Multivariable Calculus- Semester III

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Limits

Definition

Limit for multi-variable functions is given as,
$\lim_{(x,y)\to(a,b)} f(x) = l$, given any $\epsilon > 0$, $\exists\delta > 0$ such that $\mod f(x) - l < \epsilon$
for $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$
If a limit exists then it is unique.

Example:
Q. Find limit.
a) $\lim_{(x,y)\to(2,3)} \frac{x^2y}{x^2+y^2}$
$\rightarrow$
$\lim_{(x,y)\to(2,3)} \frac{x^2y}{x^2+y^2} = \frac{(2)^2(3)}{(2)^2+(3)^2} = \frac{4(3)}{4+9} = \frac{12}{13}$

$$\boxed{\lim_{(x,y)\to(2,3)} \frac{x^2y}{x^2+y^2} = \frac{12}{13}}$$

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b) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$
$\rightarrow$
We know that, if a limit exists then it is unique along all paths.
Along, x = 0
$\frac{x-y}{x+y} = \frac{-y}{y} = -1$
$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = -1$
Along, y = 0
$\frac{x-y}{x+y} = \frac{x}{x} = 1$
$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = 1$
Along two different paths there are two different limits. But, we know if limit exists then it is unique.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} \text{does not exist.}$$

Important

Open disk or open ball

Open disk or open ball is given as,
$B((a,b),\delta) = \left\{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-a)^2+(y-b)^2 < \delta}\right\}$

Theorem

Let $\lim_{(x,y)\to(a,b)} f(x,y) = 0$ and g(x,y) is bounded in some $B((a,b),\delta)$ then,
$\lim_{(x,y)\to(a,b)} f(x,y).g(x,y) = 0$

Example:
Q. Find limit if exists.
a) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$
$\rightarrow$
Here,
$\frac{x^2y^2}{x^2+y^2} = (\frac{x^2}{x^2+y^2})y^2$
We know,
$\lim_{(x,y)\to(0,0)} y^2 = 0$
$0 < y^2 \rightarrow x^2 \leq x^2+y^2 \rightarrow 0 \leq \frac{x^2}{x^2+y^2} \leq 1$
By the theorem,
Let $\lim_{(x,y)\to(a,b)} f(x,y) = 0$ and g(x,y) is bounded in some $B((a,b),\delta)$ then,
$\lim_{(x,y)\to(a,b)} f(x,y).g(x,y) = 0$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0}$$

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b) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$
$\rightarrow$
Here,
$\frac{x^3+y^3}{x^2+y^2} = \frac{x^3}{x^2+y^2} + \frac{y^3}{x^2+y^2} = x(\frac{x^2}{x^2+y^2}) + y(\frac{y^2}{x^2+y^2})$
We know,
$\lim_{(x,y)\to(0,0)} x = 0$ and $\lim_{(x,y)\to(0,0)} y = 0$

$0 \leq \frac{x^2}{x^2+y^2} \leq 1$ and $0 \leq \frac{y^2}{x^2+y^2} \leq 1$
By the theorem,
Let $\lim_{(x,y)\to(a,b)} f(x,y) = 0$ and g(x,y) is bounded in some $B((a,b),\delta)$ then,
$\lim_{(x,y)\to(a,b)} f(x,y).g(x,y) = 0$
$\lim_{(x,y)\to(0,0)} x(\frac{x^2}{x^2+y^2}) = 0$ and $\lim_{(x,y)\to(0,0)} y(\frac{y^2}{x^2+y^2}) = 0$

$$\boxed{\therefore \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0}$$

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c) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$
$\to$
Use, polar co-ordinates, $x = r\cos\theta$ and $y = r\sin\theta$
$\therefore r = \sqrt{x^2+y^2}$
$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r} = \lim_{r\to 0} r\cos\theta\sin\theta = 0$

$$\boxed{\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0}$$

Continuity

Definition

A function f(x,y) is said to be continuous at (a,b) if,
$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Example: Q. Discuss the continuity of the following:
a) $f(x,y) = \frac{xy}{x^2+y^2}, (x,y) \neq (0,0), f(0,0) = 0$
$\rightarrow$
We check limit along various paths,
Along x = 0,
$f(x,y) = 0 \implies \lim_{(x,y)\to(0,0)} f(x,y) = 0$
Along x = y,
$f(x,y) = \frac{x^2}{2x^2} = \frac{1}{2} \implies \lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$
Along two different paths there are two different limits. But, we know if limit exists then it is unique.
$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.
$\boxed{\therefore f(x,y) \text{is discontinuous at (0,0)}}$

Partial Derivatives

Definition

First order partial derivatives are given as,
$f_x = \frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b)-f(a,b)}{h}$
$f_y = \frac{\partial f}{\partial y}(a,b) = \lim_{k \to 0} \frac{f(a,b+k)-f(a,b)}{k}$

Second order partial deriavatives are given as,
$f_{xx} = \frac{\partial^2 f}{\partial x^2}(a,b) = \lim_{h \to 0} \frac{f_x(a+h,b)-f_x(a,b)}{h}$
$f_{xy} = \frac{\partial^2 f}{\partial y^2}(a,b) = \lim_{k \to 0} \frac{f_y(a,b+k)-f_y(a,b)}{k}$
$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}(a,b) = \lim_{h \to 0} \frac{f_y(a,b+k)-f_y(a,b)}{h}$
$f_{yy} = \frac{\partial^2 f}{\partial y^2}(a,b) = \lim_{k \to 0} \frac{f_y(a,b+k)-f_y(a,b)}{k}$

Example:
Q. Find first order partial derivatives:
a) $f(x,y) = \tan^{-1}(\frac{x}{y})$
$\rightarrow$
We know,derivative of;
$\tan^{-1}(x) = \frac{1}{1+x^2}$
So,
$f_x = \frac{\partial f}{\partial x} = \frac{y}{x^2+y^2}$
$f_y = \frac{\partial f}{\partial y} = \frac{(-x)}{x^2+y^2}$

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b) $f(x,y) = \frac{xy}{x^2+y^2}, (x,y) \neq (0,0), f(0,0) = 0$
$\rightarrow$
First order derivatives are,
$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(h,0)-f(0,0)}{h} = 0$
$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(0,k)-f(0,0)}{k} = 0$

$\therefore f_x = 0, f_y = 0$

Q. Find all first and second order partial derivatives of f at (1,2).
$f = 3x^3 + 2xy^2 + y^3$
$\rightarrow$
$f = 3x^3 + 2xy^2 + y^3$
$f_x = 9x^2 + 2y^2 \implies f_x(1,2) = 9 + 8 = 17$
$f_{xx} = 18x \implies f_{xx}(1,2) = 18$
$f_{xy} = 4y \implies f_{xy}(1,2) = 8$
$f_y = 4xy + 3y^2 \implies f_y(1,2) = 8 + 12 = 20$
$f_{yy} = 4x + 6y \implies f_{yy}(1,2) = 4 + 12 = 16$
$f_{yx} = 4y \implies f_{yx}(1,2) = 4$

Q. Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2}{x+y}$ where, $u = \log(x^3 + y^3 - x^2y - xy^2)$
$\rightarrow$
$u = \log(x^3 + y^3 - x^2y - xy^2)$
$\frac{\partial u}{\partial x} = \frac{3x^2-2xy-y^2}{x^3+y^3-x^2y-xy^2}$
$\frac{\partial u}{\partial y} = \frac{3y^2-x^2-2xy}{x^3+y^3-x^2y-xy^2}$
Now,
LHS $= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$
$= \frac{3x^2-2xy-y^2+3y^2-2xy-x^2}{x^3+y^3-x^2y-xy^2}$
$= \frac{2(x-y)^2}{(x+y)(x-y)^2}$
$= \frac{2}{(x+y)}$
$=$ RHS
$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2}{x+y}$

Laplace equation

Laplace equation is given as,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

–**Harmonic function:**
If u(x,y) satisfies the Laplace equation, then u is called as a harmonic function.

Example:

Q. Check whether the following are harmonic or not:

a) $u(x, y) = x^2 + y^2$

$\rightarrow$

$f_x = \frac{\partial u}{\partial x} = 2x$

$f_{xx} = \frac{\partial^2 u}{\partial x^2} = 2$

$f_y = \frac{\partial u}{\partial y} = 2y$

$f_{yy} = \frac{\partial^2 u}{\partial y^2} = 2$

Now,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 = 4 \neq 0$

$\therefore u(x, y) = x^2 + y^2$ is not a harmonic function.

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b) $u(x, y) = \log(x^2 + y^2)$

$\rightarrow$

$f_x = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$

$f_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$

$f_y = \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$

$f_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$

Now,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)+2(x^2-y^2)}{(x^2+y^2)^2}$

$\therefore u(x, y) = \log(x^2 + y^2)$ is a harmonic function.

Q. Show that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ where $f = \tan^{-1}(\frac{y}{x})$.

$\rightarrow$

$\frac{\partial f}{\partial x} = \frac{(-y)}{x^2+y^2} \implies \frac{\partial^2 f}{\partial y \partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \implies \frac{\partial^2 f}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Q. Show that $x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = y^2 v^3$ where $v = (1 - 2xy + y^2)^{-1/2}$

$\rightarrow$

$\frac{\partial v}{\partial x} = y(1 - 2xy + y^2)^{-3/2}$

$\frac{\partial v}{\partial y} = (x - y)(1 - 2xy + y^2)^{-3/2}$

Now,

LHS $= x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = xy(1 - 2xy + y^2)^{-3/2} - y(x - y)(1 - 2xy + y^2)^{-3/2}$

$= xy(1-2xy+y^2)^{-3/2} - (xy-y^2)(1-2xy+y^2)^{-3/2}$
$= xy(1-2xy+y^2)^{-3/2} - xy(1-2xy+y^2)^{-3/2} + y^2(1-2xy+y^2)^{-3/2}$
$= y^2(1-2xy+y^2)^{-3/2}$
$= y^2v^3$
$=$ RHS
$\therefore x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = y^2v^3$

Q. Suppose $z = \sin(nx).\cos(y^2)$ satisfies,
$y^3.\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = 0$ for all (x,y). Find n.
$\rightarrow$
Here,
$\frac{\partial z}{\partial x} = \cos(nx).n.\cos(y^2) = n\cos(nx).\cos(y^2)$
$\therefore \frac{\partial^2 z}{\partial x^2} = -n^2.\sin(nx).\cos(y^2)$
$\frac{\partial z}{\partial y} = -\sin(nx).\sin(y^2).2y = -2y\sin(nx).\sin(y^2)$
$\therefore \frac{\partial^2 z}{\partial y^2} = -2\sin(nx)\sin(y^2) - 4y^2\sin(nx)\cos(y^2)$
$y^3.\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = y^3(-n^2.\sin(nx).\cos(y^2)) - y(-2\sin(nx)\sin(y^2) - 4y^2\sin(nx)\cos(y^2)) +$
$(-2y\sin(nx).\sin(y^2)) = 0$
Hence, proved.

Q. $f(x,y) = \frac{xy^3}{x^2+y^2}$ if (x,y) $\neq$ 0, f(0,0) = 0.
Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$
$\rightarrow$
$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$
$\implies f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = k$
$\implies f_x(0,k) = k$
$\implies f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$
$\therefore \frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0} \frac{k-0}{k} = 1$
$\therefore \frac{\partial^2 f}{\partial y \partial x}(0,0) = 1$
Now,
$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$
$\implies f_y(h,0) = \lim_{k\to 0} \frac{f(h,k)-f(0,k)}{k} = 0$
$\implies f_y(h,0) = 0$
$\implies f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$
$\therefore \frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$
$\therefore \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$

Differentiability

Definition

We say that $f(x,y)$ is differentiable at (a,b) if there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - (hc_1 + kc_2)}{\sqrt{h^2+k^2}} = 0$$

Theorem

If $f(x,y)$ is differentiable at (a,b) then:

- f is continuous at (a,b)
- Both first-order partial derivatives exist at (a,b)

Proof:
I. To prove continuity:
Given differentiability, we add and subtract $(hc_1 + kc_2)$:

$$\implies \lim_{(h,k)\to(0,0)} \left[\frac{f(a+h,b+k) - f(a,b) - (hc_1 + kc_2)}{\sqrt{h^2+k^2}} \cdot \sqrt{h^2+k^2} + (hc_1 + kc_2) \right] = 0$$

$\therefore f$ is continuous at (a,b)
II. To prove existence of partial derivatives:
Put $k = 0$:

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = c_1 \Rightarrow \frac{\partial f}{\partial x}(a,b) = c_1$$

Put $h = 0$:

$$\lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = c_2 \Rightarrow \frac{\partial f}{\partial y}(a,b) = c_2$$

$\therefore$ Both partial derivatives exist.
Example:

Is $f(x,y) = \frac{x^4+y^4}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$ differentiable at $(0,0)$
I. Continuity:

$$\frac{x^4 + y^4}{x^2 + y^2} = x \cdot \frac{x^3}{x^2 + y^2} + y \cdot \frac{y^3}{x^2 + y^2}$$

Since $x, y \to 0$ and both fractions are bounded, by the product limit theorem:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2} = 0$$

II. Partial derivatives:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^4}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k^4}{k} = 0$$

III. Differentiability check:
Since $c_1 = c_2 = 0$, we check:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0)}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{h^4+k^4}{h^2+k^2}}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{h^4 + k^4}{(h^2 + k^2)\sqrt{h^2 + k^2}} = 0$$

$\boxed{\therefore f(x,y) \text{ is differentiable at } (0,0)}$

Applications

Find approximate values
We know that if f is differentiable at (a,b),

$$\lim_{(h,k)\to(0,0)} [\frac{f(a+h, b+k) - f(a,b) - (f_x(a,b)h + f_y(a,b)k)}{\sqrt{h^2 + k^2}}]$$

For very small values of h and k in comparison to a and b,
$f(a+h, b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k$
Example:

Find approximate value of:
$\sqrt{(5.98)^2 + (8.01)^2}$
$\rightarrow$
Let, $f(x, y) = \sqrt{x^2 + y^2}$
Here, a = 6, b = 8, h = -0.02, k = 0.01
Now,
$f(a+h, b+k) \approx f(a, b) + f_x(a, b)h + f_y(a, b)k$
So,
$f(a, b) = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$
$f_x(a, b) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{6}{10}\frac{3}{5}$
$f_y(a, b) = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \frac{8}{10}\frac{4}{5}$
$\therefore f(a+h, b+k) \approx 10 + \frac{3}{5} \times (-0.02) + \frac{4}{5} \times (0.01)$
$\approx 10 - 0.012 + 0.008$
≈ 9.996
$\boxed{\therefore \sqrt{(5.98)^2 + (8.01)^2} \approx 9.996}$

Chain rule

Rule I:
If f(x,y) is differentiable function of x and y; x = ϕ(t) and y = ψ(t) are differentiable functions of t then,
For u = f(ϕ(t), ψ(t)),

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

Example:
Find $\frac{du}{dt}$ at $t = \frac{\pi}{2}, \pi$
$u = e^{xy^2}, x = t\cos t, y = t\sin t$
$\rightarrow$
By chain rule I,
If f(x,y) is differentiable function of x and y; x = ϕ(t) and y = ψ(t) are differentiable functions of t then,
For u = f(ϕ(t), ψ(t)),

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$\frac{du}{dt} = (y^2e^{xy^2}).(-t\sin t + \cos t) + (2xye^{xy^2}).(t\cos t + \sin t)$
$\frac{du}{dt} = \sin t(2xye^{xy^2} - ty^2e^{xy^2}) + \cos t(y^2.e^{xy^2} + 2txye^{xy^2})$
$\therefore \frac{du}{dt} = e^{xy^2}(\sin t(2xy - ty^2) + \cos t(y^2 + 2txy))$
Now,
At $t = \frac{\pi}{2}$,
$\frac{du}{dt} = \frac{-\pi}{2}^3$
At $t = \pi$,
$\frac{du}{dt} = 0$

Rule II:
If $w = f(u,v)$ is a differentiable function of u & v and u = ϕ(x,y) & v = ψ(x,y) are differentiable functions of x and y then,
For w = f(ϕ(x,y), ψ(x,y)) =,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}.\frac{\partial v}{\partial y}$$

Example:
If z = $\tan^{-1}\frac{x}{y}$, x = u + v, y = u - v. Show that,

$$\frac{\partial z}{\partial u} + \frac{\partial w}{\partial v} = \frac{u-v}{u^2+v^2}$$

$\rightarrow$
By chain rule II,
If $w = f(u,v)$ is a differentiable function of u & v and u = ϕ(x,y) & v = ψ(x,y) are differentiable functions of x and y then,
For w = f(ϕ(x,y), ψ(x,y)) =,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}.\frac{\partial v}{\partial y}$$

$\frac{\partial z}{\partial u} = \frac{y}{x^2+y^2} + \frac{-x}{x^2+y^2}$
$\frac{\partial z}{\partial v} = \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}$
Now,
$LHS = \frac{\partial z}{\partial u} + \frac{\partial w}{\partial v} = \frac{2y}{x^2+y^2} = \frac{u-v}{u^2+v^2} = RHS$
Hence, proved.

Homogeneous functions

f(x,y) is called "homogeneous function" of degree n if it can be expressed as,

$$f(tx, ty) = t^n.f(x, y)$$

Example:
f(x,y) $= x^2 - xy + y^2$
$\rightarrow$
Here,
f(tx,ty) $= (tx)^2 - (tx)(ty) + (ty)^2$
$= t^2(x^2 - xy + y^2)$
$= t^2.f(x, y)$
f(tx,ty) $= = t^2.f(x, y)$
$\therefore$ f(x,y) is a homogeneous function.

Euler's theorem

Statement:
If f(x,y) is a homogeneous function of x and y of degree n in an open region having continuous partial derivatives then,

$$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{de\partial y} = nf$$

Proof:
Since, f(x,y) is a homogeneous function of x and y of degree n,

$$f(x, y) = x^n.f(\frac{y}{x})$$

Differentiate the above equation wrt x,

$$\frac{\partial f}{\partial x} = nx^{n-1}.f(\frac{y}{x}) + x^n.f'(\frac{y}{x})$$

$$\frac{\partial f}{\partial x} = nx^{n-1}.f(\frac{y}{x}) - x^{n-2}y.f'(\frac{y}{x})$$

Multiply both sides by x,

$$x.\frac{\partial f}{\partial x} = nx^n.f(\frac{y}{x}) - y.x^{n-1}.f'(\frac{y}{x})$$

Differentiate partially wrt y,

$$\frac{\partial f}{\partial y} = x^n.f'(\frac{y}{x}).\frac{1}{x}$$

$$\therefore \frac{\partial f}{\partial y} = x^{n-1}.f'(\frac{y}{x})$$

Multiply both sides by y,

$$y.\frac{\partial f}{\partial y} = y.x^{n-1}.f'(\frac{y}{x})$$

Add both,

$$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nx^n f(\frac{y}{x})$$

$$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$$

Hence, proved.
Example:
Find $x.f_x + y.f_y$

f(x,y) $= \frac{x^2+y^2}{x-y}$
$\rightarrow$
First,
f(tx,ty) $= \frac{(tx)^2+(ty)^2}{(tx)-(ty)} = \frac{t^2(x^2+y^2)}{t(x-y)}$
$\therefore f(tx,ty) = t.f(x,y)$
So, f is a homogeneous function of degree 1.
$\therefore x.f_x + y.f_y = f$

Critical point

The point (a,b) is called as a critical point of f if,
$f_x(a,b) = 0$ and $f_y(a,b) = 0$

Important

Let f be a real valued function defined on D containing point (a,b),
– If f(a,b) $\geq$ f(x,y), $\forall (x,y) \in D \implies$ f(a,b) is an absolute maximum.
– If f(a,b) $\leq$ f(x,y), $\forall (x,y) \in D \implies$ f(a,b) is an absolute minimum.
– If f(a,b) $\geq$ f(x,y), $\forall (x,y) \in N_\delta(a,b) \implies$ f(a,b) is an local maximum.
– If f(a,b) $\leq$ f(x,y), $\forall (x,y) \in N_\delta(a,b) \implies$ f(a,b) is an local minimum.

Second derivative test

Suppose a function f(x,y) and it's first & second order partial derivatives are continuous on $N_\delta(a,b)$ and $f_x(a,b) = 0 \& f_y(a,b) = 0$ then,
Let, $A = f_{xx}(a,b), B = f_{xy}(a,b), C = f_{yy}(a,b)$
$\Delta = AC - B^2$
If,
$\Delta > 0$ and A $< 0 \rightarrow$ f has local maximum at (a,b)
$\Delta > 0$ and A $> 0 \rightarrow$ f has local minimum at (a,b)
$\Delta < 0 \rightarrow$ f has saddle point at (a,b)
$\Delta = 0 \rightarrow$ Test fails

Lagrange's Method

It is used to find extreme values of a function f(x,y,z) subject to another function g(x,y,z) = 0
Consider the equation,

$$\nabla f = \lambda \nabla g$$

$$\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \lambda(\frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} + \frac{\partial g}{\partial z}\vec{k})$$

Directional Derivatives

$$(\partial f)_{\vec{u}}(a,b) = \lim_{t \to 0} \frac{f(a+tu_1, b+tu_2) - f(a,b)}{t}$$

Formula:

$$(\partial f)_{\vec{u}}(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$$

Partial derivatives are specific directional derivatives: When $\vec{u} = (1,0)$, we get:

$$(\partial f)_{\vec{u}}(a,b) = \frac{\partial f}{\partial x}(a,b)$$

When $\vec{u} = (0,1)$, we get:

$$(\partial f)_{\vec{u}}(a,b) = \frac{\partial f}{\partial y}(a,b)$$

Integration

To find area,

$$A = \iint_R 1\, dA$$

To find volume,

$$V = \iiint_R 1\, dV$$

Example:
Q. Find area of the region bounded by,
$y = x$ and $y = x^2$
$\rightarrow$
These curves intersect at: (0,0) and (1,1)

$$Area = \iint_R 1\, dA = \int_0^1 \int_{x^2}^x 1\, dydx = \int_0^1 [x - x^2]dx = \frac{1}{6}$$

$\therefore A = \frac{1}{6}$

Q. Find volume of a tetrahedron bounded by $x = 0, y = 0, z = 0, x+y+z = 1$

$$Volume = \iiint_R 1\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-y-x} 1\, dzdydx = \frac{1}{6}$$

$\therefore V = \frac{1}{6}$

Sometimes the process can be lengthy due to difficult or complex limits so substitute the following,
Polar coordinates:

When the given region is a disc or part of a disc; use polar co-ordinates;

$$x = r\cos\theta, y = r\sin\theta$$

Jacobian is:

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

$$\therefore \partial(x, y) = r\partial(r, \theta)$$

Spherical polar coordinates:
When the given region is a sphere or a part of sphere use,

$$x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta, z = r\cos\phi$$

Here, $0 \le \phi \le \pi, 0 \le \theta \le 2\pi, 0 \le r \le a$
Jacobian is:

$$J = \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = r^2\sin\phi$$

$$\partial(x, y, z) = r^2\sin\phi\partial(r, \phi, \theta)$$

Cylindrical polar coordinates:
When the given region is a cylinder use,

$$x = r\cos\theta, y = r\sin\theta, z = z$$

Here, $0 \le \theta \le 2\pi, 0 \le r \le a$
Jacobian is:

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

$$\partial(x, y, z) = r\partial(r, \theta, z)$$

Example:
Q. Find volume of the sphere $x^2 + y^2 + z^2 = a^2$
$\rightarrow$

Using spherical polar coordinates,

$$Volume = \iiint_R 1 \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^2 \sin\phi \, dr d\phi d\theta = \frac{4\pi a^3}{3}$$

$\therefore V = \frac{4\pi a^3}{3}$

Algorithms

Limits

Find limit:
→ If not a indeterminate form, then substitute directly.
→ Check along various paths → If same value obtained then the limit exists and that value is the answer.
→ Use the product theorem
→ Use polar coordinates (mainly when $\sqrt{x^2 + y^2}$ is present)

Continuity

Check if given function is continuous or not:
→ Write the definition
→ Find $\lim_{(x,y)\to(a,b)} f(x, y)$
→ Compare with f(a,b) $\implies$ If equal then continuous

Partial Derivatives

Find 1st order partial derivatives:
→ Directly differentiate if possible
→ Use definition
→ If asked put specific points and find values

Find 2nd order partial derivatives:
→ Find first order partial derivatives
→ Use them to find 2nd order by above methods.

Check if a function is harmonic or not:
→ Write the Laplace equation
→ Check if the given function satisfies it.
→ If satisfies then it is harmonic

"Show that" questions:
→ Write the given function
→ Find all the required elements of LHS and RHS
→ Calculate LHS and RHS
→ Show that LHS = RHS

Find "n":
→ Use steps from above
→ Will get some equation → Solve it and get n

Show that $f_{xy} \neq f_{yx}$***:***
→ Find both separately using the methods mentioned above → Compare

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Differentiability

Check if differentiable at some point (a,b):
→ Check continuity at (a,b) → If continuous move further
→ Check if both first order partial derivatives exist or not → If exists then move further
→ Check if the definition is satisfied:

$$\lim_{(x,y)\to(a,b)} \frac{f(a+h,b+k) - f(a,b) - (hc_1 + kc_2)}{\sqrt{h^2+k^2}} = 0$$

→ If all of these satisfied then it is differentiable at (a,b)

Find approximate value:
→ Represent the given question in general terms
→ Use $f(a+h,b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k$
→ Find the values required
→ Get the required value

Questions on chain rule I:
→ Here the question contains 1 function of 2 variables and 2 functions of 1 variable
→ Use the definition:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

→ Find using the definition
→ If asked at a specific point t then substitute and obtain the required value.

Questions based on chain rule 2:
→ Here the question contains 1 function of 2 variables and 2 functions of 2 variable
→ Use the definition:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}.\frac{\partial v}{\partial y}$$

→ Find both

Check if homogeneous or not:
→ Consider the definition:

$$f(tx, ty) = t^n f(x, y)$$

→ Check if the given function satisfies this → If yes, then it is homogeneous

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Questions based on Euler's theorem:
Proof:
→ Write the statement → Differentiate partially wrt x and y → Add both
→ Make some changes and prove it.
Find $x.f_x + y.f_y$***:***
→ Check if homogeneous → Find n → Use $x.f_x + y.f_y = nf$

Find critical points:
Write definition → Find first order partial derivatives → Equate to 0 → Get equations → Solve them → Get solution → Put in f_y → Get critical points.

Find local minima or maxima using 2nd derivative test:
Find all first and second order partial derivatives → Find critical points → Consider points one by one and find Δ → Check the second derivative conditions

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Word problems based on Lagrange's theorem:
Assume variables and form equations according to the conditions → Classify g(x,y,z) = 0 and f(x,y,z) from these → Find ∇f and ∇g → Use $\nabla f = \lambda \nabla g$ → Find x,y,z in terms of λ → Find λ by re-substitution → Finally, get x,y and z.

Rectangular box problem:
Volume = xyz and Surface area = xy + 2xz + 2yz → Find 1st order partial derivatives → Find critical points → Use 2nd derivative test → Find minimum value → Dimensions are obtained.

Find directional derivative:
Use

$$(\partial f)_{\vec{u}}(a) = \nabla f 9a).\vec{u}$$

→ Make sure that $\vec{u}$ is unit vector

Find tangent plane:
A function and a point is given → Check if the point is on that plane → Find $\nabla f(a)$ → It is normal to the tangent plane → Use $\nabla f(a).((x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k} = 0$ → Gives the tangent plane equation

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Integration

Evaluate limits:
Similar to single integrals → First solve inner limit then the outer one → If order is changed then choose the appropriate limits

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Find area:

Use: A = $\iint_R 1\, dA$ → Find limits of R from a rough diagram → Put them and solve A

Find volume:
Use: V = $\iiint_R 1\, dA$ → Find limits of R from a rough diagram → Put them and solve V

If polar coordinates used:
Proper substitution of function → Remember the Jacobians correctly → Remaining process similar to above

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