Linear Algebra Notes- Semester III

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November 6, 2025

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Vector Spaces.

System of Linear Equations.

General form:

Consider a system in n variables, containing m equations;

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Matrix representation for this system is:
Ax = b

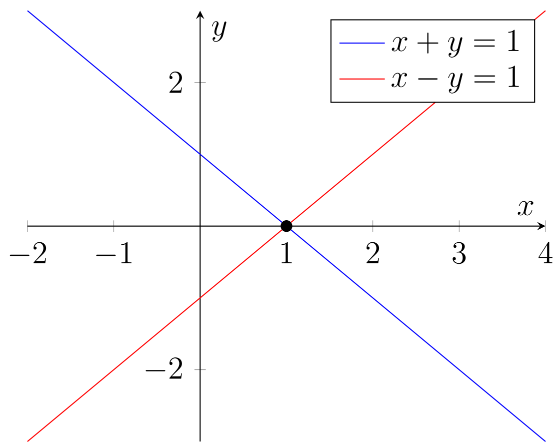
$$
\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

This system can be consistent or inconsistent:

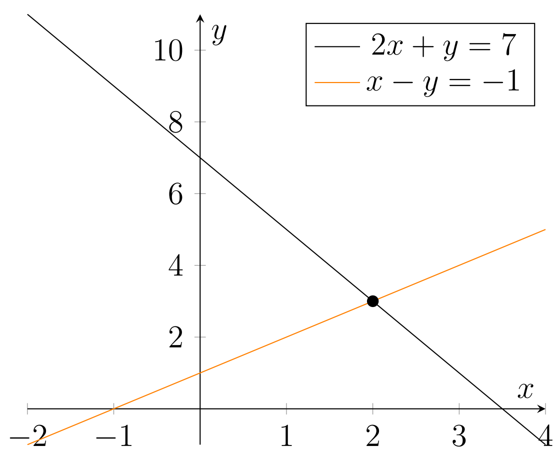
Consistent:

A system of equations is said to be consistent if it has at least one solution.
⋆ **Unique solution:** A system is said to have unique solution when the equations are linearly independent.
• *Example:*
x + y = 1, x - y = 1.
By solving we get the solution, (x,y) = (1,0).

Graphically,

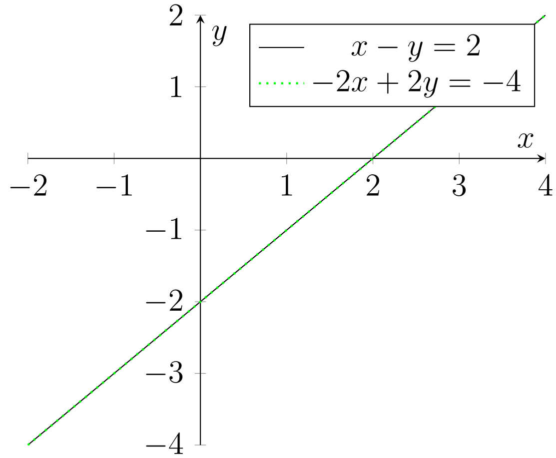


• *Example:*
2x + y = 7, x - y = -1.
By solving we get the solution, (x,y) = (2,3).
Graphically,



⋆ **Infinite solutions:** A system is said to have infinite solutions when the equations are linearly dependent.
• *Example:*
x - y = 2, -2x + 2y = -4.

Graphically,



Inconsistent:

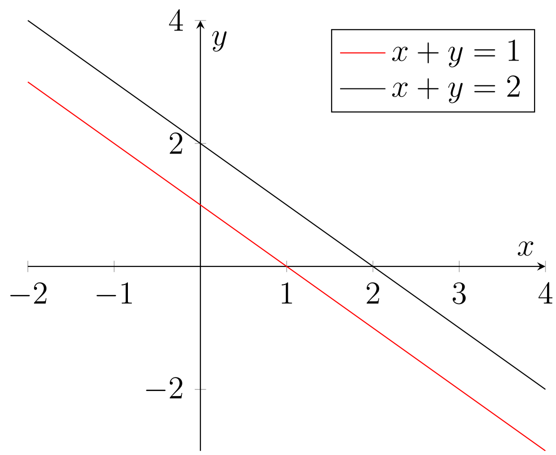
A system is said to be inconsistent if it doesn't have any solutions.

⋆ **No solution:** A system is said to have no or zero solution if it's equations represent parallel lines or parallel planes.

• *Example:*

x + y = 1, x + y = 2.

Graphically,



Row Echelon Form(REF):

A matrix is said to be in REF if:
– All zero rows are at the bottom.
– The leading entry of each row appears to the right of the leading entry in the row directly above it.
– All entries below a leading entry are zero.

Note: If the leading entry is 1 and is the only non-zero entry in its column, then it is called "Reduced Row Echelon Form (RREF)."

Rank and Nullity:

$\star$ **Rank:**
Rank is the number of linearly independent rows in the matrix.

$\boxed{\text{Rank} = \text{Number of non-zero rows in REF}}$

$\star$ **Nullity:**
Nullity is the number of linearly dependent rows in a matrix.

$\boxed{\text{Nullity} = \text{Number of zero rows in REF}}$

$\star$ **Inverse:** A matrix is invertible if its nullity = 0. That is, if it has full rank.

$\boxed{\therefore \text{Rank} = \text{Order}}$

• **Examples:**
Q1.Convert the following into row echelon form and check invertibility.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$\to$

$$R_2 - 4R_1 \ \& \ R_3 - 7R_1 \to A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

$$R_2/(-3) \ \& \ R_3/(-6) \to A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_3 - R_2 \rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\therefore \text{The row echelon form of A is: } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

$\boxed{\text{Since A is not in full rank; A is not invertible.}}$

Q2.Consider the given matrix and solve the following:

1. $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

a. Find det(B).

$$\rightarrow det(B) = 1 \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} - 2 \begin{bmatrix} 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{bmatrix} + 3 \begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} = 0$$

$\boxed{\therefore det(B) = 0}$

b.Reduce B to row echelon form.

$\rightarrow$

$$R_2 - 2R_1, R_3 - 3R_1, R_4 - 4R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$$R_2/(-1), R_3/(-2), R_4/(-3) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$R_3 - R_2, R_4 - R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\therefore \text{The row echelon form is: } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

c. State the rank and nullity.

$\rightarrow$ $\boxed{\text{Rank} = 2 \text{ and Nullity} = 2.}$

d. Is it invertible?

$\rightarrow$ $\boxed{\text{Since, B is not in full rank; it is not invertible.}}$

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2. $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

a. Find det(C).

$$\rightarrow \det(\mathrm{B}) = 1 \begin{bmatrix} 4 & 6 & 8 \\ 6 & 9 & 12 \\ 5 & 6 & 7 \end{bmatrix} - 2 \begin{bmatrix} 2 & 6 & 8 \\ 3 & 9 & 12 \\ 4 & 6 & 7 \end{bmatrix} + 3 \begin{bmatrix} 2 & 4 & 8 \\ 376 & 12 & \\ 4 & 5 & 7 \end{bmatrix} - 4 \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 5 & 6 \end{bmatrix} = 0$$

$\boxed{\therefore \det(\mathrm{C}) = 0}$

b. Reduce C to row echelon form.

$\rightarrow$

$$R_2 - 2R_1, R_3 - 3R_1, R_4 - 4R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -12 & -9 \end{bmatrix}$$

$$R_4/(-3), R_2 \leftrightarrow R_4 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\therefore \text{The REF is: } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

c. State the rank and nullity.

$\rightarrow$ $\boxed{\text{Rank = 2 and Nullity = 2.}}$

d. Is it invertible?
$\rightarrow$ $\boxed{\text{Since, B is not in full rank; it is not invertible.}}$

$\star$ **Find Rank using determinants of sub-matrices:**
– Examine the largest possible square sub-matrices.
– Calculate the determinant.
– If its determinant is non-zero then, the rank of original matrix is equal to the size of that sub-matrix.
– If determinant is zero then, move to smaller sub-matrices and repeat the process.
–Continue until you find a sub-matrix with a non-zero determinant.
• *Example:* Find the rank of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$\rightarrow$
Largest possible sub-matrix is of order 3x3, that is the original matrix.
det(A) = 1(45-48) -2(36-42) + 3(32-35) = 0
$\therefore$ det(A) = 0.
Now, consider the 2x2 matrix,

$$B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

det(B) = 1(5) - 2(4) = -3 $\neq$ 0
$\boxed{\therefore \text{Rank} = 2}$

Important:

To find solutions we can analyze rank as,
Consider a augmented matrix [A|b],
– If rank[A|b] $\neq$ rank[A] then, the system is inconsistent. $\rightarrow$ No solution.
– If rank[A|b] = rank[A] then, the system is consistent.
$\implies$ rank[A|b] = rank[A] = Number of variables $\rightarrow$ Unique solution.
$\implies$ rank[A|b] = rank[A] < Number of variables $\rightarrow$ Infinite solutions.

$\star$ **Examples:**

Q1. Find rank and possible solutions of the following:

a) x + 2y + 3z = 0, 4x + 5y + 6z = 0, 7x + 8y + 9z = 0.

$\rightarrow$

Augmented matrix is,

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix}$$

$$R_2 - 4R_1, R_3 - 7R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{bmatrix}$$

$$R_2/(-3), R_3/(-6) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$R_3 - R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank[A|b] = rank(A) = 2 < Number of variables

$\therefore$ This system has infinite solutions.

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b) x + y = 2, x + y = 5. $\rightarrow$

Augmented matrix is,

$$[A \mid b] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 5 \end{bmatrix}$$

$$R_2 - R_1 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

rank[A | b] $\neq$ rank(A)

$\boxed{\therefore \text{System has no solution.}}$

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c) x + 2y + 3z = 10, 4x + 5y + 6z = 20, 7x + 8y + 9z = 30

$\rightarrow$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 20 \\ 7 & 8 & 9 & 30 \end{bmatrix}$$

$$R_2 - 4R_1, R_3 - 7R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -20 \\ 0 & -6 & -9 & -40 \end{bmatrix}$$

$$R_3 - 2R_2, (-1)R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & 3 & 6 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the echelon form,
rank[A | b] = rank(A) = 2 < Number of variables = 3.
$\therefore$ This system has infinitely many solutions.
We get,
x + 2y + 3z = 10, 3y + 6z = 20.
Variable z is a free variable so, let z = t.

$$\boxed{x = \frac{-10+3t}{3}, y = \frac{20-6t}{3}, z = t; t \in \mathbb{R}}$$

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d) 4x + 2y + 2z = 1, 2x + 4y + 2z = 1, 2x + 2y + 4z = 1.
$\rightarrow$
Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 2 & 1 \\ 2 & 2 & 4 & 1 \end{bmatrix}$$

$$2R_2 - R_1, 2R_3 - R_1 \rightarrow \begin{bmatrix} 4 & 2 & 2 & 1 \\ 0 & 6 & 2 & 1 \\ 0 & 2 & 6 & 1 \end{bmatrix}$$

$$3R_3 - R_2 \rightarrow \begin{bmatrix} 4 & 2 & 2 & 1 \\ 0 & 6 & 2 & 1 \\ 0 & 0 & 16 & 2 \end{bmatrix}$$

From echelon form,
rank[A | b] = rank(A) = Number of variables.
$\therefore$ This system has a unique solution.
So, by solving simultaneously we get,

$$\boxed{x = y = z = \frac{1}{8}}$$

Q2. Find values of λ and μ to get no/unique/infinite.
a) $-2x_1 + 3x_2 - x_3 = 2, 3x_1 + x_2 + 2x_3 = -3, x_1 + 4x_2 + \lambda x_3 = \mu.$
$\rightarrow$
Matrix form of given system is,

$$\begin{bmatrix} -2 & 3 & -1 \\ 3 & 1 & 2 \\ 1 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ \mu \end{bmatrix}$$

Consider the augmented matrix [A | b] = $\begin{bmatrix} -2 & 3 & -1 & 2 \\ 3 & 1 & 2 & -3 \\ 1 & 4 & \lambda & \mu \end{bmatrix}$

$2R_2 + 3R_1, 2R_3 + R_1 \rightarrow \begin{bmatrix} -2 & 3 & -1 & 2 \\ 0 & 11 & 1 & 0 \\ 0 & 11 & 2\lambda - 1 & 2\mu + 2 \end{bmatrix}$

$R_3 - R_2 \rightarrow \begin{bmatrix} -2 & 3 & -1 & 0 \\ 0 & 11 & 1 & 0 \\ 0 & 0 & 2\lambda - 2 & 2\mu + 2 \end{bmatrix}$

$R_3/2 \rightarrow \begin{bmatrix} -2 & 3 & -1 & 0 \\ 0 & 11 & 1 & 0 \\ 0 & 0 & \lambda - 1 & \mu + 1 \end{bmatrix}$

$\therefore$ The row echelon form is: $\begin{bmatrix} -2 & 3 & -1 & 0 \\ 0 & 11 & 1 & 0 \\ 0 & 0 & \lambda - 1 & \mu + 1 \end{bmatrix}$

I. For no solution, rank[A | b] $\neq$ rank(A)

λ - 1 = 0 and $\mu + 1 \neq 0$

$\boxed{\therefore \lambda = 1, \mu \neq -1}$

II. For unique solution, rank[A | b] = rank(A) = Number of variables

$\lambda - 1 \neq 0$ and $\mu \in \mathbb{R}$

$\boxed{\therefore \lambda \neq 1, \mu \in \mathbb{R}}$

III. For infinite solutions, rank[A | b] = rank(A) $<$ Number of variables

$\lambda - 1 = 0$ and $\mu + 1 = 0$

$\boxed{\therefore \lambda = 1, \mu = -1}$

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b) $-x_1 + x_2 - x_3 = -2, \lambda x_1 + x_2 - x_3 = \mu, 2x_1 - x_2 + x_3 = 3.$

$\rightarrow$

Interchange rows 2 and 3, take x_1 as last column.

Matrix representation after the change,

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 2 \\ 1 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ \mu \end{bmatrix}$$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 1 & -1 & -1 & -2 \\ -1 & 1 & 2 & 3 \\ 1 & 4 & \lambda & \mu \end{bmatrix}$$

$$R_2 + R_1, R_3 - R_1 \rightarrow \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda + 1 & \mu + 2 \end{bmatrix}$$

$\therefore$ The row echelon form is: $\begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda + 1 & \mu + 2 \end{bmatrix}$

I. For no solution, rank[A | b] $\neq$ rank(A)

$\lambda + 1 = 0$ and $\mu + 2 \neq 0$

$\boxed{\therefore \lambda = -1, \mu \neq -2}$

II. For unique solution, rank[A | b] = rank(A) = Number of variables

$\lambda + 1 \neq 0$ and $\mu \in \mathbb{R}$

$\boxed{\therefore \lambda \neq -1, \mu \in \mathbb{R}}$

III. For infinite solutions, rank[A | b] = rank(A) $<$ Number of variables

$\lambda + 1 = 0$ and $\mu + 2 = 0$

$\boxed{\therefore \lambda = -1, \mu = -2}$

Geometric interpretation of solution of system of linear equations:

The intersection of the equations in a system leads to a point or line or plane.

• *Example:* Find the intersection set for the following system;

$x_1 - x_2 + x_3 = 0, 2x_1 + x_2 - x_3 = 0$

$\rightarrow$

Matrix representation is,

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix} \rightarrow R_2 - 2R_1 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

From the echelon form we get,

$x_1 = 0, x_1 = t, x_2 = t$

Since, x_2 and x_3 are free variables.

$\boxed{L = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2 = t, x_3 = t \text{is the required intersection. It is a line.}}$

Field

A Field is a set of numbers (or other mathematical objects) on which addition, subtraction, multiplication or division($\neq 0$) is possible similarly to real numbers.
$\implies$ Vector spaces are defined over these fields.
$\star$ **Properties of a Field:** $\mathbb{K}$ is said to be a field if it satisfies;
– Closure:
Let x,y $\in \mathbb{K}$ then x + y and xy $\in \mathbb{K}$
– Commutativity:
For x and y $\in \mathbb{K}$; x + y = y + x and xy = yx
– Associativity:
For x, y, and z $\in \mathbb{K}$; (x + y) + z = x + (y + z) and (xy)z = x(yz)
– Distributivity:
For x, y, and z $\in \mathbb{K}$; x(y + z) = xy + xz
–Identity elements:
Unique additive(0) and multiplicative(1) identity elements should be present in $\mathbb{K}$. That is, x + 0 = x and x*1 = x $\forall x \in \mathbb{K}$
–Inverse:
Additive(-x) and multiplicative($\frac{1}{x}$) inverse should exists. That is, x + (-x) = 0 and x*($\frac{1}{x}$) = 1.

Some examples of Field are:
$\mathbb{R}, \mathbb{C}, M_{2\times 2}(\mathbb{R})$,etc.

Vector Spaces

$\mathbb{V}$ is said to be a vector space over field $\mathbb{K}$ if it satisfies the following conditions:
For $\vec{u}, \vec{v}, \vec{w} \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{K}$,
✓ Closure under addition: $\vec{u} + \vec{v} \in \mathbb{V}$.
✓ Commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
✓ Associative: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
✓ Existence of zero element: $\exists \vec{0}$ such that, $\vec{u} + \vec{0} = \vec{u}$
✓ Additive inverse: For any $\vec{u} \in \mathbb{V}, \exists \vec{v}$ such that $\vec{u} + \vec{v} = 0$.
✓ Scalar product closure: $\alpha.\vec{u} \in \mathbb{V}$.
✓ Distributivity for vectors: $\alpha(\vec{u} + \vec{v}) = \alpha.\vec{u} + \alpha.\vec{v}$.

✓ Distributivity for scalars (in addition): $(\alpha+\beta)\vec{u} = \alpha.\vec{u} + \beta.\vec{u}$.
✓ Distributivity for scalars (in multiplication): $\alpha(\beta.\vec{u}) = (\alpha.\beta).\vec{u}$
✓ Existence of 1: For $1 \in \mathbb{K}$, $1.\vec{u} = \vec{u}$

$\star$ **Examples:**

Q1. Check whether the following are vector spaces or not:

a) $\mathbb{V} = \mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R}^2 \right\}$ **over** $\mathbb{R}$

$\rightarrow$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ be arbitrary elements of $\mathbb{R}^2$.
α and β be two scalars from $\mathbb{R}$.
Check the conditions,

I. Closure under addition:
$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$
Since, $x_i, y_i \in \mathbb{R}; x_i + y_i \in \mathbb{R}$.
$\therefore \vec{x} + \vec{y} \in \mathbb{V}$.

II. Commutative:
$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ and $\vec{y} + \vec{x} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \end{bmatrix}$
Since, $x_i, y_i \in \mathbb{R}; x_i + y_i = y_i + x_i$.
$\therefore \vec{x} + \vec{y} = \vec{y} + \vec{x}$.

III. Associative:
$(\vec{x} + \vec{y}) + \vec{z} = \begin{bmatrix} (x_1 + y_1) + z_1 \\ (x_2 + y_2) + z_2 \end{bmatrix}$ and $\vec{x} + (\vec{y} + \vec{z}) = \begin{bmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2) \end{bmatrix}$
Since, $x_i, y_i, z_i \in \mathbb{R}; (x_i + y_i) + z_i = x_i + (y_i + z_i)$.
$\therefore (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$.

IV. Existence of zero element:
$\vec{x} + \vec{0} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
Since, $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x_i + 0 = x_i$.
$\therefore \exists \vec{0}$ such that, $\vec{x} + \vec{0} = \vec{x}$

V. Additive inverse:
$\vec{x} + \vec{-x} = \begin{bmatrix} x_1 + (-x_1) \\ x_2 + (-x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$
$\therefore$ For any $\vec{x} \in \mathbb{V}, \exists(\vec{-x})$ such that $\vec{x} + \vec{-x} = 0$.

VI. Scalar product closure:

$\alpha.\vec{x} = \begin{bmatrix} \alpha.x_1 \\ \alpha.x_2 \end{bmatrix}$
Since, $x_i \in \mathbb{R}$ and $\alpha \in \mathbb{R} \implies \alpha.x_i \in \mathbb{R}$
$\therefore \alpha.\vec{x} \in \mathbb{V}$.
VII. Distributivity for vectors:
$\alpha(\vec{x}+\vec{y}) = \begin{bmatrix} \alpha.(x_1+y_1) \\ \alpha.(x_2+y_2) \end{bmatrix} = \begin{bmatrix} \alpha.x_1+\alpha.y_1 \\ \alpha.x_2+\alpha.y_2 \end{bmatrix} = \alpha.\vec{x}+\alpha.\vec{y}$
$\therefore \alpha(\vec{x}+\vec{y}) = \alpha.\vec{x}+\alpha.\vec{y}$.
VIII. Distributivity for scalars (in addition):
$(\alpha+\beta)\vec{x} = \begin{bmatrix} (\alpha+\beta)x_1 \\ (\alpha+\beta)x_2 \end{bmatrix} = \begin{bmatrix} \alpha.x_1+\beta.x_1 \\ \alpha.x_2+\beta.x_2 \end{bmatrix}$
$\therefore (\alpha+\beta)\vec{x} = \alpha.\vec{x}+\beta.\vec{x}$.
IX. Distributivity for scalars (in multiplication):
$\alpha(\beta.\vec{x}) = \begin{bmatrix} \alpha(\beta.x_1) \\ \alpha(\beta.x_2) \end{bmatrix} = (\alpha.\beta).\vec{x}$
$\therefore \alpha(\beta.\vec{x}) = (\alpha.\beta).\vec{x}$
X. Existence of 1:
$\vec{1}.\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
Since, $\vec{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
For $1 \in \mathbb{R}, 1.\vec{x} = \vec{x}$
Since, all the conditions are satisfied.

$\boxed{\text{The given set } \mathbb{V} = \mathbb{R}^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \text{is a vector space over } \mathbb{R}}$

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b) $\mathbb{V} = \{a_0 + a_1.x + a_2.x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ **over** $\mathbb{R}$
$\rightarrow$
Let $\vec{a} = a_0 + a_1.x + a_2.x^2$, $\vec{b} = b_0 + b_1.x + b_2.x^2$, $\vec{c} = c_0 + c_1.x + c_2.x^2$ be arbitrary elements of $\mathbb{V}$.
α and β be two scalars from $\mathbb{R}$.
Check the conditions,
I. Closure under addition:
$\vec{a}+\vec{b} = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$
Since, $a_i, b_i \in \mathbb{R}; a_i+b_i \in \mathbb{R}$.
$\therefore \vec{a}+\vec{b} \in \mathbb{V}$.
II. Commutative:

$\vec{a}+\vec{b}=(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$ and $\vec{b}+\vec{a}=(b_0+a_0)+(b_1+a_1)x+(b_2+a_2)x^2$

Since, $a_i, b_i \in \mathbb{R}; a_i+b_i=b_i+a_i$.

$\therefore \vec{a}+\vec{b}=\vec{b}+\vec{a}$.

III. Associative:

$(\vec{a}+\vec{b})+\vec{c}=((a_0+b_0)+c_0)+((a_1+b_1)+c_1)x+((a_2+b_2)+c_2)x^2$ and $\vec{a}+(\vec{b}+\vec{c})=(a_0+(b_0+c_0))+(a_1+(b_1+b_2))x+(a_2+(b_2+b_3))x^2$

Since, $a_i, b_i, c_i \in \mathbb{R}; (a_i+b_i)+c_i=a_i+(b_i+c_i)$.

$\therefore (\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.

IV. Existence of zero element:

$\vec{a}+\vec{0}=(a_0+0)+(a_1+0)x+(a_2+0)x^2$

Since, $\vec{0}=0+0x+0x^2$ and $a_i+0=a_i$.

$\therefore \exists\vec{0}$ such that, $\vec{a}+\vec{0}=\vec{a}$

V. Additive inverse:

$\vec{a}+\vec{-a}=(a_0-a_0)+(a_1-a_1)x+(a_2-a_2)x^2=0$

$\therefore$ For any $\vec{a}\in\mathbb{V}, \exists(\vec{-a})$ such that $\vec{a}+\vec{-a}=0$.

VI. Scalar product closure:

$\alpha.\vec{a}=\alpha.a_0+\alpha.a_1x+\alpha.a_2x^2$

Since, $a_i\in\mathbb{R}$ and $\alpha\in\mathbb{R} \implies \alpha.a_i\in\mathbb{R}$

$\therefore \alpha.\vec{a}\in\mathbb{V}$.

VII. Distributivity for vectors:

$\alpha(\vec{a}+\vec{b})=\alpha(a_0+b_0)+\alpha(a_1+b_1)x+\alpha(a_2+b_2)x^2=(\alpha.a_0+\alpha.a_1x+\alpha.a_2.x^2)+(\alpha.b_0+\alpha.b_1.x+\alpha.b_2.x^2)=\alpha,\vec{a}+\alpha.\vec{b}$

$\therefore \alpha(\vec{a}+\vec{b})=\alpha.\vec{a}+\alpha.\vec{b}$.

VIII. Distributivity for scalars (in addition):

$(\alpha+\beta)\vec{a}=(\alpha+\beta)a_0+(\alpha+\beta)a_1.x+(\alpha+\beta)a_2.x^2$

$\therefore (\alpha+\beta)\vec{a}=\alpha.\vec{a}+\beta.\vec{a}$.

IX. Distributivity for scalars (in multiplication):

$\alpha(\beta.\vec{a})=\alpha(\beta.a_0+\beta.a_1.x+\beta.a_2.x^2)=\alpha.\beta(a_0+a_1.x+a_2.x^2)=\alpha.\beta.\vec{a}$

$\therefore \alpha(\beta.\vec{a})=(\alpha.\beta).\vec{a}$

X. Existence of 1:

$\vec{1}.\vec{a}=a_0+a_1.xa_2.x^2$

Since, $\vec{1}=1$

For $1\in\mathbb{R}, 1.\vec{a}=\vec{a}$

Since, all the conditions are satisfied.

$\boxed{\text{The given set } \mathbb{V}=\left\{a_0+a_1.x+a_2.x^2 \mid a_0,a_1,a_2\in\mathbb{R}\right\} \text{ is a vector space over } \mathbb{R}}$

Vector Subspace

For a vector space($\mathbb{V}$) over a filed($\mathbb{K}$). A subset $\mathbb{S}$ of $\mathbb{V}$ is said to be a vector subspace of $\mathbb{V}$ if $\mathbb{S}$ is a vector-subspace over the same field $\mathbb{R}$.

$\star$ **Properties of a vector subspace:**

For $\vec{u}, \vec{v} \in \mathbb{S}$ and $\alpha \in \mathbb{K}$.

– Closure under vector addition: $\vec{u} + \vec{v} \in \mathbb{S}$

– Closure under scalar multiplication: $\alpha.\vec{u} \in \mathbb{S}$.

Note: Every vector space is a subspace of itself. Zero set is a subspace of all vector spaces.

Examples:

Q1. Show that the following are vector subspace of $\mathbb{V}$ over $\mathbb{R}$.

$\mathbb{S} = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \right\}$ over $\mathbb{V} = M_{2\times2}(\mathbb{R})$.

$\rightarrow$

Here the given vector space, its subset and field is;

$\mathbb{V} = M_{2\times2}(\mathbb{R})$, $\mathbb{S} = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \right\}$ and $\mathbb{R}$

Now consider, $\vec{u} = \begin{bmatrix} u & u \\ u & u \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v & v \\ v & v \end{bmatrix} \in \mathbb{S}$.

Consider some scalar, $\alpha \in \mathbb{R}$.

I. Closure under vector addition:

$\vec{u} + \vec{v} = \begin{bmatrix} u+v & u+v \\ u+v & u+v \end{bmatrix}$

Since, $u, v \in \mathbb{R}; u + v \in \mathbb{R}$

$\therefore \vec{u} + \vec{v} \in \mathbb{S}$.

II. Closure under scalar multiplication:

$\alpha.\vec{u} = \begin{bmatrix} \alpha.u & \alpha.u \\ \alpha.u & \alpha.u \end{bmatrix}$

Since, $\alpha, u \in \mathbb{R}; \alpha.u \in \mathbb{R}$

$\therefore \alpha.\vec{u} \in \mathbb{S}$.

Since both the conditions are satisfied.

$\boxed{\therefore \mathbb{S} \text{ is a vector subspace of } \mathbb{V} \text{ over } \mathbb{R}}$.

Q2. State the subspaces of the following.

a) $P_n(\mathbb{K}) = \{a_0 + a_1x + ... + a_nx^n \mid a_i \in \mathbb{K}\}$

$\rightarrow$
Subspaces of $P_n(\mathbb{K})$ are:
– $P_n(\mathbb{K})$
– $\vec{0} = \{0 + 0x + ... + 0x^n\}$
– Even functions: $S_0 = \{p(x) \in P(\mathbb{K}) \mid p(-x) = p(x)\}$
– Odd functions: $S_1 = \{p(x) \in P(\mathbb{K}) \mid p(-x) = -p(x)\}$

b) $\mathbb{V} = \mathbb{R}^2$
$\rightarrow$
Subspaces of $\mathbb{R}^2$ are:
– $\mathbb{R}^2$
– $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
– Line passing through origin: $S_0 = \{cv \mid c \in \mathbb{R}\} = \{(ca, cb) \mid c \in \mathbb{R}\}$

c) $\mathbb{V} = \mathbb{R}^3$
$\rightarrow$
Subspaces of $\mathbb{R}^3$ are:
– $\mathbb{R}^3$
– $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
– Line passing through origin: $S_0 = \{tv \mid t \in \mathbb{R}\} = \{(ta, tb, tc) \mid t \in \mathbb{R}\}$
– Plane through origin: $S_1 = \{c_1v_1 + c_2v_2 \mid c_1, c_2 \in \mathbb{R}\}$
Here, v_1, v_2 are non-collinear vectors that lie on the plane.

Span of Vectors

The span of two vectors means all the different results you can get by scaling and adding them together. That is, the set of all possible linear combinations.
$span\,\{\vec{u}, \vec{v}\} = \{\alpha\vec{u} + \beta\vec{v} \mid \alpha, \beta \in \mathbb{R}\}$

$\star$ **Examples:**
Q1. Find $span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$
$\rightarrow$

Let, S = $span\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \left\{\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}1\\-1\end{bmatrix} \mid \alpha,\beta\in\mathbb{R}\right\}\subseteq\mathbb{R}^2$

Now, consider an arbitrary point in $\mathbb{R}^2$, $\begin{bmatrix}a\\b\end{bmatrix}$

We need to find values of α and β such that,

$\begin{bmatrix}a\\b\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}1\\-1\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\alpha+\beta\\\alpha-\beta\end{bmatrix}$

So, $\alpha+\beta=a$ and $\alpha-\beta=b$

$\therefore \alpha=\frac{a+b}{2}, \beta=\frac{a-b}{2}$

Thus, $\begin{bmatrix}a\\b\end{bmatrix} = \left(\frac{a+b}{2}\right)\begin{bmatrix}1\\1\end{bmatrix}+\left(\frac{a-b}{2}\right)\begin{bmatrix}1\\-1\end{bmatrix}$

So, any arbitrary point in $\mathbb{R}^2$ can be represented in linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}1\\-1\end{bmatrix}$

$\therefore \begin{bmatrix}a\\b\end{bmatrix} \in span\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

$\therefore \mathbb{R}^2\subseteq S$

From above, S = $\mathbb{R}^2$

$$\boxed{\therefore span\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2}$$

Q2. Find $span\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$

$\rightarrow$

Let, $S = span\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\} = \left\{\alpha_1\begin{bmatrix}1\\2\end{bmatrix}+\alpha_2\begin{bmatrix}1\\1\end{bmatrix}+\alpha_3\begin{bmatrix}0\\1\end{bmatrix}\right\}\subseteq\mathbb{R}^2$

Now, consider an arbitrary element in $\mathbb{R}^2$, $\begin{bmatrix}a\\b\end{bmatrix}$

We need to find values of α_1, α_2 and α_3 such that,

$\begin{bmatrix}a\\b\end{bmatrix} = \alpha_1\begin{bmatrix}1\\2\end{bmatrix}+\alpha_2\begin{bmatrix}1\\1\end{bmatrix}+\alpha_3\begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\alpha_1+\alpha_2\\\alpha_1+2\alpha_2+\alpha_3\end{bmatrix}$

So, $\alpha_1+\alpha_2=a, \alpha_1+2\alpha_2+\alpha_3=b$

Since α_3 is a free variable, let $\alpha_3=t$

By solving we get,

$\alpha_1=2a-b+t, \alpha_2=b-a-t, \alpha_3=t$

Thus, $\begin{bmatrix} a \\ b \end{bmatrix} = (2a - b + t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b - a - t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
So, any arbitrary point in $\mathbb{R}^2$ can be represented in linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\therefore \begin{bmatrix} a \\ b \end{bmatrix} \in span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$
$\therefore \mathbb{R}^2 \subseteq S$
From above, S = $\mathbb{R}^2$

$$\boxed{\therefore span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2}$$

Linear independence and dependence of vectors:

Linear Independence:

Vectors $\vec{u_1}, \vec{u_2}, ..., \vec{u_n}$ in a vector space $\mathbb{V}$ over field $\mathbb{K}$ are said to be linearly independent if,
$\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + ... + \alpha_n\vec{u_n} = 0 \implies \alpha_1 = \alpha_2 = ... = \alpha_n = 0$

Linear Dependence:

Vectors $\vec{u_1}, \vec{u_2}, ..., \vec{u_n}$ in a vector space $\mathbb{V}$ over field $\mathbb{K}$ are said to be linearly dependent if,
$\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + ... + \alpha_n\vec{u_n} = 0 \implies$ at least one $\alpha_i \neq 0$

$\star$ **Examples:**
Q1. Check if following are linearly independent or not:
1] $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\}$
$\rightarrow$
Let, $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$
Consider, $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} = 0$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$\alpha_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + \alpha_2 \begin{bmatrix}4\\5\\6\end{bmatrix} + \alpha_3 \begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Matrix representation of this is,

$\begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$R_2 - 2R_1, R_3 - 3R_1 \rightarrow \begin{bmatrix}1 & 4 & 7\\0 & -3 & -6\\0 & -6 & -12\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$R_2/(-3), R_3/(-6), R_3 - R_2 \rightarrow \begin{bmatrix}1 & 4 & 7\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Since α_3 is a free variable, let $\alpha_3 = t$.

By solving we get,

$\alpha_1 = t, \alpha_2 = -2t, \alpha_3 = t$

$\alpha_1, \alpha_2, \alpha_3$ are non-zero (except when t = 0),

$\boxed{\therefore \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix} \right\} \text{are linearly dependent. Dependence relation is : } t\vec{u_1} - 2t\vec{u_2} + t\vec{u_3} = 0}$

2] $\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}4\\5\end{bmatrix} \right\}$

$\rightarrow$

Let, $\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}4\\5\end{bmatrix} \right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$

Consider, $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} + \alpha_4\vec{u_4} = 0$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$

$\alpha_1 \begin{bmatrix}1\\2\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\3\end{bmatrix} + \alpha_3 \begin{bmatrix}3\\4\end{bmatrix} + \alpha_4 \begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Matrix representation is,

$\begin{bmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 5\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$R_2 - 2R_1 \rightarrow \begin{bmatrix}1 & 2 & 3 & 4\\0 & -1 & -2 & -3\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$R_2/(-1) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since α_3, α_4 is a free variable, let $\alpha_3 = a, \alpha_4 = b$.
By solving we get,
$\alpha_1 = a + 2b, \alpha_2 = -2a - 3b, \alpha_3 = a, \alpha_4 = b$
At least one $\alpha_i \neq 0$.

$$\boxed{\therefore \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\} \text{ are linearly dependent.}}$$

$$\boxed{\text{Dependence relation is:} (a + 2b)\vec{u_1} + (-2a - 3b)\vec{u_2} + a\vec{u_3} + b\vec{u_4} = 0}$$

3] $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

$\rightarrow$

Let, $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

Consider, $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} = 0$ for some $\alpha_1, \alpha_2, \alpha_3$

$$\alpha_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2R_2 - R_1, 2R_3 - R_1 \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$3R_3 - R_1 \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

By solving we get,
$\alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\boxed{\therefore \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \text{ are linearly independent.}}$$

4] $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\rightarrow$

Let, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

Consider, $\alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \alpha_3 \vec{u_3} = 0$ for some $\alpha_1, \alpha_2, \alpha_3$

Matrix representation is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1, R_3 - R_1 \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$(-1)R_2, (-1)R_3, R_2 \leftrightarrow R_3 \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

By solving we get,

$\alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\boxed{\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{are linearly independent.}}$$

Basis

A subset S is said to be a basis of a vector space $\mathbb{V}$ if:

– Vectors in S are linearly independent.

– Vectors in S span vector space $\mathbb{V}$

Dimension of a vector space:

It is the size of basis or maximum number of linearly independent vectors or minimum number of spanning vectors.

⋆ **Example:**
Q1. Check if the following form a basis or not:
$\{1, x, x^2\}$
$\rightarrow$
Let, $\{1, x, x^2\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$
Consider $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} = \vec{0}$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.
$\alpha_1(1) + \alpha_2(x) + \alpha_3(x^2) = 0 + 0x + 0x^2$
$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$
$\therefore \{1, x, x^2\}$ are linearly independent.
Now, consider, $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} = \vec{a}$ for some $\alpha_1, \alpha_2, \alpha_3$ and $\vec{a}$ is an arbitrary element of $\mathbb{V}$.
$\alpha_1(1) + \alpha_2(x) + \alpha_3(x^2) = a_1 + a_2x + a_3x^2$
$\therefore \alpha_1 = a_1, \alpha_2 = a_2, \alpha_3 = a_3$
Since an arbitrary element can be represented in terms of $\vec{u_1}, \vec{u_2}, \vec{u_3}$.
$\therefore \{1, x, x^2\}$ span $\mathbb{V}$.
$\boxed{\therefore \left\{1, x, x^2\right\} \text{ forms a basis of } P_2(x)\text{. Its dimension is 3}}$

Q2. Show that the vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ form a basis of vefctor space $\mathbb{R}^3$.
$\rightarrow$
Let $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$
Consider, $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} = \vec{0}$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

$$\alpha_1\begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\1\end{bmatrix} + \alpha_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2 - R_1 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 - R_2 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

By solving this, we get,

$\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent.

Now, consider, $\alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \alpha_3 \vec{u_3} = \vec{a}$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\vec{a}$ is an arbitrary element of $\mathbb{V}$.

$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Matrix representation is,

$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$R_2 - R_1 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 - a_1 \\ a_3 \end{bmatrix}$

$R_3 - R_2 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 - a_1 \\ a_3 - (a_2 - a_1) \end{bmatrix}$

By solving,

$2\alpha_3 = a_3 - (a_2 - a_1) \implies \alpha_3 = (\frac{a_3 - a_2 + a_1}{2})$

$\alpha_2 - \alpha_3 = (a_2 - a_1) \implies \alpha_2 = (a_2 - a_3) + (\frac{a_3 - a_2 + a_1}{2}) = (\frac{3a_2 - 3a_3 - a_1}{2}) \implies \alpha_2 = (\frac{3a_2 - 3a_3 - a_1}{2})$

$\alpha_1 + \alpha_3 = a_1 \implies \alpha_1 = a_1 - (\frac{3a_3 - 3a_2 - a_1}{2}) = \frac{5a_1 - 3a_2 + a_3}{2} \implies \alpha_3 = \frac{5a_1 - 3a_2 + a_3}{2}$

Since an arbitrary element can be represented in terms of $\vec{u_1}, \vec{u_2}, \vec{u_3}$

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$

$\boxed{\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{forms a basis of vector space } \mathbb{R}^3\text{. Its dimension is 3}}$

Questions:

Q1. Find rank and nullity of the following matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 1 & 0 \\ 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & 1 & 1 & 3 \end{bmatrix}$$

$\rightarrow$

$$R_2 - 2R_1, R_3 - 3R_1, R_4 + R_1 \rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 & -6 \\ 0 & 0 & -3 & -3 & -6 \\ 0 & 0 & 3 & 3 & 6 \end{bmatrix}$$

$$R_3 - R_2, R_4 + R_2 \rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2/(-3) \rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required echelon form, from this,

$\boxed{\therefore \text{Rank} = 2, \text{ Nullity} = 2.}$

Q2. Solve the following system of equations:

$x + y + z = 1, 2x + 2y + 2z = 2, x + y + z = 3$

$\rightarrow$

Matrix representation of the given system is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - R_1 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$R_2 \leftrightarrow R_3 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

From the echelon form,
rank(A) $\neq$ rank[A | b]

$\boxed{\therefore \text{The given system has no solution.}}$

Q3. Solve the following system of equations:
$x + 2y + 3z = 0, 4x + 5y + 6z = 0, 7x + 8y + 9z = 7$
$\rightarrow$
Matrix representation of given system is,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 7 \end{bmatrix}$$

$R_2 - 4R_1, R_3 - 7R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 7 \end{bmatrix}$

$2R_2 - R_3, R_3/(-6) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & (\frac{-7}{6}) \end{bmatrix}$

$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & (\frac{-7}{6}) \\ 0 & 0 & 0 & 0 \end{bmatrix}$

From the echelon form,
rank(A) = rank[A | b] < Number of variables.

$\boxed{\therefore \text{The given system has infinitely many solutions.}}$

Q4. Find the infinitely many solutions:
a) $x_2 - 2x_3 + x_4 = 2, 3x_1 - 2x_2 - 4x_3 = 4, 2x_1 + 3x_3 - x_4 = 5.$
$\rightarrow$
Matrix form of the given system,

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 3 & -2 & -4 & 0 \\ 2 & 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 0 & 1 & -2 & 1 & 2 \\ 3 & -2 & -4 & 0 & 4 \\ 2 & 0 & 3 & -1 & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_1 \rightarrow \begin{bmatrix} 3 & -2 & -4 & 0 & 4 \\ 0 & 1 & -2 & 1 & 2 \\ 2 & 0 & 3 & -1 & 5 \end{bmatrix}$$

$$3R_3 - 2R_1 \rightarrow \begin{bmatrix} 3 & -2 & -4 & 0 & 4 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 4 & -17 & -3 & 7 \end{bmatrix}$$

$$R_3 - 4R_2 \rightarrow \begin{bmatrix} 3 & -2 & -4 & 0 & 4 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 25 & -7 & -1 \end{bmatrix}$$

From the echelon form,

rank(A) = rank[A | b] < Number of variables.

$\therefore$ The given system has infinitely many solutions.

Here x_4 is a free variable so put $x_4 = t$

By solving we get,

$$\boxed{x_1 = \frac{192 - 50t}{25}, x_2 = \frac{48 - 39t}{25}, x_3 = \frac{-1 + 7t}{25}, x_4 = t}$$

---

b) $x_1 - 2x_2 - 3x_3 - x_4 = 2, 2x_1 - x_4 = 5.$

$\rightarrow$

Matrix form of the given system,

$$\begin{bmatrix} 1 & -2 & -3 & -1 \\ 2 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 1 & -2 & -3 & -1 & 2 \\ 2 & 0 & 0 & -1 & 5 \end{bmatrix}$$

$$R_2 - 2R_1 \rightarrow \begin{bmatrix} 1 & -2 & -3 & -1 & 2 \\ 0 & 4 & 6 & 1 & -8 \end{bmatrix}$$

From the echelon form,
rank(A) = rank[A | b] < Number of variables,
$\therefore$ The given system has infinitely many solutions.
Here, x_3, x_4 are free variables so let, $x_3 = a, x_4 = b$.
By solving we get,

$$\boxed{x_1 = \frac{-4+b}{2}, x_2 = \frac{-8-6a-b}{4}, x_3 = a, x_4 = b}$$

---

c) $2x_1 - x_2 + 3x_4 = 2, x_1 - 2x_3 - x_4 = 3$
$\rightarrow$
Matrix form of the given system is,

$$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 2 & -1 & 0 & 3 & 2 \\ 1 & 0 & -2 & -1 & 3 \end{bmatrix}$$

$$2R_2 - R_1 \rightarrow \begin{bmatrix} 2 & -1 & 0 & 3 & 2 \\ 0 & 1 & -4 & -5 & 4 \end{bmatrix}$$

From the echelon form,
rank(A) = rank[A | b] < Number of variables,
$\therefore$ The given system has infinitely many solutions.
Here, x_3, x_4 are free variables so let, $x_3 = a, x_4 = b$.
By solving we get,

$$\boxed{x_1 = 3 + 2a + b, x_2 = 4 + 4a + 5b, x_3 = a, x_4 = b.}$$

Q5. Find the values of λ and μ for following conditions:
$2x + 3y - z = 5, x - y + 2z = 1, -x + 2y + \lambda z = \mu$

- No solution.
- Unique solution.
- Infinitely many solutions.

$\rightarrow$
Matrix form of given system is,

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ \mu \end{bmatrix}$$

Consider the augmented matrix,

$$[A \mid b] = \begin{bmatrix} 2 & 3 & -1 & 5 \\ 1 & -1 & 2 & 1 \\ -1 & 2 & \lambda & \mu \end{bmatrix}$$

$$2R_2 - R_1, 2R_3 - R_1 \rightarrow \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -5 & 5 & -3 \\ 0 & 7 & 2\lambda - 1 & 2\mu + 5 \end{bmatrix}$$

$$5R_3 + 7R_2 \rightarrow \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -5 & 5 & -3 \\ 0 & 0 & 10\lambda + 30 & 10\mu + 4 \end{bmatrix}$$

From the echelon form,

I. No solution: rank(A) $\neq$ rank[A | b]

So, $10\lambda + 30 = 0$ and $10\mu + 4 \neq 0$

$10\lambda = -30$ and $10\mu \neq -4$

$$\boxed{\therefore \lambda = -3, \mu \neq \frac{-2}{5}}$$

II. Unique solution: rank(A) = rank[A | b] = Number of variables.

So, $10\lambda + 30 \neq 0$ and $\mu \in \mathbb{R}$

$10\lambda \neq -30$ and $\mu \in \mathbb{R}$

$$\boxed{\therefore \lambda \neq -3, \mu \in \mathbb{R}}$$

III. Infinite solutions: rank(A) = rank[A | b] $<$ Number of variables.

So, $10\lambda + 30 = 0$ and $10\mu + 4 = 0$

$10\lambda = -30$ and $10\mu = -4$

$$\boxed{\therefore \lambda = -3, \mu = \frac{-2}{5}}$$

Q6. Check if given vectors are linearly independent or not:

a) $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$

$\rightarrow$

Let, $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$

Consider, $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} + \alpha_4\vec{u_4} = 0$ for some $\alpha_i \in \mathbb{R}$

$$\alpha_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - 4R_1, R_3 - 7R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 3 \end{bmatrix}$$

$$R_3 - 2R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Here, α_4 is a free variable, consider $\alpha_4 = t$.

So, there is at least one $\alpha_i \neq 0$(except when t = 0)

$$\boxed{\therefore \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} \text{ are linearly dependent.}}$$

---

b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\rightarrow$

Let, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

Consider, $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3} + \alpha_4\vec{u_4} = 0$ for some $\alpha_i \in \mathbb{R}$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By the echelon form we get, $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

$$\boxed{\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ are linearly independent.}}$$

Q7. Check whether the following spans $\mathbb{R}^3$

a) $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

$\rightarrow$

Consider, $\vec{u_0} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, an arbitrary point in $\mathbb{R}^3$

We need to represent $\vec{u_0}$ in linear combination of $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$.

Matrix representation is,

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 3R_1 \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c-3a \end{bmatrix}$$

$$R_2/(-3), R_3/(-6), R_3 - R_2 \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ (\frac{b-2a}{(-3)}) \\ (\frac{c-3a}{(-6)} - \frac{b-2a}{(-3)}) \end{bmatrix}$$

Here α_3 is a free variable so let, $\alpha_3 = t$.

So by solving we get,

$\alpha_1 = \frac{11a+4b-19t}{3}, \alpha_2 = \frac{-b+2a-6t}{3}, \alpha_3 = t$

$\therefore$ Any point in $\mathbb{R}^3$ is in $span\,\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

$$\boxed{\therefore span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\} = \mathbb{R}^3}$$

---

b) $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}$

$\rightarrow$

Since, $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ are elements in $\mathbb{R}^2$ they don't span $\mathbb{R}^3$.

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\} \text{do not span } \mathbb{R}^3}$$

c) $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

$\rightarrow$

Consider, $\vec{u_0} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, an arbitrary point in $\mathbb{R}^3$

We need to represent $\vec{u_0}$ in linear combination of $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$.
Matrix representation is,

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$2R_2 - R_1, 2R_3 - R_1 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ 2b - a \\ 2c - a \end{bmatrix}$$

$$3R_3 - R_2 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ 2b - a \\ 6c - 2b - 2a \end{bmatrix}$$

By solving we get,
$\alpha_1 = \frac{3a-b-c}{4}, \alpha_2 = \frac{3b-a-c}{4}, \alpha_3 = \frac{3c-b-a}{4}$
$\therefore$ Any point in $\mathbb{R}^3$ is in $span\,\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

$$\boxed{\therefore span \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} = \mathbb{R}^3}$$

---

d) $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\rightarrow$

Consider, $\vec{u_0} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, an arbitrary point in $\mathbb{R}^3$

We need to represent $\vec{u_0}$ in linear combination of $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$.
Matrix representation is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$R_2 - R_1, R_3 - R_1 \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b - a \\ c - a \end{bmatrix}$$

$R_2 \leftrightarrow R_3, R_2/(-1), R_3/(-1) \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ a-c \\ a-b \end{bmatrix}$

So by solving we get,

$\alpha_1 = c, \alpha_2 = b-c, \alpha_3 = a-b$

$\therefore$ Any point in $\mathbb{R}^3$ is in $span\,\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

$$\boxed{\therefore span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^3}$$

---

e) $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\rightarrow$

Consider, $\vec{u_0} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, an arbitrary point in $\mathbb{R}^3$

We need to represent $\vec{u_0}$ in linear combination of $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$.

Matrix representation is,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

So by solving we get,

$\alpha_1 = a, \alpha_2 = b, \alpha_3 = c$

$\therefore$ Any point in $\mathbb{R}^3$ is in $span\,\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

$$\boxed{\therefore span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3}$$

Q8. Check if the following is a basis of $\mathbb{P}_2[x]$

$y_1 = 1 + x, y_2 = x + x^2, y_3 = 1 + x^2$

$\rightarrow$

I. Linear independence:

Consider, $\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0$ for some $\alpha_i \in \mathbb{R}$.

$\alpha_1(1 + x) + \alpha_2(x + x^2) + \alpha_3(1 + x^2) = 0$

Matrix representation is,

$$\begin{bmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$R_2-R_1, R_3-R_2 \rightarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By solving we get, $\alpha_1=\alpha_2=\alpha_3=0$

$\therefore y_1, y_2, y_3$ are linearly independent.

II. Span:

Consider, $\vec{u_0}=\begin{bmatrix}a\\b\\c\end{bmatrix}$, an arbitrary point in $\mathbb{R}^3$

We need to represent $\vec{u_0}$ in linear combination of $\{y_1, y_2, y_3\}$.

Matrix representation is,

$$\begin{bmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$R_2-R_1, R_3-R_2 \rightarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}a\\b-a\\c-b+a\end{bmatrix}$$

By solving we get, $\alpha_1=a-\frac{c}{2}, \alpha_2=b+\frac{c}{2}, \alpha_3=\frac{c}{2}$

$\therefore y_1, y_2, y_3$ spans $\mathbb{R}^3$.

$$\boxed{\therefore \left\{y_1=1+x, y_2=x+x^2, y_3=1+x^2\right\} \text{ forms a basis of } \mathbb{P}_2[x]}$$

Linear Transformations.

Definition

Let $\mathbb{V}_1$ and $\mathbb{V}_2$ be two vector spaces over the same field $\mathbb{K}$. Let $T : \mathbb{V}_1 \to \mathbb{V}_2$ be a function from $\mathbb{V}_1$ to $\mathbb{V}_2$. The map T is said to be a linear transformation from $\mathbb{V}_1$ to $\mathbb{V}_2$ if,
– $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}), \forall \vec{u}, \vec{v} \in \mathbb{V}_1$
– $T(\alpha\vec{u}) = \alpha T(\vec{u}), \forall \vec{u} \in \mathbb{V}_1$

Examples:

Q. Check whether the following map is a linear transformation or not:
a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x, y) = (3x + y, 2y)$
$\to$
Let $\vec{u} = (x_1, y_1) \implies T(\vec{u}) = (3x_1 + y_1, 2y_1)$ and $\vec{v} = (x_2, y_2) \implies T(\vec{v}) = (3x_2 + y_2, 2y_2)$ be two arbitrary points $\in \mathbb{R}^2$ and α be a scalar from $\mathbb{R}$
I. $T(\vec{u} + \vec{v}) = T((x_1, y_1) + (x_2, y_2))$
$= T(x_1 + x_2, y_1 + y_2)$
$= (3(x_1 + x_2) + (y_1 + y_2), 2(y_1 + y_2))$
$= (3x_1 + 3x_2 + y_1 + y_2, 2y_1 + 2y_2)$
$= (3x_1 + y_1 + 3x_2 + y_2, 2y_1 + 2y_2)$
$= (3x_1 + y_1, 2y_1) + (3x_2 + y_2, 2y_2)$
$= T(\vec{u}) + T(\vec{v})$
$\therefore T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
II. $T(\alpha\vec{u}) = T(\alpha(x_1, y_1))$
$= T(\alpha x_1, \alpha y_1)$
$= (3(\alpha x_1) + (\alpha y_1), 2(\alpha y_1))$
$= (\alpha)(3x_1 + y_1, 2y_1)$

$= \alpha T(\vec{u})$
$\therefore T(\alpha\vec{u}) = \alpha T(\vec{u})$
Both conditions are satisfied.

$$\boxed{\therefore T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } T(x, y) = (3x + y, 2y) \text{ is a linear transformation.}}$$

---

b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix}$

$\rightarrow$

Let $\vec{u} = (x_1, x_2) \implies T(\vec{u}) = \begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix}$ and $\vec{v} = (x_3, x_4) \implies T(\vec{v}) = \begin{pmatrix} 2x_4 \\ 3x_3 + x_4 \end{pmatrix}$ be two arbitrary points $\in \mathbb{R}^2$ and α be a scalar from $\mathbb{R}$

I. $T(\vec{u} + \vec{v}) = T((x_1, x_2) + (x_3, x_4))$
$= T(x_1 + x_3, x_2 + x_4)$
$= \begin{pmatrix} 2(x_2 + x_4) \\ 3(x_1 + x_3) + (x_1 + x_4) \end{pmatrix}$
$= \begin{pmatrix} 2x_2 + 2x_4 \\ 3x_1 + 3x_3 + x_2 + x_4 \end{pmatrix}$
$= \begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix} + \begin{pmatrix} 2x_4 \\ 3x_3 + x_4 \end{pmatrix}$
$= T(\vec{u}) + T(\vec{v})$
$\therefore T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
II. $T(\alpha\vec{u}) = T(\alpha(x_1, x_2))$
$= \begin{pmatrix} 2\alpha x_2 \\ 3\alpha x_1 + \alpha x_2 \end{pmatrix}$
$= \alpha \begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix}$
$= \alpha T(\vec{u})$
$\therefore T(\alpha\vec{u}) = \alpha T(\vec{u})$
Both conditions are satisfied.

$$\boxed{\therefore T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix} \text{ is a linear transformation.}}$$

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c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_3 \\ 3x_1 + x_2 + 4x_3 \end{pmatrix}$

$\rightarrow$

Let $\vec{u} = (x_1, x_2, x_3) \implies T(\vec{u}) = \begin{pmatrix} 2x_2 + x_3 \\ 3x_1 + x_2 + 4x_3 \end{pmatrix}$ and $\vec{v} = (x_4, x_5, x_6) \implies T(\vec{v}) = \begin{pmatrix} 2x_5 + x_6 \\ 3x_4 + x_5 + 4x_6 \end{pmatrix}$ be two arbitrary points $\in \mathbb{R}^2$ and α be a scalar from $\mathbb{R}$

I. $T(\vec{u} + \vec{v}) = T((x_1, x_2, x_3) + (x_4, x_5, x_6))$

$= T(x_1 + x_4, x_2 + x_5, x_3 + x_6)$

$= \begin{pmatrix} 2(x_2 + x_5) + (x_3 + x_6) \\ 3(x_1 + x_4) + (x_2 + x_5) + 4(x_3 + x_6) \end{pmatrix}$

$= \begin{pmatrix} 2x_2 + 2x_5 + x_3 + x_6 \\ 3x_1 + 3x_4 + x_2 + x_5 + 4x_3 + 4x_6 \end{pmatrix}$

$= \begin{pmatrix} (2x_2 + x_3) + (2x_5 + x_6) \\ (3x_1 + x_2 + 4x_3) + (3x_4 + x_5 + 4x_6) \end{pmatrix}$

$= \begin{pmatrix} 2x_2 + x_3 \\ 3x_1 + x_2 + 4x_3 \end{pmatrix} + \begin{pmatrix} 2x_5 + x_6 \\ 3x_4 + x_5 + 4x_6 \end{pmatrix}$

$= T(\vec{u}) + T(\vec{v})$

$\therefore T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

II.$T(\alpha\vec{u}) = T(\alpha(x_1, x_2, x_3))$

$= T \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$

$= \begin{pmatrix} 2\alpha x_2 + \alpha x_3 \\ 3\alpha x_1 + \alpha x_2 + 4\alpha x_3 \end{pmatrix}$

$= \begin{pmatrix} \alpha(2x_2 + x_3) \\ \alpha(x_1 + x_2 + 4x_3) \end{pmatrix}$

$= \alpha \begin{pmatrix} 2x_2 + x_3 \\ 3x_1 + x_2 + 4x_3 \end{pmatrix}$

$= \alpha T(\vec{u})$

$\therefore T(\alpha\vec{u}) = \alpha T(\vec{u})$

Both conditions are satisfied.

$\boxed{\therefore T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_3 \\ 3x_1 + x_2 + 4x_3 \end{pmatrix} \text{ is a linear transformation.}}$

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d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1.x_2 \end{pmatrix}$

$\rightarrow$

Counter-example:

Let $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$T(\vec{u} + \vec{v}) = T(\begin{pmatrix} 1 \\ 5 \end{pmatrix}) = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$T(\vec{u}) + T(\vec{v}) = T(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) + T(\begin{pmatrix} 0 \\ 3 \end{pmatrix}) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$\therefore T(\vec{u} + \vec{v}) \neq T(\vec{u}) + T(\vec{v})$

$\boxed{T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1.x_2 \end{pmatrix} \text{ is not a linear transformation.}}$

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e) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \end{pmatrix}$

$\to$

Let $\vec{u} = (x_1, x_2, x_3) \implies T(\vec{u}) = \begin{pmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \end{pmatrix}$ and $\vec{v} = (x_4, x_5, x_6) \implies T(\vec{v}) = \begin{pmatrix} \alpha_{11}x_4 + \alpha_{12}x_5 + \alpha_{13}x_6 \\ \alpha_{21}x_4 + \alpha_{22}x_5 + \alpha_{23}x_6 \end{pmatrix}$ be two arbitrary points $\in \mathbb{R}^2$ and α be a scalar from $\mathbb{R}$

I. $T(\vec{u} + \vec{v}) = T((x_1, x_2, x_3) + (x_4, x_5, x_6))$

$= T(x_1 + x_4, x_2 + x_5, x_3 + x_6)$

$= \begin{pmatrix} \alpha_{11}(x_1 + x_4) + \alpha_{12}(x_2 + x_5) + \alpha_{13}(x_3 + x_6) \\ \alpha_{21}(x_1 + x_4) + \alpha_{22}(x_2 + x_5) + \alpha_{23}(x_3 + x_6) \end{pmatrix}$

$= \begin{pmatrix} \alpha_{11}x_1 + \alpha_{11}x_4 + \alpha_{12}x_2 + \alpha_{12}x_5 + \alpha_{13}x_3 + \alpha_{13}x_6 \\ \alpha_{21}x_1 + \alpha_{21}x_4 + \alpha_{22}x_2 + \alpha_{22}x_5 + \alpha_{23}x_3 + \alpha_{23}x_6 \end{pmatrix}$

$= \begin{pmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \end{pmatrix} + \begin{pmatrix} \alpha_{11}x_4 + \alpha_{12}x_5 + \alpha_{13}x_6 \\ \alpha_{21}x_4 + \alpha_{22}x_5 + \alpha_{23}x_6 \end{pmatrix}$

$= T(\vec{u}) + T(\vec{v})$

$\therefore T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

II.$T(\alpha\vec{u}) = T(\alpha(x_1, x_2, x_3))$

$= T \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$

$= \begin{pmatrix} \alpha\alpha_{11}x_1 + \alpha\alpha_{12}x_2 + \alpha\alpha_{13}x_3 \\ \alpha\alpha_{21}x_1 + \alpha\alpha_{22}x_2 + \alpha\alpha_{23}x_3 \end{pmatrix}$

$= \begin{pmatrix} \alpha(\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3) \\ \alpha(\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3) \end{pmatrix}$

$= \alpha \begin{pmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \end{pmatrix}$
$= \alpha T(\vec{u})$
$\therefore T(\alpha\vec{u}) = \alpha T(\vec{u})$
Both conditions are satisfied.

$\boxed{\therefore T : \mathbb{R}^3 \to \mathbb{R}^2 \text{ such that } T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \end{pmatrix} \text{ is a linear transformation.}}$

Q. Find linear transformation to find 3× of a given image:
$\rightarrow$
Consider the standard basis elements, $T(\vec{e_1}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, T(\vec{e_2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
So, $T(\vec{e_1}) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, T(\vec{e_2}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$
Now for for any point on the image,
$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\therefore T(\begin{pmatrix} x \\ y \end{pmatrix}) = T(x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix})$
$\therefore T(\begin{pmatrix} x \\ y \end{pmatrix}) = T(x \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + T(y \begin{pmatrix} 0 \\ 1 \end{pmatrix})$
$\therefore T(\begin{pmatrix} x \\ y \end{pmatrix}) = xT(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + yT(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$
$\therefore T(\begin{pmatrix} x \\ y \end{pmatrix}) = x \begin{pmatrix} 3 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$\boxed{\therefore T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 3x \\ 3y \end{pmatrix} \text{ is the required linear transformation.}}$

Kernel of a Linear Transformation

Let $\mathbb{V}_1$ and $\mathbb{V}_2$ be vector spaces over same field $\mathbb{K}$.
Let $T : \mathbb{V}_1 \to \mathbb{V}_2$ be a linear transformation, then, kernel of T is denoted as $Ker\, T$ and is given by,
$\star Ker\, T = \left\{ \vec{u} \in \mathbb{V}_1 \mid T(\vec{u}) = \vec{0} \in \mathbb{V}_2 \right\}$

$\star Ker\, T = \vec{0} \implies$ Transformation is one-one
$\star$ If both vector spaces are finite and have same dimensions + If the given transformation is one-one $\implies$ Transformation T is onto.

Examples:

Q. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ **such that** $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix}$

Is T a linear transformation?
– Find $Ker\, T$
– Is T one-one? Is T onto?
$\rightarrow$
I. Linear transformation.
Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be two arbitrary elements in $\mathbb{R}^3$.
Let α be a scalar in $\mathbb{R}$

– $T(\vec{u} + \vec{v}) = T \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$

$= \begin{bmatrix} (u_1 + v_1) - (u_2 + v_2) \\ (u_2 + v_2) - (u_3 + v_3) \\ (u_1 + v_1) - (u_3 + v_3) \end{bmatrix}$

$= \begin{bmatrix} (u_1 - u_2) + (v_1 - v_2) \\ (u_2 - u_3) + (v_2 - v_3) \\ (u_1 - u_3) + (v_1 - v_3) \end{bmatrix}$

$= \begin{bmatrix} u_1 - u_2 \\ u_2 - u_3 \\ u_1 - u_3 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ v_2 - v_3 \\ v_1 - v_3 \end{bmatrix}$

$= T(\vec{u}) + T(\vec{v})$
$\therefore T(\vec{u} + \vec{u}) = T(\vec{u}) + T(\vec{v})$

– $T(\alpha\vec{u}) = T \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{bmatrix}$

$= \begin{bmatrix} \alpha u_1 - \alpha u_2 \\ \alpha u_2 - \alpha u_3 \\ \alpha u_1 - \alpha u_3 \end{bmatrix}$

$$= \begin{bmatrix} \alpha(u_1 - u_2) \\ \alpha(u_2 - u_3) \\ \alpha(u_1 - u_3) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} u_1 - u_2 \\ u_2 - u_3 \\ u_1 - u_3 \end{bmatrix}$$

$= \alpha T(\vec{u})$

$\therefore T(\alpha\vec{u}) = \alpha T(\vec{u})$

$$\boxed{\therefore T : \mathbb{R}^3 \to \mathbb{R}^3 \text{ such that } T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix} \text{ is a linear trasformation.}}$$

II. $Ker\,T$

$Ker\,T = \left\{\vec{u} \in \mathbb{R}^3 \mid T(\vec{u}) = \vec{0} \in \mathbb{R}^3\right\}$

Let $\vec{u} \in \mathbb{R}^3$ such that $T(\vec{u}) = \vec{0} \in \mathbb{R}^3$

$T(\vec{u}) = \vec{0}$

$$T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 - u_2 \\ u_2 - u_3 \\ u_1 - u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - R_1 \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - R_2 \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since u_3 is a free variable let, $u_3 = t$.

By solving we get,

$u_1 = u_2 = u_3 = t$

$$\boxed{\therefore Ker\, T = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}}$$

III. One-one or onto:
Since, $Ker\, T \neq \left\{ \vec{0} \right\}$

$$\boxed{\therefore T : \mathbb{R}^3 \to \mathbb{R}^3 \text{ such that } T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix} \text{ is neither one-one nor onto.}}$$

Q. Find the kernel of the given transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_3 \\ x_2 - x_3 \end{bmatrix}$
Is it one-one? Is it onto? Justify.

$\to$

$Ker\, T = \left\{ \vec{u} \in \mathbb{R}^3 \mid T(\vec{u}) = \vec{0} \in \mathbb{R}^3 \right\}$

Let $\vec{u} \in \mathbb{R}^3$ such that $T(\vec{u}) = \vec{0} \in \mathbb{R}^3$

$T(\vec{u}) = \vec{0}$

$$T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 - u_2 + u_3 \\ u_3 \\ u_2 - u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the echelon form, we get,

$u_1 = u_2 = u_3 = 0$

$$\boxed{\therefore Ker\, T = \left\{ \vec{0} \right\} \text{ .So, T is one-one and onto.}}$$

Matrix Representation:

$\star$ Used to check if the transformation is one-one or onto.
$\rightarrow$ det $\neq 0$ or full-rank matrix.
$\star$ If inverse of the representation exists then the inverse obtained is the inverse map of the given transformation.

Examples:

Q. Find matrix representation of the following linear transformations with respect to given bases.

a) Linear transformation: $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

Bases: $B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\rightarrow$

Let, $B_1 = \{\vec{u_1}, \vec{u_2}\}, B_2 = \{\vec{v_1}, \vec{v_2}\}$

Now,

$T(\vec{u_1}) = T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

We need to find $\alpha_1, \alpha_2 \in \mathbb{R}$ such that,

$T(\vec{u_1}) = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2}$

$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 \end{bmatrix}$

Matrix representation is,

$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

$R_2 - R_1 \rightarrow \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

By the echelon form,

$\alpha_1 = -1, \alpha_2 = 4$

$\therefore T(\vec{u_1}) = (-1)\vec{v_1} + (4)\vec{v_2} ... \star$

$T(\vec{u_2}) = T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

We need to find $\beta_1, \beta_2 \in \mathbb{R}$ such that,
$T(\vec{u_2}) = \beta_1\vec{v_1} + \beta_2\vec{v_2}$
$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \beta_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_1 + \beta_2 \\ \beta_1 \end{bmatrix}$
Matrix representation is,
$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$
$R_2 - R_1 \to \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$
By the echelon form,
$\beta_1 = 1, \beta_2 = 2$
$\therefore T(\vec{u_2}) = (1)\vec{v_1} + (2)\vec{v_2} ... \star$

$\boxed{\therefore \text{The matrix representation is: } \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}}$

---

b) Linear transformation: $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$
Bases: $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
$\rightarrow$
Let, $B_1 = \{\vec{u_1}, \vec{u_2}\}, B_2 = \{\vec{v_1}, \vec{v_2}\}$
Now,
$T(\vec{u_1}) = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
We need to find $\alpha_1, \alpha_2 \in \mathbb{R}$ such that,
$T(\vec{u_1}) = \alpha_1\vec{v_1} + \alpha_2\vec{v_2}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$
Matrix representation is,
$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$
By the echelon form,
$\alpha_1 = 1, \alpha_2 = 1$
$\therefore T(\vec{u_1}) = (1)\vec{v_1} + (1)\vec{v_2} ... \star$

$T(\vec{u_2}) = T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$
We need to find $\beta_1, \beta_2 \in \mathbb{R}$ such that,
$T(\vec{u_2}) = \beta_1\vec{v_1} + \beta_2\vec{v_2}$
$\begin{bmatrix}1\\-1\end{bmatrix} = \beta_1\begin{bmatrix}1\\0\end{bmatrix} + \beta_2\begin{bmatrix}0\\1\end{bmatrix}$
$\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}\beta_1\\\beta_2\end{bmatrix}$
Matrix representation is,
$\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\end{bmatrix}$
By the echelon form,
$\beta_1 = 1, \beta_2 = -1$
$\therefore T(\vec{u_2}) = (1)\vec{v_1} + (-1)\vec{v_2} ... \star$

$\boxed{\therefore \text{The matrix representation is: } \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}$

---

c) Linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1 - 2x_2 + x_3\\x_2 - x_3\\x_3\end{bmatrix}$

Bases: $B_1 = \left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}\right\}, B_2 = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}4\\0\\0\end{bmatrix}\right\}$
$\to$
Let, $B_1 = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}, B_2 = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$
Now,
$T(\vec{u_1}) = T\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$
We need to find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that,
$T(\vec{u_1}) = \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \alpha_3\vec{v_3}$
$\begin{bmatrix}1\\0\\1\end{bmatrix} = \alpha_1\begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2\begin{bmatrix}2\\2\\0\end{bmatrix} + \alpha_3\begin{bmatrix}4\\0\\0\end{bmatrix}$
$\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}\alpha_1 + 2\alpha_2 + 4\alpha_3\\\alpha_1 + 2\alpha_2\\\alpha_1\end{bmatrix}$
Matrix representation is,

$$\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\1 & 2 & 0\\1 & 0 & 0\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$$

$$R_2 - R_1, R_3 - R_1 \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\0 & 0 & -4\\0 & -2 & -4\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\0 & -2 & -4\\0 & 0 & -4\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$$

$$R_2/(-2), R_3/(-4) \begin{bmatrix}1\\0\\\frac{1}{4}\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$$

By the echelon form,

$\alpha_1 = 1, \alpha_2 = \frac{-1}{2}, \alpha_3 = \frac{1}{4}$

$\therefore T(\vec{u_1}) = (1)\vec{v_1} + \frac{-1}{2}\vec{v_2} + \frac{1}{4}\vec{v_3}$

$$T(\vec{u_2}) = T \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-2\\1\\1\end{bmatrix}$$

We need to find $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that,

$T(\vec{u_2}) = \beta_1\vec{v_1} + \beta_2\vec{v_2} + \beta_3\vec{v_3}$

$$\begin{bmatrix}-2\\1\\1\end{bmatrix} = \beta_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta_2 \begin{bmatrix}2\\2\\0\end{bmatrix} + \beta_3 \begin{bmatrix}4\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}\beta_1 + 2\beta_2 + 4\beta_3\\\beta_1 + 2\beta_2\\\beta_1\end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\1 & 2 & 0\\1 & 0 & 0\end{bmatrix} \begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$

$$R_2 - R_1, R_3 - R_1 \begin{bmatrix}-2\\3\\3\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\0 & 0 & -4\\0 & -2 & -4\end{bmatrix} \begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix}-2\\3\\3\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\0 & -2 & -4\\0 & 0 & -4\end{bmatrix} \begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$

$$R_2/(-2), R_3/(-4)\ \begin{bmatrix} -2 \\ \frac{-3}{2} \\ \frac{-3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

By the echelon form,
$\beta_1 = 1, \beta_2 = 0, \beta_3 = \frac{-3}{4}$
$\therefore T(\vec{u_2}) = (1)\vec{v_1} + (0)\vec{v_2} + \frac{-3}{4}\vec{v_3}$

$$T(\vec{u_3}) = T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

We need to find $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that,
$T(\vec{u_3}) = \gamma_1\vec{v_1} + \gamma_2\vec{v_2} + \gamma_3\vec{v_3}$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \gamma_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \gamma_2 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \gamma_3 \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \gamma_1 + 2\gamma_2 + 4\gamma_3 \\ \gamma_1 + 2\gamma_2 \\ \gamma_1 \end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

$$R_2 - R_1, R_3 - R_1\ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -4 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3\ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -4 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

$$R_2/(-2), R_3/(-4)\ \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

By the echelon form,
$\gamma_1 = 2, \gamma_2 = \frac{-3}{2}, \gamma_3 = \frac{1}{2}$
$\therefore T(\vec{u_3}) = (2)\vec{v_1} + (\frac{-3}{2})\vec{v_2} + \frac{1}{2}\vec{v_3}$

$$\boxed{\therefore \text{The matrix representation is: } \begin{bmatrix} 1 & 1 & 2 \\ \frac{-1}{2} & 0 & \frac{-3}{2} \\ \frac{1}{4} & \frac{-3}{4} & \frac{1}{2} \end{bmatrix}}$$

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d) Linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_2 - x_3 \\ x_3 \end{bmatrix}$

Bases: $B_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$\rightarrow$

Let, $B_1 = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}, B_2 = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

Now,

$T(\vec{u_1}) = T \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

We need to find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that,

$T(\vec{u_1}) = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3}$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Matrix representation is,

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

$R_2 + R_1, R_3 + R_1 \rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

$R_3 \leftrightarrow R_2 \rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

By the echelon form,

$\alpha_1 = \frac{1}{2}, \alpha_2 = 1, \alpha_3 = \frac{1}{2}$

$\therefore T(\vec{u_1}) = (\frac{1}{2})\vec{v_1} + (1)\vec{v_2} + (\frac{1}{2})\vec{v_3}$

$T(\vec{u_2}) = T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

We need to find $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that,

$T(\vec{u_2}) = \beta_1 \vec{v_1} + \beta_2 \vec{v_2} + \beta_3 \vec{v_3}$

$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \beta_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Matrix representation is,
$$\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$
$$R_2+R_1, R_3+R_1 \to \begin{bmatrix}-2\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\0 & 0 & 2\\0 & 2 & 0\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$
$$R_3 \leftrightarrow R_2 \to \begin{bmatrix}-2\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$
By the echelon form,
$\beta_1 = 1, \beta_2 = \frac{-1}{2}, \beta_3 = \frac{-1}{2}$
$\therefore T(\vec{u_2}) = (1)\vec{v_1} + (\frac{-1}{2})\vec{v_2} + (\frac{-1}{2})\vec{v_3}$
$$T(\vec{u_3}) = T\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}$$
We need to find $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that,
$T(\vec{u_3}) = \gamma_1\vec{v_1} + \gamma_2\vec{v_2} + \gamma_3\vec{v_3}$
$$\begin{bmatrix}1\\-1\\2\end{bmatrix} = \gamma_1\begin{bmatrix}-1\\1\\1\end{bmatrix} + \gamma_2\begin{bmatrix}1\\-1\\1\end{bmatrix} + \gamma_3\begin{bmatrix}1\\1\\-1\end{bmatrix}$$
Matrix representation is,
$$\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}\gamma_1\\\gamma_2\\\gamma_3\end{bmatrix}$$
$$R_2+R_1, R_3+R_1 \to \begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\0 & 0 & 2\\0 & 2 & 0\end{bmatrix}\begin{bmatrix}\gamma_1\\\gamma_2\\\gamma_3\end{bmatrix}$$
$$R_3 \leftrightarrow R_2 \to \begin{bmatrix}-1\\3\\0\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\gamma_1\\\gamma_2\\\gamma_3\end{bmatrix}$$
By the echelon form,
$\gamma_1 = \frac{5}{2}, \gamma_2 = \frac{3}{2}, \gamma_3 = 0$
$\therefore T(\vec{u_3}) = (\frac{5}{2})\vec{v_1} + (\frac{3}{2})\vec{v_2} + (0)\vec{v_3}$
$$\boxed{\therefore \text{The matrix representation is: } \begin{bmatrix}\frac{1}{2} & 1 & \frac{5}{2}\\1 & \frac{-1}{2} & \frac{3}{2}\\\frac{1}{2} & \frac{-1}{2} & 0\end{bmatrix}}$$

Q. Find the matrix representation of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3 T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix}$ **with respect to the bases** $B_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Is T invertible? If yes then find T^{-1}

Is T one-one? Is T onto?

Find $Ker\, T$

$\to$

I. Let, $B_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

Now,

$$T(\vec{u_1}) = T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

We need to find, $\alpha_1, \alpha_2, \alpha_3$ such that,

$T(\vec{u_1}) = \alpha_1 \vec{v_1} + \alpha_2 \vec{e_2} + \alpha_3 \vec{v_3}$

$$\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$R_3 - R_2 \to \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

From the echelon form,

$\alpha_1 = -1, \alpha_2 = -1, \alpha_3 = 1$

$\therefore T(\vec{u}) = (-1)\vec{v_1} + (-1)\vec{v_2} + (1)\vec{v_3}$

Now,

$$T(\vec{u_2}) = T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

We need to find $\beta_1, \beta_2, \beta_3$ such that,

$$T(\vec{u_2}) = \beta_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \beta_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$

$$R_3 - R_2 \rightarrow \begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&1\\0&0&-1\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$

From the echelon form,

$\beta_1 = 1, \beta_2 = 0, \beta_3 = -1$

$\therefore T(\vec{u_2}) = (1)\vec{v_1} + (0)\vec{v_2} + (-1)\vec{v_3}$

$$T(\vec{u_3}) = T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

We need to find $\gamma_1, \gamma_2, \gamma_3$ such that,

$$T(\vec{u_3}) = \gamma_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \gamma_2\begin{bmatrix}0\\1\\1\end{bmatrix} + \gamma_3\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Matrix representation is,

$$\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}\gamma_1\\\gamma_2\\\gamma_3\end{bmatrix}$$

$$R_2 - R_3 \rightarrow \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&1\\0&0&-1\end{bmatrix}\begin{bmatrix}\gamma_1\\\gamma_2\\\gamma_3\end{bmatrix}$$

From the echelon form,

$\gamma_1 = 0, \gamma_2 = 1, \gamma_3 = 0$

$T(\vec{u_3}) = (0)\vec{v_1} + (1)\vec{v_2} + (0)\vec{v_3}$

$$\boxed{\therefore \text{The matrix representation is:} A = \begin{bmatrix}-1&1&0\\-1&0&1\\1&-1&0\end{bmatrix}}$$

II. Now,

$$det(A) = \begin{vmatrix}-1&1&0\\-1&0&1\\1&-1&0\end{vmatrix}$$

$= (-1)(0+1) - (1)(0-1) + 0 = 0$

$\boxed{\therefore \text{T is not invertible.}}$

III. Since, T is not full-rank.

$\boxed{\therefore \text{T is not one-one or onto.}}$

IV. Now,

$Ker\,T = \left\{\vec{u} \in \mathbb{R}^3 \mid T(\vec{u}) = \vec{0} \in \mathbb{R}^3\right\}$

$T(\vec{u}) = 0 \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 - R_1 \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 - R_2 \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{\therefore Ker\,T = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}}$

Q. $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ **such that,** $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_2 - x_3 \end{bmatrix}$ **and** $B_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

Find matrix representation.

$\rightarrow$

Let, $B_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

Now,

$T(\vec{u_1}) = T \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

$R_2 - 2R_1 \rightarrow \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

From the echelon form,

$\alpha_1 = -3, \alpha_2 = \frac{7}{2}$
$\therefore T(\vec{u_1}) = (-3)\vec{v_1} + (\frac{7}{2})\vec{v_2}$

$$T(\vec{u_2}) = T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$R_2 - 2R_1 \rightarrow \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

From the echelon form,
$\beta_1 = -1, \beta_2 = \frac{5}{2} \therefore T(\vec{u_2}) = (-1)\vec{v_1} + (\frac{5}{2})\vec{v_2}$

$$T(\vec{u_3}) = T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

$$R_2 - 2R_1 \rightarrow \begin{bmatrix} 4 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

From the echelon form,
$\beta_1 = 4, \beta_2 = -4 \therefore T(\vec{u_3}) = (-4)\vec{v_1} + (4)\vec{v_2}$

$$\boxed{\therefore \text{The matrix representation is: } \begin{bmatrix} -3 & -1 & -4 \\ \frac{7}{2} & \frac{5}{2} & 4 \end{bmatrix}}$$

Now,
$$Ker\,T = \left\{ \vec{u} \in \mathbb{R}^3 \mid T(\vec{u}) = \vec{0} \in \mathbb{R}^2 \right\}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the echelon form,
$u_1 = \frac{-3}{2}, u_2 = \frac{1}{2}, u_3 = 1$

$$\boxed{\therefore Ker\,T = \left\{ \begin{bmatrix} \frac{-3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}}$$

Q. $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_2 - x_3 \end{bmatrix}$ **and** $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Find the matrix representation.

$\rightarrow$

Since, the given bases are standard.

$$\boxed{\therefore \text{The matrix representation of T is: } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}}$$

Q. $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ **be a linear transformation such that,** $T\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ **then**

Find $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Find $T\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$

$\rightarrow$

I. Let, $\vec{u_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{u_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$B_1 = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ is basis of $\mathbb{R}^3$

Let, $a = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$a = \delta_1\vec{u_1} + \delta_2\vec{u_2} + \delta_3\vec{u_3}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

$$2R_2 - R_1, 2R_3 - R_1 \rightarrow \begin{bmatrix} x_1 \\ 2x_2 - x_1 \\ 2x_3 - x_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

$$3R_3 - R_2 \rightarrow \begin{bmatrix} x_1 \\ 2x_2 - x_1 \\ 6x_3 - 2x_2 - 2x_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

From the echelon form,

$\delta_1 = \frac{3x_1 - x_2 - x_3}{de4}, \delta_2 = \frac{3x_3 - x_2 - x_1}{4}, \delta_3 = \frac{3x_3 - x_2 - x_1}{4}$

$$\boxed{\therefore T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{6x_3 - 2x_2}{4} \\ \frac{6x_3 - 2x_2}{4} \end{bmatrix}}$$

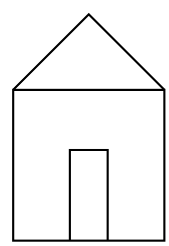
II. Now,

$$T \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 10 \\ 35 \\ 35 \end{bmatrix}$$

Geometry of Linear transformations

Original House

Consider the image of an house as given we will study the transformations with help of this image:



Original House

Rotations

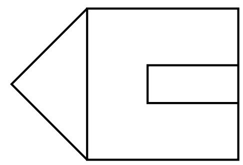
The transformation to rotate an image counterclockwise about the origin by angle θ is given as,
$T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ such that,

$$T_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For our image, the result is,

Rotated 90° Counterclockwise

When rotated by 90 degree counterclockwise:



Rotated 90°

Reflection

About X-axis

The transformation to reflect an image about X-axis is given as,
$T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ such that,

$$T_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

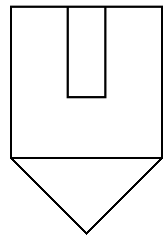
About Y-axis

The transformation to reflect an image about Y-axis is given as,
$T_3 : \mathbb{R}^3 \to \mathbb{R}^3$ such that,

$$T_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

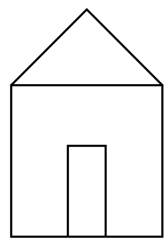
Reflected over x-axis

When reflected about X-axis,



Reflected over x-axis

Reflected over y-axis



Reflected over y-axis

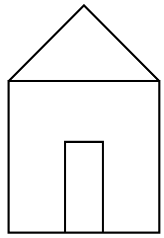
Translation

The transformation to translate an image about (t_x, t_y), that is move the image by t_x along x and t_y along y is given as,
$T_4 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that,

$$T_4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translated by (3,1)

When translated by (3,1):



Translated (3,1)

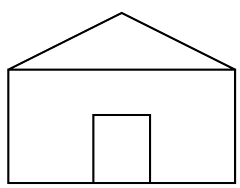
Dilation

The transformation to dilate an image by (d_x, d_y), that is scalr the image by d_x units along x and d_y units along y is given as,
$T_5 : \mathbb{R}^3 \to \mathbb{R}^3$ such that,

$$T_5 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Dilated by (1.5, 0.75)

When dilated by (1.5,0.75):



Dilated (1.5, 0.75)

Algorithm

Vector Spaces

Convert to REF:
Check entries below the 1st leading entry $\rightarrow$ Convert them to zero using row transformations $\rightarrow$ Check entries below the 2nd leading entry $\rightarrow$ Convert them to zero using row transformations $\rightarrow$ ... $\rightarrow$ Repeat till all conditions are satisfied

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Find rank/nullity + Check if invertible:
Convert to REF $\rightarrow$ Rank = Number of non-zero rows and Nullity = Number of zero rows $\rightarrow$ If it is full-rank then it is invertible.

Find rank using determinants of sub-matrices:
Consider the largest sub-matrix $\rightarrow$ Find it's determinant $\rightarrow$ If det $\neq 0 \implies$ Rank = Size of that sub-matrix OR det $= 0 \implies$ Move to smaller sub-matrix.

Analysis of a system:
rank[A|b] $\neq$ rank(A) $\rightarrow$ Inconsistent system $\rightarrow$ No solution
rank[A|b] = rank(A) $\rightarrow$ Consistent system
– = Number of variables $\rightarrow$ Unique solution $\rightarrow$ Solve simultaneously and find the solution.
– < Number of variables $\rightarrow$ Infinite solutions $\rightarrow$ Put free variable and solve.

Find λ or/and μ:
Consider matrix representation $\rightarrow$ [A|b] $\rightarrow$ Convert to REF $\rightarrow$ Consider given cases and find values accordingly.

Check if it is a Field:

Conditions:
→ Closure under add and multiplication
→ Commutativity in add and multiplication
→ Associativity in add and multiplication
→ Distributivity in addition
→ Check if identity elements exist (0 and 1)
→ Check if additive and multiplicative inverse exist (-x and (1/x))

Check if it is a vector space:
Conditions:
→ Closure in vector addition
→ Commutative
→ Associative
→ Existence of 0 element
→ Existence of additive inverse
→ Closure in scalar multiplication
→ Distributivity of vectors in addition
→ Distributivity of scalars in addition
→ Distributivity of scalars in multiplication
→ Existence of 1 element

Check if it is a vector sub-space:
Conditions:
→ Closure in vector addition
→ Closure in scalar multiplication
IMP:
A vector space is a vector sub-space of itself. 0 is a vector sub-space of every vector space.

Find span of vectors:
Take an arbitrary point → Show it in terms of given vectors → Span is the set from which the point was taken

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Check for linear independence or dependence:
Dependence relation → $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + ... + \alpha_n\vec{u_n} = 0$
Independent $\implies$ All $\alpha_i = 0$
Dependent $\implies$ At least one $\alpha_i \neq 0$

Check if a set forms a basis:
Vectors in that set $\rightarrow$ Linearly independent + Spans the vector space

---

Find dimension of a vector space:
Dimension of vector space = Size of basis / Maximum number of linearly independent vectors / Minimum number of spanning vectors

---

Linear Transformations

Check if it is a linear transformation:
Consider two arbitrary points in V_1 and a scalar in k $\rightarrow$ $\mathrm{T}(\vec{u}+\vec{v}) = \mathrm{T}(\vec{u}) + \mathrm{T}(\vec{v})$ and $\mathrm{T}(\alpha\vec{u}) = \alpha\mathrm{T}(\vec{u})$ both must be satisfied $\rightarrow$ To prove that T is NOT a linear transformation: Take a counter example.

Find transformation for 2x or 3x:
Consider standard basis $\rightarrow$ Find their images $\rightarrow$ Show arbitrary point in terms of these vectors $\implies$ Gives the required transformation.

Find ker T:
Write definition $\rightarrow$ Consider an arbitrary element $\vec{u}$ $\rightarrow$ $\mathrm{T}(\vec{u}) = 0$ $\rightarrow$ Convert to REF and find solutions $\implies$ These are elements of ker T.

Check if it is one-one or onto:
ker T = 0 $\implies$ T is one-one
One-one + Both spaces have same dimension $\implies$ T i onto

---

Find matrix representation:
$B_1 = \{\vec{u_1}, \vec{u_2}\}, B_2 = \{\vec{v_1}, \vec{v_2}\}$ $\rightarrow$ Consider $T(\vec{u_1}) = \alpha_1\vec{v_1} + \alpha_2\vec{v_2}$ and $T(\vec{u_2}) = \beta_1\vec{v_1} + \beta_2\vec{v_2}$ $\rightarrow$ Find $\alpha_1, \alpha_2, \beta_1, \beta_2$ $\rightarrow$ Get matrix representation $\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$

Invertible or not:
det(Matrix representation) = 0 $\implies$ T is not invertible
det(Matrix representation) $\neq 0$ $\implies$ T is invertible
$\rightarrow$ Full column rank $\implies$ One-one
$\rightarrow$ Full row rank $\implies$ Onto

Images of specific elements is given:
Given : $\mathrm{T}(\vec{u_1}) = i_1$, $\mathrm{T}(\vec{u_2}) = i_2$, $\mathrm{T}(\vec{u_3}) = i_3 \rightarrow \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ form a basis $\rightarrow$ Consider an arbitrary point and represent it in terms of $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} \rightarrow$ Fin constants and re-substitute $\rightarrow$ Get required transformation.

Some important transformation

To find a transformation:
Draw a rough diagram $\rightarrow$ Consider standard basis $\rightarrow$ Find their image using the diagram $\rightarrow T(\vec{u_1}) = \alpha_1\vec{v_1} + \alpha_2\vec{v_2}$ and $T(\vec{u_2}) = \beta_1\vec{v_1} + \beta_2\vec{v_2} \rightarrow$ Find $\alpha_1, \alpha_2, \beta_1, \beta_2 \rightarrow$ Get matrix representation $\begin{bmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{bmatrix}$
