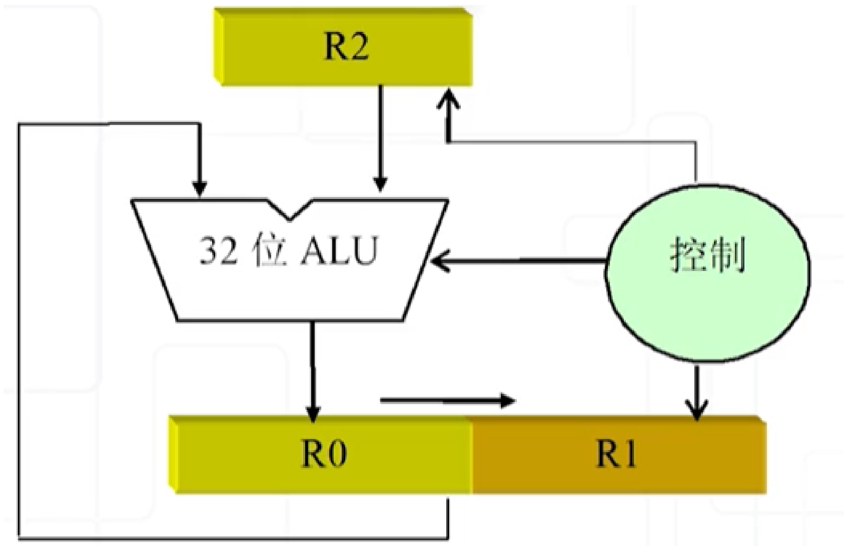
Multiplication Operation with Signed magnitude

When consider only one digit, we can use an *and-gate* to get the answer simply, so we can use *ALU* and *and-gate* to implement the multiplication operation.

> We can also design a more specific circuit to finish the task.
>
> - If the last digit of R1 is 0, do nothing (R0 plus 0)
> - If the last digit of R1 is 1, then set R0 to R0 plus R2.
>
> After addition operation end, do the *right-shift* to R0 and R1.
>
> **NOTE**: when doing *right-shift*, treat R0 and R1 as a total!



With the method above, we can design a circuit to do 32-bit multiplication operation of signed magnitude.



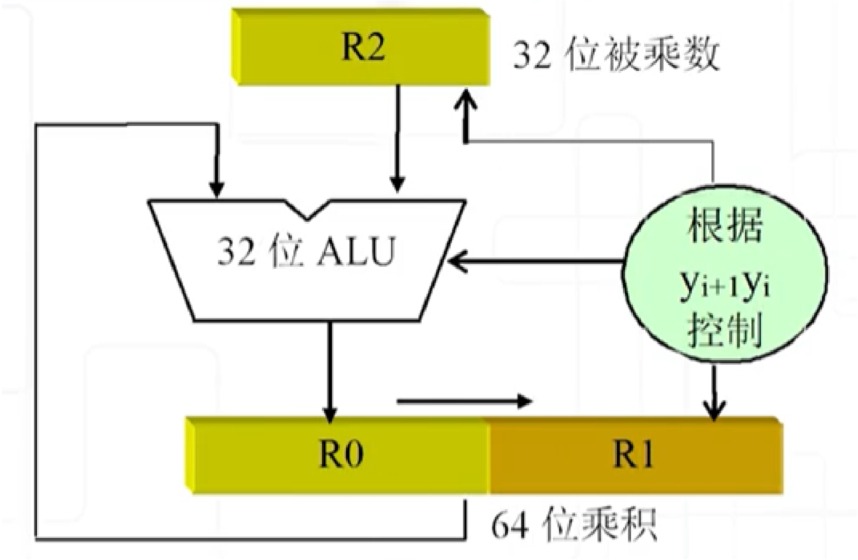
NOTE: the sign digit is **NOT** involved in the calculation above!

Multiplication Operation with Two's Complement: **Booth Algorithm**

With the method above, we use the last two digit of R1 to control the operation:

- If the last two digit of R1 $y_{i+1}y_i$ is 00 or 11, do nothing.
- If the last two digit of R1 $y_{i+1}y_i$ is 01, then set R0 to R0 plus R2.
- If the last two digit of R1 $y_{i+1}y_i$ is 10, then set R0 to R0 minus R2.

We can use the same structure above to complete the multiplication operation.



> **Correctness Proof**
>
> Suppose $[x]_{TC} = x_n x_{n-1} \dots x_1 x_0, [y]_{TC} = y_n y_{n-1} \dots y_1 y_0$, then
>
> $$result = (0 - y_0)x \times 2^0 + (y_0 - y_1)x \times 2^1 + (y_1 - y_2)x \times 2^2 + \dots + (y_{30} - y_{31})x \times 2^{31}$$
>
> So,
>
> $$result = x(-y_{31} \times 2^{31} + y_{30} \times 2^{30} + y_{29} \times 2^{29} + \dots + y_0 \times 2^0)$$
>
> y_{31} is the sign digit of y, so whether y is positive or negative, we have:
>
> $$y = (-y_{31} \times 2^{31} + y_{30} \times 2^{30} + y_{29} \times 2^{29} + \dots + y_0 \times 2^0)$$
>
> As a result, $result = xy$, the algorithm is correct!

Suppose the digit numbers of the operands are w.

Unsigned Multiplication Operation

Unsigned multiplication operation is same as multiplication operation under modulo 2^w, that is,

$$UMult_w(u, v) = (u \cdot v) \ mod \ 2^w$$

Signed Multiplication Operation (TC)

We can find that the formula still works:

$$Tmult_w(u, v) = (u \cdot v) \ mod \ 2^w$$

Compiler Optimization in Multiplication Operation

Use `<<` or `>>` and `+` or `-` to optimize:

> [*Example*] The following code can be optimized.
>
> ```
> long mul12(long x) {
>     return x * 12;
> }
> ```
>
> After compiling with optimization, we can get:

> ```
> leaq (%rax, %rax, 2), %rax
> salq $2, %rax
> ```
>
> which means:
>
> ```
> t = x + x * 2;
> return (t << 2);
> ```

> [*Example*] When the compiler optimize division operation, a correct term may be added.
>
> We have mentioned before that the result of `x >> y` and `x / (1 << y)` may be different. The compiler will automatically optimize the code `x / (1 << y)` as `(x + (1 << y) - 1) >> y`, that is, $\lfloor (x + (2^y - 1)) / 2^y \rfloor$, then the two results are the same.

Overflow

The multiplication operation usually **DO NOT** have an overflow-check feature, and the compiler **MAY NOT** check the overflow problem of multiplying.