Week-1 [Essential math for ML]

Q) what is first step in machine learning?

↓

Get data → learn pattern from data.

↓

data would be a vector

↓

$$\left\{ \begin{bmatrix} \text{Capinde} \\ 28 \\ \vdots \end{bmatrix}, \ldots \right\}$$

Vectors:

vectors are 1D representation of data. ⇒ $\begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}$

those attributes

↑

* Features

& dimensions are same.

Basic vector algebra: →

R^n ⇒ n dimensional vector ⇒ $\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$

Vector addition: →

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a+x \\ b+y \\ c+z \end{bmatrix}$$

eg: ①
$$\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix}$$

②
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Scalar multiplication:

$$2 \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

vector multiplication - Dot product (Inner product): →

$$[a\ b\ c] \cdot [x\ y\ z] = ax + by + cz \Rightarrow \langle \bar{A}, \bar{B} \rangle$$

$$\underset{x}{[1, 2, 3]} \cdot \underset{y}{[1\ 0\ 2]} = 1\times1 + 2\times0 + 3\times2$$
$$= 7$$

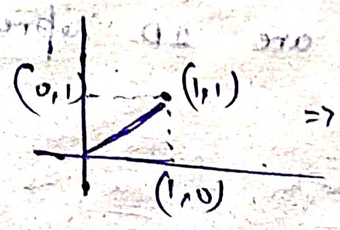
we can also write it as

$x^T y$ which is same as $x \cdot y$.

This is just to get satisfaction of multiplying col & row.

Length of a vector (Norm)

Suppose $R^2 = \begin{bmatrix}1\\1\end{bmatrix} \Rightarrow$



$\Rightarrow \sqrt{1^2+1^2} = \sqrt{2}$

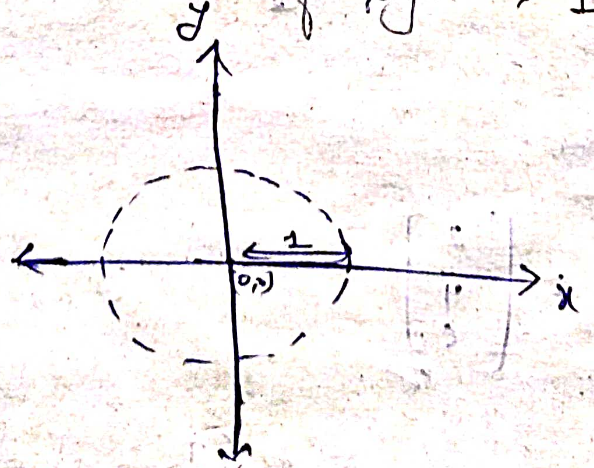
L_2 norm (Euclidean norm)

$x = (x_1, x_2, x_3, \ldots, x_n)$

$$||x||_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2}$$

Eg: $||(1,2,3)||_2 = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

Locate all those vectors of length $\rightarrow 1$



l-1 norm:

$x = (x_1, x_2, ---, x_n)$

$$\|x\|_1 = |x_1| + |x_2| + |x_3| + --- + |x_n|.$$

$$\|(1,2,3)\|_1 = |1| + |2| + |3| = 6$$

l-p norm:

$x = (x_1, x_2, ---, x_n).$

$$\|x\|_p = (x_1^p + x_2^p + x_3^p + --- + x_n^p)^{1/p}$$

3 norm $\Rightarrow$ $\|x\|_3 = (x_1^3 + x_2^3 + --- + x_n^3)^{1/3}$

= Euclidean length

(or)

Euclidean norm

Angle b/w vectors $\Rightarrow$

x & y $\Rightarrow$ $x \cdot y = |x|_2 \, |y|_2 \cos\theta$

$$\cos\theta = \frac{x \cdot y}{|x|_2 \cdot |y|_2} \Rightarrow \theta = \cos^{-1}\left(\frac{x \cdot y}{|x|_2 \, |y|_2}\right)$$

Perpendicular vectors: $\Rightarrow$ $\theta = 90°$

(or)

orthogonal.

orthogonal set $\Rightarrow \{x_1, x_2, x_3, ---, x_n\}$

this is orthogonal set iff $x_i \cdot x_j = 0 \;\forall\; i,j \in R$

orthonormal vectors :

Two vectors $\vec{x}$ & $\vec{y}$ are orthonormal if $\vec{x}.\vec{y} = 0$

and $||x||_2, ||y||_2 = 1$

suppose $\{\bar{a}, \bar{b}, \bar{c}\}$ → it's a orthogonal set

⇓ 3 vectors

⇓ $\bar{a}.\bar{b} = 0, \bar{b}.\bar{c} = 0$ & $\bar{c}.\bar{a} = 0$

you were asked to find its orthonormal set

⇓

it's orthonormal set is $\left\{\frac{a}{|a|}, \frac{b}{|b|}, \frac{c}{|c|}\right\}$

Note:

(1) If 2 vectors $\bar{a}$ & $\bar{b}$ are at $90°$ & if you change their length but not direction → then they're still in $90°$. → $\bar{a}.\bar{b} = 0$ Then $2\bar{a}.3\bar{b} = 0$ & $\frac{\vec{a}}{2}.\frac{\vec{b}}{3} = 0$

Eg. $\bar{a} = 3\hat{i} + x\hat{j}$, $\bar{b} = 4\hat{i} + 5\hat{j}$

$12 + 5x = 0 \Rightarrow x = \frac{-12}{5}$

⇓

$\bar{a} = 3\hat{i} - \frac{12}{5}\hat{j}$, $\bar{b} = 4\hat{i} + 5\hat{j}$

Now $2\bar{a}.3\bar{b} \Rightarrow \left(6\hat{i} - \frac{24}{5}\hat{j}\right).\left(12\hat{i} + 15\hat{j}\right)$

$= 72 - \frac{24}{5} \times 15$

$= 72 - 72$

$= 0$

proved

(2) If we're given with a orthogonal set &
we're asked to find its orthonormal set.
then, divide all those vectors with their
corresponding 2-norm.

ith
$(a, b, c) \in$ orthogonal set

ith
$\left(\frac{a}{|a|}, \frac{b}{|b|}, \frac{c}{|c|}\right) \in$ orthonormal set.

Q) What is the geometric meaning of dot product?

generally dot product b/w $\vec{a}$ & $\vec{b}$ gives you some scaling of
the cosine angle b/w vectors $\vec{a}$ & $\vec{b}$

Q) How are these relevant to Machine learning?

→ Representing features ⟶ [ ]

→ Normalising features

→ Similarity measure for applications like word2vec,
CNN, etc.

→ Input in deep learning

→ And many more.

Let's say we're given with some features a data
of a person and it contains marks in class 10,
class 12 and so on. Now we want to predict

whether he'd get a job or not.

$$f\left(\begin{bmatrix} 10^{th} \\ 12^{th} \\ \vdots \\ \vdots \end{bmatrix}\right) = \text{get a job.}$$

if we consider the dataset

$$\begin{bmatrix} 11^{th} \\ 12^{th} \\ \vdots \\ \vdots \end{bmatrix} \begin{matrix} \to x_1 \\ \to x_2 \\ \to x_3 \\ \to x_n \end{matrix}$$

the these $x_1, x_2, \ldots, x_n$ are marks right and they can be out of 10 & out of 100.

But here out of 100 wala feature dominate the most → if this type of variation is there then we kinda normalise all those features.

→ Normalisation in data transformation.

* It is used to scale and standardize the features of a dataset.

* we make 'em lie in a particular range

* Primary goal is to bring all the features to a similar scale, typically b/w 0 and 1.

① Min-Max normalization technique.

② Z-score normalisation technique.

Now, Min-Max Technique

House	Square Feet	Bedrooms	Price (in Lakh).
1	1200	$3 \to \frac{3-2}{4-2} \to 0.5$	50
2	1500	$4 \to \frac{4-2}{4-2} \to 1$	60
3	1000	$2 \to \frac{2-2}{4-2} \to 0$	40
4	1800	$5 \to \frac{5-2}{4-2} \to \frac{3}{2} \to 1.5$	80

formula, $x' = \frac{x - min}{max - min}$

$x' \rightarrow$ the new value
$x \rightarrow$ old value.
min $\rightarrow$ minimum value among those values under a particular feature..
max $\rightarrow$ max.. .. ..

Square foot

$1200 \rightarrow \frac{1200 - 1000}{1800 - 1000} = \frac{200}{800} = 0.25$

$1500 \rightarrow \frac{1500 - 1000}{1800 - 1000} = \frac{500}{800} \Rightarrow 0.625$

$1000 \rightarrow \frac{1000 - 1000}{800} = 0$

$1800 \rightarrow \frac{1800 - 1000}{800} \Rightarrow 1$

min = 1000
max = 1800

→ Z - Score technique

Student	Height (In Inches)
1	$64 \rightarrow 1.5$
2	$70 \rightarrow 0$
3	$72 \rightarrow 0.5$
4	$68 \rightarrow -0.5$
5	$76 \rightarrow 1.5$

$x' = \frac{\underset{\text{old}}{x} - \underset{\text{mean}}{\mu}}{\underset{\text{SD}}{\sigma}}$ new

$\mu = \frac{350}{5} \Rightarrow 70$

$\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$
$= 4$

$64' = \frac{64 - 70}{4} \Rightarrow \frac{6}{4} = 1.5$

$\frac{76 - 70}{4} \Rightarrow \frac{6}{4} = \frac{3}{2}$
$= 1.5$

$70' = \frac{70 - 70}{4} \Rightarrow 0$

$72' = \frac{72 - 70}{4} \Rightarrow 0.5$

$68' = \frac{68 - 70}{4} \Rightarrow \frac{-2}{4} \Rightarrow -0.5$

Linear Combination of Vectors :

Let's say we have vectors $\vec{u}, \vec{v}$ then $\alpha\vec{u} + \beta\vec{v}$ is a linear combination of vectors $\vec{u}$ & $\vec{v}$ where α & β are scalar.

Q: given 2 vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Ans: linear combination of these 2 is $2 \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \times \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 11 \\ 16 \end{bmatrix}$$

Span of a set :

Suppose you've a set 'S' containing 2 vectors $\vec{u}$ & $\vec{v}$

$$S = \{\vec{u}, \vec{v}\}$$

Span $(S) \equiv$ {set of all linear combinations of elements of S}

$$= \{ a\vec{u} + b\vec{v}, \text{where } a \in R, b \in R \}$$

Eg: $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

$$\text{Span}(S) = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} + \begin{bmatrix} 3\beta \\ 4\beta \end{bmatrix} \rightarrow \begin{bmatrix} \alpha + 3\beta \\ 2\alpha + 4\beta \end{bmatrix}$$

So, $\text{Span}(S) = \left\{ \begin{bmatrix} \alpha + 3\beta \\ 2\alpha + 4\beta \end{bmatrix}, \text{where } \alpha, \beta \in R \right\}$

Eg. $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$span(S) = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ where } \alpha \in R \right\}$

Linear dependent vectors (LD) :→

proper set of vectors :

lets say you've 3 vectors $\{\vec{a}, \vec{b}, \vec{c}\}$
we call this set as linearly independent if
you can ~~write~~ write any 1 vector in that set
as linear combination of other 2 vectors.

i.e if $\vec{c} = \alpha\vec{a} + \beta\vec{b} \rightarrow ①$
(or)
$\vec{a} = \alpha\vec{b} + \beta\vec{c} \rightarrow ②$
(or)
$\vec{b} = \alpha\vec{c} + \beta\vec{a} \rightarrow ③$

$\boxed{1\ I \rightarrow OP}$

2

Eg: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$
a b c

we can write $c = 2b - a$,

So, a, b, c are linearly dependent.

Consider Eq①, $\alpha\vec{a} + \beta\vec{b} - \vec{c} = 0$

we can write this as

$\boxed{\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0}$

with this Equation all above 3 equation would cover.

we call $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ to be L.I

if there exists $\alpha_1, \alpha_2, \dots, \alpha_n$
$\exists$

such that $\boxed{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_n x_n = 0}$

condition: with $\alpha \neq 0$

Linearly Independent Vectors (LI) :→

Closure ← Property:

opposite to LD.

A set which is not L.D is L.I

$$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$$

$$\alpha \begin{bmatrix}1\\0\end{bmatrix} + \beta \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Here you can make those 2 vectors linear combination $= 0$ unless both α & β equals 0.

$$\alpha = \beta = 0$$

Note:

$S = \{\bar{x}_1, \bar{x}_2, \dots, \begin{bmatrix}0\\0\end{bmatrix}\}$

If we're given a set with $\begin{bmatrix}0\\0\end{bmatrix}$ vector then would be automatically a linear dependent set.

How ?

$$\alpha_1 \bar{x}_1 + \beta \bar{x}_2 + \gamma x_3 + \dots + Z \begin{bmatrix}0\\0\end{bmatrix}$$

if put $\alpha_1 = \beta = \gamma = \dots = 0$ & $Z \neq 0$

then that linear combination would equal to 0

So, S would be L.D since $Z \neq 0$

Vector Space (V) : → for a vector space you need a set of vectors (V), Set of Scalars (F), addition (+), Multiplication (.)

$\Downarrow$ Field $\in R, C$

Addition properties:

$\bar{x}, \bar{y} \in V$

Closure Property:

1. When you add 2 vectors in Vector space, you'll get a vector within that Vector space only. $\Rightarrow \bar{x} + \bar{y} \in V$

2. Same for scalar multiplication, i.e, when you multiply a scalar to a vector, you'll get a vector $\in$ to that same Vector space. $\Rightarrow c\vec{x} \in V$

3. Commutative property $\longrightarrow \vec{x} + \vec{y} = \vec{y} + \vec{x}$.

4. Associative property $\longrightarrow (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$.

5. Existence of 0 : $\vec{x} + 0 = \vec{x}$. Addition Identity

6. Existence of Inverse: $\vec{x} + \vec{y} = 0$. Here $\vec{y}$ is Inverse of $\vec{x}$.

Multiplication Properties:

7. multiplicative Identity $\rightarrow 1\vec{x} = \vec{x}$. Here 1 is multiplicative Identity.

8. $(ab)x = a(bx)$, $x \in V$, $a, b \in F$ ----> Associative property

9. $a(x + y) = ax + ay$ $a \in F$, $x, y \in V$.

10. $(a+b)x = ax + bx$, $a, b \in F$, $x \in V$

distributive property

Q) Which of the following is not a subspace of $R^3(R)$?

(a) $W_1 = \{(x, y, z) : x - y - z = 0\}$

(b) $W_2 = \{(x, y, z) : y \geq x\}$

(c) $W_3 = \{(x, y, z) : z = 2x - 3y\}$

(d) $W_4 = \{(x, y, 0) : x, y \in R\}$

Sol (a)

$W_1 = \{(x, y, z) : x - y - z = 0\}$

let,

2 vectors: $(a, b, c) \in W_1 \Rightarrow a - b - c = 0.$

$(d, e, f) \in W_1 \Rightarrow d - e - f = 0.$

Linear combination of above 2 vectors should also $\in$ to W, i.e they should satisfy that condition $x - y - z = 0$

$\alpha(a, b, c) + \beta(d, e, f)$

$= (\alpha a + \beta d, \alpha b + \beta e, \alpha c + \beta f)$ — ①

Condition $\Rightarrow$ $(\alpha a + \beta d) - (\alpha b + \beta e) - (\alpha c + \beta f) = 0$

$\alpha(a - b - c) + \beta(d - e - f) = \alpha(0) + \beta(0)$

$\alpha a - \alpha b - \alpha c + \beta d - \beta e - \beta f = 0$

$(\alpha a + \beta d) - (\alpha b + \beta e) - (\alpha c + \beta f) = 0$ — ②

eq ① = eq ② also, option ⓐ correct.

W_1 is a subspace of R^3

(b) $W_2 = \{(x, y, z) : y \geq x\}$

Let,

$(a, b, c) \in W_2 : b \geq a$

$(d, e, f) \in W_2 : e \geq d$

$\alpha, \beta \in F$ ✓ $b \geq a \Rightarrow \alpha b \geq \alpha a$ $\beta e \geq \beta d$

$\alpha(a, b, c) + \beta(d, e, f)$ $\alpha b + \beta e \geq \alpha a + \beta d$

$(\alpha a, \alpha b, \alpha c) + (\beta d, \beta e, \beta f)$

$(\alpha a + \beta d, \alpha b + \beta e, \alpha c + \beta f)$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}1\\3\\4\end{pmatrix} \Rightarrow \begin{pmatrix}2\\5\\7\end{pmatrix} ✓$$

$$\begin{pmatrix}-2\\-1\\3\end{pmatrix} + \begin{pmatrix}-3\\-2\\4\end{pmatrix} \Rightarrow \begin{pmatrix}-5\\-3\\7\end{pmatrix} ✓$$

$$\begin{pmatrix}-2\\-1\\3\end{pmatrix} + \begin{pmatrix}3\\4\\5\end{pmatrix} \Rightarrow \begin{pmatrix}1\\3\\8\end{pmatrix} ✓$$

$(x, y, z) : y \geq x$

But consider $(-x, -y, -z)$ then $-y \leq -x$.

$$\begin{pmatrix}1\\2\\3\end{pmatrix} ✓ \quad \begin{pmatrix}-1\\-2\\-3\end{pmatrix} \Rightarrow$$ this guy satisfies $-y \leq -x$ but not $y \geq x$ ✓

$-\begin{pmatrix}1\\2\\3\end{pmatrix} + \alpha\begin{pmatrix}0\\?\\?\end{pmatrix}$ ~~$-2 \geq -1$~~

$-2 \leq -1$ ✓ $x, y, z \in \mathbb{R}$ — mention

R^{+} ✓ R^{-} ✗

So, W_2 is not subspace of R^3 yes if R^{+} ✓

(c) $W_3 = \{(x,y,z) : z = 2x - 3y\}$

Let, $(a,b,c) : c = 2a - 3b$

$(d,e,f) : f = 2d - 3e$

Linear Combination??

$\alpha(a,b,c) + \beta(d,e,f) \Rightarrow$

$\alpha c = 2a\alpha - 3b\alpha$ → ①

$\beta f = 2\beta d - 3\beta e$ → ②

$(\alpha a + \beta d, \alpha b + \beta e, \alpha c + \beta f)$

Eq ① + Eq ②

$\alpha c + \beta f = 2a\alpha - 3b\alpha + 2\beta d - 3\beta e$

$(\alpha c + \beta f) = (2a\alpha + 2\beta d) - (3b\alpha + 3\beta e) \Rightarrow$ LHS

$\alpha c + \beta f = 2(\alpha a + \beta d) - 3(\alpha b + \beta e)$

$(\alpha c + \beta f) = (2a\alpha + 2\beta d) - (3b\alpha + 3\beta e) \Rightarrow$ RHS

We're got LHS = RHS

So, W_3 is a subspace of R^3

(d) $W_4 = \{(x,y,0) : x, y \in R\}$

$(x_1, y_1, 0)$; $x_1, y_1 \in R$

$(x_2, y_2, 0)$; $x_2, y_2 \in R$

$\alpha(x_1, y_1, 0) + \beta(x_2, y_2, 0)$

$(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, 0)$

$W_4 \in R^3$ //

it's a subs of R^3

Note: Elements of Vector Space are Called Vectors.

Basis & dimension of Vector Space:

We Know,

$\text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\} \Rightarrow$ is whole of R^2

Basis $\Rightarrow$ Here we want to find a set 'S' such that

(a) Span of that set 'S' $\Rightarrow$ Span(S) = that Complete Vector space 'V'

(b) If you remove any subspace of Vector Space 'V' then span of that would n't give Vector space 'V'.

Eg:

$\text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\} = R^2 \Rightarrow V$

remove this subspace $\rightarrow$ Scalar $\times \begin{bmatrix}3\\4\end{bmatrix}$

$\text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} \Rightarrow R^2$ but won't Equal to Vector space V.

~~Small guy~~ Tailor under & Tailor aunty for Tailor Vector space

So, Here our basis is that Complete vectors set on spanning we'll be getting Complete vector space.

(c) Basis is not unique but no. of Elements in the basis are same.

$\boxed{\text{* dimension of vector space.}}$

Q: $\mathbb{R}^n(\mathbb{R})$

Standard basis:

for $\mathbb{R}^2(\mathbb{R}) \Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ → dimension = 2

for $\mathbb{R}^n(\mathbb{R}) = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \right\}$ → dimension $= n$

$\boxed{P_n(\mathbb{R})}$ → INR set, polynomial upto degree n.

$x^3 + x$, $x^2 + 1$ ----→ for a vector space of polynomials with degree less than equal to 3.

Basis $= \{1, x, x^2, \ldots, x^n\}$ → this is the basis for polynomials upto degree n.

⇓

dimension $= n+1$

bcz their linear combination (span) would give diffent types of polynomials

from Professor Gilbert Stein from MIT

Matrix Multiplication (a different approach - if time permits) →

(a) $\begin{bmatrix} 1 & 1 & 1 \\ c_1 & c_2 & c_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a \begin{bmatrix} 1 \\ c_1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ c_2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ c_3 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ c_1 & c_2 & c_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}$

$$d_1 = 1\begin{bmatrix}1\\c_1\\1\end{bmatrix} + b_1\begin{bmatrix}1\\c_2\\1\end{bmatrix} + 1\begin{bmatrix}1\\c_3\\1\end{bmatrix}$$

$$d_2 = 1\begin{bmatrix}1\\c_1\\1\end{bmatrix} + b_2\begin{bmatrix}1\\c_2\\1\end{bmatrix} + 1\begin{bmatrix}1\\c_3\\1\end{bmatrix}.$$

$$d_3 = 1\begin{bmatrix}1\\c_1\\1\end{bmatrix} + b_3\begin{bmatrix}1\\c_2\\1\end{bmatrix} + 1\begin{bmatrix}1\\c_3\\1\end{bmatrix}$$