Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences

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Global dynamics in adaptive models of collective choice with social influence

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Summary. In this chapter we present a unified approach for modelling the diffusion of alternative choices within a population of individuals in the presence of social externalities, starting from two particular discrete-time dynamic models – Galam's model of rumors spreading [10] and a formalization of Schelling's binary choices [7]. We describe some peculiar properties of discrete-time (or event-driven) dynamic processes and we show how some long-run (asymptotic) outcomes emerging from repeated short time decisions can be seen as emerging properties, sometimes unexpected, or difficult to be forecasted.

1 Introduction

A classical theme in the mathematical modelling of social systems is the description of how the collective behavior affects the individual choices and, vice versa, how repeated choices of interacting individuals give rise, in the long run, to the emergence of collective behaviors and social structures. In other words, the mutual dependencies between "Micromotives and Macrobehavior" [25] or between "Individual Strategy and Social Structure" [27].

In this chapter we present two models describing the diffusion of alternative choices within a population of individuals in the presence of social externalities. The first model we consider is the Galam's model of rumors spreading, as published in [10]. The diffusion of a given opinion by word-of-mouth mechanisms can be considered as a particular case of individual decisions (believing or not in a given opinion) affected by the opinion prevailing in the society. Several mathematical models have been used in sociological, mathematical, and physical literature (see, e.g., [12,15,26]). Starting from the pioneering works by Galam (see, e.g., [13]) opinion dynamics has become one of the main streams of sociophysics, an expanding field with hundreds of papers published in leading physical (and nonphysical) journals. It emerged in the 70s of last century and since then the number of topics covered has been increasing, see, e.g., [12] for a review.

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The second model we present is a mathematical formalization of a pioneering qualitative model proposed in [24], where a class of binary choice games with externalities is considered to model how individual choices are influenced by social externalities. In this famous seminal paper Schelling assumes that each agent's payoff depends only on the number of agents who choose one way or the other and not on their identities; he provides qualitative explanations of several every-day life situations, as well as a general framework that includes some well known game-theoretic situations, such as the n-players prisoner's dilemma or the minority games.

Following this approach, in [7] an explicit discrete-time dynamic model is proposed; there a population of bounded rationality players is assumed to be engaged in a game where they repeatedly choose between two strategies through an adaptive adjustment process. This allowed the authors to study the effects on the dynamic behavior of different kinds of payoff functions that represent social externalities, as well as the qualitative changes of the asymptotic dynamics induced by variations of the main parameters of the model. The adaptive process by which agents switch their decisions depends on the difference observed between their own payoffs and those associated with the opposite choice in the previous turn; the switching intensity is modulated by a parameter λ representing the speed of reaction of agents: small values of such λ imply more inertia while, on the contrary, larger values of λ imply more reactive agents. This one-dimensional discrete dynamic model gives rise to different asymptotic dynamics, including convergence to steady states, periodic and even chaotic oscillations, as well as particular the structure of basins of attraction when several coexisting attractors are present. This latter effect is obtained in the case of non-monotonic payoff functions, a quite interesting situation in sociological applications (as explained by [24] and [15]) that in our mathematical framework leads to a dynamical system represented by an iterated noninvertible map. Moreover, in the limiting case of impulsive agents, represented by $\lambda \to \infty$, a discontinuous dynamical system is obtained. In this case, border collision bifurcations cause the creation and destruction of periodic attractors as some parameters are varied.

We finally illustrate a recent unified dynamic model, proposed in [8], which embeds these two models into a general discrete-time dynamic model for studying individual interactions in variously sized groups. In fact, while Galam's model of rumors spreading considers a majority rule for interactions in several groups, Schelling's framework considers individuals interacting in one large group, with payoff functions that describe how collective choices influence individual preferences. The general model proposed in [8] incorporates these two approaches and allows one to analyze how the social dynamics may differ depending on the size of the group they are taking place in.

The chapter is organized as follows. In Sect. 2 Galam's model of rumors is described, in Sect. 3 the formalization of Schelling's model of binary choices, as given in [7], is summarized, together with the main results obtained in the case of monotonic and nonmonotonic payoff functions. In Sect. 4 the same model is considered in the limiting case of impulsive agents; some of the results concerning the global dynamic properties of piecewise linear discontinuous maps with the related problem of border collision bifurcations are provided. Finally, in Sect. 5, the unified model proposed by [8] is described; Sect. 6 concludes.

2 Galam's model

The model Galam proposes in [10] explains the so called *Pentagon French hoax*, according to which, on September the 11th, no plane crashed on the Pentagon. In this model individuals try to choose an opinion (true or false) on this rumor on the basis of repeated discussions in social gatherings. At each iteration, small groups of people get together and within each group they line up with a consensual opinion in which everyone agrees with the majority inside the group. The process is formalized as follows. The probability to be sitting at a group of size i is denoted by a_i, $i = 1, 2, \ldots, L$; obviously the constraint

$$\sum_{i=1}^{L} a_i = 1 \tag{1}$$

holds. In Galam's model, the inclusion of one-person groups makes the assumption "everyone gathering simultaneously" realistic. At each social meeting, given the social spaces, individuals distribute among them according to probabilities a_i

Consider a N person population and assume two possible opinions, denoted[3] by "A" and "B"; assume that at time t everyone is holding an opinion, i.e., $N_A(t)$ individuals are believing to opinion A and $N_B(t)$ persons are sharing the opinion B; it holds $N_A(t) + N_B(t) = N$. It is immediate to compute the probabilities to hold on A or B, respectively

$$P_A(t) = \frac{N_A(t)}{N} \text{ and } P_B(t) = 1 - P_A(t). \tag{2}$$

From this initial configuration, people discuss the issue at each social meeting; each cycle of multisize discussions is marked by an unitary time increment. The opinion modification process is the following. In any group of size k with j agents sharing opinion A and $(k - j)$ sharing opinion B, if $j > k/2$ then all k members adopt opinion A; vice versa, if $j < k/2$, then everybody adopts opinion B. Finally, in the symmetric case $j = k/2$ the outcome is determined assuming a bias in favor of one of the two opinions. For a generic group of size k, the majority rule dynamics is formalized as:

$$P_A^k(t+1) = \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_A(t)^j (1 - P_A(t))^{k-j}, \tag{3}$$

[3] Actually, in [10] opinions are denoted by + and −; the notation we use here is to uniform the notation across all the models.

where $\lfloor k/2+1 \rfloor$ indicates the Integer Part of $k/2+1$ and $C_j^k = k!/(k-j)!j!$ are binomial coefficients. Considering as initial index of the sum $\lfloor k/2+1 \rfloor$, models the bias in favor of opinion B in case of a local doubt is obtained. As a matter of fact, when k is odd the sum starts considering the minimum majority, while when k is even the sum starts at $k/2+1$.

Aggregating all groups of size $k = 1, ..., L$, the overall updating process becomes

$$P_A(t+1) = \sum_{k=1}^{L} a_k P_A^k(t+1) \tag{4}$$

with $P_A^k(t+1)$ given by (3), that is, the dynamics of P_A is

$$P_A(t+1) = \sum_{k=1}^{L} a_k \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_A(t)^j (1-P_A(t))^{k-j}. \tag{5}$$

A single step of the opinion dynamics can be illustrated as the example provided in Fig. 1. Assume there are three individuals among which two have opinion A and one opinion B; they can gather at a 2-size and a 1-size groups. For each individual the probability of sitting at a size 2 is $2/3$ and the probability of sitting at the size 1 group $1/3$.

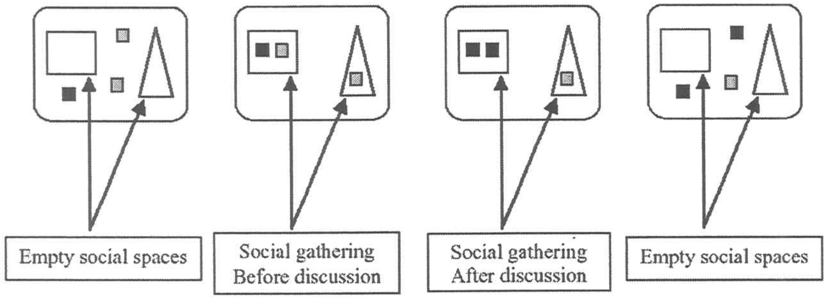

Fig. 1. A one-step opinion dynamics. *First stage*, people sharing the two opinions are moving around. *Grey* have A opinion while *black* have opinion B. No discussion is occurring with 2 *grey* and 1 *black*. *Second stage* right, people is partitioned in groups of various sizes from one to two and no change of opinion occurs. *Third stage*, within each group consensus has been reached. As a result, they are now 1 *grey* and 2 *black*. Last stage, people are again moving around with no discussion

In this case, Galam's formalization provides

$$P_A^2(1) = \sum_{j=\lfloor \frac{2}{2}+1 \rfloor}^{2} C_j^2 P_A(0)^j (1-P_A(0))^{2-j} = C_2^2 P_A(0)^2 (1-P_A(0))^{2-2} = \frac{4}{9} \tag{6}$$

and

$$P_A^1(1) = \sum_{j=\lfloor \frac{1}{2}+1 \rfloor}^{1} C_j^1 P_A(0)^j (1-P_A(0))^{1-j} = C_1^1 P_A(0)^1 (1-P_A(0))^{1-1} = \frac{2}{3} \tag{7}$$

And finally

$$P_A(1) = \sum_{k=1}^{2} a_k P_A^k(1) = \frac{2}{3}\frac{4}{9} + \frac{1}{3}\frac{2}{3} = \frac{14}{27}. \tag{8}$$

In the course of time, the same people keeps meeting in different groups in the same cluster configuration of size groups. The process and (5) is iterated to follow the time evolution of P_A.

This model, explains how the propagation of "absurd" rumors from initial tiny minorities may be explained by the bias driven by the tie effect. In particular, in [11] provides an analysis of how group shared belief may favor one opinion against the other.

Finally, it must be observed that Galam's model assumes implicitly that the number of agents is large. This is evident when considering the example discussed in Fig. 1, and labeling the agents as illustrated in Fig. 2.

Here, the three individuals are identified as I_1, I_2, and I_3; they have respective opinions A, A, and B, therefore, the probability $P_A(0) = 2/3$. They can gather at a 2-size and a 1-size groups. For each individual the probability of sitting at a size 2 is $2/3$ and the probability of sitting at the size 1 group $1/3$.

All the possible evolutions depend on the gathering configurations yet the final probabilities are, respectively, $2/3$, $1/3$, and $1/3$, therefore, we have

$$P_A(1) = \frac{1}{3}\frac{2}{3} + \frac{1}{3}\frac{1}{3} + \frac{1}{3}\frac{1}{3} = \frac{1}{3}\left(\frac{2}{3} + \frac{1}{3} + \frac{1}{3}\right) = \frac{4}{9}$$

which is different from the result predicted by (8). By simple computations it is easy to find that when individuals are six, four with opinion A and two with opinion B, it holds $P_A(1) = 22/45$.

When the number of agents is large the model becomes more accurate; when the number of agents is small, [9] provides a different formula, yet, according to the same authors, the formula addresses this issue only partially.

3 A mathematical formalization of Schelling model

In [24], a simple model which analyzes how individual choices are influenced by social interactions (social externalities), is proposed. In this model, agents face binary choices and are assumed to interact *impersonally*, i.e., each agent's payoff depends only on the number of agents who choose one way or the other

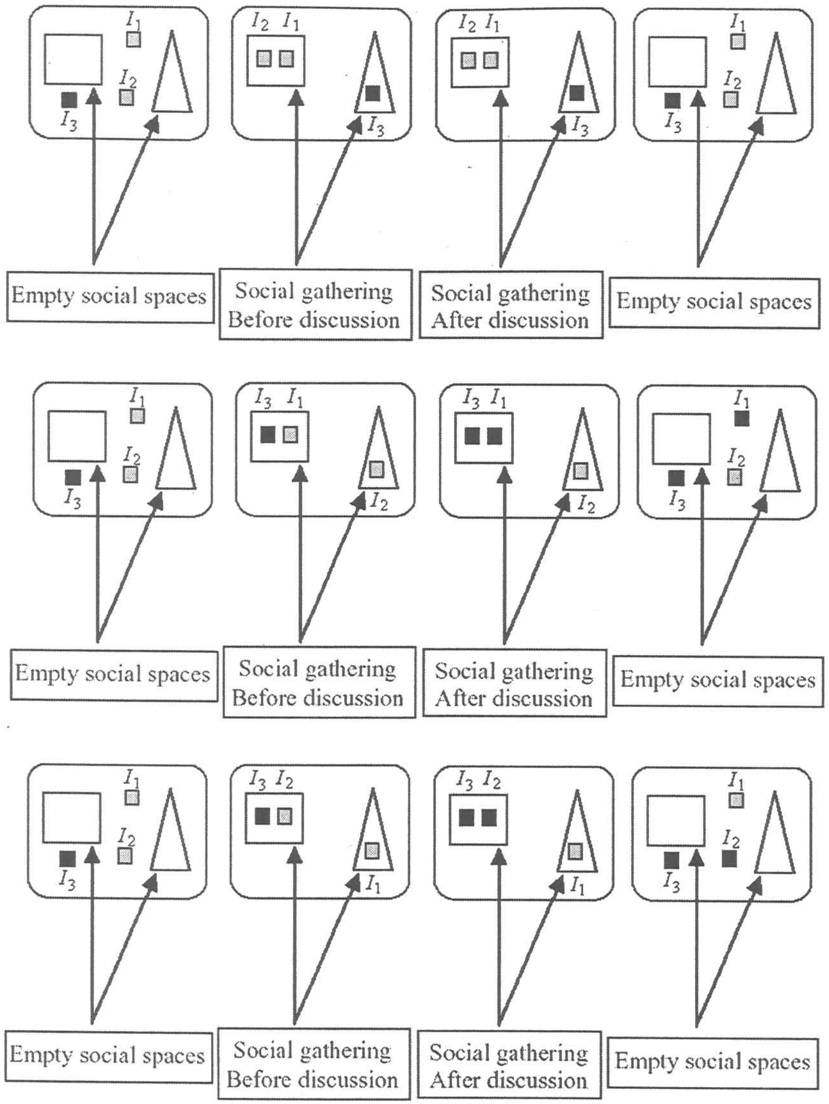

Fig. 2. A one-step opinion dynamics, when individuals follow different path. Each path has probability 1/3

and not on their identities. This model provides a qualitative explanation of a wealth of every-day life situations, and is general enough to include several games, such as the minority game and the well known n-players prisoner's dilemma.

A formalization of Schelling's approach is proposed in [7]; there, a population of players is assumed to be engaged in a game where they have to choose between two strategies, A and B, respectively.[4] The interaction is repeated over time in a discrete-time dynamic model.

A continuum of agents is normalized to the interval $[0, 1]$, and the real variable $x \in [0, 1]$ denotes the fraction of players choosing strategy A. The payoffs associated to strategies A and B are function of x, say $A : [0, 1] \to \mathbb{R}$, $B : [0, 1] \to \mathbb{R}$; they are denoted by $A(x)$ and $B(x)$, respectively. As binary choices are considered, when fraction x is playing A, then fraction $1 - x$ is playing B. Therefore, $x = 0$ means that the whole population is playing B and $x = 1$ means that all the agents are playing A.

Agents are homogeneous and myopic, that is, each of them is only interested to increase its own next period payoff. The dynamic adjustment is modeled assuming that x *increases whenever* $A(x) > B(x)$ *whereas it decreases when the opposite inequality holds.* In this process all the agents update their binary choice at each time period $t = 0, 1, 2, \ldots$, and x_t represents the fraction of those playing strategy A at time period t.

At time $(t + 1)$, x_t becomes common knowledge, therefore each agent is able to compute (or observe) payoffs $A(x_t)$ and $B(x_t)$. If $A(x_t) > B(x_t)$ then a fraction δ_A of the $(1 - x_t)$ agents that are playing B will switch to strategy A in the following turn; similarly, if $A(x_t) < B(x_t)$ then a fraction δ_B of the x_t players that are playing A will switch to strategy B. Formally, at any time period t, agents decide their action for period $t + 1$ according to:

$$x_{t+1} = f(x_t) = \begin{cases} x_t + \delta_A g\left[\lambda\left(A(x_t) - B(x_t)\right)\right](1 - x_t) & \text{if } A(x_t) \geq B(x_t) \\ x_t - \delta_B g\left[\lambda\left(B(x_t) - A(x_t)\right)\right] x_t & \text{if } A(x_t) < B(x_t), \end{cases} \tag{9}$$

where

- $g : \mathbb{R}_+ \to [0, 1]$ is a continuous and increasing function such that $g(0) = 0$ and $\lim_{z\to\infty} g(z) = 1$; it modulates how the difference between the previous turn payoffs affects the fraction of switching agents.
- δ_A, $\delta_B \in [0, 1]$ represent the fraction of agents switching to A and B, respectively. When $\delta_A = \delta_B$, the propensity to switch to either strategies is the same. On the contrary, $\delta_A \neq \delta_B$ represents a form of bias; in fact, given any payoff difference $|A(x) - B(x)| > 0$, $\delta_A > \delta_B$ implies that when $A(x) > B(x)$ switching from choice B to choice A is favored over switching from A to B when $A(x) < B(x)$.
- λ is a positive real number that represents the switching intensity (or speed of reaction) of agents as a consequence of the difference between payoffs. Small values of λ imply more inertia, i.e., anchoring attitude, of the actors involved, while, on the contrary, larger values of λ can be interpreted in terms of impulsiveness, see Sect. 4.

[4] In [24] and [7] the two choices are denoted, respectively, R and L.

The dynamics of model (9) has been examined in [7]. In particular, when the payoff functions have a single internal intersection point x^*, depending on which choice is preferred on the left and right neighborhood of x^*, the following propositions can be proved.

Proposition 1. *Assume that* $A : [0,1] \to \mathbb{R}$ *and* $B : [0,1] \to \mathbb{R}$ *are continuous functions such that*

- $A(0) < B(0)$
- $A(1) > B(1)$
- *there exists unique* $x^* \in (0,1)$ *such that* $A(x^*) = B(x^*)$,

then dynamical system (9) has three fixed points, $x = 0$, $x = x^*$, *and* $x = 1$, *where* x^* *is unstable and constitutes the boundary that separates the basins of attraction of the stable fixed points* 0 *and* 1*. All the dynamics generated by (9) converge to one of the two stable fixed points monotonically, decreasing if* $x_0 < x^*$, *increasing if* $x_0 > x^*$.

This proposition mathematically formalizes the qualitative results Schelling provides in [24]. By contrast, in the other case, the behavior is different; the discrete-time setting may give rise to oscillating behavior.

Proposition 2. *If* $A : [0,1] \to \mathbb{R}$ *and* $B : [0,1] \to \mathbb{R}$ *are continuous functions such that*

- $A(0) > B(0)$
- $A(1) < B(1)$
- *there exists unique* $x^* \in (0,1)$ *such that* $A(x^*) = B(x^*)$,

then the dynamical system (9) has only one fixed point at $x = x^*$, *which is stable if* $f'_-(x^*) > -1$ *and* $f'_+(x^*) > -1$, *and is unstable (in the sense of Lyapunov) if at least one of these two slopes is smaller than* -1*. Both slopes decrease as* λ *or* δ_A *or* δ_B *increase, i.e., if the propensity to switch to the opposite choice increases.*

This result differs from the description given in [24], where oscillations are ruled out as continuous time is implicitly assumed. Instead, in a discrete-time setting, overshooting (or overreaction) phenomena may occur for sufficiently large values of λ. Schelling [24] also describes interesting cases in which the payoff functions are nonmonotonic; in this case there may be more than one intersection and also two or more interior equilibria may exist. Several examples are discussed in [24]; more recently, [7] provides an analysis of the resistance to antibiotics phenomena in terms of payoff functions with two intersection points.

Nonmonotonic payoff functions may lead to the existence of more than one intersection, i.e., two or more interior equilibria may exist, see Fig. 3 and the dynamic scenery may become more complicated.

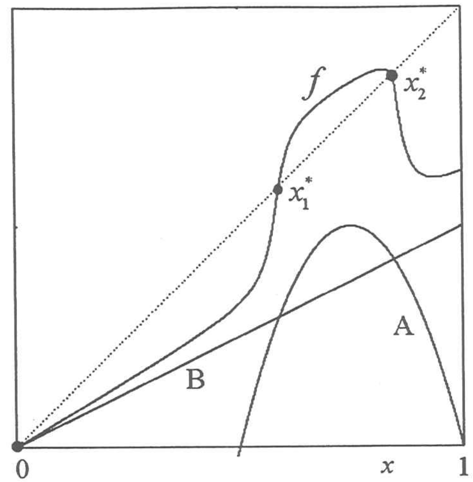

Fig. 3. Payoff functions with two intersections, and the relative map obtained with $g(\cdot) = 2/\pi \arctan(\cdot)$, $\delta_A = \delta_B = 0.5$, $\lambda = 6$

Some interesting examples are discussed in [24]; among the others we mention the one about the local use of insecticides. While everyone benefits the use of insecticides by the others, the value of insecticides gets dissipated unless some neighbors use insecticides too. When others use it moderately it becomes cost-effective but when almost everybody uses there are not enough bugs to spray it, and they become cost-ineffective. The importance of nonmonotonic payoff functions is also highlighted in [15].

In this case, the results of the analysis can be formalized as

Proposition 3. *If* $R : [0,1] \to \mathbb{R}$ *and* $L : [0,1] \to \mathbb{R}$ *are continuous functions such that*

- $A(0) < B(0)$
- $A(1) < B(1)$
- *there exist two points* $x_1^* < x_2^*$ *both in* $(0,1)$ *such that* $A(x_i^*) = B(x_i^*)$, $i = 1, 2$

then dynamical system (9) has three fixed points $x = 0$, $x = x_1^*$, *and* $x = x_2^*$, *where* 0 *is always stable,* x_1^* *is always unstable, and* x_2^* *may be stable or unstable. When* x_2^* *is unstable, then a cyclic (periodic or chaotic) attractor* $S(x_2^*) \subseteq [f(c_{\text{Max}}), c_{\text{Max}}]$ *exists around it and is bounded inside the trapping set* $[f(c_{\text{Max}}), c_{\text{Max}}]$, *provided that* $f(c_{\text{Max}}) > x_1^*$*. The unstable fixed point* x_1^* *is both the upper boundary of the immediate basin of the stable fixed point* 0, *and the lower boundary of the immediate basin of* x_2^* *(or* $S(x_2^*)$ *if it exists); furthermore, if* $(1 - \delta_L)\, x_2^* > x_1^*$ *then as* λ *increases non-connected portions of the basins are created.*

The proof of this result goes beyond the scope of this chapter; it is reported in [7]. Intuitively, this can be explained observing that the map f is noninvertible, i.e., there exist at least a pair of distinct points that are mapped into the same point, see Fig. 5. Following the notation introduced in [20], we denote by Z_k the subset of points, in the range of f, that have k preimages.

In the particular case of the map f represented in Fig. 4, $Z_1 = [0, c_{\min})$, $Z_2 = (c_{\min}, c_{Max}]$, and $Z_0 = (c_{Max}, 1)$, where $c_{\min}$ and c_{Max}, respectively, represent the relative minimum and maximum values. In addition, as it concerns the unstable fixed point x_1^* (located on the boundary that separates the two basins) it can be observed that $x_1^* < c_{\min}$; as a consequence $x_1^* \in Z_1$ and the point itself is its unique preimage as in Proposition 1. This is the reason why x_1^* is the unique point that forms the boundary separating the two basins of attraction. However, any parameter variation such that $c_{\min}$ becomes lower than x_1^* as illustrated in Fig. 5a, brings x_1^* in region Z_2. Therefore, there exists more than one preimage, say $x_1^{*(-1)}$, belonging to the basin boundary as well. Therefore, any initial condition $x_0 \in \left(x_1^{*(-1)}, 1\right)$ is first mapped below $(0, x_1^*)$ and then converges to the fixed point $x = 0$. In other words, the basin of the stable fixed point $x = 0$ (everybody is choosing L) is now $\mathfrak{B}(0) = (0, x_1^*) \cup \left(x_1^{*(-1)}, 1\right)$, i.e., a nonconnected set, with the "hole" $\mathfrak{B}(x_2^*) = \left(x_1^*, x_1^{*(-1)}\right)$ "nested" inside. We recall that the widest component of the basin that contains the attractor is called immediate basin of the attractor.

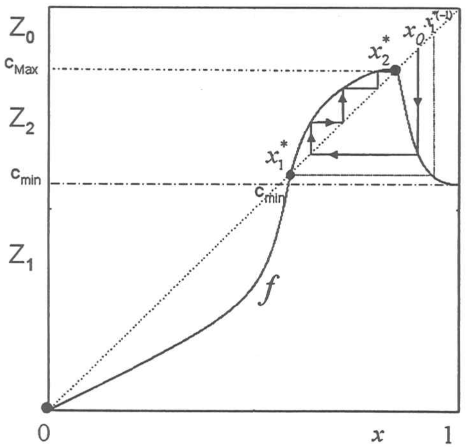

Fig. 4. Map f obtained with the same parameter values as Fig. 3 but $\lambda = 10$; the trajectory starting from $x_0 = 0.9$ converges to x_2^*

For this kind of map there exists a value λ such that the topological structure of the basins exhibits a qualitative change; this value is characterized by

the contact $c_{\min} = x_1^*$ between a critical point (relative minimum value) and an unstable fixed point. This global bifurcation leads to a counterintuitive behavior of the system. In fact, as the initial fraction of the populations of players choosing strategy R (the initial condition x_0) increases from 0 to 1, we first move from the basin of the lower equilibrium $x = 0$ into the basin of the upper one x_2^*, to finally re-enter into the basin of attraction of 0; that is, while when many players initially choose R the process will evolve toward a final equilibrium such that a large fraction of population chooses R, on the contrary when even more players initially choose R then nobody will end up playing R in the long run.

The situation may become even more involved when the position of the minimum is shifted horizontally so that global shape of the map f implies a new zone Z_3. This is illustrated in Fig. 5b, where parameters are $\delta_R = \delta_L = 0.4$, $\lambda = 40$. In this case $x_1^* \in Z_3$, actually as λ increases a global bifurcation occurs: from $c_{\min} > x_1^*$ to $c_{\min} = x_1^*$ where the contact bifurcation occurs and, finally, $c_{\min} < x_1^*$ as depicted. At this stage, there exist three distinct preimages of the boundary point x_1^*: x_1^* itself and two more preimages denoted by $x_1^{*(-1),1}$ and $x_1^{*(-1),2}$ in the same figure. The result is that so that both basins consist of two disjoint portions:

$$\mathfrak{B}(0) = (0, x_1^*) \cup \left(x_1^{*(-1),1}, x_1^{*(-1),2}\right)$$

and

$$\mathfrak{B}(A(x_2^*)) = \left(x_1^*, x_1^{*(-1),1}\right) \cup \left(x_1^{*(-1),2}, 1\right).$$

In Fig. 6 three different trajectories can be observed; they are generated by initial conditions $x_0 = 0.8$, $x_0 = 0.91$, and $x_0 = 0.95$, respectively (of course, any $x_0 < x_1^*$ generates a trajectory converging to 0). We also notice that, for this set of parameters, the larger fixed point x_2^* is not stable as around it there may exist a chaotic or high-period periodic attractor. However, we can observe that the occurrence of the global bifurcation that changes the topological structure of the attractors is not influenced by the kind of coexisting attractors.

Furthermore, the global analysis of the dynamic properties of the model reveals the occurrence of a global bifurcation that causes the transition from connected to nonconnected basins of attraction. This implies that several basin boundaries are suddenly created; they may be seen as a possible mathematical description of an extreme form of path dependence, observed in social systems, which is responsible of irreversible transitions from one equilibrium to another (and distant) one as final outcome.

The choice of a discrete time scale, allows the occurrence of overshooting and cyclic phenomena in social systems. In particular, with monotonic payoff functions, the model proposed in this paper allowed us to study the occurrence of oscillatory time series (periodic or chaotic). As it is well known,

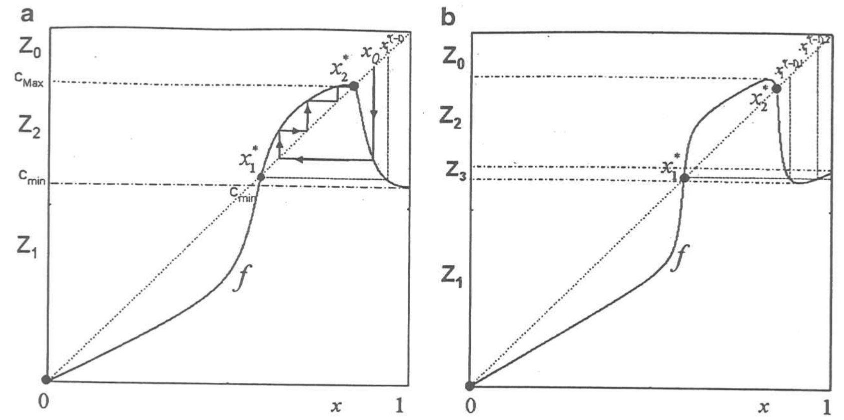

Fig. 5. Map f with different parameter values. (a) The parameters are the same as Fig. 4; the trajectory starting from $x_0 = 0.9$ converges to x_2^*. (b) $\delta_R = \delta_L = 0.4$, $\lambda = 40$. In this case $x_1^* \in Z_3$

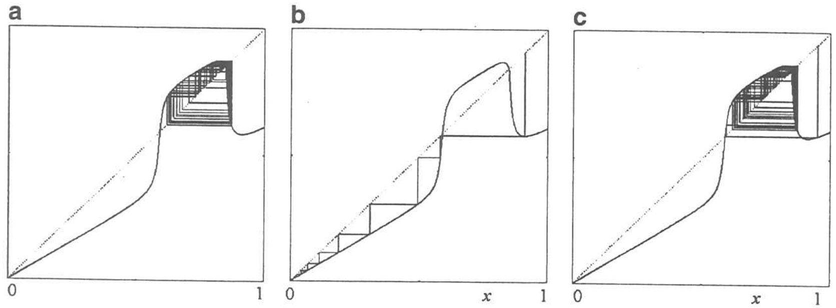

Fig. 6. Three trajectories for the map illustrated in Fig. 5 (b) with respective initial condition $x_0 = 0.8$ (a), $x_0 = 0.91$ (b), and $x_0 = 0.95$ (c)

discrete-time adaptive processes may lead to oscillations, often related to overshooting effects that are quite common in the presence of emotional human decisions.

4 The case of impulsive agents

The effects of impulsivity are examined in [5] as the parameter λ increases, up to the limiting case obtained as $\lambda \to +\infty$. In fact, according to the Clinical Psychology literature [23], impulsiveness can be separated in different components such as lack of planning and acting on the spur of the moment.

This case is equivalent to consider $g(\cdot) = 1$, i.e., the switching rate only depends on the sign of the difference between payoffs, no matter how much

they differ. The dynamical system assumes then the form of the following discontinuous map

$$x_{t+1} = f_\infty(x_t) = \begin{cases} \delta_A + (1-\delta_A)\,x_t & \text{if } B(x_t) < A(x_t) \\ x_t & \text{if } B(x_t) = A(x_t) \\ (1-\delta_B)x_t & \text{if } B(x_t) > A(x_t)\,. \end{cases} \tag{10}$$

Any interior intersection between the payoff curves $A(x)$ and $B(x)$ generates a point of discontinuity.

In the following, we summarize the results of the analysis of 1-discontinuity and 2-discontinuity cases.

4.1 Piecewise linear maps with one discontinuity

In [5] the case of impulsive agents with payoff functions $A(x)$ and $B(x)$ with one and only one internal intersection $x^* \in (0,1)$ has been examined. The family of iterated maps $f : [0,1] \to [0,1]$ we analyze in the case of impulsive agents can be expressed either

$$x' = T_1(x) = \begin{cases} (1-\delta_A)x & \text{if } x < d \\ x & \text{if } x = d \\ (1-\delta_B)x + \delta_B & \text{if } x > d \end{cases} \tag{11}$$

or

$$x' = T_2(x) = \begin{cases} (1-\delta_A)x + \delta_A & \text{if } x < d \\ x & \text{if } x = d \\ (1-\delta_B)x & \text{if } x > d \end{cases} \tag{12}$$

according to the situations described in Propositions 1 and 2, respectively. The parameter $d \in (0,1)$ represents the discontinuity point located at the interior equilibrium, i.e., $d = x^*$, and, as usual, the parameters δ_A, δ_B are subject to the constraints $0 \le \delta_A \le 1$, $0 \le \delta_B \le 1$. It is worth noticing that the value of the map in the discontinuity point, $x = d$, is not important for the analysis which follows, therefore it will often be omitted.

The study of the dynamic properties of iterated piecewise linear maps with one or more discontinuity points has been rising increasing interest in recent years, as witnessed by the high number of papers and books devoted to this topic, both in the mathematical literature (references to this huge literature are provided in [5] and [6]).

The bifurcations involved in discontinuous maps are often described in terms of the so called *border-collision bifurcations*, that can be defined as due to contacts between an invariant set of a map with the border of its region of definition. The term *border-collision bifurcation* was introduced for the first time in [22] and it is now widely used in this context. However the study and description of such bifurcations was started several years before by Leonov in [16] and [17], who described several bifurcations of that kind and gave a recursive relation to find the analytic expression of the sequence of bifurcations occurring in a one-dimensional piecewise linear map with one discontinuity point. His results are also described and used by Mira in [18] and [19].

By applying the methods suggested in [16] and [17], see also [18] and [19], to the map T_2, it is possible to give the analytical equation of the bifurcation curves where stable periodic cycles are created or destroyed. The boundaries that separate two adjacent periodicity regions in the space of parameters are also called *periodicity tongues*. They are characterized by the occurrence of a *border-collision*, involving the contact between a periodic point of the cycles existing inside the regions and the discontinuity point. To better formalize and explain our results it is suitable to label the two components of our map $x' = T_2(x)$ as follows:

$$x' = T_2(x) = \begin{cases} T_L(x) = m_L x + (1 - m_L) & \text{if } x < d \\ T_R(x) = m_R x & \text{if } x > d, \end{cases} \tag{13}$$

where $m_L = (1 - \delta_A)$ and $m_R = (1 - \delta_B)$.

First of all, notice that all the possible cycles of the map T_2 of period $k > 1$ are always stable. In fact, the stability of a k-cycle is given by the slope (or eigenvalue) of the function $T_2^k = T_2 \circ ... \circ T_2$ (k times) in the periodic points of the cycle, which are fixed points for the map T_2^k, so that, considering a cycle with p points on the left side of the discontinuity and $(k - p)$ on the right side, the eigenvalue is given by $m_L^p m_R^{(k-p)}$ which, in our assumptions, is always positive and less than 1.

In [5], the study of the conditions for the existence of the periodic cycles are limited to the analysis of the bifurcation curves of the so-called "principal tongues,"or "main tongues" [4] or "tongues of first degree" [16–19], which are the cycles of period k having one point on one side of the discontinuity point and $(k - 1)$ points on the other side (for any integer $k > 1$). Let us begin with the conditions to determine the existence of a cycle of period k having one point on the left side L and $(k-1)$ points on the right side R. The condition (i.e., the bifurcation) that marks its creation is that *the discontinuity point $x = d$ is a periodic point to which we apply, in the sequence, the maps $T_L, T_R, ..., T_R$*. For example, the condition for the creation of a 3-cycle, i.e., $k = 3$, is $T_R \circ T_R \circ T_L(d) = d$. Then the k-cycle with periodic points $x_1, ..., x_k$, numbered with the first point on the left side, satisfies $x_2 = T_L(x_1)$, $x_3 = T_R(x_2), ..., x_1 = T_R(x_k)$, and this cycle ends to exist when the last point (x_k) merges with the discontinuity point, that is, $x_{k-1} = d$ which may be stated as *the point $x = d$ is a periodic point to which we apply, in the sequence, the maps $T_R, T_L, T_R, ..., T_R$*. The closing condition related with the 3-cycle is, $T_R \circ T_L \circ T_R(d) = d$. Notice that both these conditions express the occurrence of a border collision bifurcation, being related to a contact between a periodic point and the boundary (or border) of the region of differentiability of the

corresponding branch of the map. In general, for a cycle of period $k > 1$, the equation of one boundary of the corresponding region of periodicity is:

$$m_L = m_{Li} = \frac{m_R^{(k-1)} - d}{(1-d)m_R^{(k-1)}} \tag{14}$$

while the other boundary, i.e., the closure of the periodicity tongue of the same cycle, is given by:

$$m_L = m_{Lf} = \frac{m_R^{(k-2)} - d}{(1 - m_R d)m_R^{(k-2)}}. \tag{15}$$

The proof of these two equations is reported in [5]. Thus the k-cycle exists for $m_R^{k-2} > d$ and m_L in the range

$$m_{Li} \le m_L \le m_{Lf} \tag{16}$$

and the periodic points of the k-cycle, say $(x_1^*, x_2^*, ..., x_k^*)$ where $x_1^* < d$ and $x_i^* > d$ for $i > 1$, can be obtained explicitly as:

$$\begin{aligned} x_1^* &= \tfrac{m_R^{(k-1)}(1-m_L)}{1-m_L m_R^{(k-1)}} \\ x_2^* &= T_L(x_1^*) = m_L x_1^* + 1 - m_L \\ x_3^* &= T_R(x_2^*) = m_R(m_L x_1^* + 1 - m_L) \\ x_4^* &= T_R(x_3^*) = m_R^2(m_L x_1^* + 1 - m_L) \\ &\dots \\ x_k^* &= T_R(x_{k-1}^*) = m_R^{(k-2)}(m_L x_1^* + 1 - m_L). \end{aligned} \tag{17}$$

An illustration of the periodicity tongues together with their boundaries is provided in Fig. 7.

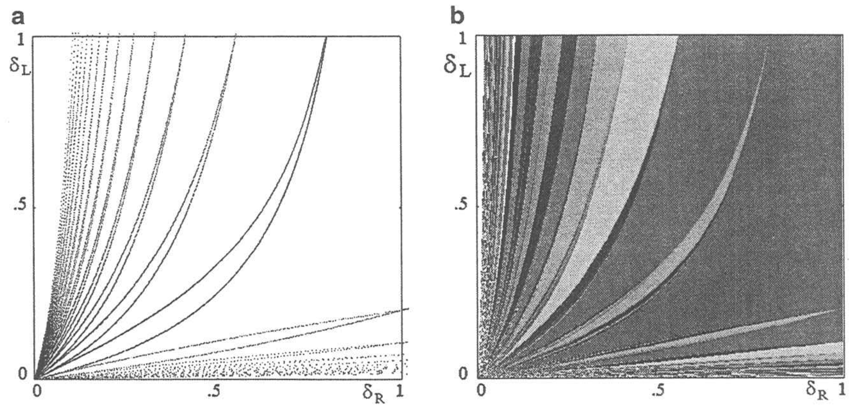

Fig. 7. (a) The bifurcation curves obtained by the analytical expressions calculated for the map T, with the discontinuity point d=0.8, for cycles of periods 2, . . . , 15. (b) The tongues of periodicity relative to case considered in (a)

4.2 Piecewise linear maps with two discontinuities

As discussed in Sect. 3, when the payoff functions are nonmonotonic there may be more then one intersection point. In [6] the case of two internal intersection has been examined. In this case the family of iterated maps $T : [0,1] \to [0,1]$ has the form:

$$x' = T(x) = \begin{cases} (1-\delta_B)x & \text{if } x < d_1 \text{ or } x > d_2 \\ x & \text{if } x = d \\ (1-\delta_A)x + \delta_A & \text{if } d_1 < x < d_2 \end{cases}$$

with parameters subject to the following constraints:

$$0 < d_1 < d_2 < 1$$

$$0 < \delta_A < 1, 0 < \delta_B < 1.$$

It is easy to realize that the origin is always an attractor; it may be the only one or it may coexist with an attracting cycle of period $k > 1$, according to the values of the four parameters d_1, d_2, δ_A, and δ_B. Also in this case it is possible to prove the existence of stable periodic cycles in analytically defined regions of the space of the parameters, as well as the analytic conditions for their creation or destruction through border collision bifurcations (see [1–4]).

In the case of coexisting attractors, it is also important to bound the sets of initial conditions that generate trajectories converging to either one or to the other, i.e., the respective basins of attraction. Indeed, we only have the two following possibilities:

1. Each of the two basins consists of a single interval, separated by the discontinuity point d_1; as a consequence, the basin of the origin is the first interval: $B(O) = [0, d_1[$ while the other points converge to the attracting cycle: $B(C) =]d_1, 1]$;
2. Each of the two basins consists of two or more intervals, separated by the two discontinuity points d_1 and d_2 and their preimages. The immediate basin of the origin is clearly the segment $[0, d_1[$ so that the whole basin is given by this segment and all its preimages: $B(0) = \cup_{j\geq 0} T^{-j}([0, d_1[)$, while the complementary region in $[0,1]$ gives the basin of the cycle: $B(C) = [0,1] \setminus B(0)$.

In [6], the authors provide the analytic description of the bifurcations occurring in a piecewise linear map T: $[0,1] \to [0,1]$ formed by three portions with two different slopes separated by two discontinuity points $0 \leq d_1 < d_2 \leq 1$. The problem is approached as the generalization of a simpler map with only one discontinuity. They show how both the bifurcation diagram and the analytic expression of the periodicity tongues of first degree maintain some important aspects of the map with only one discontinuity point; also the effects, on the structure of the border collision bifurcation curves – induced

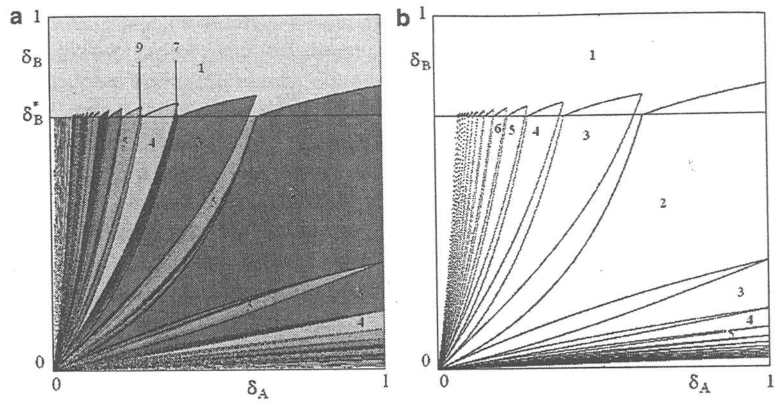

Fig. 8. (a) The tongues of periodicity, in the parameter plane (δ_A, δ_B), for cycles of periods $2, \ldots, 15$ when $d_1 = 0.2$ and $d_2 = 0.7$. (b) The bifurcation curves relative to the tongues depicted in (a) obtained by the analytical expressions calculated for the map T

by the introduction of the second discontinuity point – are examined. An illustration is provided in Fig. 8. The methods followed to obtain the analytic expressions are quite general and can be easily generalized to cases with different linear functions, several discontinuities, and with slopes different from the ones considered in the model studied in this chapter.

Furthermore, differently from the case of only one discontinuity, when considering two discontinuities, cases of multistability can be obtained, i.e., the coexistence of a stable fixed point and a stable periodic cycle, each with its own basin of attraction. These basins may be either connected intervals, separated by a discontinuity point, or nonconnected sets formed by the union of several disjoint intervals, separated by a discontinuity point and its preimages of any rank.

Finally, the discontinuous map – which can be interpreted as a model of the social behavior of impulsive agents – can be seen as the limit case of the continuous map, that models agents with some degree of inertia in making their choices. It is possible to show how the bifurcation curves of the limiting case, characterized by periodic cycles only, can be obtained from those of the continuous model, that also exhibit chaotic behavior, with a high value of the parameter λ.

5 A general model with different switching propensity

In [8] the two models described in Sects. 2 and 3 are merged into a single one which includes each of them as a particular case. Furthermore, this general model allows us to describe other situations which cannot be studied by any of the original models described above.

For each group of size k, define two payoff functions $A_k(j)$ and $B_k(j)$, where $j \in 0, 1, \ldots, k$ is the number of people in the group choosing A. As in Galam's model, at each time t individuals gather in group and, after the interaction, leave the group with an updated preference which depends on the outcome of the interaction, in the sense that in each group a fraction of the agents with lower payoff will switch to the choice that gives higher payoff.

This dynamic adjustment can be formalized as follows. At each time t, consider a size k group where $0 \leq j_t \leq k$ agents are choosing A; after the social gathering, as a result of the intragroup interaction, individuals may switch opinion. That is, at time $t+1$, the number of individuals choosing A will be:

$$
\begin{aligned}
j_{t+1} &= h_k(j_t, \delta_A, \delta_B) \\
&= \begin{cases} j_t + \lfloor \delta_A (k - j_t) \rfloor = \lfloor k\delta_A + (1-\delta_A) j_t \rfloor & \text{if } A_k(j_t) > B_k(j_t) \\ j_t & \text{if } A_k(j_t) = B_k(j_t) \\ j_t - \lfloor \delta_B j_t \rfloor = \lfloor j_t (1-\delta_B) \rfloor & \text{if } A_k(j_t) < B_k(j_t), \end{cases}
\end{aligned}
\tag{18}
$$

where δ_A,$\delta_B \in [0, 1]$ represent the probability according to which agents may switch to A and B, respectively, depending on to the greater payoff observed. Furthermore, since in each group the number of agents is integer, it makes little sense to consider fraction of agents who are switching choices, therefore we have introduced integer parts in the right hand side of (18) are introduced to consider integer number of agents.

In formulation (18) the Integer part has been considered; this indicates the behavioral momentum of agents, that is, how difficult is for agents to switch choices when facing inferior payoffs. Behavioral momentum is a well known psychological construct which has been examined for example in [14]. Finally, in this model the switching mechanism has no bias in the sense of [10].[5] In order to obtain a bias toward B it is sufficient to consider, for example,

$$
j_{t+1} = h_{k,B}(j_t, \delta_A, \delta_B) = \begin{cases} \lfloor k\delta_A + (1-\delta_A) j_t \rfloor & \text{if } A_k(j_t) > B_k(j_t) \\ \lfloor j_t (1-\delta_B) \rfloor & \text{if } A_k(j_t) \leq B_k(j_t). \end{cases}
\tag{19}
$$

Symmetrically, it is immediate to obtain a switching function biased toward A. In this formulation, such a bias – which is related to how local uncertainty $A_k(j_t) = B_k(j_t)$ is resolved- is combined to the bias, related to the asymmetric switching propensity $\delta_A \neq \delta_B$ as described in Sect. 3.

[5] Recall that in Sect. 2, in the case of local doubt the choice was biased toward B.

As in [10], it is possible to obtain the dynamics of the probability of choosing A:

$$
P_A(t+1) = \sum_{k=1}^{L} a_k \sum_{j=0}^{k} C_j^k P_A(t)^j (1 - P_A(t))^{k-j} \frac{h_k(j, \delta_A, \delta_B)}{k},
\tag{20}
$$

where the last term is the relative number of agents choosing option A in a group of size k.

This model assumes that individuals are impulsive in the sense described in [5]. However, it is possible to model inertia in the agents' reaction, as in [7], by introducing a modulating function g as described in Sect. 3.

In [8] by the unified model, different situations besides the original ones proposed in [10] and [24] have been considered. Two of them are particularly interesting.

The first one considers the case of a single large group. This situation is important from the theoretical point of view as it represents the case in which all the groups merge into one. Furthermore, it illustrates how the size of the group and the payoffs affect the dynamics. In fact, in the case of the majority rule [10], it takes just one iteration for the population to reach unanimity on the majority's opinion, and the killing point for this model coincides with the floor (i.e., the Integer Part) of $N/2$; by contrast, when considering different payoffs the dynamics may become quite interesting. In fact, recall that, given the necessity to maintain an integer number of agent in each group, the fraction of agents switching choices must be rounded. In [8] we consider *floor*, where $\lfloor x \rfloor$ defines the largest integer $n \leq x$; *ceiling*, where $\lceil x \rceil$ defines the smallest integer $n \geq x$. Finally, we consider *nearest integer*, where $\lfloor x \rceil$, defines the integer closest to x; as usual, to avoid ambiguity we adopt the convention according which half-integer are always rounded to even numbers. A situation similar to the one presented in [10] is analyzed and it is possible to observe that, considering different rounding functions, the dynamics can have a different evolution from the one predicted by Galam's original model.

The second example consists of payoff functions with – using Schelling's terminology, see [24] – both contingent internality and contingent externality. In this case it is possible to observe that an A choice benefits those who choose B, and a B benefits choice those who choose A. Among the vivid real life examples provided in [24], we quote[6] one situation related to traffic congestion after a blizzard: "... let A be staying home and B using the car right after a blizzard. The radio announcer gives dire warnings and urges everybody to stay home. Many do, and those who drive are pleasantly surprised by how empty the roads are; if the others had known, they would surely have driven. If they had, they would all be at the lower left extremity of the B curve." ([24], page 405). This kind of games has been examined in [21] as compounded dispersion game. The dynamics in dispersion games when agent

[6] Recall that we use, respectively, A and B instead of R and L.

are allowed in finite size groups can be quite interesting depending on which rounding function is considered. In particular in [8] the payoff functions are assumed to be

$$\begin{aligned} A_k(j) &:= j \\ B_k(j_{t:}) &:= k - j \end{aligned}$$

with switching propensities $\delta_A = 0.3$ and $\delta_B = 0.9$. Also it is assumed that, when payoff are identical, agents have a bias toward A, that is those who have played B will switch choice with propensity δ_A. Finally, the probabilities to be sitting at the different sizes groups are defined as follows:

$$a_2 = 0.1,\ a_3 = 0.1,\ a_k = 0.8, \text{ where } k \text{ is fixed as any value greater than } 3$$

all other groups have probability 0. The dynamics of choices as k varies in the range $[4, 276]$ have been examined; considering the different rounding functions the differences in the dynamics are stark, yet these differences are to be interpreted not as numerical artifacts rather as the result of the different biases implicitly modeled by the rounding functions.

6 Conclusion

In this chapter we have described some discrete-time dynamic models used to represent adaptive mechanisms through which individuals perform repeated binary choices in the presence of social externalities. The long-run (asymptotic) outcomes emerging from repeated short time decisions can be seen as an emerging property, sometimes unexpected, or difficult to be forecasted. Moreover, when there are several coexisting attractors each with its own basin of attraction, i.e., in the presence of so called multistability, the adaptive dynamics can be seen as an equilibrium-selection device. In this case the long-run outcome becomes path dependent, and historical accidents may play a crucial role in the selection of the social emerging behavior. This path dependence can also be seen as an evolutionary approach to the explanation of collective behavior as the result of repeated individual and myopic choices.

Starting from two particular discrete-time dynamic models – Galam's model of rumors spreading [10] and a formalization of Schelling's binary choices [7] – we have described some peculiar properties of discrete-time (or event-driven) dynamic processes. In particular, from a mathematical point of view, the kind of global dynamic analysis of the discrete time dynamics illustrated in this chapter, obtained through a continuous dialogue among analytic, geometric and numerical methods, is based on the properties of non-invertible one-dimensional maps, see, e.g., [20]. Moreover, in the limiting case of impulsive agents, the global dynamic properties of piecewise linear discontinuous maps allowed us to illustrate some results concerning border collision bifurcations, a topic that has been recently been at the center of a flourishing

stream of literature, see, e.g., [1–4]. In other words, the models described in this chapter allowed us to get some insight into the two kinds of complexity, related to complex attracting sets and complex structure of the basins of attraction. Both these kinds of complexity are related to the presence of overshooting phenomena, typical of discrete-time adaptive systems, see, e.g., [21]. However, overshooting should not be seen as an artificial effect or a distortion of reality due to discrete time scale. In fact, as stressed in [25], overshooting and over-reaction arise quite naturally in social systems, due to emotional attitude, excess of prudence, or lack of information. The extreme form of agents' impulsivity, represented by the limiting case of a switching intensity that tends to infinity, i.e., actors that decide to switch the strategy choice even when the discrepancy between the payoffs observed in the previous period is extremely small, may even be interpreted as the automatic change of an electrical or mechanical device that changes its state according to a measured difference between two indexes of performance.

Finally, the general model described at the end of the chapter, constitutes a generalization and a synthesis of two models of social interaction of Galam and Schelling, respectively. It may be further used to analyze other situations, such as some of those considered in [24] for a single population that can be extended to small groups. Other aspects of this model that can be investigated concern, for example, the role of impulsiveness in small groups, and how some social parameters, such as agents' impulsiveness and population size, affect the dynamics. In particular, it would be interesting to investigate what makes a group large and how its dynamic behavior is different from that of a small one.

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