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The Problems Submitted by Ramanujan to the Journal of the Indian Mathematical Society

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To Jerry Lange on his 70th birthday

1. Introduction

Between the years 1911 and 1919 Ramanujan submitted a total of 58 problems, several with multiple parts, to the *Journal of the Indian Mathematical Society.* For the first five, the spelling *Ramanujam* was used. Several of the problems are elementary and can be attacked with a background of only high school mathematics. For others, significant amounts of hard analysis are necessary to effect solutions, and a few problems have not been completely solved. Every problem is either interesting or curious in some way. All 58 problems can be found in Ramanujan's *Collected Papers* [**172, pp. 322–334**]. As is customary in problems sections of journals, editors normally prefer to publish solutions other than those given by the proposer. However, if no one other than the proposer has solved the problem, or if the proposer's solution is particularly elegant, then the proposer's solution is published. This was likely the practice followed by the editors of the *Journal of the Indian Mathematical Society*, but naturally the editors of Ramanujan's *Collected Papers* chose different criteria; only those printed solutions by Ramanujan were reproduced in his *Collected Papers.*

The publication of the *Collected Papers* in 1927 brought Ramanujan's problems to a wider mathematical audience. Several problems have become quite famous and have attracted the attention of many mathematicians. Some problems have spawned a plethora of papers, many containing generalizations or analogues. It thus seems appropriate to provide a survey of all 58 problems indicating the activity generated by the problems since 1927.

In referring to these problems, we follow the numbering given in the *Journal of the Indian Mathematical Society.* Although the division of problems into categories is always somewhat arbitrary, we have decided to place the 58 problems in nine subsets as follows:

Solutions of Equations: 283, 284, 507, 666, 722
Radicals: 289, 524, 525, 682, 1070, 1076
Further Elementary Problems: 359, 785

Number Theory: 427, 441, 464, 469, 489, 584, 629, 661, 681, 699, 770, 723, 784
Integrals: 295, 308, 353, 386, 463, 739, 783
Series: 260, 327, 358, 387, 546, 606, 642, 700, 724, 768, 769
Continued Fractions: 352, 541, 1049
Other Analysis: 261, 294, 526, 571, 605, 738, 740, 753, 754
Geometry: 662, 755

Some of Ramanujan's problems have been slightly rephrased by the editors of his *Collected Papers.* Generally, we quote either Ramanujan's formulation of each problem, or the version in the *Collected Papers.* However, we have taken the liberty of replacing occasional archaic spelling by more contemporary spelling, and most often we have employed summation notation in place of the more elaborate notation $a_1 + a_2 + \cdots$. After the number of the question, the volume and page number(s) where the problem first appeared in the *Journal of the Indian Mathematical Society*, which we abbreviate by JIMS, are stated, and these are followed by the volume(s) and page number(s) where solutions, partial solutions, or comments are given. We do not cite problems individually in the references of this paper. We also do not separately list in our references the solvers of the problems cited in Ramanujan's *Collected Papers* [**172**]. However, if a solution was published *after* the publication of the *Collected Papers* in 1927, then we record it as a separate item in the bibliography. Many of the problems, or portions thereof, can be found in Ramanujan's notebooks [**171**]. Normally in such a case, we cite where a problem can be located in the notebooks and where it can also be found in Berndt's accounts of the notebooks [**20**]–[**24**].

2. Solutions of Equations

Question 283 (JIMS 3, p. 89; 3, pp. 198–200; 4, p. 106). *Show that it is possible to solve the equations*

$$\begin{aligned} x+y+z&=a, & px+qy+rz&=b,\\ p^2x+q^2y+r^2z&=c, & p^3x+q^3y+r^3z&=d,\\ p^4x+q^4y+r^4z&=e, & p^5x+q^5y+r^5z&=f, \end{aligned}$$

where x,y,z,p,q,r are the unknowns. Solve the above when $a=2, b=3, c=4, d=6, e=12,$ and $f=32$.

Question 283 is a special case of the more general system

$$\begin{aligned} x_1+x_2+\cdots+x_n&=a_1,\\ x_1y_1+x_2y_2+\cdots+x_ny_n&=a_2,\\ x_1y_1^2+x_2y_2^2+\cdots+x_ny_n^2&=a_3,\\ &\vdots\\ x_1y_1^{2n-1}+x_2y_2^{2n-1}+\cdots+x_ny_n^{2n-1}&=a_{2n}, \end{aligned} \tag{2.1}$$

where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are $2n$ unknowns, ingeniously solved by Ramanujan in his third published paper [**164**], [**172, pp. 18–19**]. Implicit assumptions were made in Ramanujan's solution, and thus it should be emphasized that (2.1) is not always solvable. For a sketch of Ramanujan's solution, see a paper by Berndt and S. Bhargava [**25**]. Another derivation of the general solution for (2.1) was found by M. T. Naraniengar [**148**]. The more general system (2.1) is also found on page 338 in Ramanujan's second notebook [**171**]. We quote the discussion of (2.1) from Berndt's book [**23, p. 30**].

"It is easy to see that the system (2.1) is equivalent to the single equation

$$\sum_{i=1}^{n} x_i(y_i s+t)^{2n-1} = \sum_{j=0}^{2n-1} \binom{2n-1}{j} a_{j+1} s^j t^{2n-1-j}.$$

Thus, Ramanujan's problem is equivalent to the question: When can a binary $(2n-1)$–ic form be represented as a sum of n $(2n-1)$th powers? In 1851, J. J. Sylvester [**196**], [**197**], [**198, pp. 203–216, 265–283**] found the following necessary and sufficient conditions for a solution: The system of n equations,

$$\begin{aligned} a_1u_1 + a_2u_2 + \cdots + a_{n+1}u_{n+1} &= 0,\\ a_2u_1 + a_3u_2 + \cdots + a_{n+2}u_{n+1} &= 0,\\ &\vdots\\ a_nu_1 + a_{n+1}u_2 + \cdots + a_{2n}u_{n+1} &= 0, \end{aligned}$$

must have a solution $u_1, u_2, \ldots, u_{n+1}$ such that the n–ic form

$$p(w,z) := \sum_{j=0}^{n} u_{j+1} w^j z^{n-j}$$

can be represented as a product of n distinct linear forms. Thus, the numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are related to the factorization of $p(w,z)$. Sylvester's theorem belongs to the subject of invariant theory, which was developed in the late nineteenth and early twentieth centuries. For a contemporary account, but with classical language, see a paper by J. P. S. Kung and G.–C. Rota [**113**]."

QUESTION 284 (JIMS 3, P. 89; 4, P. 183). *Solve*

$$\frac{x^5-6}{x^2-y} = \frac{y^5-9}{y^2-x} = 5(xy-1).$$

This problem is the special case $a=6, b=9$ of the more general system

$$\frac{x^5-a}{x^2-y} = \frac{y^5-b}{y^2-x} = 5(xy-1),$$

recorded by Ramanujan on page 338 in his second notebook and solved by Berndt in [**23, pp. 27–29**]. The solutions comprise 25 values for both x and y. As pointed out in [**23, pp. 28–29**], Ramanujan's published solution, which is reproduced in [**172, pp. 322–323**], contains some mistakes. We very briefly describe the solutions. Let

$$x = \alpha+\beta+\gamma \qquad \text{and} \qquad y = \alpha\beta+\beta\gamma+\gamma\alpha,$$

where $\alpha\beta\gamma = 1$. Then [**23, pp. 28–29**] α^5, β^5, and γ^5 are roots of the equation

$$t^3 - at^2 + bt - 1 = 0.$$

Ramanujan then listed the values of x as

$$\alpha+\beta+\gamma, \quad \alpha+\beta\rho+\gamma\rho^4, \quad \alpha+\beta\rho^2+\gamma\rho^3, \quad \alpha\rho+\beta\rho+\gamma\rho^3,$$
$$\alpha\rho+\beta\rho^2+\gamma\rho^2, \quad \alpha\rho^2+\beta\rho^4+\gamma\rho^4, \quad \alpha\rho^3+\beta\rho^3+\gamma\rho^4,$$

where ρ is a primitive fifth root of unity. However, observe that the third member of this set may be derived from the second member by replacing ρ by ρ^2. Also, the seventh can be obtained from the fourth by replacing ρ by ρ^3. Lastly, the sixth arises from the fifth when ρ is replaced by ρ^2. Ramanujan missed the values

$$\alpha\rho + \beta + \gamma\rho^4, \quad \alpha\rho + \beta\rho^4 + \gamma, \quad \alpha\rho + \beta\rho^3 + \gamma\rho.$$

QUESTION 507 (JIMS 5, P. 240; 6, PP. 74–77). *Solve completely*

$$x^2 = y + a, \qquad y^2 = z + a, \qquad z^2 = x + a \tag{2.2}$$

and hence show that

$$\sqrt{8 - \sqrt{8 + \sqrt{8 - \cdots}}} = 1 + 2\sqrt{3}\sin 20^\circ, \tag{a}$$

$$\sqrt{11 - 2\sqrt{11 + 2\sqrt{11 - \cdots}}} = 1 + 4\sin 10^\circ, \tag{b}$$

$$\sqrt{23 - 2\sqrt{23 + 2\sqrt{23 + 2\sqrt{23 - \cdots}}}} = 1 + 4\sqrt{3}\sin 20^\circ. \tag{c}$$

In parts (a)–(c), the signs under the outward-most radical sign have period three; they are, respectively, $-,+,-;-,+,-;-,+,+$.

Ramanujan's published solution in the *Journal of the Indian Mathematical Society* is correct. However, there are four sign errors in the solution published in his *Collected Papers* [**172, pp. 327–329**], which we now relate. The equations in (2.2) easily imply that x satisfies a polynomial equation of degree 8. This polynomial factors over $\mathbb{Q}(\sqrt{4a-7})$ into a quadratic polynomial $x^2 - x - a$ and two cubic polynomials, which are given in the *Collected Papers* near the end of Ramanujan's solution on page 328 and just before the verifications of the three examples (a)–(c). Each of these two cubic polynomials has two sign errors. The polynomials are correctly given in Ramanujan's original solution and in Berndt's book [**23, pp. 10, 11, eqs. (4.2), (4.3)**].

The formulation of Question 507 indicates that the three examples can be deduced from the solutions of (2.2). However, in his published solution, Ramanujan did not do this but established each identity *ad hoc*. In his solution to Question 507, M. B. Rao [**176**] derived each of the examples, (a)–(c), from the general solution of (2.2).

On pages 305–307 in his second notebook [**171**], Ramanujan offers a more extensive version of Question 507 [**23, pp. 10–20**]. By taking successive square roots in (2.2), it is not difficult to see that we can approximate the roots by an infinite sequence of nested radicals

$$\sqrt{a}, \quad \sqrt{a + \sqrt{a}}, \quad \sqrt{a + \sqrt{a + \sqrt{a}}}, \quad \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a}}}}, \ldots.$$

However, we must be careful in taking these square roots, for there are three different square roots to be taken and there are two choices for the sign of the square root in each case. The 2^3 different sequences of nested radicals correspond to the eight roots of the octic polynomial arising from (2.2). Of course, one must determine the values of a for which the eight infinite sequences converge. In [**23, pp. 14–16**], it was shown that the infinite sequences converge for $a \geq 2$, which is not the best possible result. For example, it was indicated in [**23, p. 15**] that the sequence

$$a_1 = \sqrt{a}, \quad a_2 = \sqrt{a - \sqrt{a}}, \quad a_3 = \sqrt{a - \sqrt{a + \sqrt{a}}},$$
$$a_4 = \sqrt{a - \sqrt{a + \sqrt{a + \sqrt{a}}}}, \dots,$$

with signs $+, -, +$ of period 3, likely converges at least for $a \geq 1.9408$. In our discussion of Question 289, we will return to the question of the convergence of infinite sequences of nested radicals.

For a particular value of a, one can numerically check which infinite sequence of nested radicals corresponds to a given root. In general, for the two infinite sequences of nested radicals arising from the two roots of the quadratic polynomial, the identification is easy. However, for the remaining six roots, the problem is more difficult. On pages 305–306 in his second notebook, Ramanujan made these general identifications [**23, p. 17, Entry 5**]. In [**23, p. 18**], asymptotic expansions of the six roots and the six sequences of nested radicals, as $a \to \infty$, were established in order to obtain the desired matchings.

There is also a very brief discussion of the system (2.2) in F. Cajori's book [**53, pp. 196–197**].

QUESTION 722 (JIMS 8, P. 240). *Solve completely*

$$x^2 = a + y, \quad y^2 = a + z, \quad z^2 = a + u, \quad u^2 = a + x;$$

and deduce that, if

$$x = \sqrt{5 + \sqrt{5 + \sqrt{5 - \sqrt{5 + x}}}},$$

then

$$x = \frac{1}{2}\left(2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}}\right);$$

and that, if

$$x = \sqrt{5 + \sqrt{5 - \sqrt{5 - \sqrt{5 + x}}}},$$

then

$$x = \frac{1}{4}\left(\sqrt{5} - 2 + \sqrt{13 - 4\sqrt{5}} + \sqrt{50 + 12\sqrt{5} - 2\sqrt{65 - 20\sqrt{5}}}\right).$$

Question 722 is obviously an analogue of Question 507. The first complete solution to this problem was given by M. B. Rao [**176**] in 1925. He also solved Question 507, examined the analogue for five equations of the same type as Questions 507 and 722, and presented several examples. The second solution was given in 1929 by G. N. Watson [**210**], who also derived Ramanujan's two examples. A considerably shorter solution was found by A. Salam [**183**] in 1943. Question 722 can also be found in Ramanujan's third notebook [**171, vol. 2, p. 367**], and a fourth solution is given in Berndt's book [**23, pp. 42–47, Entry 32, Corollary**].

QUESTION 666 (JIMS 7, P. 120; 8, P. 31). *Solve in positive rational numbers*

$$x^y = y^x.$$

For example, $x = 4, y = 2; x = 3\frac{3}{8}, y = 2\frac{1}{4}$.

Since the published solution by J. C. Swaminarayan and R. Vythynathaswamy is short and elegant and was not published in Ramanujan's *Collected Papers*, we reproduce it here.

Put $x = ky$. It follows that

$$y^{k-1} = k.$$

It is easy to see that k is a rational solution if and only if $k = 1 + 1/n$ for some positive integer n. Thus,

$$y = \left(1 + \frac{1}{n}\right)^n \qquad \text{and} \qquad x = \left(1 + \frac{1}{n}\right)^{n+1}.$$

When $n = 1, x = 4$ and $y = 2$; when $n = 2, x = \frac{27}{8}$ and $y = \frac{9}{4}$.

3. Radicals

QUESTION 289 (JIMS 3, P. 90; 4, P. 226). *Find the value of*

(i) $$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}},$$

(ii) $$\sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \cdots}}}.$$

The values of (i) and (ii) are 3 and 4, respectively. Ramanujan's solutions in volume 4 [**172, p. 323**] are not completely rigorous. A note by T. Vijayaraghavan at the end of Appendix 1 in Ramanujan's *Collected Papers* [**172, p. 348**] justifies Ramanujan's formal procedure. This note was considerably amplified in a letter from Vijayaraghavan to B. M. Wilson on January 4, 1928 [**36, pp. 275–278**]. If $a_j > 0, 1 \leq j < \infty$, then Vijayaraghavan proved that a sufficient condition for the convergence of the sequence

$$t_n := \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n}}}$$

is that

$$\overline{\lim_{n\to\infty}} \frac{\log a_n}{2^n} < \infty.$$

Vijayaraghavan's criterion for the convergence of infinite nested radicals is a problem in Pólya and Szegö's book [**156, Prob. 162, pp. 37, 214**]. A. Herschfeld [**96**] also proved Vijayaraghavan's criterion. In a later question submitted to the *Journal of the Indian Mathematical Society*, Vijayaraghavan [**204**] claimed a stronger theorem for the convergence of t_n, which he said was best possible. However, the problem is flawed, and no correction or solution was evidently published. However, Problem 163 in Pólya and Szegö's book [**156, pp. 37, 214**] is likely the criterion that Vijayaraghavan had in mind. A sufficient condition for convergence when the sequence $\{a_j\}$ is complex has been established by G. Schuske and W. J. Thron

[**189**]. Convergence criteria for certain infinite nested radicals of pth roots have been given by J. M. Borwein and G. de Barra [**42**].

Part (i) appeared as a problem in the William Lowell Putnam competition in 1966 [**141**]. The values of (i) and (ii) appear as examples for a general theorem of Ramanujan on nested radicals in Section 4 of Chapter 12 in his second notebook [**21, p. 108, Entry 4**]. Further references for Question 289 and related work are found in Section 4, and additional examples of infinite nested radicals appear in Entries 5 and 6 in that same chapter [**21, pp. 109–112**].

QUESTION 524 (JIMS 6, P. 39; 6, PP. 190–191). *Show that*

$$\text{(i)} \qquad \sqrt[3]{\cos \tfrac{2}{7}\pi} + \sqrt[3]{\cos \tfrac{4}{7}\pi} + \sqrt[3]{\cos \tfrac{8}{7}\pi} = \sqrt[3]{\tfrac{1}{2}(5 - 3\sqrt{7})},$$

$$\text{(ii)} \qquad \sqrt[3]{\cos \tfrac{2}{9}\pi} + \sqrt[3]{\cos \tfrac{4}{9}\pi} + \sqrt[3]{\cos \tfrac{8}{9}\pi} = \sqrt[3]{\tfrac{1}{2}(3\sqrt[3]{9} - 6)}.$$

Part (i) appears on the last page of Ramanujan's second notebook [**171, p. 356**]. The proofs of (i) and another companion result by Berndt in [**23, p. 39**] use a general result of Ramanujan [**23, p. 22, Entry 10**] on the sum of the cube roots of the three roots of a cubic polynomial. The proofs of (i) and (ii) by N. Sankara Aiyar in volume 6 of the *Journal of the Indian Mathematical Society* are similar.

QUESTION 682 (JIMS 7, P. 160; 10, P. 325). *Show how to find the cube roots of surds of the form* $A + \sqrt[3]{B}$, *and deduce that*

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{\tfrac{1}{9}} - \sqrt[3]{\tfrac{2}{9}} + \sqrt[3]{\tfrac{4}{9}}.$$

QUESTION 525 (JIMS 6, P. 39; 6, PP. 191–192). *Show how to find the square roots of surds of the form* $\sqrt[3]{A} + \sqrt[3]{B}$, *and hence prove that*

$$\text{(i)} \qquad \sqrt{\sqrt[3]{5} - \sqrt[3]{4}} = \tfrac{1}{3}\left(\sqrt[3]{2} + \sqrt[3]{20} - \sqrt[3]{25}\right),$$

$$\text{(ii)} \qquad \sqrt{\sqrt[3]{28} - \sqrt[3]{27}} = \tfrac{1}{3}\left(\sqrt[3]{98} - \sqrt[3]{28} - 1\right).$$

QUESTION 1070 (JIMS 9, P. 160; 16, PP. 122–123). *Show that*

$$\text{(i)} \quad \left(\sqrt[5]{\tfrac{1}{5}} + \sqrt[5]{\tfrac{4}{5}}\right)^{1/2} = \left(1 + \sqrt[5]{2} + \sqrt[5]{8}\right)^{1/5} = \sqrt[5]{\tfrac{16}{125}} + \sqrt[5]{\tfrac{8}{125}} + \sqrt[5]{\tfrac{2}{125}} - \sqrt[5]{\tfrac{1}{125}};$$

(ii)

$$\left(\sqrt[5]{\tfrac{32}{5}} - \sqrt[5]{\tfrac{27}{5}}\right)^{1/3} = \sqrt[5]{\tfrac{1}{25}} + \sqrt[5]{\tfrac{3}{25}} - \sqrt[5]{\tfrac{9}{25}};$$

(iii)

$$\left(\frac{3 + 2\sqrt[4]{5}}{3 - 2\sqrt[4]{5}}\right)^{1/4} = \frac{\sqrt[4]{5} + 1}{\sqrt[4]{5} - 1}.$$

QUESTION 1076 (JIMS 11, P. 199). *Show that*

(i) $$\left(7\sqrt[3]{20}-19\right)^{1/6}=\sqrt[3]{\tfrac{5}{3}}-\sqrt[3]{\tfrac{2}{3}};$$

(ii) $$\left(4\sqrt[3]{\tfrac{2}{3}}-5\sqrt[3]{\tfrac{1}{3}}\right)^{1/8}=\sqrt[3]{\tfrac{4}{9}}-\sqrt[3]{\tfrac{2}{9}}+\sqrt[3]{\tfrac{1}{9}}.$$

Of course, one can establish each of the nine identities in the four preceding problems by taking the appropriate power of each side of each equality, applying the multinomial theorem, and simplifying. However, such a proof provides no insight whatsoever into such an equality, nor does it indicate how Ramanujan might have discovered it. Both the left and right sides of each of the equalities are units in some algebraic number field. Although Ramanujan never used the term *unit*, and probably did not formally know what a unit was, he evidently realized their fundamental properties. He then recognized that taking certain powers of units often led to elegant identities.

Berndt, Chan, and Zhang [**33**] have found generalizations of the identities above. For example, for any real number a,

$$\left(\frac{(a+4)\sqrt[3]{a}+(1-2a)\sqrt[3]{4}}{9}\right)^{1/2}=\frac{\sqrt[3]{2}+\sqrt[3]{4a}-\sqrt[3]{a^2}}{3}$$

and

$$\left(\frac{(a+2)\sqrt[3]{4a}+(1-4a)}{9}\right)^{1/2}=\frac{\sqrt[3]{2a^2}-\sqrt[3]{4a}-1}{3},$$

which when we set $a=5$ in the former equality and $a=7$ in the latter yield (i) and (ii), respectively, of Question 525. As another example, for any real number a,

$$\left((a^2-7a+1)+(6a-3)\sqrt[3]{a}+(6-3a)\sqrt[3]{a^2}\right)^{1/3}=\sqrt[3]{a^2}-\sqrt[3]{a}-1,$$

which with $a=2$ yields the equality in Question 682.

In both the original formulation and the *Collected Papers* [**172, p. 334**], the exponents 1/6 and 1/8 on the left sides of (i) and (ii) in Question 1076 were unfortunately inverted. The inversion likely was not immediately discovered, as the first solution to the corrected problems was not given until 1927 by S. Srinivasan [**194**]. In 1929, a second, shorter solution was given by R. Kothandaraman [**110**].

On page 341 in his second notebook [**171**], Ramanujan stated two general radical identities involving cube roots. For example,

$$\begin{aligned}&\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}}\\&=\frac{1}{3}\left\{\sqrt[3]{(4m+n)^2}+\sqrt[3]{4(m-2n)(4m+n)}-\sqrt[3]{2(m-2n)^2}\right\}.\end{aligned}$$

See [**23, pp. 34–36**] for these two identities and several additional examples of the sorts we have discussed here.

Many papers have been written on simplifying radicals. In particular, we mention papers by R. Zippel [**224**] and T. J. Osler [**151**]. S. Landau [**116**], [**117**] has given Galois explanations for several types of radical identities.

4. Further Elementary Problems

QUESTION 785 (JIMS 8, PP. 159–160; 8, P. 232). *Show that*

$$\left(3\left\{(a^3+b^3)^{1/3}-a\right\}\left\{(a^3+b^3)^{1/3}-b\right\}\right)^{1/3}=(a+b)^{2/3}-(a^2-ab+b^2)^{1/3}. \tag{4.1}$$

This is analogous to

$$\left(2\left\{(a^2+b^2)^{1/2}-a\right\}\left\{(a^2+b^2)^{1/2}-b\right\}\right)^{1/2}=a+b-(a^2+b^2)^{1/2}. \tag{4.2}$$

The proof by K. K. Ranganatha Aiyar, R. D. Karve, G. A. Kamtekar, L. N. Datta, and L. N. Subramanyam is clever and short, and so we give it.

In the identity

$$(a+b-r)^3=(a+b)^3-r^3-3r(a+b)^2+3r^2(a+b),$$

put $r^3=a^3+b^3$. Thus,

$$\begin{aligned}(a+b-r)^3=&3ab(a+b)-3r(a+b)^2+3r^2(a+b)\\ =&3(a+b)(r-a)(r-b).\end{aligned}$$

Hence,

$$\{3(r-a)(r-b)\}^{1/3}=(a+b)^{2/3}-\left(\frac{a^3+b^3}{a+b}\right)^{1/3},$$

from which (4.1) follows.

Equality (4.2) can be proved in a similar fashion.

QUESTION 359 (JIMS 4, P. 78; 15, PP. 114–117). *If*

$$\sin(x+y)=2\sin\left(\tfrac{1}{2}(x-y)\right), \qquad \sin(y+z)=2\sin\left(\tfrac{1}{2}(y-z)\right),$$

prove that

$$\left(\tfrac{1}{2}\sin x\cos z\right)^{1/4}+\left(\tfrac{1}{2}\cos x\sin z\right)^{1/4}=(\sin 2y)^{1/12},$$

and verify the result when

$$\sin 2x=(\sqrt{5}-2)^3(4+\sqrt{15})^2, \qquad \sin 2y=\sqrt{5}-2, \qquad \sin 2z=(\sqrt{5}-2)^3(4-\sqrt{15})^2.$$

Note that it took over ten years before a solution was submitted. More recently, another solution was given by V. R. Thiruvenkatachar and K. Venkatachaliengar [**203, pp. 2–9**], but their solution is also quite lengthy. It would seem desirable to have a briefer, more elegant solution, but perhaps this is not possible.

5. Number Theory

QUESTION 441 (JIMS 5, P. 39; 6, PP. 226–227). *Show that*

$$(6a^2-4ab+4b^2)^3=(3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3+(5a^2-5ab-3b^2)^3 \tag{5.1}$$

and find other quadratic expressions satisfying similar relations.

This problem gives a two–parameter family of solutions to Euler's diophantine equation

$$A^3+B^3+C^3=D^3. \tag{5.2}$$

The published solution by S. Narayanan, in fact, gives a more general family of solutions; if $\ell = \lambda(\lambda^3+1), m = 2\lambda^3-1, n = \lambda(\lambda^3-2)$, and $p = \lambda^3+1$, then

$$(\ell a^2 - nab + nb^2)^3 = (pa^2 + mab - mb^2)^3 + (na^2 - nab + \ell b^2)^3 + (ma^2 - mab - pb^2)^3.$$

Equation (5.1) is obtained by setting $\lambda = 2$ above.

The equality (5.1) is mentioned by Hardy and Wright [**93, p. 201**]. C. Hooley [**99**] employed (5.1) to obtain a lower bound for the number of integers less than x that can be represented as a sum of two cubes.

Question 441 can also be found on page 266 in Ramanujan's second notebook [**23, p. 56**]. In his second notebook, Ramanujan gave further families of solutions to (5.2). First, he gave two families of solutions which include (5.1) as special cases [**23, p. 54, Entry 1; p. 55, Entry 3**]. Second, Ramanujan gave another two–parameter family of solutions to (5.2) and several examples [**22, pp. 197–199, Entry 20(iii)**].

According to Dickson [**73, p. 550**], the problem of finding rational or integral solutions to (5.2) can be traced back to Diophantus [**74**]. Euler [**77**], [**79, pp. 428–458**] found the most general family of rational solutions to (5.2). Ramanujan, in his third notebook [**171, vol. 2, p. 387**], [**23, pp. 107, 108**] also gave the most general solution of (5.2), but in a different formulation. Both Hardy [**92, p. 11**] and Watson [**213, p. 145**] were unaware of this entry in the notebooks and so did not realize that Ramanujan had found the most general solution of (5.2). There are, in fact, several forms of the general solution; Hardy and Wright [**93, p. 200**] present one due to A. Hurwitz. Further references to general solutions of (5.2) can be found in Dickson's *History* [**73, pp. 550–561**].

The complete characterization of all *integral* solutions to (5.2) is an open problem. However, C. Sándor [**186**] has solved the problem if one restricts the solutions to be represented by quadratic forms in 2 variables; note that Ramanujan's solutions in Question 441 are represented by quadratic forms. Sándor's paper also contains a summary of further (especially recent) progress on the problem of finding all integral solutions to (5.2).

QUESTION 661 (JIMS 7, P. 119; 13, PP. 15–17; 14, PP. 73–77). *Solve in integers*

$$x^3 + y^3 + z^3 = u^6 \tag{5.3}$$

and deduce the following:

$$\begin{aligned} 6^3 - 5^3 - 3^3 &= 2^6, & 8^3 + 6^3 + 1^3 &= 3^6, \\ 12^3 - 10^3 + 1^3 &= 3^6, & 46^3 - 37^3 - 3^3 &= 6^6, \\ 174^3 + 133^3 - 45^3 &= 14^6, & 1188^3 - 509^3 - 3^3 &= 34^6. \end{aligned}$$

Obviously, (5.3) is a special case of Euler's diophantine equation (5.2).

The solutions in both volumes 13 and 14 of the *Journal of the Indian Mathematical Society* are by N. B. Mitra. In volume 13, Mitra presented four methods for obtaining solutions of (5.3). The first family of solutions contains the special case $8^3 + 6^3 + 1^3 = 3^6$. The second family contains as special cases all examples in the first column above, and the second example in the second column. In volume 14, Mitra established the general rational solution of (5.3). However, the most general solution in *integers* is not known.

QUESTION 681 (JIMS 7, P. 160; 13, P. 17; 14, PP. 73–77). *Solve in integers*

$$x^3 + y^3 + z^3 = 1, \tag{5.4}$$

and deduce the following:

$$6^3 + 8^3 = 9^3 - 1, \quad 9^3 + 10^3 = 12^3 + 1, \quad 135^3 + 138^3 = 172^3 - 1,$$

$$791^3+812^3 = 1010^3-1, \;\; 11161^3+11468^3 = 14258^3+1, \;\; 65601^3+67402^3 = 83802^3+1.$$

Clearly, (5.4) is a particular instance of Euler's diophantine equation (5.2).

Observe that the second example given above gives the famous "taxi–cab" representations of 1729 [**172, p. xxxv**], [**22, p. 199**].

As in the previous problem, the solutions in both volumes 13 and 14 of the *Journal of the Indian Mathematical Society* are by N. B. Mitra. In volume 13, Mitra established a one–parameter family of solutions yielding the first two examples above. In volume 14, he derived the general rational solution of (5.4). M. Venkata Rama Ayyar [**15**] found other methods for examining Questions 441, 661, and 681. In an earlier paper [**16**], he and M. B. Rao found further solutions to (5.4). Some very special solutions to (5.4) are found in Ramanujan's lost notebook [**173, p. 341**]. M. Hirschhorn [**97**], [**98**] has supplied a plausible argument for how Ramanujan might have discovered these particular solutions.

In contrast to Question 661, Question 681 has attracted considerable attention in the literature. R. D. Carmichael [**55**] raised the problem of finding the general integral solution of (5.4), and H. C. Bradley [**44**] asked if the one–parameter family

$$x = 3r - 9r^4, \qquad y = 1 - 9r^3, \qquad z = 9r^4, \tag{5.5}$$

of solutions (also found by Mitra) constitutes all integral solutions to (5.4). They furthermore asked for a general solution, if (5.5) does not give all integral solutions. N. B. Mitra [**142**] negatively answered the first question by constructing another family of solutions. Unaware of the work of Ramanujan, Carmichael, Bradley, and Mitra, in a letter to K. Mahler, L. J. Mordell asked if (5.4) had other solutions besides the trivial ones $x = 1, y = -z$. In response, Mahler [**136**] constructed the family of solutions (5.5). D. H. Lehmer [**128**] showed how to construct infinite sequences of solutions to (5.4). In particular, starting from the parametric solution (5.5), he derived an infinite sequence of parametric solutions, the simplest being

$$x = 2^4 3^5 r^{10} - 2^4 3^4 r^7 - 3^4 r^4 + 3r, \;\; y = 2^4 3^5 r^9 + 2^3 3^4 r^6 - 3^2 r^3 + 1, \;\; z = 2^4 3^5 r^{10} - 3^3 5 r^4.$$

Supplementing the work of Lehmer, H. J. Godwin [**86**] and V. D. Podsypanin [**155**] found further families of solutions. Finding a complete description of all *integral* solutions to (5.4) appears to be an unsolved problem.

It is natural to generalize Question 681 by asking which positive integers n are the sum of three cubes. For early references, consult Dickson's *History* [**73, pp. 726–727**]. Many papers have been written on this question, and we cite only a few of them. It is conjectured that, if C is the set of all integers representable as a sum of three cubes, then C has positive density. The best result to date is by R. C. Vaughan [**202**] who showed that

$$\#\left(C \cap [1, x]\right) \gg x^{19/21-\epsilon},$$

for each $\epsilon > 0$. Extensive computer searches for solutions of

$$n = x^3 + y^3 + z^3,$$

for various n, have been conducted by several mathematicians, including D. R. Heath–Brown, W. M. Lioen, and H. J. J. te Riele [**95**] and A. Bremner [**45**]. Up to the present time, the most extensive calculations have been performed by K. Koyama, Y. Tsuruoka, and H. Sekigawa [**111**].

QUESTION 464 (JIMS 5, P. 120; 5, PP. 227–228). *$2^n - 7$ is a perfect square for the values $3, 4, 5, 7, 15$ of n. Find other values.*

The equation

$$2^n - 7 = x^2 \tag{5.6}$$

is called Ramanujan's diophantine equation, or the Ramanujan–Nagell diophantine equation, and is perhaps the most famous of the 58 problems that Ramanujan submitted to the *Journal of the Indian Mathematical Society.* It should be emphasized that the "solution" by K. J. Sanjana and T. P. Trivedi in volume 5 offers a systematic derivation of the five given solutions but does not show that these are the only solutions (as the authors make clear).

Unaware of Ramanujan's problem, W. Ljunggren [**132**] proposed the same problem in 1943. T. Nagell's name is attached to (5.6) because in 1948 he solved Ljunggren's problem and therefore was the first to prove Ramanujan's "conjecture" that no other solutions exist. However, since his paper was written in Norwegian in a relatively obscure Norwegian journal [**145**], very few mathematicians were aware of his solution. Thus, after Th. Skolem, S. D. Chowla, and D. J. Lewis [**193**] used Skolem's p–adic method to prove Ramanujan's conjecture in 1959, Nagell republished his proof in English in a more prominent journal [**146**] and pointed out that he had solved the problem much earlier than had Skolem, Chowla, and Lewis. Also, in 1959, unaware of Nagell's work, H. S. Shapiro and D. L. Slotnik [**192**], in their work on error correcting codes, found all solutions of an equation easily seen to be equivalent to (5.6). Significantly improving the methods in [**193**], Chowla, M. Dunton, and Lewis [**66**] gave in 1960 another proof of Ramanujan's conjecture. In 1962, using the arithmetic of certain cubic fields, L. J. Mordell [**143**] gave another, less elementary proof. M. F. Hossain [**100**], W. Johnson [**105**], G. Turnwald [**201**], and P. Bundschuh [**52**] are some of the authors who have also found proofs. In his book [**144**], Mordell gave Hasse's simplification [**94**] of Nagell's proof. Instructive surveys of all known proofs with a plethora of references have been written by E. Cohen [**68**] and A. M. S. Ramasamy [**175**].

Many generalizations of (5.6) are found in the literature. We confine further remarks to the *generalized Ramanujan–Nagell* equation

$$x^2 + D = p^n. \tag{5.7}$$

J. Browkin and A. Schinzel [**49**] proved that $2^n - D = y^2$ has at most one solution if $D \not\equiv 0, 4, 7 \pmod 8$. Furthermore, if any solution exists, then $n \leq 2$. They also pointed out that, in fact, in 1956, they [**48**] had completely solved a diophantine equation, easily shown to be equivalent to (5.6), and so had given the second solution to Ramanujan's problem. R. Apéry [**8**] showed that, if p is an odd prime not dividing $D > 0$, then (5.7) has at most two solutions. Ljunggren [**133**] proved that when $D = 7$, (5.7) has no solutions when p is odd. More recently, J. H. E. Cohn [**70**] has shown that for 46 values of $D \leq 100$, (5.7) has no solutions. H. Hasse [**94**] and F. Beukers [**37**] have written excellent surveys on (5.7). Beukers' thesis [**37**] and his two papers [**38**] and [**39**] contain substantial new results as well. In [**38**], Beukers studied (5.7) for $p = 2$ and proved, among other things, a conjecture of Browkin and Schinzel on the number of solutions when $D > 0$. In his second paper [**39**], assuming that D is a negative integer and p is an odd prime not dividing D, he showed that there are at most four solutions. Moreover, he gave a family of equations having exactly three solutions. In both papers, hypergeometric functions play a central role. M. H. Le has written several papers on (5.7), most of them improving results of Beukers. To describe some of his results, assume in the sequel that $D < 0$, that $(D, p) = 1$, p is prime, and that $N(D, p)$ denotes the number of solutions of (5.7). In [**124**], Le considered the case when $p = 2$ and showed that, in various cases, $N(D, 2) \leq 2, \leq 3, = 4$. In [**123**] and [**125**], Le considered the case of an odd prime p and showed that if $\max(D, p)$ is sufficiently large (which is made precise), then $N(D, p) \leq 3$. There are many further generalizations of (5.7) studied by Le and others, but we will not discuss these here.

Question 469 (JIMS 5, p. 159; 15, p. 97). *The number* $1 + n!$ *is a perfect square for the values* $4, 5, 7$ *of* n. *Find other values.*

In volume 15, citing Dickson's *History* [**73, p. 681**], M. B. Rao wrote that the question was originally posed by H. Brocard [**46**] in 1876, and then again in 1885 [**47**]. (The problem was also mentioned on the Norwegian radio program, Verdt aa vite (Worth knowing).) Furthermore, Dickson [**73, p. 682**] reported that A. Gérardin [**83**] remarked that, if further solutions of

$$1 + n! = m^2 \tag{5.8}$$

exist, then m has at least 20 digits. H. Gupta [**88**] found no solutions in the range $8 \leq n \leq 63$ and thereby concluded that m has at least 45 digits. Recent calculations [**35**] have shown that there are no further solutions up to $n = 10^9$.

In 1993, M. Overholt [**152**] proved that (5.8) has only finitely many solutions if the weak form of Szpiro's conjecture is true, but this remains unproved. To state the weak form of Szpiro's conjecture, which is a special case of the ABC conjecture, first set

$$N_0(n) = \prod_{p|n} p,$$

where p denotes a prime. Let a, b, and c denote positive integers, relatively prime in pairs and satisfying the equality $a + b = c$. Then the weak form of Szpiro's conjecture asserts that there exists a constant s such that

$$|abc| \leq N_0^s(abc).$$

For a further discussion of Szpiro's conjecture, see a paper by S. Lang [**119, pp. 44–45**]. The more general equation

$$n! + A = m^2 \tag{5.9}$$

was examined by A. Dabrowski [**72**] in 1996. By a short, elementary argument, he proved that, if A is not a square, then there are only finitely many solutions of (5.9) in positive integers m and n. He also showed that, if A is a square, then (5.9) has only finitely many solutions, provided that the weak form of Szpiro's conjecture is true.

The problem of finding solutions to (5.8) also appears in R. K. Guy's book [**89, pp. 193–194**].

QUESTION 723 (JIMS 7, P. 240; 10, PP. 357–358). *If* $[x]$ *denotes the greatest integer in* x, *and* n *is any positive integer, show that*

$$\text{(i)} \qquad \left[\frac{n}{3}\right]+\left[\frac{n+2}{6}\right]+\left[\frac{n+4}{6}\right]=\left[\frac{n}{2}\right]+\left[\frac{n+3}{6}\right],$$

$$\text{(ii)} \qquad \left[\tfrac{1}{2}+\sqrt{n+\tfrac{1}{2}}\right]=\left[\tfrac{1}{2}+\sqrt{n+\tfrac{1}{4}}\right],$$

$$\text{(iii)} \qquad \left[\sqrt{n}+\sqrt{n+1}\right]=\left[\sqrt{4n+2}\right].$$

All three parts of Question 723 may be found on the first page of Ramanujan's third notebook [**171, vol. 2, p. 361**], and proofs can be found in Berndt's book [**23, pp. 76–78**]. A. A. Krishnaswami Aiyangar [**2**] later posed a problem giving analogues, one involving fourth roots and one involving fifth roots, of all three parts of Question 723. In a subsequent paper [**3**], he established theorems generalizing the results in his problem, and so further generalized Ramanujan's Question 723. K.–J. Chen [**60**] has also established extensions of (ii) and (iii). Part (iii) appeared on the William Lowell Putnam exam in 1948 [**85**].

QUESTION 770 (JIMS 8, P. 120). *If* $d(n)$ *denotes the number of divisors of* n *(e.g.,* $d(1)=1, d(2)=2, d(3)=2, d(4)=3,\ldots$*) show that*

$$\text{(i)} \qquad \sum_{n=0}^{\infty}\frac{(-1)^n d(2n+1)}{2n+1}$$

is a convergent series; and that

$$\text{(ii)} \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1} d(n)}{n}$$

is a divergent series in the strict sense (i.e., not oscillating).

There are two published solutions to Question 770. The first is by S. D. Chowla [**63**] and uses two theorems of E. Landau. The second, by G. N. Watson [**212**], is completely different and uses Dirichlet's well–known asymptotic formula

$$\sum_{n\le x} d(n)=x\log x+(2\gamma-1)x+O(\sqrt{x})$$

(where γ denotes Euler's constant), as x tends to ∞, as well as similar formulas for the sum over odd n and for sums over $n\equiv 1,3 \pmod 4$, due to T. Estermann [**76**].

QUESTION 784 (JIMS 8, P. 159). *If* $F(x)$ *denotes the fractional part of* x *(e.g.,* $F(\pi)=0.14159\ldots$*), and if* N *is a positive integer, show that*

$$\text{(i)} \qquad \varliminf_{N\to\infty} NF(N\sqrt{2})=\frac{1}{2\sqrt{2}}, \qquad \varliminf_{N\to\infty} NF(N\sqrt{3})=\frac{1}{\sqrt{3}},$$

$$\varliminf_{N\to\infty} NF(N\sqrt{5})=\frac{1}{2\sqrt{5}}, \qquad \varliminf_{N\to\infty} NF(N\sqrt{6})=\frac{1}{\sqrt{6}},$$

$$\varliminf_{N\to\infty} NF(N\sqrt{7})=\frac{3}{2\sqrt{7}};$$

$$\text{(ii)} \qquad \varliminf_{N\to\infty} N(\log N)^{1-p}F(Ne^{2/n})=0,$$

where n is any integer and p is any positive number; show further that in (*ii*) *p cannot be zero.*

Question 784 is concerned with the approximation of irrational numbers by rational numbers. The proposed equalities should be compared with Hurwitz's theorem [**150, p. 304, Thm. 6.11**]: Given any irrational number ξ, there are infinitely many distinct rational numbers m/n such that

$$\left|\xi - \frac{m}{n}\right| < \frac{1}{\sqrt{5}n^2}.$$

The constant $1/\sqrt{5}$ is best possible. To see how Question 784 relates to Hurwitz's theorem, we more closely examine the first assertion in Question 784, which implies that there exists a sequence of positive integers $\{N_k\}$ tending to ∞ such that

$$N_k\sqrt{2} - [N_k\sqrt{2}] \sim \frac{1}{2\sqrt{2}N_k},$$

as $N_k \to \infty$. Setting $M_k = [N_k\sqrt{2}]$, we may write the last formula in the equivalent form

$$\sqrt{2} - \frac{M_k}{N_k} \sim \frac{1}{2\sqrt{2}N_k^2} \qquad (N_k \to \infty),$$

which makes clear the relation to Hurwitz's theorem.

A partial solution was given by A. A. Krishnaswamy Aiyangar [**1**], and a complete solution was found by T. Vijayaraghavan and G. N. Watson [**205**].

Question 427 (JIMS 4, p. 238; 10, pp. 320–321). *Express*

$$(Ax^2 + Bxy + Cy^2)(Ap^2 + Bpq + Cq^2)$$

in the form $Au^2 + Buv + Cv^2$; and hence show that, if

$$(2x^2 + 3xy + 5y^2)(2p^2 + 3pq + 5q^2) = 2u^2 + 3uv + 5v^2,$$

then one set of the values of u and v is

$$u = \tfrac{5}{2}(x + y)(p + q) - 2xp, \qquad v = 2qy - (x + y)(p + q).$$

Question 427 appears in Ramanujan's second notebook [**171, p. 266**]. In Berndt's book [**23, pp. 9–10**] it is shown that Question 427 is a consequence of a more general lemma. The two published solutions in the *Journal of the Indian Mathematical Society* are more complicated.

In fact, Question 427 is a special case of Gauss' theory of composition of binary quadratic forms [**69, p. 212**], [**51, Chap. 7**], which we very briefly describe. Suppose that $Q_1(x, y)$ and $Q_2(x, y)$ are integral, positive definite quadratic forms of discriminant d. Then if Q_3 is a form of discriminant d,

$$Q_3(x_3, y_3) = Q_1(x_1, y_1)Q_2(x_2, y_2),$$

where, for certain integral coefficients $A_j, B_j, 1 \le j \le 4$,

$$\begin{aligned} x_3 =& A_1x_1x_2 + A_2x_1y_2 + A_3x_2y_1 + A_4y_1y_2, \\ y_3 =& B_1x_1x_2 + B_2x_1y_2 + B_3x_2y_1 + B_4y_1y_2. \end{aligned}$$

Thus, Ramanujan considered a special case of Gauss' theory for $Q_3 = Q_1 = Q_2$, and explicitly determined the coefficients $A_j, B_j, 1 \le j \le 4$ for u, v, or x_3, y_3, in the notation above.

QUESTION 489 (JIMS 5, P. 200; 7, P. 104). *Show that*

$$\left(1+e^{-\pi\sqrt{55}}\right)\left(1+e^{-3\pi\sqrt{55}}\right)\left(1+e^{-5\pi\sqrt{55}}\right)\cdots=\frac{1+\sqrt{3+2\sqrt{5}}}{\sqrt{2}}e^{-\pi\sqrt{55}/24}.$$

As A. C. L. Wilkinson observed in volume 7, Question 489 gives the value of Weber's class invariant $\mathfrak{f}(\sqrt{-55})$ [**219, p. 723**], which is also found (with different notation) in both Ramanujan's first and second notebooks [**24, p. 192**]. Recall [**24, p. 183**] that, for each positive real number n, Ramanujan's class invariant G_n is defined by

$$G_n := 2^{-1/4}q^{-1/24}\prod_{k=0}^{\infty}(1+q^{2k+1}), \tag{5.10}$$

where $q=\exp(-\pi\sqrt{n})$. In the notation of Weber [**219**], $G_n =: 2^{-1/4}\mathfrak{f}(\sqrt{-n})$.

Weber [**219**] calculated a total of 105 class invariants, or the monic irreducible polynomials satisfied by them, for the primary purpose of generating Hilbert class fields. Without knowledge of Weber's work, Ramanujan calculated a total of 116 class invariants, or the monic irreducible polynomials satisfied by them. Not surprisingly, many of these had also been calculated by Weber. Using, most likely, methods considerably different from those of Weber in his calculations, Ramanujan was motivated by the connections of class invariants with the explicit determinations of values of theta–functions and the Rogers–Ramanujan continued fraction. After arriving in England, he learned of Weber's work, and so when he wrote his famous paper on modular equations, class invariants, and approximations to π [**165**], [**172, pp. 23–39**], he gave a table of 46 new class invariants. Since Ramanujan did not supply any proofs in his paper [**165**], [**172, pp. 23–39**] or notebooks, Watson took up the task of calculating class invariants and wrote seven papers on calculating invariants, with three of them [**214**], [**215**], [**216**] specifically directed at verifying Ramanujan's class invariants. After Watson's work, a total of 18 of Ramanujan's class invariants remained to be verified up to recent times. Using four distinct methods, Berndt, H. H. Chan, and L.–C. Zhang completed the verification of Ramanujan's class invariants in two papers [**28**], [**30**]. This work can also be found in Chapter 34 of Berndt's book [**24**]. For expository, less technical accounts on Ramanujan's class invariants, their applications, and attempts to prove them, see two further papers by Berndt, Chan, and Zhang [**31**], [**32**]. Not all of Watson's verifications of Ramanujan's class invariants are rigorous. Zhang [**222**], [**223**] has given rigorous derivations of the invariants calculated by Watson by means of his "empirical method," while Chan [**59**] has taken Watson's empirical method, employed class field theory to put it on a firm foundation, and determined several new invariants as well.

QUESTION 699 (JIMS 7, P. 160). *Show that the roots of the equations*

$$x^6-x^3+x^2+2x-1=0, \tag{i}$$

$$x^6+x^5-x^3-x^2-x+1=0 \tag{ii}$$

can be expressed in terms of radicals.

First observe that

$$x^6-x^3+x^2+2x-1=(x+1)(x^5-x^4+x^3-2x^2+3x-1) \tag{5.11}$$

and that

$$x^6 + x^5 - x^3 - x^2 - x + 1 = (x-1)(x^5 + 2x^4 + 2x^3 + x^2 - 1). \tag{5.12}$$

Thus, Ramanujan's problem can be reduced to solving two quintic polynomials. It is doubtful that Ramanujan had actually solved these two quintic polynomials. It is unclear why Ramanujan introduced these two linear factors. In an unpublished lecture on solving quintic polynomials, Watson [**218**] remarked, "I do not know why Ramanujan inserted the factor $x+1$; it may have been an attempt at frivolity, or it may have been a desire to propose an equation in which the coefficients were as small as possible, or it may have been a combination of the two."

Watson [**211**] observed that $2^{-1/4}G_{47}$, where G_n is defined by (5.10), is a solution of

$$x^5 - x^3 - 2x^2 - 2x - 1 = 0, \tag{5.13}$$

a result which is also found in both Ramanujan's first and second notebooks [**23, p. 191**]. The class equation for G_{79} is not found in Weber's book [**219**]. However, on pages 263 and 300 in his second notebook [**171**], Ramanujan claimed that $2^{1/4}/G_{79}$ is a root of

$$x^5 - x^4 + x^3 - 2x^2 + 3x - 1 = 0. \tag{5.14}$$

This result can be deduced from equivalent results due to R. Russell [**182**] and later by Watson [**217**]. See also Berndt's book [**24, pp. 193, 275**]. Watson [**211**] furthermore pointed out that (5.13) was explicity solved by G. P. Young [**221**] in a paper devoted to the general problem of explicitly finding solutions to solvable quintic polynomials and to working out many examples, the first of which is (5.13). Young remarks that (5.13) was "brought under the notice of the writer by a mathematical correspondent," whom Watson conjectured was A. G. Greenhill, who had also studied (5.13). A few years later, A. Cayley [**58**] considerably simplified Young's calculations. Since the solutions of (5.14) had not been given in the literature, Watson [**211**] explicitly determined them. The work of Ramanujan and Watson is summarized in a paper by S. Chowla [**65**].

D. Dummit [**75**] and V. M. Galkin and O. R. Kozyrev [**82**] have also examined (5.13) and (5.14) and provided some insights into the Galois theory behind the two equations. For example, the Galois groups of the Hilbert class fields over $\mathbb{Q}$ are dihedral groups of order 10. Unaware of the work of Russell, Watson, and others, Galkin and Kozyrev rederived the class equation (5.14).

QUESTION 629 (JIMS 7, P. 40; 8, PP. 25–30). *Prove that*

$$\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\pi n^2 x} \cos\left\{\pi n^2 \sqrt{1-x^2}\right\} = \frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}} \sum_{n=1}^{\infty} e^{-\pi n^2 x} \sin\left\{\pi n^2 \sqrt{1-x^2}\right\}; \tag{5.15}$$

and deduce the following:

(i)
$$\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\pi n^2} = \sqrt{5\sqrt{5}-10}\left(\frac{1}{2} + \sum_{n=1}^{\infty} e^{-5\pi n^2}\right),$$

(ii)
$$\sum_{n=1}^{\infty} e^{-\pi n^2}\left(\pi n^2 - \tfrac{1}{4}\right) = \tfrac{1}{8}.$$

In volume 8, three solutions are given. In the first, the solver erroneously claimed that

$$\sum_{n=1}^{\infty} e^{-\pi n^2}\pi n^2 = \tfrac{1}{4} \qquad \text{and} \qquad \sum_{n=1}^{\infty} e^{-\pi n^2} = \tfrac{1}{2}.$$

Chowla [**62**] pointed out these mistakes and evaluated each series above in terms of gamma functions. The second and third solutions by N. Durai Rajan and M. Bhimasena Rao, respectively, are correct.

Equality (5.15) can be found in Section 23 of Chapter 18 in Ramanujan's second notebook [**22, p. 209**], where it is a corollary of a more general transformation formula, Entry 23 [**22, p. 208**]. Part (ii) is given as Example (iv) in Section 7 of Chapter 17 in Ramanujan's second notebook [**22, p. 104**]. After Ramanujan, set

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad |q| < 1.$$

In that same Section 7, Ramanujan offers the values of $\varphi(e^{-\pi}), \varphi(e^{-\pi\sqrt{2}})$, and $\varphi(e^{-2\pi})$ [**22, pp. 103–104**], all of which are classical. In particular [**22, p. 103**],

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{?\left(\frac{3}{4}\right)}. \tag{5.16}$$

Observe that (i) can be written in the form

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5\sqrt{5} - 10}. \tag{5.17}$$

In view of (5.16), we see that (5.17) explicitly determines $\varphi(e^{-5\pi})$. Part (i) is found in the second notebook, where it is an example in the aforementioned Section 23 [**22, p. 210**]. In Berndt's book [**22, p. 210**], one can find a short proof that uses the same generalized theta transformation formula that leads to a proof of (5.15).

Part (i), in the form (5.17), is also recorded in Ramanujan's first notebook [**171, p. 285**]. The values $\varphi(e^{-3\pi}), \varphi(e^{-7\pi}), \varphi(e^{-9\pi})$, and $\varphi(e^{-45\pi})$ are also recorded in the first notebook and were first proved by Berndt and Chan [**27**]; see also Berndt's book [**24, pp. 327–328**], where several "easier" values of φ are also established [**24, p. 325, Entry 1**]. Explicit values of φ yield at once explicit values for the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$ and for the complete elliptic integral of the first kind $K(k)$, for certain values of the modulus k [**24, p. 323, eq. (0.4)**].

We shorten Ramanujan's formulation of Question 584, in part, by using the notation

$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \qquad (a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad |q| < 1. \tag{5.18}$$

Question 584 (JIMS 6, p. 199). *Examine the correctness of the following results:*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}; \tag{5.19}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{5.20}$$

The identities (5.19) and (5.20) are called the Rogers–Ramanujan identities, and they have a long, interesting history, which we briefly relate here. As the wording of the problem intimates, Ramanujan had not proved these identities when he submitted them. Also, as their name suggests, they were, in fact, first discovered and proved by L. J. Rogers [**180**] in 1894. Ramanujan had evidently stated them in one of his initial letters to Hardy, for Hardy [**172, p. 314**] later claimed that, "They were rediscovered nearly 20 years later by Mr Ramanujan, who communicated them to me in a letter from India in February 1913." (For a discussion of Hardy's assertion, see Berndt and Rankin's book [**36, pp. 43–44**].) Hardy informed several mathematicians about (5.19) and (5.20), but he obviously failed to consult with Rogers. The "unproved" identities became well–known, and P. A. MacMahon stated them without proof and devoted an entire chapter to them in volume 2 of his treatise *Combinatory Analysis* [**135**]. In 1917, Ramanujan was perusing old volumes of the *Proceedings of the London Mathematical Society* and accidently came across Rogers' paper [**180**]. Shortly thereafter, Ramanujan [**170**], [**172, pp. 214–215**] found his own proof, and Rogers [**181**] published a second proof. There now exist many proofs, which have been classified and discussed by G. E. Andrews in a very informative paper [**6**].

The Rogers–Ramanujan identities are recorded as Entries 38(i), (ii) in Chapter 16 in Ramanujan's second notebook [**22, p. 16**]. It is ironic that they are, in fact, limiting cases of Entry 7 in the same chapter, as observed by R. A. Askey [**22, pp. 77–78**]. Entry 7 is a limiting case of Watson's transformation for ${}_8\varphi_7$, and, in view of the complexity of Entry 7, it is inconceivable that Ramanujan could have found it without having a proof of it.

The Rogers–Ramanujan identities have interesting combinatorial interpretations, first observed by MacMahon [**135**]. The first, (5.19), implies that: The number of partitions of a positive integer n into distinct parts, each two differing by at least 2, is equinumerous with the number of partitions of n into parts that are congruent to either 1 or 4 modulo 5. The second implies that: The number of partitions of a positive integer n into distinct parts, with each part at least 2 and each two parts differing by at least 2, is equinumerous with the number of partitions of n into parts that are congruent to either 2 or 3 modulo 5.

For further history and information on the Rogers–Ramanujan identities, see Andrews' paper [**6**] and books [**4, pp. 103–105**], [**5**] and Berndt's book [**22, pp. 77–79**].

6. Integrals

After Ramanujan (e.g., see [**22, p. 36, Entry 22(ii)**]), set

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad |q| < 1.$$

Using this notation and the notation (5.18), we abbreviate Ramanujan's Question 386.

QUESTION 386 (JIMS 4, P. 120; 7, PP. 143–144). *Show that*

$$\int_0^\infty \frac{dx}{(-x^2;q^2)_\infty} = \frac{\pi}{2\psi(q)}. \tag{6.1}$$

The evaluation (6.1) can be deduced from the theorem

$$\int_0^\infty \frac{(-axq;q)_\infty x^{n-1}}{(-x;q)_\infty}dx = \frac{\pi}{\sin(n\pi)} \frac{(q^{1-n};q)_\infty (aq;q)_\infty}{(q;q)_\infty (aq^{1-n};q)_\infty}, \tag{6.2}$$

which is stated by Ramanujan in his paper [**168, eq. (19)**], [**172, p. 57**]. The deduction of (6.1) from (6.2) is given on the following page in that paper [**168, eq. (24)**], [**172, p. 58**]. Ramanujan evidently never had a rigorous proof of (6.2), for he wrote [**168**], [**172, p. 57**] "My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's theorem will be found in Mr Hardy's note which follows this paper." (That paper is [**90**], [**91, pp. 594–597**].) The special case $a = 0$ of (6.2), which, of course, contains (6.1), is stated by Ramanujan in his *Quarterly Reports* to the University of Madras, the focus of which is Ramanujan's "Master Theorem." To see how Ramanujan deduced the special case $a = 0$ from his "Master Theorem," see Hardy's book [**92, p. 194**] or Berndt's book [**20, p. 302**]. The solution of Question 386 by N. Durai Rajan in volume 7 employs a partial fraction decomposition of the integrand.

R. A. Askey [**9**], [**10**], [**11**], [**12**] has shown that (6.2) is a q–analogue of the beta integral. For references to extensions and further related work, see the aforementioned papers by Askey. A nice discussion of (6.2) may also be found in his book [**7, Chap. 10**] with Andrews and R. Roy. The evaluation (6.2) also appears as Entry 14 in Chapter 16 of Ramanujan's second notebook. See Berndt's book [**22, p. 29**], where several references for (6.2) and kindred integrals can also be found.

QUESTION 783 (JIMS 8, P. 159; 10, PP. 397–399). *If*

$$x = y^n - y^{n-1},$$
$$J_n = \int_0^1 \frac{\log y}{x} dx,$$

show that

$$\text{(i)} \qquad J_0 = \tfrac{1}{6}\pi^2, \qquad J_{1/2} = \tfrac{1}{10}\pi^2, \qquad J_1 = \tfrac{1}{12}\pi^2, \qquad J_2 = \tfrac{1}{15}\pi^2;$$

$$\text{(ii)} \qquad J_n + J_{1/n} = \tfrac{1}{6}\pi^2.$$

This is a very beautiful result which has been greatly generalized by Berndt and R. J. Evans [**34**] in the following theorem.

THEOREM 1. *Let g be a strictly increasing, differentiable function on $[0, \infty)$ with $g(0) = 1$ and $g(\infty) = \infty$. For $n > 0$ and $t \geq 0$, define*

$$v(t) := \frac{g^n(t)}{g(t^{-1})}.$$

Suppose that

$$\varphi(n) := \int_0^1 \log g(t) \frac{dv}{v}$$

converges. Then

$$\varphi(n) + \varphi(1/n) = 2\varphi(1).$$

To deduce (ii) of Question 783, let $g(t) = 1+t$. Observing that $v(t) = t(1+t)^{n-1}$, setting $u = 1+t$, and using the value $\varphi(1) = \pi^2/12$, we find that Theorem 1 reduces to (ii). The proof by N. Durai Rajan and "Zero" in volume 10 is longer than the proof by Berndt and Evans of the more general result. Question 783 can be found on page 373 of Ramanujan's third notebook [**22, pp. 326–329, Entry 41**].

QUESTION 308 (JIMS 3, P. 168; 3, P. 248). *Show that*

(i) $$\int_0^{\pi/2} \theta \cot\theta \log(\sin\theta) d\theta = -\frac{\pi^3}{48} - \frac{\pi}{4}\log^2 2,$$

(ii) $$\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{x} dx - \frac{1}{2}\int_0^1 \frac{\tan^{-1} x}{x} dx = \frac{\pi}{8}\log 2.$$

Observe that

$$\int_0^1 \frac{\tan^{-1} x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} =: C, \tag{6.3}$$

which is called Catalan's constant. The arithmetical nature of C is unknown, and this is a long outstanding, famous problem. It is conjectured that C is transcendental. The function

$$\varphi(x) := \int_0^x \frac{\tan^{-1} t}{t} dt$$

was studied by Ramanujan in his paper [**166**], [**172, pp. 40–43**] and in his notebooks [**20, pp. 265–267**]. For related results in the notebooks, see [**20, pp. 268–273, 285–290**] and [**24, pp. 457–461**]. The function

$$\psi(x) := \int_0^x \frac{\sin^{-1} t}{t} dt$$

was also examined by Ramanujan in his notebooks [**20, pp. 264–265, 268, 285–288**]. In fact, (ii) is a special case of Entry 16 of Chapter 9 in the second notebook, which we can write in the form

$$\psi(\sin(\pi x)) = \pi x \log|2\sin(\pi x)| + \frac{1}{2}\sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k^2} \tag{6.4}$$

(see [**20, p. 264, penultimate line of p. 285**]). Setting $x = 1/4$ in (6.4) and using (6.3), we deduce (ii).

For (i), observe that an integration by parts gives

$$\int_0^{\pi/2} \theta \cot\theta \log(\sin\theta) d\theta = -\frac{1}{2}\int_0^{\pi/2} \log^2(\sin\theta) d\theta, \tag{6.5}$$

which can be found in the tables of A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev [**157, p. 544, formula 7**]. The solution by K. J. Sanjana in volume 3 proceeds similarly. Part (i) is Example 2 in Section 13 of Chapter 10 in Ramanujan's second notebook, and the proof given in [**21, pp. 31–32**] is quite different from the aforementioned proof. For values of other integrals akin to that on the right side of (6.5), see the aforementioned tables [**157, pp. 543–544**].

QUESTION 295 (JIMS 3, P. 128; 5, P. 65). *If $\alpha\beta = \pi$, show that*

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}dx}{\cosh(\alpha x)} = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}dx}{\cosh(\beta x)}. \tag{6.6}$$

Ramanujan's solution involving double integrals and Fourier transforms in volume 5 is very short and clever. His idea is examined in greater generality in his paper [**168, Sect. 4**], [**172, pp. 53–58**], where the example above and further examples are given. The identity (6.6) was communicated by Ramanujan in his first letter to Hardy [**172, p. 350**], [**36, p. 27**]. It is also found in Chapter 13 of Ramanujan's second notebook [**21, p. 225**]. In both the letter and notebooks, (6.6) is given to again illustrate the same general idea used by Ramanujan in his paper [**168**].

QUESTION 353 (JIMS 4, P. 40; 8, PP. 106–110; 16, PP. 119–120). *If n is any positive odd integer, show that*

$$\int_0^\infty \frac{\sin(nx)}{\cosh x + \cos x}\frac{dx}{x} = \frac{\pi}{4}; \tag{6.7}$$

and hence prove that

$$\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)\left(\cosh\frac{(2k+1)\pi}{2n} + \cos\frac{(2k+1)\pi}{2n}\right)} = \frac{\pi}{8}. \tag{6.8}$$

The proofs of (6.7) and (6.8) by A. C. L. Wilkinson in volume 8 employ contour integration, but his proof of (6.8) is very long. In volume 16, Chowla showed that a much shorter proof of (6.8) can be effected by using the Poisson summation formula for Fourier sine transforms.

Related results can be found in Section 22 of Chapter 14 in Ramanujan's second notebook [**21, pp. 278–280**]. In particular [**21, p. 79, Corollary**], if $\alpha, \beta > 0$ with $\alpha\beta = \pi^3/4$, then

$$\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)\left(\cosh\sqrt{\alpha(2k+1)} + \cos\sqrt{\alpha(2k+1)}\right)}$$
$$+\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)\cosh((2k+1)\pi/2)\cosh^2(\beta(2k+1)^2)} = \frac{\pi}{8}.$$

QUESTION 463 (JIMS 5, P. 120). *If*

$$\int_0^\infty \frac{\cos(nx)}{e^{2\pi\sqrt{x}}-1}dx = \varphi(n),$$

then

$$\int_0^\infty \frac{\sin(nx)}{e^{2\pi\sqrt{x}}-1}dx = \varphi(n) - \frac{1}{2n} + \varphi\left(\frac{\pi^2}{n}\right)\sqrt{\frac{2\pi^3}{n^3}}.$$

Find $\varphi(n)$, and hence show that

$$\varphi(0) = \frac{1}{12} \qquad \varphi(\tfrac{1}{2}\pi) = \frac{1}{4\pi}, \qquad \varphi(\pi) = \frac{2-\sqrt{2}}{8}, \qquad \varphi(2\pi) = \frac{1}{16}, \qquad \varphi(\infty) = 0.$$

A complete discussion of these results can be found in Ramanujan's paper [**169**], [**172, pp. 59–67**] and in Berndt's account of Ramanujan's notebooks [**23, pp. 296–303**].

QUESTION 739 (JIMS 8, P. 40; 8, PP. 218–219). *Show that*

$$\int_0^\infty e^{-nx}(\cot x+\coth x)\sin(nx)dx=\frac{\pi}{2}\left(\frac{1+e^{-n\pi}}{1-e^{-n\pi}}\right)^{(-1)^n}$$

for all positive integral values of n.

Wilkinson's solution in volume 8 uses contour integration.

7. Series

QUESTION 260 (JIMS 3, P. 43; 3, PP. 86–87). *Show, without using calculus, that*

$$1+2\sum_{n=1}^{\infty}\frac{1}{(4n)^3-4n}=\frac{3}{2}\log 2. \tag{7.1}$$

Question 260 is the first question submitted by Ramanujan to the *Journal of the Indian Mathematical Society.* In fact, the first two questions were communicated by P. V. Seshu Aiyar, Ramanujan's mathematics instructor at the Government College of Kumbakonam.

The equality (7.1) is a corollary of Entry 4 of Chapter 2 in Ramanujan's second notebook [**20, pp. 28–29**]. Ramanujan was not the first to pose (7.1) as a problem; Lionnet [**131**] offered (7.1) as a problem in 1879. The problem can also be found in G. Chrystal's book [**67, p. 322**].

QUESTION 327 (JIMS 3, P. 209). *Show that Euler's constant, namely [the limit of]*

$$\sum_{k=1}^{n}\frac{1}{k}-\log n$$

when n *is infinite, is equal to*

$$\log 2-1\left(\frac{2}{3^3-3}\right)-2\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$
$$-3\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots+\frac{2}{39^3-39}\right)-\cdots,$$

the first term in the n*th group being*

$$\frac{2}{\left\{\frac{1}{2}(3^n+3)\right\}^3-\left\{\frac{1}{2}(3^n+3)\right\}}.$$

We reformulate Question 327. Let $A_k=\frac{1}{2}(3^k-1), k\geq 0$. Then

$$\gamma=\log 2-2\sum_{k=1}^{\infty}k\sum_{j=A_{k-1}+1}^{A_k}\frac{1}{(3j)^3-3j},$$

where γ denotes Euler's constant. Evidently, a solution was never published in the *Journal of the Indian Mathematical Society.* However, the problem appears as

Entry 16 in Chapter 8 in Ramanujan's second notebook, and a proof can be found in [**20, p. 196**].

Question 724 (JIMS 7, p. 240; 8, pp. 191–192; 16, p. 121). *Show that*

$$\text{(i)} \qquad \sum_{k=0}^{n-1} \tan^{-1} \frac{1}{2n+2k+1} = \sum_{k=0}^{n-1} \tan^{-1} \frac{1}{(2k+1)(1+2(2k+1)^2)},$$

$$\text{(ii)} \qquad \sum_{k=0}^{n-1} \tan^{-1} \frac{1}{(2n+2k+1)\sqrt{3}} = \sum_{k=0}^{n-1} \tan^{-1} \frac{1}{((2k+1)\sqrt{3})^3}.$$

Mehr Chand Suri showed in volume 16 that (i) and (ii) are the special cases $x=1$ and $x=1/\sqrt{3}$ of the identity

$$\sum_{k=0}^{n-1} \tan^{-1} \frac{x}{2n+2k+1} = \sum_{k=0}^{n-1} \tan^{-1} \frac{x(x^2+1)}{4(2k+1)^3+(3x^2-1)(2k+1)},$$

which can be readily established by induction. The original formulation of (ii) is incorrect.

Ramanujan recorded further identities for $\tan^{-1}$ sums in Chapter 2 of his second notebook [**20, pp. 25–40**].

Question 768 (JIMS 8, p. 119; 8, p. 227). *If*

$$\psi(x) = \frac{x+2}{x^2+x+1},$$

show that

$$\text{(i)} \qquad \sum_{n=1}^{\infty} \frac{1}{3^n} \psi(x^{1/3^n}) = \frac{1}{\log x} + \frac{1}{1-x}$$

for all positive values of x; *and that*

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} \frac{1}{3^n} \psi(x^{1/3^n}) = \frac{1}{1-x}$$

for all negative values of x.

The sign of the right side of (ii) is incorrect in the original formulation. Part (i) can be found in Ramanujan's third notebook [**23, pp. 399–400, Entry 30**]. A companion result is given on the same page [**23, p. 399, Entry 29**].

Question 769 (JIMS, 8, p. 120; 9, pp. 120–121). *Show that*

$$\log 2 \sum_{n=2}^{\infty} \frac{1}{n \log n \log(2n)} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n} = \frac{1}{\log 2}.$$

The original formulation contains an obvious misprint. In volume 9, K. B. Madhava, M. K. Kewalramani, N. Durairajan, and S. V. Venkatachala Aiyar offered the generalization

$$\log r \sum_{n=2}^{\infty} \frac{1}{n \log n \log(rn)} + \sum_{n=2}^{\infty} \frac{\theta(n)}{n \log n} = \frac{1}{\log r},$$

where

$$\theta(n) = \begin{cases} r-1, & \text{if } r|n, \\ -1, & \text{if } r \nmid n. \end{cases}$$

Question 769 coincides with Entry 11(iii) in Chapter 13 of Ramanujan's second notebook [**21, p. 217**].

QUESTION 387 (JIMS 4, P. 120). *Show that*

$$\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k}-1} = \frac{1}{24} - \frac{1}{8\pi}. \tag{7.2}$$

Although no solutions were published, (7.2) has been rediscovered several times in the literature. Its home is in the theory of elliptic functions, as Ramanujan himself indicated when he proved (7.2) in his paper [**165, p. 361**], [**172, p. 34**]. However, Ramanujan was not the first to establish (7.2). The first mathematician known to us to have proved (7.2) is O. Schlömilch [**187**], [**188**] in 1877. Although not explicitly stated, (7.2) was also established by A. Hurwitz [**101**], [**102**] in his thesis in 1881. Others who discovered (7.2) include C. Krishnamachari [**112**], S. L. Malurkar [**137**], M. B. Rao and M. V. Ayyar [**177**], H. F. Sandham [**184**], and C.–B. Ling [**130**]. The discovery of (7.2) in Ramanujan's paper [**165**] or notebooks [**171**] motivated the proofs by Watson [**207**], Grosswald [**87**], and Berndt [**19**]. More precisely, (7.2) is an example in Section 8 of Chapter 14 in Ramanujan's second notebook. The discussion in Berndt's book [**21, p. 256**] contains many further references.

Schlömilch and several others cited above, in fact, proved a more general formula than (7.2). Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then

$$\alpha \sum_{k=1}^{\infty} \frac{k}{e^{2\alpha k}-1} + \beta \sum_{k=1}^{\infty} \frac{k}{e^{2\beta k}-1} = \frac{\alpha+\beta}{24} - \frac{1}{4}. \tag{7.3}$$

The special case $\alpha = \beta = \pi$ of (7.3) yields (7.2). Equality (7.3) is Corollary (i) in Section 8 of Chapter 14 in the second notebook [**21, p. 255**]. There is a further generalization. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, and let n be an integer exceeding 1. Then

$$\alpha^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\alpha k}-1} - (-\beta)^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\beta k}-1} = \{\alpha^n - (-\beta)^n\} \frac{B_{2n}}{4n}, \tag{7.4}$$

where $B_k, k \geq 0$, denotes the kth Bernoulli number. Observe, by (7.3), that (7.4) is *not* valid for $n = 1$. The first proof of (7.4) known to us is by Rao and Ayyar [**177**]. Ramanujan also discovered (7.4), and it can be found as Entry 13 in Chapter 14 in his second notebook [**21, p. 261**]. If we set $\alpha = \beta = \pi$ in (7.4), assume that n is odd, and replace n by $2n+1$, we find that, for each positive integer n,

$$\sum_{k=1}^{\infty} \frac{k^{4n+1}}{e^{2\pi k}-1} = \frac{B_{4n+2}}{8n+4}, \tag{7.5}$$

which is also in Ramanujan's notebooks [**21, p. 262, Cor. (iv)**]. Apparently the first proof of (7.5) is by J. W. L. Glaisher [**84**] in 1889. Many proofs of (7.4) and (7.5) can be found in the literature, and readers should consult Berndt's book [**21, pp. 261–262**] for many of these references.

Analogues of (7.3) and (7.4) wherein negative odd powers of k appear in the summands are also very famous. Although we do not state the primary formula here, it is generally called "Ramanujan's formula for $\zeta(2n+1)$." It is recorded by Ramanujan as Entry 21(i) in his second notebook [**21, pp. 275–276**], and there exist many proofs of it. See Berndt's book [**21, p. 276**] for almost two dozen references.

Most of the proofs of (7.2)–(7.5) do not involve elliptic functions or modular forms. However, Berndt [**19**] has shown that all of the identities discussed above, and others as well, can be derived from one general modular transformation formula for a large class of functions generalizing the logarithm of the Dedekind eta–function.

QUESTION 358 (JIMS 4, P. 78; 7, PP. 99–101). *If n is a multiple of 4, excluding 0, show that*

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)^{n-1}\operatorname{sech}\left(\tfrac{1}{2}(2k+1)\pi\right)=0. \tag{7.6}$$

Although not stated, n must be a *positive* integer.

In fact, (7.6) is originally due to Cauchy [**57, pp. 313, 362**]. Other proofs of (7.6) have been given by Rao and Ayyar [**178**], Chowla [**64**], Sandham [**185**], Riesel [**179**], Ling [**130**], and K. Narasimha Murthy Rao [**149**].

Question 358 has a beautiful generalization. Let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$, and let n be a positive integer. Then

$$\alpha^n\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^{2n-1}}{\cosh(\alpha(2k+1)/2)}+(-\beta)^n\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^{2n-1}}{\cosh(\beta(2k+1)/2)}=0. \tag{7.7}$$

If n is even and $\alpha=\beta=\pi$ in (7.7), then (7.6) arises. Equality (7.7) is Entry 14 in Chapter 14 in Ramanujan's second notebook [**21, p. 262**], and the corollary (7.6) is recorded immediately thereafter. The first proof of (7.7) appears to be by Malurkar [**137**] in 1925. Proofs have also been given by Nanjundiah [**147**] and Berndt [**19, p. 177**].

Analogues of (7.6) and (7.7) exist for negative powers of $2k+1$ in the summands. Such results were also found by Malurkar [**137**], Nanjundiah [**147**], and Berndt [**19**]. These formulas, (7.6), and (7.7) can be deduced from the same general transformation formula; see [**19**] for details.

QUESTION 546 (JIMS 6, P. 80; 7, PP. 107–109, 136–141). *Show that*

$$\sum_{k=0}^{\infty}\frac{2^{2k}(k!)^2}{(2k)!(2k+1)^2}\left(\frac{1}{3}-\frac{1}{4^{k+1}}\right)=\frac{\pi}{12}\log(2+\sqrt{3}), \tag{i}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k2^{2k}(k!)^2}{(2k)!(2k+1)^2}=\frac{\pi^2}{8}-\frac{1}{2}\log^2(1+\sqrt{2}). \tag{ii}$$

The evaluations (i) and (ii) are Examples (v) and (vi), respectively, in Section 32 of Chapter 9 in Ramanujan's second notebook [**20, p. 289**]. These and several other evaluations of this sort were derived in [**20, pp. 288–290**] from the following two related corollaries given in the same section. If $|x| < 1$, then

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(2k)!(2k+1)^2} \left(\frac{4x}{(1+x)^2}\right)^k = (1+x) \sum_{k=0}^{\infty} \frac{(-x)^k}{(2k+1)^2}.$$

If $|x| < \pi/4$, then

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}(k!)^2 \tan^{2k+1}(2x)}{(2k)!(2k+1)^2} = 2 \sum_{k=0}^{\infty} \frac{\tan^{2k+1} x}{(2k+1)^2}.$$

In turn, these corollaries may be deduced from Whipple's quadratic transformation for a well poised generalized hypergeometric function ${}_3F_2$.

QUESTION 606 (JIMS 6, P. 239; 7, PP. 136–141, 192). *Show that*

$$\sum_{k=0}^{\infty} \frac{(\sqrt{5}-2)^{2k+1}}{(2k+1)^2} = \frac{\pi^2}{24} - \frac{1}{12} \log^2(2+\sqrt{5}). \tag{7.8}$$

Question 606 gives the value of $\chi_2(\sqrt{5}-2)$, where

$$\chi_2(z) := \tfrac{1}{2}\{\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)\} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^2}, \qquad |z| \leq 1,$$

and where $\mathrm{Li}_n(z), n \geq 2$, denotes the polylogarithm

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \qquad |z| \leq 1.$$

The value of $\chi_2(\sqrt{5}-2)$ is recorded in a slightly different form in Chapter 9 of Ramanujan's second notebook [**20, p. 248, Ex. (vi)**]. The evaluation (7.8) is also in Lewin's book [**129, p. 19, eq. (1.70)**], but it is originally due to Landen [**118**] in 1780.

QUESTION 642 (JIMS 7, P. 80; 7, PP. 232–233). *Show that*

$$\text{(i)} \qquad \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{2k+1}\right) \frac{5^{-n}}{2n+1} = \frac{\pi^2}{4\sqrt{5}},$$

$$\text{(ii)} \qquad \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{2k+1}\right) \frac{9^{-n}}{2n+1} = \frac{\pi^2}{8} - \frac{3}{8} \log^2 2.$$

The original formulation of (i) is incorrect, as pointed out by M. B. Rao in his solution. Ramanujan made the same mistake when he recorded it as Example (ii) in Section 8 of Chapter 9 in his second notebook. Part (ii) in Question 642 is given as Example (i) in that same section [**20, p. 250**]. For other examples of this sort, see Catalan's paper [**56**].

We have slightly reformulated the next question.

QUESTION 700 (JIMS 7, P. 199; 8, P. 152). *Sum the series*

$$\sum_{k=1}^{n}(a+b+2k-1)\frac{(a)_k^2}{(b)_k^2},$$

where, for $k \geq 1$,

$$(c)_k = c(c+1)(c+2)\cdots(c+k-1).$$

The sum of the series is

$$\frac{1}{b-a-1}\left(a^2 - \frac{(a)_{n+1}}{(b)_n}\right).$$

The two published proofs are elementary. The first, by K. R. Rama Aiyar, uses Euler's elementary identity

$$\sum_{k=0}^{n}(1-a_{k+1})a_1a_2\cdots a_k = 1 - a_1a_2\cdots a_{n+1}.$$

8. Continued Fractions

QUESTION 352 (JIMS 4, P. 40). *Show that*

$$\text{(i)}\qquad \frac{1}{1+}\,\frac{e^{-2\pi}}{1}\,+\,\frac{e^{-4\pi}}{1}\,+\,\frac{e^{-6\pi}}{1}\,+\cdots = \left[\sqrt{\tfrac{1}{2}(5+\sqrt{5})} - \tfrac{1}{2}(\sqrt{5}+1)\right]e^{2\pi/5},$$

$$\text{(ii)}\qquad \frac{1}{1-}\,\frac{e^{-\pi}}{1}\,+\,\frac{e^{-2\pi}}{1}\,-\,\frac{e^{-3\pi}}{1}\,+\cdots = \left[\sqrt{\tfrac{1}{2}(5-\sqrt{5})} - \tfrac{1}{2}(\sqrt{5}-1)\right]e^{\pi/5}.$$

In both the original formulation and Ramanujan's *Collected Papers* [**172, p. 325**], (i) has an obvious misprint.

Question 352 gives the values of the Rogers–Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1}\,+\,\frac{q}{1}\,+\,\frac{q^2}{1}\,+\,\frac{q^3}{1}\,+\cdots, \qquad |q|<1,$$

when $q = e^{-2\pi}, -e^{-\pi}$, respectively. As the name suggests, $R(q)$ was first studied by L. J. Rogers in 1894 [**180**]. In his first letter to Hardy, Ramanujan stated both (i) and (ii) [**172, p. xxvii**], [**36, p. 29**], and in his second letter, he gave the value of $R(e^{-2\pi/\sqrt{5}})$ [**172, p. xxviii**], [**36, p. 57**]. In both letters, Ramanujan wrote that $R(e^{-\pi\sqrt{n}})$ "can be exactly found if n be any positive rational quantity" [**172, p. xxvii**], [**36, pp. 29, 57**]. (We have quoted from the first letter; the statement in the second letter is similar but is omitted from the excerpts of Ramanujan's letters in the *Collected Papers*.) In both his first notebook [**171, p. 311**] and lost notebook [**173, pp. 204, 210**], Ramanujan offered several further values for $R(q)$. On page 210 of [**173**], Ramanujan planned to list fourteen values, but only three are actually given. Since the lost notebook was written in the last year of his life, his illness and subsequent death obviously prevented him from determining the values he intended to compute. In several papers [**159**]–[**163**], K. G. Ramanathan derived some of Ramanujan's values for $R(q)$. All of the values in the first notebook were systematically computed by Berndt and Chan in [**26**], while all the values, including the eleven omitted values, were established by Berndt, Chan, and Zhang in [**29**]. Moreover, in the latter paper, the authors demonstrated for the first time the meaning and truth of Ramanujan's claim that $R(e^{-\pi\sqrt{n}})$ "can be exactly found" when n is a positive rational number. More precisely, they used modular equations to derive some general formulas for $R(e^{-\pi\sqrt{n}})$ in terms of class invariants. Thus, if the requisite class invariants are known, $R(e^{-\pi\sqrt{n}})$ can be determined exactly. A brief expository account of this work can be found in [**31**]. S.–Y. Kang [**106**] has proved a formula in the lost notebook [**173**] that likely was used by Ramanujan to compute values of $R(e^{-\pi\sqrt{n}})$.

Question 541 (JIMS 6, p. 79; 8, pp. 17–20). *Prove that*

$$1+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\frac{1}{1\cdot 3\cdot 5\cdot 7}+\cdots+\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{2}{1\,+}\,\frac{3}{1\,+}\,\frac{4}{1\,+}\cdots=\sqrt{\tfrac{1}{2}\pi e}. \tag{8.1}$$

K. B. Madhava offered an informative discussion of this problem in volume 8. His solution employs Prym's identity [**158**] for the incomplete gamma function

$$\sum_{n=0}^{\infty}\frac{x^n}{(a)_n}=e^x x^{-a}\int_0^x e^{-t}t^{a-1}dt, \qquad a,x>0,$$

where $(a)_0=1$ and $(a)_n=a(a+1)(a+2)\cdots(a+n-1), n\geq 1$, with $a=x=1/2$, and the continued fraction of Legendre [**127, p. 509**]

$$x^{a-1}e^x\int_x^{\infty}e^{-t}t^{-a}dt=\frac{1}{x\,+}\,\frac{a}{1\,+}\,\frac{1}{x\,+}\,\frac{a+1}{1\,+}\,\frac{2}{x\,+}\,\frac{a+2}{x\,+}\,\frac{3}{x\,+}\cdots, \quad a \text{ real}, x>0,$$

with $x=a=1/2$.

Question 541 is a special case of the first part of Entry 43 in Chapter 12 in Ramanujan's second notebook [**21, p. 166**],

$$\sum_{n=0}^{\infty}\frac{x^n}{1\cdot 3\cdot 5\cdots(2n+1)}=\sqrt{\frac{\pi}{2x}}e^{x/2}-\frac{1}{x\,+}\,\frac{1}{1\,+}\,\frac{2}{x\,+}\,\frac{3}{1\,+}\,\frac{4}{x\,+}\,\frac{5}{1\,+}\cdots,$$

where x is any complex number outside $(-\infty,0]$. In turn, Entry 43 is a corollary of Entry 42 [**21, p. 165**], which gives a continued fraction of Legendre [**126**] for certain generalized hypergeometric functions ${}_1F_1(a;b;x)$. More precisely, if n is a nonnegative integer, and $x\notin(-\infty,0]$, then

$$\begin{aligned}&{}_1F_1(1;n+1;x)\\ =&\frac{e^x?\,(n+1)}{x^n}-\frac{n}{x\,+}\,\frac{1-n}{1\,+}\,\frac{1}{x\,+}\,\frac{2-n}{1\,+}\,\frac{2}{x\,+}\,\frac{3-n}{1\,+}\,\frac{3}{x\,+}\cdots.\end{aligned}$$

Lastly, we remark that Question 541 is closely related to a result communicated by Ramanujan in his first letter to Hardy [**172, p. xxvii**], [**36, p. 28**], and given also as Corollary 1 of Entry 43 in Chapter 12 of the second notebook [**21, p. 166**], namely,

$$\int_0^a e^{-x^2}dx=\frac{\sqrt{\pi}}{2}-\frac{e^{-a^2}}{2a\,+}\,\frac{1}{a\,+}\,\frac{2}{2a\,+}\,\frac{3}{a\,+}\,\frac{4}{2a\,+}\cdots, \tag{8.2}$$

which Watson [**208**] found in Laplace's treatise on celestial mechanics [**121, pp. 253–256**]. However, the first rigorous proof is due to Jacobi [**103**].

QUESTION 1049 (JIMS 11, P. 120). *Show that*

$$\text{(i)} \qquad \int_0^\infty \frac{\sin(nx)dx}{x+\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots} = \frac{\sqrt{\frac{1}{2}\pi}}{n+\frac{1}{n+}\frac{2}{n+}\frac{3}{n+}\cdots};$$

$$\text{(ii)} \qquad \int_0^\infty \frac{\sin(\frac{1}{2}\pi nx)dx}{x+\frac{1^2}{x+}\frac{2^2}{x+}\frac{3^2}{x+}\cdots} = \frac{1}{n+}\frac{1^2}{n+}\frac{2^2}{n+}\frac{3^2}{n+}\cdots.$$

The formulas above are valid for $n>0$.

If we set

$$f(x) := \frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots$$

and

$$g(x) := \frac{1}{x+}\frac{1^2}{x+}\frac{2^2}{x+}\frac{3^2}{x+}\cdots,$$

then (i) and (ii) can be respectively written in the forms

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(nx)dx = f(n)$$

and

$$\int_0^\infty g(x)\sin(\tfrac{1}{2}\pi nx)dx = g(n).$$

Thus f and g are self-reciprocal with respect to Fourier sine transforms.

The first solution to Question 1049 was given by E. G. Phillips [**154**]. In his proof of (i), Phillips used (8.2), and in his proof of (ii), he utilized the continued fraction

$$\int_0^\infty \frac{e^{-xt}}{\cosh t}dt = \frac{1}{x+}\frac{1^2}{x+}\frac{2^2}{x+}\frac{3^2}{x+}\cdots, \qquad x>0,$$

a special case of a continued fraction of T. J. Stieltjes [**195**]. L. J. Lange [**120**] independently found a similar solution. (It is curious that immediately following Phillips' paper is the obituary of M. J. M. Hill, the first mathematician whom Ramanujan wrote from India [**36, pp. 15–19**]. Hill did not fully understand Ramanujan's work, and there is no mention of Ramanujan in the obituary.)

W. N. Bailey [**17**] generalized (8.2) and thereby generalized (i), with, *inter alia*, the Bessel function $J_\nu(xt), -1<\nu<3/2$, appearing in the integrand. He gave a simpler proof of this result in a later paper [**18**].

9. Other Analysis

QUESTION 261 (JIMS 3, P. 43; 3, PP. 124–125). *Show that*

$$\text{(a)} \qquad \prod_{n=1}^\infty \left(1+\frac{1}{n^3}\right) = \frac{1}{\pi}\cosh(\tfrac{1}{2}\pi\sqrt{3}),$$

$$\text{(b)} \qquad \prod_{n=2}^\infty \left(1-\frac{1}{n^3}\right) = \frac{1}{3\pi}\cosh(\tfrac{1}{2}\pi\sqrt{3});$$

and prove from first principles that (b) $= \frac{1}{3}$(a).

Question 261 was communicated to the *Journal of the Indian Mathematical Society* by P. V. Seshu Aiyar.

The more general product

$$\prod_{n=0}^{\infty}\left(1+\left(\frac{x}{a+nd}\right)^3\right)$$

was studied by Ramanujan in his paper [**167**], [**172, pp. 50–52**].

In Chapter 2 of Ramanujan's second notebook, the product evaluations (a) and (b) can be found as Examples 3 and 4 in Section 11 [**20, p. 41**]. In Chapter 13 of his second notebook [**21, pp. 230–231, Entry 27**], Ramanujan evaluated the product

$$\prod_{k=1}^{\infty}\left(1+\left(\frac{x}{k}\right)^n\right),$$

where n is an even positive integer. Further product evaluations of the same kind are located on pages 279 and 287 in the second notebook [**23, pp. 335–337, Entries 1–4**].

Question 571 (JIMS 6, p. 160; 7, p. 32). *If*

$$\tfrac{1}{2}\pi\alpha = \log\tan\{\tfrac{1}{4}\pi(1+\beta)\},$$

show that

$$\left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\cdots = e^{\pi\alpha\beta/2}.$$

This result was proved by Ramanujan in his paper [**166**], [**172, pp. 40–43**]. An equivalent formulation can be seen on page 286 in his second notebook [**24, p. 461**].

Question 294 (JIMS 3, p. 128; 4, pp. 151–152). *Show that* [*if x is a positive integer*]

$$\frac{1}{2}e^x = \sum_{n=0}^{x-1}\frac{x^n}{n!} + \frac{x^x}{x!}\theta,$$

where θ lies between $\frac{1}{2}$ and $\frac{1}{3}$.

In volume 4, Ramanujan gave only a partial solution of this ultimately famous problem. A variant of Question 294 appears in Section 48 in Chapter 12 of Ramanujan's second notebook [**21, p. 181**]. The problem was completely solved independently by G. Szegö [**199**], [**200, pp. 143–152**] in 1928 and by Watson [**209**] in 1929. In his first letter to Hardy [**172, p. xxvi**], [**36, p. 27**], Ramanujan claimed the stronger result

$$\theta = \frac{1}{3} + \frac{4}{135(x+k)},$$

where k lies between $\frac{8}{45}$ and $\frac{2}{21}$. Partial evidence for this claim arises from the asymptotic expansion [**21, p. 182**]

$$\theta = \frac{1}{3} + \frac{4}{135x} - \frac{8}{2835x^2} + \cdots,$$

as x tends to ∞. In his paper [**209**], Watson remarked, "I shall also give reasons, which seem to me to be fairly convincing, for believing that k lies between $\frac{8}{45}$ and $\frac{2}{21}$." To the best of our knowledge, however, it appears that this claim has never been firmly established. However, K. P. Choi [**61**] has shown that, for all $x \geq 1$, $\theta < \frac{1}{3} + \frac{8}{243}$.

Question 294 has the following connection with probability [**139**], [**140**]. Suppose that each of the n independent random variables $X_k, 1 \leq k \leq n$, has a Poisson distribution with parameter 1. Then $S_n := \sum_{k=1}^{n} X_k$ has a Poisson distribution with parameter n. Thus,

$$P(S_n \leq n) = e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Upon applying the central limit theorem, we find that

$$\lim_{n \to \infty} P(S_n \leq n) = \tfrac{1}{2}.$$

For further connections to probability, see papers by K. O. Bowman, L. R. Shenton, and G. Szekeres [**43**], D. F. Lawden [**122**], E. S. Key [**107**], and P. Flajolet, P. J. Grabner, P. Kirschenkofer, and H. Prodinger [**81**].

Question 294 is also related to the famous "birthday surprise problem;" see papers by M. L. Klamkin and D. J. Newman [**108**], and by M. Blaum, I. Eisenberger, G. Lorden, and R. J. McEliece [**40**].

E. T. Copson [**71**] considered the analogous problem for e^{-x}. For further ramifications of Question 294, including applications and analogues, see papers by J. C. W. Marsaglia [**138**], J. D. Buckholtz [**50**], R. B. Paris [**153**], L. Carlitz [**54**], and K. Jogdeo and S. M. Samuels [**104**].

C. Y. Yilbrim [**220**] has studied the zeros of the partial sum in Question 294; a certain sum involving these zeros arises in formulas for certain mean values of $|\zeta(\frac{1}{2} + it)|^2$, where ζ denotes the Riemann zeta–function.

Results related to Question 294 can be found in D. E. Knuth's book [**109, pp. 112–117**]. For further discussions of this problem see the commentary in Szegö's *Collected Papers* [**200, pp. 151–152**] and Berndt's book [**21, pp. 181–184**].

Question 738 (JIMS 8, p. 40). *If*

$$\phi(x) = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1} x^n}{n!} e^{-(n+1)x}, \tag{9.1}$$

show that $\phi(x) = 1$ *when* x *lies between 0 and 1; and that* $\phi(x) \neq 1$ *when* $x > 1$. *Find the limit of*

$$\{\phi(1+\epsilon) - \phi(1)\} / \epsilon$$

as $\epsilon \to 0$ *through positive values.*

The first part of Question 738 can be found as a special case of a corollary in Section 13 of Chapter 3 in Ramanujan's second notebook [**20, p. 70**]. Ramanujan also communicated a version of this corollary in his *Quarterly Reports* to the University of Madras [**20, p. 306, eq. (1.14)**]. In fact, (9.1) has a long history. It can be traced back to papers of J. H. Lambert [**115**] in 1758, J. L. Lagrange [**114**] in 1770, and Euler [**78**], [**80**] in 1783. The most common proof utilizes the Lagrange inversion formula. Indeed, in Pólya and Szegö's treatise [**156, pp. 125, 135, 301,**

316–317], (9.1) is given as a problem to illustrate this formula. For many further references, see Berndt's book [**20, pp. 72, 307**].

The first complete solution to Problem 738 was given by Szegö [**199**], [**200, pp. 143–152**]. The limit queried by Ramanujan equals -2.

F. C. Auluck [**13**] and Auluck and Chowla [**14**] conjectured that $\phi(x)$ is completely monotonic for $x > 1$, that is, $(-1)^k \phi^{(k)}(x) \geq 0, k = 0, 1, 2, \dots$. S. M. Shah and U. C. Sharma [**191**] established this inequality for $k = 0, 1, 2, 3, 4$. On the other hand, Shah [**190**] proved that $x\phi(x)$ and $e^x \phi(x)$ are not completely monotonic. Finally, R. P. Boas, Jr. [**41**] proved that $\phi(x)$ is not completely monotonic on any interval (c, ∞).

QUESTION 526 (JIMS 6, P. 39). *If n is any positive quantity, show that*

$$\sum_{k=1}^{\infty} \frac{k^{k-2}}{(n+k)^k} < \frac{1}{n}; \tag{9.2}$$

and find approximately the difference when n is great. Hence show that

$$\sum_{k=1}^{\infty} \frac{k^{k-2}}{(1000+k)^k} < \frac{1}{1000}$$

by approximately 10^{-440}.

The editors of Ramanujan's *Collected Papers* slightly reworded the problem.

Using Question 738, Szegö [**199**], [**200, pp. 143–152**] established the inequalities posed by Ramanujan in Question 526. If we let Δ_n denote the difference of the right and left sides in (9.2), Szegö claimed to have shown that

$$\Delta_{1000} < 10^{-439.98084} = 1.0451 \times 10^{-440}.$$

It was pointed out by S. S. Macintyre [**134**] that Szegö actually showed that

$$\Delta_{1000} < 1.0132 \times 10^{-440}.$$

Macintyre [**134**] improved Szegö's result by proving that

$$\Delta_{1000} = 1.0125 \times 10^{-440},$$

correct to 5 significant figures. More generally, she proved that

$$\Delta_n = e^{-n} \left\{ \frac{2n+1}{n(n+1)^2} - \frac{7}{3(n+1)^3} + \frac{17}{3(n+1)^4} + O\left(\frac{1}{n^5}\right) \right\}.$$

QUESTION 740 (JIMS 8, P. 40; 8, PP. 220–221). *If*

$$\phi(x) = \left\{ \frac{e^x [x]!}{x^{[x]}} \right\}^2 - 2\pi x,$$

where $[x]$ denotes the greatest integer in x, show that $\phi(x)$ is a continuous function of x for all positive values of x, and oscillates from $\frac{1}{3}\pi$ to $-\frac{1}{6}\pi$ when x becomes infinite. Also differentiate $\phi(x)$.

Question 753 (JIMS 8, p. 80; 10, pp. 395–397). *If*

$$\phi(x) = \tfrac{1}{2}\log(2\pi x) - x + \int_1^x \frac{[t]}{t}dt,$$

where $[t]$ *denotes the greatest integer in* t, *show that*

$$\overline{\lim_{x\to\infty}}\, x\phi(x) = \frac{1}{24}, \qquad \underline{\lim_{x\to\infty}}\, x\phi(x) = -\frac{1}{12}.$$

Questions 740 and 753 are very closely related, with Stirling's formula playing a key role in the solutions.

Question 754 (JIMS 8, p. 80; 12, p. 101; 13, p. 151). *Show that*

$$e^x x^{-x}\pi^{-1/2}?\,(1+x) = (8x^3+4x^2+x+E)^{1/6},$$

where E *lies between* $\frac{1}{100}$ *and* $\frac{1}{30}$ *for all positive values of* x.

K. B. Madhava's partial solution in volume 12 does not yield the bounds for E proposed by Ramanujan. In volume 13, E. H. Neville and C. Krishnamachary pointed out a couple of numerical errors in Madhava's solution, and consequently Madhava's bounds for E are actually better than what Madhava originally claimed, but still not as sharp as those posed by Ramanujan. Neville and Krishnamachary conclude their remarks by writing, "Mr Ramanujam's assertion is seen to be credible, but more powerful means must be used if it is to be proved." The problem still appears to be open.

Question 605 (JIMS 6, p. 239; 7, pp. 191–192). *Show that, when* $x=\infty$,

$$\frac{(x+a-b)!(8x+2b)!(9x+a+b)!}{(3x+a-c)!(3x+a-b+c)!(12x+3b)!} = \sqrt{\frac{2}{3}}.$$

This result is a straightforward application of Stirling's formula.

On page 346 in his second notebook, Ramanujan stated a much more general result. Under certain prescribed conditions on m, n, A_k, B_j, a_k, and $b_j, 1\le k\le m,\ 1\le j\le n$,

$$\lim_{x\to\infty}\frac{\prod_{k=1}^m ?\,(A_kx+a_k+1)}{\prod_{j=1}^n ?\,(B_jx+b_j+1)} = (2\pi e)^{(m-n)/2}\frac{\prod_{k=1}^m A_k^{a_k+1/2}}{\prod_{j=1}^n B_j^{b_j+1/2}}. \tag{9.3}$$

Question 605 is easily seen to be a special case of (9.3), and is one of several examples offered by Ramanujan on page 346. For proofs of (9.3) and the aforementioned corollaries, see Berndt's book [**23, pp. 340–341**].

10. Geometry

Question 662 (JIMS 7, pp. 119–120). *Let* AB *be a diameter and* BC *be a chord of a circle* ABC. *Bisect the minor arc* BC *at* M; *and draw a chord* BN *equal to half of the chord* BC. *Join* AM. *Describe two circles with* A *and* B *as centers and* AM *and* BN *as radii, cutting each other at* S *and* S', *and cutting the given circle again at the points* M' *and* N' *respectively. Join* AN *and* BM *intersecting at* R, *and also join* AN' *and* BM' *intersecting at* R'. *Through* B *draw a tangent to the given circle, meeting* AM *and* AM' *produced at* Q *and* Q' *respectively. Produce* AN *and* $M'B$ *to meet at* P, *and also produce* AN' *and* MB *to meet at* P'. *Show* *that the eight points $P, Q, R, S, S', R', Q', P'$ are cyclic, and that the circle passing through these eight points is orthogonal to the given circle ABC.*

This result appears in Entry 7(iv) of Chapter 19 in Ramanujan's second notebook, and a proof can be found in Berndt's book [**22, pp. 244–246**]. The problem was reproduced in *Mathematics Today* [**174**], a journal for students of mathematics in Indian high schools and colleges. A total of 24 solutions were received.

QUESTION 755 (JIMS 8, P. 80). *Let p be the perimeter and e the eccentricity of an ellipse whose center is C, and let CA and CB be a semi–major and a semi–minor axis. From CA cut off CQ equal to CB, and also produce AC to P making CP equal to CB. From A draw AN perpendicular to CA (in the direction of CB). From Q draw QM making with QA an angle equal to ϕ (which is to be determined) and meeting AN at M. Join PM and draw PN making with PM an angle equal to half of the angle APM, and meeting AN at N. With P as center and PA as radius describe a circle, cutting PN at K, and meeting PB produced at L. Then, if*

$$\frac{\text{arc } AL}{\text{arc } AK} = \frac{p}{4AN},$$

trace the changes in ϕ when e varies from 0 to 1. In particular, show that $\phi = 30^\circ$ when $e = 0; \phi \to 30^\circ$ when $e \to 1; \phi = 30^\circ$ when $e = 0.99948$ nearly; ϕ assumes the minimum value of about $29^\circ 58\frac{3}{4}'$ *when e is about* 0.999886; *and ϕ assumes the maximum value of about* $30^\circ 44\frac{1}{4}'$ *when e is about* 0.9589.

Question 755 appears as Corollary (ii) in Section 19 of Chapter 18 in Ramanujan's second notebook; a proof can be found in Berndt's book [**22, p. 190**]. M. B. Villarino [**206**] has obtained improvements for the approximations given in the last part of the problem.

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