# MTH091

# Algebraic Literacy

Chemeketa Mathematics Department

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# 1.1 Subsets of Real Numbers

## Introduction

The history of numbers is a long and fascinating one, with roots stretching back into prehistoric times.

Whenever people have wanted to count or measure objects they have created a type of number to suit their needs. In some cultures counting on fingers or toes was sufficient. In others, tally marks were made on wood or bone to record quantities.

As the concept of numbers evolved over time, symbols were created to represent specific values. The numerals we commonly use today, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ were developed in India about 1500 years ago and spread throughout Southeast Asia, China, Africa, and Europe along Arab trade routes.

As science, technology, and commerce have advanced, mathematical thought has expanded the types of numbers available and the operations for combining them. The numbers discussed in this section are all part of the set of **real numbers**.

## Natural Numbers

**Natural numbers** are the numbers we usually count with: $1, 2, 3, 4, \ldots$ and so on. Natural numbers are one of the oldest and most studied sets of real numbers. No doubt you already know a few facts about natural numbers.

You probably know that when you add or multiply natural numbers, changing the order does not change the answer: $3 + 5 = 5 + 3$ and $6 \times 10 = 10 \times 6$, for example.

You might also know that most natural numbers are either prime or composite. Prime numbers, like $2, 3, 5, 7, 11,$ and $13$ can only be divided evenly by $1$ and themselves. Composite numbers, on the other hand, can be divided by several numbers. The number $8$, for instance, can be divided evenly by $1, 2, 4,$ and $8$. (*Fun fact: the number $1$ is in a special category of its own, it is considered neither prime nor composite.*)

What you may not have considered is that adding two natural numbers always creates another natural number, no matter how many times you do it. However, when subtracting it's possible to end up with $0$ or with a negative value, neither of which are natural numbers.

If we add $0$ to the natural numbers we call that new set the whole numbers.

## Integers

Today the number $0$ and negative values are ideas we are familiar with and encounter regularly: a microwave timer beeps when it gets down to $0$ seconds, the temperature outside might be $-15$ degrees, etc.

What may seem common now were actually significant advances in thought that took mathematicians hundreds of years of work. In fact, it wasn't until Newton's work on calculus in the 17th century that they became a fundamental part of the real number system.

When you include all the natural numbers, along with their negatives and the number $0$, you create a set matter the **integers**, which looks something like this: $\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

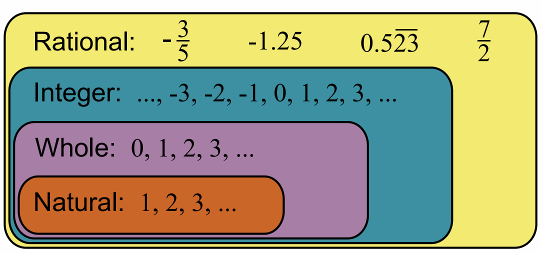
Any combination of adding, subtracting, and multiplying integers always results in another integer. Division, however, can sometimes produce numbers that are not integers.

## Rational Numbers

Any number that can be written as a fraction, such as $\frac{1}{2}$, $\frac{2}{3}$, or $\frac{3}{4}$, is considered a **rational number**. Every rational number can also be expressed as a decimal that either terminates or repeats. For example, $\frac{3}{4}$ as a decimal is $0.75$, which terminates, and $\frac{2}{3}$ as a decimal is $0.666\ldots$, which repeats.

Notice that this includes all whole numbers and integers since they can be written as fractions with a denominator of $1$ or as decimals that terminate.

In fact, all of our sets so far have included the earlier sets. Natural numbers are part of the set of whole numbers, which are part of the set of integers, and integers are a subset of rational numbers.



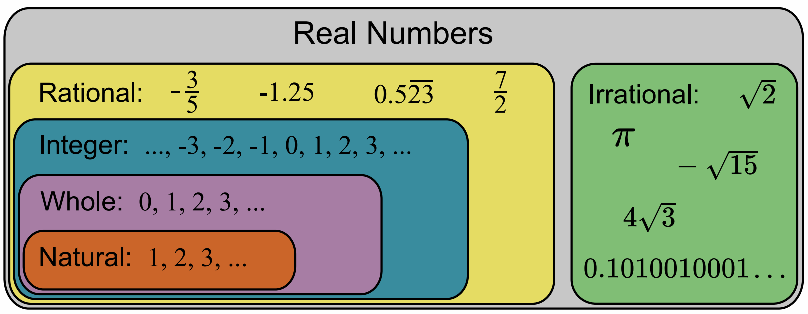

And since we can add, subtract, multiply, and divide rational numbers as much as we want (provided we don't divide by $0$), you might assume the story is over. If so, you're in good company (the famous Greek mathematician Pythagoras thought the same thing), but it turns out that there's still more to explore.

## Irrational Numbers

You might be surprised to learn that some quantities cannot be expressed as whole numbers or as fractions. For instance, the length of the diagonal of a square is often not a rational number.

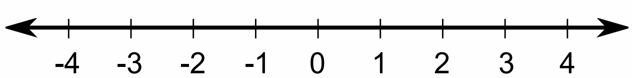
Since these numbers cannot be expressed as a ratio of whole numbers they are matter **irrational numbers**. If you try to find the decimal representation of an irrational number you'll find a never-ending decimal that goes on forever and doesn't repeat.

Many irrational numbers are the result of taking square roots, like $\sqrt{2}$, but several more come from other mathematical concepts and techniques. Perhaps the most famous irrational number is $\pi$, which is the ratio of the circumference of a circle to its diameter.

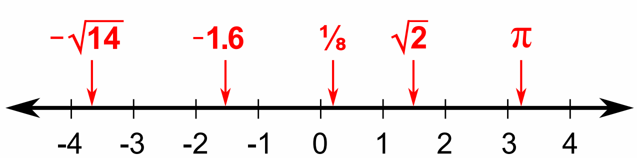



## The Real Number Line

A number line is a useful tool for visualizing real numbers. It is nothing more than a straight line with marks that correspond to specific numbers. Positive numbers are positioned to the right of zero while negative numbers are placed to the left with arrows on either end indicating the line extends in both directions.




Every real number has a place on the number line, and every location on a number line is assumed to correspond to a real number. Here are a few real numbers and their relative locations on the number line.




## Inequality

Real numbers can be ordered based on their position on the number line. Numbers on the left are considered smaller than numbers on the right.

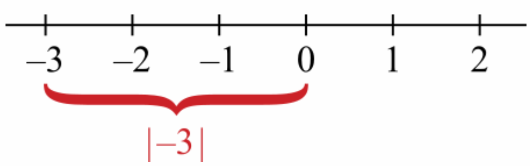
The symbols $>$ and $<$ are used to represent "greater than" and "less than" relations, respectively. The wide side of those symbols opens toward the larger value while the other side points to the smaller value.

The symbol $>$ means that the value on the left is greater than the value on the right. For example, $10 > 5$ shows that $10$ is greater than $5$.

The symbol $<$ means that the value on the left is less than the value on the right. For example, $4 < 8$ indicates that $4$ is less than $8$.

## Absolute Value

The number line also helps us find the size, or magnitude, of a number. The distance between $0$ and a number is matter the absolute value of the number and is written as $|x|$. For example, $|3| = 3$ since the number $3$ is three spaces away from $0$. Similarly, $|-3| = 3$ because $-3$ is also three spaces from $0$.




In practice, finding the "absolute value" means removing any negative sign in front of a number.

## Looking Ahead

Now that we've explored the different types of real numbers and how they relate to each other, we're ready to start working with them. In the next sections we'll focus on operations with integers and fractions. Refreshing these skills now will form a solid foundation for all the algebraic work that follows.

# 1.2 Working with Integers

## Introduction

When working with positive integers, all of the basic operations of addition, subtraction, multiplication, and division work exactly as you've learned in elementary school and usually pose no problems.

It's when we incorporate negative integers that things sometimes get confusing and even seasoned mathematicians occasionally make mistakes.

In this section, we will go over the basic concepts so that you can generally avoid these issues.

## Adding & Subtracting Integers

It's always helpful to link a concept in math with something you can visualize that has similar behavior.

Here's an image that might be useful. It is a box that has balloons lifting it up into the air and bags of sand pulling it down.

Suppose you wanted the box to go down, what could you do? There are actually two options. You can add sandbags or remove balloons. Either of those actions would make the box go down.

But what if you wanted the box to go higher? In that case, removing sandbags or adding balloons would do the trick.

If we imagine the balloons represent positive integers and the sandbags represent negative integers, then the rules for addition and subtraction will follow the same pattern.

Adding a negative integer (sandbag) is the same as subtracting a positive integer (balloon). In symbols, $a + (-b) = a - b$.

Subtracting a negative integer (sandbag) is the same as adding a positive integer (balloon). In symbols, $a - (-b) = a + b$.

If the balloon and sandbag analogy doesn't make sense to you, you can use similar scenarios like gaining or losing money, riding up or down an elevator, or walking forward or backward to help illustrate and remember the rules.

It can sometimes be helpful to see a few concrete examples, so here are a few calculations involving positive and negative $6$ and $8$.

**Example 1**

a. $6+(-8)=6-8=-2$

b. $6-(-8)=6+8=14$

c. $-6+8=2$

d. $-6+(-8)=-6-8=-14$

As you encounter exercises with negative integers be very careful to write down the signs of the numbers correctly. Keeping track of the signs of the integers is crucial to getting the correct answers.

## Multiplying and Dividing Integers

The key to multiplying integers is to remember that multiplication is simply a shorthand notation for repeated addition. For example, $2\times 4$ is shorthand for two $4$'s added together, while $3\times 4$ is short for three $4$'s added together.

Using this concept, we can find the product $2\times(-4)$ by adding $-4$ two times.

$$\begin{aligned} 2\times(-4) &= (-4)+(-4) && \text{using the definition of multiplication} \\ &= -8 && \text{using the rule for adding two negative integers} \end{aligned}$$

Notice that the result is a negative number. Anytime we multiply a positive and a negative together the result will always be negative. This is because it is a repeated addition of negative numbers.

But what if we multiply two negative integers? Consider the product $(-2)(-4)$. If we rewrite negative $2$ as the opposite of $2$, taking advantage of the fact that $-2=-(2)$, then the answer is easier to see.

$$\begin{aligned} (-2)(-4) &= -(2)(-4) && \text{since } (-2)=-(2) \\ &= -(2(-4)) && \text{focusing on multiplying } (2(-4)) \\ &= -(-8) && \text{since } 2(-4)=-8 \\ &= 8 \end{aligned}$$

As this example demonstrates, the product of two negative numbers is always positive.

Another way to understand why the product of two negative numbers is positive is to observe a pattern. Let's look at what happens as we multiply $-10$ by numbers that get smaller.

$$\begin{aligned} -10 \times (3) &= -30 \\ -10 \times (2) &= -20 \\ -10 \times (1) &= -10 \\ -10 \times (0) &= 0 \end{aligned}$$

Notice that each time we decrease the number by $1$, the answer increases by $10$. If we continue this pattern, the next steps would be:

$$\begin{aligned} -10 \times (-1) &= 10 \\ -10 \times (-2) &= 20 \\ -10 \times (-3) &= 30 \end{aligned}$$

Here again we arrive at the same conclusion that a negative times a negative results in a positive value.

**Rules for Multiplying Integers**

- The product of two numbers with the same sign is positive.
- The product of two numbers with different signs is negative.

Here are a few more examples:

**Example 2**

a. $7 \times -5 = -35$

b. $-7 \times -5 = 35$

c. $(-1)(9) = -9$

d. $(-2)(-4)(-5) = -40$

We've been focusing on multiplication, but the process for division is the same. The reason is that division of one number by another is equivalent to multiplying by the reciprocal of the second number. For example, $8 \div 2 = 4$ is equivalent to $8 \cdot \frac{1}{2} = 4$.

Since any division $a \div b$ can be written as the multiplication $a \cdot \frac{1}{b}$, the sign rules for multiplication also apply to division.

**Rules for Dividing Integers**

- The quotient of two numbers with the same sign is positive.
- The quotient of two numbers with different signs is negative.

Here are a few examples:

**Example 3**

a.$40 \div -10 = -4$

b. $\frac{24}{-8} = -3$

c. $\frac{-15}{3} = -5$

d. $\frac{-45}{-9} = 5$

## Division Involving $0$

Since $0$ is an integer, we will take a moment to discuss two situations where division involves $0$. The two cases are easily confused, but you can always check your answer by remembering that any division can be rewritten as multiplication. For example, $\frac{15}{3} = 5$ because $5(3) = 15$.

To see how this helps, consider $\frac{0}{2}$, what would it equal? Since $0 \cdot 2 = 0$ it must be that $\frac{0}{2} = 0$. As a rule, $0$ divided by any nonzero number is always $0$.

But how about $\frac{2}{0}$, what does it equal? To find an answer we would need to identify the exact number $n$ so that $n \cdot 0 = 2$. But no such number exists since $0$ times anything is still $0$. We must conclude, therefore, that division by $0$ is undefined.

**Rules for Division Involving Zero**

- $\frac{0}{b} = 0$ provided $b \neq 0$
- $\frac{b}{0}$ is undefined for all values of $b$

## Powers of Integers

We've seen that multiplication is a shorthand notation for repeated addition. There is also a shorthand for repeated multiplication: exponents. For instance, if we wanted to multiply the number $5$ by itself $6$ times, we can write either write it longhand as $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ or use the much shorter notation $5^6$.

In the example $5^6$, the number $5$ is the base and $6$ is the exponent. The exponent tells us how many times the base is multiplied by itself. We might also say that $5^6$ is $6\text{th}$ power of $5$.

**Definition of an Exponent**

For any natural number $n$, $b^n = b \cdot b \cdot b \cdot \ldots \cdot b$ where $b$ is a factor $n$ times.

exponent

base

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$$

We call $b$ the base and $n$ the exponent and say that $b$ is being raised to the $n\text{th}$power.

Notice that $b^1$ means there is just one $b$, so we do not usually write the exponent. Two powers have specific names that you will hear frequently. We often refer to $b^2$as "$b$ squared" and $b^3$ as "$b$ cubed". These names come from geometry formulas for the area of a square and the volume of a cube.

When a positive integer is raised to a positive exponent the result is always a positive number, so there's nothing special to be aware of. However, when a negative integer is raised to an exponent the result could be positive or it could be negative—it depends on whether the exponent is an even number or an odd number.

**Exponents with Negative Bases**

- If the base is negative and the exponent is even, the final value will always be positive.

$$(-5)^2 = (-5)(-5) = 25$$

- If the base is negative and the exponent is odd, the final value will always be negative.

$$(-5)^3 = (-5)(-5)(-5) = -125$$

Most calculators have an exponent button (which often looks like $x^y$ or ^ depending on the model) to help you evaluate powers quickly. When using a calculator, you will want to take note of the parentheses. If there are parentheses around the negative base then the power applies to the entire integer.

If there are no parentheses then the power does not apply to the negative sign. It's very easy to confuse expressions like $-3^2$ and $(-3)^2$, but the two are different: $-3^2 = -(3^2) = -9$ while $(-3)^2 = (-3)(-3) = 9$.

This is one of the most common mistakes in algebra. Always slow down and double check your work when negatives are involved.

Use the examples below to practice evaluating powers of integers with your calculator.

**Example 4**

a. $(-7)^2 = 49$

b. $-7^2 = -49$

c. $-5^3 = -125$

d. $(-3)^4 = 81$

## Roots of Integers

We know that powers are used to evaluate expressions like $5^2 = 25$, but what if we wanted to go in the reverse direction, start with $25$ and getting back $5$? That is what **roots** do. Just as subtraction is the opposite of addition, roots are the inverses of powers.

**Definition of a Root**

The $n$-th root of a number $a$, written as $\sqrt[n]{a}$, is the value $r$ for which $r^n = a$.

In other words, for any natural number $n$, the root $\sqrt[n]{a}$ is the number that equals $a$ when it is multiplied by itself $n$ times.

$$\underbrace{\sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \sqrt[n]{a} \ \cdots \ \sqrt[n]{a}}_{n \text{ times}} = a$$

We call $n$ the *index* or *degree* of the root.

In particular, a root of index $2$ is called the **square root** and is usually written as $\sqrt{x}$, without specifying the index.

Using this definition, we can see that $\sqrt{9} = 3$ because $3 \times 3 = 9$, and $\sqrt[3]{8} = 2$ because $2 \times 2 \times 2 = 8$.

Here are a some other examples of roots.

**Example 5**

a. $\sqrt{16} = 4$

b. $\sqrt{25} = 5$

c. $\sqrt{36} = 6$

d. $\sqrt[3]{27} = 3$

Special care should be taken when evaluating roots of negative integers. For instance, if we were to try to find $\sqrt{-9}$, then we're essentially asking, *"What number, when multiplied by itself, equals $-9$?"*

Since the product of any two real numbers is always positive, we'll never find one that, when squared, equals $-9$. Square roots of negative numbers lead to the realm of imaginary and complex numbers, which are not addressed in this course.

While square roots of negative numbers lead to imaginary numbers, cube roots do not pose the same challenge.

Consider, as an example, $\sqrt[3]{-8}$. Here we are asking *"What number, when multiplied by itself three times, equals -8?"* In this case, there is a real number solution because $(-2) \cdot (-2) \cdot (-2) = -8$. Unlike square roots, cube roots of negative numbers do not require the introduction of imaginary numbers or complex numbers.

## Conclusion

In this section we've covered basic operations with integers. In the coming chapters, we'll use those operations to form and simplify expressions and solve equations. But first, we'll turn our attention to operations with fractions.

# 1.3 Working with Fractions

## Introduction

In section 1.1 we reintroduced you to several different sets of numbers that we use regularly in math. Of all these sets of numbers, it is quite possible that the fractions are the ones that have caused the most frustration for students.

In fact, a 2008 report to the US Department of Education found that "the most important foundational skill not presently developed appears to be proficiency with fractions". So if you struggle with fractions, just know you are not alone.

It is widely believed that the ancient Greek mathematician Pythagoras was the first person to rigorously explore fractions; though he did so in a purely geometric way, often with triangles.

We've come a long way since then. No doubt you have a calculator or smart phone that can do all sorts of things with fractions without you having to think much about it. Even so, having a basic understanding of how fractions work, *without* having to rely on technology, will speed you on your way to success.

## Reviewing Basic Fraction Concepts

Before we get into the nuts and bolts of working with fractions, let's start by reviewing some basic concepts and terminology.

Fractions are a way of representing parts of a whole. They consist of a **numerator** (the top number) and a **denominator** (the bottom number). For example, in the fraction $\frac{3}{4}$, the number $3$ is the numerator and $4$ is the denominator.

Fractions that represent values less than 1 are matter **proper fractions**, and those with values greater than or equal to 1 are referred to as **improper fractions**.

Earlier in your education you may have come across *mixed numbers*, ones in which there is a whole number part and a fraction part, like $3\frac{1}{2}$. Almost without exception, mixed numbers are **not used** in algebra, calculus or statistics.

And now on to some examples of things you might be asked to do.

## Simplifying Fractions

Sometimes the numerator and denominator of a fraction share a common factor, in other words, they are both multiples of the same smaller number. If we divide both of them by that number, we end up with an equivalent fraction that is said to be **simpler** than the original fraction.

To completely **simplify** we should divide out the largest piece they both share, which is matter the **greatest common factor (GCF)**. Some books call this the greatest common divisor (GCD), since it is what we divide by, but they mean the same thing.

The result is the equivalent fraction where the numerator and denominator have been reduced to their smallest possible values.

Let's say we want to simplify the fraction $\frac{20}{45}$. Here's how we do that.

1. Find the GCF: Since $20 = 4 \cdot 5$ and $45 = 9 \cdot 5$, the GCF of $20$ and $45$ is $5$.
2. Divide the numerator by the GCF: $20 \div 5 = 4$
3. Divide the denominator by the GCF: $45 \div 5 = 9$
4. Rewrite the fraction: $\frac{20}{45} = \frac{4}{9}$.

and now we know $\frac{20}{45}$ simplifies to $\frac{4}{9}$. Both fractions represent the same number, just in different formats.

It's standard practice to reduce fractions as much as possible as you work through a problem.

**Example 1**

Simplify $\frac{18}{24}$.

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The GCF of $18$ and $24$ is $6$, so:

$$\begin{aligned} \frac{18}{24} &= \frac{18 \div 6}{24 \div 6} && \text{Divide top and bottom by GCF 6} \\ &= \frac{3}{4} \end{aligned}$$

If you don't instantly recognize the GCF don't worry, you can begin the simplification by dividing by any common divisor, and repeating that process until no common factors remain.

**Example 2**

Simplify $\frac{144}{180}$.

$$\begin{aligned} \frac{144}{180} &= \frac{72}{90} && \text{Dividing top and bottom by 2} \\ &= \frac{36}{45} && \text{Dividing top and bottom again by 2} \\ &= \frac{12}{15} && \text{Dividing top and bottom by 3} \\ &= \frac{4}{5} && \text{Dividing top and bottom again by 3} \\ &= \frac{4}{5} \end{aligned}$$

The same result could have been found in one step if we'd known $36$ was the GCF.

## Multiplying Fractions

Multiplying fractions is very straightforward and has a simple rule. To multiply fractions, you simply multiply the numerators together to get the new numerator and multiply the denominators together to get the new denominator.

**Multiplying Fractions**

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Let's consider a few examples.

**Example 3a**

Find $\frac{1}{3} \cdot \frac{5}{4}$

$$\begin{aligned} \frac{1}{3} \cdot \frac{5}{4} &= \frac{1 \cdot 5}{3 \cdot 4} && \text{Multiply across} \\ &= \frac{5}{12} \end{aligned}$$

**Example 3b**

Find $\frac{2}{3} \cdot \frac{5}{7}$

$$\begin{aligned} \frac{2}{3} \cdot \frac{5}{7} &= \frac{2 \cdot 5}{3 \cdot 7} && \text{Multiply across} \\ &= \frac{10}{21} \end{aligned}$$

And, of course, we should keep our eye out for opportunities to simplify.

**Example 4**

Multiply $\frac{4}{9} \cdot \frac{3}{8}$.

$$\begin{aligned} \frac{4}{9} \cdot \frac{3}{8} &= \frac{12}{72} && \text{Multiply across} \\ &= \frac{1}{6} && \text{Simplify by dividing top and bottom by 12} \end{aligned}$$

## Dividing Fractions

Unlike multiplication, when dividing we do not divide corresponding parts of each fraction. Instead, we first flip the second fraction upside down to create its reciprocal, and then multiply the first fraction by the reciprocal of the second one.

When we want to divide by a fraction like $\frac{2}{5}$, we actually multiply by its reciprocal, $\frac{5}{2}$. The reciprocal of a fraction is created by flipping the numerator and denominator.

**Dividing Fractions**

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

**Example 5a**

Calculate $\frac{5}{6} \div \frac{2}{3}$. Simplify if needed.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

$$\begin{aligned} \frac{5}{6} \div \frac{2}{3} &= \frac{5}{6} \cdot \frac{3}{2} && \text{Multiply by reciprocal} \\ &= \frac{15}{12} && \text{Multiply across} \\ &= \frac{5}{4} && \text{Simplify by dividing by 3} \end{aligned}$$

**Example 5b**

Divide $\frac{5}{6} \div \frac{10}{9}$. Simplify if needed.

$$\begin{aligned} \frac{5}{6} \div \frac{10}{9} &= \frac{5}{6} \cdot \frac{9}{10} && \text{Multiply by reciprocal} \\ &= \frac{45}{60} && \text{Multiply across} \\ &= \frac{3}{4} && \text{Simplify by dividing by 15} \end{aligned}$$

## Adding and Subtracting Fractions with the Same Denominator

When adding or subtracting fractions with the same denominator, you simply add or subtract the numerators while keeping the denominator unchanged.

**Adding and Subtracting Fractions with the Same Denominators**

$$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$$

As long as you *do not* add or subtract the denominators, you should be alright. Again, to add or subtract just combine the numerators and leave the denominator unchanged.

Let's try a few examples, starting with addition.

**Example 6a**

Add $\frac{1}{5} + \frac{2}{5}$

$$\begin{aligned} \frac{1}{5} + \frac{2}{5} &= \frac{1+2}{5} \\ &= \frac{3}{5} \end{aligned}$$

**Example 6b**

Add $\frac{9}{4} + \frac{7}{4}$

$$\begin{aligned} \frac{9}{4} + \frac{7}{4} &= \frac{9+7}{4} && \text{Add numerators, keep denominator} \\ &= \frac{16}{4} \\ &= 4 \end{aligned}$$

The next examples are both subtraction.

**Example 7a**

Subtract $\frac{7}{8} - \frac{3}{8}$

$$\begin{aligned} \frac{7}{8} - \frac{3}{8} &= \frac{7-3}{8} \\ &= \frac{4}{8} \\ &= \frac{1}{2} \end{aligned}$$

**Example 7b**

Subtract $\frac{2}{7} - \frac{5}{7}$

$$\begin{aligned} \frac{2}{7} - \frac{5}{7} &= \frac{2-5}{7} && \text{Subtract numerators, keep denominator} \\ &= \frac{-3}{7} \end{aligned}$$

## Adding and Subtracting Fractions with Different Denominators

If we need to add fractions that do not have the same denominator, what do we do? The answer is that we rewrite them so that they do have the same denominator. The process is kind of like simplifying but in reverse.

A quick example will illustrate the idea, then we'll go through the details. For this example, all you need to know is that $\frac{20}{40} = 12$ (which you can verify by simplifying.)

$$\begin{aligned} \frac{1}{2} + \frac{13}{40} &= \frac{20}{40} + \frac{13}{40} && \text{replacing } \frac{1}{2} \text{ with } \frac{20}{40} \\ &= \frac{20+13}{40} && \text{add the numerators} \\ &= \frac{33}{40} \end{aligned}$$

The trick was being able to rewrite $\frac{1}{2}$ as $\frac{20}{40}$ so that both fractions had a denominator of $40$. In this case, $40$ was the smallest multiple that both denominators share, or **least common multiple** (LCM).

The LCM can always be used as the common denominator, but so can any common multiple of the denominators. As long as we can rewrite one (or both) of the fractions so that they have the same denominator, we can do the addition or subtraction.

The fastest way to rewrite both fractions so they have a common denominator is to multiply each part of the fraction by the denominator of the other one. Let's see how that works.

$$\begin{aligned} \frac{1}{2}+\frac{4}{3} &= \frac{1}{2}\cdot\frac{3}{3}+\frac{4}{3}\cdot\frac{2}{2} && \text{multiplying each part by the denominator of the other fraction} \\ &= \frac{3}{6}+\frac{8}{6} && \text{now our factions have the same denominator} \\ &= \frac{11}{6} \end{aligned}$$

We summarize this technique below.

**Adding and Subtracting Fractions with Different Denominators**

$$\frac{a}{b} \pm \frac{c}{d} = \frac{a\cdot d \pm c\cdot b}{b\cdot d}$$

Let's try a few examples.

**Example 8a**

Add $\frac{5}{12}+\frac{7}{18}$

$$\begin{aligned} \frac{5}{12}+\frac{7}{18} &= \frac{15}{36}+\frac{14}{36} && \text{Rewrite with LCM 36} \\ &= \frac{29}{36} \end{aligned}$$

**Example 8b**

Subtract $\frac{11}{6}-\frac{7}{4}$

$$\begin{aligned} \frac{11}{6}-\frac{7}{4} &= \frac{22}{12}-\frac{21}{12} && \text{Rewrite with LCM 12} \\ &= \frac{1}{12} \end{aligned}$$

## Powers and Roots of Fractions

Lastly, we'll touch on something that will come up again in a later section: powers and roots of fractions.

The rules for these are straightfoward: we simply apply the power or root to both parts of the fraction.

**Powers and Roots of Fractions**

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

and

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Let's see how this works with powers first.

**Example 9a**

Evaluate $\left(\frac{5}{3}\right)^2$

$$\begin{aligned}\left(\frac{5}{3}\right)^2 &= \frac{5^2}{3^2} \\ &= \frac{25}{9}\end{aligned}$$

**Example 9b**

Evaluate $\left(\frac{2}{5}\right)^3$

$$\begin{aligned}\left(\frac{2}{5}\right)^3 &= \frac{2^3}{5^3} && \text{Apply exponent to numerator and denominator} \\ &= \frac{8}{125}\end{aligned}$$

Now let's try roots of fractions.

**Example 11**

Simplify $\sqrt{\frac{49}{100}}$

$$\begin{aligned}\sqrt{\frac{49}{100}} &= \frac{\sqrt{49}}{\sqrt{100}} && \text{Take square root of numerator and denominator}\\ &= \frac{7}{10}\end{aligned}$$

**Example 11a**

Simplify $\sqrt[3]{\frac{8}{27}}$

$$\begin{aligned}\sqrt[3]{\frac{8}{27}} &= \frac{\sqrt[3]{8}}{\sqrt[3]{27}} && \text{Take cube root of numerator and denominator}\\ &= \frac{2}{3}\end{aligned}$$

## Unit Conversion

One common use of fractions is converting between different units of measurement. This always involves multiplying by fractions that equal $1$, once you take into consideration the units of the numerator and denominator. Such fractions are known as **conversion factors**.

For example, since $1\text{foot} = 12\text{inches}$, the fractions $\frac{1 \text{ foot}}{12 \text{ inches}}$ and $\frac{12 \text{ inches}}{1 \text{ foot}}$ are both equal to $1$ and can be used as conversion factors. Since they equal $1$, multiplying by them doesn't change the value of what we're measuring, it only changes the units.

The general strategy is to choose a conversion factor that will cancel out the units we don't want and leave us with the units we do want.

**Example 12**

Convert 8 feet to inches.

$$\begin{aligned}8 \text{ feet} &= 8 \text{ feet} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} \\ &= \frac{8 \cdot 12 \text{ inches}}{1} \\ &= 96 \text{ inches}\end{aligned}$$

Notice how "feet" cancels out, leaving us with inches.

**Example 13**

A recipe calls for 2.5 cups of flour, but you only have a measuring spoon that measures tablespoons. How many tablespoons of flour do you need if there are 16 tablespoons in 1 cup?

$$\begin{aligned}2.5 \text{ cups} &= 2.5 \text{ cups} \cdot \frac{16 \text{ tablespoons}}{1 \text{ cup}} \\ &= 2.5 \cdot 16 \text{ tablespoons} \\ &= 40 \text{ tablespoons}\end{aligned}$$

## Conclusion

Fractions are fundamental building blocks in math and we'll frequently encounter them as parts of larger problems. And while calculators can handle operations with fractions, understanding how the underlying concepts work gives you insight and reasoning skills that technology cannot replace, and help you spot when the technology makes a mistake!

# 2.1 The Language of Algebra

## Introduction

In English, it's not uncommon for a word to have more than one meaning. Sometimes a double meaning can make you smile, as with the classic joke "Why was the math book sad? Because it had too many problems." However, multiple meanings can also leave you confused, like when a friend asks if you saw a green tank on your way to school. Are they asking about a large container, or a military vehicle?

In mathematics, we try our best to use precise language with agreed-upon meanings for specific words and phrases to avoid confusion. Spending time early in the course to become familiar with this terminology will make the rest of the term go much more smoothly.

## Constants and Variables

As you might be able to guess, a **constant** is something that does not change; while a **variable** is a quantity whose value will change.

For example, the distance from my home to work is 7 miles. That distance doesn't change. It is constant. The time it takes to travel from home to work is variable, however. It can change from one day to the next based on the speed of traffic.

In algebra letters can be used to represent constants and variables. Generally speaking, we usually use letters and the beginning of the alphabet $a, b, c$ to represent constants and letters toward the end of the alphabet $x, y, z$ to represent variables.

That rule is not set in stone. Sometimes it makes sense to use a letter that matches the context of a particular quantity. For example, we could use $d$ to represent the constant distance from your home to work and $t$ to represent the variable time it takes to travel.

## Algebraic Expression and Equations

Expressions and equations are two important concepts in algebra that are related, but different, and easy to confuse.

When constants and variables are combined together using **operations** (like addition, multiplication, etc.) we have created an **expression**. An **equation**, on the other hand, is a statement about the equality of two expressions. An equation will always have an equal sign $=$ in it. For example, $13 + 5$ is an expression whereas $13 + 5 = 18$ is an equation.

Here are a few more examples to help you see the difference.

| **Expressions** | **Equations** |
|---|---|
| $2 - 7$ | $2 - 7 = -5$ |
| $3x + 5$ | $y = 3x + 5$ |
| $2t + 3t$ | $2t + 3t = 5t$ |

Note that expressions and equations can both involve a combination of letters and numbers, which is perhaps why they are easily mistaken for each other. Again, the key difference is that equations always have an equal sign but expressions never do. For example, $2x + 5$ and $11x$ are both expressions while $2x + 5 = 11x$ is an equation.

## Evaluating and Solving

It's important to recognize whether you are working with an expression or an equation because we do different things with them.

Because equations have an equal sign, we can solve for the value of a variable that makes an equation true. For instance, the equation $x + 1 = 4$ can be solved by realizing that if the variable $x = 3$ then both sides of the equation would have the same value of $4$. In this case, we would say that $x = 3$ is the solution to the equation. In Chapters 3 and 4 we will explore a number of ways to systematically solve equations.

Since expressions do not involve an equal sign, we cannot solve them. We can, however, evaluate expressions when we are given specific values. If an expression contains variables, then we will need to replace those letters with numbers before finding the value represented by the expression.

If, for instance, tacos cost $\$2$ and burritos are $\$5$, then the expression $2t + 5b$ could tell us how much was spent on an order of tacos and burritos, but only if we knew how many of each were purchased. If someone orders one taco and two burritos, then substituting $t = 1$ and $b = 2$ gives

$$\begin{aligned} 2t + 5b &= 2 \cdot 1 + 5 \cdot 2 \\ &= 2 + 10 \\ &= 12 \end{aligned}$$

and we now know our expression has a value of $\$12$.

## Equivalent Expressions

Equivalent expressions represent two different ways of calculating the same value. You are likely already familiar with some basic rules that produce equivalent expressions, even if you don't know their names, and we will discover others later in this chapter.

Suppose, for instance, that two employees are planning a company event. They have $12$ tables and each table seats $8$ people. One employee uses the expression $12 \cdot 8$ to calculate the number of chairs needed. The other calculates the total number of chairs with the expression $8 \cdot 12$. These both produce the same result of $96$ so they are equivalent.

This is an example of the commutative property for multiplication, which says that $a \cdot b$ and $b \cdot a$ always produce the same result. And since $a + b$ and $b + a$ always produce the same result, there is also a commutative property for addition.

**Commutative Property for Addition:**

$$a + b = b + a$$

**Commutative Property for Multiplication:**

$$a \cdot b = b \cdot a$$

Notice that the commutative property only works with addition or multiplication, it does not apply to subtraction or division. You can switch the order of the numbers you are adding or multiplying and still get the same answer, but if you change the order of subtraction or division the final value will not be the same.

## Combining Like Terms

Expressions can be simple like $3x$ or more complicated like $8x^2 - 11x + 4x^2 + 5x$. When given a complicated expression, we often try to write it in a simpler, equivalent form. Here we'll focus on simplifying by combining like terms.

First, let's define some key terminology. When we add or subtract quantities, each individual quantity is called a **term**. When we multiply quantities together, each individual quantity is called a **factor**. When a variable is multiplied by a constant, that constant is the **coefficient** of the variable. In $5x + 3y$, the coefficient of $x$ is $5$ and the coefficient of $y$ is $3$.

Two terms are considered **like terms** if they have the exact same variables raised to the exact same powers.

- The expressions $32x$ and $3x$ are like terms because they both involve the variable $x$.
- The expressions $-5.6x^2y$ and $31x^2y$ are like terms because they are both multiples of $x^2y$.
- The expressions $21x^2$ and $7x$ are <u>not</u> like terms. While they involve the same variable, they do not have the same powers.

**Combining Like Terms:**

Like terms can be combined by adding or subtracting the coefficients and keeping the variable parts.

Using this process we are able to return to the earlier examples and combine the like terms.

**Example**

a. Combine the like terms $32x + 3x$

b. Combine the like terms $-5.6x^2y + 31x^2y$

**Solution**

a. $32x + 3x = (32 + 3)x = 35x$

b. $-5.6x^2y + 31x^2y = (-5.6 + 31)x^2y = 25.4x^2y$

When there are several like terms, the commutative property can be used to move like terms next to each other so that they are easier to combine. This is the case with the more complicated example we saw at the start of this subsection.

$$\begin{aligned} 8x^2 - 11x + 4x^2 + 5x &= 8x^2 + 4x^2 - 11x + 5x \\ &= (8+4)x^2 + (-11+5)x \\ &= 12x^2 - 6x \end{aligned}$$

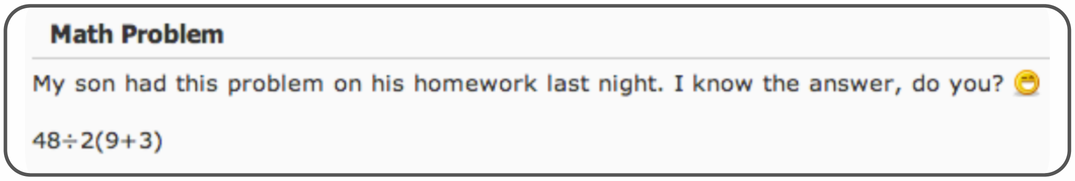
## Looking Ahead

Now that we can build algebraic expressions and combine like terms, we need to learn how to evaluate these expressions correctly when we replace the variables with specific values. Which brings up an important question: when an expression has multiple operations, which do we perform first? That's exactly the question we'll investigate in the next section.

# 2.2 Using Algebraic Expressions

## Introduction

According to the website Know Your Meme, one of the oldest math-related memes got its start on April 7, 2011 when a parent posted the following question from their son's homework to a local message board.




The answers were split, with nearly half saying $48 \div 2(9+3) = 2$ and the other half thinking $48 \div 2(9+3) = 288$. Who was right? And how can we avoid this type of confusion ourselves? That's one of the major things we'll examine in this section.

## Applying the Order of Operations

As we saw in the previous section, to evaluate an expression means to determine its value when specific values have been used for each variable.

**Example 1**

Evaluate the expression $x - y$ for the values $x = 2$ and $y = -3$.

$$\begin{aligned} x - y &= 2 - (-3) && \text{Substitute the values.} \\ &= 2 + 3 && \text{Using the rule for subtracting negative integers.} \\ &= 5 && \text{Simplify.} \end{aligned}$$

We had to be a little bit careful in this example because it involved subtracting a negative integer, but there was only one operation (the subtraction) taking place.

Often expressions will involve multiple operations. Take, for instance, the expression $5x - 3$. It involves two operations: a multiplication and a subtraction. Which one do we do first?

Fortunately, mathematicians have agreed on a set of rules that tell us which to do first. These rules are called the **standard order of operations**.

**The Standard Order of Operations**

1. Perform any operations within groupings. Groupings are typically shown within parentheses or brackets. Less commonly, groupings will involve an operation in an exponent or in the numerator (top) or denominator (bottom) of a fraction.
2. Apply any exponents.
3. Apply any multiplication or division operations from left to right.
4. Apply any addition or subtraction operations from left to right.

To remember the order of operations, people around the world use several different acronyms, such as PEMDAS, BODMAS, and GEMS. For our purposes, we'll be using the **GEMDAS** acronym.

- **G** stands for **G**roupings: Perform calculations inside grouping symbols first.
- **E** stands for **E**xponents: Calculate any exponents or powers next.
- **MD** stands for **M**ultiplication and **D**ivision: Do multiplication and division from left to right since they have the same priority.
- **AS** stands for **A**ddition and **S**ubtraction: Finally, perform addition and subtraction from left to right since they have the same priority.

If an expression has more than one operation with equal priority (like multiplication/division or addition/subtraction) then you work with them from left to right.

The following example should help illustrate this rule.

**Example 2**

Use the order of operations to evaluate

$$20 - 8 \div 2 \times 3 + 5$$

---

$$\begin{aligned} 20 - 8 \div 2 \times 3 + 5 &= 20 - 4 \times 3 + 5 && \text{First, divide: } 8 \div 2 = 4 \\ &= 20 - 12 + 5 && \text{Then, multiply: } 4 \times 3 = 12 \\ &= 8 + 5 && \text{After that, subtract: } 20 - 12 = 8 \\ &= 13 && \text{Finally, add: } 8 + 5 = 13 \end{aligned}$$

We are now ready to see which answer to the meme is correct.

**Example 3**

Use the order of operations to evaluate

$$48 \div 2(9+3)$$

$$\begin{aligned} 48 \div 2(9+3) &= 48 \div 2 \times 12 && \text{First, add: } 9+3=12 \\ &= 24 \times 12 && \text{Then, divide: } 48 \div 2 = 24 \\ &= 288 && \text{Finally, multiply: } 24 \times 12 = 288 \end{aligned}$$

Notice that we started with what was inside the parenthesis since those are a grouping symbol. That left us with a division and a multiplication. Since those are operations with the same level of importance, we evaluated them from left to right: dividing next and lastly multiplying.

More importantly, the confusion could have been avoided entirely by using proper grouping symbols. It should have been written as $\frac{48}{2}(9+3)$ if they wanted the answer to be $288$ or as $\frac{48}{2(9+3)}$ if they wanted an answer of $2$. In algebra and beyond, it is very rare to see the $\div$ symbol used which prevents this type of ambiguity.

## Evaluating Algebraic Expressions with the Order of Operations

When substituting values into an exprssion with variables, it is helpful to place parenthesis around any inserted values. Doing so will help maintain the order of operations in the expression, as we'll see in the next two examples.

**Example 4**

Carefully evaluate $7(2-x)$ for $x=-5$.

$$\begin{aligned} 7(2-x) &= 7(2-(-5)) && \text{Substitute } -5 \text{ for } x \\ &= 7(7) && \text{Subtract } 2-(-5)=7 \\ &= 49 && \text{Simplify.} \end{aligned}$$

Here $2-(-5)$ was grouped together with the parentheses, so we evaluated that subtraction first before multiplying by $7$.

**Example 5**

Carefully evaluate $4 - 5x^2$ for $x = -3$.

$$\begin{aligned} 4 - 5x^2 &= 4 - 5(-3)^2 && \text{Substitute } -3 \text{ for } x \\ &= 4 - 5 \cdot 9 && \text{Evaluate the exponent first}(-3)^2 = 9 \\ &= 4 - 45 && \text{Multiply next.} \\ &= -41 && \text{Subtract.} \end{aligned}$$

In this last example there were no groupings so we evaluated the exponent first.

## Creating and Using Algebraic Expressions

We are now ready to create expressions to match real-life situations. While there's no set process for this, if we can recognize which operations match the language used and choose our variables appropriately, we should be fine. To help, here are some common phrases that often indicate specific operations, with examples of how they translate to expressions:

**Addition**

- **sum**: The sum of two exam scores: $e_1 + e_2$
- **total**: The total cost of a shirt and pants: $s + p$
- **combined**: The combined number of apples and bananas: $a + b$

**Subtraction**

- **decreased by**: The price decreased by 3 dollars: $p - 3$
- **fewer than**: The team scored 10 fewer points: $s - 10$
- **difference**: The difference between the high and low temperatures: $H - L$

**Multiplication**

- **product**: The product of length, width, and height: $l \cdot w \cdot h$
- **each**: Each movie ticket costs 12 dolars: $12t$
- **of**: A used car is 2/3 the price of a new one: $\frac{2}{3}c$
- **per**: A mechanic charges 80 dollars per hour: $80h$

**Division**

- **ratio**: The ratio of wins to losses: $\frac{w}{l}$
- **divided by**: Distance divided by time: $\frac{D}{T}$
- **quotient**: The quotient of cookies and people: $\frac{c}{p}$
- **split evenly**: The cost of dinner is split evenly among a number of friends: $\frac{c}{n}$

Often situations will involve more than one operations and you will have to be careful to place them in the proper order when creating a representative expression.

**Example 6**

The cost to hire a plumber involves a $\$65$ call-out fee on top of a labor charge of $\$85$ per hour. Write an expression that would give the total cost to hire a repair technician for $x$ hours. Use your expression to calculate the cost for the technician for $5$ hours.

**Solution**

The $\$85$ rate per hour should multiply the number of hours worked. The $\$65$ call-out fee is then added to that total. So the cost of hiring this plumber for $x$ hours could be calculated with the expression: $85x + 65$.

The total cost for 5 hours would be

$$85 \cdot (5) + 65 = 425 + 65 = \$490$$

Expressions can also be written when there are several different quantities that need to be represented by variables.

**Example 7**

A store earns $\$25$ for every shirt sold, $\$40$ for every pair of jeans and $\$10$ for belts. Write an expression for the total amout of money earned based on the number of shirts, jeans and belts that were sold.

---

We need to multiply the number of items sold by their individual prices and then add the results. This gives us an expression of

$$25s + 40j + 10b$$

if we use $s$, $j$ and $b$ for the number of shirts, jeans and belts sold.

In our final example we will see two different, but equivalent, expressions that both represent the same thing.

**Example 8**

An employee earns $\$18$ per hour. They get paid every two weeks and worked $27$ hours last week. Write an expression for what their total pay will be if they work $x$ hours this week. Use your expression to calculate the employee's pay if they work $30$ hours this week.

There are two different ways to think about this. We could either calculate their pay each week and add the results, or we could total up their hours for both weeks and then calculate the pay. The first method gives the expression

$$18 \cdot 27 + 18x$$

while the other is

$$18(27 + x)$$

Both expressions are equivalent so we can use either one to calculate their total pay if they work another $30$ hours. With the first expresson we get

$$18 \cdot 27 + 18 \cdot 30 = 486 + 540 = \$1026$$

and with the second one the result is

$$18(27 + 30) = 18(57) = 1026$$

In either case, the employee would earn $\$1026$.

## Looking Ahead

Thinking about our previous example, it is easy to see where the need to solve an equation comes from. If the employee wanted to earn a certain amount they would have to figure out what to plug in for $x$ to have this expression equal that amount. In the following chapter we'll go through methods for finding answers to questions like that.

But first we'll need to learn and practice more techniques for working with and simplifying expressions, and that's what the next few sections are for.

# 2.3 Exponents

## Introduction

Earlier in this chapter we saw that terms can be added and subtracted only when they have the same variables and the same powers. We matter those **like terms**.

Things get a bit more interesting when we start multiplying and dividing terms because there is no requirement that the powers and variables match.

## Definition of an Exponent

The nice thing about every rule for exponents is that if you ever forget one you can always figure it out by going back to the basic definition of an exponent.

**Definition of an Exponent**

For any natural number $n$, the expression $b^n$ means $b \cdot b \cdot b \cdot \ldots \cdot b$ where $b$ is a factor $n$ times.

$$\underset{\text{base}}{b}^{\overset{\text{exponent}}{n}} = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$$

We call $b$ the base and $n$ the exponent and say that $b$ is being raised to the $n\text{th}$ power.

By using this definition we'll be able to spot the patterns that make up the shortcut rules we can use to simplify complicated exponential expressions.

## The Product Rule

Consider the product $b^3 \cdot b^4$. Both terms have the same base, $b$, but are raised to different exponents. Notice what happens when we expand each of them using the definition of an exponent.

$$\begin{aligned} b^3 \cdot b^4 &= (b \cdot b \cdot b)(b \cdot b \cdot b \cdot b) \\ &= b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \\ &= b^7 \end{aligned}$$

Is there a pattern we could use as a shortcut to go from $b^3 \cdot b^4$ to $b^7$ without having to expand each term? Notice that $3 + 4 = 7$. That's the shortcut!

When multiplying expressions with the same base, we simply keep the base and add the two powers. This is the **product rule** of exponents and can be written as $b^m \cdot b^n = b^{m+n}$. We will use this in our next example.

**Example 1**

Use the product rule to simplify $2^3 \cdot 2^8$.

$$\begin{aligned} 2^3 \cdot 2^8 &= 2^{3+8} \\ &= 2^{11} \\ &= 2048 \end{aligned}$$

**Example 2**

Simplify $x^5 \cdot x^3$.

$$\begin{aligned} x^5 \cdot x^3 &= x^{5+3} \\ &= x^8 \end{aligned}$$

## The Quotient Rule

Let's try a similar technique on the quotient of two terms. We'll expand $\frac{b^5}{b^3}$ just as we did above, simplify, and look for a pattern.

$$\begin{aligned}\frac{b^5}{b^3} &= \frac{b \cdot b \cdot b \cdot b \cdot b}{b \cdot b \cdot b} \\ &= \frac{b}{b} \cdot \frac{b}{b} \cdot \frac{b}{b} \cdot b \cdot b \\ &= 1 \cdot 1 \cdot 1 \cdot 1 \cdot b \cdot b \\ &= b \cdot b \\ &= b^2\end{aligned}$$

Notice that the final exponent is $2$, which also happens to be the difference of the original powers $5 - 3 = 2$. When dividing exponential expressions with the same base, the result has the same base and the power is the top exponent minus the bottom exponent.

We call this the **quotient rule** for exponents and it can be expressed as $\dfrac{b^m}{b^n} = b^{m-n}$.

**Example 3**

Use the quotient rule to simplify $\dfrac{5^6}{5^2}$.

$$\begin{aligned}\frac{5^6}{5^2} &= 5^{6-2} \\ &= 5^4 \\ &= 625\end{aligned}$$

**Example 4**

Simplify $\dfrac{y^9}{y^4}$.

$$\begin{aligned}\frac{y^9}{y^4} &= y^{9-4} \\ &= y^5\end{aligned}$$

## Power of a Power Rule

Next, let's explore raising a term with a power to another power. For example, what happens if we raise $b^3$ to the 4th power? This would mean repeating $b^3$ as a factor $4$ times and then expanding each of those $b^3$.

$$\begin{aligned}(b^3)^4 &= b^3 \cdot b^3 \cdot b^3 \cdot b^3 \\ &= (b \cdot b \cdot b)(b \cdot b \cdot b)(b \cdot b \cdot b)(b \cdot b \cdot b) \\ &= \underbrace{b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b}_{12\ times} \\ &= b^{12}\end{aligned}$$

That was a lot of expanding but, hopefully you can spot the shortcut. The shortcut is simply to multiply the exponents: $(b^m)^n = b^{m \cdot n}$. This is called the **power of a power rule**.

**Example 5**

Simplify $(5^2)^3$.

$$\begin{aligned}(5^2)^3 &= 5^{2 \cdot 3} \\ &= 5^6 \\ &= 15625\end{aligned}$$

**Example 6**

Simplify $(a^4)^6$.

$$\begin{aligned}(a^4)^6 &= a^{4 \cdot 6} \\ &= a^{24}\end{aligned}$$

## Power of a Product Rule

When a product of variables and/or real numbers is in parentheses and raised to a power, then all the factors are raised to that power. This rule, like the others, becomes apparent by going back to the definition of an exponent and making use the commutative property of multiplication.

$$\begin{aligned}(4b)^3 &= (4b)(4b)(4b) && \text{Use the definition of an exponent} \\ &= 4 \cdot 4 \cdot 4 \cdot b \cdot b \cdot b && \text{Apply the commutative property of multiplication} \\ &= 4^3 b^3 && \text{Use the definition of an exponent} \\ &= 64b^3\end{aligned}$$

The shortcut here, of course, is that the power could have been applied to each part of the product: $(a \cdot b)^n = a^n \cdot b^n$. This is how we deal with the **power of a product**.

**Example 7**

Simplify $(5xy)^3$.

$$\begin{aligned}(5xy)^3 &= 5^3\,x^3\,y^3 \\ &= 125\,x^3\,y^3\end{aligned}$$

**Example 8**

Simplify $(3xy^2)^4$.

$$\begin{aligned}(3xy^2)^4 &= 3^4 \cdot x^4 \cdot (y^2)^4 \\ &= 81x^4y^8\end{aligned}$$

## Power of a Quotient Rule

Let's see if a similar thing works if we take the power of a quotient.

$$\begin{aligned}\left(\frac{2}{b}\right)^5 &= \left(\frac{2}{b}\right)\left(\frac{2}{b}\right)\left(\frac{2}{b}\right)\left(\frac{2}{b}\right)\left(\frac{2}{b}\right) \\ &= \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{b \cdot b \cdot b \cdot b \cdot b} \\ &= \frac{2^5}{b^5} \\ &= \frac{32}{b^5}\end{aligned}$$

Notice how it would have been much shorter to simply apply a power of $5$ to all parts of the quotient.

Thus, the rule for finding the **power of a quotient** is $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. In simple terms, when a quotient is raised to a power, then that power applies to all parts of the quotient.

**Example 9**

Simplify $\left(\dfrac{2x^3}{y}\right)^5$.

$$\begin{aligned}\left(\frac{2x^3}{y}\right)^5 &= \frac{(2x^3)^5}{y^5}\\ &= \frac{2^5\cdot(x^3)^5}{y^5}\\ &= \frac{32x^{15}}{y^5}\end{aligned}$$

## Application: Unit Conversion

The rules for powers of products and quotients are useful in real-world situations where you need to convert between units for area or volume. In such cases, rather than finding a new conversion factor, we simply start with the regular conversion factor and then apply the appropriate power to the entire factor—both the numbers and the units.

**Example 10**

A box has a volume of $1000\ \text{in}^3$. Convert this to cubic centimeters. Use $1\ \text{in} = 2.54\ \text{cm}$.

$$\begin{aligned}&1000\ \text{in}^3 \times \left(\frac{2.54\ \text{cm}}{1\ \text{in}}\right)^3 && \text{Cube the conversion factor}\\ &= 1000\ \text{in}^3 \times \frac{16.387\ \text{cm}^3}{1\ \text{in}^3} && \text{Apply the power}\\ &= 1000 \times 16.387\ \text{cm}^3 && \text{Simplify the units}\\ &= 16{,}387\ \text{cm}^3 && \text{Multiply}\end{aligned}$$

**Example 11**

A rectangular room measures $12$ feet by $15$ feet. Find its area in square meters. Use $1\ \text{ft} = 0.3048\ \text{m}$.

First, multiply the two given dimensions to get the area in square feet, then convert to square meters:

$$\begin{aligned}
12\ \text{ft} \times 15\ \text{ft} &= 180\ \text{ft}^2 && \text{Area in square feet} \\
&= 180\ \text{ft}^2 \times \left(\frac{0.3048\ \text{m}}{1\ \text{ft}}\right)^2 && \text{Square the conversion factor} \\
&= 180\ \text{ft}^2 \times \frac{0.0929\ \text{m}^2}{1\ \text{ft}^2} && \text{Apply the power} \\
&= 180 \ \ \times 0.0929\ \text{m}^2 && \text{Simplify the units} \\
&\approx 16.7\ \text{m}^2 && \text{Multiply}
\end{aligned}$$

Alternatively, we could convert both measurements to meters first, then multiply to find the area:

$$\begin{aligned}
&\left(12\ \text{ft} \times \frac{0.3048\ \text{m}}{1\ \text{ft}}\right) \cdot \left(15\ \text{ft} \times \frac{0.3048\ \text{m}}{1\ \text{ft}}\right) \\
&= (12 \times 0.3048\ \text{m}) \cdot (15 \times 0.3048\ \text{m}) \\
&\approx 16.7\ \text{m}^2
\end{aligned}$$

Both methods produce the same result, so choose whichever approach feels more natural to you.

## Simplifying Exponents

While our examples so far have focused on each property individually, there's no reason to expect that will always be the case. In many instances we will need to deploy multiple rules in order to fully simplify an expression with exponents.

Before we finish with one of these more complicated examples, let's list all of the rules we've discovered so far.

**Rules of Exponents**

| Exponent Rule | Formula | Example |
|---|---|---|
| Product of Powers | $b^m \cdot b^n = b^{m+n}$ | $2^3 \cdot 2^4 = 2^{3+4}$ |
| Quotient of Powers | $\dfrac{b^m}{b^n} = b^{m-n}$ | $\dfrac{5^7}{5^2} = 5^{7-2}$ |
| Power of a Power | $(b^m)^n = b^{m \cdot n}$ | $(3^2)^4 = 3^{2 \cdot 4}$ |
| Power of a Product | $(a \cdot b)^n = a^n \cdot b^n$ | $(2 \cdot 4)^3 = 2^3 \cdot 4^3$ |
| Power of a Quotient | $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ | $\left(\dfrac{6}{3}\right)^4 = \dfrac{6^4}{3^4}$ |

Now that we have all of the rules available for quick reference, we'll dive into our final example.

**Example 12**

Simplify $\left(\dfrac{-3w^3}{5w}\right)^4$.

$$\begin{aligned}
\left(\frac{-3w^3}{5w}\right)^4 &= \left(\frac{-3w^2}{5}\right)^4 && \text{Use the Quotient Rule to simplify } \frac{w^3}{w} = w^2 \\
&= \frac{(-3w^2)^4}{5^4} && \text{Use the Power of a Quotient Rule.} \\
&= \frac{(-3)^4(w^2)^4}{5^4} && \text{Use the Power of a Product Rule.} \\
&= \frac{(-3)^4 w^8}{5^4} && \text{Use the Power of a Power Rule.} \\
&= \frac{81w^8}{625} && \text{Evaluate the powers.}
\end{aligned}$$

It's important to emphasize that there is no fixed sequence for simplifying exponents. In many cases, especially when dealing with intricate expressions, multiple pathways can lead to the final answer. And when in doubt, a reliable approach is to revert back to the fundamental definition of an exponent, expand the expression, and then simplify.

## Looking Ahead

The strategies we've learned for working with positive exponents will continue to serve us well in the next section as we tackle negative exponents. We'll discover that the same fundamental principles apply, even when the exponents themselves are negative numbers.

# 2.4 Negative Exponents

## Introduction

Building on our experience using definition of an exponent to find rules for positive powers, in this section we'll look into zero powers and negative exponents.

Then, equipped with those new tools, we'll discuss a special notation used in science that allows us to describe both values of immense measure and values of minuscule size with the same level of precision.

## The Zero Exponent Rule

Let's start this section by reminding you of a very basic rule of arithmetic that we've used many times before: *anything (except* $0$*) divided by itself equals* $1$.

You're used to using that rule to simplify expressions with numbers like $\dfrac{13}{13}=1$, but you may not have realized it works for any type of expression, provided it doesn't equal $0$. For instance, $\dfrac{2x+3}{2x+3}=1$ as long as $2x+3\neq 0$ (since division by $0$ is undefined).

So then, what can we make of an expression like $\dfrac{b^3}{b^3}$? As long as $b\neq 0$, this has to equal $1$. That seems simple enough, but how does it help us find a rule for a zero exponent?

In the last section we learned about the quotient rule. Look at what happens when we simplify the same expression $\dfrac{b^3}{b^3}$ using the quotient rule.

$$\begin{aligned}\frac{b^3}{b^3}&=b^{3-3} && \text{Use the quotient rule: } \frac{b^m}{b^n}=b^{m-n}\\ &=b^0 && \text{Simplify}\end{aligned}$$

We've just seen that $\dfrac{b^3}{b^3}=b^0$ but we also have $\dfrac{b^3}{b^3}=1$. Putting these two answers together we see that $b^0=1$. This is our rule for a zero exponent and is true as long as $b$ is not zero.

Just in case you were wondering, $0^0$ does not have a defined value. It's considered "indeterminate" and is studied more in a calculus course.

Let's see how the zero exponent rule applies in different situations.

**Example 1a**

Simplify $\dfrac{6^4}{11^0}$. Assume all variables are nonzero.

$$\begin{aligned} \frac{6^4}{11^0} &= \frac{6^4}{1} && \text{Since } 11^0 = 1 \\ &= 1296 \end{aligned}$$

**Example 1b**

Simplify $\dfrac{x^3y^5}{x^3}$. Assume all variables are nonzero.

$$\begin{aligned} \frac{x^3y^5}{x^3} &= x^0y^5 && \text{Use the quotient rule} \\ &= 1 \cdot y^5 && \text{Use the zero exponent rule} \\ &= y^5 \end{aligned}$$

## The Negative Power Rule

The strategy of simplifying expression two different ways and equating the results is a major way identities and rules are discovered in math. We just used that method to uncover the rule for a zero power, and we will continue to use it to find rules for negative powers.

Consider, for example, the expression $\dfrac{b^4}{b^7}$. One way to simplify this is to use the definition of an exponent.

$$\begin{aligned} \frac{b^4}{b^7} &= \frac{b \cdot b \cdot b \cdot b}{b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b} && \text{Use the definition of an exponent} \\ &= \frac{b}{b} \cdot \frac{b}{b} \cdot \frac{b}{b} \cdot \frac{b}{b} \cdot \frac{1}{b \cdot b \cdot b} && \text{Use the commutative property of multiplication} \\ &= 1 \cdot 1 \cdot 1 \cdot 1 \frac{1}{b \cdot b \cdot b} && \text{Since } \frac{b}{b} = 1 \\ &= \frac{1}{b \cdot b \cdot b} \end{aligned}$$

Now watch what happens when we simplify the same expression using the quotient rule for exponents.

$$\begin{aligned} \frac{b^4}{b^7} &= b^{4-7} && \color{blue}{\text{Use the quotient rule}} \\ &= b^{-3} && \color{blue}{\text{Simplify}} \end{aligned}$$

Putting the answers together, we have $b^{-3} = \dfrac{1}{b^3}$. This is an example of the rule for negative powers. Simply put, $b^{-n} = \dfrac{1}{b^n}$, provided $b \neq 0$.

Whenever we are asked to "simplify" an expresssion with negative exponents, it's standard practice to rewrite the expression so that the final answer only has positive exponents.

**Example 2a**

Simplify $z^{-1} \cdot w^3$ and write the answer with positive exponents.

$$z^{-1}w^3 = \frac{w^3}{z} \qquad \color{blue}{\text{Since } z^{-1} = \frac{1}{z}}$$

**Example 2b**

Simplify $b^2 \cdot b^{-8}$ and write the answer with positive exponents.

$$\begin{aligned} b^2 \cdot b^{-8} &= b^{2+(-8)} && \color{blue}{\text{Use the product rule}} \\ &= b^{-6} && \color{blue}{\text{Simplify}} \\ &= \frac{1}{b^6} && \color{blue}{\text{Since } b^{-6} = \frac{1}{b^6}} \end{aligned}$$

Occasionally we might come across an expression that has a negative exponent on the bottom of a fraction, something like $\dfrac{1}{b^{-3}}$. How should we deal with that? A careful application of the zero exponent rule and the quotient will give the answer.

$$\begin{aligned} \frac{1}{b^{-3}} &= \frac{b^0}{b^{-3}} && \color{blue}{\text{Since } b^0 = 1} \\ &= b^{0-(-3)} && \color{blue}{\text{Use the quotient rule}} \\ &= b^3 && \color{blue}{\text{Simplify}} \end{aligned}$$

How about that? The result has a positive exponent! In general, $\dfrac{1}{b^{-n}} = b^n$. This is a matching counterpart to the usual rule $b^{-n} = \dfrac{1}{b^n}$ for negative exponents.

An easy way to remember both rules is to notice that a term with a negative exponent transforms into a term with a positive exponent if we shift it up or down across the fraction line.

**Example 3**

Simplify $\dfrac{4a^{-5}}{b^{-3}}$ and write the answer with positive exponents.

$$\frac{4a^{-5}}{b^{-3}} = \frac{4b^3}{a^5}$$

You might be wondering why the $4$ didn't get moved to the bottom. The reason is that in $4a^{-5}$ it's only the $a$ that has a negative exponent. If it had been $(4a)^{-5}$ then the power would have applied to both and we would have moved it all to the denominator.

With this result in hand, we are finally ready to make a complete list of all the rules of exponents.

**Rules of Exponents**

| Exponent Rule | Formula | Example |
|---|---|---|
| Product of Powers | $b^m \cdot b^n = b^{m+n}$ | $2^3 \cdot 2^4 = 2^{3+4}$ |
| Quotient of Powers | $\dfrac{b^m}{b^n} = b^{m-n}$ | $\dfrac{5^7}{5^2} = 5^{7-2}$ |
| Power of a Power | $(b^m)^n = b^{m \cdot n}$ | $(3^2)^4 = 3^{2 \cdot 4}$ |
| Power of a Product | $(a \cdot b)^n = a^n \cdot b^n$ | $(2 \cdot 4)^3 = 2^3 \cdot 4^3$ |
| Power of a Quotient | $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ | $\left(\dfrac{6}{3}\right)^4 = \dfrac{6^4}{3^4}$ |
| Zero Exponent Rule | $b^0 = 1$ (as long as $b \neq 0$) | $7^0 = 1$ |
| Negative Exponent Rule | $b^{-n} = \dfrac{1}{b^n}$ and $\dfrac{1}{b^{-n}} = b^n$ | $2^{-3} = \dfrac{1}{2^3}$ |

All of the rules we learned in the last section still apply when working with negative exponents.

In the next examples we will need to use several of these rules.

**Example 4**

Simplify $\left(\dfrac{x^8 \cdot y^{-5}}{z^9}\right)^{-2}$ and write the answer with positive exponents.

$$\begin{aligned}
\left(\frac{x^8 \cdot y^{-5}}{z^9}\right)^{-2} &= \frac{\left(x^8\right)^{-2} \cdot \left(y^{-5}\right)^{-2}}{\left(z^9\right)^{-2}} && \text{Use the power of a quotient rule} \\
&= \frac{x^{-16} \cdot y^{10}}{z^{-18}} && \text{Using the power of powers rule} \\
&= \frac{y^{10} \cdot z^{18}}{x^{16}} && \text{Using negative power rule}
\end{aligned}$$

## Scientific Notation

We've been exploring the theoretical framework around exponents, and it's now time to step into a common application: scientific notation.

Scientific notation is a compact way of writing very small numbers and very large numbers that are too long to comfortably write normally. For instance, the distance from Earth to the nearest star, Alpha Centauri, is $40,208,000,000,000,000$ meters away. Or, on the other end of the spectrum, diameter of a hydrogen atom is about $0.000000000106$ meters. Imagine how easy it would be to make a mistake copying down one of those numbers, not to mention the difficulty in doing arithmetic with them.

Scientific notation makes the numbers easier to write and easier to compare, by separating them into two parts: a number and a power of $10$.

**Scientific Notation**

A number is in scientific notation if it is written in the form

$$a \times 10^n$$

where $1 < a < 10$ and $n$ is an integer.

Notice that the number part always has exactly one nonzero digit on the left of the decimal. It sounds more complicated than it actually is. This is one of those things that's easier to see than to read about. The following numbers are written in standard decimal format and in scientific notation.

| Original Number | Scientific Notation |
|---|---|
| $53200$ | $5.32 \times 10^4$ |
| $40,208,000,000,000,000$ | $4.0208 \times 10^{16}$ |
| $0.0000077$ | $7.7 \times 10^{-6}$ |
| $0.00138$ | $1.38 \times 10^{-3}$ |

Again, there are two key things to notice. First, the number part must be a decimal number between $1$ and $10$. If your starting number isn't between $1$ and $10$, then you move the decimal place until it is. Second, count the number of places, $n$, that you moved the decimal point and multiply the decimal number by $10$ raised to a power of $n$. The sign of $n$ should be positive if you started with a large number and negative if you started with a small number.

Let's start with a few examples where we begin with a number in standard format and convert it into scientific notation.

**Example 5a**

Change $0.00000000000047$ into scientific notation.

$$0.00000000000047 = 4.7 \times 10^{-13}$$

**Example 5b**

Change $92,960,000$ into scientific notation.

$$92,960,000 = 9.296 \times 10^7$$

In these final examples we will do the reverse, starting with a number in scientific notation and turning it back into a number in standard form.

**Example 6a**

Change $5.88 \times 10^{12}$ into standard notation.

$$5.88 \times 10^{12} = 5,880,000,000,000$$

**Example 6b**

Change $7.53 \times 10^{-10}$ into standard notation.

$$7.53 \times 10^{-10} = 0.000000000753$$

## Conclusion

Zero and negative exponents complete our foundation of exponent rules. What started as simple repeated multiplication now extends to handle any integer exponent. In fact, though we will not prove this, the rules apply to all nonzero real number exponents.

This mathematical elegance, where patterns extend logically to new situations, is a hallmark of algebra. And as we continue through algebra, using patterns like these will be essential for simplifying expressions and solving equations.

# 2.5 The Distributive Property

## Introduction

Suppose a store is offering $10\%$ off every item in stock. This discount could be applied two different ways: either by reducing the price of each item individually or by calculating the total price of everything and then applying the discount.

As you probably know from experience, both methods give the same final result. What you might not know, however, is *why* the results are the same. That's what we'll explore in this section.

## The Distributive Property

It's important to point out that the example in the introduction is an example of a special property that doesn't hold true for all mathematical expressions. In most cases, we can't simply rearrange the order of addition and multiplication without changing the outcome.

However, as the example illustrates, in the special case where the same number is multiplying several different values, there are two equivalent ways to evaluate the expression. This relationship is matter the **distributive property**.

**The Distributive Property:**

For any numbers $a$, $b$, and $c$,

$$a(b+c) = a \cdot b + a \cdot c$$

Many people find it helpful to draw arrows to indicate the multiplications that flow from applying the distributive property.

$$a\,(b+c) = ab + ac$$

The distributive property also holds for subtraction: $a(b-c) = a \cdot b - a \cdot c$ and can be extended to have multiple values inside the parenthesis including variables. It's also possible to distribute not just single numbers but entire expressions! We'll walk through examples of all of those below.

## Distributing Numbers

Consider the expression $3(4-9)$. Let's evaluate it the standard way and also using the distributive property, just to show the answers are the same.

Option 1. Use the order of operations:

$$\begin{aligned} 3(4-9) &= 3(-5) \\ &= -15 \end{aligned}$$

Option 2. Use the distributive property:

$$\begin{aligned} 3(4-9) &= 3(4) - 3(9) \\ &= 12 - 27 \\ &= -15 \end{aligned}$$

When distributing an integer, be extra careful with any negative signs.

$$\begin{aligned} -2(-3+5-11) &= -2(-3) - 2(5) - 2(-11) \\ &= 6 - 10 + 22 \\ &= 18 \end{aligned}$$

While using distribution might not offer a clear advantage over the standard order of operations when dealing with only numbers, its power becomes evident when expressions within parentheses involve variables. Consider the example:

$$\begin{aligned} -2.5(3x-4) &= -2.5(3x) - (-2.5)(4) \\ &= -7.5x - (-10) \\ &= -7.5x + 10 \end{aligned}$$

In this case, distributing allows us to simplify the expression by applying the multiplication to term individually.

## Distributing Monomials

By combining distribution with the product rule of exponents we can distribute monomials. A **monomial** is a single term containing a coefficient and possibly one or more variables raised to non-negative integer powers. For instance, $-2x$ and $3x^2$ are monomials.

Multiplying monomials involves multiplying the number parts (the coefficients) and multiplying the variable parts (using the product rule). To multiply $-2x$ and $3x^2$ we would multiply $-2 \cdot 3 = -6$ and multiply $x \cdot x^2 = x^3$, yielding the answer $-6x^3$.

When distributing monomials, we follow the same principle as before: each term inside the parentheses is multiplied by the monomial outside.

**Example 1a**

Use distribution to simplify $3x(-2x+5)$.

$$\begin{aligned} 3x(-2x+5) &= 3x(-2x)+3x(5) && \text{Distribute } 3x \\ &= -6x^2+15x && \text{Multiply} \end{aligned}$$

**Example 1b**

Use distribution to simplify $-4x^2(5x^5-7x+1)$.

$$\begin{aligned} -4x^2(5x^5-7x+1) &= -4x^2(5x^5)-4x^2(-7x)-4x^2(1) && \text{Distribute } -4x^2 \\ &= -20x^7+28x^3-4x^2 && \text{Multiply} \end{aligned}$$

In these examples we've been distributing a monomial to a combination of numbers and variables within parentheses. But what if the monomial was replaced with a larger expression? Could we use the same approach? Let's find out.

## Distributing Binomials

A **binomial** is an expression containing two monomial terms. For instance, $3x-4$ and $x+6$ are both binomials.

If we need to find the product $(3x-4)(x+6)$ we can apply the same process we used with monomials, but this time each term in the first binomial gets distributed to the entire second binomial, essentially making this a double distribution. The process will look like this:

$$(3x-4)(x+6) = 3x(x+6) - 4(x+6)$$

Let's walk through those steps and see what we end up with.

$$\begin{aligned} (3x-4)(x+6) &= 3x(x+6)-4(x+6) && \text{Distribute each term to the second binomial} \\ &= 3x^2+18x-4x-24 && \text{Distribute } 3x \text{ and } -4 \\ &= 3x^2+14x-24 && \text{Combine like terms} \end{aligned}$$

To shorten the process, it can be helpful to draw arrows indicating all of the multiplications that need to be done. Here's the general template:

$$(a+b)(c+d) = ac+ad+bc+bd$$

You might recognize this as the so-matter "FOIL method". FOIL is a popular acronym some people find useful for remembering the steps for multiplying two binomials. It is matter FOIL because we multiply the First terms, the Outer terms, the Inner terms, and then the Last terms of each binomial.

$$(a+b)(c+d) = ac + ad + bc + bd$$

Since the FOIL method comes from the distributive property, if you are comfortable with distribution then it is not necessary to memorize what each FOIL letter stands for, but please use it if you find it helpful.

Let's try a few examples.

**Example 2a**

Use distribution to find the product: $(x+3)(x+2)$

$$\begin{aligned}
(x+3)(x+2) &= x(x) + x(2) + 3(x) + 3(2) && \text{Apply distribution/FOIL} \\
&= x^2 + 2x + 3x + 6 && \text{Multiply} \\
&= x^2 + 5x + 6 && \text{Combine like terms}
\end{aligned}$$

**Example 2b**

Use distribution to find the product: $(2x-9)(3x+4)$

$$\begin{aligned}
(2x-9)(3x+4) &= 2x(3x) + 2x(4) - 9(3x) - 9(4) && \text{Apply distribution/FOIL} \\
&= 6x^2 + 8x - 18x - 36 && \text{Multiply} \\
&= 6x^2 - 10x - 36 && \text{Combine like terms}
\end{aligned}$$

In our final example, we will multiply a binomial $x+2$ by a trinomial $3x^2 - 7x + 6$. There is no FOIL acronym for this situation, but drawing distribution arrows can certainly help you keep track of all the multiplications regardless of the size of the two expressions.

$$(x+2)(3x^2 - 7x + 6)$$

With that as our guide, let's finish this example.

**Example 3**

Use distribution to find the product: $(x+2)(x^2-7x+6)$

$$\begin{aligned}(x+2)(3x^2-7x+6) &= x(3x^2-7x+6)+2(3x^2-7x+6)\\ &= x(3x^2)+x(-7x)+x(6)+2(3x^2)+2(-7x)+2(6)\\ &= 3x^3-7x^2+6x+6x^2-14x+12\\ &= 3x^3-x^2-8x+12\end{aligned}$$

## Conclusion

The distributive property is a fundamental tool that connects multiplication with addition and subtraction. Whether you're distributing a single number, a monomial, or even a binomial, the underlying principle remains the same: multiply each term inside the parentheses by the expression outside.

# 2.6 Factoring

## Introduction

In the previous section, we spent time using the distributive property to expand expressions into multiple terms that could then be simplified. In this section we'll use those insights to do the reverse: convert an expanded expression back into its original factors.

## Identifying Factors.

We should start by saying exactly what a **factor** is. When two quantities are combined through multiplication, each of those quantities is matter a factor. For example, consider the expression $3xy$. This expression has three factors: $3$, $x$ and $y$ since they are quantites being multiplied together to create the expression.

Sometimes the factors within an expression may not be immediately apparent. To identify factors it is helpful to think about what can divide evenly into the expression.

An example with numbers will illustrate this idea. We might ask if $3$ is a factor of $15$. To answer this question, we should ask ourselves if $15$ can be divided by $3$ without leaving a remainder. Alternatively, we can try to write $15$ as the product of $3$ and another number. Since $\frac{15}{3} = 5$ and $3 \cdot 5 = 15$, it becomes clear that $3$ is a factor of $15$.

The same type of reasoning can be applied to expressions with variables and powers. For instance, is $7x$ a factor of $-21x^2$? In other words, can we divide $-21x^2$ by $7x$ without any remainder?

The answer is yes, we can do that division and $\frac{-21x^2}{7x} = -3x$ with no remainder. Using this result we can see that $-21x^2 = (7x)(-3x)$.

By expressing $-21x^2$ as $(7x)(-3x)$ we are putting it in a **factored form** and we can say that we have **factored** the original expression.

There can be several different, yet equally valid ways to factor $6x^2$.

$$6x^2 = 6 \cdot x^2$$
$$6x^2 = 6 \cdot x \cdot x$$
$$6x^2 = 3x \cdot 2x$$
$$6x^2 = 2 \cdot 3 \cdot x^2$$

You can probably come up with additional ways to factor $6x$ beyond the four written here. While all of these representations differ in appearance, they all represent the same quantity and can be used interchangeably.

It is critical to point out that factoring involves expressing a quantity as the product of other quantities multiplied together. Although we could write $6x^2 = 4x^2 + 2x^2$, that would not be considered a way to factor $6x^2$ because it involves adding terms rather than multiplying them.

## Factoring the Greatest Common Factor

The examples we've seen so far contained a single term. Oftentimes, factoring comes into play when we are dealing with expression that have multiple terms. In such cases, it can be helpful to identify factors that are common to all of the terms. In particular, we will be looking for the largest common factor shared by the terms, matter the **greatest common factor** or GCF.

Think about the expressions$7x^2$ and $14x$, what is their greatest common factor? The largest thing we can divide into both terms is $7x$. How do we know that is the largest and that there isn't something bigger? If we divide both terms by $7x$ we get:

$$\frac{7x^2}{7x} = x$$

$$\frac{14x}{7x} = 2$$

Since $x$ and $2$ don't have any common factors, we can be confident that we've found the GCF.

**Example 1**

Find the GCF of $6x^4$ and $9x^2$.

---

Each of the coefficients can be divided by $3$ and each of the variable parts can be divided by $x^2$, so the GCF is $3x^2$. To double check, we divide both by $3x^2$

$$\frac{6x^4}{3x^2} = 2x^2$$

$$\frac{9x^2}{3x^2} = 3$$

and notice that the results don't have anything in common.

Now that we can identify the greatest common factor between terms, let's move on to factoring the GCF out of an expression. Factoring out the greatest common factor involves dividing each term by the GCF and rewriting the expression in factored form.

If we take the expression $6x^4 + 9x^2$ then we can use the results from our previous example since we already know the GCF is $3x^2$.

$$\begin{aligned} 6x^4 + 9x^2 &= 3x^2\left(\frac{6x^4}{3x^2}\right) + 3x^2\left(\frac{9x^2}{3x^2}\right) \\ &= 3x^2\left(2x^2 + 3\right) \end{aligned}$$

Notice that if we were to take our result and distribute the $3x^2$ we would exactly reverse our steps and end up back with the original expression.

$$\begin{aligned} 3x^2(2x^2 + 3) &= 3x^2(2x^2) + 3x^2(3) \\ &= 6x^4 + 9x^2 \end{aligned}$$

We can always check our factoring by multiplying.

## Factoring Quadratic Trinomials of the form $x^2 + bx + c$

Although we should always begin by looking for a GCF, expressions may not have one. This is often the case with polynomials of the form $x^2 + bx + c$, which are sometimes matter quadratic (due to the exponent of $2$) trinomials (because there are 3 terms). Quadratics play a pivotal role in various mathematical contexts and you will continue to make use of them in every math class from here on.

However, lack of a GCF does not mean that an expression cannot be factored. Take, for example, $x^2 + 5x + 6$ which has a GCF of $1$ but can be factored as $(x + 2)(x + 3)$. We can check this using the FOIL method or distribution.

$$\begin{aligned} (x + 2)(x + 3) &= x^2 + 2x + 3x + 6 \\ &= x^2 + 5x + 6 \end{aligned}$$

This check is more than a trival thing. If we look at it closely we can unlock the process for factoring quadratic trinomials of the form $x^2 + bx + c$. Notice that the two numbers in our factors add up to $5$ and multiply to $6$, which are the coefficients $b$ and $c$ in $x^2 + 5x + 6$. That's the trick.

To factor trinomials of the form $x^2 + bx + c$ we need to find two numbers that add up to $b$ and multiply to $c$. Once we have those, the factored form will be $(x + \text{Number1})(x + \text{Number2})$.

In practice, we usually examine pairs of factors of $c$ and see if any of them add up to $b$. That is often quicker and easier than a guess-and-check method.

**Example 2**

Let's see this in work with a few examples.

Factor $x^2 + 10x + 16$.

---

Since $c = 16$, we should list all the factor pairs that multiply to $16$.

$$\begin{aligned} 16 &= 1 \cdot 16 \\ &= 2 \cdot 8 \\ &= 4 \cdot 4 \end{aligned}$$

Of those pairs, $2 + 8 = 10$ so $2$ and $8$ are the numbers we are looking for and our answer is that

$$x^2 + 10x + 16 = (x + 2)(x + 8)$$

**Example 3**

Factor $x^2 + 2x - 15$.

---

Since $c = -15$, we should list all the factor pairs that multiply to $-15$.

$$\begin{aligned} -15 &= -1 \cdot 15 \\ &= -3 \cdot 5 \\ &= -5 \cdot 3 \\ &= -15 \cdot 1 \end{aligned}$$

Of those pairs, $-3 + 5 = 2$ so $-3$ and $5$ are the numbers we are looking for and our answer is that

$$x^2 + 2x - 15 = (x - 3)(x + 5)$$

**Example 4**

Factor $x^2 - 4x - 21$.

Since $c = -21$, we should list all the factor pairs that multiply to $-21$.

$$\begin{aligned} -21 &= -1 \cdot 21 \\ &= -3 \cdot 7 \\ &= -7 \cdot 3 \\ &= -21 \cdot 1 \end{aligned}$$

Of those pairs, $-7 + 3 = -4$ so $-7$ and $3$ are the numbers we are looking for and our answer is that

$$x^2 - 4x - 21 = (x - 7)(x + 3)$$

Our final two examples showcase situations where it can be a bit harder, or even impossible, to factor a trinomial.

**Example 5**

Factor $x^2 + 5x - 2$.

Since $c = -2$, we should list all the factor pairs that multiply to $-2$.

$$\begin{aligned} -2 &= -1 \cdot 2 \\ &= -2 \cdot 1 \end{aligned}$$

None of those pairs adds up to $5$, which means this cannot be factored.

When an expression cannot be factored, we say that it is **prime**. In real life, most quadratic trinomials are prime. In class, however, the excercises are designed to allow you to practice the skills taught and most can be factored. So don't give up too early. Just because you can't find factors at first doesn't mean they don't exist. Don't declare an expression "prime" until you have tried all possible values.

**Example 6**

Factor $x^2 - 13x + 30$, if possible.

Since $c = 30$, we should list all the factor pairs that multiply to $30$.

$$\begin{aligned} 30 &= 1 \cdot 30 \\ &= 2 \cdot 15 \\ &= 3 \cdot 10 \\ &= 5 \cdot 6 \end{aligned}$$

Of those pairs, none add up to $-13$. But there are still some factor pairs we haven't checked: the ones where both values are negative.

$$\begin{aligned} 30 &= (-1) \cdot (-30) \\ &= (-2) \cdot (-15) \\ &= (-3) \cdot (-10) \\ &= (-5) \cdot (-6) \end{aligned}$$

Of those pairs, $-3 - 10 = -13$ so $-3$ and $-10$ are the numbers we are looking for and our answer is that

$$x^2 - 13x + 30 = (x - 3)(x - 10)$$

As the last example illustrates, it's only after we've considered both positive and negative factors that we can be sure we've tested every possible combination.

## Conclusion

You might be wondering why we devote time to factoring when the result is equivalent to the original expression. The truth is that factored expressions offer distinct advantages in terms of simplicity. This will become especially evident when we tackle quadratic equations in Chaper 4, but it's also apparent when evaluating quadratic expressions here.

Suppose you were asked to evaluate $(x - 1)(x + 3)$ and $x^2 + 2x - 3$ for $x = 2$ by hand. Which would you find easier to work with?

Factored expressions are often more manageable for calculations because they frequently involve fewer terms or operations. Let's go through both evaluations side-by-side and see for ourselves.

$$\begin{aligned} (x-1)(x+3) &= (2-1)(2+3) \\ &= (1)(5) \\ &= 5 \end{aligned} \qquad \begin{aligned} x^2+2x-3 &= 2^2+2(2)-3 \\ &= 4+4-3 \\ &= 5 \end{aligned}$$

While neither calculation was difficult, the factored expression was substantially simpler to evaluate. Lastly, if you hadn't noticed, the results are the same, because one expression is the factored version of the other!

# 3.1 One-Step Equations

## Introduction

Imagine you are regularly setting aside $\$75$ into a savings account every month. The total amount of money in the account can be determined by the expression $75m$, where $m$ represents the number of months.

Now, picture this: you have a plan to go on a trip that requires $\$800$. The big question is, how many months do you need to save before you have enough for your adventure? The answer to this question involves solving an equation like $75m = 800$.

Whenever we want to find unknown values, solving equations will be a reliable tool for finding the answers.

In this chapter and the next we will learn to write equations, use them to represent real life scenarios and solve them for unknown values.

## Vocabulary

As we begin, it will be helpful if we review some common vocabulary that was introduced back in section 2.1.

- **Constant:** A quantity that does not change. Numbers are constants. Letters can also be used for constants when their value is unknown.
- **Variable:** A quantity whose value can change. Variables are usually represented by letters like $x$, $y$, etc.
- **Operation:** A mathematical way of combining two numbers. Addition, subtraction, multiplication and division are the four basic operations.
- **Expression:** A collection of constants and/or variables that are combined by operations.
- **Equation:** A mathematical sentence stating that two things are exactly the same, or equal.
- **Solution:** The specific value(s) for a variable that make an equation true.
- **Solving:** The process of finding the solution(s) of an equation.
- **Equivalent Equations:** Equations that have the exact same solution(s).
- **Inverse Operations:** Operations that have the opposite effect and undo each other. Adding $5$ and subtracting $5$ would be inverse operations, for instance.

Informally, we can think of an equation as any math sentence that includes an equal sign. The equal sign $=$ means "is the same as".

The equal sign was invented in 1557 by the Welsh mathematician Robert Recorde who was tired of having to write the words "is equal to" over and over. Up until then, equations were more like sentences with actual words mixed with numbers!

For example, the equation $3 + 2 = 5$ tells us that $2 + 3$ *is the same as* $5$. This equation is true since $3 + 2$ really does equal as $5$. True equations are known as **identities**.

On the other hand, the equation $2 + 3 = 8$ is not true. It is a false statement or a **contradiction**.

Equations with variable quantities are intriguing. For example, $x + 7 = 12$ is true when $x$ is $5$ but false for every other value. Equations like this, that could be true or could be false depending on the value of the variable, are called **conditional** equations.

Since the equation $x + 7 = 12$ is true when $x$ is replaced with $5$, we refer to $5$ as the solution of the equation. Solutions are typically written in equation form, such as $x = 5$

It is important to note that not every equation that contains a variable is conditional or has a single solution. For example, $x - 4 = x - 4$ is always true (it's an identity) and any value for $x$ is a solution. But $x + 7 = x + 8$ is always false (it's a contradiction) and no value will ever be a solution.

One of the most important things we will learn in Algebra is how to find the solution(s) of an equation when they exist. We turn our attention for the remainder of this section to finding solutions to equations where there is a single operation. That will lay the foundation for solving more complex equations in the following sections.

## Mental Math vs. Algebraic Solving

Some conditional equations contain simple numbers and operations where we might be able to spot the solution simply by looking at it and applying our own number sense.

For example, consider this equation: $3x = 6$. It's a basic fact of multiplication that $3(2) = 6$ so the solution must be $x = 2$.

While we were able to solve that equation mentally, it certainly would be more challenging to mentally solve an equation like $3.835x = 62.894$. Because of that, it's important that we develop a systematic process for solving equations algebraically.

That process will allow us to transform a given equation with an uncertain solution into an equivalent equation that explicitly states the solution.

Since the goal is to get an equivalent equation, whatever we do to solve an equation must maintain the equality of both sides. There are a number of properties that do not impact the equality of an equation. We list several of them below.

**Properties of Equality**

| | |
|---|---|
| **Reflexive Property:** | Any quantity is equal to itself. $a = a$ |
| **Symmetric Property:** | If $a = b$, then $b = a$ |
| **Transitive Property:** | If $a = b$ and $b = c$, then $a = c$ |
| **Addition Property of Equality:** | If $a = b$, then $a + c = b + c$ |
| **Subtraction Property of Equality:** | If $a = b$, then $a - c = b - c$ |
| **Multiplication Property of Equality:** | If $a = b$, then $a \cdot c = b \cdot c$ |
| **Division Property of Equality:** | If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$ |
| **Substitution Property:** | In any expression, if $a = b$, you can replace $a$ with $b$ or $b$ with $a$ |

Generally speaking, these rules can be summarized as "whatever you do to one side of an equation you must do to the other side".

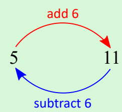
## Inverse Operations

Now that we've seen the importance of performing operations on both sides of an equation, we want to make sure we always know which operation to perform when solving. Since our goal is to end up with an equivalent equation in which the variable is isolated, we need to be able to get rid of any number that appears with the variable. This is done by using inverse operations.

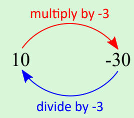
You are likely already familiar with some inverse operations. Addition and subtraction, for instance, are usually thought of as inverse operations, as are multiplication and division. Powers and roots are also inverse operations. In more advanced algebra courses, you will make use of another pair of inverse operations: exponentials and logarithms.

One method of testing if two operations are inverses is by picking a number, applying the first operation, then applying the second operation and checking to see if it returns the initial value. Here are a few examples.

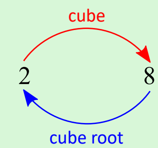
**Inverse Operation Examples**




adding and subtracting $6$ are inverse operations




multiplying and dividing by $-3$ are inverse operations




cubing a number and taking the cube root are inverse operations

There are, however, some finer points that need to be discussed. The first is that addition/subtraction and multiplication/division are only inverse operations when using the same number. Subtracting $6$ is not the inverse of adding $200$, even though one is addition and the other is subtraction.

The second point of interest is that division by $0$ is not defined, so multiplying and dividing by $0$ are not inverse operations.

The third point concerns powers and roots. Exponents and roots are fairly straightforward if the exponent and/or root is odd. For example, an exponent of $3$ and a $3$rd root are inverse operations. As a demonstration of that, a quick calcultion shows $(-4)^3 = -64$ and $\sqrt[3]{-64} = -4$, exactly the behavior we expect from inverse operations.

Things are more complicated with even exponents and roots, however. For instance, $(-3)^2 = 9$ but $\sqrt{9} = 3$, which is not the number we started with.

Does this mean that an exponent of $2$ and a $2$nd root are never inverse operations? Not exactly. If we had started with positive $3$, then $(3)^2 = 9$ and $\sqrt{9} = 3$ and this time the operations do indeed undo each other.

So what went wrong with $-3$? Well, the issue is that there are two numbers that equal $9$ when you square them, $3$ and $-3$, but the square root will only return the positive (or principle) one. This is true for all even roots and is something we will need to take into consideration later when solving equations with even powers.

| | | | |
|---|---|---|---|
| **Inverse Operations** | $x \underset{\text{subtract } a}{\overset{\text{add } a}{\rightleftarrows}} x+a$ | $x \underset{\text{divide by } b}{\overset{\text{multiply by } b}{\rightleftarrows}} bx$ | $x \underset{\text{take the } n^{\text{th}} \text{ root}}{\overset{\text{raise to the } n^{\text{th}} \text{ power}}{\rightleftarrows}} x^n$ |
| **Exclusions** | none | $b \neq 0$ | if $n$ is even, then $x$ cannot be negative |

**Example 1**

Give the inverse of each operation:

a. Adding $81$
b. Dividing by $-5$
c. Applying a power of $4$
d. Subtracting $-23$
e. Applying the $5$th root

---

a. Subtracting $81$
b. Multiplying by $-5$
c. Applying the $4$th root
d. Adding $-23$
e. Applying a power of $5$

## Solving with Inverse Operations

We now have the foundational for solving equations: identify the operation applied to the variable and then apply the inverse operation.

To see this process in action, let's revisit the equation $3x = 6$. How can we solve it using inverse operations? Since $x$ is being multiplied by $3$, we should undo that multiplication by dividing by $3$. If we divide both sides of the equation by $3$, the resulting equation will show us the solution:

$$\begin{aligned} 3x &= 6 && \text{Original equation} \\ \frac{3x}{3} &= \frac{6}{3} && \text{Divide both sides by 3} \\ x &= 2 && \text{Simplify} \end{aligned}$$

Notice that we had to divide *both* sides of the equation by $3$ in order to get an equivalent equation. It's critical to remember to do the same thing to both sides to maintain the equality of the equation.

**Example 2a**

Solve $x - 13 = 35$

Notice that that $13$ is being subtracted from $x$, so we need to add $13$ to both sides.

$$\begin{aligned} x - 13 &= 35 && \text{Notice 13 is being subtracted from } x \\ x - 13 + 13 &= 35 + 13 && \text{Add 13 to both sides} \\ x &= 48 && \text{Simplify} \end{aligned}$$

**Example 2b**

Solve $\frac{n}{4} = 24$

Since $n$ is being divided by $4$, we will multiply both sides by $4$.

$$\begin{aligned} \frac{n}{4} &= 24 && \text{Notice } n \text{ is being divided by 4} \\ 4 \cdot \frac{n}{4} &= 4 \cdot 24 && \text{Multiply both sides by 4} \\ n &= 96 && \text{Simplify} \end{aligned}$$

**Example 2c**

Solve $75m = 800$

Since $m$ is being multiplied by $75$, we will divide both sides by $75$.

$$\begin{aligned} 75m &= 800 && \text{Notice } m \text{ is being multiplied by 75} \\ \frac{75m}{75} &= \frac{800}{75} && \text{Divide by 75} \\ m &\approx 10.667 && \text{Evaluate} \end{aligned}$$

This is the equation from our introduction. By solving it we now know you would need to save for almost $11$ months to have enough for the $\$800$ trip.

The final two examples involve powers and roots, so we will proceed with caution. If an equation has an even power then the opposite operation would be the corresponding even root, and there are three possible outcomes.

If $x^n = A$ and $n$ is an even number then:

- If $A$ is positive, then there are two solutions: the positive (principle) root $x = \sqrt[n]{A}$ and the negative (secondary) root $x = -\sqrt[n]{A}$.
- If $A = 0$ then there is one solution: $x = 0$
- If $A$ is negative, then there is no real number solution.

The "plus/minus" symbol $\pm$ can be used to indicate the two solutions in the case where $A$ is positive.

Also, remember that for a square root (or $2$nd root) we don't write the root index $2$ with the radical symbol. So rather than writing $\sqrt[2]{25}$ just write $\sqrt{25}$.

**Example 3a**

Solve $x^2 = 49$

In the equation $x^2 = 49$, $x$ is raised to an even power and the result is positive, so there are two solutions. Since the power is $2$ we will use the square root to solve.

$$\begin{aligned} x^2 &= 49 \\ x &= \pm\sqrt{49} \\ x &= \pm 7 \end{aligned}$$

We can either list the solutions separately as $x = 7$, $x = -7$ or, as we've done here, use the $\pm$ symbol to indicate both values.

**Example 3b**

Solve $p^6 = -729$

In the equation $p^6 = -729$, $p$ is raised to an even power and the result is negative. Therefore, the equation has no real number solution.

**Example 3c**

Solve $n^5 = -5.37824$

In the equation $n^5 = -5.37824$ the power is $5$ which is odd, so there is a single solution and we will a use $5$th root to find it.

$$\begin{aligned} n^5 &= -5.37824 \\ n &= \sqrt[5]{-5.37824} \\ n &= -1.4 \end{aligned}$$

### Tip: Evaluating Roots on a Calculator

In this last example, it was necessary to calculate a 5th root. Every calculator should have an easy-to-find square root button, but a different button must be used for higher roots. If you are using a TI-83 or TI-84, here are the steps for calculating a 5th root:

1. Begin your calculation by entering the number 5, since you want to do a 5th root.
2. Press the [MATH] button
3. Select $\boxed{\sqrt[x]{\;}}$
4. Enter the number you're finding the 5th root of. For the example above, that would be -5.37824.
5. Press [ENTER] to get the result.

Roots can also be entered as fractional exponents. For instance, the square root is the same as a power of $1/2$ and a cube root is the same as a power of $1/3$. If you use fractional exponents, be sure they are enclosed in parenthesis.

We'll conclude by looking at equations with roots. As discussed earlier, odd roots aren't particularly complicated. However, we need to be careful with equations involving even roots. When we are given an even root of a number, it is understood to be the positive (or principal) root. Because of that, it can't be set equal to a negative number.

If $\sqrt[n]{x} = A$ and $n$ is an even number then:

- If $A$ is a non-negative number, then there is one solution: $x = A^n$
- If $A$ is negative, then there is no solution.

**Example 4a**

Solve $\sqrt[3]{x} = -5$

Since the equation $\sqrt[3]{x} = -5$ inovles an odd root we can solve it by raising both sides to the $3$ rd power

$$\begin{aligned}\sqrt[3]{x} &= -5 \\ x &= (-5)^3 \\ x &= -125\end{aligned}$$

**Example 4b**

Solve $\sqrt[4]{w} = 3.5$

The equation $\sqrt[4]{w} = 3.5$ involves an even root and is equal to a positive number, so we can solve it by raising both sides to the $4$th power.

$$\begin{aligned}\sqrt[4]{x} &= 3.5 \\ x &= (3.5)^4 \\ x &= 150.0625\end{aligned}$$

**Example 4c**

Solve $\sqrt{n} = -7$

The equation $\sqrt{n} = -7$ has a square root equal to a negative number, so there is no real number solution to this equation.

## Conclusion

In this section we've learned to Solve Six Types of Equations using inverse operations.

In the next stage of our journey we will explore equations like $3x - 12 = 4$ that involve more than one operation.

# 3.2 Solving Multi-Step Equations by Inverse Operations

## Introduction

In the previous section we discussed inverse operations and how to perform an operation *on both sides* of an equation in order to solve it.

All of the examples in that section had a single operation, so it was fairly straightforward to know which inverse operation to use. Now, however, we will encounter equations with multiple operations. When solving these types of equations not only will we need to find the inverse operations, but we will have to deploy them in the proper order.

In this section, we will delve into the process of solving multi-step equations using inverse operations.

## Identifying the Order of Operations

Before we discuss the theory behind solving multi-step equations, we need to make sure we know the correct order of operations. This was covered in detail in the previous chapter (see section 2.2), but here's a quick review:

**The Standard Order of Operations**

1. Perform any operations within groupings. Groupings are typically shown within parentheses or brackets. Less commonly, groupings will involve an operation in an exponent or in the numerator (top) or denominator (bottom) of a fraction.
2. Apply any exponents.
3. Apply any multiplication or division operations from left to right.
4. Apply any addition or subtraction operations from left to right.

Not only is it important to be able to *perform* operations in the correct order, it's also important to be able to *identify* the operations that are taking place in an equation, in the correct order. To do this, focus on what is happening to the variable.

**Example 1a**

What's the order of operations for the variable $x$ in the expression $5x - 48$.

Since multiplication happens before subtraction, the order of operations is :

1. Multiply by $5$
2. Subtract $48$

**Example 1b**

Identify the correct order of operations for the variable $n$ in the expression $\frac{3n+2}{7}$.

The fraction bar acts as a grouping symbol, so the correct order of operations is:

1. Multiply by $3$
2. Add $2$
3. Divide by $7$

**Example 1c**

List the proper order of operations for the variable $p$ in $(p-16)^2+9$.

Since operations in parentheses should be done first, the proper order is:

1. Subtract $16$
2. Exponent of $2$
3. Add $9$

## Solving Multi-Step Equations

As the name implies, a multi-step equation requires that we perform multiple operations in order to find the solution. We discussed inverse operations in the previous section. Now, it's time to learn about *inverse processes*.

It will help if we start with something everyone is familiar with: putting on socks and shoes.

When getting dressed, socks must be put on before shoes. But what do you do when getting undressed? The *inverse process* is to remove shoes first and then socks.

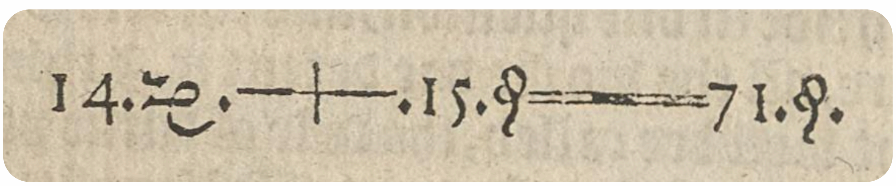
Notice that this process requires undoing each step in the reverse order. Shoes were the last things to be put on, for instance, but they are the first things that must be taken off.

| $\downarrow$ | **Actions** | **Inverse Actions** | $\uparrow$ |
| --- | --- | --- | --- |
| | Put on Socks | Take off Socks | |
| | Put on Shoes | Take off Shoes | |
| | $\longrightarrow$ | | |

This illustrates the notion of an inverse process: In order to undo a multi-step process, the steps must be undone in the *reverse order* of how they were originally done.

This is exactly how we will solve multi-step equations. Once we have identified the operations taking place and the order in which they occur, we will apply the inverse operations in the reverse order.

There's no better place to start than with the first equation ever printed. It comes from Robert Recorde's *The Whetstone of Witte* (1557), on the very same page where he introduced the equal sign.



He wrote it a bit fancier than we do today, but the equation is $14x + 15 = 71$. Let's solve it!

**Example 2**

Use inverse operations to solve $14x + 15 = 71$ for $x$.

The operations happening to $x$ are

1. Multiply by $14$
2. Add $15$

So in order to solve for $x$ we should

1. Subtract $15$
2. Divide by $14$

Remembering to do those operations on both sides, we get the following:

$$\begin{aligned} 14x + 15 &= 71 && \text{Original equation} \\ 14x + 15 - 15 &= 71 - 15 && \text{Subtract 15} \\ 14x &= 56 && \text{Simplify} \\ \frac{14x}{14} &= \frac{56}{14} && \text{Divide by 14} \\ x &= 4 && \text{Simplify} \end{aligned}$$

And just like that we've solved the first equation ever printed!

In the following examples we will solve equations involving the three expressions we analyzed in Examples 1-3.

**Example 3**

Use inverse operations to solve $5x - 48 = -23$ for $x$.

We analyzed the operations of $5x - 48$ in Example 1. All that remains is to do the opposite operations in the reverse order.

$$\begin{aligned} 5x - 48 &= -23 && \text{Original equation} \\ 5x - 48 + 48 &= -23 + 48 && \text{Add 48} \\ 5x &= 25 && \text{Simplify} \\ \frac{5x}{5} &= \frac{25}{5} && \text{Divide by 5} \\ x &= 5 && \text{Simplify} \end{aligned}$$

**Example 4**

Use inverse operations to solve $\dfrac{3n+2}{7} = 5$

We analyzed the operations of $\dfrac{3n+2}{7}$ in Example 2. All that remains is to do the opposite operations in the reverse order.

$$
\begin{aligned}
\frac{3n+2}{7} &= 5 && \text{Original equation} \\
7 \cdot \frac{3n+2}{7} &= 7 \cdot 5 && \text{Multiply by 7} \\
3n+2 &= 35 && \text{Simplify} \\
3n+2-2 &= 35-2 && \text{Subtract 2} \\
3n &= 33 && \text{Simplify} \\
\frac{3n}{3} &= \frac{33}{3} && \text{Divide by 3} \\
n &= 11 && \text{Simplify}
\end{aligned}
$$

**Example 5**

Use inverse operations to solve $(p-16)^2 + 9 = 73$

We analyzed the operations of $(p-16)^2 + 9$ in Example 3. All that remains is to do the opposite operations in the reverse order.

$$
\begin{aligned}
(p-16)^2 + 9 &= 73 && \text{Original equation} \\
(p-16)^2 + 9 - 9 &= 73 - 9 && \text{Subtract 9} \\
(p-16)^2 &= 64 && \text{Simplify} \\
\sqrt{(p-16)^2} &= \pm\sqrt{64} && \text{Square root} \\
p - 16 &= \pm 8 && \text{Simplify} \\
p - 16 + 16 &= \pm 8 + 16 && \text{Add 16}
\end{aligned}
$$

$$
\begin{aligned}
p = 8 + 16 = 24 & \qquad \text{Calculate both solutions} \\
\text{and} & \\
p = -8 + 16 = 8 &
\end{aligned}
$$

Note that at the end of the solving process in Example 7 we had two equations to solve. We did not simply find one value and say the answer was $\pm$ of that number. We had to evaluate both $8 + 16$ and $-8 + 16$.

In our final example we'll look at an equation that comes from a realistic scenario.

**Example 6**

Suppose the cost to hire a plumber involves a $\$65$ call-out fee on top of a labor charge of $\$85$ per hour. Let's say that $c$ represents the total cost to hire a plumber. Then the equation for the total cost would be $c = 85h + 65$ for $h$ hours. If you receive an invoice for $\$660$, how many hours did the plumber put on the bill?

---

To find the answer we need to solve the equation when $c = 660$.

$$\begin{aligned} 660 &= 85h + 65 && \text{Starting equation} \\ 595 &= 85h && \text{Subtract 65} \\ \frac{595}{85} &= \frac{85h}{85} && \text{Divide by 85} \\ h &= 7 && \text{Simplify} \end{aligned}$$

Let's check our answer.

$$\begin{aligned} c &= 85h + 65 && \text{Starting equation} \\ &= 85 \cdot 7 + 65 && \text{Substitute } h = 7 \\ &= 595 + 65 \\ &= 660 \end{aligned}$$

We can now say that the plumber must have billed $7$ hours of labor if the total cost was $660.

## Conclusion

It is important to note that using inverse processes to solve multi-step equations is a good approach to use if the variable only occurs once in the equation. In more complicated equations, such as $x^2 + 7x - 12 = 6$, where the variable appears more than once, we'll have to use other strategies. Those techniques will be explored in detail in Chapter 4.

# 3.3 Solving Literal Equations and Formulas

## Introduction

If you were to stop a random person on the street and ask them to give you a fact from math, there's a good chance they'd give you an equation with lots of letters in it that they memorized in school, like $a^2 + b^2 = c^2$.

Equations like this, that have more than one variable, are called **literal equations**. Literal equations that express well-defined rules in specific contexts, like geometry, physics, or finance, are called **formulas.**

From adjusting recipes and paying taxes to predicting planetary orbits and biochemical reactions, formulas are versatile tools used in a wide range of fields. In this section, we'll explore some of these practical formulas and learn how to work with and solve literal equations.

## Solving Literal Equations

In this section, we'll take the strategies and rules we've already learned about solving equations and transfer them over to solving formulas and other literal equations.

Formulas are often written with one variable isolated to make the calculations straightforward. For instance, the classic formula for calculating distance traveled $d$ using rate $r$ and time $t$ is

$$d = rt$$

This formula quickly gives the distance for any rate and time.

But what if we needed to find $t$? If we knew that $d = 100$ and $r = 50$, for instance, then solving $100 = 50t$ once would be manageable.

$$\begin{aligned} d &= rt && \text{Original formula} \\ 100 &= 50t && \text{Substituted} d = 100 \text{ and } r = 50 \\ \frac{100}{50} &= \frac{50t}{50} && \text{Divide to isolate } t \\ 2 &= t && \text{Simplify} \end{aligned}$$

But if we needed to find $t$ for several different values it would be more convenient if we had a formula for that specific purpose. To create a formula like that we can take the distance formula and solve for $t$ without putting in any values for $d$ and $r$. We follow the same steps as in the specific case above, dividing to isolate $t$ and simplifying.

$$\begin{aligned} d &= rt && \text{Original formula} \\ \frac{d}{r} &= \frac{rt}{r} && \text{Divide to isolate } t \\ \frac{d}{r} &= t && \text{Simplify} \\ t &= \frac{d}{r} && \text{Final answer} \end{aligned}$$

As we can see from this example, solving formulas with generic values transforms specific cases into universal rules. Unlike single-variable equations where the answer is a number, solutions for formulas usually don't simplify to a single value. Isolating a variable in a formula creates a new formula!

And since we already know how to use inverse operations to solve, the primary challenge with formulas lies in maintaining clarity and organization.

**Example 1**

Solve the literal equation $ax + b = c$ for $x$

$$\begin{aligned} ax + b &= c && \text{Original equation} \\ ax &= c - b && \text{Subtract } b \\ x &= \frac{c-b}{a} && \text{Divide by } a \end{aligned}$$

This solution cannot be simplified further since $a$, $b$, and $c$ are distinct variables that cannot be combined.

## Practical Examples

**Example 2**

The formula $I = Prt$ gives the amount of simple interest produced by a principal balance $P$ earning an interest rate of $r$ for $t$ years. Solve the formula for $P$

$$\begin{aligned} I &= Prt && \text{Original formula} \\ \frac{I}{rt} &= P && \text{Divide by } rt \\ P &= \frac{I}{rt} && \text{New formula} \end{aligned}$$

**Example 3**

The perimeter of a rectangle is given by the formula $P = 2L + 2W$ where $L$ is the length and $W$ is the width. Solve the formula for the width $W$.

$$\begin{aligned} P &= 2L + 2W && \text{Original formula} \\ P - 2L &= 2W && \text{Subtract } 2L \\ \frac{P - 2L}{2} &= W && \text{Divide by } 2 \\ W &= \frac{P - 2L}{2} && \text{New formula} \end{aligned}$$

**Example 4**

Newton's Second Law of Motion states that force $F$ is equal to the product of mass $m$ and acceleration $a$, i.e., $F = ma$. Solve this formula for $m$.

$$\begin{aligned} F &= ma && \text{Original formula} \\ \frac{F}{a} &= m && \text{Divide by } a \\ m &= \frac{F}{a} && \text{New formula} \end{aligned}$$

**Example 5**

The formula for calculating the area $A$ of a triangle is $A = \frac{1}{2}bh$, where $b$ is the base length and $h$ is the height. Solve this formula for $h$.

$$\begin{aligned} A &= \frac{1}{2}bh && \text{Original formula} \\ 2A &= bh && \text{Multiply by 2} \\ \frac{2A}{b} &= h && \text{Divide by } b \\ h &= \frac{2A}{b} && \text{New formula} \end{aligned}$$

**Example 6**

The formula for converting temperature from degrees Celsius $C$ to degrees Fahrenheit $F$ is $F = \frac{9}{5}C + 32$. Solve this formula for $C$.

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{Original formula} \\ F - 32 &= \frac{9}{5}C && \text{Subtract 32} \\ \frac{5}{9}(F - 32) &= C && \text{Divide by } \frac{9}{5} \text{ or multiply by } \frac{5}{9} \\ C &= \frac{5}{9}(F - 32) && \text{New formula} \end{aligned}$$

**Example 7**

The Pythagorean Theorem states that in a right triangle, the square of the length of the hypotenuse $c$ is equal to the sum of the squares of the other two sides $a$ and $b$, i.e., $c^2 = a^2 + b^2$. Solve this formula for $a$.

$$\begin{aligned} c^2 &= a^2 + b^2 && \text{Original formula} \\ c^2 - b^2 &= a^2 && \text{Subtract } b^2 \\ \sqrt{c^2 - b^2} &= a && \text{Take the square root} \\ a &= \sqrt{c^2 - b^2} && \text{New formula} \end{aligned}$$

Normally when using a square root to solve we would take the $\pm$ root. In this case, we are solving for the side of a triangle which must be positive.

**Example 8**

The formula for calculating the volume $V$ of a cylinder is $V = \pi r^2 h$, where $r$ is the radius and $h$ is the height. Solve this formula for $r$.

$$
\begin{aligned}
V &= \pi r^2 h && \text{Original formula} \\
\frac{V}{\pi h} &= r^2 && \text{Divide by } \pi h \\
\sqrt{\frac{V}{\pi h}} &= r && \text{Take the square root} \\
r &= \sqrt{\frac{V}{\pi h}} && \text{New formula}
\end{aligned}
$$

## Conclusion

Later in Chapter 5 we will explore graphing equations containing variables $x$ and $y$. Having equations solved for $y$ often makes them easier to graph by hand or input into a graphing calculator.

# 4.1 Solving through Simplification

## Introduction

Up until now we have explored "elementary" equations that could be solved using inverse operations alone.

In this chapter, we begin looking at "intermediate" equations where the variable occurs in multiple terms, with different powers, or where other complications arise that require additional techniques.

We'll start by using the skills we developed in Chapter 2 (combining like terms, distribution, etc.) to collect all the variable terms together into simplified equations that can be solved the same way we did in Chapter 3.

## Solving Equations by Combining Like Terms

When an equation involves multiple terms with the same variable on the same side of the equation, we should try to combine those terms before solving. Combining any like terms can simplify the equation, making it easier to solve.

**Example 1**

Solve the equation $5x + 4 - 7x = 15$ by first combining like terms.

$$\begin{aligned} 5x + 4 - 7x &= 15 && \text{Original equation} \\ -2x + 4 &= 15 && \text{Combine } 5x - 7x = -2x \\ -2x &= 11 && \text{Subtract } 4 \\ x &= \frac{11}{-2} && \text{Divide by } -2 \end{aligned}$$

When combining like terms, remember that it's only possible to combine terms that have the same variables raised to the same powers. We cannot combine $2x$ and $3y$ (different variables) nor can we combine $2x$ with $3x^2$ (different powers).

**Example 2**

Solve the equation $3x^2 - 2x + 5 + 6x^2 + 2x = 86$ by first combining like terms.

$$\begin{aligned} 3x^2 - 2x + 5 + 6x^2 + 2x &= 86 && \text{Original equation} \\ 9x^2 + 5 &= 86 && \text{Combine like terms} \\ 9x^2 &= 81 && \text{Subtract 5} \\ x^2 &= 9 && \text{Divide by 9} \\ x &= \pm\sqrt{9} && \text{Square root} \\ x &= \pm 3 && \text{Simplify} \end{aligned}$$

## Solving Equations with the Variable on Both Sides

Up until now, all of our equations only had the variable on one side. But what if the variable appears on both sides of the equation?

If we could move all the variable terms to one side of the equation then we could combine the like terms and solve, so that's what we'll do!! It doesn't matter which side the variable terms end up as long as they are on the same side.

This is done by adding or subtracting the term we want to move. For instance, if $5x + 6 = 2x$ and we wanted to move $2x$ to the left side, we would subtract $2x$ from both sides

$$\begin{aligned} 5x + 6 &= 2x \\ 5x - 2x + 6 &= 2x - 2x && \text{Subtract } 2x \\ 3x + 6 &= 0 && \text{Simplify} \end{aligned}$$

and then we could proceed to solve like normal. Or, if we wanted to move $5x$ over to the right side we would subtract it from both sides.

$$\begin{aligned} 5x + 6 &= 2x \\ 5x - 5x + 6 &= 2x - 5x && \text{Subtract } 5x \\ 6 &= -3x && \text{Simplify} \end{aligned}$$

Both of those equations have the same solution, so it doesn't mater which we use. The important thing is to get all of the variable terms on the same side.

**Example 3**

Solve the equation: $4x + 5 = 3x + 10$.

We will solve this both ways to show the answers are identical. First we'll subtracting $3x$ so that all of the variables end up on the left.

$$\begin{aligned} 4x + 5 &= 3x + 10 && \\ 4x - 3x + 5 &= 3x - 3x + 10 && \text{Subtract } 3x \\ x + 5 &= 10 && \text{Simplify} \\ x &= 5 && \text{Subtract } 5 \end{aligned}$$

Now we will solve it again, this time by subtracting $4x$ so that all the variable terms end up on the right.

$$\begin{aligned} 4x + 5 &= 3x + 10 && \\ 4x - 4x + 5 &= 3x - 4x + 10 && \text{Subtract } 4x \\ 5 &= -x + 10 && \text{Simplify} \\ -5 &= -x && \text{Subtract } 10 \\ 5 &= x && \text{Divide by } -1 \end{aligned}$$

**Example 4**

Consider the equation: $2x + 3 - x = 5x - 2$. Let's combine the like terms on each side:

$$\begin{aligned} 2x + 3 - x &= 5x - 2 && \\ x + 3 &= 5x - 2 && \text{Combine like terms on the left} \\ -4x + 3 &= -2 && \text{Subtract } 5x \\ -4x &= -5 && \text{Subtract } 3 \\ x &= \frac{5}{4} && \text{Divide by } -4 \end{aligned}$$

## Solving Equations Using Distribution

Distribution, also known as the distributive property, allows us to simplify equations by distributing a value across terms within parentheses allowing us to move terms around and combine them.

**Example 5**

Use distribution to solve the equation $2(x+3)=10$.

$$\begin{aligned} 2(x+3) &= 10 & & \\ 2x+6 &= 10 & & \text{Distribute the 2} \\ 2x &= 4 & & \text{Subtract 6} \\ x &= 2 & & \text{Divide by 2} \end{aligned}$$

The equation in Example 5 could have been solved using inverse operations and bypassing distribution entirely.

$$\begin{aligned} 2(x+3) &= 10 & & \\ x+3 &= 5 & & \text{Divide by 2} \\ x &= 2 & & \text{Subtract 3} \end{aligned}$$

So why don't we do it that way all of the time? Early division certainly can be an effective option, especially if doing so will simplify both sides, but frequently it will generate fractions which can make the equation unnecessarily complicated.

**Example 6**

Use distribution to solve the equation $5(x-2)=3x-14$.

$$\begin{aligned} 5(x-2) &= 3x-14 & & \\ 5x-10 &= 3x-14 & & \text{Distribute the 5} \\ 2x-10 &= -14 & & \text{Subtract } 3x \\ 2x &= -4 & & \text{Add 10} \\ x &= -2 & & \text{Divide by 2} \end{aligned}$$

Notice that if we had chose to divide by $5$ as our first step, then the resulting equation $x-2=\frac{3x}{2}$ — 7 would now have a fraction in it and we would need to be extra careful with the rest of the solving steps.

**Example 7**

Use distribution to solve the equation $3-2(x+8)=23-11x$.

$$\begin{aligned}
3-2(x+8) &= 23-11x & \\
3-2x-16 &= 23-11x & \text{Distribute the } -2 \\
-2x-13 &= 23-11x & \text{Combine } 3-16=-13 \\
9x-13 &= 23 & \text{Add } 11x \\
9x &= 36 & \text{Add } 13 \\
x &= 4 & \text{Divide by } 9
\end{aligned}$$

In this problem, distributing first helped us avoid accidentally combining the $3$ and $-2$, which is a very common mistake. It's tempting to do that subtraction, but it cannot be done since $-2$ is multiplying $x+8$.

Another common mistake that is unique to distribution is attempting to distribute when there is a power or a root around an expression. It is not possible to use distribution in cases like $5(x+2)^3$ or $5\sqrt{x+2}$. Distribution applies to multiplication over addition or subtraction, which doesn't directly work with powers or roots.

## Solving Equations by Clearing Fractions

This last technique can be used to remove many of the fractions in an equation, making it easier to solve. Fractions in equations can sometimes complicate the solving process, so eliminating them can be a helpful strategy.

To clear fractions from an equation, we need to multiply *every* term in the equation by the least common multiple (LCM) of the denominators of the fractions in the equation. This effectively eliminates the fractions, and you can proceed to solve the resulting equation using the methods we've discussed earlier.

Let's see how this works in a few examples.

**Example 8**

Solve the equation $\frac{3}{2}x - \frac{1}{4} = \frac{5}{6}$ by clearing fractions.

The denominators are $2$, $4$, and $6$, and their LCM is $12$. Multiplying both sides of the equation by $12$ will clear the fractions:

$$\begin{aligned}
\frac{3}{2}x - \frac{1}{4} &= \frac{5}{6} \\
12 \cdot \left(\frac{3}{2}x - \frac{1}{4}\right) &= 12 \cdot \frac{5}{6} && \text{Multiply both sides by the LCM 12} \\
12 \cdot \frac{3}{2}x - 12 \cdot \frac{1}{4} &= 12 \cdot \frac{5}{6} && \text{On the left, distribute 12} \\
\frac{36}{2}x - \frac{12}{4} &= \frac{60}{6} && \text{Perform the multiplications} \\
18x - 3 &= 10 && \text{Simplify each fraction}
\end{aligned}$$

Now, we can solve for $x$ as usual:

$$\begin{aligned}
18x - 3 &= 10 \\
18x &= 13 && \text{Add 3} \\
x &= \frac{13}{18} && \text{Divide by 18}
\end{aligned}$$

Clearing fractions might introduce larger coefficients, but that's generally a worthwhile tradeoff.

**Example 9**

Solve the equation $\frac{2}{3}x + \frac{1}{5} = \frac{x}{4} + \frac{3}{10}$ by clearing fractions.

The denominators are $3$, $5$, and $4$, and their LCM is $60$.

$$\begin{aligned}
60 \cdot \left(\frac{2}{3}x + \frac{1}{5}\right) &= 60 \cdot \left(\frac{x}{4} + \frac{3}{10}\right) && \text{Multiply side by the LCM 60} \\
60 \cdot \frac{2}{3}x + 60 \cdot \frac{1}{5} &= 60 \cdot \frac{x}{4} + 60 \cdot \frac{3}{10} && \text{Distribute 60 to each term} \\
\frac{120}{3}x + \frac{60}{5} &= \frac{60x}{4} + \frac{180}{10} && \text{Perform the multiplications} \\
40x + 12 &= 15x + 18 && \text{Simplify each fraction} \\
40x - 15x + 12 &= 18 && \text{Subtract } 15x \\
40x - 15x &= 18 - 12 && \text{Subtract } 12 \\
25x &= 6 && \text{Combine like terms} \\
x &= \frac{6}{25} && \text{Divide by } 25
\end{aligned}$$

We now have a fairly robust set of tools for solving equations.

**Equation Solving Process**

- Use distribution to remove any parenthesis.
- Add/subtract to put all variable terms on one side of the equation.
- If desired, multiply by the LCM of the denominators to eliminate fractions.
- Combine any like terms.
- Use inverse operations in the reverse order to isolate the variable.

## Looking Ahead

The techniques we've learned work well when the variable appears only once, or when we can combine terms so that the variable only appears once.

But what if we come across equations like $x^2 + 5x + 6 = 0$, where we can't combine $x^2$ with $5x$? As we'll see in the next section, we combine what we've learned here with an old skill: factoring.

# 4.2 Solving by Factoring

## Introduction

A surprising number of real-world applications involve equations where $x^2$ is the highest power. These *quadratic equations* help with everyting from minimizing costs to finding the height of a rocket or the distance traveled by a baseball and even creating smooth computer animations.

We've solved simple quadratic equations before by taking a square root. While square roots will still play a role, when equations that have both linear $x$ and quadratic $x^2$ terms factoring will be one of our first tools.

## Solving with the Zero-Product Property

The reason factoring is useful as a solving tool has to do with an important fact that you might be able to figure out own your own as you think about the following question:

Suppose we have two quantities, $A$ and $B$, and that their product is $A \cdot B = 0$. What does that tell us about $A$ and $B$?

If $A \cdot B = 0$, then it must be that $A = 0$ or $B = 0$ or, perhaps, both are equal to zero. There's no alternative; one (or both) must be equal to zero, guaranteed. This is called the *zero-product property*.

**Zero-Product Property**

If $A \cdot B = 0$, then $A = 0$ or $B = 0$.

It's important to point out that this *only* works for the number $0$, hence the name ***zero**-product property*. For instance, if $A \cdot B = 6$ then there's no guarantee that either $A = 6$ or $B = 6$, it could easily just as easily be that $A = 2$ and $B = 3$.

But when the product equals zero we can, with confidence, say that at least one of the factors **must** be zero. Setting each factor in a quadratic equation equal to zero will create two simple linear equations. Solving each of those will give us the solutions to the original problem.

Let's walk through a few examples.

**Example 1a**

Solve $(x+5)(x-7)=0$

Since the product of the two factors is zero, we can use the zero-product property to find the solution by setting each equal to zero and solving.

$$\begin{aligned} x+5&=0 \qquad \text{or} \qquad & x-7&=0 \\ x&=-5 \qquad \text{or} \qquad & x&=7 \end{aligned}$$

The two solutions are $x=-5$ and $x=7$.

**Example 1b**

Solve $2x(3x-4)=0$

Using the zero-product property, we set each factor equal to zero:

$$\begin{aligned} 2x&=0 \qquad \text{or} \qquad & 3x-4&=0 \\ x&=0 \qquad \text{or} \qquad & x&=\frac{4}{3} \end{aligned}$$

The two solutions are $x=0$ and $x=\frac{4}{3}$.

This process of solving factored equations by the zero-product property can be extended naturally to equations with more than two factors. We simply set each factor, no matter how many we have, equal to zero and then solve, as we'll see in the next example.

**Example 2**

Solve $(x+2)(x-6)(x-9)=0$

Even though we have three factors, we still set each factor equal to zero and solve:

$$\begin{aligned} x+2&=0 \qquad \text{or} \qquad & x-6&=0 \qquad \text{or} \qquad & x-9&=0 \\ x&=-2 \qquad \text{or} \qquad & x&=6 \qquad \text{or} \qquad & x&=9 \end{aligned}$$

The three solutions are $x=-2$, $x=6$, and $x=9$.

## Solving a Quadratic Equation by Factoring

In the prior examples, the equations were already factored. Most of the time, however, we will have to find the factored form ourselves.

As a refresher, recall that factoring entails finding expressions that can be multiplied together to give the original expression. We learned about two types of factoring in section 2.6 and a given problem could require one or a combination of both.

The first type was factoring out the greatest common factor (GCF). This should be used when all terms share a common factor. For instance, in $3x^2 + 6x$, the GCF is $3x$ allowing us to factor it as $3x(x+2)$.

The second method deals with the special case of a trinomial in the form $x^2 + bx + c$. In such cases, we try to find two numbers that multiply to $b$ and add up to $c$. Take $x^2 + 5x + 6$ as an example. It factors as $(x+2)(x+3)$ since $2 \cdot 3 = 6$ and $2 + 3$=5.

If we can factor an expression that's equal to zero, then we can finish solving it using the zero-product priciple as we did earlier.

**Example 3**

Solve the equation $x^2 + x - 6 = 0$ by factoring.

---

To factor, we need two numbers that multiply to $-6$ and add up to $1$. These numbers are $3$ and $-2$.

$$\begin{aligned} x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \end{aligned}$$

By the zero-product property:

$$\begin{aligned} x + 3 &= 0 \qquad \text{or} \qquad & x - 2 &= 0 \\ x &= -3 \qquad \text{or} \qquad & x &= 2 \end{aligned}$$

The solutions are $x = -3$ and $x = 2$.

This next example can actually be solved two ways. Let's solve it first by factoring.

**Example 4**

Solve $x^2 - 100 = 0$.

The left side can be factored as a difference of squares.

$$\begin{aligned} x^2 - 100 &= 0 \\ (x+10)(x-10) &= 0 \end{aligned}$$

By the zero-product property:

$$\begin{aligned} x + 10 &= 0 \qquad \text{or} \qquad & x - 10 &= 0 \\ x &= -10 \qquad \text{or} \qquad & x &= 10 \end{aligned}$$

The solutions are $x = -10$ and $x = 10$.

This example could also have been solved using square roots if we had first rewritten it at $x^2 = 100$.

It's not uncommon to be able to solve a problem in multiple ways. Mathematicians are always looking for new, elegant methods to solve problems, even old ones that have been solved before. The Pythagorean Theorem, for instance, which is likely almost 4000 years old, is known to have over 300 different proofs. So feel free to explore different techniques and ideas when solving equations.

Throughout math, whenever we encounter a new problem we always want to turn it into a problem we've solved before.

In the next example the equation is not equal to zero, so what do we do? We move terms to the other side of the equation so that it does equal zero. After doing that it will look similar to problems we've solved earlier and we'll proceed as we did before.

**Example 5**

Solve $4x^2 = 28x$.

---

We'll have to set the equation equal to $0$ by subtracting $28x$ from both sides of the equation.

$$\begin{aligned} 4x^2 &= 28x \\ 4x^2 - 28x &= 0 \\ 4x(x-7) &= 0 \end{aligned}$$

By the zero-product property:

$$\begin{aligned} 4x &= 0 \qquad \text{or} \qquad & x - 7 &= 0 \\ x &= 0 \qquad \text{or} \qquad & x &= 7 \end{aligned}$$

The solutions are $x = 0$ and $x = 7$.

In this last example, it might appear that the equation is beyond our skills. But if we work with it carefully, we can turn it into something that looks more familiar.

**Example 6**

Solve $5x^3 + 10x^2 - 75x = 0$.

---

$$\begin{aligned} 5x^3 + 10x^2 - 75x &= 0 \\ 5x(x^2 + 2x - 15) &= 0 \\ 5x(x+5)(x-3) &= 0 \end{aligned}$$

By the zero-product property:

$$\begin{aligned} 5x &= 0 \qquad \text{or} \qquad & x + 5 &= 0 \qquad \text{or} \qquad & x - 3 &= 0 \\ x &= 0 \qquad \text{or} \qquad & x &= -5 \qquad \text{or} \qquad & x &= 3 \end{aligned}$$

The solutions are $x = 0, x = -5$ and $x = 3$.

We now have a process for trying to solve an equation by factoring. Here's a summary of the steps:

**Steps for Solving Factorable Equations**

1. **Set Up:** Use addition and/or subtraction to move all terms to one side of the equation so the equation is equal to zero.
2. **Factor:** Factor the expression.
3. **Zero-Product Property:** Set each factor equal to zero separately.
4. **Solve:** Solve for the variable in each equation from step 3.

## Looking Ahead

Factoring is a powerful tool for solving quadratic equations, but it has one significant limitation: not all quadratic expressions can be factored using integers. When factoring fails or becomes too difficult to spot, we need a method that works for every quadratic equation. That final tool, the quadratic formula, is our next section.

# 4.3 The Quadratic Formula

## Introduction

The highly trained atheletes of the Red Bull Cliff Diving World Series execute incredible acrobatics from heights of up to $88$ feet above the water. Such dives require focus and skill, along with timing.

How do the divers know how much time they have to perform the twists and flips that make up their dives? From those heights you don't want to guess, you want to be sure you know how long it takes to hit the water.

In this section will cover a formula that can answer that question. What's more, this magic formula can solve every possible quadratic equation.


Photo by madprime
(CC BY-ND 2.0)

## The Quadratic Formula

We know that while many quadratic equations can be solved by factor, some quadratics are "prime" and cannot be factord. Additionally, there might be times when a quadratic could be factored but we simply can't spot the factors.

Fortunately, there is a formula that will solve all quadratic equations, always. The foundations of this formula stretch back thousands of years beginning, we think, with ancient Babylonian and Egyptian mathematicians and continuing on through developments by Greek, Chinese, Indian, Persian, and European scholars.

The **quadratic formula**, in the form we know it today, is given below.

**The Quadratic Formula**

If $ax^2 + bx + c = 0$ where $a \neq 0$, then the two solutions are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula is something you should memorize. There are lots of ways to do that, and you should pick a method that suits your learning style, but many find that singing the formula works.

## Using the Quadratic Formula

Simply knowing the quadratic formula is not enough, however, we need to be able to insert values and evaluate the formula without making errors. Here are a few tips for avoiding common mistakes.

**Tips for Using the Quadratic Formula**

- **Arrange the Equation:** It must be in standard form and equal to zero: $ax^2 + bx + c = 0$.
- **Mind the Signs:** The values $a$, $b$, and $c$ do not include the variable $x$ but they do keep their signs. Put parenthesis around each one and remember that $b^2$ is always positive.
- **Evaluate Systematically:** Work on the square root first. Then simplify the rest of the numerator. Don't forget the $\pm$ sign and write two equations if needed. Lastly, divide everything in the numerator by $2a$.

Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses, especially when inserting a negative number. Also make sure that both terms in the numerator are divided by the denominator.

Since there are lots of places were simple mistakes can be made, we'll walk through a number of examples together.

**Example 1**

Solve $x^2 + 2x - 15 = 0$ for $x$ using the quadratic formula.

The equation is in the standard form, equal to zero, and the coefficients are $a = 1$, $b = 2$, and $c = -15$. We can solve this with the quadratic formula.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\
x &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-15)}}{2(1)} && \text{Substitute } a = 1, b = 2, c = -15 \\
x &= \frac{-2 \pm \sqrt{64}}{2} && \text{Simplify inside the square root} \\
x &= \frac{-2 \pm 8}{2} && \text{Evaluate the square root} \\
x = \frac{-2 + 8}{2} \quad \text{or} \quad x &= \frac{-2 - 8}{2} && \text{Write as two equations} \\
x = \frac{6}{2} \quad \text{or} \quad x &= \frac{-10}{2} && \text{Evaluate} \\
x = 3 \quad \text{or} \quad x &= -5 && \text{Simplify}
\end{aligned}$$

The equation used in Example 1 can also be solved by factoring and applying the zero-product property.

$$\begin{aligned}
x^2 + 2x - 15 &= 0 && \text{Original equation} \\
(x - 3)(x + 5) &= 0 && \text{Factor} \\
x - 3 = 0 \quad \text{or} \quad x + 5 &= 0 && \text{Use zero factor principle} \\
x = 3 \quad \text{or} \quad x &= -5 && \text{Solve}
\end{aligned}$$

Since we obtain the same solutions using either method, you can choose which ever technique you prefer. However, if a quadratic equation can be factored, that will always be faster than using the quadratic formula.

In the next example, however, the quadratic equation that cannot be factored, making the quadratic formula our best option for solving.

**Example 2**

Solve $3x^2 = 9x - 4$ for $x$ using the quadratic formula.

This equation is not in the standard form, so that will be the first step.

$$\begin{aligned} 3x^2 &= 9x - 4 && \text{Original equation} \\ 3x^2 - 9x &= -4 && \text{Subtract } 9x \\ 3x^2 - 9x + 4 &= 0 && \text{Add } 4 \end{aligned}$$

Now that the equation is in the standard form it is easier to identify the coefficients: $a = 3, b = -9, c = 4$ and we are ready to start using the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\ x &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(3)(4)}}{2(3)} && \text{Substitute } a = 3, b = -9, c = 4 \\ x &= \frac{9 \pm \sqrt{33}}{6} && \text{Simplify inside the square root.} \\ x = \frac{9 + \sqrt{33}}{6} \quad \text{or} \quad x &= \frac{9 - \sqrt{33}}{6} && \text{Write as two equations.} \\ x = 2.457 \quad \text{or} \quad x &= 0.543 && \text{Decimal approximation} \end{aligned}$$

Notice in the second to last step, the fraction bar is extended below both terms in the numerator: both are divided by 6. When using a calculator to simplify an expression such as this, we must use two separate steps or put parentheses around the entire numerator to get the correct value.

**Example 3**

Solve $4x^2 - 20x + 25 = 0$ for $x$ using the quadratic formula.

The equation is in the standard form, equal to zero, and the coefficients are $a = 4, b = -20$, and $c = 25$. We can solve this with the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\ x &= \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)} && \text{Substitute } a = 4, b = -20, c = 25 \\ x &= \frac{20 \pm \sqrt{0}}{8} && \text{Simplify inside the square root} \\ x &= \frac{20}{8} && \text{Simplify} \\ x &= \frac{5}{2} && \text{Reduce fraction} \end{aligned}$$

In Examples 1 and 2 we had two solutions, but this time, in Example 3, there was only one solution: $x = \frac{5}{2}$. In the next example, there will not be any real number solutions.

**Example 4**

Solve $-5x^2 - 5 = -8x$ for $x$ using the quadratic formula.

If we rearrange the equation to be in the standard form: $-5x^2 + 8x - 5 = 0$ then we can use the quadratic formula with $a = -5, b = 8$, and $c = -5$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\ x &= \frac{-(8) \pm \sqrt{(8)^2 - 4(-5)(-5)}}{2(-5)} && \text{Substitute } a = -5, b = 8, c = -5 \\ x &= \frac{-8 \pm \sqrt{-36}}{-10} && \text{Simplify inside the square root} \end{aligned}$$

We stop at this point because $\sqrt{-36}$ is not a real number.

Since $\sqrt{-36}$ is not a real number, the solutions will involve imaginary and complex numbers, which are not part of this course. The solutions exist, we simply don't have the tools to work with them.

So, in this course, anytime your quadratic formula involves the square root of a negative number then we can skip to the end and say there are no real solutions.

## A Practial Application


Now that we've see a few variations of what can happen, let's return to the situation that started this section.

Let's suppose that for a particular dive the quadratic equation $y = -16x^2 + 7x + 88$ gives the height (in feet) of the diver above the surface of the water $x$ seconds after leaving the platform.

Since the surface of the water corresponds to a height of $y = 0$, solving $0 = -16x^2 + 7x + 88$ with the quadratic formula will help us determine how much time the diver has before they hit the water.

**Example 5**

Solve $0 = -16x^2 + 7x + 88$ for $x$ using the quadratic formula.

**Step 1:** Apply the quadratic formula with $a = -16$, $b = 7$, and $c = 88$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(7) \pm \sqrt{(7)^2 - 4(-16)(88)}}{2(-16)}$$

**Step 2:** Simplify and approximate.

$$x = \frac{-7 \pm \sqrt{5681}}{-32}$$
$$x = \frac{-7 \pm 75.372}{-32}$$

**Step 3:** Solve for both solutions.

$$x = \frac{-7 + 75.372}{-32} = -2.137 \quad \text{or} \quad x = \frac{-7 - 75.372}{-32} = 2.574$$

In practial applications, it's not unusual for only one of the solutions to make sense in the given context. Since $x$ represents the time in seconds *after* the diver left the platform, the only value of the two that makes sense is $x = 2.574$. This would indicate that the diver will hit the water in about $2.5$ seconds. They will need to plan, practice and train with that time limit in mind.

## Conclusion

In the world of mathematics it is somewhat rare to find a universal key that unlocks solutions, and that's exactly what the quadratic formula does. While factoring and other methods might be simpler and faster, they do not have the universal application of the quadratic formula. It's widely used in both theoretical and practial settings, and there's no doubt you'll run into it again.

As we move into the next chapter, where graphing comes into play, we'll uncover another lens through which to view these equations and interpret their solutions.

# 4.4 Beyond the Basics

## Introduction

In this section, we will introduce three types of equations that are just a little bit different than what we've seen before.

These new equations can easily be confused for one of the earlier types, so we will have to proceed with caution to identify them correctly.

## Equations with Negative Coefficients

When solving an equation for a variable like $x$, we sometimes run into the expression $-x$. Since $-x$ can be viewed as $-1 \cdot x$, if we are able to isolate $-x$, then both sides could be divide by $-1$ to give us the value for positive $x$.

When solving an equation where the variable is negative, there are a few common errors that should be avoided. The first error is simply copying the equation incorrectly. For instance, $2 - x = 9$ is not the same as $x - 2 = 9$.

**Example 1**

Solve $2 - x = 9$ for $x$.

---

$$\begin{aligned} 2 - x &= 9 && \text{Original equation} \\ -x &= 7 && \text{Subtract 2} \\ x &= -7 && \text{Divide by } -1 \end{aligned}$$

The second error is not following the proper order of operations. For example, it is very tempting to simplify $5-2x$ as $3x$, but that places subtraction before multiplication, which is not the correct order.

**Example 2**

Solve $5-2x=21$ for $x$.

$$\begin{aligned} 5-2x &= 21 && \text{Given equation} \\ -2x &= 16 && \text{Subtract } 5 \\ x &= -8 && \text{Divide by } -2 \end{aligned}$$

Some prefer to rewrite the left side of the equation as $-2x+5$ to make the operations easier to see.

$$\begin{aligned} -2x+5 &= 21 && \text{Rewritten equation} \\ -2x &= 16 && \text{Subtract } 5 \\ x &= -8 && \text{Divide by } -2 \end{aligned}$$

Both produce the same result.

This same error is also possible in situations where distribution could be used. In the next example we will solve $7-3(x-1)=34$. Note that the $7$ and the $-3$ cannot be combined since the $-3$ is multiplying $(x-1)$ and $7$ is not.

**Example 3**

Solve $7-3(x-1)=34$ for $x$.

$$\begin{aligned} 7-3(x-1) &= 34 && \text{Given equation} \\ -3(x-1) &= 27 && \text{Subtract } 7 \\ x-1 &= -9 && \text{Divide by } -3 \\ x &= -8 && \text{Add } 1 \end{aligned}$$

Or, if you prefer to use distribution:

$$\begin{aligned} 7-3(x-1) &= 34 && \text{Given equation} \\ 7-3x+3 &= 34 && \text{Distribute } -3 \\ 10-3x &= 34 && \text{Combine constants} \\ -3x &= 24 && \text{Subtract } 10 \\ x &= -8 && \text{Divide by } -3 \end{aligned}$$

## Equations with a Variable in the Denominator

The next type of equation to watch out for is one where the variable is in the denominator. It's critical to recognize that dividing something by $x$ is not the same as dividing $x$ by something. For instance, $\frac{5}{x}$ is not the same as $\frac{x}{5}$.

Luckily, equations with a variable in the denominator can usually be converted into ones where the variable isn't in the denominator.

This is done by "clearing the fractions" by multiplying each term by the least common denominator (LCD). We did a similar step to simplify equations in section 4.1.

**Example 4**

Solve $\frac{5}{x} = 8$ for $x$.

---

Since the denominators are $x$ and $1$, the LCD is $x$.

$$\begin{aligned} \frac{5}{x} &= 8 && \text{Given equation} \\ x \cdot \frac{5}{x} &= x \cdot 8 && \text{Multiply by } x \\ 5 &= 8x && \text{Simplify} \\ x &= \frac{5}{8} && \text{Divide by } 8 \end{aligned}$$

Equations where both sides are fractions are known as **proportions**. With proportions, the process of clearing the fractions can be viewed as **cross multiplication**.

To cross multiply fractions, multiply the top of each fraction with the bottom of the other fraction. This converts an equation like $\frac{a}{b} = \frac{c}{d}$ into one of the form $a \cdot d = b \cdot c$.

$$\frac{a}{b} = \frac{c}{d} \longrightarrow a \cdot d = b \cdot c$$

**Example 5**

Solve $\frac{10}{x} = \frac{2}{7}$ for $x$.

$$\begin{aligned} \frac{10}{x} &= \frac{2}{7} && \text{Given equation} \\ 10 \cdot 7 &= 2 \cdot x && \text{cross multiply to eliminate the denominator} \\ 70 &= 2x && \text{Simplify} \\ 35 &= x && \text{Divide by 2} \end{aligned}$$

Sometimes cross multiplication will lead to situations where distribution and/or combining like terms is necessary.

**Example 6**

Solve $\frac{3}{x+1} = \frac{4}{3x}$ for $x$.

$$\begin{aligned} \frac{3}{x+1} &= \frac{4}{3x} && \text{Given equation} \\ 9x &= 4(x+1) && \text{cross multiply to eliminate the denominator} \\ 9x &= 4x + 4 && \text{Distribute 4} \\ 5x &= 4 && \text{Subtract } 4x \\ x &= \frac{4}{5} && \text{Divide by 5} \end{aligned}$$

Cross multiplication is a handy tool, but don't forget that it can only be used when both sides are single fractions.

If one side has two or more terms then we cannot use cross multiplication and need to fall back on the method of multiplying by a common denominator. That's what we'll encounter in the next example.

**Example 7**

Solve $\frac{4}{x}+3=\frac{23}{5x}$ for $x$.

Since the left side is not a single fraction, we cannot cross multiply. Instead, we'll multiply by the common denominator, which is $5x$.

$$\begin{aligned} \frac{4}{x}+3 &= \frac{23}{5x} && \text{Given equation} \\ 5x\left(\frac{4}{x}+3\right) &= 5x\cdot\frac{23}{5x} && \text{Multiply both sides by } 5x \\ 5x\cdot\frac{4}{x}+5x\cdot 3 &= 5x\cdot\frac{23}{5x} && \text{On the left, distribute } 5x \\ \frac{20x}{x}+15x &= \frac{115x}{5x} && \text{Multiply} \\ 20+15x &= 23 && \text{Simplify fractions} \\ 15x &= -3 && \text{Subtract 20 from both sides} \\ x &= \frac{-3}{15} && \text{Divide both sides by 15} \\ x &= \frac{-1}{5} && \text{Reduce fraction} \end{aligned}$$

## Equations with a Variable in the Exponent

This final type of equation might look familiar, and that's part of the challenge. It takes a careful eye to notice that expressions like $x^2$ and $2^x$ are not the same. The first has a variable base with power of $2$. The other has a base of $2$ and a variable exponent.

When the variable appears in the exponent, we call it an **exponential equation**. Again, the key feature of an exponential equation is that the variable is in the exponent, while the base remains constant.

**Example 8**

For each equation, determine whether it is an exponential equation. If it is exponential, identify the base.

**a.** $x^3 = 27$

**b.** $3^x = 27$

**c.** $2^{x+1} = 16$

**d.** $x^2 + 5x = 14$

**e.** $4 \cdot 5^x = 80$

---

**a.** Not exponential. The variable $x$ is the base, not in the exponent.

**b.** Exponential equation with base $3$. The variable $x$ is in the exponent.

**c.** Exponential equation with base $2$. The variable expression $x + 1$ is in the exponent.

**d.** Not exponential. This is a quadratic equation with the variable in the base positions.

**e.** Exponential equation with base $5$. The coefficient $4$ is not part of the base.

In exponential equations, the exponent is a variable, not a fixed number like $2$ or $3$, so we cannot use roots to solve them. Sometimes, however, we can use our background knowledge of numbers to try a guess-and-check method of solving.

**Example 9**

**a.** Solve $7^x = 49$

**b.** Solve $10^x = 1000$

**c.** Solve $3^x = 81$

**d.** Solve $5^x = 125$

---

Since the variables are in the exponent, these are all exponential equation. We will use guessing and checking to find the solutions.

**a.** $x = 2$ because $7^2 = 49$.

**b.** $x = 3$ because $10^3 = 1000$.

**c.** $x = 4$ because $3^4 = 81$.

**d.** $x = 3$ because $5^3 = 125$.

## Logarithms

When we can't guess-and-check the solution to an exponential equation, an inverse operation called a **logarithm** can be used.

Just as subtraction undoes addition and division undoes multiplication, logarithms undo exponentiation. Logarithms allow you to find the exponent if you know what the base is.

The base of a logarithm is always written as a subscript. For example in the expression $\log_2(16)$, the base is the number $2$. And when no subscript is shown, like $\log(100)$, the base is assumed to be $10$.

The value of a logarithm is the exponent you need to put on the base in order to get the number in the parenthesis. With this in mind, it makes sense that $\log_2(8) = 3$ since $2^3 = 8$, for example, or that $\log(100) = 2$ since $10^2 = 100$.

**Key Idea**

A logarithm is the missing exponent.

$$\log_b(y) = \square \quad \text{only if} \quad b^{\,\square} = y$$

**Example 10**

Use your understanding of what a logarithm is to find the following values mentally.

**a.** $\log_7(49) = \square$

**b.** $\log(1000) = \square$

**c.** $\log_3(81) = \square$

**d.** $\log_5(125) = \square$

---

For each logarithm, we need to figure out which exponent to put on the base to get the number in parentheses. That exponent will be our answer.

**a.** $\log_7(49) = 2$ because $7^2 = 49$

**b.** $\log(1000) = 3$ because $10^3 = 1000$

**c.** $\log_3(81) = 4$ because $3^4 = 81$

**d.** $\log_5(125) = 3$ because $5^3 = 125$

You might recognize that these are the same exact answers we had in **Example 9**, which shows that logarithms are a tool for solving exponential equations. Together, these examples illustrate the basic connection between logarithms and exponents: a logarithmic equation of the form $\log_b(y) = x$ is equivalent to the exponential equation $b^x = y$.

While the logarithms in this example were ones we were able to sort out mentally from our knowledge of exponents, finding the value of most logarithms is not a trivial exercise. Thankfully, all scientific calculators have a `LOG` button that finds logarithms when the base is $10$. For example, if you enter `LOG(100)` your calculator will return `2` since $10^2 = 100$.

**Example 11**

If possible, use the LOG button on your calculator evaluate each expression. If needed, round to $3$ decimal places and check your answers.

**a.** $\log(200)$

**b.** $\log(649)$

**c.** $\log(937)$

**d.** $\log(-100)$

---

**a.** $\log(200) \approx 2.301$ which we check by seeing $10^{2.301} = 199.986 \approx 200$

**b.** $\log(649) \approx 2.812$ which we check by seeing $10^{2.812} = 648.634 \approx 649$

**c.** $\log(937) \approx 2.972$ which we check by seeing $10^{2.972} = 937.562 \approx 937$

**d.** $\log(-100)$ does not exist since no power of $10$ gives a negative number. We can only put positive numbers into logarithms.

If you remember that $\log$ of a number is the exponent you need to put on a $10$ to get that number, you can see how it helps us solve equations like $10^x = 100$. In general, if your equation is $10^x = y$ then the solution is $x = \log(y)$.

Keep in mind that the LOG button does not allow you to solve equations like $3^x = 24$ directly since the base is $3$, not $10$. Methods for solving exponential equations like that are covered in a precalculus course. For now we'll stick to equations where the base is $10$.

**Example 12**

If possible, use the LOG button on your calculator to solve each equation. If needed, round to $3$ decimal places.

**a.** Solve $10^x = 80$

**b.** Solve $10^x = 5000$

**c.** Solve $2^x = 720$

**d.** Solve $10^x = 0.0242$

---

**a.** $x = \log(80) \approx 1.903$

**b.** $x = \log(5000) \approx 3.699$

**c.** Cannot be solved directly with $\log$ since the base is not $10$.

**d.** $x = \log(0.0242) \approx -1.616$

The negative answer in Example 12(d) might surprise you after the comment in Example 11(d). It's perfectly fine for a negative number to come out of a logarithm, we just cannot put negative numbers into a logarithm. In fact, whenever the input is less than $1$, the logarithm has to be negative because we need a negative exponent to make $10^x$ small.

## Conclusion

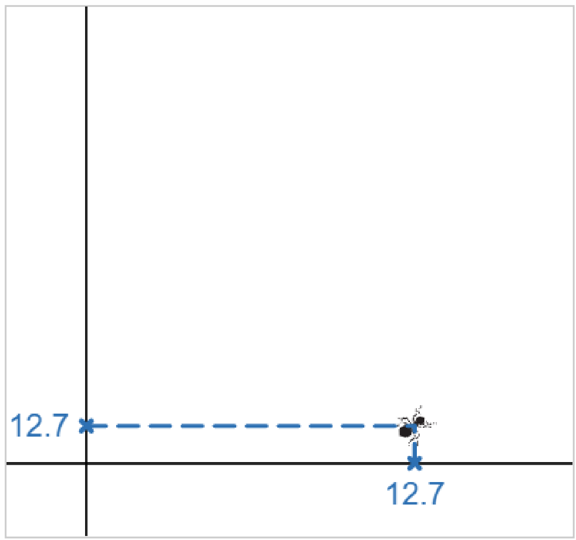
In this section we've expanded our equation-solving toolkit to handle several challenging variations: equations with negative coefficients, variables in denominators, and exponential equations. Being able to recognize the specific type of equation is critical so we can choose the appropriate solving technique.

While we've developed algebraic methods for solving many different types of equations, there's another powerful approach we haven't explored yet: solving equations graphically. In Chapter 5, we'll discover how graphs can help us visualize equations and their solutions.

# 5.1 The Cartesian Coordinate System

## Introduction

Legend has it that René Descartes, the 17th-century French mathematician and philosopher renowned for coining the phrase "I think, therefore I am," once lay in bed observing a bug's journey across his ceiling. In a moment of inspiration, he realized he could pinpoint the insect's location by defining its distance from the two adjoining walls that formed a corner on the ceiling.




This spark of ingenuity forever transformed the way we visualize mathematics.

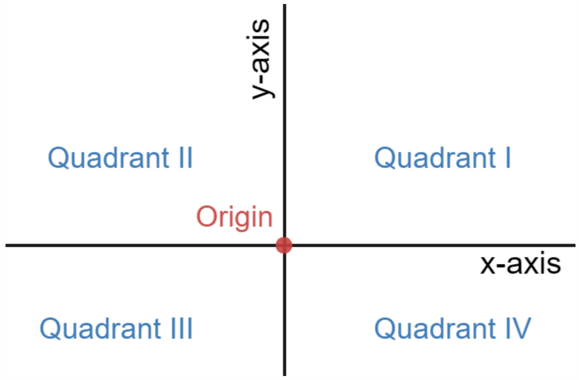
## The Cartesian Plane

The coordinate system that came from Descartes' work is now called the Cartesian coordinate system or the Cartesian coordinate plane.

This system is formed by two perpendicular number lines, each one called an **axis**. The horizontal axis is often referred to as the $x$-axis and the vertical axis is known as the $y$-axis.

The point where the two axes meet is called the **origin**. The origin serves as the fixed point of reference for all other points on the coordinate plane.

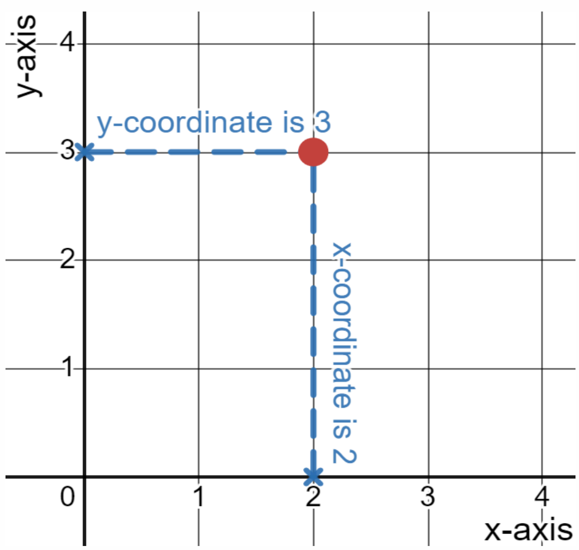
The point where the two axes meet is called the origin. The two axes together divide the plane into 4 rectangular quarters. We call these quadrants and refer to them as quadrant I, II, III, and IV.




## Cartesian Coordinates

Graphing on the Cartesian plane hinges on knowing two values: one for the $x$-axis and another for the $y$-axis. Together, these two coordinates identify a unique point on the plane.

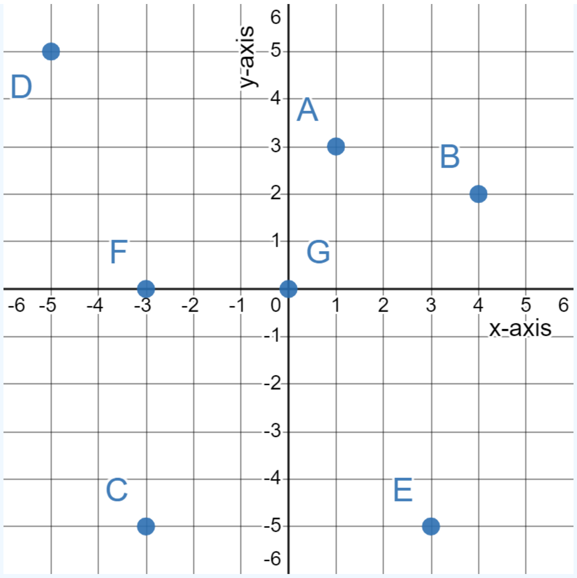
The coordinates of a point are always written as an ordered $(x, y)$ pair, with the $x$-coordinate first and the $y$-coordinate second. So a pair of numbers like $(2, 3)$ would represent the point that has an $x$-coordinate of $2$ and a $y$-coordinate of $3$.




Again, the pair of numbers is ordered, so the $x$-coordinate must be listed first.

**Example 1**

Use an ordered pair to give the coordinates of each point labeled in this graph. Also identify the quadrant the point is in.




A is the point $(1, 3)$. This point is in quadrant I.

B is the point $(4, 2)$. This point is in quadrant I.

C is the point $(-3, -5)$. This point is in quadrant III.

D is the point $(-5, 5)$. This point is in quadrant II.

E is the point $(3, -5)$. This point is in quadrant IV.

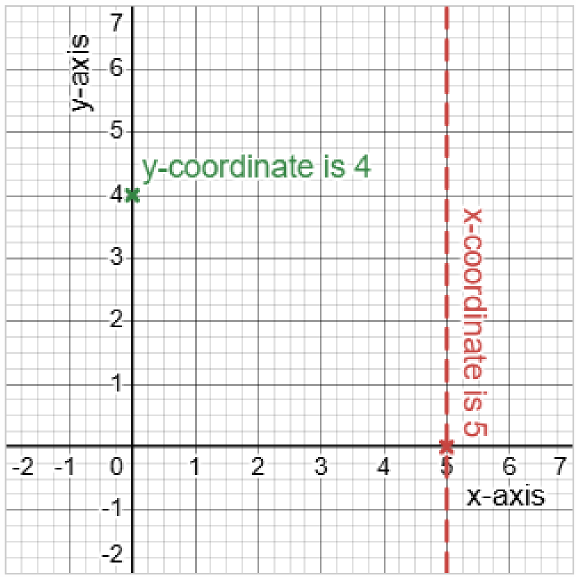
F is the point $(-3, 0)$. This point is on the x-axis.

G is the point $(0, 0)$. This point is the origin.

One of the most common mistakes people make with ordered pairs is to list the coordinates in reverse order. So always remember, an ordered pair has the form $(x, y)$.

## Plotting Points

The key to ploting points on the Cartesian plane is remembering that coordinates are an ordered $(x, y)$ pair. To plot the point $(5, 4)$, for instance, find $5$ on the $x$-axis and draw an imaginary vertical line through it. Then find $4$ on the $y$-axis and sketch an imaginary horizontal line through it. The point $(5, 4)$ is located at the spot where those two lines cross. The animation below illustrates this process.




## Scatterplots

Plotting ordered pairs can be very helpful in visualizing concepts and relationships. This is especially true when the information is given as a table of values.

By viewing each pair of data in the table as an ordered pair, we can convert the table into plotted points, creating what is known as a **scatterplot**. The shape of the scatterplot can suggest patterns that are difficult to see from the numerical data alone.

**Example 2**

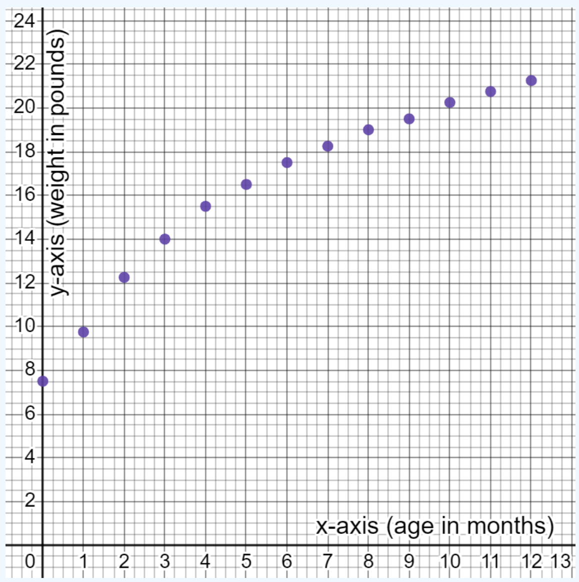
As part of a research study, a newborn baby was weighed every month for 12 months to track its development. The weights are recorded in the table below.

| **Age (months)** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Weight (pounds)** | 7.5 | 9.75 | 12.25 | 14 | 15.5 | 16.5 | 17.5 | 18.25 | 19 | 19.5 | 20.25 | 20.75 | 21.25 |

Create a scatterplot of the data and describe any patterns you see.

---

We will view each age/weight pairs as ordered pairs, using age as the $x$-coordinate and weight as the $y$-coordinate.




By graphing these points, we can quickly see visually that the baby's weight increased faster in the first 6 months than it did in the second six months.

## Graphing Linear Equations

Whenever we encounter an equation of any type a process similar to the one we used above can be empolyed to produce a graph of that equation. The only difference is the values in the table will need to be calculated using the equation.

This is done by putting random values into the equation for the $x$-variable and evaluating the equation to get the corresponding $y$-value. The points are then plotted and a smooth line or curve is drawn through them to indicate the other values we did not compute.

As an example, let's consider the linear equation $F = \frac{9}{5}C + 32$ which we saw in section 3.3. This formula converts temperature in degrees Celsius to degrees Fahrenheit, something that would be helpful for a US native traveling abroad.

If the temperature forecast was for $15$ degrees Celsius, then the formula gives the corresponding temperature in degrees Fahrenheit.

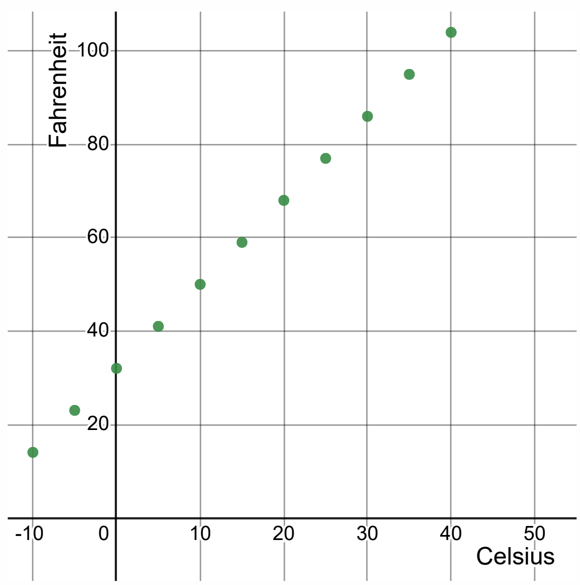
$$\begin{aligned} F &= \frac{9}{5} \cdot 15 + 32 \\ &= 27 + 32 \\ &= 59 \end{aligned}$$

This means that a temperature of $15$ degrees Celsius is equivalent to $59$ degrees Fahrenheit. If you were headed outside, you might want to take a light jacket.

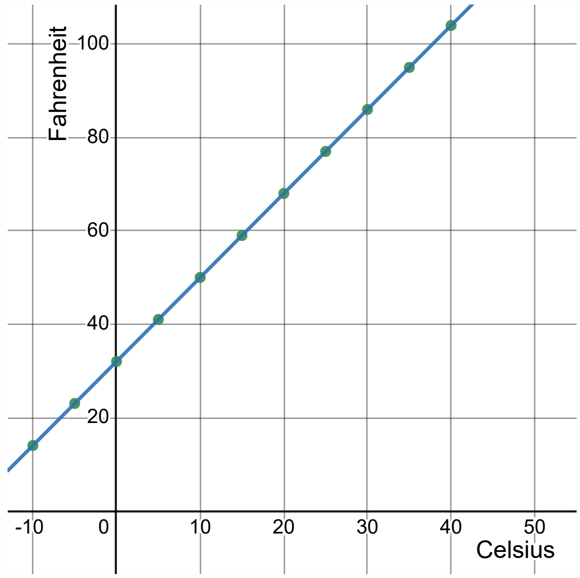
In the table below we've computed several more Celsius/Fahrenheit temperatures.

| **Celsius** | -10 | -5 | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Fahrenheit** | 14 | 23 | 32 | 41 | 50 | 59 | 68 | 77 | 86 | 95 | 104 |

With this table, we can now graph the ordered pairs to visualize the relationship between Celsius and Fahrenheit.



Since we only plotted some of the potential Celsius/Fahrenheit pairs (there are other we could have computed), we'll connect the dots with a line to show all the possible pairs.




Any equation like this one, which produces the graph of a line, is called a **linear equation**. In linear equations the variables are only added, subtracted, or multiplied by constants—no roots or powers higher than $1$. Linear equations will be the focus of the next two sections.

## Graphing Nonlinear Equations

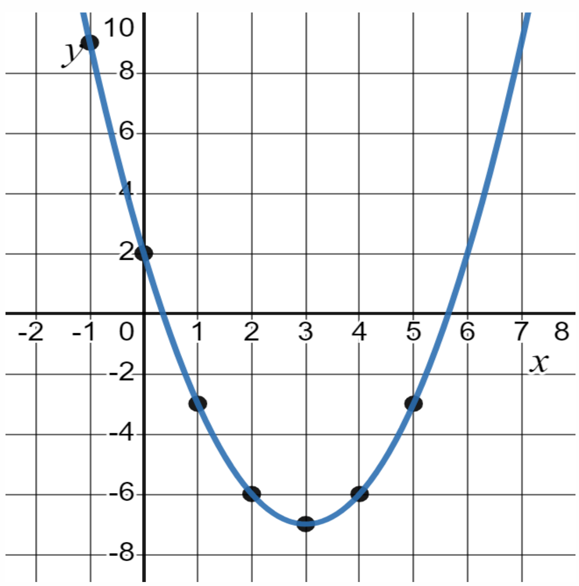
Not every equation that we encounter will result in the graph of a line. When variables have powers of $2$ or higher, or appear in other more complex ways like square roots, the graphs will be curved rather than straight, like the graph from the baby data earlier.

Consider the nonlinear equation: $y = x^2 - 6x - 2$. Even though the equation is more complicated than a linear equation, we start the same way by using the equation to calculate the y-values for several different x-values.

| $x$ | $y = x^2 - 6x - 2$ |
|---|---|
| $-1$ | $(-1)^2 - 6(-1) + 2 = 9$ |
| $0$ | $(0)^2 - 6(0) + 2 = 2$ |
| $1$ | $(1)^2 - 6(1) + 2 = -3$ |
| $2$ | $(2)^2 - 6(2) + 2 = -6$ |
| $3$ | $(3)^2 - 6(3) + 2 = -7$ |
| $4$ | $(4)^2 - 6(4) + 2 = -6$ |
| $5$ | $(5)^2 - 6(5) + 2 = -3$ |

Then we plot these $(x, y)$ points and draw a smooth curve between them. Keep in mind that while linear equations can be draw with just a few points, with nonlinear graphs it is best to plot several points before drawing the curve.

Plotting those points creates this U-shaped curve which is called a **parabola**




A defining feature of every parabola is that it can be split into two mirror image halves. If the two halves of your parabola do not match, it's a good idea to go back and double check your calculations, in particular places where you square a negative.

Parabolas and the equations that create them, known as **quadratic equation**, are covered in greater detail toward the end of this chapter.

## Conclusion

In this section, we explored the Cartesian coordinate system, plotting ordered pairs, and visualizing the relationships between variables through scatterplots. We learned how linear equations result in straight-line graphs, while nonlinear equations can give rise to intriguing curves like parabolas.

As we move forward into the next section we will consider the concept of **slope**, which plays a pivotal role in understanding the steepness and direction of lines.

# 5.2 Understanding Slope

## Introduction

If you've ever been walking, hiking or biking in a hilly area then you know that not all trails are flat! Trails have various inclines; some extremely steep and others relatively level.

Lines are the same; they can rise rapidly or drop gradually or anything in between. That change in elevation is measured and understood using the concept of **slope**.

## The Concept of Slope

In the case of our trail, how could you describe its steepness? One way would be to measure how much the trail rises or falls over a fixed distance, like $100$ meters. If you found that one section rises $15$ meters for every $100$ meters you move forward then you'd know it is steeper than a section that only rise $5$ meters over the same distance. This would allow you to compare the steepness of different trail segments.

Slope is a measure of how rapidly lines change and it works exactly the same way. We even use similar words. Slope is often described as "rise-over-run", where "rise" means the vertical change and "run" is the horizontal change.

In our trail example, "rise" is the vertical change in elevation while the "run" is the horizontal distance covered. If we divide the rise by the run (which is what "rise-**over**-run" literally means) then we get a ratio indicating the vertical change for every unit change horizontally.

Because of this, slope is often referred to as a "rate of change" or a "per unit change". Things like miles *per* gallon, servings *per* package, dollars *per* hour, or feet *per* second are examples of rates of change.

## Calculating Slope

Usually, we use the letter $m$ to represent slope and, if we know the rise and run, then $m = \frac{\text{rise}}{\text{run}}$.

For instance, if a trail rises $20$ meters for every $100$ meters you move forward, the slope would be

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{20}{100} \\ &= \frac{1}{5} \end{aligned}$$

If the rise and run are not provided, which is generally the case, we will have to find them. The best way to do this is to identify the coordinates of two points $(x_1, y_1)$ and $(x_2, y_2)$ on the line. Since rise is a vertical change, it could be found by subtracting the $y$-coordinates. Run could be calculated in a similar way by subtracting the two $x$-coordinates.

**Slope Formula:**

If $(x_1, y_1)$ and $(x_2, y_2)$ are two points a line, then the slope of that line is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

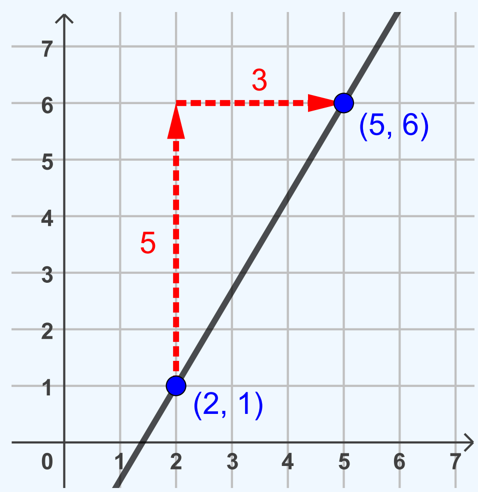
Let's walk through an examaple that can be done both ways.

**Example 2**

Find the slope of the line that passes through the points $(2, 1)$ and $(5, 6)$.

From the graph it's clear that the rise is $5$ and the run is $3$, giving us a slope of

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{5}{3} \end{aligned}$$




Now let's use the slope formula.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6 - 1}{5 - 2} \\ &= \frac{5}{3} \end{aligned}$$

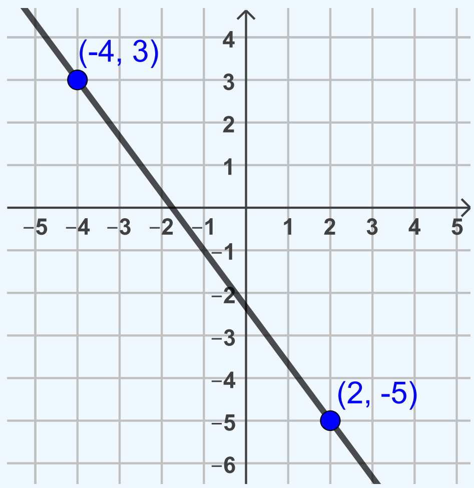
The slope between these two points is $m = \frac{5}{3}$. This means that for every $3$ units of horizontal change, there is a corresponding $5$ units of vertical change.

In this example we used $(2, 1)$ as the first point $(x_1, y_1)$ and $(5, 6)$ as the second point $(x_2, y_2)$. In practice, it makes no difference which point you label as the first one. What is vital, however, is that once the points are labeled, the coordinates go into their proper places in the formula.

**Example 3**

Find the slope of the line that passes through the points $(-4,3)$ and $(2,-5)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-5-3}{2-(-4)} \\ &= \frac{-8}{6} \\ &= \frac{-4}{3} \end{aligned}$$




The slope between these two points is $m = \frac{-4}{3}$. This means that for every $3$ units of horizontal change, there is a corresponding drop of $-4$ units in the vertical direction.

Comparing the graphs from these last two examples will give us an extra insight into slope and the behavior of lines.

A positive slope (like in Example 1) corresponds to an upward slant from left to right - just like a trail that climbs upward as you hike forward. A negative slope (like Example 2) indicates that the line has a downward slant, similar to a trail that descends as you move ahead.

## Zero Slope and Undefined Slope

Since the slope formula has subtraction, there's a chance we might end up with $0$ in either the numerator or the denominator.

If $y_1 = y_2$ then the slope is zero ($m = 0$), which means the line is horizontal.

If $x_1 = x_2$ then the slope is undefined (since division by $0$ is undefined), which means the line is vertical.

Let's explore a few more examples where this comes into play.

**Example 4**

Find the slope of the line through the points $(-2, 4)$ and $(3, 4)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-4}{3-(-2)} \\ &= \frac{0}{5} \\ &= 0 \end{aligned}$$

In this case, since the slope was $0$, the line is horizontal. This could have been seen by noticing the two $y$-coordinates were both $4$.

**Example 5**

Find the slope the line through $(-1, 2)$ and $(-1, -3)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-3-2}{-1-(-1)} \\ &= \frac{-5}{0} \\ &= \text{undefined} \end{aligned}$$

Since division by $0$ is undefined, the slope is undefined. This means the line is vertical, which we could have seen by noticing the two $x$-coordinates were the same.

## Using Slope to Graph Lines

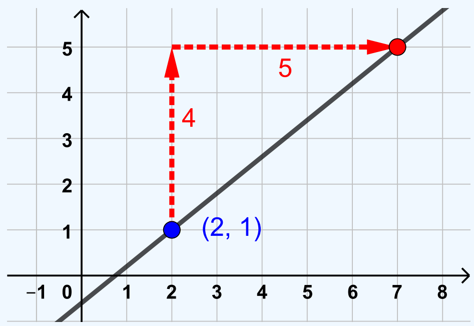
Now that we know the variety of values slope can have, we can begin to use it to graph lines.

If you know the slope of a line and the coordinates of one point, then other points can be found by adding the rise to the $y$-coordinate and the run to the $x$ coordinate.

**Example 6a**

Use slope to graph the line through $(2, 1)$ with a slope of $m = \frac{4}{5}$.

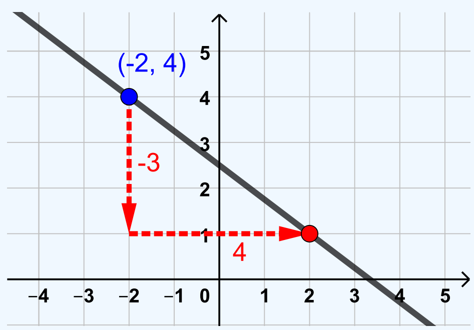
To graph the line, start at the point (2,1) and move up 4 spaces and right 5 spaces to make another point. Then draw the line through those points. Extra points can be added using that same up 4, right 5 pattern if desired.




**Example 6b**

Use slope to graph the line through $(-2, 4)$ with a slope of $m = \frac{-3}{4}$.

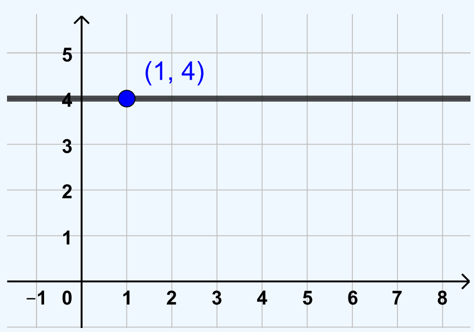
For this line we start at $(-2, 4)$ and then move *down* $3$ spaces and right $4$ to create second point, and then draw the line.




**Example 6c**

Use slope to graph the line through $(1, 4)$ with a slope of $m = 0$.

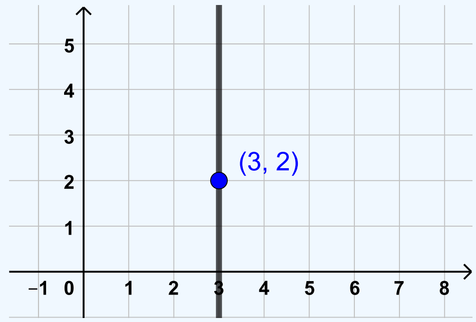
Since the slope is $m = 0$, we know this line is horizontal. To graph it we start at $(1, 4)$ and pick any other point that also has a $y$-coordinate of $4$. All the points on a horizontal line always have the same $y$-coordinate.



**Example 6d**

Use slope to graph the line through $(3, 2)$ with undefined slope.

Since the slope is undefined, this line is vertical. To graph it we start at $(3, 2)$ and pick any other point with an $x$-coordinate of $3$. All the points on a vertical line always have the same $x$-coordinate.




## Practical Application of Slope

Slope isn't just a mathematical concept; it has real-world applications. From architecture and urban planning (such as determining the slope of wheelchair ramps) to calculating rates (like steps per minute or dollars per hour), slope gets used in many places.

In real world scenarios the information may not be given as points, so we might have to extract the coordinates from the description before we can compute the slope.

**Example 7**

Suppose a scuba diver breathes air at a constant rate. After $15$ minutes of diving they have $2225$psi of air. After $30$ minutes their gauge shows $1400$psi. Find and interpret the slope in this scenario.

---

If we use time as the $x$-coordinates and psi of air as the $y$-coordinates, then our two points are $(15, 2225)$ and $(30, 1400)$.

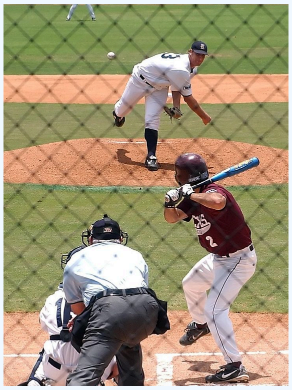
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1400 - 2225}{30 - 15} \\ &= \frac{-825}{15} \\ &= -55 \end{aligned}$$

Viewing this as a rate, it would mean that for every minute of diving the amount of air in the diver's tank decreases by $55$ psi.

Since scuba divers need to save enough air to slowly ascend (to avoid decompression), monitoring the rate at which they consume air is vital. Most divers use color coded pressure gauges so they don't have to do algebra when diving.

**Example 8**

In baseball, the batter at home plate is $60.5$ feet away from the pitcher's mound. Suppose on a particular pitch the ball travels that distance from the mound to home plate in $0.5$ seconds. Find and interpret the slope in this scenario.



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If we use time as the $x$-coordinates and distance in feet from home plate as the $y$-coordinates, then our two points are $(0, 0)$ and $(0.5, 60.5)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{60.5 - 0}{0.5 - 0} \\ &= \frac{60.5}{0.5} \\ &= 121 \end{aligned}$$

If we include the units this tell us the ball is traveling at a rate of $121$ feet per second. In other words, for this example the slope is the speed of the pitch.

That sounds fast, but it's hard to compare with other things we might be familiar with that are measured in miles per hour. But if we know that $1 \text{ mile} = 5280 \text{ feet}$ and $1 \text{ hour} = 3600 \text{ seconds}$, we can convert the units.

$$\begin{aligned} &\frac{121 \text{ feet}}{1 \text{ second}} \times \frac{1 \text{ mile}}{5280 \text{ feet}} \times \frac{3600 \text{ seconds}}{1 \text{ hour}} && \text{Apply conversion factors} \\ &\quad = 121 \times \frac{1}{5280} \times \frac{3600}{1} \ \frac{\text{miles}}{\text{hour}} && \text{Simplify units} \\ &\quad = 82.5 \ \frac{\text{miles}}{\text{hour}} && \text{Multiply} \end{aligned}$$

We now have the speed of the ball in miles per hour and can compare it more easily with the speed of a car or a person running.

## Conclusion

Now that we have a clear understanding of what slope signifies and how to compute it, we need to spend more time seeing how it helps us graph lines by hand, and how it might be incorporated into linear equations.

That's the focus of the next section where we'll investigate the slope-intercept form of a line.

# 5.3 Graphing Linear Equations

## Introduction

In the last section, we had to create a list of points to graph a line. In this section, we will introduce a new tool, called the **slope-intercept form** of a line, that lets us to extract information without having to calculate any points.

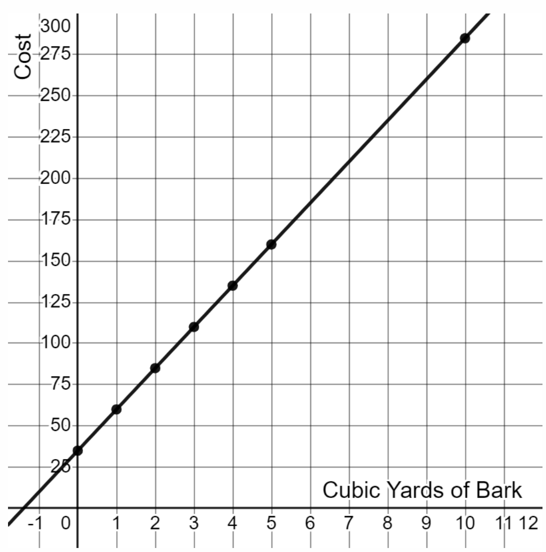
## The Slope Intercept Form

Let's begin with a practical situation. As we walk through this carefully we'll uncover exactly what this slope intercept form is and what makes it useful.

Imagine a landscape supply company charges $\$25$ per cubic yard for bark mulch and an additional flat $\$35$ delivery fee. When ordering bark, the total cost will start at $\$35$ and will go up at a rate of $\$25$ for each cubic yard ordered.

If we let $x$ represent the number of cubic yards of bark and let $y$ represent the total cost, then the equation for total cost becomes $y = 25x + 35$. Let's make a table of values for this equation and plot those points.

| $x$ (cubic yards) | 0 | 1 | 2 | 3 | 4 | 5 | 10 |
|---|---|---|---|---|---|---|---|
| $y = 25x + 35$ (total cost in $) | $35 | $60 | $85 | $110 | $135 | $160 | $285 |



This graph is clearly a line and the point where the line crosses the $y$-axis is $(0, 35)$. Notice that the $y$-coordinate of this point is the constant in our equation $y = 25x + 35$.

What is the slope of the line? We can use any two points from the table to find out. Let's use $(5, 160)$ and $(10, 285)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{285 - 160}{10 - 5} \\ &= \frac{125}{5} \\ &= 25 \end{aligned}$$

Remarkably, this value is also in our equation as the coefficient of $x$. In our scenario, it represents the rate of change of the cost: $\$25$ *per* cubic yard of bark.

The fact that both the slope and $y$-intercept can both be identified just from the equation is an amazing shortcut. This holds true for any equation in the form $y = mx + b$.

**Linear Equations:**

Any equation that can be put in the form $y = mx + b$ is a **linear equation**. Its graph will be a straight line. Additionally,

- $m$ represents the slope or rate of change of the line.
- The point $(0, b)$ is the $y$-intercept of the line.

The format $y = mx + b$ is called the **slope-intercept** form of a line

The slope-intercept form is a powerful tool for understanding and working with linear equations and their graphs.

**Example 1**

Identify the slope and $y$-intercept of the following lines.

**a.** $y = -2x + 31$

**b.** $y = \frac{2}{5}x - 7$

---

Since both equations are in the slope-intercept form, the slope and $y$-intercept can be identified from the equation without any calculations.

**a.** $y = -2x + 31$ has a slope of $m = -2$ and a $y$-intercept at $(0, 31)$.

**b.** $y = \frac{2}{5}x - 7$ has a slope of $m = \frac{2}{5}$ and a $y$-intercept at $(0, -7)$.

**Example 2**

**a.** Write the equation of a line with a slope of $3$ and a $y$-intercept at $(0, 12)$.

**b.** Write the equation of a line with a slope of $\frac{-5}{6}$ and a $y$-intercept at $(0, 8)$.

---

If we use the slope intercept form, the values for $m$ and $b$ can be substituted in directly.

**a.** $y = 3x + 12$ has a slope of $m = 3$ and a $y$-intercept at $(0, 12)$.

**b.** $y = \frac{-5}{6}x + 8$ has a slope of $m = \frac{-5}{6}$ and a $y$-intercept at $(0, 8)$.

## Graphing Linear Equations in Slope-Intercept Form

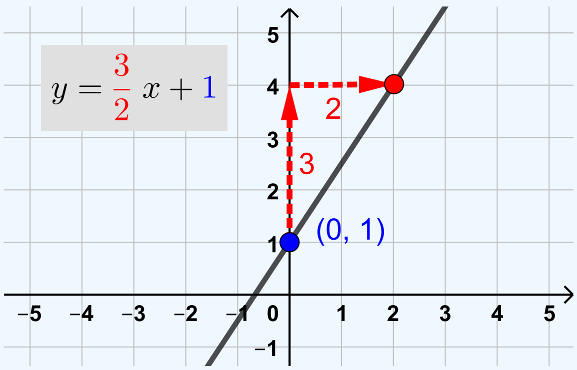
Graphing linear equations becomes intuitive when they're in slope-intercept form.

To graph a line in slope-intercept form, first plot the point $(0, b)$, which is the $y$-intercept. Then, starting from the $y$-intercept, use the rise of the slope to move up or down and the run of the slope to move side-to-side and plot a second point. Use those two points to draw the line.

**Example 3**

Graph the line $y = \frac{3}{2}x + 1$.

Since this is in the slope-intercept form we know the $y$-intercept is $(0, 1)$. Starting from that point we'll use the slope $\frac{3}{2}$ and move up $3$ spaces and right $2$ to plot a second point.



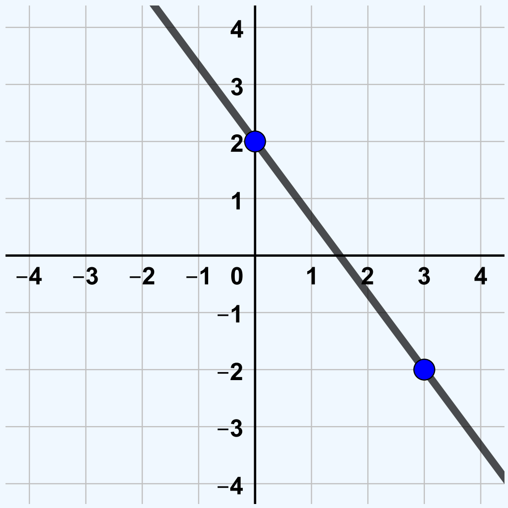

## Writing the Slope-Intercept Form of a Line Given Its Graph

If you're given the graph of a line, it's possible to extract the slope and $y$-intercept directly from it. The $y$-intercept should be straightforward to spot, since it is where the line crosses the $y$-aixis. The slope is not much harder, but we need to be precise.

To find the slope, find another point on the line where you are confident of it's coordinates. Then carefully count the rise and run between that point and your $y$-intercept. Make sure the sign of your final slope value matches the direction of the line. As we saw in the last section, lines with positive slope rise when moving left to right while lines with negative slope go down.

**Example 4a**

Find the equation of the line graphed below.

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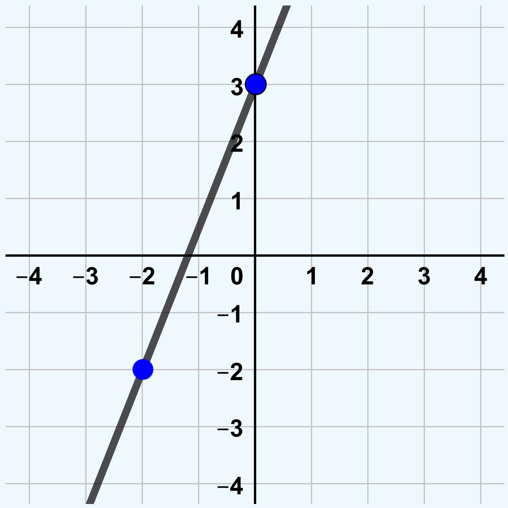
The $y$-intercept is $(0, 2)$, so $b = 2$. The slope appears to be $m = \frac{-4}{3}$. From this we conclude the equation of the line must be

$$y = \frac{-4}{3}x + 2$$

**Example 4b**

Find the equation of the line graphed below.

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The $y$-intercept is $(0, 3)$, so $b = 3$. The slope appears to be $m = \frac{5}{2}$. From this we conclude the equation of the line must be

$$y = \frac{5}{2}x + 3$$

## Finding the Slope-Intercept Form Given Two Points

If we are given two points instead of a graph then it's still possible to find the equation of the line that goes through them. It is an involved process, so we'll walk through the process together.

For our example, let's use the points $(2, 5)$ as $(x_1, y_1)$ and $(6, -1)$ as $(x_2, y_2)$. We'll start with the slope formula.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{slope formula} \\ &= \frac{-1 - 5}{6 - 2} && \text{insert coordinates of the two points} \\ &= \frac{-6}{4} && \text{evaluate the subtractions} \\ &= -\frac{3}{2} && \text{reduce the fraction} \end{aligned}$$

The only part of the slope-intercept form that is missing from our equation is the $y$-intercept $b$. Since neither of our points was the $y$-intercept, we will need to put the coordinates of one of our points, along with the slope we just calculated, into the slope-intercept equation and solve for $b$. We will use the point $(2, 5)$ and, of course, $m = \frac{-3}{2}$.

$$\begin{aligned} y &= mx + b && \text{slope-intercept form} \\ 5 &= -\frac{3}{2}(2) + b && \text{insert the slope and the coordinates of the point} \\ 5 &= -3 + b && \text{simplify} \\ 8 &= b && \text{add 3 to both sides} \end{aligned}$$

With both $m$ and $b$ in hand, we can now say that $y = \frac{-3}{2}x + 8$ is the line that goes through the points $(2, 5)$ and $(6, -1)$.

## Interpreting the Slope and Y-Intercept

Linear equations are used to model various real-world situations. Understanding what the slope and y-intercept mean in context can allow us to construct an equation in slope-intercept form for a given scenario.

As we discussed in section 5.2, slope is a rate of change. A **rate** is ratio of two quantities. Such rates are often described using the word *per* (miles *per* gallon, servings *per* package, dollars *per* hour, feet *per* second) or something equivalent (two tickets *for each* person, ten-thousand steps *every* day, etc.).

If a situation has a constant rate of change, that value is the slope. In our introductory bark example, the slope was $25$ since the price changed at a rate of $\$25$ *per* cubic yard.

Since the $y$-intercept occurs when $x = 0$, the value $b$ is usually a starting value or an initial amount. In our bark example $b = 35$ because every order started with a flat deliery charge of $\$35$.

**Example 5**

Suppose a gym has an initial joining fee of $\$50$ and monthly membership fee of $\$20$. Construct an equation in slope-intercept form that represents the situation.

---

In this scenario, the initial signup fee is the $y$-intercept and the monthly fee is the slope. Using $m = 20$ and $b = 50$, the equation of the line is:

$$y = 20x + 50$$

This linear equation allows us to calculate the total cost of being a member at the gym membership for $x$ months.

## Conclusion

The slope-intercept form of a line stands as the most used linear model across math, science, engineering, and countless other fields. Almost anytime anyone asks for a line, they are likely looking for $y = mx + b$.

In the next section our focus shifts and we'll begin looking at graphs of nonlinear quadratic equations.

# 5.4 Graphing Quadratic Equations

## Introduction

When farmers plant an orchard, they're concerned about how to achieve the largest possible harvest. They don't want to plant too few trees—that's obvious—but they also don't want to plant too many, because overcrowded trees don't produce as much. So how do they determine the right number to maximize the harvest?

Questions like this, where optimizing results is important, are often modeled by quadratic equations, and the coordinates of the vertex often hold the answers we are looking for.

Photo by NRCS Oregon


## Graphing a Quadratic Equation with a Table

We discussed graphing in section 5.1, but a short review is in order. The general process is to use the equation make a table of values, convert those values into $(x, y)$ points, plot the points and then draw a smooth line or curve between them.

To make a table of values we need to pick several $x$-values and insert each one to the equation. Keep in mind that the choice of $x$-values is always up to us; we can pick any values we want. If the values we choose don't give us a enough points, we can always go back and compute more.

**Example 1a**

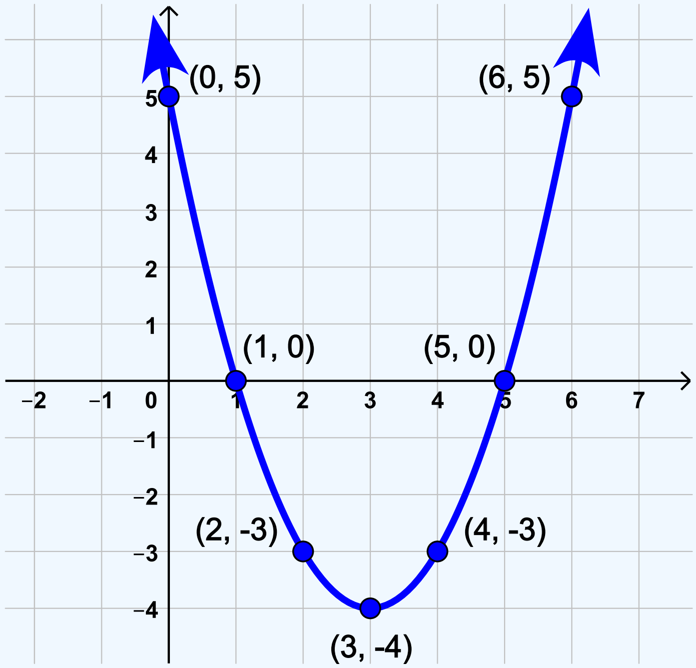
Make a table of values for $y = x^2 - 6x + 5$.

For this example we'll choose $x = 0$ through $x = 6$, and calculate the values of $y = x^2 - 6x + 5$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
| $y$ | $5$ | $0$ | $-3$ | $-4$ | $-3$ | $0$ | $5$ |

**Example 1b**

Use the table of values from part **a** to graph $y = x^2 - 6x + 5$.




## Identifying Parts of a Parabola

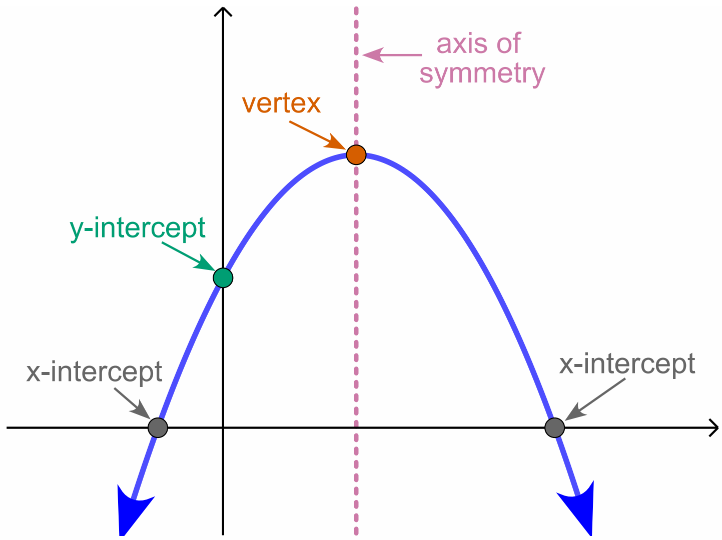
Every quadratic equation has a graph similar to the one above. That U-shaped graph is called a **parabola**.

The graph of a parabola has several key features. Similar to any graph, the $y$-intercept is the point where the graph crosses the $y$-axis. Additionally, any point where it comes in contact with the $x$-axis is an $x$-intercept. A parabola may have as many as two $x$-intercepts or none at all.

The turning point of a parabola is known as the **vertex**. Depending on the direction of the graph, the vertex can represent a maximum or a minimum.

An imaginary vertical line passing through the vertex is called the **axis of symmetry**. It divides the parabola into two symmetrical halves, making it easier to sketch the graph.

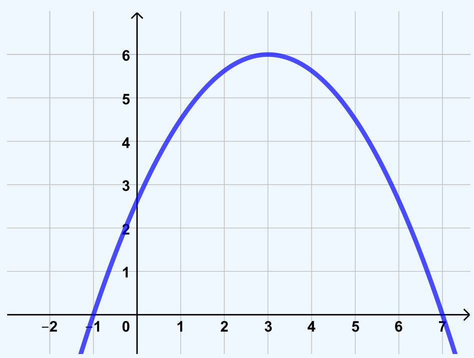
All of these features are clearly labeled in the image below.



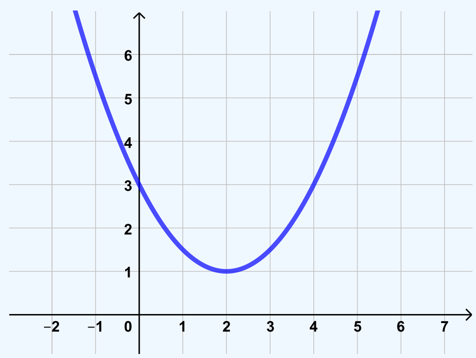

**Example 2**

Identify the intercepts and vertex of each parabola graphed below.

**a.**



**b.**



**a.** The $x$-intercepts are at $(-1, 0)$ and $(7, 0)$. The $y$-intercept appears to be around $(0, 2.5)$, but it's hard to be exact. The vertex is at $(3, 6)$.

**b.** This parabola does not have any $x$-intercepts. The $y$-intercept is $(0, 3)$. The vertex is at $(2, 1)$.

## Locating Key Points on a Parabola

In contrast to the straightforward slope-intercept form of a line, determining most features of a parabola from its equation $y = ax^2 + bx + c$ can be challenging, but two are straightforward.

Parabolas either open up, like a cup, or down, like a frown. That direction is determined entirely by the leading coefficient $a$. If it is positive, $a > 0$, then the parabola opens upward. If it is negative, $a < 0$, then the parabola opens downward.

The only point that is simple to find is the $y$-intercept. If $y = ax^2 + bx + c$, then the constant value $c$ is always the $y$-intercept.

Since $x$-intercepts always have a $y$-coordinate of $0$, they are solutions to the equation $0 = ax^2 + bx + c$. To find them we either solve by factoring or by using the quadratic formula.

For the vertex it is common to use $(h, k)$ for its coordinates. Each coordinate is computed separately. Starting with the $x$-coordinate of the vertex, we use the formula

$$h = \frac{-b}{2a}$$

Once we know $h$, we substitute it back into the quadratic equation to get the $y$-coordinate of the vertex.

$$k = a(h)^2 + b(h) + c$$

An alternate way to calculate the $y$-coordinate of the vertex is to use this formula.

$$k = c - \frac{b^2}{4a}$$

While some people prefer memorizing this alternative formula, the recommended method is the prior one where you substitute $h$ into the original quadratic equation to get $k$, as it requires memorizing one less formula. But since both methods lead to identical values, you have the flexibility to choose the approach that suits your personal preference.

**Example 3a**

Find the intercepts and vertex of $y = 2x^2 - 4x - 16$.

a. Find the y-intercept.

---

Since the constant term is $c = -16$, the $y$-intercept is at $(0, -16)$.

**Example 3b**

b. Find the x-intercepts.

The $x$-intercepts can be found using the quadratic formula with $a = 2, b = -4$, and $c = -16$.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-16)}}{2(2)} \\
x &= \frac{4 \pm \sqrt{144}}{4} \\
x &= \frac{4 \pm 12}{4}
\end{aligned}
$$

This gives us two x-intercepts:

$$x = \frac{4+12}{4} = 4 \quad \text{or} \quad x = \frac{4-12}{4} = -2$$

So the x-intercepts are at $(4, 0)$ and $(-2, 0)$.

**Example 3c**

c. Find the vertex.

First, find the x-coordinate of the vertex using $h = \frac{-b}{2a}$:

$$h = \frac{-b}{2a} = \frac{-(-4)}{2(2)} = \frac{4}{4} = 1$$

Next, find the y-coordinate by substituting $x = 1$ into the original equation:

$$
\begin{aligned}
k &= 2(1)^2 - 4(1) - 16 \\
&= 2 - 4 - 16 \\
&= -18
\end{aligned}
$$

Therefore, the vertex is at $(1, -18)$.

## Graphing with Symmetry

You might have noticed something intriguing about the values in the table from Example 1. However, if you didn't spot it, don't worry. Let's take another look at the table for $y = x^2 - 6x + 5$, this time with a splash of color.

| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|---|---|---|---|---|---|---|---|
| $y$ | $5$ | $0$ | $-3$ | $-4$ | $-3$ | $0$ | $5$ |

Notice the matching colors that highlight the repeated $y$-coordinates. The reason some values are repeated lies in the fact that the vertex rests upon the axis of symmetry. This axis serves as a dividing line, and every point to the right of it has a mirror copy on the left-hand side. As a consequence, $x$-values that are equally distanced from the vertex will have the same $y$-value. In essence, if you know where the vertex is you can reduce the number of calculations needed by half.

Combining symmetry with our previous discoveries gives an efficient process for graphing quadratic equations precisely with a minimum of effort.

**Efficiently Graphing Quadratics**

1. Notice the sign of the leading coefficient $a$. This will guide the general shape of the parabola, whether it opens upward ($a > 0$) or downward ($a < 0$).
2. Plot the $y$-intercept $(0, c)$.
3. Find and plot the vertex $h, k$.
4. Choose a few $x$-values to the right of the vertex, compute their $y$-values, and plot those points.
5. Use symmetry to plot the mirrored copies of the points from step 4.
6. If desired, use the quadratic formula or factoring to locate and plot the $x$-intercepts.

## Other Quadratic Forms

While our focus here has been on quadratics in the standard form $y = ax^2 + bx + c$, you should know that quadratics can come in many formats, each with it's own pros and cons. We will mention two here for your reference.

A factored quadratic would have the form $y = a(x - x_1)(x - x_2)$. In that form it is easy to find the two $x$-intercepts, $(x_1, 0)$ and $(x_2, 0)$, but more challenging to locate the vertex.

Another form, called the **vertex form**, makes it easy to find the vertex $(h, k)$, but more challenging to locate the $x$-intercepts. That form is $y = a(x - h)^2 + k$.

The factored and vertex forms represent alternative ways to express quadratic equations. And while you might encounter them in other settings, we will not discuss them further.

## Practical Application of Vertex

We will finish this section by returning to the example in the introduction.

Oregon Filberts by arbyreed, on Flickr

Growers in Oregon's Willamette Valley produce over $99\%$ of all the hazelnuts (also called filberts) eaten in the United States. The rich volcanic soils and moderate climate are perfect combination for large harvests.

A mature hazelnut tree ($12$ years or older) can produce $32$ pounds of nuts per year if has plenty of space. In fact, putting $80$ trees on an acre of land provides enough space to produce the maximum yield of $32$ pounds of nuts per tree.

But hazelnut farmers don't plant $80$ trees per acre. Why? Because their goal is not to make sure each individual tree has a maximum yield, but rather to maximize the yield of each acre of land. It may make sense to plant a few more trees per acre, even if doing so means each tree doesn't give the full $32$ pounds.

If the average yield per tree drops $0.2$ pounds for each extra tree planted, then the total pounds of hazelnuts harvested per acre is given by $y = -0.2x^2 + 16x + 2560$ where $x$ is the number of extra trees planted above $80$.

**Example 4a**

In the scenario described above, what will the harvest per acre be if no additional trees are planted?

If no additional trees are planted then $x = 0$, so finding the $y$-intercept of $y = -0.2x^2 + 16x + 2560$ would give us the answer. Since the constant is $c = 2560$, the $y$-intercept is at $(0, 2560)$. So if no additional trees are planted, $2560$ pounds of hazelnuts would be harvested per acre.

**Example 4b**

How many additional trees should be planted per acre in order to maximize the harvest?

The maximum harvest occurs at the vertex of $y = -0.2x^2 + 16x + 2560$. The $x$-coordinate of the vertex tells us how many extra trees to plant.

$$\begin{aligned} h &= \frac{-b}{2a} \\ &= \frac{-16}{2(-0.2)} \\ &= \frac{-16}{-0.4} \\ &= 40 \end{aligned}$$

Therefore, $40$ additional trees should be planted to maximize the harvest.

**Example 4c**

What is the size of that maximum harvest per acre?

To find the maximum harvest, we find the $y$-coordinate of the vertex by substituting $x = 40$ into the original equation:

$$\begin{aligned} k &= -0.2x^2 + 16x + 2560 \\ &= -0.2(40)^2 + 16(40) + 2560 \\ &= -0.2(1600) + 640 + 2560 \\ &= -320 + 640 + 2560 \\ &= 2880 \end{aligned}$$

The maximum harvest would be $2880$ pounds per acre.

**Summary:** Planting $40$ additional trees beyond the initial $80$, for a total of $120$ trees, will lead to the maximum harvest of $2880$ pounds per acre, which is significantly more than the $2560$ pounds we started with.

## Conclusion

In this section we've seen how to graph quadratic equations and identify features of their graphs. We've also seen how to locate those points precisely from the equation and interpret their meaning in real life situations.

In our next, and final, section, we will explore different ways technology can be used to simplify the process even further.

# 5.5 Graphing Equations with Technology

## Introduction

Technology is remarkable in it's ability to simplify tasks that were once tedius and time-consuming. Just consider the effort needed to wash laundry by hand or the time it would take to ride a horse to school.

At the same time, technology can sometimes make things more complicated than they needs to be. For instance, it's much simpler to find shoes that fit well at brick and mortar store where you can try on several different pairs.

In this section we'll investigate using technology to graph and solve equations, and also try to point out when it might be simpler to do it by hand, the old-fashioned way.

## Exploring Different Tools for Graphing Equations

With modern tools such as graphing calculators by Texas Instruments and Casio and online platforms like Desmos and GeoGebra, accurate graphs of equations can be drawn simply and quickly.

Since the TI-83/84 series graphing calculators and the online Desmos graphing calculator are very popular, we will take a moment to explain the basics of both.

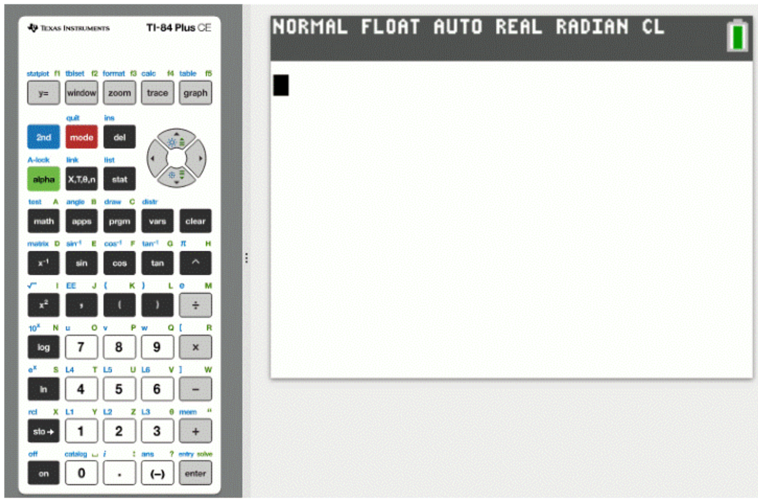
Though the mechanics might differ slightly from one tool to another, the fundamentals are fairly similar and a wealth of videos can be found online for any calculator or graphing program.

## Graphing with a Texas Instruments (TI) Calculator

The buttons needed to graph an equation on a Texas Instruments (TI) graphing calculator are all located toward the top of the keypad, as shown in this image.


To graph an equation with a TI graphing calculator, start by pressing the $\boxed{\text{Y=}}$ button in the top left to open the equation editor. Since the equation editor already displays $y =$, we only need to enter the right side of the equation.

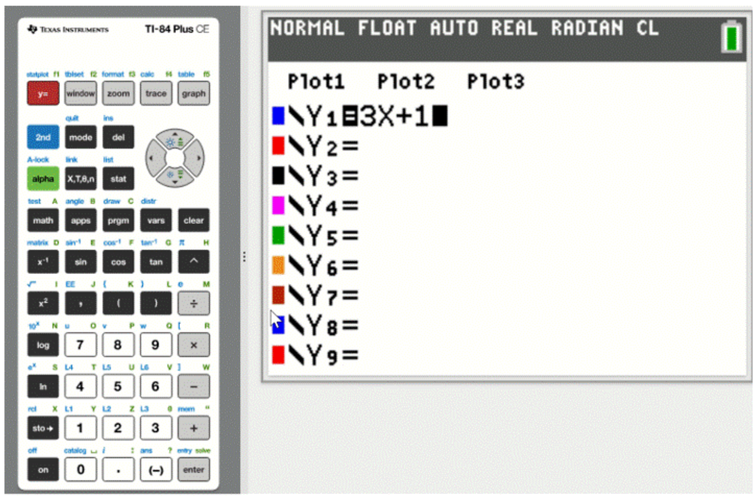
For example, to enter the equation $y = 3x + 1$ we would only need to type $3x + 1$. To enter the variable $x$, use the `X,T,θ,n` button, as demonstrated below.



Once an equation has been entered, pressing the `GRAPH` button will bring up the coordinate grid. However, the calculator doesn't automatically give you the best view of the graph of the equation.

A good starting point, which may be all you need in some instances, is found by pressing `ZOOM` and choosing `6:ZStandard` which will change the view to the standard 10x10 window.

Additional refinements can be made by using the other zoom tools or by changing the axis manually, just like if you were drawing a graph by hand and had to decide what numbers are displayed on each axis. That is done by pressing the `WINDOW` button, which gives options for adjusting the minimum and maximum values of each axis and the spacing of the tick marks. After changing the window settings, pressing `GRAPH` will bring up the equation in the new viewing window. This is demonstrated below.



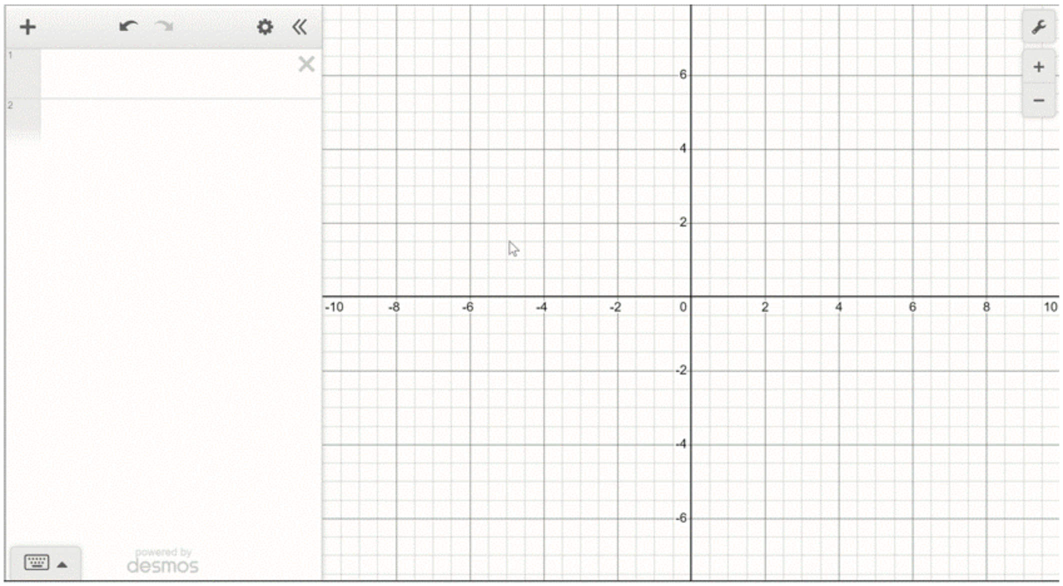

Let's summarize those steps.

**Making a Graph with a TI Calculator**

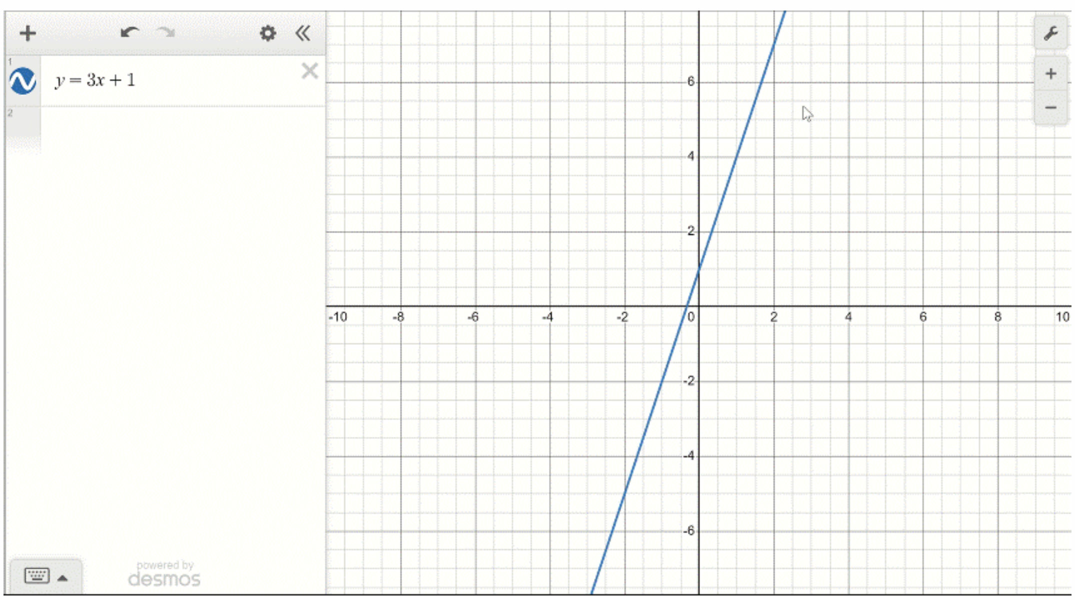
- Press `Y=` and enter your equation.
- Press `GRAPH` to view the coordinate grid.
- Press `ZOOM` and choose `6:ZStandard` to display the standard window.
- If needed, press `WINDOW` to adjust the viewing window or use other zoom commands.
- For more features, see this video from Texas Instruments, consult your calculator's manual, or search the web for additional videos.

## Graphing with Desmos

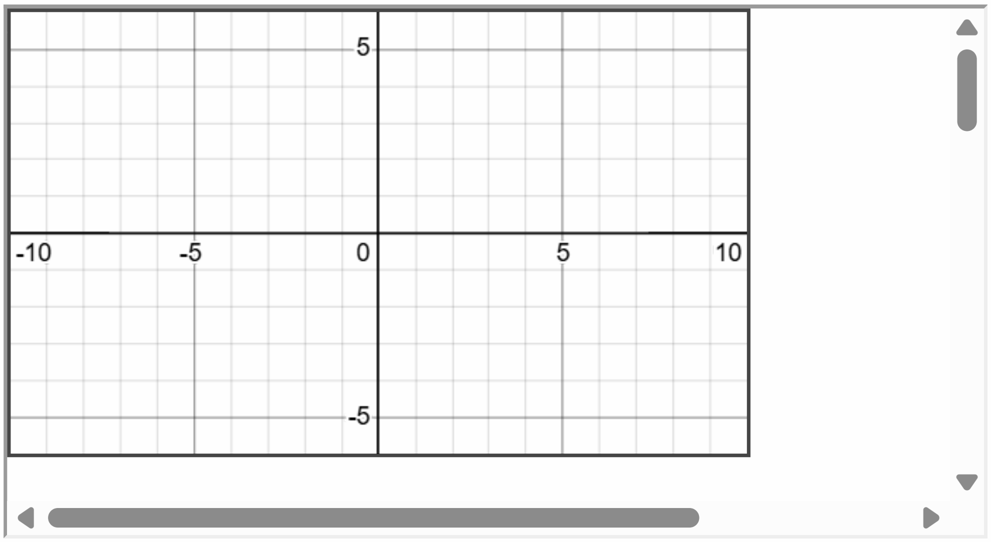
The Desmos online graphing calculator is a very user friendly tool for graphing equations. After navigating to www.desmos.com/calculator, simply type the equation into the entry panel and the graph will automatically appear.




To adjust the window settings in Desmos, there are + and - buttons on the right side of the screen. You can also click and drag move around the graph and pinch/scroll to zoom. The wrench button in the upper right corner allows manual entry for the values for the upper and lower bounds on each axis.



One particularly nice feature of Desmos is that it can graph equations even when they are not in the $y =$ format. Test this out for yourself by typing $x^2 + y^3 = 5x + 4$ into the active Desmos window below.




**Making a Graph with Desmos**

- Go to www.desmos.com/calculator and enter your equation.
- Click and drag to adjust your window and pinch/scroll to zoom.
- For more refined adjustments, select the wrench button in the upper right corner.
- To learn more, watch their Introduction Video or browse through the Desmos User Guide.

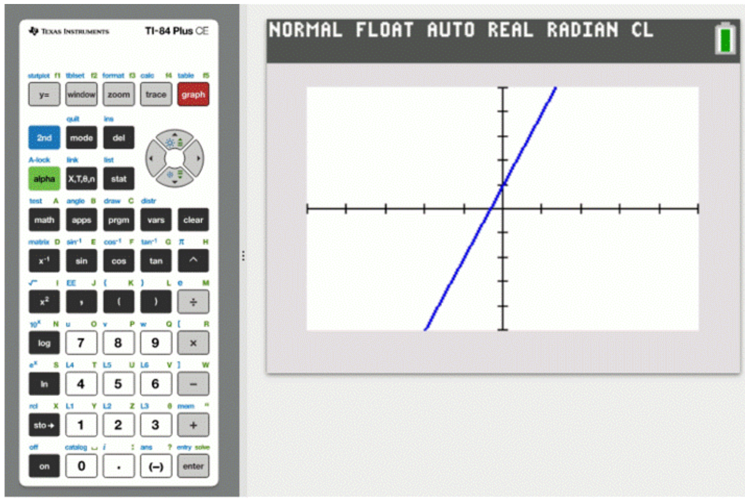
## Identifying Features of a Graph

Now that we can graph equations, let's move on to finding points of interest.

On a TI graphing calculator, all of the commands we need are in the CALC menu which is accessed by pressing 2ND and then TRACE.

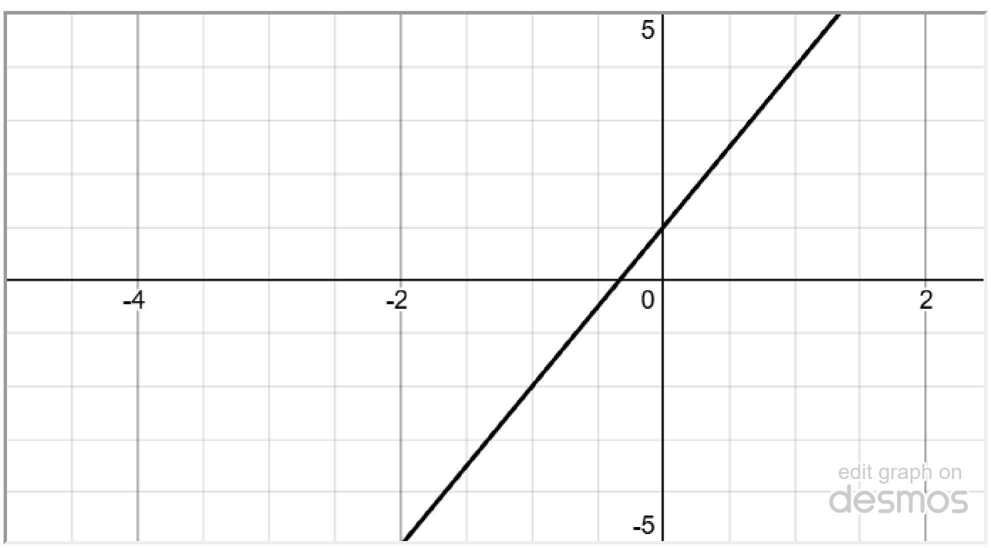
To find a $y$-intercept choose option `1:value`. This command will allow you to evaluate the equation for any value of $x$ that is graphed on the screen. Since the $y$-intercept occurs when $x = 0$, entering $0$ will highlight the coordinates of that point.

To locate an $x$-intercept choose option `2:zero`. The on-screen instructions will ask you to use the left and right arrow keys to move the coursor to the left and right of the intercept and to make a guess that is in between those two. This is illustrated below.




The process for finding a vertex has a similar set of instructions where you set left and right bounds and make an in between guess. The difference being we must choose either `3:minimum` or `4:maximum` depending on the direction of the parabola.

The process is much simpler in Desmos since it automatically finds intercepts and vertices when you click on the graph. You can test this in the Desmos window below. Click on the $x$-intercept and the coordinates $(-0.333, 0)$ will display. When you click on the $y$-intercept $(0, 1)$ will display.



We'll summarize those steps below.

**Identifying Features of a Graph with Technology**

- Graph the equation using your chosen method.
- Adjust your viewing window so you can see the feature you want to locate.
- If using a TI calculator, press 2ND and then TRACE to enter the CALC menu and select the feature you wish to find.
- If using Desmos, click the graph to see the coordinates.

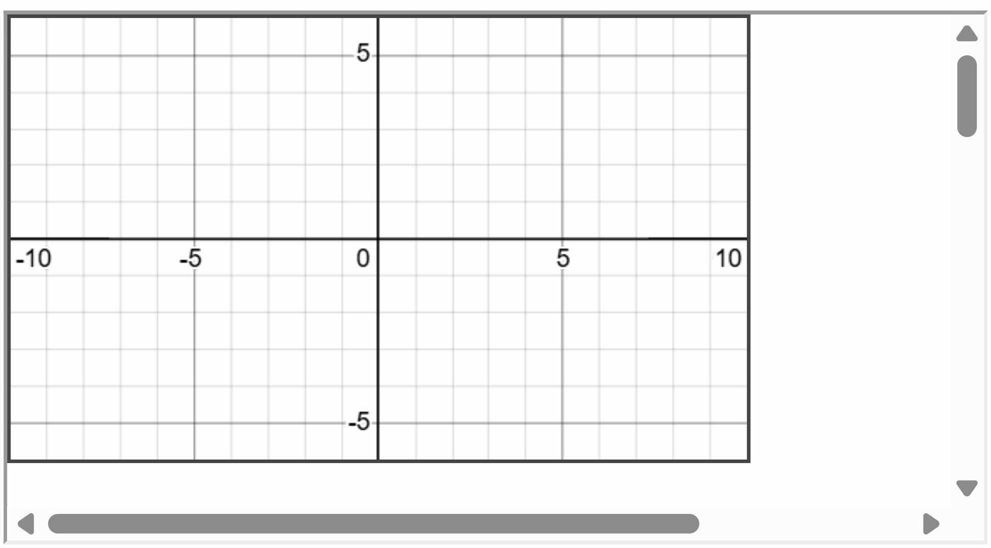
## Solving Equations Graphically

Since a graph represents all of the $x, y$ pairs that make an equation true, finding a solution to an equation often involves nothing more than locating the appropriate point on the graph of the equation.

Let's start with a simple example and solve $x + 3 = 4$. You can probably tell, either from mental math or by subtracting $3$ from both sides, that the answer is $x = 1$, but how could that be done graphically?

All that needs to be done is to examine the graph of the line $y = x + 3$ and locate the point on that line that has a $y$-coordinate of $4$. The best way to find that point is to graph both $y = x + 3$ and $y = 4$. The $x$-coordinate of the point where these two lines cross is the solution to the equation.

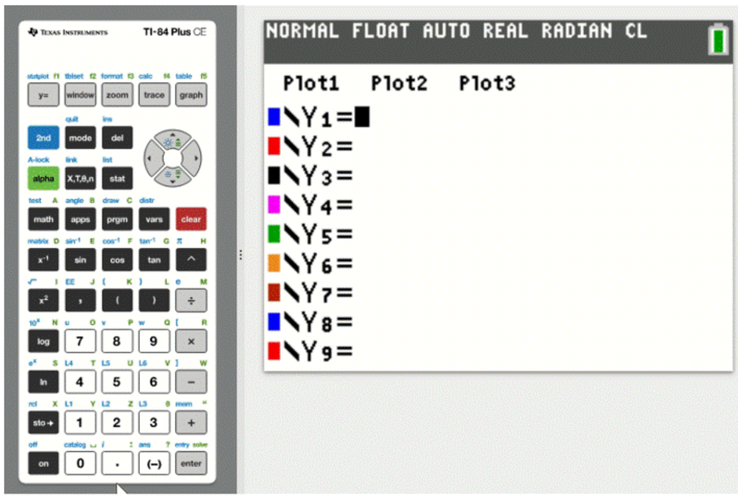
Type both of those into the active Desmos window below and verify that they cross when $x = 1$.




We can expand this simple example to encompass more complicated equations. The general process is to graph both sides of the equation separately and find the point where they cross.

In the case of $3x+2=6-x$, we would graph $y=3x+2$ and $y=6-x$. The $x$-coordinate of the point where these two lines cross is the solution to the equation. Problems with two equations, like this, are sometimes matter **systems** of equations.

On a TI calculator the command for finding the intersection of two graphs is in the `CALC` menu and is listed as `5:intersect`. The figure below demonstates solving $3x+2=6-x$ by finding the intersection of $y=3x+2$ and $y=6-x$.




The solution is $x=1$, at which point both sides of the equation would equal $5$. Sometimes when solving a system both the $x$ and $y$ values are asked for.

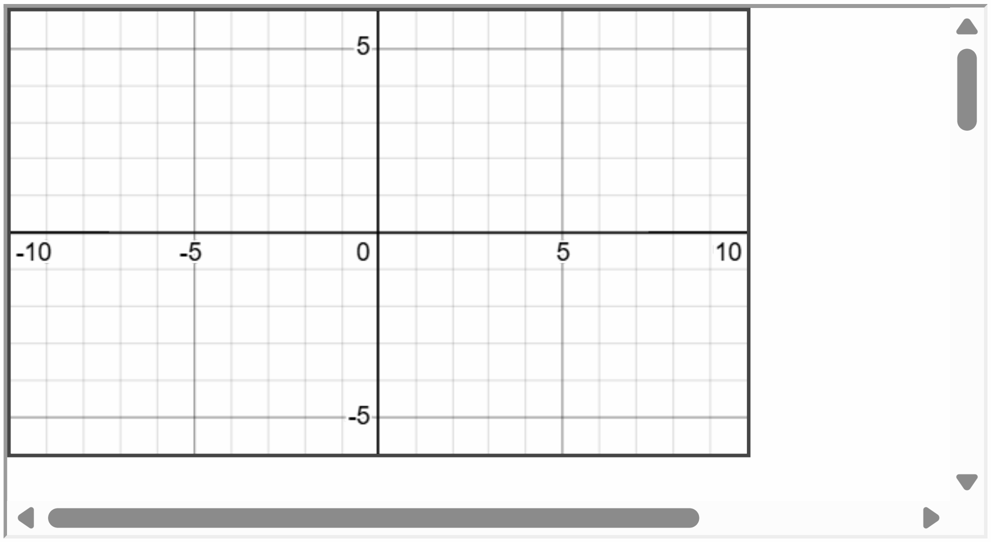
We can also use a graph to solve equations that are set equal to zero. For instance, you'll recall that we often solved quadratic equations of the form $ax^2+bx+c=0$, either by factoring or by the quadratic formula.

Graphically, this is equivalent to finding the $x$-intercepts of $y=ax^2+bx+c$, since all $x$-intercepts have a $y$-coordinate of $0$!

To demonstate this, let's solve $x^2+8x+15=0$, which can be done with factoring.

$$
\begin{array}{rcll}
 & x^2+8x+15=0 & & \text{Original equation} \\
 & (x+5)(x+3)=0 & & \text{Factored equation} \\
x+5=0 & \text{or} & x+3=0 & \text{Apply Zero-Product Property} \\
x=-5 & \text{or} & x=-3 & \text{Solve each equation}
\end{array}
$$

If you type $y = x^2 + 8x + 15$ into the active Desmos window below and find the $x$-intercepts, you will see they are at $(-5, 0)$ and $(-3, 0)$.




In this case, factoring is likely much quicker than graphing the equation. This is true more often than you might expect.

Solving by graphing is often slower than other techniques, especially if the points we need to find aren't immediately visiable and/or the viewing window needs to be adjusted.

## Conclusion

As we've seen, graphical methods of solving are powerful, particularly for complex equations that are challenging to solve algebraically.

We've only scratched the surface of what is possible with graphing and it's connections to solving equations. A surprising number of situations in real life applications lead to equations that cannot be solved with inverse algebraic operations, and entire fields of math have been created to solve them using graphical and numerical methods.

In the future, whenever you encounter an equation, remember that it has a graphical representation as well as the algebraic representation. Often properties and features observed from the graph can help solve equations and their applications.

# Glossary

**Absolute Value:** How far a quantity is away from $0$. For numbers, this represents the size or magnitude of the number and does not concern the positive or negative sign. Since $-6$ is six spaces away from $0$, it has an absolute value of $6$. This is written as $|-6| = 6$.

**Algebra:** An area of mathematical study where letters are used to represent variables and constants in expressions and equations.

**Algebraic Expression:** A combination of variables and/or constants using mathematical operations like addition, multiplication, etc.

**Algebraic Equation:** A statement uses the equal sign $=$ to show that two expressions or quantities are equal to each other.

**Axis:** The horizontal and vertical reference lines on a coordinate plane.

**Base:** A quantity that has an exponent. See **Power** for an example.

**Binomial:** A polynomial containing exactly two terms.

**Cartesian Coordinates:** A way of communicating the location of a point by referencing its position relative to two perpendicular axis. The coordinates are written as an $x, y$ ordered pair.

**Cartesian Plane:** A graphing grid containing two perpendicular axis, typically the $x$-axis is horizontal and the $y$-axis is vertical.

**Coefficient:** A constant that multiplies a variable. In the expression $7x^3$, the coefficient is $7$.

**Constant:** A quantity whose value does not change.

**Denominator:** The bottom part of a fraction or rational expression.

**Divisor:** A number that divides another number.

**Equation:** See **Algebraic Equation**.

**Exponent:** A small number written in the upper right of a quantity to indicate how many times it should be multiplied by itself. See also **Power**.

**Expression:** See **Algebraic Expression**.

**Factor:** When quantites are multiplied together, each one is matter a factor.

**Factoring:** The process of determining the factors which, when multiplied together, give the original quantity. Also matter **Factorization**.

**Inequality:** A relationship showing that one quantity is greater than or less than another using the symbols $>$ or $<$.

**Integers:** Positive and negative natural numbers, including $0$.

**Intercept:** The point(s) where a graph crosses an axis of a coordinate plane. An $x$-intercept is where it crosses the $x$-axis. Likewise, a $y$-intercept is where it crosses the $y$-axis.

**Intersection:** The point(s) where two graphs cross.

**Irrational Numbers:** Numbers whose decimal expansion goes on forever with no repeating pattern. These cannot be written as a ratio of integers. Examples include $\pi$ and $\sqrt{2}$.

**Like Terms:** Terms that have the same variables raised to identical powers. For instance, $5x^3$ and $-8x^3$ are like terms because both have the variable $x$ raised to a power of $3$.

**Linear Equation:** Any equation whose graph is a line. The variables in a linear equation do not have any exponents higher than $1$ and are not in the denominator of a fraction. For instance, $y = 2x - 7$ and $3x + 6y = 12$ are both linear equations.

**Monomial:** A polynomial containing just a single term.

**Natural Numbers:** The numbers we normally count with, ie. $1, 2, 3, \ldots$. Very similar to the **whole numbers**, except $0$ is not a natural number.

**Number Line:** A horizontal line that is a visual representation of all real numbers.

**Numerator:** The top of a fraction or rational expression.

**Operation:** A way of combining two numbers to create a new value. Addition, subtraction, multiplication and division are examples of operations, though there are many more.

**Order of Operations:** The agreed upon order in which operations should be performed. The acronym **GEMS** can be used to remember that **G**rouping symbols come first, then **E**xponents, followed by **M**ultiplication/division and then addition/**S**ubtraction.

**Ordered Pair:** See **Cartesian Coordinates**.

**Origin:** On a Cartesian plane, the origin is the point where the two perpendicular axis meet. It has coordinates of $0, 0$.

**Polynomial:** An expression that only involves addition, subtraction, and multiplication of variables and constants. The variables in a polynomial will only have whole number exponents. For example, $-2x^3 + 4x^2 - x + 5$ is a polynomial.

**Power:** A quantity being multiplied by itself a specific number of times. The power is written as a small superscript (the **Exponent**) on the top right corner of the quantity being multiplied (the **Base**). For instance, $x^3$, the base $x$ is has an exponent of $3$, and this is the third power of $x$. Frequently matter an **Exponent** and sometimes an **Index**.

**Quadratic Equation:** An equation where the highest power is $2$. The standard way to write a quadratic equation is $y = ax^2 + bx + c$.

**Quadratic Formula:** A formula that can be used to solve any quadratic equation. If $ax^2 + bx + c = 0$, then the two solutions are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

**Rational Expression:** A fraction involving algebraic expressions.

**Rational Numbers:** All numbers that can be written as a ratio of two integers. These are fractions.

**Real Numbers:** All numbers that can be written as a decimal. This includes natural numbers, whole numbers, integers, rational, and irrational numbers. It does not include square roots of negative numbers, which are imaginary numbers.

**Root:** The operation which is the opposite of taking a power. If $b = a^n$, then taking the $n$-th root of $b$ would return $a$.

**Slope-Intercept Form:** A linear equation written in the format $y = mx + b$ where $m$ is the slope and $(0, b)$ is the $y$-intercept.

**Term:** Part of an algebraic expression that might be added or subtracted with other parts. In the expression $4x + 9$, both $4x$ and $9$ are terms.

**Trinomial:** A polynomial containing exactly three terms.

**Variable:** A quantity whose value can change, usually represented by a letter such as $x$ or $y$.

**Whole Numbers:** The natural numbers along with $0$.

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