QUANTUM CIRCUITS FOR MAXIMALLY ENTANGLED STATES CREATION

INTRODUCTION:

Quantum correlations depend on the delicate balance of the coefficients of the wave function. It is natural to expect that quantum computers will have to be very refined to achieve such a good description of multipartite correlations along the successive action of gates. Entanglement is at the heart of quantum efficiency. Again, if a quantum computer is not able to generate faithful large entanglement, it will remain inefficient. A fundamental factor in quantum computing is the ability to generate large entangled states, such as area law violating states. However, such ability has to be accomplished by a sufficiently large coherence time for such multipartite maximally entangled states. Note that GHZ-like states are highly entangled and useful to test violate qubit Bell inequalities, but even more entangled are the absolute maximally entangled (AME) states, which are maximally entangled in every bipartition of the system.

This work is organized as follows. We review the basic properties of AME states and show explicit examples. We present the quantum circuits that generate AME states by using the properties of graph states. We also propose the simulation of AME states having local dimension larger than 2 by using qubits instead of qudits. We analyze the entanglement majorization criteria in the proposed circuits and find further optimal circuits for experimental implementation by imposing a majorization arrow in terms of entanglement. In Sec. V, we implement GHZ and AME states for five qubit systems in IBM quantum computers, quantifying the state preparation quality by testing maximal violation of suitably chosen Bell inequalities. Finally, We discuss and summarize the main results of the paper.

A. General properties of AME states:

AME states, also known in some references as maximally multipartite entangled states, are n qudit quantum states with local dimension d such that every reduction to n/2 parties is maximally mixed, where · is the floor function. Such states are maximally entangled when considering the average entropy of reductions as a measure of multipartite entanglement, that is, when the average von Neumann entropy S(ρ) = Tr [ρ log ρ], taken over all reductions to n/2 parties, achieves the global maximum value S(ρ) = n/2, where logarithm is taken in basis d. For instance, Bell states and GHZ states are AME states for bipartite and three partite systems, respectively, for any number of internal levels d. The existence of AME states for n qudit systems composed by d levels each, denoted AME (n, d), is a hard open problem in general. This problem is fully solved for any number of qubits: an AME (n, 2) exists only for n = 2, 3, 5, 6 [21–23]. Among all existing AME states, there is one special class composed by minimal support states. These states are defined as follows: an AME (n, d) state has minimal support if it can be written as the superposition of dn/2 fully separable orthogonal pure states. Here, we consider superposition at the level of vectors, in such a way that the linear combination of pure states always produces another pure state. For example, generalized Bell states for two-qudit systems and generalized GHZ states for three-qudit systems have minimal support. It is simple to show that all coefficients of every AME state having minimal support can be chosen to be identically equal to d−n/2/2, i.e., identical positive numbers. By contrast, AME states having nonminimal support are required to be composed by nontrivial phases in their entries in order to have all reduced density matrices proportional to the identity. In other words, nonminimal support AME states require destructive interference.

AME states define an interesting mathematical problem itself but also they define attractive practical applications. These include quantum secret sharing, open destination quantum teleportation, and quantum error correcting codes, the last one being a fundamental ingredient for building a quantum computer.

B.Explicit expressions of AME states:

The simplest AME(n, d) states, denoted n,d , having minimal support are the Bell and GHZ states

$$\Omega_{2,d} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1}$$

$$\Omega_{3,d} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |$$

respectively. These states are AME for any number of internal levels d 2. That is, every single particle reduction in both states 2,d and 3,d produces the maximally mixed state. On the other hand, it is not obvious to prove that there is no AME state for n = 4 qubits. The AME (5,2) state can be written as

$$|\Upsilon_{5,2}\rangle = \frac{1}{4\sqrt{2}} \sum_{i=1}^{32} c_i$$

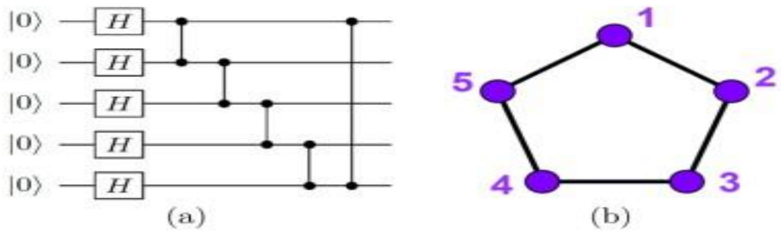
where the five-digits binary decomposition of i should be considered inside the ket and ci = {1, 1, 1, 1, 1, −1, −1, 1, 1, −1, −1, 1, 1, 1, 1, 1, 1, 1, − 1, −1, 1, −1, 1, −1, −1, 1, −1, 1, −1, −1, 1, 1}.

By using local unitary operations, the same state can be reduced to any of the following states:

$$
\begin{aligned}
|0_{L1}\rangle = \tfrac{1}{4}(&|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\
&+ |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\
&- |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\
&- |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle), \\
|1_{L1}\rangle = \tfrac{1}{4}(&|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle \\
&+ |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\
&- |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle \\
&- |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle).
\end{aligned}
$$

B. QUANTUM CIRCUITS TO CONSTRUCT AME STATES:

AME states can be constructed in different ways. For our purpose, we consider graph states formalism. Graph states are represented by an undirected graph, where each vertex corresponds to a |+> state and each edge with a control-Z (CZ) gate. We can easily construct the quantum circuit for a graph state by considering a simple rule, as we will see later. In addition, a graph can be transformed into another—equivalent—one by applying local unitary operations [32]. This kind of transformation modifies the number of edges of a graph but not its entanglement properties. This property could allow us to adapt the circuit to different quantum chip architectures, in order to reduce as much as possible the number of gates required to physically implement the state.



Quantum circuit to generate AME(5,2) (a) and its corresponding graph (b).

A. Graph states

Graph states are n partite pure quantum states constructed from an undirected graph composed by n vertices V = {vi} and connected by edges E = {ei j = {vi, vj}}. Each graph has associated an adjacency matrix A, whose entries satisfy that Ai j = 1 if an edge ei j exists and Ai j = 0 otherwise. Selfinteractions are forbidden, meaning that diagonal entries of A vanish.

A graph state for n qudits can be constructed as follows is the generalized controlled-Z gate, ω = e2πi/d, and

$$|G\rangle = \prod_{i<j}^{n} \mathrm{CZ}_{ij}^{A_{ij}} (F_d|\bar{0}\rangle)^{\otimes n},$$

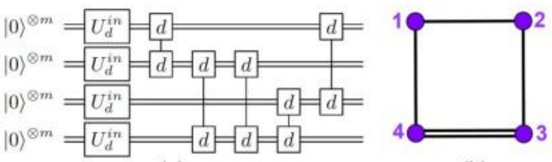
$$\mathrm{CZ}_{ij} = \sum_{k=0}^{d-1} \omega^{kl} |\bar{k}\rangle\langle\bar{k}|_i \otimes |\bar{l}\rangle\langle\bar{l}|_j$$

$$F_d = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{kl} |\bar{k}\rangle\langle\bar{l}|,$$

is the Fourier qudit gate. From now on, we distinguish between qubits and qudits states by writing a bar over symbols associated to qudit states, e.g., |0¯ , keeping the usual notation with no bar for qubits, e.g., |0>. B. AME states from graph states:

We are interested in finding optimal AME graph states, in the sense of having the minimum number of edges and coloring index. The smaller the number of edges the smaller the number of operations required to generate AME states. Coloring index is related with the number of operations that can be performed in parallel, so it is proportional to the circuit depth. It worth mentioning that graph AME states are hard to construct in general, especially for large values of local dimension d and number of parties n. Fortunately, there are suitable tools useful to simplify the construction of graphs for specific values of d and n.

For a nonprime local dimension there exists some methods to find AME graph states. One of those consists on taking the prime factorization d = d1d2 ··· dm and looking for every AME(n, di) state. The AME(n, d) is just given by the tensor product of the AME(n, di) states, followed by a suitable relabeling of symbols. When prime factorization of d includes a power of some factor, we can construct an AME state by artificially defining each party, i.e., by using qudits in lower dimension m < d and then performing the suitable set of CZ gates between the m level qudit systems. For instance, this method can be used to find the AME (4,4) state from qubits instead of ququarts (qudits with d = 4 levels each).

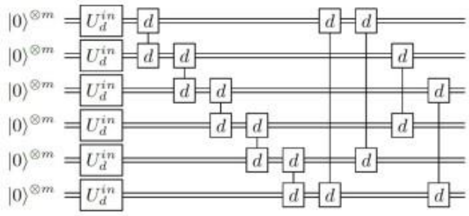


(a) (b)

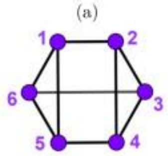
Graph state that generates an AME(4,d) state for any prime dimension d 3 (b) and its corresponding circuit (a) by using qubits instead of qudits.

C. AME states circuits using qubits

The construction of a qubit quantum circuit from a graph state is straightforward since we just have to perform Hadamard gates on all qubits initialized at |0 state and CZ gates, according to graph edges. These quantum gates are commonly used in current quantum devices, e.g., in quantum computing [38]. However, in order to implement an AME state for d > 2 internal levels we require a qudit quantum



(a)

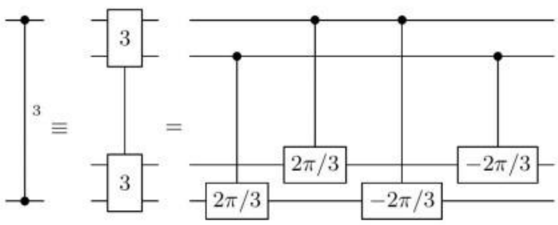


(b)

D. AME states circuits of minimal support:

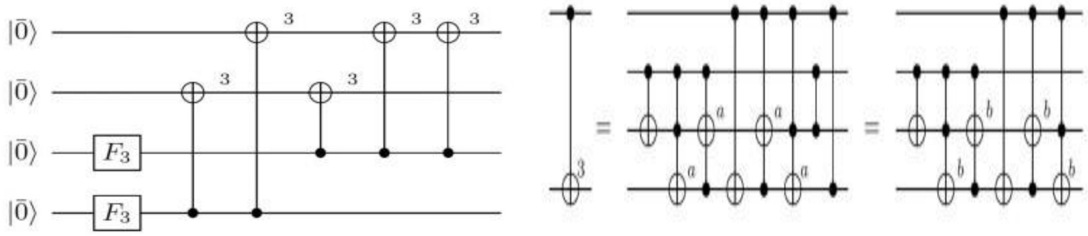
For qutrits, the AME(4,3) state of Eq. (10) has minimal support. The quantum circuit that generates this state is shown in Fig. 9 [14]. The quantum gates required to construct this circuit are the Fourier transform gate for qutrits F3 and the C3–adder gate C3|i>|
j>=|i>|i + j>

which is the generalization of controlled-NOT (CNOT) gate for qutrits. It is represented with the CNOT symbol with the superscript;



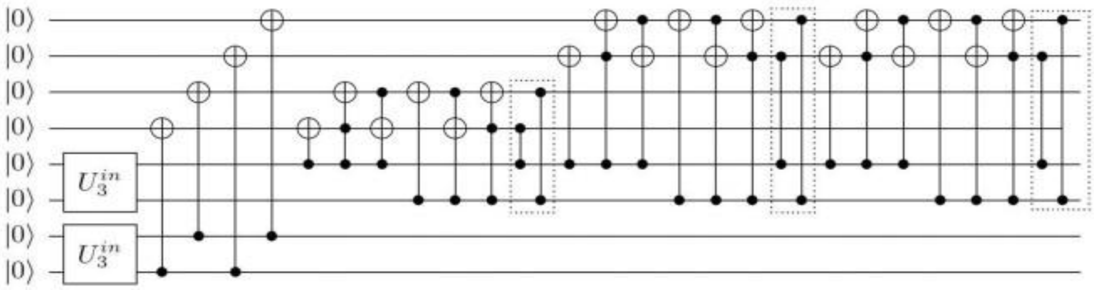
D.ENTANGLEMENT MAJORIZATION:

Majorization has deep implications in quantum information theory [40]. In particular, quantum algorithms obey a majorization arrow, which means that majorization could be at the core of their efficiency [41,42]. Following this idea, we wonder whether the above quantum circuits designed to construct AME states obey majorization. If not, it is interesting to ask whether more efficient circuits obeying majorization exist.

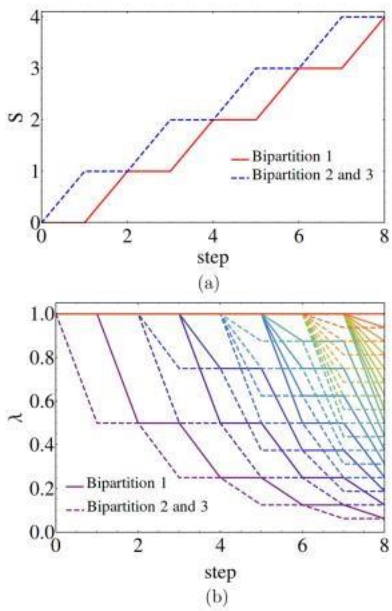


E. EXPERIMENTAL IMPLEMENTATION:

The experimental implementation of an AME state is a highly demanding task for a quantum computer. It requires the consideration of some figure of merit in order to test the quality of the preparation state. For qubit AME states of bipartite and three-partite systems one can consider Mermin Bell inequalities as a figure of merit, as they are maximally violated by these states.



Circuit for the construction of the AME(4,3) state by using two qubits to represent each qutrit. The controlled-Z gates (framed with dots) are only necessary when we use the approximation of Toffoli gate CCNOTa.



Majorization in AME(4,4) state circuit of Fig. 5. Entropy increases at each step s in all bipartitions until it reaches the maximum value S = 2 log2 4 = 4 (a). Majorization in terms of eigenvalues of the reduced density matrix. At the end of the computation, all eigenvalues are the same, which leads to a density matrix proportional to the identity (b).

CONCLUSIONS:

Quantum computing is a challenging field of research in quantum mechanics that could change the way we do computations in the future. The ultimate goal of a quantum computer is to coherently control a relatively large number of qubits in such a way that a multipartite quantum protocol can be successfully implemented, despite the inherent decoherence of quantum information. It is natural to expect that quantum over classical advantage in computing is directly related to the amount of quantum correlations existing in the involved qubits. It is thus a remarkably important task to understand the behavior of quantum computers when multipartite correlations take extreme values, e.g., when the system is a genuinely multipartite maximally entangled state.

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