First Order PDEs

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Outline

Introduction

In this presentation, we discuss first-order equations using geometrical considerations and the method of characteristics. We begin with semilinear equations, and then show how to treat quasilinear and even general non- linear equations. Along the way, we encounter weak solutions and several applications to physical examples.

Cauchy Problem for Quasilinear Equations

Let us recall a simple fact from the theory of ordinary differential equations: the equation $du/dt = f(t, u)$ can be uniquely solved (at least for small values of t) for each initial condition $u(0) = u_0$, provided that f is continuous in t and Lipschitz continuous in the variable u. Recall that the solution may exist globally in time, or may blow up at some finite time.

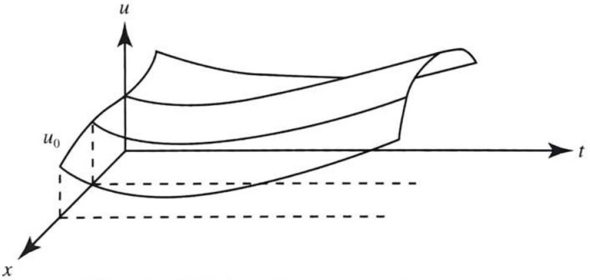

Figure 1. ODE depending on parameter x.

Cauchy Problem ... Contd.

If we allow the equation and the initial condition to depend on a parameter x, then the solution u depends on x and may be written $u(x,t)$. In fact, u becomes a solution of $u_t = f(x,t,u)$, $u(x,0) = u(x)$ that may be thought of as an initial value problem for a PDE in which u_x does not appear. Assuming f and u_0 are continuous functions of x, the solution $u(x,t)$ will be continuous in x (as well as t).
Geometrically, the graph $z = u(x,t)$ is a surface in $\mathbb{R}^3$ that contains the curve $(x, 0, u_0(x))$ (see Figure 1). This surface may be defined for all $t > 0$, or may blow up at some finite t_0 (which may depend on x).
However, if the surface remains bounded, then it will continue as a graph for all $t > 0$. In particular, the surface cannot fold over on itself and thereby fail to be the graph of a function.
These elementary ideas from ODE theory lie behind the method of characteristics which applies to general quasilinear first-order PDEs, as we shall discover in this section.

An Example: The Transport Equation

Consider the initial value problem for the transport equation

$$u_t + au_x = 0, \quad u(x,0) = h(x),$$

where a is a constant. Let us try to reduce this problem to an ODE along some curve $x(t)$; that is, find $x(t)$ so that

$$\frac{d}{dt}u(x(t),t) = au_x + u_t.$$

By the chain rule, we simply require $dx/dt = a$, i.e., $x = at + x_0$ where x_0 is the x-intercept of the curve. Along this curve we have $u_t = 0$ (i.e., $u =$ const); since u has the value $h(x_0)$ at the x-intercept, we must have $u(x,t) = h(x_0) = h(x - at)$. Indeed, if h is C^1, then we can check that $u(x,t) = h(x - at)$ satisfies the PDE and the initial condition. Notice that the solution corresponds to "transporting" (without change) the initial data $h(x)$ along the x-axis at a speed $dx/dt = a$ (see Figure 2). The lines $x = at + x_0$ are called the characteristic curves for $u_t + au_x = 0$.

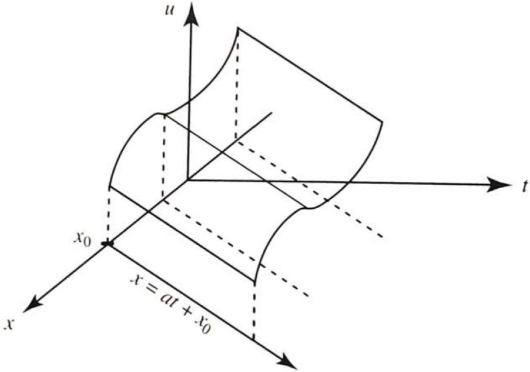

Figure 2. Solution of the transport equation.

The reduction of a PDE to an ODE along its characteristics is called the method of characteristics and applies to much more complicated equations. For example, the semilinear transport equation $u_t + au_x = f(u)$ reduces to the ODE $u_t = f(u)$ along the characteristic curve $x(t) = at + x_0$ and can therefore easily be integrated. Let us now see how and why this method applies to quasilinear PDEs.

Let us consider the quasilinear equation for a function u of two variables x and y,

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{1}$$

where the coefficient functions a, b, and c, are continuous in x, y, and u. If $u(x, y)$ is a solution of (1), let us consider the graph $z = u(x, y)$. This surface has normal vector $\vec{N}_0 = (-u_x(x_0, y_0), -u_y(x_0, y_0), 1)$ at the point $(x_0, y_0, u(x_0, y_0))$. But if we let $z_0 = u(x_0, y_0)$, then equation (1) implies that the vector

$$\vec{V}_0 = (a(x_0, y_0, z_0), b(x_0, y_0, z_0), c(x_0, y_0, z_0))$$

is perpendicular to this normal vector and hence must lie in the tangent plane to the graph of $z = u(x, y)$ at the point z_0 (see Figure 3).

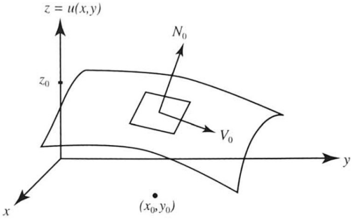

Figure 3. The vector field V is tangent to the graph.

In other words, $\vec{V}(x, y, z) = (a(x, y, z), b(x, y, z), c(x, y, z))$ defines a vector field in $\mathbb{R}^3$, to which graphs of solutions must be tangent at each point. Surfaces that are tangent at each point to a vector field in $\mathbb{R}^3$, are called integral surfaces of the vector field, just as curves that are tangent to vector fields are called integral curves. Thus to find a solution of (1), we should try to find integral surfaces. Of course, there may be many integral surfaces of $\vec{V}$, so we might try to be more specific and find the integral surface containing a given curve $\Gamma \subset \mathbb{R}^3$. Thus we are led to formulating the following Cuachy Problem.

The Cauchy Problem.

Given a curve Γ in $\mathbb{R}^3$, can we find a solution u of the first-order partial differential equation whose graph contains Γ? In the special case that Γ is the graph $(x, h(x))$ in the xz-plane of a function h, the Cauchy problem is just an initial value problem with the obvious interpretation of the variable y as "time."

How can we construct integral surfaces? We might try using the characteristic curves that are the integral curves of the vector field $\vec{V}$. That is, $\chi = (x(t), y(t), z(t))$ is a characteristic if it satisfies the following system of ordinary differential equations called the characteristic equations:

$$\frac{dx}{dt} = a(x, y, z),\ \frac{dy}{dt} = b(x, y, z),\ \frac{dz}{dt} = c(x, y, z). \tag{2}$$

We can solve (2) uniquely for small $|t - t_0|$ if we are given initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0,$$

and we assume that the functions a, b, c are all continuously differentiable in x, y, z. If the graph $z = u(x, y)$ is a smooth surface S that is a union of such characteristic curves, then at each point (z_0, y_0, z_0) the tangent plane contains the vector $\vec{V}(x_0, y_0, z_0)$; hence S must be an integral surface. In other words, a smooth union of characteristic curves is an integral surface. So how can we obtain a smooth union of characteristic curves? If the given curve Γ is noncharacteristic (i.e., Γ is nowhere tangent to the vector field $\vec{V}$), then a simple procedure for solving the Cauchy problem is to flow out from each point of Γ along the characteristic curve through that point, thereby sweeping out an integral surface (see Figure 4). This is the method of characteristics.

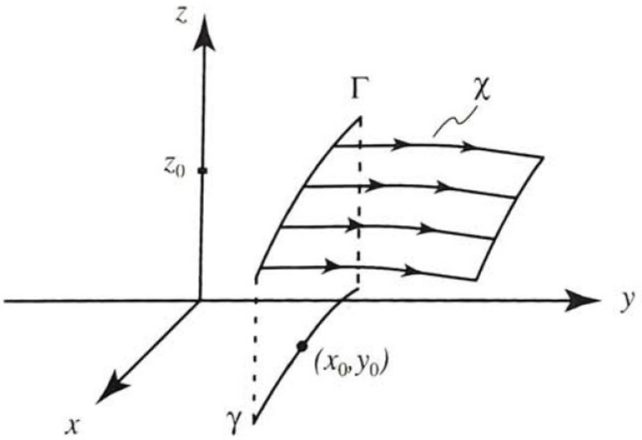

Figure 4. Flowing out along characteristics.

Analytically, this construction of an integral surface containing Γ can be achieved by first writing Γ as the graph of a curve $(f(s), g(s), h(s))$ parameterized by s and then solving the equations (2) for each s using $x_0 = f(s),\ y_0 = g(s),\ z_0 = h(s)$, as initial conditions. In this way, we obtain our integral surface S parameterized by s and t. In fact, it is also true that an integral surface S of the vector field $\vec{V} = (a, b, c)$ is always a union of characteristic curves: this can be proved using the uniqueness theorem for solutions of ordinary differential equations (see Exercise 1). Since the solutions of the characteristic equations are unique, we find that the integral surface S is unique. This means that we have proved the following.

Theorem.

If Γ is noncharacteristic. then the vector field $\vec{V}$ admits a unique integral surface S containing Γ.

To find the solution $u(x.y)$ of (1), it remains only to replace the variables s and t by expressions involving x and y. Theoretically, this can be achieved exactly when the integral surface S is the graph of a function.

The method of characteristics is of more than theoretical value since it actually produces a formula for the solution. provided of course that we can explicitly solve the system of ordinary differential equations (2) and solve for s, t in terms of x, y. We shall discuss several examples of semilinear and quasilinear equations in two variables, but first let us discuss how to generalize the foregoing procedure to an equation in n variables.

We now replace (1) with the equation

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n, u)u_{x_i} = c(x_1, x_2, \ldots, x_n, u). \tag{3}$$

The characteristic curves are now the integral curves of the system of $n+1$ equations in $n+1$ unknowns

$$\frac{dx_i}{dt} = a_i(x_1, x_2, \ldots, x_n, z), \quad \frac{dz}{dt} = c(x_1, x_2, \ldots, x_n, u), \tag{4}$$

which can be solved if we are given initial conditions on an $n-1$-dimensional manifold Γ :

$$x_i = f_i(s_1, \ldots, s_{n-1}),\ z = h(s_1, \ldots, s_{n-1}).$$

This generates an n-dimensional integral manifold M parameterized by $(s_1, \ldots, s_{n-1}, t)$. The solution $u(x_1, \ldots, x_n)$ is obtained by solving for $(s_1, \ldots, s_{n-1}, t)$ in terms of the variables $(x_1, \ldots, x_n)$. Thus we can apply the method of characteristics to solve initial value problems, or more generally Cauchy problems, for equations in n variables.

c. Semilinear Equations

Let us consider the Cauchy problem for the semilinear equation in two variables

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u), \tag{5}$$

with Γ parameterized by $(f(s), g(s), h(s))$. The characteristic equations

$$\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y), \quad \frac{dz}{dt} = c(x,y,z), \tag{6}$$

with initial conditions

$$x(s,0) = f(s), \quad y(s,0) = g(s), \quad z(s,0) = h(s). \tag{7}$$

Notice that the first two equations form a system (decoupled from z), which may be solved to obtain a curve $(x(t), y(t))$ in the xy-plane; such a curve is sometimes called a projected characteristic curve since it is simply the projection into the xy-plane of the characteristic curve χ. If we first find the projected characteristics, we can then integrate the remaining characteristic equation to find z.

Moreover, regarding the problem of solving for s and t in terms of x and y, the inverse function theorem tells us that this can be achieved provided the Jacobian matrix is nonsingular:

$$J \equiv \det \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} \equiv x_s y_t - y_s x_t \neq 0. \tag{8}$$

Notice that this condition is independent of the behavior of z. In particular, at $t = 0$ we obtain the condition

$$f'(s)b(f(s), g(s)) - g'(s)a(f(s), g(s)) \neq 0, \tag{9}$$

which geometrically means that the projection of Γ into the xy-plane is a curve $\gamma = (f(s), g(s))$ that is nowhere parallel to the vector field (a, b). But (9) implies by continuity that (8) holds at least for small values of t, so we have the following result: Provided the (projected) initial curve $\gamma = (f(s), g(s))$ satisfies (9), there exists a unique solution $u(x, y)$ of (5) in a neighborhood of γ.

However, away from γ (i.e. for larger values of t) the solution may develop a singularity where $J = 0$; in fact, even if (8) holds for all values of s and t, the solution may develop a blow up type of singularity if the equation for dz/dt is nonlinear.

Example 1. Let us solve the initial value problem $u_x + 2u_y = u^2$ with $u(x, 0) = h(x)$. As an initial value problem, Γ is the graph $(x, h(x))$ in the xz-plane, so we may parameterize Γ by $(s, 0, h(s))$, and we see that (9) holds: $2 \neq 0$. The equations (6) become

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = z^2.$$

We may integrate the first two equations (treating s as a constant) to find $x(s, t) = t + c_1(s)$ and $y(s, t) = 2t + c_2(s)$, where the functions $c_1(s)$ and $c_2(s)$ may be determined from the initial conditions: $x(s, 0) = c_1(s) = s$ and $y(s, 0) = c_2(s) = 0$ so that $x = t + s$ and $y = 2t$.

Notice that (8) holds for all s and t, and we can explicitly solve for s and t to find $s = x - (y/2)$ and $t = y/2$. We may integrate the equation for to find $z(s.t) = (t + c_3(s))^{-1}$ and use the initial condition $z(s.0) = h(s)$ to evaluate $c_3 = -1/h(s)$. so

$$z = \frac{h(s)}{1 - th(s)}.$$

Finally, we may eliminate s and t to express our solution as

$$u(x,y) = \frac{h(x - (y/2))}{1 - (y/2)h(x - (y/2))}.$$

Notice that $u(x.0) = h(x)$ and the solution $u(x.y)$ is certainly well defined for small enough values of y (assuming h is bounded); however, u may become infinite if y becomes large enough to cause the denominator to vanish. Even though the equations for dx/dt and dy/dt are linear and the solutions exist for all s and t. the equation for dz/dt is nonlinear and may produce a singularity.

In the semilinear case, it is possible to take a different perspective on the Cauchy problem. Consider the curve γ parameterized by $(f(s), g(s))$ as given and then impose Cauchy data $h(s)$ along. Provided γ is noncharacteristic i.e. $(f(s), g(s))$ satisfies (9)], then there is a unique solution of (5) satisfying

$$u(f(s), g(s)) = h(s). \tag{10}$$

This means that the Cauchy data $u|_\gamma = h$ also determines the derivatives $u_x|_\gamma$ and $u_y|_\gamma$: this is true theoretically because the solution u is uniquely defined in a neighborhood of γ, but in fact may be found explicitly by solving the system of algebraic equations along $\gamma(s) = (f(s), g(s))$:

$$a(\gamma(s))u_x(\gamma(s)) + b(\gamma(s))u_y(\gamma(s)) = c(f(s), g(s), h(s)),$$

$$f'(s)u_x(\gamma(s)) + g'(s)u_y(\gamma(s)) = h'(s), \tag{11}$$

which admit unique solutions by (9).

Moreover, the Cauchy data also enables us to find all partial derivatives of u on γ, theoretically by differentiating the unique solution near γ, but explicitly by using equation (11) (see Example 2). Thus, in a well-posed Cauchy problem, the Cauchy data will determine all derivatives of the solution on the initial curve.

On the other hand, if $J \equiv x_s y_t - y_s x_t = 0$ along γ, then the method of characteristics breaks down. Is there any hope of solving the Cauchy problem? Yes, but only if $h(s)$ is chosen to make Γ a characteristic curve (which does not turn vertical). In fact, if $(f')^2 + (g')^2 \neq 0$ along γ, then the condition $J \equiv 0$ shows $(dx/a) = (dy/b)$ so that γ must be a projected characteristic. But this means the Cauchy data h cannot be chosen arbitrarily. Moreover, with this particular choice of h, there is an infinite number of solutions to the Cauchy problem since any smooth union of characteristics gives an integral surface, i.e., the solution is no longer unique. Thus, for characteristic γ the Cauchy problem is certainly not well posed.

Example 2

Consider the equation $u_x + xu_y = u^2$. We find the projected characteristics by solving $(dx/1) = (dy/x)$ to find the parabolas

$$y = \frac{x^2}{2} + C.$$

Provided we take γ nowhere tangent to the parabolas, then the Cauchy problem is well posed, i.e., admits a unique solution with Cauchy data $u|_\gamma = h$ for arbitrary (smooth) h. For example, the Cauchy problem is well posed if the Cauchy data h is prescribed on the y-axis. Let us verify that $u|_{x=0} = h$ determines all derivatives of the solution on the y-axis. First of all, by solving (11), we can find $u_x|_{x=0}$ and $u_y|_{x=0}$. Differentiating these functions with respect to y will determine $u_{xy}|_{x=0}$ and $u_{yy}|_{x=0}$. To determine $u_{xx}|_{x=0}$ we differentiate the equation with respect to x:

$$u_{xx} = 2uu_x - u_y - xu_{xy}.$$

Now all terms on the right-hand side are known on the y-axis, so we have found u_{xx}.

Similarly, we may proceed to find the other higher-order derivatives of u. (Exercise 9 illustrates this procedure.) On the other hand, let us take γ to be the parabola $y = x^2/2$. Then in order for Γ to be characteristic, we must have

$$\frac{dz}{z^2} = dx.$$

We may integrate this equation to find $-(1/z) = x + c$ where c is an arbitrary constant. In fact the constant c is determined by picking a point over for Γ to pass through. For example, if Γ passes through $(0, 0, z_0)$, then $c = -1/z_0$ and Γ is given by

$$z = \frac{z_0}{1 - z_0 x}.$$

In order to find the infinite family of solutions of this Cauchy problem, let us use the general solution discussed in Section 1.1.e. If we let $\phi(x, y, z) = y - (x^2/2)$ and $\psi(x, y, z) = x + z^{-1}$, then the general solution may be written as $\phi = f(\psi)$ where f is an arbitrary C^1 function.

Therefore

$$y - \frac{x^2}{2} = f\left(x + \frac{1}{z}\right).$$

Along Γ, $y - (x^2/2) = 0$ and $x + z^{-1} = (1/z_0)$, so the solution will pass through Γ provided $f(1/z_0) = 0$; with this sole restriction on f we see that there is an infinite number of solutions.

With obvious modifications, the preceding observations also pertain to semilinear equations in n variables. In particular, integrating (4) yields functions $x_i(s_1, \ldots, s_{n-1}, t)$ for $i = 1, \ldots, n$ and $z(s_1, \ldots, s_{n-1}, t)$ that may be used to determine the solution $u(x_1, \ldots, x_n)$ by the inverse function theorem provided

$$\det \left(\frac{\partial x_i}{\partial(s_j, t)}\right) \neq 0, \quad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n-1.$$

Let us consider a linear example with $n = 3$.

Example 3

Let us solve the Cauchy problem $u_x + xu_y - u_z = u$ with $u(x, y, 1) = x + y$. To convert to the notation used in (3), let us replace $x.y, z$ by x_1, x_2, x_3 so our equation becomes $u_{x_1} x_1 u_{x_2} - u_{x_3} = u$ and the initial condition becomes $u(x_1, x_2, 1) = x_1 + x_2$ and we are now free to use the variable z for u and write the characteristic equations as

$$\frac{dx_1}{dt} = 1,\ \frac{dx_2}{dt} = x_1,\ \frac{dx_3}{dt} = -1,\ \frac{dz}{dt} = z.$$

The initial surface Γ is just the hyperplane $x_3 = 1$, $z = x_1 + x_2$, so it is certainly noncharacteristic (since any curve in Γ must have $dx_3/dt = 0$. We parameterize Γ by $x_1 = s_1,\ x_2 = s_2,\ x_3 = 1,\ z = s_1 + s_2$.

We use these as initial conditions to solve our characteristic equations (being careful to integrate x_1 before we attempt x_2) to find

$$x_1 = t + s_1,\ x_2 = (t^2/2) + s_1 t + s_2,\ x_3 = -t + 1,\ z = (s_1 + s_2)e^t.$$

We can then solve for $s_1,\ s_2$, and t and plug into z to find

$$u(x_1, x_2, x_3) = \left(x_1 + x_2 + (x_3 - 1)\left[1 + x_1 + \frac{1}{2}(x_3 - 1)\right]\right) e^{1-x_3}.$$

Notice that the solution exists for all values of $x_1,\ x_2$, and x_3.

For simplicity we restrict our attention in this and the following section to $n = 2$. To solve the Cauchy problem for the quasilinear equation (1) with Γ parameterized by $(f(s), g(s), h(s))$, we solve the characteristic equations (2) with initial conditions (7) to find the integral surface S. The only difference from the semilinear case is that the characteristic equations for dx/dt and dy/dt need not decouple from the dz/dt equation; this means that we must take the z values into account even to find the projected charcteristic curves in the xy-plane. In particular, this allows for the possibility that the projected characteristics may cross each other.

The condition for solving for s and t in terms of x and y is again expressed as (8). In particular, at $t = 0$ this now takes the form

$$f'(s)b(f(s), g(s), h(s)) - g'(s)a(f(s), g(s), h(s)) \neq 0, \tag{13}$$

which is geometrically the condition that the tangent to Γ and the vector field (a, b, c) along Γ project to vectors in the xy-plane that are nowhere parallel. By continuity, we have the following: Provided Γ satisfies (13), there is a unique solution $u(x, y)$ of the Cauchy problem for (1), at least in a neighborhood of Γ. As in the semilinear case, away from Γ (i.e. for larger values of t) the solution may develop singularities. Geometrically. this may be due to the integral surface folding over on itself at some point (1.1) (see Figure 5). In such a case, the solution experiences a "gradient catastrophe" (i.e., u_x, becomes infinite) as $(x, y) \to (x_1, y_1)$, and thereafter the solution u cannot be both single valued and continuous.

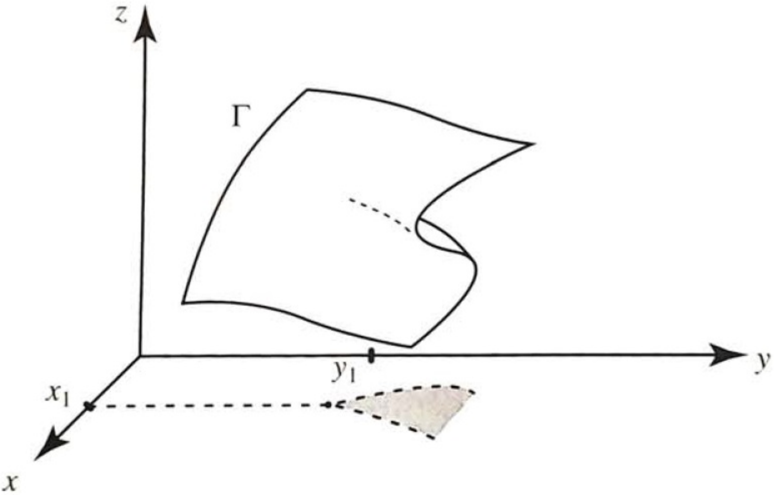

Figure 5. The integral surface experiences a gradient catastrophe.

Example 4

Let us solve the Cauchy problem $uu_x + yu_y = x$ with $u(x,1) = 2x$. The characteristic equations are

$$\frac{dx}{dt} = z, \ \frac{dy}{dt} = y, \ \frac{dz}{dt} = x,$$

and Γ may be parameterized by $(s, 1, 2s)$. We easily check that (13) is satisfied: $1 \neq 0$. Notice that the characteristic equation for y happens to decouple and may be integrated to obtain $y = c(s)e^t$, and the initial condition then yields $y = e^t$. The equations for x and z form a 2×2 system, which may be solved by finding eigenvalues and eigenvectors, or we can simply observe that

$$\frac{d(x+z)}{dt} = x + z, \quad \frac{d(x-z)}{dt} = -(x-z),$$

which yield

$$x + z = c_1(s)e^t, \quad x - z = c_2(s)e^{-t}.$$

Using the initial conditions, we evaluate c_1 and c_2, then solve for x and z:

$$x = \frac{3}{2}se^t - \frac{1}{2}se^{-t},\ y = e^t,\ z = \frac{3}{2}se^t + \frac{1}{2}se^{-t}.$$

Notice that z is defined for all s and t, but if we eliminate s and t in favor of x and y we obtain our solution

$$u(x, y) = x\frac{3y^2 + 1}{3y^2 - 1},$$

which exists for $|y| < 1/\sqrt{3}$: a blow-up singularity has developed at $y = 1/\sqrt{3}$, which is where $x_s y_t - y_s x_t$ vanishes.

We shall encounter in the next subsection an example where the solution develops a gradient catastrophe type of singularity.

e. General Solutions

In ordinary differential equations, an initial value problem is often solved by finding a general solution that depends on an arbitrary constant and then using the initial condition to evaluate the constant. For quasilinear first-order PDEs, a similar process may be achieved by the method of Lagrange, which produces a general solution that depends on an arbitrary function; the Cauchy problem is then solved by evaluating the function.

In order to describe our general solution, let us consider a function $\phi(x, y, z)$ for which $\phi(x, y, z) = \text{const}$ is an integral surface of $\vec{V} = (a, b, c)$. This means, of course, that ϕ is constant along the characteristics (2), and so differentiating $\phi(x(t), y(t), z(t)) = \text{const}$ with respect to t shows that ϕ satisfies

$$a\phi_x + b\phi_y + c\phi_z = 0. \tag{14}$$

Suppose we have another function $\psi(x, y, z)$ which is also constant along the characteristics (2), but is independent of ϕ (i.e., $\nabla\phi$ and $\nabla\psi$ are nowhere colinear, where $\nabla\phi = (\phi_x, \phi_y, \phi_z)$, and $\nabla\psi = (\psi_x, \psi_y, \psi_z)$). Consider the equation

$$F(\phi, \psi) = 0, \tag{15}$$

where F is an arbitrary C^1 function with $F_\phi^2 + F_\psi^2 \neq 0$. Now (15) defines a curve $\mathcal{C}$ in (ϕ, ψ)-space. For each point $(\phi_0, \psi_0) \in \mathcal{C}$, consider the surfaces $S_\phi = \{(x, y, z) : \phi(x, y, z) = \phi_0\}$ and $S_\psi = \{(x, y, z) : \psi(x, y, z) = \psi_0\}$. Suppose S_ϕ and S_ψ intersect; by the independence of ϕ and ψ the intersection will be a curve $\chi \subset \mathbb{R}^3$. Moreover, the characteristic through any point of χ will lie in both S_ϕ and S_ψ and so must coincide with χ; i.e., χ is a characteristic.

As we let (ϕ_0, ψ_0) vary over $\mathcal{C}$ we see that (15) defines a surface in $\mathbb{R}^3$ that is a union of characteristic curves χ of (2) and hence is an integral surface for $\vec{V}$. Therefore, (15) implicitly defines a solution $z = u(x, y)$ of (1).

This means that the problem of finding a general solution of (1) is reduced to finding two independent functions ϕ and ψ that are constant along the characteristics, whose equations we shall write in nonparameterized form

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}.$$

This may require some tricky manipulations, as illustrated in the following.

Example 4 (Revisited)

Let us find a general solution for the equation $uu_x + yu_y = x$. We must find two independent functions constant along the characteristic curves

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}.$$

Notice that the first and third expressions do not contain a y term and so may be solved by integration to find $z^2 = x^2 + C$. The function $\phi = z^2 - x^2$ is clearly constant along the characteristic curve; in fact it is easy to check that $z\phi_x + y\phi_y + x\phi_z = 0$. Thus we have found one of our functions. To find a second independent solution, let us use $\phi = z^2 - x^2 = C_1$ to write $z = \sqrt{x^2 + C_1}$. Substituting into the first equation, we have eliminated z and so we obtain the ordinary differential equation

$$\frac{dx}{\sqrt{x^2 + C_1}} = \frac{dy}{y}.$$

This may be integrated to find $\ln|x+\sqrt{x^2+C_1}|+C_2=\ln|y|$ or

$$C_3=\left|\frac{y}{x+\sqrt{x^2+C_1}}\right|.$$

But recalling $z=\sqrt{x^2+C_1}$ we have found that $\psi(x,y,z)=y(x+z)^{-1}$ is a constant. It is easy to check that $z\psi_x+y\psi_y+x\psi_z=0$, so we have found our second independent solution. The general solution can then be written

$$F\left(u^2-x^2,\frac{y}{x+u}\right)=0,$$

where $F(\phi,\psi)$ is an arbitrary C^1 function satisfying $F_\phi^2+F_\psi^2\neq 0$. Assuming $F_\phi\neq 0$. we may use the implicit function theorem to write (15) as $\phi=f(\psi)$, where f is an arbitrary C^1 function.
In this example, then, we could also write the general solution as

$$u^2=f\left(\frac{y}{x+u}\right)+x^2.$$

We shall next consider an example of a quasilinear equation that arises in applications; it also gives the simplest example of a gradient catastrophe.

Example 5 (Invisid Burgers Equation)

Suppose a one-dimensional stream of particles is in motion, each particle having constant velocity. We may interpret this as a velocity field $u(x, y)$: y denotes time, and $u(x, y)$ gives the velocity of the particle at position x at time y. If we follow an individual particle, we get a function $x(y)$ for which $u(x(y), y)$ remains constant. Differentiating this with respect to y, we obtain a quasilinear equation

$$uu_x + u_y = 0. \tag{17}$$

Suppose the velocity field is known initially: $u(x, 0) = h(x)$. We want to solve the Cauchy problem for (17) with Γ parameterized as $(s, 0, h(s))$. The characteristic equations are

$$\frac{dx}{dt} = z, \ \frac{dy}{dt} = 1, \ \frac{dz}{dt} = 0.$$

We may integrate these equations to find $z(s, t) = c_1(s)$, $y(s, t) = c_2(s) + t$, and $x(s, t) = c_1(s)t + c_3(s)$.

Invoking the initial conditions, we find that $x = h(s)t + s$, $y = t$, and $z = h(s)$. Thus $s = x - zy$ so that

$$z = h(x - zy) \tag{18}$$

defines our solution $z = u(x, y)$ implicitly.
Of course we can also approach the problem by finding the general solution of (17) and then invoking the initial condition to find the particular solution. In this approach, we must find ϕ and ψ constant on the curves

$$\frac{dx}{z} = \frac{dy}{1} = \frac{dz}{0}.$$

The last expression implies that z is constant along the characteristics, so we can take $\phi(x, y, z) = z$ (notice that $u\phi_x + \phi_y = 0$). Treating z as a constant in the first equation, $dx/z = dy$ implies $x = zy + C$. If we take $\psi(x, y, z) = x - zy$, we find that ψ is constant along characteristics.

If we solve for in $F(\phi, \psi) = 0$, we can write $\phi = f(\psi)$ where f is an arbitrary function. In other words, our general solution is $z = f(x - zy)$. which defines u implicitly. The initial condition shows that $f(\psi) = h(\psi)$ so we again obtain (18).

Let us now investigate the possibility that the solution experiences a gradient catastrophe type of singularity. The projected characteristic curve beginning at the point s_1 on the x-axis is parameterized by $(h(s_1)t + s_1, t))$: this is a straight line in the xy-plane having "slope" $dx/dy = h(s_1)$: moreover, z has the constant value $h(s)$ along this line. If s_2 is another point on the x-axis with $s_1 < s_2$ but $h(s_1) > h(s_2)$, then the projected characteristic curves beginning from s_1 and s_2 will intersect at

$$y = t = \frac{s_2 - s_1}{h(s_1) - h(s_2)} > 0.$$

Since z is constant along both curves but has different initial values, the integral surface has folded over on itself. Thus if $h'(s_0) < 0$ for any s_0, the solution $u(x, y)$ fails to exist globally, and at the point where (8) fails. the solution suffers a gradient catastrophe type of singularity. On the other hand. if $h'(s) > 0$ for all s, then the characteristics emanating from distinct points s_1 and s_2 on the x-axis will not intersect for positive values of $y = t$. and so the solution $u(x, y)$ will exist globally for $y > 0$.
Given the original physical model behind the equation (17), we can interpret these results as follows: If the initial velocities of the particles form a nondecreasing function of position. then the particles will spread out in a smooth fashion: but if the initial velocities are somewhere decreasing, then the stream of particles will undergo a "shock" that corresponds to collisions of converging particles.

Referrence:

R.C. McOwen: Partial Differential Equations: Methods and Applications, Prentice Hall, 1996.