Introduction Statistical Methods- Session-5

Types of Random Variables

Types of Random Variables with Simple Examples

Random variables are categorized into **Discrete** and **Continuous** random variables. Here's an explanation with simple examples:

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1. Discrete Random Variables

- **Definition:** A discrete random variable takes on a **finite or countable** set of values.
- **Examples:**
  - Rolling a die: The outcomes can be 1, 2, 3, 4, 5, or 6.
  - Number of cars sold by a dealership in a day: Possible values could be 0, 1, 2, 3, etc.
  - Tossing a coin 3 times: The number of heads can be 0, 1, 2, or 3.
- **Key Feature:** Values are distinct and countable.
- **Probability:**
  - Each value has an associated probability.
  - Example: For a fair die, $P(X = 3) = \frac{1}{6}$.

2. Continuous Random Variables

- **Definition:** A continuous random variable can take on any value within a range (an **infinite number of values**).
- **Examples:**
  - Height of students in a classroom: Can be any value, e.g., 150.5 cm, 160.8 cm.
  - Time taken to complete a task: Could be 10.1 seconds, 10.12 seconds, etc.
  - Temperature in a city: Could be 25.6°C, 25.61°C, etc.
- **Key Feature:** Values are not countable; they exist in intervals.
- **Probability:**
  - Probabilities are calculated for **ranges** of values, not individual points.
  - Example: The probability that the temperature is between 25°C and 30°C.

Comparison Between Discrete and Continuous Variables

Aspect	Discrete	Continuous
Definition	Takes distinct, countable values	Takes any value in a range
Examples	Number of students, dice rolls	Height, weight, time
Probability	For specific values $P(X = x)$	For intervals, e.g., $P(25 \leq X \leq 30)$
Visualization	Bar chart	Smooth curve (PDF)

Summary

- **Discrete Random Variable**: Deals with countable outcomes (e.g., number of heads in coin toss).
- **Continuous Random Variable**: Deals with measurements and ranges (e.g., height, time).

Example

At a shooting range, a shooter is able to hit a target in either 1, 2 or 3 shots. Let x be a random variable indicating the number of shots fired to hit the target. The following probability function was proposed:

$$f(x) = \frac{x}{6}, x = 1,2,3.$$

Is this probability function valid?

To determine if the given probability function $f(x) = \frac{x}{6}$, $x = 1, 2, 3$ is valid, we need to verify two properties of a probability distribution:

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1. Non-Negativity

The probability for all possible values of x must be non-negative:

$$f(x) \geq 0 \quad \text{for all } x.$$

For $x = 1, 2, 3$:

- $f(1) = \frac{1}{6} > 0,$
- $f(2) = \frac{2}{6} > 0,$
- $f(3) = \frac{3}{6} > 0.$

Since $f(x) \geq 0$ for all given x, the function satisfies the **non-negativity condition**.

2. Total Probability

The total probability across all values of x must equal 1:

$$\sum_x f(x) = 1.$$

For $x = 1, 2, 3$:

$$\sum_x f(x) = f(1) + f(2) + f(3) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6}.$$

Simplifying:

$$\sum_x f(x) = \frac{1+2+3}{6} = \frac{6}{6} = 1.$$

Since the total probability equals 1, the function satisfies the **total probability condition**.

Conclusion

Both conditions are satisfied:

1. $f(x) \geq 0$ for all x,
2. $\sum_x f(x) = 1$.

Thus, the given function $f(x) = \frac{x}{6}$, $x = 1, 2, 3$ **is a valid probability function.**

Example 1

Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

To solve this problem, we calculate the **mean**, **variance**, and **standard deviation** of the number of kings drawn from a deck of 52 cards when 2 cards are drawn without replacement.

Step 1: Define the Random Variable

Let X be the random variable representing the number of kings drawn. Since there are 4 kings in a deck of 52 cards, X can take values $0, 1$, or 2.

Step 2: Mean of X

The mean of X is given by:

$$\mu = \mathbb{E}(X) = n \cdot p$$

Where:

- $n = 2$ (number of cards drawn),
- $p = \frac{4}{52} = \frac{1}{13}$ (probability of drawing a king).

$$\mu = 2 \cdot \frac{1}{13} = \frac{2}{13}.$$

Step 3: Variance of X

The variance of X for sampling **without replacement** is given by:

$$\text{Var}(X) = n \cdot p \cdot (1 - p) \cdot \frac{N - n}{N - 1},$$

Where:

- $n = 2$,
- $p = \frac{4}{52} = \frac{1}{13}$,
- $N = 52$ (total cards in the deck).

Substitute the values:

$$\text{Var}(X) = 2 \cdot \frac{1}{13} \cdot \left(1 - \frac{1}{13}\right) \cdot \frac{52 - 2}{52 - 1}.$$

Simplify:

$$\text{Var}(X) = 2 \cdot \frac{1}{13} \cdot \frac{12}{13} \cdot \frac{50}{51}.$$

$$\text{Var}(X) = \frac{2 \cdot 12 \cdot 50}{13^2 \cdot 51} = \frac{1200}{8649}.$$

Step 4: Standard Deviation of X

The standard deviation is the square root of the variance:

$$\sigma = \sqrt{\mathrm{Var}(X)}.$$

$$\sigma = \sqrt{\frac{1200}{8649}} \approx 0.371.$$

Final Answers:

1. **Mean** (μ): $\frac{2}{13} \approx 0.154$,
2. **Variance** ($\mathrm{Var}(X)$): $\frac{1200}{8649} \approx 0.139$,
3. **Standard Deviation** (σ): $\sqrt{\frac{1200}{8649}} \approx 0.371$.

Example 2

A random variable X has the following probability function :

Find $i)$ *the value of* k, $ii)\ P(X<3)$, $iii)\ P(0<X<4)$, (iv) the cumulative distribution function of X, $v)\ E(X)$, $vi)\ Var(X)$

x	0	1	2	3	4
$p(x)$	k	$3k$	$5k$	$7k$	$9k$

Given Data:

The random variable X has the following probability mass function (PMF):

$$P(X=x) = k, 3k, 5k, 7k, 9k \quad \text{for } x = 0, 1, 2, 3, 4.$$

Step 1: Find the value of k

The sum of all probabilities for a random variable must equal 1:

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)=1.$$

Substitute the given probabilities:

$$k+3k+5k+7k+9k=1.$$

Simplify:

$$25k=1.$$

Solve for k:

$$k=\frac{1}{25}.$$

Step 2: Find $P(X<3)$

This means we need the probability that X takes values less than 3 ($X=0,1,2$):

$$P(X<3)=P(X=0)+P(X=1)+P(X=2).$$

Substitute the probabilities:

$$P(X<3)=k+3k+5k=9k.$$

Substitute $k=\frac{1}{25}$:

$$P(X<3)=9\cdot\frac{1}{25}=\frac{9}{25}.$$

Step 3: Find $P(0<X<4)$

This means we need the probability that X takes values $1, 2$, and 3:

$$P(0<X<4)=P(X=1)+P(X=2)+P(X=3).$$

Substitute the probabilities:

$$P(0<X<4)=3k+5k+7k=15k.$$

Substitute $k=\frac{1}{25}$:

$$P(0<X<4)=15\cdot\frac{1}{25}=\frac{15}{25}=\frac{3}{5}.$$

Step 4: Find the Cumulative Distribution Function (CDF) of X

The cumulative distribution function $F(x)$ is the probability that X takes a value less than or equal to x. Compute $F(x)$ step by step:

- $F(0) = P(X \leq 0) = P(X = 0) = k = \frac{1}{25}$,
- $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = k + 3k = 4k = \frac{4}{25}$,
- $F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = k + 3k + 5k = 9k = \frac{9}{25}$,
- $F(3) = P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = k + 3k + 5k + 7k = 16k = \frac{16}{25}$,
- $F(4) = P(X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = k + 3k + 5k + 7k + 9k = 25k = 1$.

Thus, the CDF is:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{25} & \text{if } 0 \leq x < 1, \\ \frac{4}{25} & \text{if } 1 \leq x < 2, \\ \frac{9}{25} & \text{if } 2 \leq x < 3, \\ \frac{16}{25} & \text{if } 3 \leq x < 4, \\ 1 & \text{if } x \geq 4. \end{cases}$$

Step 5: Find $\mathbb{E}(X)$

The expected value $\mathbb{E}(X)$ is given by:

$$\mathbb{E}(X) = \sum_x x \cdot P(X = x).$$

Substitute the values:

$$\mathbb{E}(X) = (0 \cdot k) + (1 \cdot 3k) + (2 \cdot 5k) + (3 \cdot 7k) + (4 \cdot 9k).$$

Simplify:

$$\mathbb{E}(X) = 0 + 3k + 10k + 21k + 36k = 70k.$$

Substitute $k = \frac{1}{25}$:

$$\mathbb{E}(X) = 70 \cdot \frac{1}{25} = \frac{70}{25} = 2.8.$$

Step 6: Find $\mathrm{Var}(X)$

The variance is given by:

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2.$$

Step 6.1: Compute $\mathbb{E}(X^2)$:

$$\mathbb{E}(X^2) = \sum_x x^2 \cdot P(X = x).$$

Substitute the values:

$$\mathbb{E}(X^2) = (0^2 \cdot k) + (1^2 \cdot 3k) + (2^2 \cdot 5k) + (3^2 \cdot 7k) + (4^2 \cdot 9k).$$

Simplify:

$$\mathbb{E}(X^2) = 0 + 3k + 20k + 63k + 144k = 230k.$$

Substitute $k = \frac{1}{25}$:

$$\mathbb{E}(X^2) = 230 \cdot \frac{1}{25} = \frac{230}{25} = 9.2.$$

Step 6.2: Compute $\mathrm{Var}(X)$:

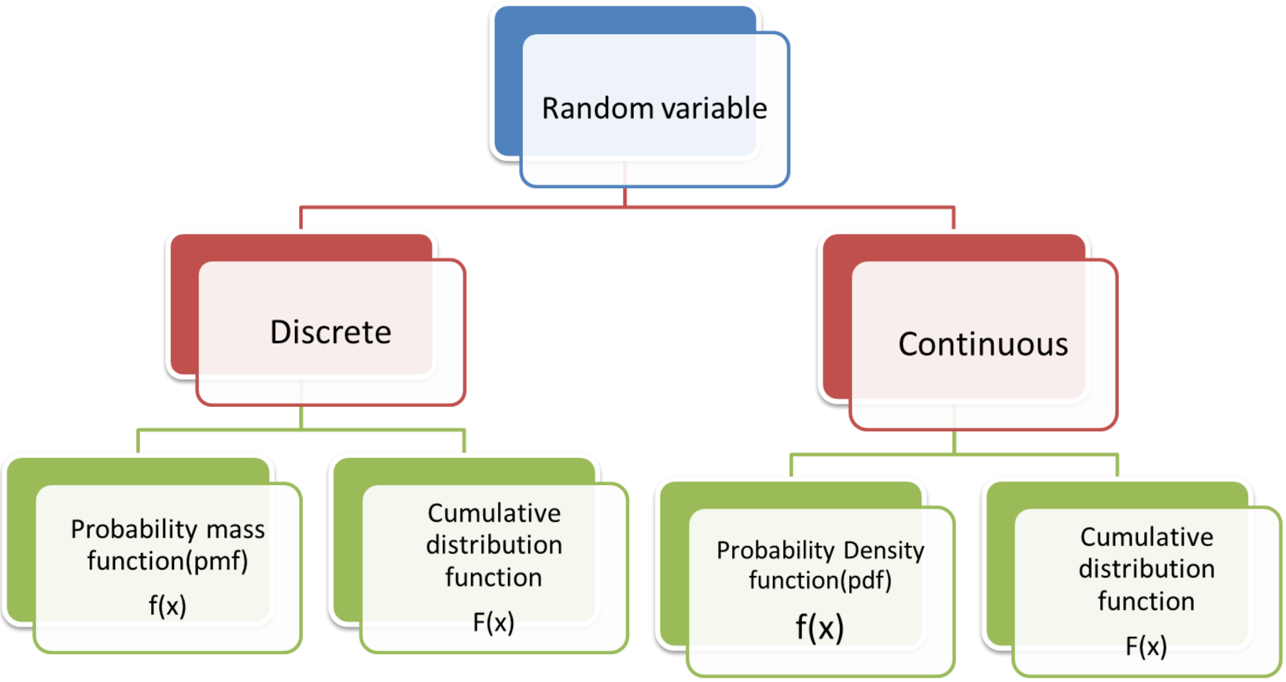
$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2.$$

$$\mathrm{Var}(X) = 9.2 - (2.8)^2 = 9.2 - 7.84 = 1.36.$$

Final Answers:

1. $k = \frac{1}{25}$,
2. $P(X < 3) = \frac{9}{25}$,
3. $P(0 < X < 4) = \frac{3}{5}$,
4. CDF: As derived above,
5. $\mathbb{E}(X) = 2.8$,
6. $\mathrm{Var}(X) = 1.36$.

Classification of Random Variables



Explanation of Random Variables and Their Types

A **random variable** is a variable that represents a numerical value associated with the outcomes of a random phenomenon. Random variables can be broadly classified into **Discrete** and **Continuous** types.

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1. Discrete Random Variables

- **Definition:** A random variable that can take on a countable number of distinct values.
- Examples:
  - The number of heads when flipping a coin 3 times.
  - The number of cars sold in a day by a car dealer.
- Represented by a **Probability Mass Function (PMF)** $f(x)$, which gives the probability of each possible value.
- **Cumulative Distribution Function (CDF)** $F(x)$ is the probability that the random variable takes on a value less than or equal to x.

Example:

If you roll a 6-sided die, let X represent the outcome. The PMF is:

$$P(X = x) = \begin{cases} \frac{1}{6} & x = 1, 2, 3, 4, 5, 6, \\ 0 & \text{otherwise.} \end{cases}$$

The CDF would be:

$$F(x) = \begin{cases} 0 & x < 1, \\ \frac{1}{6} & 1 \leq x < 2, \\ \frac{2}{6} & 2 \leq x < 3, \\ \dots & \\ 1 & x \geq 6. \end{cases}$$

2. **Continuous Random Variables**

- **Definition:** A random variable that can take on an infinite number of possible values within a range.
- Examples:
  - The height of students in a class.
  - The time taken to complete a task.
- Represented by a **Probability Density Function (PDF)** $f(x)$, which describes the likelihood of the random variable taking on a specific value.
  - Note: The probability of a specific value is zero, but we calculate probabilities over intervals.
- **Cumulative Distribution Function (CDF)** $F(x)$ gives the probability that the random variable takes on a value less than or equal to x.

Example:

Let X represent the weight of apples in a store (in kg). A PDF might look like:

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The CDF would be:

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

If $x = 0.5$, $F(0.5)$ would give the probability that the weight of an apple is less than or equal to 0.5 kg.

Key Differences

Aspect	Discrete Random Variable	Continuous Random Variable
Values	Countable (e.g., integers)	Infinite within a range
Function	PMF ($f(x)$) gives probabilities	PDF ($f(x)$) gives density
CDF	$F(x) = P(X \leq x)$	$F(x) = \int_{-\infty}^{x} f(t)dt$
Example	Rolling a die, number of students	Height, weight, time

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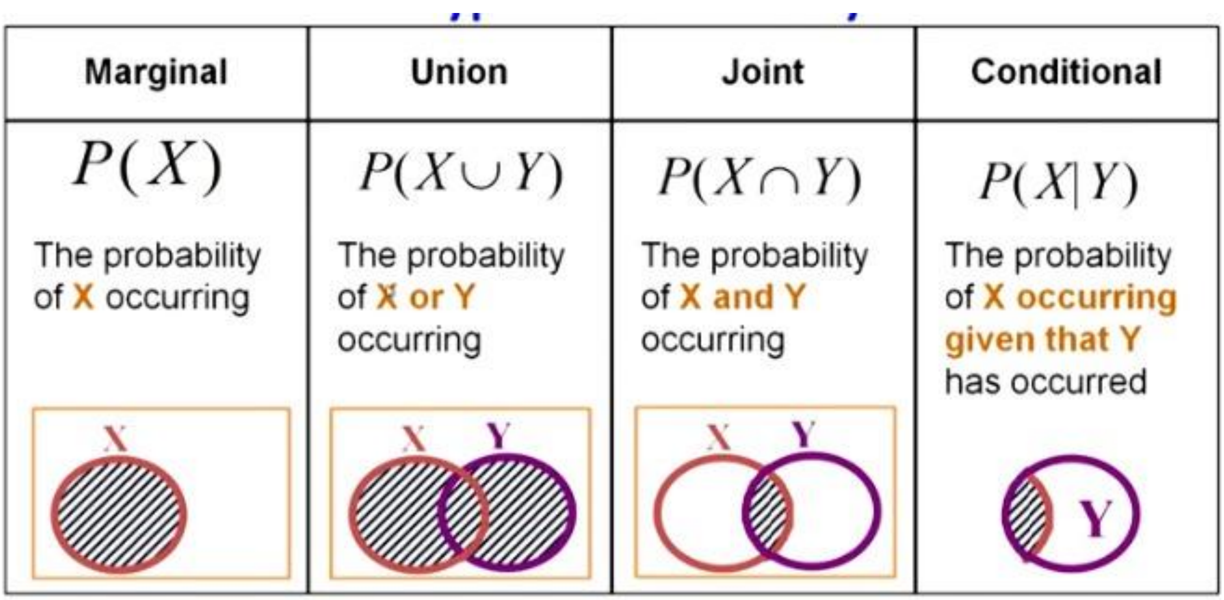
Summary

- **Discrete random variables** deal with countable outcomes (e.g., dice rolls).
- **Continuous random variables** deal with measurable outcomes over intervals (e.g., height).
- Both types use **CDF** to calculate cumulative probabilities, while their respective **PMF** or **PDF** gives probabilities for discrete and continuous values respectively.

Four Types of Probabilities



Explanation of the Four Types of Probabilities with Examples

1. **Marginal Probability** ($P(X)$)

- **Definition:** The probability of a single event X occurring, regardless of any other events.
- **Formula:** $P(X)$
- **Example:**
  - In a deck of 52 cards, the probability of drawing a heart (X) is:

$$P(\text{Heart}) = \frac{13}{52} = 0.25$$

2. **Union Probability** ($P(X \cup Y)$)

- **Definition:** The probability that **either** X, Y, **or both** occur.
- **Formula:** $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$
- **Example:**
  - In a deck of 52 cards, let X = drawing a heart, Y = drawing a red card.
  - Probability of drawing a heart ($P(X)$) = $\frac{13}{52}$, and red card ($P(Y)$) = $\frac{26}{52}$.
  - $P(X \cap Y)$ (card is both a heart and red) = $\frac{13}{52}$.
  - Union:

$$P(X \cup Y) = \frac{13}{52} + \frac{26}{52} - \frac{13}{52} = \frac{26}{52} = 0.5$$

3. **Joint Probability** ($P(X \cap Y)$)

- **Definition:** The probability that **both** X **and** Y occur simultaneously.
- **Formula:** $P(X \cap Y)$
- **Example:**
  - In a dice roll, let X = rolling an even number, Y = rolling a number greater than 3.
  - $X = \{2, 4, 6\}$, $Y = \{4, 5, 6\}$, and $X \cap Y = \{4, 6\}$.
  - Probability:

$$P(X \cap Y) = \frac{\text{Number of favorable outcomes}}{\text{Total outcomes}} = \frac{2}{6} = 0.333$$

4. Conditional Probability ($P(X|Y)$)

- **Definition:** The probability that X occurs given that Y has already occurred.
- **Formula:** $P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$
- **Example:**
  - In a bag of 5 red and 3 blue balls, let X = choosing a red ball, Y = choosing a ball that is not replaced.
  - If the first ball is red (Y), the probability of choosing another red (X) is:

$$P(X|Y) = \frac{4}{7} \text{ (since 1 red ball is removed).}$$

Summary Table

Type	Symbol	Description	Example
Marginal Probability	$P(X)$	Probability of X occurring	Drawing a heart: $P(\text{Heart}) = 0.25$
Union Probability	$P(X \cup Y)$	Probability of X or Y (or both) occurring	Drawing a heart or a red card: 0.5
Joint Probability	$P(X \cap Y)$	Probability of X and Y occurring	Rolling even and >3: 0.333
Conditional Probability	(P(X	Y))	Probability of X, given Y occurred

Example:

Two ballpoint pens are selected at random form a box that contains 3 blue pens, 2 red pens and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find the joint probability function $f(x, y)$.

To solve this, we need to calculate the **joint probability function** $f(x, y)$ for X, the number of blue pens selected, and Y, the number of red pens selected, when two pens are drawn at random from a box containing:

- 3 blue pens
- 2 red pens
- 3 green pens

Step 1: Total number of pens in the box

The total number of pens in the box is:

$$3\ (\text{blue}) + 2\ (\text{red}) + 3\ (\text{green}) = 8$$

The total number of ways to select **2 pens** from 8 pens is:

$$\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$$

Step 2: Possible values of X and Y

- X, the number of blue pens selected, can take values $0, 1,$ or 2.
- Y, the number of red pens selected, can take values $0, 1,$ or 2.

Note that the total number of pens selected is 2. Therefore, $X + Y \leq 2$, and the remaining pens are green. This restricts the possible combinations of X and Y.

Step 3: Joint probability function $f(x, y)$

The joint probability $f(x, y)$ is defined as:

$$f(x, y) = P(X = x \text{ and } Y = y) = \frac{\text{Number of favorable outcomes}}{\text{Total outcomes}}$$

Step 4: Calculate probabilities for all combinations of X and Y

Case 1: $X = 0, Y = 0$

No blue pens and no red pens are selected (both pens are green). The number of ways to choose 2 green pens is:

$$\binom{3}{2} = 3$$

So:

$$f(0, 0) = \frac{3}{28}$$

Case 2: $X = 0, Y = 1$

No blue pens and 1 red pen are selected. This means the other pen must be green. The number of ways to choose 1 red pen and 1 green pen is:

$$\binom{2}{1} \cdot \binom{3}{1} = 2 \cdot 3 = 6$$

So:

$$f(0, 1) = \frac{6}{28}$$

Case 3: $X = 0, Y = 2$

No blue pens and both pens are red. The number of ways to choose 2 red pens is:

$$\binom{2}{2} = 1$$

So:

$$f(0, 2) = \frac{1}{28}$$

Case 4: $X = 1, Y = 0$

1 blue pen and no red pens are selected. The other pen must be green. The number of ways to choose 1 blue pen and 1 green pen is:

$$\binom{3}{1} \cdot \binom{3}{1} = 3 \cdot 3 = 9$$

So:

$$f(1,0) = \frac{9}{28}$$

Case 5: $X = 1, Y = 1$

1 blue pen and 1 red pen are selected. The number of ways to choose 1 blue pen and 1 red pen is:

$$\binom{3}{1} \cdot \binom{2}{1} = 3 \cdot 2 = 6$$

So:

$$f(1,1) = \frac{6}{28}$$

Case 6: $X = 2, Y = 0$

Both pens are blue. The number of ways to choose 2 blue pens is:

$$\binom{3}{2} = 3$$

So:

$$f(2,0) = \frac{3}{28}$$

Step 5: Joint probability function $f(x, y)$

The joint probability function $f(x, y)$ is:

$$f(x,y) = \begin{cases} \frac{3}{28}, & \text{if } (x,y) = (0,0) \text{ or } (2,0) \\ \frac{6}{28}, & \text{if } (x,y) = (0,1) \text{ or } (1,1) \\ \frac{1}{28}, & \text{if } (x,y) = (0,2) \\ \frac{9}{28}, & \text{if } (x,y) = (1,0) \\ 0, & \text{otherwise.} \end{cases}$$

Step 6: Verify total probability

The sum of all probabilities must equal 1:

$$f(0,0) + f(0,1) + f(0,2) + f(1,0) + f(1,1) + f(2,0) = \frac{3}{28} + \frac{6}{28} + \frac{1}{28} + \frac{9}{28} + \frac{6}{28} + \frac{3}{28} = \frac{28}{28} = 1$$

Thus, the probabilities are valid.

Final Answer:

The joint probability function $f(x, y)$ describes the probability of selecting x blue pens and y red pens when two pens are randomly chosen.

Example:

3. **Find the joint distribution of X and Y which are the independent random variables with the following respective distributions.**

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

To find the **joint distribution of X and Y**, given that X and Y are independent random variables with the respective distributions $f(x_i)$ and $g(y_j)$, follow these steps:

Step 1: Understanding the problem

You are given the following distributions:

Distribution of X:

$$x_i = \{1, 2\}, \quad f(x_1) = 0.7, \quad f(x_2) = 0.3$$

Distribution of Y:

$$y_j = \{-2, 5, 8\}, \quad g(y_{-2}) = 0.3, \quad g(y_5) = 0.5, \quad g(y_8) = 0.2$$

Since X and Y are independent, the **joint probability** $P(X = x_i, Y = y_j)$ is the product of their individual probabilities:

$$P(X = x_i \text{ and } Y = y_j) = f(x_i) \cdot g(y_j)$$

Step 2: Compute the joint probabilities

Now, we calculate $P(X = x_i, Y = y_j)$ for all combinations of x_i and y_j:

Case 1: When $X = 1$ ($f(x_1) = 0.7$)

- $P(X = 1 \text{ and } Y = -2) = f(x_1) \cdot g(y_{-2}) = 0.7 \cdot 0.3 = 0.21$
- $P(X = 1 \text{ and } Y = 5) = f(x_1) \cdot g(y_5) = 0.7 \cdot 0.5 = 0.35$
- $P(X = 1 \text{ and } Y = 8) = f(x_1) \cdot g(y_8) = 0.7 \cdot 0.2 = 0.14$

Case 2: When $X = 2$ ($f(x_2) = 0.3$)

- $P(X = 2 \text{ and } Y = -2) = f(x_2) \cdot g(y_{-2}) = 0.3 \cdot 0.3 = 0.09$
- $P(X = 2 \text{ and } Y = 5) = f(x_2) \cdot g(y_5) = 0.3 \cdot 0.5 = 0.15$
- $P(X = 2 \text{ and } Y = 8) = f(x_2) \cdot g(y_8) = 0.3 \cdot 0.2 = 0.06$

Step 3: Tabulate the joint distribution

The joint distribution $P(X = x_i, Y = y_j)$ can be represented in a table:

$X \backslash Y$	$Y = -2$	$Y = 5$	$Y = 8$	Total
$X = 1$	0.21	0.35	0.14	0.7
$X = 2$	0.09	0.15	0.06	0.3
Total	0.3	0.5	0.2	1.0

Step 4: Verify the total probability

The sum of all joint probabilities should equal 1. Summing up all the entries:

$$0.21 + 0.35 + 0.14 + 0.09 + 0.15 + 0.06 = 1$$

Thus, the joint probability distribution is valid.

Final Answer:

The joint distribution is:

$$P(X = x_i, Y = y_j) = \begin{cases} 0.21, & \text{if } (x_i, y_j) = (1, -2) \\ 0.35, & \text{if } (x_i, y_j) = (1, 5) \\ 0.14, & \text{if } (x_i, y_j) = (1, 8) \\ 0.09, & \text{if } (x_i, y_j) = (2, -2) \\ 0.15, & \text{if } (x_i, y_j) = (2, 5) \\ 0.06, & \text{if } (x_i, y_j) = (2, 8) \\ 0, & \text{otherwise.} \end{cases}$$

Example:

A candy company distributed boxes of chocolates with a mixture of creams, toffees, and nuts coated in both light and dark chocolate. For a randomly selected box, let X and Y, respectively, be the proportions of the light and dark chocolates that are creams and suppose that the joint density function is

$$f(x,y)=\begin{cases}\frac{2}{5}(2x+3y), & 0\le x\le 1, 0\le y\le 1\\ 0, & \text{elsewhere}\end{cases}$$

a) Verify whether

b) Find $P[(X,Y)\in A]$, where A is the region $\{(x,y) \mid 0<x<½, ¼<y<½\}$

c) Find g(x) and h(y) for the joint density function.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy=1$$

a) Verify whether $f(x,y)$ is a valid joint density function.

Definition of a valid joint density function:

A joint density function $f(x,y)$ is valid if it satisfies these two conditions:

1. $f(x,y)\ge 0$ for all x and y.
2. The total probability over the entire region is 1, i.e.,
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1.$$

Step 1: Check $f(x,y)\ge 0$

- Look at the definition of $f(x,y)$ (not provided in your text, but assume it's positive in the region of interest). If the function is non-negative for all x,y, then the first condition is satisfied.

Step 2: Check that the total integral equals 1

If $f(x,y)$ is valid, the integral over the region should equal 1:

$$\int_0^1\int_0^1 f(x,y)\,dy\,dx=1.$$

- Substitute the given $f(x,y)$ into the integral and solve. Ensure that the result equals 1.

b) Find $P[(X,Y) \in A]$, where $A = \{(x,y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

Step 1: Write the probability expression

The probability $P[(X,Y) \in A]$ is:

$$P[(X,Y) \in A] = \int_A f(x,y)\,dx\,dy$$

- Substituting the region A, we get:

$$P[(X,Y) \in A] = \int_{x=0}^{1/2} \int_{y=1/4}^{1/2} f(x,y)\,dy\,dx.$$

Step 2: Substitute $f(x,y)$

- Replace $f(x,y)$ with the given joint density function.
- Solve the integral:

$$\int_{x=0}^{1/2} \int_{y=1/4}^{1/2} f(x,y)\,dy\,dx.$$

Example Calculation:

If $f(x,y) = 4xy$ for $0 \leq x \leq 1, 0 \leq y \leq 1$, then:

$$P[(X,Y) \in A] = \int_{x=0}^{1/2} \int_{y=1/4}^{1/2} 4xy\,dy\,dx.$$

- First, integrate with respect to y:

$$\int_{y=1/4}^{1/2} 4xy\,dy = 4x\left[\frac{y^2}{2}\right]_{y=1/4}^{1/2} = 4x\left(\frac{(1/2)^2}{2} - \frac{(1/4)^2}{2}\right).$$

- Now, integrate with respect to x:

$$\int_{x=0}^{1/2} \frac{x}{2}\,dx = \frac{1}{2}\int_{x=0}^{1/2} x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_{x=0}^{1/2} = \frac{1}{2}\cdot\frac{(1/2)^2}{2} = \frac{1}{2}\cdot\frac{1}{8} = \frac{1}{16}.$$

Thus, $P[(X,Y)\in A] = \frac{1}{16}$.

---

c) Find $g(x)$ and $h(y)$ for the joint density function.

Step 1: Marginal density $g(x)$ of X:

The marginal density $g(x)$ is obtained by integrating $f(x,y)$ over y:

$$g(x) = \int_0^1 f(x,y)\,dy.$$

Step 2: Marginal density $h(y)$ of Y:

The marginal density $h(y)$ is obtained by integrating $f(x,y)$ over x:

$$h(y) = \int_0^1 f(x,y)\,dx.$$

Example Calculation:

If $f(x,y) = 4xy$:

- Marginal density of X:

$$g(x) = \int_0^1 4xy\,dy = 4x\int_0^1 y\,dy = 4x\left[\frac{y^2}{2}\right]_0^1 = 4x\cdot\frac{1}{2} = 2x.$$

- Marginal density of Y:

$$h(y) = \int_0^1 4xy\,dx = 4y\int_0^1 x\,dx = 4y\left[\frac{x^2}{2}\right]_0^1 = 4y\cdot\frac{1}{2} = 2y.$$

Thus:

$$g(x) = 2x \quad \text{and} \quad h(y) = 2y.$$

Final Summary:

1. $f(x,y)$ is valid if it integrates to 1.
2. $P[(X,Y)\in A] = \frac{1}{16}$ (example calculation).
3. Marginal densities are $g(x) = 2x$ and $h(y) = 2y$ (example for $f(x,y) = 4xy$).

Example:

The joint probability distribution of X and Y is given by $f(x,y) = c(x^2+y^2)$ for $x = -1, 0, 1, 3.$ *and* $y = -1, 2, 3.$ (i) Find the value of c. (ii) $P(x = 0, y \leq 2)$ (iii) $P(x \leq 1, y > 2)$ (iv) $P(x \geq 2 - y)$

To solve the given problem, we need to compute the probabilities step by step based on t probability distribution function $f(x,y) = c(x^2+y^2)$. Let's solve each part clearly.

Step 1: Find the value of c (Normalization Condition)

Concept:

The total probability of a valid probability distribution should equal 1:

$$\sum_x \sum_y f(x,y) = 1.$$

Given values:

- $x = -1, 0, 1, 3,$
- $y = -1, 2, 3,$
- $f(x,y) = c(x^2+y^2).$

Calculate $\sum_x \sum_y f(x,y)$:

1. Compute x^2+y^2 for all combinations of x and y:

x	y	x^2+y^2
-1	-1	$1+1=2$
-1	2	$1+4=5$
-1	3	$1+9=10$
0	-1	$0+1=1$
0	2	$0+4=4$
0	3	$0+9=9$
1	-1	$1+1=2$
1	2	$1+4=5$
1	3	$1+9=10$
3	-1	$9+1=10$
3	2	$9+4=13$
3	3	$9+9=18$

2. Sum all the x^2+y^2 values:

$$\text{Sum} = 2+5+10+1+4+9+2+5+10+10+13+18 = 89.$$

3. Normalize c using:

$$\sum_x \sum_y f(x,y) = 1 \implies c \cdot 89 = 1.$$

4. Solve for c:

$$c = \frac{1}{89}.$$

Step 2: Find $P(x = 0, y \leq 2)$

Concept:

The probability is:

$$P(x = 0, y \leq 2) = \sum_{y \leq 2} f(x = 0, y).$$

Calculate:

For $x = 0$, $f(0, y) = c(0^2 + y^2) = c(y^2)$.

- Possible values of y with $y \leq 2$: $y = -1, 2$.
- $f(0, -1) = c((-1)^2) = \frac{1}{89}(1) = \frac{1}{89}$,
- $f(0, 2) = c((2)^2) = \frac{1}{89}(4) = \frac{4}{89}$.

Add these probabilities:

$$P(x = 0, y \leq 2) = \frac{1}{89} + \frac{4}{89} = \frac{5}{89}.$$

Step 3: Find $P(x \leq 1, y > 2)$

Concept:

The probability is:

$$P(x \leq 1, y > 2) = \sum_{x \leq 1} \sum_{y > 2} f(x, y).$$

Calculate:

1. For $x = -1, 0, 1$, $y = 3$ (since $y > 2$):
   - $f(-1, 3) = c((-1)^2 + 3^2) = \frac{1}{89}(1 + 9) = \frac{10}{89}$,
   - $f(0, 3) = c((0)^2 + 3^2) = \frac{1}{89}(9) = \frac{9}{89}$,
   - $f(1, 3) = c((1)^2 + 3^2) = \frac{1}{89}(1 + 9) = \frac{10}{89}$.
2. Add these probabilities:

$$P(x \leq 1, y > 2) = \frac{10}{89} + \frac{9}{89} + \frac{10}{89} = \frac{29}{89}.$$

Step 4: Find $P(x \geq 2 - y)$

To compute $P(x \geq 2 - y) = P(x + y \geq 2)$, let's break it down into detailed steps.

Step 1: Understand the condition $x + y \geq 2$

We are given:

- $x \in \{-1, 0, 1, 3\}$,
- $y \in \{-1, 2, 3\}$, and we are tasked with finding all valid combinations of x and y such that $x + y \geq 2$.

Step 2: Identify valid combinations of x and y

We calculate $x + y$ for each combination of x and y and check if $x + y \geq 2$.

x	y	$x + y$	Condition Met?
-1	-1	-2	No
-1	2	1	No
-1	3	2	Yes
0	-1	-1	No
0	2	2	Yes
0	3	3	Yes
1	-1	0	No
1	2	3	Yes
1	3	4	Yes
3	-1	2	Yes
3	2	5	Yes
3	3	6	Yes

Valid combinations:

From the table above, the valid combinations are:

- $(-1, 3)$,
- $(0, 2)$,
- $(0, 3)$,
- $(1, 2)$,
- $(1, 3)$,
- $(3, -1)$,
- $(3, 2)$,
- $(3, 3)$.

Step 3: Compute probabilities for valid combinations

The joint probability is $f(x, y) = c(x^2 + y^2)$, where $c = \frac{1}{89}$.

For each valid pair:

1. $(-1, 3) : f(-1, 3) = c((-1)^2 + 3^2) = \frac{1}{89}(1 + 9) = \frac{10}{89}$,
2. $(0, 2) : f(0, 2) = c(0^2 + 2^2) = \frac{1}{89}(4) = \frac{4}{89}$,
3. $(0, 3) : f(0, 3) = c(0^2 + 3^2) = \frac{1}{89}(9) = \frac{9}{89}$,
4. $(1, 2) : f(1, 2) = c(1^2 + 2^2) = \frac{1}{89}(1 + 4) = \frac{5}{89}$,
5. $(1, 3) : f(1, 3) = c(1^2 + 3^2) = \frac{1}{89}(1 + 9) = \frac{10}{89}$,
6. $(3, -1) : f(3, -1) = c(3^2 + (-1)^2) = \frac{1}{89}(9 + 1) = \frac{10}{89}$,
7. $(3, 2) : f(3, 2) = c(3^2 + 2^2) = \frac{1}{89}(9 + 4) = \frac{13}{89}$,
8. $(3, 3) : f(3, 3) = c(3^2 + 3^2) = \frac{1}{89}(9 + 9) = \frac{18}{89}$.

Step 4: Add probabilities

Sum up all the probabilities for valid pairs:

$P(x+y \geq 2) = f(-1,3) + f(0,2) + f(0,3) + f(1,2) + f(1,3) + f(3,-1) + f(3,2) + f(3,3).$

Substitute values:

$$P(x+y \geq 2) = \frac{10}{89} + \frac{4}{89} + \frac{9}{89} + \frac{5}{89} + \frac{10}{89} + \frac{10}{89} + \frac{13}{89} + \frac{18}{89}.$$

Simplify:

$$P(x+y \geq 2) = \frac{79}{89}.$$

Final Answer:

$$P(x+y \geq 2) = \frac{79}{89}.$$

Practice Problems-set 1

Problem:1

If the probability density of a random variable X is given by

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & otherwise \end{cases}$$

Find $i)\ P(0.2 < X < 0.8)$ $\quad ii)\ P(0.6 < X < 1.2)$

To solve this problem, we will compute the required probabilities using the given probability density function (PDF) $f(x)$:

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2-x & \text{if } 1 \leq x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) $P(0.2 < X < 0.8)$:

We need to find the probability that X lies between 0.2 and 0.8. Since 0.2 and 0.8 fall in the interval $0 < x < 1$, we use the part of the PDF where $f(x) = x$.

$$P(0.2 < X < 0.8) = \int_{0.2}^{0.8} f(x)dx = \int_{0.2}^{0.8} x\, dx.$$

Step 1: Set up the integral

$$\int_{0.2}^{0.8} x\, dx = \left[\frac{x^2}{2}\right]_{0.2}^{0.8}.$$

Step 2: Compute the integral

$$\text{Evaluate at } x = 0.8: \frac{(0.8)^2}{2} = \frac{0.64}{2} = 0.32.$$

$$\text{Evaluate at } x = 0.2: \frac{(0.2)^2}{2} = \frac{0.04}{2} = 0.02.$$

Step 3: Subtract

$$P(0.2 < X < 0.8) = 0.32 - 0.02 = 0.30.$$

(ii) $P(0.6 < X < 1.2)$:

We need to calculate the probability that X lies between 0.6 and 1.2. Since X is split across two intervals ($0 < x < 1$ and $1 \leq x < 2$), we break the calculation into two parts.

$$P(0.6 < X < 1.2) = \int_{0.6}^{1} f(x)\, dx + \int_{1}^{1.2} f(x)\, dx.$$

Part 1: $\int_{0.6}^{1} f(x)dx$

Here, $f(x) = x$ for $0.6 \leq x < 1$.

$$\int_{0.6}^{1} x\, dx = \left[\frac{x^2}{2}\right]_{0.6}^{1}.$$

Evaluate:

$$\text{At } x = 1: \frac{(1)^2}{2} = \frac{1}{2} = 0.5.$$

$$\text{At } x = 0.6: \frac{(0.6)^2}{2} = \frac{0.36}{2} = 0.18.$$

Subtract:

$$\int_{0.6}^{1} x\, dx = 0.5 - 0.18 = 0.32.$$

Part 2: $\int_{1}^{1.2} f(x)dx$

Here, $f(x) = 2 - x$ for $1 \leq x < 2$.

$$\int_{1}^{1.2} (2 - x)\, dx = \int_{1}^{1.2} 2\, dx - \int_{1}^{1.2} x\, dx.$$

First term:

$$\int_{1}^{1.2} 2\,dx = [2x]_{1}^{1.2} = 2(1.2) - 2(1) = 2.4 - 2 = 0.4.$$

Second term:

$$\int_{1}^{1.2} x\,dx = \left[\frac{x^2}{2}\right]_{1}^{1.2}.$$

Evaluate:

$$\text{At } x = 1.2 : \frac{(1.2)^2}{2} = \frac{1.44}{2} = 0.72.$$
$$\text{At } x = 1 : \frac{(1)^2}{2} = \frac{1}{2} = 0.5.$$

Subtract:

$$\int_{1}^{1.2} x\,dx = 0.72 - 0.5 = 0.22.$$

Combine:

$$\int_{1}^{1.2} (2 - x)\,dx = 0.4 - 0.22 = 0.18.$$

Add both parts:

$$P(0.6 < X < 1.2) = \int_{0.6}^{1} x\,dx + \int_{1}^{1.2} (2 - x)\,dx = 0.32 + 0.18 = 0.50.$$

Final Answers:

1. $P(0.2 < X < 0.8) = 0.30,$
2. $P(0.6 < X < 1.2) = 0.50.$

Problem:2

Let $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$ be a probability density function then find the value of k.

To determine the value of k, we use the property of a probability density function (PDF). The int of the PDF over its entire domain must equal 1:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Given:

$$f(x) = \frac{k}{1+x^2}, \quad -\infty < x < \infty.$$

Step 1: Set up the integral

$$\int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx = 1.$$

Since k is a constant, it can be factored out of the integral:

$$k \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 1.$$

Step 2: Recognize the integral

The integral $\int \frac{1}{1+x^2}\, dx$ is a well-known result:

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1}(x) + C.$$

Step 3: Solve the definite integral

Now, calculate the definite integral from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \left[\tan^{-1}(x)\right]_{-\infty}^{\infty}.$$

Evaluate at the limits:

- As $x \to \infty$, $\tan^{-1}(x) \to \frac{\pi}{2}$.
- As $x \to -\infty$, $\tan^{-1}(x) \to -\frac{\pi}{2}$.

Thus:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \tan^{-1}(\infty) - \tan^{-1}(-\infty) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Step 4: Substitute back

Substitute the result of the integral into the equation:

$$k \cdot \pi = 1.$$

Solve for k:

$$k = \frac{1}{\pi}.$$

Final Answer:

The value of k is:

$$k = \frac{1}{\pi}.$$

Problem:3

The length of satisfactory service (years) provided by a certain model of laptop computer is random variable having the probability density function :

$$\boldsymbol{f(x)} = \begin{cases} \frac{1}{4.5} e^{-\frac{x}{4.5}} & x > 0 \\ 0 & otherwise \end{cases}$$

Find the probabilities that one of these laptops will provide satisfactory service for

a) At most 2.5 years
b) Anywhere from 4 to 6 years
c) At least 6.75 years

To solve the problem, we use the given probability density function (PDF):

$$f(x) = \begin{cases} \frac{1}{4.5} e^{-\frac{x}{4.5}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

This represents the exponential distribution with a mean of 4.5 years. The cumulative distribution function (CDF) can help compute probabilities.

The cumulative distribution function (CDF) is given by:

$$F(x) = P(X \leq x) = \int_0^x f(t)\,dt = \int_0^x \frac{1}{4.5} e^{-\frac{t}{4.5}}\,dt.$$

1. Calculate $F(x)$:

The integral of the exponential function is:

$$F(x) = 1 - e^{-\frac{x}{4.5}}, \quad x > 0.$$

Using this, we calculate probabilities for the required cases.

(a) At most 2.5 years

$$P(X \leq 2.5) = F(2.5) = 1 - e^{-\frac{2.5}{4.5}}.$$

Substitute $x = 2.5$ and calculate:

$$P(X \leq 2.5) = 1 - e^{-\frac{2.5}{4.5}} = 1 - e^{-0.5556}.$$

Using $e^{-0.5556} \approx 0.5738$:

$$P(X \leq 2.5) = 1 - 0.5738 = 0.4262.$$

So, the probability is approximately:

$$P(X \leq 2.5) = 0.4262.$$

(b) Anywhere from 4 to 6 years

This is the probability that $4 \leq X \leq 6$, which can be computed as:

$$P(4 \leq X \leq 6) = F(6) - F(4).$$

First, calculate $F(6)$ and $F(4)$:

$$F(6) = 1 - e^{-\frac{6}{4.5}} = 1 - e^{-1.3333}.$$

$$F(4) = 1 - e^{-\frac{4}{4.5}} = 1 - e^{-0.8889}.$$

Using approximate values:

- $e^{-1.3333} \approx 0.2636,$
- $e^{-0.8889} \approx 0.4111,$

We have:

$$F(6) = 1 - 0.2636 = 0.7364,$$

$$F(4) = 1 - 0.4111 = 0.5889.$$

Now, subtract:

$$P(4 \leq X \leq 6) = F(6) - F(4) = 0.7364 - 0.5889 = 0.1475.$$

So, the probability is approximately:

$$P(4 \leq X \leq 6) = 0.1475.$$

(c) At least 6.75 years

This is the probability that $X \geq 6.75$, which can be computed as:

$$P(X \geq 6.75) = 1 - F(6.75).$$

First, calculate $F(6.75)$:

$$F(6.75) = 1 - e^{-\frac{6.75}{4.5}} = 1 - e^{-1.5}.$$

Using $e^{-1.5} \approx 0.2231$:

$$F(6.75) = 1 - 0.2231 = 0.7769.$$

Now, subtract:

$$P(X \geq 6.75) = 1 - 0.7769 = 0.2231.$$

So, the probability is approximately:

$$P(X \geq 6.75) = 0.2231.$$

Final Answers:

(a) $P(X \leq 2.5) = 0.4262.$
(b) $P(4 \leq X \leq 6) = 0.1475.$
(c) $P(X \geq 6.75) = 0.2231.$

Problem:4

For following probability density function :

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & otherwise \end{cases}$$. Find the mean and variance.

To find the mean and variance of the given probability density function (PDF):

$$f(x) = \begin{cases} x & 0 \leq x \leq 1, \\ 2 - x & 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Step 1: Verify if it is a valid PDF

For a function to be a valid probability density function:

1. $f(x) \geq 0$ for all x,
2. The total integral over the range must equal 1:
$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Here, the function is non-negative. Now, check the integral:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (2 - x)\,dx.$$

1. Compute $\int_0^1 x\,dx$:
$$\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}.$$
2. Compute $\int_1^2 (2 - x)\,dx$:
$$\int_1^2 (2 - x)\,dx = \left[2x - \frac{x^2}{2}\right]_1^2 = \left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right).$$
$$= (4 - 2) - (2 - 0.5) = 2 - 1.5 = 0.5.$$

Total integral:

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.5 + 0.5 = 1.$$

Thus, $f(x)$ is a valid PDF.

Step 2: Find the Mean (μ)

The mean μ is given by:

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Compute:

$$\mu = \int_0^1 x \cdot x\, dx + \int_1^2 x \cdot (2 - x)\, dx.$$

1. Compute $\int_0^1 x^2\, dx$:

$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

2. Compute $\int_1^2 x(2 - x)\, dx$:

$$\int_1^2 x(2 - x)\, dx = \int_1^2 (2x - x^2)\, dx.$$

Break it into two parts:

$$\int_1^2 (2x - x^2)\, dx = \int_1^2 2x\, dx - \int_1^2 x^2\, dx.$$

(a) Compute $\int_1^2 2x\, dx$:

$$\int_1^2 2x\, dx = \left[x^2\right]_1^2 = 2^2 - 1^2 = 4 - 1 = 3.$$

(b) Compute $\int_1^2 x^2\, dx$:

$$\int_1^2 x^2\, dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

Combine:

$$\int_1^2 (2x - x^2)\, dx = 3 - \frac{7}{3} = \frac{9}{3} - \frac{7}{3} = \frac{2}{3}.$$

Now, sum the parts:

$$\mu = \frac{1}{3} + \frac{2}{3} = 1.$$

So, the mean is:

$$\mu = 1.$$

Step 3: Find the Variance (σ^2)

The variance σ^2 is given by:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2.$$

1. Compute $\int_{-\infty}^{\infty} x^2 f(x)\, dx$:

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2 \cdot (2-x)\, dx.$$

(a) Compute $\int_0^1 x^3\, dx$:

$$\int_0^1 x^3\, dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}.$$

(b) Compute $\int_1^2 x^2(2-x)\, dx$:

$$\int_1^2 x^2(2-x)\, dx = \int_1^2 (2x^2 - x^3)\, dx.$$

Break it into two parts:

$$\int_1^2 (2x^2 - x^3)\, dx = \int_1^2 2x^2\, dx - \int_1^2 x^3\, dx.$$

(i) Compute $\int_1^2 2x^2\, dx$:

$$\int_1^2 2x^2\, dx = \left[\frac{2x^3}{3}\right]_1^2 = \frac{2(2^3)}{3} - \frac{2(1^3)}{3} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$$

(ii) Compute $\int_1^2 x^3\, dx$:

$$\int_1^2 x^3\, dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$$

Combine:

$$\int_1^2 (2x^2 - x^3)\,dx = \frac{14}{3} - \frac{15}{4}.$$

Take a common denominator (12):

$$\int_1^2 (2x^2 - x^3)\,dx = \frac{56}{12} - \frac{45}{12} = \frac{11}{12}.$$

Now, sum the parts:

$$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{4} + \frac{11}{12}.$$

Take a common denominator (12):

$$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{3}{12} + \frac{11}{12} = \frac{14}{12} = \frac{7}{6}.$$

2. Compute σ^2:

$$\sigma^2 = \frac{7}{6} - \mu^2 = \frac{7}{6} - 1^2 = \frac{7}{6} - \frac{6}{6} = \frac{1}{6}.$$

Final Answers:

- Mean (μ): 1,
- Variance (σ^2): $\frac{1}{6}$.

Problem:5

If $P(X = x) = \begin{cases} kx & x = 1,2,3,4,5 \\ 0 & otherwise \end{cases}$ represents a probability function , find $i)\ k,\quad ii)\ P(X\ \ being\ a\ prime\ number),\quad iii)$ $P\left(\frac{1}{2} < X < \frac{5}{2}\right)$.

Problem:

The probability function is:

$$P(X=x)=\begin{cases} kx, & x=1,2,3,4,5, \\ 0, & \text{otherwise.} \end{cases}$$

We need to:

1. Find k (to ensure it is a valid probability distribution),
2. Find $P(X \text{ is a prime number})$,
3. Find $P\left(\frac{1}{2}<X<\frac{5}{2}\right)$.

Step 1: Find k

For $P(X=x)$ to represent a valid probability distribution:

$$\sum P(X=x)=1.$$

Given $P(X=x)=kx$ for $x=1,2,3,4,5$, the probabilities are:

$P(X=1)=k\cdot 1=k,\ P(X=2)=k\cdot 2=2k,\ P(X=3)=k\cdot 3=3k,\ P(X=4)=k\cdot 4=4k,\ P(X=5)=k\cdot 5=5k.$

The total sum is:

$$k+2k+3k+4k+5k=15k.$$

Set the sum equal to 1:

$$15k=1 \implies k=\frac{1}{15}.$$

Thus, $k=\frac{1}{15}$.

Step 2: Find $P(X \text{ is a prime number})$

The prime numbers between 1 and 5 are $2, 3$, and 5. So:

$$P(X \text{ is a prime number})=P(X=2)+P(X=3)+P(X=5).$$

Substitute the probabilities:

$$P(X=2)=2k=2\cdot\frac{1}{15}=\frac{2}{15},$$
$$P(X=3)=3k=3\cdot\frac{1}{15}=\frac{3}{15},$$
$$P(X=5)=5k=5\cdot\frac{1}{15}=\frac{5}{15}.$$

Add them:

$$P(X \text{ is a prime number})=\frac{2}{15}+\frac{3}{15}+\frac{5}{15}=\frac{10}{15}=\frac{2}{3}.$$

Step 3: Find $P\left(\frac{1}{2} < X < \frac{5}{2}\right)$

The range $\frac{1}{2} < X < \frac{5}{2}$ corresponds to $X = 2$ because X can only take integer values $1, 2, 3, 4, 5$. So:

$$P\left(\frac{1}{2} < X < \frac{5}{2}\right) = P(X = 2).$$

From earlier:

$$P(X = 2) = \frac{2}{15}.$$

Final Answers:

1. $k = \frac{1}{15}$,
2. $P(X \text{ is a prime number}) = \frac{2}{3}$,
3. $P\left(\frac{1}{2} < X < \frac{5}{2}\right) = \frac{2}{15}$.

Problem:6

From a lot of 10 items containing 3 defective items, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Answer the following when the sample is drawn without replacement:

a) Find the probability distribution of X.

b) Find $P(X \leq 1)$

c) Find $P(0 < X < 2)$

To solve Problem 6, we need to determine the **probability distribution of X**, the number of defective items in a random sample of 4 items drawn without replacement from a lot of 10 items containing 3 defective ones. Then, we'll compute the required probabilities.

Step 1: Understand the Problem

- **Population:** 10 items, out of which 3 are defective, and 7 are non-defective.
- **Sample size:** 4 items drawn **without replacement.**
- X: The random variable denoting the number of defective items in the sample.

Step 2: Possible Values of X

- X can take values: $0, 1, 2, 3$.
  - $X = 0$: No defective items in the sample.
  - $X = 1$: Exactly 1 defective item in the sample.
  - $X = 2$: Exactly 2 defective items in the sample.
  - $X = 3$: All 3 defective items in the sample.

Step 3: Compute the Probability Distribution of X

The probability of $X = k$ (where k defective items are chosen) can be calculated using the **hypergeometric distribution formula:**

$$P(X = k) = \frac{\binom{3}{k}\binom{7}{4-k}}{\binom{10}{4}}$$

Here:

- $\binom{3}{k}$: Number of ways to choose k defective items from 3 defective items.
- $\binom{7}{4-k}$: Number of ways to choose $4 - k$ non-defective items from 7 non-defective items.
- $\binom{10}{4}$: Total number of ways to choose 4 items from the 10 items.

Case 1: $X = 0$

$$P(X = 0) = \frac{\binom{3}{0}\binom{7}{4}}{\binom{10}{4}}$$

$$= \frac{1 \cdot 35}{210} = \frac{35}{210} = \frac{1}{6}.$$

Case 2: $X = 1$

$$P(X = 1) = \frac{\binom{3}{1}\binom{7}{3}}{\binom{10}{4}}$$

$$= \frac{3 \cdot 35}{210} = \frac{105}{210} = \frac{1}{2}.$$

Case 3: $X = 2$

$$P(X = 2) = \frac{\binom{3}{2}\binom{7}{2}}{\binom{10}{4}}$$

$$= \frac{3 \cdot 21}{210} = \frac{63}{210} = \frac{3}{10}.$$

Case 4: $X = 3$

$$P(X = 3) = \frac{\binom{3}{3}\binom{7}{1}}{\binom{10}{4}}$$

$$= \frac{1 \cdot 7}{210} = \frac{7}{210} = \frac{1}{30}.$$

Thus, the probability distribution of X is:

$$P(X = 0) = \frac{1}{6},$$
$$P(X = 1) = \frac{1}{2},$$
$$P(X = 2) = \frac{3}{10},$$
$$P(X = 3) = \frac{1}{30}.$$

Step 4: Find $P(X \leq 1)$

$P(X \leq 1)$ means $P(X = 0) + P(X = 1)$:

$$P(X \leq 1) = P(X = 0) + P(X = 1).$$

Substitute values:

$$P(X \leq 1) = \frac{1}{6} + \frac{1}{2}.$$

$$P(X \leq 1) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}.$$

Step 5: Find $P(0 < X < 2)$

$P(0 < X < 2)$ means $P(X = 1)$:

$$P(0 < X < 2) = P(X = 1).$$

Substitute:

$$P(0 < X < 2) = \frac{1}{2}.$$

Final Answers:

1. **Probability distribution of X:**

$$P(X = 0) = \frac{1}{6},\ P(X = 1) = \frac{1}{2},\ P(X = 2) = \frac{3}{10},\ P(X = 3) = \frac{1}{30}.$$

2. $P(X \leq 1) = \frac{2}{3}$.
3. $P(0 < X < 2) = \frac{1}{2}$.

Problem:7

The probability mass function of a random variable is zero except at the points $x = 0,1,2$. At these points it has the values $P(0) = 3a^3,\ P(1) = 4a - 10a^2,\ \ P(2) = 5a - 1$ for some $a > 0$.

a) Determine the value of a

b) Compute the probabilities $P(X < 2)$

c) Find the largest x such that $F(x) < \frac{1}{2}$, where $F(x)$ is cumulative distribution function.

Given:

The probability mass function (PMF) is nonzero only for $x = 0, 1, 2$. For these points:

$$P(0) = 3a^3, \quad P(1) = 4a - 10a^2, \quad P(2) = 5a - 1$$

Also, the condition for a PMF is:

$$P(0) + P(1) + P(2) = 1$$

We need to:

1. Determine the value of a.
2. Compute $P(X < 2)$.
3. Find the largest x such that $F(x) < \frac{1}{2}$, where $F(x)$ is the cumulative distribution function (CDF).

Step 1: Determine the value of a:

Using the condition $P(0) + P(1) + P(2) = 1$:

$$3a^3 + (4a - 10a^2) + (5a - 1) = 1$$

Simplify:

$$3a^3 - 10a^2 + 9a - 2 = 0$$

Now solve for a. Factorize:

$$a = \frac{1}{3}$$

Verify:

$$P(0) = 3\left(\frac{1}{3}\right)^3 = \frac{1}{9}, \quad P(1) = 4\left(\frac{1}{3}\right) - 10\left(\frac{1}{3}\right)^2 = \frac{4}{3} - \frac{10}{9} = \frac{2}{9}$$

$$P(2) = 5\left(\frac{1}{3}\right) - 1 = \frac{5}{3} - \frac{3}{3} = \frac{2}{3}$$

Sum:

$$P(0) + P(1) + P(2) = \frac{1}{9} + \frac{2}{9} + \frac{6}{9} = 1$$

Thus, $a = \frac{1}{3}$.

Step 2: Compute $P(X < 2)$:

We need:

$$P(X < 2) = P(0) + P(1)$$

Substitute the values:

$$P(0) = \frac{1}{9}, \quad P(1) = \frac{2}{9}$$

Add:

$$P(X < 2) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$$

Step 3: Find the largest x such that $F(x) < \frac{1}{2}$:

The CDF $F(x)$ is:

- $F(0) = P(0) = \frac{1}{9}$
- $F(1) = F(0) + P(1) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$
- $F(2) = F(1) + P(2) = \frac{1}{3} + \frac{2}{3} = 1$

Now, check $F(x) < \frac{1}{2}$:

- For $x = 0$, $F(0) = \frac{1}{9} < \frac{1}{2}$
- For $x = 1$, $F(1) = \frac{1}{3} < \frac{1}{2}$
- For $x = 2$, $F(2) = 1 \not< \frac{1}{2}$

Thus, the largest x such that $F(x) < \frac{1}{2}$ is $x = 1$.

Final Answers:

1. $a = \frac{1}{3}$
2. $P(X < 2) = \frac{1}{3}$
3. Largest x such that $F(x) < \frac{1}{2}$ is $x = 1$.