Classification of Linear Second Order PDEs

In this note we shall consider linear second order PDEs in two and more variables.

First we consider PDEs in two variables. Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be the function of x and y we are seeking for satisfying

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y). \quad (*)$$

Here a, b, c, d, e, f and g are continuous functions from Ω into $\mathbb{R}$. Having analogy in appearance with the conic section $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ of the homogeneous equation (i.e., $g \equiv 0$) of $(*)$, we call $(*)$
hyperbolic if $b^2 - ac > 0$,
parabolic if $b^2 - ac = 0$,
elliptic if $b^2 - ac < 0$.
Clearly, the type of the equation $(*)$ depends on the coefficients of the highest derivatives u_{xx}, u_{xy} and u_{yy} only. We call $au_{xx} + 2bu_{xy} + cu_{yy}$ the **principal** part. In case the coefficients a. b and c are constants then the type remains the same in the whole of Ω but in general when the coefficients are functions of x and y the ype of the equation may be different in different parts of Ω.

Examples:
$u_{xx} - c^2 u_{yy} = 0$, c being a real constant, is hyperbolic (wave equation);
$u_x - c^2 u_{yy} = 0$, c being a real constant, is parabolic (heat equation);
$u_{xx} + u_{yy} = 0$, is elliptic (Laplace's equation);
$yu_{xx} + u_{yy} = 0$ is hyperbolic for $y < 0$ and elliptic for $y > 0$ (Tricomi's equation).

It is possible that the general equation $(*)$ can be reduced to a simple form known as the normal form.

Normal Forms:
Hyperbolic Equation:
$u_{xy} = f(x, y, u, u_x, u_y)$
or $u_{xx} - u_{yy} = f(x, y, u, u_x, u_y)$
or $u_{yy} - u_{xx} = f(x, y, u, u_x, u_y)$
Parabolic Equation:
$u_{yy} = f(x, y, u, u_x, u_y)$
or $u_{xx} = f(x, y, u, u_x, u_y)$
Elliptic Equation:
$u_{xx} + u_{yy} = f(x, y, u, u_x, u_y)$.

Now we explain the procedure to get these forms. Let

$$\xi = \phi(x,y), \quad \eta = \psi(x,y)$$

where $\phi_x\psi_y - \phi_y\psi_x \neq 0$. Then

$$u_x = u_\xi \xi_x + u_\eta \eta_x, \quad u_y = u_\xi \xi_y + u_\eta \eta_y.$$

Also,

$$u_{xx} = (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + u_\xi \xi_{xx} + (u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\eta \eta_{xx},$$
$$u_{xy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + u_\xi \xi_{xy} + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_x + u_\eta \eta_{xy},$$
$$u_{yy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + u_\xi \xi_{yy} + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\eta \eta_{yy}.$$

Putting all these in $(*)$, we get

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + Du_\xi + Eu_\eta + Fu = G,$$

where
$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$
$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,$
$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta^2,$
$D = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$
$E = a\eta_{yy} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y.$
$F = f$
$G = g.$

Also, $B^2 - AC = (b^2 - ac)(\xi_x\eta_y - \xi_y\eta_x)^2$. Hence the type remain the same after the transformation. Now, we would like to make some of A, B and C to be zero. In the hyperbolic case, i.e., $b^2 - ac > 0$, We would force $A = C = 0$ then $B \neq 0$. This would be achieved by putting

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0, \quad a\eta^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0.$$

Along the curves $y = y(x)$ these equations will hold if

$$\xi_x + \xi_y y' = 0, \quad \eta_x + \eta_y y' = 0.$$

Hence, if $a \neq 0$ at some point in Ω then by continuity $a \neq 0$ at all points in some neighborhood of that point and in this neighborhood we reduce the PDE to its normal form. We have that y' must satisfy,

$$a(y')^2 - 2by' + c = 0. \quad (**).$$

Thus,

$$y' = (b \pm \sqrt{b^2 - ac})/a.$$

If, $a \equiv 0$ and $c \equiv 0$ then we already have the normal form, hence we suppose that $c \neq 0$ and we take

$$-2b + cx' = 0, \quad (x' = dx/dy)$$

instead of $(**)$. The equation $(**)$ is called the **characteristic equation** and its solutions are called the characteristic curves. Thus, in the hyperbolic case, $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ must satisfy the characteristic equation $(**)$ for making $A \equiv C \equiv 0$. Clearly, in this case we get two real characteristic curves. In the parabolic case, $b^2 - ac = 0$, we get only one real characteristic curve and the other transformation will have to be chosen suitably. In the elliptic case, we have no real characteristic curves and we will have to choose both the transformations suitably. Let us discuss how to do this through examples.

Hyperbolic equation:

Consider the equation $u_{xx} + 4u_{xy} + u_{yy} = 0$. Here $a = 1$, $2b = 4$ and $c = 1$. Thus $b^2 - ac = 4 - 1 = 3 > 0$. The given PDE is hyperbolic. Now, the characteristic equation is $y' = 2 \pm \sqrt{3}$ and the characteristics are $\xi = (2 + \sqrt{3})x - y$ and $\eta = (2 - \sqrt{3})x - y$. Hence $\xi_x, \eta_x = 2 \pm \sqrt{3}$, $\xi_y, \eta_y = -1$. $\xi_{xx} = \xi_{xy} = \xi_{yy} = \eta_{xx} = \cdots = 0$. Now,

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = (2 + \sqrt{3})^2 - 4(2 + \sqrt{3}) + 1 = 4 + 3 + 4\sqrt{3} - 8 - 4\sqrt{3} + 1 = 0.$$

Similarly $C = 0$ and

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 4 - 3 + 2(-(2 + \sqrt{3}) - (2 - \sqrt{3})) + 1 = 4 - 3 - 8 + 1 = -6.$$

So the transformed equation is $-12u_{\xi\eta} = 0$ or $u_{\xi\eta} = 0$.

Parabolic equation:
Consider the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$. Here $a = b = c = 1$. Hence $b^2 - ac = 0$. the given PDE is parabolic. The characteristic equation is $y' = 1$ and we get only one characteristic curve $\xi = x - y$. We take the other transformation $\eta = y$ (or $\eta = x$, we should take it as simple as possible such that $\xi_x\eta_y - \xi_y\eta_x \neq 0$). Then $\xi_x, \eta_x = 1, 0$ $\xi_y, \eta_y = -1, 1$ and $\xi_{xx} = \xi_{xy} = \xi_{yy} = \eta_{xx} = \cdots = 0$. Here

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 1 - 2 + 1 = 0,$$

and

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 1 - 1 = 0,$$
$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 1.$$

Thus, the transformed equation is $u_{\eta\eta} = 0$

Elliptic Equation:
Let us consider the equation $u_{xx}+u_{xy}+u_{yy} = 0$. Here $b^2-ac = (1/4)-1 = -3/4 < 0$, hence the given PDE is elliptic. The characteristic equation is $y' = (1 \pm i\sqrt{3})/2$. The (complex) solution of this is $y = (1\pm i\sqrt{3})x/2+const$. We take the real and imaginary parts as the transformation. Thus, $\xi = x - 2y$ and $\eta = \sqrt{3}x$. Hence $\xi_x, \eta_x = 1, \sqrt{3}$ and $\xi_y, \eta_y = -2, 0$ and again $\xi_{xx} = \xi_{xy} = \xi_{yy} = \eta_{xx} = \cdots = 0$. Here

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 1 - 2 + 4 = 3,$$

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = \sqrt{3} + (1/2)(-2\sqrt{3}) = 0,$$
$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 3.$$

Hence the PDE is transformed to $3(u_{\xi\xi} + u_{\eta\eta}) = 0$ or $u_{\xi\xi} + u_{\eta\eta} = 0$.

Equation with Variable Coeeficients:
Let us consider the Tricomi equation $yu_{xx} + u_{yy} = 0$, $y < 0$. Here $a = y$, $b = 0$ and $c = 1$. For $y < 0$ this equation is hyperbolic as $b^2 - ac = -y > 0$. The characteristic equation is $y' = \pm 1/\sqrt{-y}$ which has solutions $\xi, \eta = x \pm (2/3)(-y)^{3/2}$. Here $\xi_x, \eta_x = 1$ and $\xi_y, \eta_y = \mp\sqrt{-y}$. Also, $\xi_{xx} = \eta_{xx} = \xi_{xy} = \eta_{xy} = 0$ and $\xi_{yy}, \eta_{yy} = \pm(1/2\sqrt{-y})$.

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = y - y = 0.$$

Similarly, $C = 0$ and $B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + \xi_y\eta_y = y + y = 2y$, $D = a\xi_{xx} + 2b\xi_{xy}+c\xi_{yy}+d\xi_x+e\xi_y = (1/2\sqrt{-y})$, $E = a\eta_{xx}+2b\eta_{xy}+c\eta_{yy}+d\eta_x+e\eta_y = -(1/2\sqrt{-y})$. So the given PDE is transformed as

$$4yu_{\xi\eta} + (1/2\sqrt{-y})u_\xi - (1/2\sqrt{-y})u_\eta = 0$$

or

$$-8(-y)^{3/2}u_{\xi\eta} + u_\xi - u_\eta = 0$$

or

$$u_{\xi\eta} - \frac{u_\xi - u_\eta}{6(\xi - \eta)} = 0.$$

For $y > 0$ the Tricomi equation is elliptic and in this case the characteristic equation is $y' = \pm i/\sqrt{y}$. The complex solution of this equation is $x \pm i(2/3)y^{3/2} = const$. So, we take $\xi = x$ and $\eta = (2/3)y^{3/2}$ as the transformations. Now, $\xi_x = 1$, $\xi_y = 0$, $\xi_{xx} = \xi_{xy} = \xi_{yy} = 0$ and $\eta_x = 0$, $\eta_{xx} = \eta_{xy} = 0$, $\eta_y = \sqrt{y}$ and $\eta_{yy} = 1/2\sqrt{y}$. Hence

$$A = y, \quad B = 0, \quad C = y, \quad D = 0, \quad E = 1/2\sqrt{y}.$$

So the transformed equation is

$$y(u_{\xi\xi} + u_{\eta\eta}) + \frac{1}{2\sqrt{y}} u_\eta = 0,$$

or

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{2y^{3/2}} u_\eta = 0,$$

or

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta} u_\eta = 0.$$

The classification of Linear Second order PDEs in more than two variables:
The general linear second order PDE has the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_j} + c(x)u = d(x), \quad (***)$$

$$x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n,$$

where the coefficient functions a_{ij}, b_i, c and d are continuous in the domain Ω. Again the type of the equation $(***)$ will depend on the principal part

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

We assume that $u \in C^2(\Omega)$ so that we have

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}.$$

Therefore we take both the coefficients of $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial^2 u}{\partial x_j \partial x_i}$ equal to $(a_{ij}(x)+a_{ji}(x))/2$. Thus the matrix A of coefficients of $\frac{\partial^2 u}{\partial x_i \partial x_j}$ is

$$A = \begin{bmatrix} a_{11} & (a_{12} + a_{21}/2) & \cdots & (a_{1n} + a_{n1})/2 \\ (a_{12} + a_{21}/2) & a_{22} & \cdots & (a_{2n} + a_{n2})/2 \\ \vdots & \vdots & \vdots & \vdots \\ (a_{1n} + a_{n1})/2 & (a_{2n} + a_{n2})/2 & \cdots & a_{nn} \end{bmatrix}.$$

Clearly, A is symmetric. Therefore all the eigenvalues of A are real. Let P, Z and N denote the number of positive, zero and negative eigenvalues of A counting with their multiplicity also. Then we have $P + Z + N = n$ and we classify $(***)$ as follows.
If $Z = 0$ and $P = 1$ or $P = n - 1$ then $(***)$ is called hyperbolic.
If $Z = 0$ and $1 < P < n - 1$ then $(***)$ is called ultra-hyperbolic.
If $Z = 0$ and $P = 0$ or $P = n$ then $(***)$ is called elliptic.
If $Z > 0$ $(***)$ is called parabolic.
Examples:
(i) Consider the equation $u_{xx} + u_{yy} - u_{zz} = 0$ (two dimensional wave equation). Here

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The eigenvalues are 1, 1 and -1. Hence the given PDE is hyperbolic.
(ii) Consider the equation $u_{xx} + u_{yy} - u_z = 0$ (two dimensional heat equation). Here

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are 1, 1 and 0. Hence the given PDE is parabolic.
(iii) Consider the equation $u_{xx} + u_{yy} + u_{zz} = 0$ (three dimensional Laplace equation). Here

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues are 1, 1 and 1. Hence the given PDE is elliptic.