

Matrix Operations

All Matrix Operations

Date_____ Period____

Simplify. Write "undefined" for expressions that are undefined.

1)
$$\begin{bmatrix} 2 & -1 \\ -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ -3 & -5 \end{bmatrix}$$

2)
$$\begin{bmatrix} 2 & 6 \\ -6 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 & 3 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right)$$

3)
$$\begin{bmatrix} -1 & 5 \\ 5 & -5 \end{bmatrix} \cdot \begin{bmatrix} -3 & 6 \\ -3 & 0 \end{bmatrix}$$

4)
$$\begin{bmatrix} 1 & -6 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

5)
$$\begin{bmatrix} -2 \\ -3 \\ -6 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 0 \\ -3 \end{bmatrix}$$

6)
$$-4 \cdot \left(\begin{bmatrix} -3 & -6 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 6 \\ -1 & -4 \end{bmatrix} \right)$$

7)
$$\begin{bmatrix} 3 & 1 & 3 \\ 0 & 5 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -6 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

8)
$$-5 \left(\begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \right)$$

$$9) \begin{bmatrix} -1 & -1 \\ -6 & 3 \end{bmatrix} + \begin{bmatrix} -5 & -1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 \\ 1 & 6 \end{bmatrix}$$

$$10) \begin{bmatrix} -2 \\ -6 \end{bmatrix} - 3 \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$11) -2 \begin{bmatrix} -3 & -5 & -5 \\ 0 & 5 & -6 \end{bmatrix} + \begin{bmatrix} 4 & -1 & -3 \\ 6 & 3 & 2 \end{bmatrix}$$

$$12) \begin{bmatrix} -5 & 1 \\ -4 & -5 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 & -4 & 2 \\ -6 & 3 & -6 \end{bmatrix} + \begin{bmatrix} 3 & -5 & 2 \\ 5 & 5 & 3 \end{bmatrix} \right)$$

$$13) \left(\begin{bmatrix} -4y & 2y \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2y & 6 \\ 2 & 2x \end{bmatrix} \right) \cdot \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$14) \begin{bmatrix} 6y & y^2 \\ -2y & -2y \end{bmatrix} \cdot \begin{bmatrix} -y & xy \\ -6 & x^2 \end{bmatrix} - \begin{bmatrix} 6y & -6 \\ -3y & y \end{bmatrix}$$

Critical thinking questions:

15) Give an example of a matrix expression in which you would first perform a matrix subtraction and then a matrix multiplication. Use any numbers and dimensions you would like but be sure that your expression isn't undefined.

16) A , B , and C are matrices: $A(B + C) = AB + CA$
 A) Always true B) Sometimes true
 C) False

All Matrix Operations

Simplify. Write "undefined" for expressions that are undefined.

$$1) \begin{bmatrix} 2 & -1 \\ -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ -3 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 13 \\ -27 & -29 \end{bmatrix}$$

$$2) \begin{bmatrix} 2 & 6 \\ -6 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 & 3 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right)$$

$$\begin{bmatrix} -12 & 22 \\ -52 & -22 \end{bmatrix}$$

$$3) \begin{bmatrix} -1 & 5 \\ 5 & -5 \end{bmatrix} \cdot \begin{bmatrix} -3 & 6 \\ -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -12 & -6 \\ 0 & 30 \end{bmatrix}$$

$$4) \begin{bmatrix} 1 & -6 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

Undefined

$$5) \begin{bmatrix} -2 \\ -3 \\ -6 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 0 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} -6 \\ 3 \\ -6 \\ -1 \end{bmatrix}$$

$$6) -4 \cdot \left(\begin{bmatrix} -3 & -6 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 6 \\ -1 & -4 \end{bmatrix} \right)$$

$$\begin{bmatrix} -48 & -24 \\ 24 & 40 \end{bmatrix}$$

$$7) \begin{bmatrix} 3 & 1 & 3 \\ 0 & 5 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -6 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 10 & 16 \\ 1 & -30 & -5 \end{bmatrix}$$

$$8) -5 \left(\begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \right)$$

$$\begin{bmatrix} -20 \\ 25 \\ 10 \end{bmatrix}$$

$$9) \begin{bmatrix} -1 & -1 \\ -6 & 3 \end{bmatrix} + \begin{bmatrix} -5 & -1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 \\ 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -17 & -37 \\ -16 & -9 \end{bmatrix}$$

$$10) \begin{bmatrix} -2 \\ -6 \end{bmatrix} - 3 \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16 \\ -6 \end{bmatrix}$$

$$11) -2 \begin{bmatrix} -3 & -5 & -5 \\ 0 & 5 & -6 \end{bmatrix} + \begin{bmatrix} 4 & -1 & -3 \\ 6 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 9 & 7 \\ 6 & -7 & 14 \end{bmatrix}$$

$$12) \begin{bmatrix} -5 & 1 \\ -4 & -5 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 & -4 & 2 \\ -6 & 3 & -6 \end{bmatrix} + \begin{bmatrix} 3 & -5 & 2 \\ 5 & 5 & 3 \end{bmatrix} \right)$$

$$\begin{bmatrix} -41 & 53 & -23 \\ -27 & -4 & -1 \end{bmatrix}$$

$$13) \left(\begin{bmatrix} -4y & 2y \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2y & 6 \\ 2 & 2x \end{bmatrix} \right) \cdot \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} -20y - 30 \\ 5 - 10x \end{bmatrix}$$

$$14) \begin{bmatrix} 6y & y^2 \\ -2y & -2y \end{bmatrix} \cdot \begin{bmatrix} -y & xy \\ -6 & x^2 \end{bmatrix} - \begin{bmatrix} 6y & -6 \\ -3y & y \end{bmatrix}$$

$$\begin{bmatrix} -12y^2 - 6y & 6y^2x + y^2x^2 + 6 \\ 2y^2 + 15y & -2y^2x - 2yx^2 - y \end{bmatrix}$$

Critical thinking questions:

15) Give an example of a matrix expression in which you would first perform a matrix subtraction and then a matrix multiplication. Use any numbers and dimensions you would like but be sure that your expression isn't undefined.

Many answers. Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$

16) A , B , and C are matrices: $A(B + C) = AB + CA$
 A) Always true *B) Sometimes true
 C) False

REF & RREF

Instructions: Find the reduced row echelon form of each of the following matrices

1.
$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & -2 \end{bmatrix}$$

2.
$$\begin{bmatrix} -1 & 1 \\ -1 & 2 \\ -3 & 2 \end{bmatrix}$$

3.
$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & -3 & 0 & -2 \end{bmatrix}$$

4.
$$\begin{bmatrix} -1 & -3 \\ 3 & -3 \\ -3 & -3 \\ 2 & 0 \end{bmatrix}$$

5.
$$\begin{bmatrix} 1 & 3 \\ -1 & 3 \\ 3 & 0 \\ 1 & -3 \end{bmatrix}$$

6.
$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ -1 & 0 & -1 \\ -3 & 0 & 0 \end{bmatrix}$$

7.
$$\begin{bmatrix} 3 & -3 & 2 \\ -2 & 2 & -2 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

8.
$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ -2 & -3 & -1 & -2 \\ -3 & -1 & -2 & -1 \end{bmatrix}$$

9.
$$\begin{bmatrix} 0 & 1 & 3 \\ -1 & -3 & 3 \\ 1 & -3 & 0 \end{bmatrix}$$

10.
$$\begin{bmatrix} 3 & -2 & -3 & 3 \\ 2 & 3 & 3 & 2 \end{bmatrix}$$

11.
$$\begin{bmatrix} 0 & -3 & 1 & -1 \\ -2 & 1 & 0 & 3 \end{bmatrix}$$

12.
$$\begin{bmatrix} -2 & -3 & -2 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

13.
$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$$

14.
$$\begin{bmatrix} 3 & -3 & -2 & -1 \\ 0 & 2 & -3 & -3 \\ 3 & 3 & 2 & -3 \end{bmatrix}$$

15.
$$\begin{bmatrix} -1 & -2 & 2 \\ 2 & -2 & -2 \\ -3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

16.
$$\begin{bmatrix} 1 & -2 & -3 \\ -2 & 3 & -1 \end{bmatrix}$$

17.
$$\begin{bmatrix} 1 & 3 & -3 & -3 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

18.
$$\begin{bmatrix} 3 & -3 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

19.
$$\begin{bmatrix} 2 & -3 & -2 & 3 \\ -2 & 2 & -2 & 0 \\ -1 & 3 & 3 & 2 \\ -3 & -2 & -2 & 2 \end{bmatrix}$$

20.
$$\begin{bmatrix} -1 & 1 & -1 & -2 \\ 2 & 0 & 0 & 3 \\ -1 & 0 & -1 & -2 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

21.
$$\begin{bmatrix} -3 & -1 & 3 & -3 \\ 0 & -2 & -1 & 1 \\ 3 & 2 & -3 & 0 \end{bmatrix}$$

22.
$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & -2 \\ 1 & 3 \end{bmatrix}$$

23.
$$\begin{bmatrix} 0 & -3 & 2 & -2 \\ 0 & 2 & 2 & -2 \end{bmatrix}$$

24.
$$\begin{bmatrix} -2 & -3 \\ -1 & -1 \\ -3 & -3 \end{bmatrix}$$

25.
$$\begin{bmatrix} -3 & 3 & -2 \\ 2 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

Solutions:

1.

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow -R_1$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ -1 & -2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$

$$R_4 \leftarrow R_4 + R_1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$$

$$R_2 \leftarrow -R_2$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_3 \leftarrow R_3 + R_2$$

$$R_4 \leftarrow R_4 + 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.

$$\begin{bmatrix} -1 & 1 \\ -1 & 2 \\ -3 & 2 \end{bmatrix}$$

$$R_1 \leftarrow -R_1$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -3 & 2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$

$$R_3 \leftarrow R_3 + 3R_1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_3 \leftarrow R_3 + R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & -3 & 0 & -2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -9 & -4 & -4 \end{bmatrix}$$

$$R_2 \leftarrow -\frac{1}{9}R_2$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & \frac{4}{9} & \frac{4}{9} \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & 1 & \frac{4}{9} & \frac{4}{9} \end{bmatrix}$$

4.

$$\begin{bmatrix} -1 & -3 \\ 3 & -3 \\ -3 & -3 \\ 2 & 0 \end{bmatrix}$$

$$R_1 \leftarrow -R_1$$

$$\begin{bmatrix} 1 & 3 \\ 3 & -3 \\ -3 & -3 \\ 2 & 0 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 3R_1$$

$$R_3 \leftarrow R_3 + 3R_1$$

$$R_4 \leftarrow R_4 - 2R_1$$

$$\begin{bmatrix} 1 & 3 \\ 0 & -12 \\ 0 & 6 \\ 0 & -6 \end{bmatrix}$$

$$R_2 \leftarrow -\frac{1}{12}R_2$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 6 \\ 0 & -6 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$R_3 \leftarrow R_3 - 6R_2$$

$$R_4 \leftarrow R_4 + 6R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.

$$\begin{bmatrix} 1 & 3 \\ -1 & 3 \\ 3 & 0 \\ 1 & -3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$R_4 \leftarrow R_4 - R_1$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 6 \\ 0 & -9 \\ 0 & -6 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{6}R_2$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & -9 \\ 0 & -6 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$R_3 \leftarrow R_3 + 9R_2$$

$$R_4 \leftarrow R_4 + 6R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ -1 & 0 & -1 \\ -3 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - R_1$$

$$R_3 \leftarrow R_3 + R_1$$

$$R_4 \leftarrow R_4 + 3R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 2R_2$$

$$R_3 \leftarrow R_3 - 2R_2$$

$$R_4 \leftarrow R_4 - 6R_2$$

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \\ 0 & 0 & -18 \end{bmatrix}$$

$$R_3 \leftarrow -\frac{1}{7}R_3$$

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -18 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 6R_3$$

$$R_2 \leftarrow R_2 - 3R_3$$

$$R_4 \leftarrow R_4 + 18R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

7.

$$\begin{bmatrix} 3 & -3 & 2 \\ -2 & 2 & -2 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & -2 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + 2R_1$$

$$R_3 \leftarrow R_3 - 2R_1$$

$$R_4 \leftarrow R_4 - 3R_1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{3}R_2$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_4 \leftarrow R_4 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -2 \\ 0 & 0 & \frac{-1}{3} \end{bmatrix}$$

$$R_3 \leftarrow \frac{-1}{2}R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{-1}{3} \end{bmatrix}$$

$$R_1 \leftarrow R_1 - \frac{1}{3}R_3$$

$$R_2 \leftarrow R_2 - \frac{1}{3}R_3$$

$$R_4 \leftarrow R_4 + \frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

8.

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ -2 & -3 & -1 & -2 \\ -3 & -1 & -2 & -1 \end{bmatrix}$$

$$R_1 \leftarrow -R_1$$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ -2 & -3 & -1 & -2 \\ -3 & -1 & -2 & -1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + 2R_1$$

$$R_3 \leftarrow R_3 + 3R_1$$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & -5 & -1 & -4 \\ 0 & -4 & -2 & -4 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - \frac{3}{2}R_3$$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & -4 & -2 & -4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_3 \leftarrow R_3 + 4R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 6 & 4 \end{bmatrix}$$

$$R_3 \leftarrow \frac{1}{6}R_3$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 2R_3$$

$$R_2 \leftarrow R_2 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-1}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix}$$

9.

$$\begin{bmatrix} 0 & 1 & 3 \\ -1 & -3 & 3 \\ 1 & -3 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -1 & -3 & 3 \\ 0 & 1 & 3 \\ 1 & -3 & 0 \end{bmatrix}$$

$$R_1 \leftarrow -R_1$$

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 3 \\ 1 & -3 & 0 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 3 \\ 0 & -6 & 3 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$R_3 \leftarrow R_3 + 6R_2$$

$$\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

$$R_3 \leftarrow \frac{1}{21}R_3$$

$$\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 12R_3$$

$$R_2 \leftarrow R_2 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10.

$$\begin{bmatrix} 3 & -2 & -3 & 3 \\ 2 & 3 & 3 & 2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - R_2$$

$$\begin{bmatrix} 1 & -5 & -6 & 1 \\ 2 & 3 & 3 & 2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & -5 & -6 & 1 \\ 0 & 13 & 15 & 0 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{13}R_2$$

$$\begin{bmatrix} 1 & -5 & -6 & 1 \\ 0 & 1 & \frac{15}{13} & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 5R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{-3}{13} & 1 \\ 0 & 1 & \frac{15}{13} & 0 \end{bmatrix}$$

11.

$$\begin{bmatrix} 0 & -3 & 1 & -1 \\ -2 & 1 & 0 & 3 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -2 & 1 & 0 & 3 \\ 0 & -3 & 1 & -1 \end{bmatrix}$$

$$R_1 \leftarrow \frac{-1}{2}R_1$$

$$\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \frac{-3}{2} \\ 0 & -3 & 1 & -1 \end{bmatrix}$$

$$R_2 \leftarrow \frac{-1}{3}R_2$$

$$\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \frac{-3}{2} \\ 0 & 1 & \frac{-1}{3} & \frac{1}{3} \end{bmatrix}$$

$$R_1 \leftarrow R_1 + \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{6} & \frac{-4}{3} \\ 0 & 1 & \frac{-1}{3} & \frac{1}{3} \end{bmatrix}$$

12.

$$\begin{bmatrix} -2 & -3 & -2 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_4$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 3R_1$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$R_4 \leftarrow R_4 + R_1$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -14 \\ 0 & -2 & -13 \\ 0 & -1 & 2 \end{bmatrix}$$

$$R_2 \leftarrow \frac{-1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \\ 0 & -2 & -13 \\ 0 & -1 & 2 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 2R_2$$

$$R_4 \leftarrow R_4 + R_2$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 4R_3$$

$$R_2 \leftarrow R_2 - 7R_3$$

$$R_4 \leftarrow R_4 - 9R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

13.

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 3R_1$$

$$R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -2 & -3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_3$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -7 \\ 0 & -2 & -3 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_3 \leftarrow R_3 + 2R_2$$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -7 \\ 0 & 0 & -17 \end{bmatrix}$$

$$R_3 \leftarrow \frac{-1}{17}R_3$$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 5R_3$$

$$R_2 \leftarrow R_2 + 7R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14.

$$\begin{bmatrix} 3 & -3 & -2 & -1 \\ 0 & 2 & -3 & -3 \\ 3 & 3 & 2 & -3 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - \frac{2}{3}R_3$$

$$\begin{bmatrix} 1 & -5 & \frac{-10}{3} & 1 \\ 0 & 2 & -3 & -3 \\ 3 & 3 & 2 & -3 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & -5 & \frac{-10}{3} & 1 \\ 0 & 2 & -3 & -3 \\ 0 & 18 & 12 & -6 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & -5 & \frac{-10}{3} & 1 \\ 0 & 1 & \frac{-3}{2} & \frac{-3}{2} \\ 0 & 18 & 12 & -6 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 5R_2$$

$$R_3 \leftarrow R_3 - 18R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{-65}{6} & \frac{-13}{2} \\ 0 & 1 & \frac{-3}{2} & \frac{-3}{2} \\ 0 & 0 & 39 & 21 \end{bmatrix}$$

$$R_3 \leftarrow \frac{1}{39}R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{-65}{6} & \frac{-13}{2} \\ 0 & 1 & \frac{-3}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{7}{13} \end{bmatrix}$$

$$R_1 \leftarrow R_1 + \frac{65}{6}R_3$$

$$R_2 \leftarrow R_2 + \frac{3}{2}R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-2}{3} \\ 0 & 1 & 0 & \frac{-9}{13} \\ 0 & 0 & 1 & \frac{7}{13} \end{bmatrix}$$

15.

$$\begin{bmatrix} -1 & -2 & 2 \\ 2 & -2 & -2 \\ -3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & -4 & 0 \\ 2 & -2 & -2 \\ -3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$R_3 \leftarrow R_3 + 3R_1$$

$$R_4 \leftarrow R_4 - R_1$$

$$\begin{bmatrix} 1 & -4 & 0 \\ 0 & 6 & -2 \\ 0 & -12 & 1 \\ 0 & 5 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - R_4$$

$$\begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & -3 \\ 0 & -12 & 1 \\ 0 & 5 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 4R_2$$

$$R_3 \leftarrow R_3 + 12R_2$$

$$R_4 \leftarrow R_4 - 5R_2$$

$$\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & -3 \\ 0 & 0 & -35 \\ 0 & 0 & 16 \end{bmatrix}$$

$$R_3 \leftarrow \frac{-1}{35}R_3$$

$$\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 16 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 12R_3$$

$$R_2 \leftarrow R_2 + 3R_3$$

$$R_4 \leftarrow R_4 - 16R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

16.

$$\begin{bmatrix} 1 & -2 & -3 \\ -2 & 3 & -1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + 2R_1$$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -7 \end{bmatrix}$$

$$R_2 \leftarrow -R_2$$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 7 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & 7 \end{bmatrix}$$

17.

$$\begin{bmatrix} 1 & 3 & -3 & -3 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$R_2 \leftarrow \frac{-1}{2}R_2$$

$$\begin{bmatrix} 1 & 3 & -3 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 3R_2$$

$$\begin{bmatrix} 1 & 3 & 0 & -6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

18.

$$\begin{bmatrix} 3 & -3 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & 1 \\ -2 & 1 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + 2R_1$$

$$R_3 \leftarrow R_3 + 2R_1$$

$$R_4 \leftarrow R_4 + R_1$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -1 \\ 0 & -3 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 4R_4$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -5 \\ 0 & -3 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 2R_2$$

$$R_3 \leftarrow R_3 + 3R_2$$

$$R_4 \leftarrow R_4 + R_2$$

$$\begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & -5 \\ 0 & 0 & -19 \\ 0 & 0 & -4 \end{bmatrix}$$

$$R_3 \leftarrow -\frac{1}{19}R_3$$

$$\begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 11R_3$$

$$R_2 \leftarrow R_2 + 5R_3$$

$$R_4 \leftarrow R_4 + 4R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

19.

$$\begin{bmatrix} 2 & -3 & -2 & 3 \\ -2 & 2 & -2 & 0 \\ -1 & 3 & 3 & 2 \\ -3 & -2 & -2 & 2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_3$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ -2 & 2 & -2 & 0 \\ -1 & 3 & 3 & 2 \\ -3 & -2 & -2 & 2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + 2R_1$$

$$R_3 \leftarrow R_3 + R_1$$

$$R_4 \leftarrow R_4 + 3R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 2 & 0 & 10 \\ 0 & 3 & 4 & 7 \\ 0 & -2 & 1 & 17 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 3 & 4 & 7 \\ 0 & -2 & 1 & 17 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - 3R_2$$

$$R_4 \leftarrow R_4 + 2R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 1 & 27 \end{bmatrix}$$

$$R_3 \leftarrow \frac{1}{4}R_3$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 27 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - R_3$$

$$R_4 \leftarrow R_4 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 29 \end{bmatrix}$$

$$R_4 \leftarrow \frac{1}{29}R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 7R_4$$

$$R_2 \leftarrow R_2 - 5R_4$$

$$R_3 \leftarrow R_3 + 2R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

20.

$$\begin{bmatrix} -1 & 1 & -1 & -2 \\ 2 & 0 & 0 & 3 \\ -1 & 0 & -1 & -2 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

$$R_1 \leftarrow -R_1$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 0 & 0 & 3 \\ -1 & 0 & -1 & -2 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$R_3 \leftarrow R_3 + R_1$$

$$R_4 \leftarrow R_4 + 3R_1$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -2 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & 5 & 5 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_3$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & 5 & 5 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_3 \leftarrow R_3 + R_2$$

$$R_4 \leftarrow R_4 + 3R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - 3R_4$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_3$$

$$R_2 \leftarrow R_2 + 2R_3$$

$$R_4 \leftarrow R_4 + R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$R_4 \leftarrow -\frac{1}{5}R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 6R_4$$

$$R_2 \leftarrow R_2 + 15R_4$$

$$R_3 \leftarrow R_3 + 7R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

21.

$$\begin{bmatrix} -3 & -1 & 3 & -3 \\ 0 & -2 & -1 & 1 \\ 3 & 2 & -3 & 0 \end{bmatrix}$$

$$R_1 \leftarrow -\frac{1}{3}R_1$$

$$\begin{bmatrix} 1 & \frac{1}{3} & -1 & 1 \\ 0 & -2 & -1 & 1 \\ 3 & 2 & -3 & 0 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & \frac{1}{3} & -1 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + 3R_3$$

$$\begin{bmatrix} 1 & \frac{1}{3} & -1 & 1 \\ 0 & 1 & -1 & -8 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - \frac{1}{3}R_2$$

$$R_3 \leftarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 1 & -1 & -8 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + \frac{2}{3}R_3$$

$$R_2 \leftarrow R_2 + R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

22.

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & -2 \\ 1 & 3 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -2 \\ 1 & 3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$

$$R_4 \leftarrow R_4 - R_1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{3}R_2$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 2R_2$$

$$R_3 \leftarrow R_3 + 2R_2$$

$$R_4 \leftarrow R_4 - R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

23.

$$\begin{bmatrix} 0 & -3 & 2 & -2 \\ 0 & 2 & 2 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 2R_2$$

$$\begin{bmatrix} 0 & 1 & 6 & -6 \\ 0 & 2 & 2 & -2 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{bmatrix} 0 & 1 & 6 & -6 \\ 0 & 0 & -10 & 10 \end{bmatrix}$$

$$R_2 \leftarrow \frac{-1}{10}R_2$$

$$\begin{bmatrix} 0 & 1 & 6 & -6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 6R_2$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

24.

$$\begin{bmatrix} -2 & -3 \\ -1 & -1 \\ -3 & -3 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ -3 & -3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$

$$R_3 \leftarrow R_3 + 3R_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$$

$$R_2 \leftarrow -R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -3 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

25.

$$\begin{bmatrix} -3 & 3 & -2 \\ 2 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 7 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 7 & 4 \\ 0 & -12 & -5 \\ 0 & -6 & -4 \end{bmatrix}$$

$$R_2 \leftarrow \frac{-1}{12}R_2$$

$$\begin{bmatrix} 1 & 7 & 4 \\ 0 & 1 & \frac{5}{12} \\ 0 & -6 & -4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 7R_2$$

$$R_3 \leftarrow R_3 + 6R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{13}{12} \\ 0 & 1 & \frac{5}{12} \\ 0 & 0 & \frac{-3}{2} \end{bmatrix}$$

$$R_3 \leftarrow \frac{-2}{3}R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{13}{12} \\ 0 & 1 & \frac{5}{12} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - \frac{13}{12}R_3$$

$$R_2 \leftarrow R_2 - \frac{5}{12}R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. For each of the following matrices, determine whether it is in row echelon form, reduced row echelon form, or neither.

$$(a) \begin{bmatrix} 1 & -4 & 2 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Since each row has a leading 1 that is down and to the right of the leading 1 in the previous row, this matrix is in row echelon form.

Since some of the columns with a leading 1 have other non-zero entries, it is not in reduced row echelon form.

$$(b) \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

Since each non-zero row has a leading 1 that is down and to the right of the leading 1 in the previous row, each column with a leading 1 has no other non-zero entries, and the zero row is at the bottom of the matrix, this matrix is in reduced row echelon form.

$$(c) \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Since the last row is not a zero row but does not have a leading 1, this matrix is in neither row echelon form nor reduced row echelon form.

2. Put each of the following matrices into *row echelon form*.

$$(a) \begin{bmatrix} 3 & -2 & 4 & 7 \\ 2 & 1 & 0 & -3 \\ 2 & 8 & -8 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 4 & 7 \\ 2 & 1 & 0 & -3 \\ 2 & 8 & -8 & 2 \end{bmatrix} \xrightarrow[r_3 - r_2 \rightarrow r_3]{r_1 - r_2 \rightarrow r_1} \begin{bmatrix} 1 & -3 & 4 & 10 \\ 2 & 1 & 0 & -3 \\ 0 & 7 & -8 & 5 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \rightarrow r_2} \begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 7 & -8 & -23 \\ 0 & 7 & -8 & 5 \end{bmatrix}$$

$$\xrightarrow[r_2 \rightarrow \frac{1}{7}r_2]{r_3 - r_2 \rightarrow r_3} \begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 28 \end{bmatrix} \xrightarrow{\frac{1}{28}r_3 \rightarrow r_3} \begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 3 & -1 & 2 & 4 & 1 \\ 2 & 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 2 & 4 & 1 \\ 2 & 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 & 3 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 2 & 1 & 3 & -1 & 2 \\ 3 & -1 & 2 & 4 & 1 \end{bmatrix} \xrightarrow[r_3 - 3r_1 \rightarrow r_3]{r_2 - 2r_1 \rightarrow r_2} \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & -3 & -3 & 3 & -4 \\ 0 & -7 & -7 & 10 & 8 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}r_2 \rightarrow r_2} \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & -7 & -7 & 10 & 8 \end{bmatrix} \xrightarrow{r_3 + 7r_2 \rightarrow r_3} \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 3 & \frac{4}{3} \end{bmatrix} \xrightarrow{\frac{1}{3}r_3 \rightarrow r_3} \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{bmatrix}$$

$$(c) \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \xrightarrow{\frac{1}{\cos \theta}r_1 \rightarrow r_1} \begin{bmatrix} 1 & \frac{\sin \theta}{\cos \theta} \\ -\sin \theta & \cos \theta \end{bmatrix} \xrightarrow{r_2 + \sin \theta r_1 \rightarrow r_2} \begin{bmatrix} 1 & \frac{\sin \theta}{\cos \theta} \\ 0 & \frac{\sin^2 \theta}{\cos \theta} + \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sin \theta}{\cos \theta} \\ 0 & \frac{\sin^2 \theta}{\cos \theta} + \frac{\cos^2 \theta}{\cos \theta} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{\sin \theta}{\cos \theta} \\ 0 & \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta} \end{bmatrix} = \begin{bmatrix} 1 & \tan \theta \\ 0 & \frac{1}{\cos \theta} \end{bmatrix} \xrightarrow{\cos \theta r_2 \rightarrow r_2} \begin{bmatrix} 1 & \tan \theta \\ 0 & 1 \end{bmatrix}$$

3. Put each of the matrices from the previous problem into reduced row echelon form.

(a) Continuing from above:

$$\begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 + 3r_2 \rightarrow r_1} \begin{bmatrix} 1 & 0 & \frac{4}{7} & \frac{1}{7} \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{matrix} r_2 + \frac{23}{7}r_3 \rightarrow r_2 \\ r_1 - \frac{1}{7}r_3 \rightarrow r_1 \end{matrix}} \begin{bmatrix} 1 & 0 & \frac{4}{7} & 0 \\ 0 & 1 & -\frac{8}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Continuing from above:

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{bmatrix} \xrightarrow{\begin{matrix} r_1 - 2r_2 \rightarrow r_1 \\ r_2 + r_3 \rightarrow r_2 \end{matrix}} \begin{bmatrix} 1 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 1 & 1 & 0 & \frac{16}{9} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{bmatrix}$$

(c) Continuing from above:

$$\begin{bmatrix} 1 & \tan \theta \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 - \tan \theta r_2 \rightarrow r_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. Use Gaussian Elimination to find all solutions to the following systems of linear equations.

$$(a) \begin{cases} x + 2y + 3z = 9 \\ 2x - 2z = -2 \\ 3x + 2y + z = 7 \end{cases}$$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 0 & 2 & -2 \\ 3 & 2 & 1 & 7 \end{array} \right] \xrightarrow{\begin{matrix} r_2 - 2r_1 \rightarrow r_2 \\ r_3 - 3r_1 \rightarrow r_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array} \right] \xrightarrow{\begin{matrix} r_3 - r_2 \rightarrow r_3 \\ -\frac{1}{4}r_2 \rightarrow r_2 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Then $y + 2z = 5$, so $y = 5 - 2z$, and $x + 2y + 3z = 9$, or, substituting, $x + 2(5 - 2z) + 3z = 9$. Therefore, $x = 10 - 4z + 3z = 9$, so $x = -1 + z$

Hence the solutions to this system are all of the form $(-1 + t, 5 - 2t, t)$ for some $t \in \mathbb{R}$.

$$(b) \begin{cases} x + 2y + 3z = 9 \\ 3x + 2y + z = 7 \end{cases}$$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 3 & 2 & 1 & 7 \end{array} \right] \xrightarrow{r_3 - 3r_1 \rightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \end{array} \right] \xrightarrow{-\frac{1}{4}r_2 \rightarrow r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 5 \end{array} \right]$$

Then, exactly as above, $y + 2z = 5$, so $y = 5 - 2z$, and $x + 2y + 3z = 9$, or, substituting, $x + 2(5 - 2z) + 3z = 9$. Therefore, $x = 10 - 4z + 3z = 9$, so $x = -1 + z$

Hence the solutions to this system are all of the form $(-1 + t, 5 - 2t, t)$ for some $t \in \mathbb{R}$.

5. Use Gauss-Jordan reduction to find all solutions to the following systems of linear equations.

$$(a) \begin{cases} 3x - 2y + z = -6 \\ 4x - 3y + 3z = 7 \\ 2x + y - z = -9 \end{cases}$$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & -6 \\ 4 & -3 & 3 & 7 \\ 2 & 1 & -1 & -9 \end{array} \right] \xrightarrow{\begin{matrix} r_2 - r_1 \rightarrow r_1 \\ 2r_3 - r_2 \rightarrow r_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 4 & -3 & 3 & 7 \\ 0 & 5 & -5 & -25 \end{array} \right] \xrightarrow{\begin{matrix} r_2 - 4r_1 \rightarrow r_2 \\ \frac{1}{5}r_3 \rightarrow r_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 0 & -1 & 5 & 45 \\ 0 & 1 & -1 & -5 \end{array} \right]$$

$$r_3 + r_2 \xrightarrow{r_1+r_3 \rightarrow r_1} r_3 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & -1 & 5 & 45 \\ 0 & 0 & 4 & 40 \end{array} \right] \xrightarrow{\frac{1}{4}r_3 \rightarrow r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -5 & -45 \\ 0 & 0 & 1 & 10 \end{array} \right] \xrightarrow{r_2 + 5r_3 \rightarrow r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array} \right]$$

Therefore, the unique solution is $x = -2$, $y = 5$, and $z = 10$

$$(b) \begin{cases} 3x - 2y + z = 4 \\ x + 3y - z = -3 \\ 4x - 10y + 4z = 10 \end{cases}$$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ 1 & 3 & -1 & -3 \\ 4 & -10 & 4 & 10 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & -3 \\ 3 & -2 & 1 & 4 \\ 4 & -10 & 4 & 10 \end{array} \right] \xrightarrow{\begin{array}{l} r_2 - 3r_1 \rightarrow r_2 \\ r_3 - 4r_1 \rightarrow r_3 \end{array}} \left[\begin{array}{ccc|c} 1 & 3 & -1 & -3 \\ 0 & -11 & 4 & 13 \\ 0 & -22 & 8 & 22 \end{array} \right]$$

$$r_3 - 2r_2 \rightarrow r_3 \left[\begin{array}{ccc|c} 1 & 3 & -1 & -3 \\ 0 & -11 & 4 & 13 \\ 0 & 0 & 0 & -4 \end{array} \right]$$

Therefore, this system has no solutions (the equation $0 = -4$ is inconsistent).

6. Use quadratic interpolation to find a quadratic function passing through the points $(0, 2)$, $(1, 0)$, and $(2, 4)$.

If we start with the general quadratic $p(x) = ax^2 + bx + c$ and the three conditions given above, we have the following system of equations:

$$\begin{cases} c = 2 \\ a + b + c = 0 \\ 4a + 2b + c = 4 \end{cases}$$

In matrix form, this is: $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 4 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{4r_1 - r_2 \rightarrow r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 2 & 3 & -4 \\ 0 & 0 & 1 & 2 \end{array} \right]$

$$r_2 - 3r_3 \rightarrow r_2 \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 2 & 0 & -10 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{\frac{1}{2}r_2 \rightarrow r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{r_1 - r_2 \rightarrow r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Therefore, we have $a = 3$, $b = -5$, and $c = 2$, so $p(x) = 3x^2 - 5x + 2$. Once can quickly verify that this quadratic passes through all three desired points.

7. Construct a quadratic polynomial $p(x) = ax^2 + bx + c$ satisfying the conditions: $p(1) = f(1)$, $p'(1) = f'(1)$, and $p''(1) = f''(1)$ if $f(x) = x \ln x + x^2$.

First, notice that since $f(x) = x \ln x + x^2$, then $f'(x) = \ln x + x \left(\frac{1}{x}\right) + 2x = \ln x + 2x + 1$ and $f''(x) = \left(\frac{1}{x}\right) + 2$. Then $f(1) = 1$, $f'(1) = 3$, and $f''(1) = 3$.

Next, notice that since $p(x) = ax^2 + bx + c$, then $p'(x) = 2ax + b$ and $p''(x) = 2a$. Then $p(1) = a + b + c$, $p'(1) = 2a + b$, and $p''(x) = 2a$

This gives the system of equations: $\begin{cases} a + b + c = 1 \\ 2a + b = 3 \\ 2a = 3 \end{cases}$

The matrix form of this equation is: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 3 \\ 2 & 0 & 0 & 3 \end{array} \right] \xrightarrow{\frac{1}{2}r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \frac{3}{2} \end{array} \right] \xrightarrow{\begin{array}{l} r_1 - r_2 \rightarrow r_1 \\ r_1 - r_3 \rightarrow r_1 \end{array}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \frac{3}{2} \end{array} \right]$

Thus, we have $a = \frac{3}{2}$, $b = 0$, and $c = -\frac{1}{2}$. Hence $p(x) = \frac{3}{2}x^2 - \frac{1}{2}$. One can verify this this quadratic satisfies all of the given conditions.

8. Find the elementary matrices corresponding to carrying out each of the following elementary row operations on a 3×3 matrix:

(a) $r_2 \leftrightarrow r_3$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) $-\frac{1}{4}r_2 \rightarrow r_2$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $3r_1 + r_2 \rightarrow r_2$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Find the inverse of each of the elementary matrices you found in the previous problem.

(a) $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. Write the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a product of elementary matrices. [Hint: what elementary operations are needed to put A into reduced row echelon form]

We begin by putting A into r.r.e.f.:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} r_2 - 3r_1 \rightarrow r_2 \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} r_1 + r_2 \rightarrow r_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} -\frac{1}{2}r_2 \rightarrow r_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next, we observe that the elementary matrices used in this reduction process are: $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

Then $E_3E_2E_1A = I_2$, so $A^{-1} = E_3E_2E_1$. Hence $A = (A^{-1})^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$.

$$\text{Thus } A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

11. For each of the following matrices, either find the inverse or prove that the matrix is singular.

(a) $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 & | & 1 & 0 \\ 2 & 4 & | & 0 & 1 \end{bmatrix} r_1 - r_2 \rightarrow r_1 \begin{bmatrix} 1 & -5 & | & 1 & -1 \\ 2 & 4 & | & 0 & 1 \end{bmatrix} r_2 - 2r_1 \rightarrow r_2 \begin{bmatrix} 1 & -5 & | & 1 & -1 \\ 0 & 14 & | & -2 & 3 \end{bmatrix} \frac{1}{4}r_2 \rightarrow r_2 \begin{bmatrix} 1 & -5 & | & 1 & -1 \\ 0 & 1 & | & -\frac{1}{7} & \frac{3}{14} \end{bmatrix}$$

$$r_1 + 5r_2 \rightarrow r_1 \begin{bmatrix} 1 & 0 & | & \frac{2}{7} & \frac{1}{14} \\ 0 & 1 & | & -\frac{1}{7} & \frac{3}{14} \end{bmatrix}$$

$$\text{Thus } A^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ -\frac{1}{7} & \frac{3}{14} \end{bmatrix}$$

(b) $\begin{bmatrix} 3 & -5 \\ -1 & \frac{5}{3} \end{bmatrix}$

$$\begin{bmatrix} 3 & -5 & | & 1 & 0 \\ -1 & \frac{5}{3} & | & 0 & 1 \end{bmatrix} \begin{matrix} -r_1 \rightarrow r_1 \\ r_1 \leftrightarrow r_2 \end{matrix} \begin{bmatrix} 1 & -\frac{5}{3} & | & 0 & -1 \\ 3 & -5 & | & 1 & 0 \end{bmatrix} r_2 - 3r_1 \rightarrow r_2 \begin{bmatrix} 1 & -\frac{5}{3} & | & 0 & -1 \\ 0 & 0 & | & 1 & 3 \end{bmatrix}$$

Since we encountered a row of zeros, A is a singular matrix, so A^{-1} does not exist.

(c) $\begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

$$\text{The standard row reduction algorithm yields: } A^{-1} = \begin{bmatrix} \frac{1}{7} & 0 & \frac{2}{7} \\ \frac{2}{7} & -1 & \frac{4}{7} \\ \frac{5}{14} & 0 & \frac{3}{14} \end{bmatrix}$$

$$(d) \begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 0 \\ 5 & -1 & 4 \end{bmatrix}$$

The standard row reduction algorithm yields: $A^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{bmatrix}$

12. Find all values of a for which the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & a \\ 0 & 2 & -a \end{bmatrix}$ has an inverse. Find the form of A^{-1} (in terms of a) for the cases where it exists.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & a & 0 & 1 & 0 \\ 0 & 2 & -a & 0 & 0 & 1 \end{array} \right] \xrightarrow[r_2 - 2r_1 \rightarrow r_2]{\frac{1}{2}r_3 \rightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & a-4 & -2 & 1 & 0 \\ 0 & 1 & -\frac{a}{2} & 0 & 0 & \frac{1}{2} \end{array} \right] \xrightarrow[r_2 \leftrightarrow r_3]{\frac{1}{a-4}r_3 \rightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{a}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{a-4} & \frac{1}{a-4} & 0 \end{array} \right]$$

$$\xrightarrow[r_2 + \frac{a}{2}r_3 \rightarrow r_2]{r_1 - 2r_3 \rightarrow r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{a}{a-4} & -\frac{2}{a-4} & 0 \\ 0 & 1 & 0 & -\frac{a}{a-4} & \frac{1}{2(a-4)} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{a-4} & \frac{1}{a-4} & 0 \end{array} \right]$$

$$\text{Thus } A^{-1} = \begin{bmatrix} \frac{a}{a-4} & -\frac{2}{a-4} & 0 \\ -\frac{a}{a-4} & \frac{1}{2(a-4)} & \frac{1}{2} \\ -\frac{2}{a-4} & \frac{1}{a-4} & 0 \end{bmatrix}. \text{ Notice that we must have } a \neq 4 \text{ for the inverse to exist.}$$

13. Given the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -1 & 2 & -5 \end{bmatrix}$:

- (a) Find matrix B equivalent to A having the form described by Theorem 2.12

We will solve both parts of this question at the same time by utilizing matrices of the form $[A|I_n]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -1 & 2 & -5 & 0 & 0 & 1 \end{array} \right] \xrightarrow[r_3 + r_1 \rightarrow r_3]{r_2 - r_1 \rightarrow r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 1 & 0 & 1 \end{array} \right] \xrightarrow[r_3 - 3r_2 \rightarrow r_3]{r_1 - r_2 \rightarrow r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -3 & 1 \end{array} \right]$$

Restarting our right matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{c_3 - 3c_1 \rightarrow c_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{c_3 + c_2 \rightarrow c_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Then the reduced form of } A \text{ is: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- (b) Find matrices P and Q such that $B = PAQ$ (for the matrices A and B from above).

Using the right hand sides of the reductions performed above (the row and column reductions respectively) we see that:

$$P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

14. For each permutation given below, first find the number of inversions. Then, classify the permutation as either even or odd.

(a) 53214

(b) 45132

(c) 34152

Notice that this permutation has 7 inversions, so it is an odd permutation.

Notice that this permutation has 7 inversions, so it is an odd permutation.

Notice that this permutation has 5 inversions, so it is an odd permutation.

15. Find the determinant of each of the following matrices:

$$(a) \begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -4 \\ 5 & 1 \end{vmatrix} = (2)(1) - (-4)(5) = 2 + 20 = 22.$$

$$(b) \begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 8 & -5 \end{bmatrix}$$

$$\begin{vmatrix} 3 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 8 & -5 \end{vmatrix} = (-15) + (0) + (-32) - (0) - (96) = -143.$$

$$(c) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = a_{31}A_{31} + 0 + 0 + 0 = 4 \begin{vmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -3 \end{vmatrix} = 4[0 + 0 + 0 - 0 - 0 - 6] = -24$$

$$(d) \begin{bmatrix} t & 3 \\ 2 & t+1 \end{bmatrix}$$

$$\begin{vmatrix} t & 3 \\ 2 & t+1 \end{vmatrix} = t(t+1) - 6 = t^2 + t - 6 = (t+3)(t-2)$$

16. For which values of t is the matrix in part (d) above singular? Justify your answer.

Recall that a matrix A is singular if and only if $\det(A) = 0$. For this matrix, $\det(A) = t(t+1) - 6 = t^2 + t - 6 = (t+3)(t-2)$.

Hence this matrix is singular when $t = -3$ and $t = 2$.

17. Evaluate each of the following:

$$(a) \begin{vmatrix} 4 & -2 & 3 & -5 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 4 & -2 & 3 & -5 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (4)(1)(-1)(2) = -8$$

$$(b) \begin{bmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{bmatrix}$$

$$\begin{bmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{bmatrix} = (t-1)(t-2)(t-3) = t^3 - 6t^2 + 11t - 6$$

$$(c) \begin{vmatrix} 4 & -3 & 6 & 1 \\ 4 & 0 & 4 & 1 \\ 4 & 2 & 0 & 1 \\ 4 & 1 & 2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 4 & -3 & 6 & 1 \\ 4 & 0 & 4 & 1 \\ 4 & 2 & 0 & 1 \\ 4 & 1 & 2 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & -3 & 6 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix} = 0 \text{ (Notice that the second matrix has a repeated column).}$$

18. Compute the determinant of the following matrix by putting it into triangular form: $A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 3 & 2 & -4 & -2 \\ 2 & 0 & 1 & 4 \\ 3 & 8 & -8 & -10 \end{bmatrix}$

$$\begin{vmatrix} 2 & 3 & -1 & 0 \\ 3 & 2 & -4 & -2 \\ 2 & 0 & 1 & 4 \\ 3 & 8 & -8 & -10 \end{vmatrix} \xrightarrow[r_4 - r_2 \rightarrow r_2]{r_3 - r_1 \rightarrow r_3} \begin{vmatrix} 2 & 3 & -1 & 0 \\ 3 & 2 & -4 & -2 \\ 0 & -3 & 2 & 4 \\ 0 & 6 & -4 & -8 \end{vmatrix} \xrightarrow{r_1 \leftrightarrow r_2 = (-1)} \begin{vmatrix} 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 0 & -3 & 2 & 4 \\ 0 & 6 & -4 & -8 \end{vmatrix}$$

$$r_4 + 2r_3 \rightarrow r_4 = (-1) \begin{vmatrix} 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Notice that since we encountered a row of zeros, then $\det(A) = (-1)(0) = 0$.

19. If $\det(AB) = 0$, must $\det(A)$ or $\det(B) = 0$? Justify your answer.

Suppose that $\det(AB) = 0$. Since $\det(AB) = \det(A)\det(B)$, then we have $\det(A)\det(B) = 0$. Recall that a product of two real numbers is zero if and only if at least one of the numbers is zero. Thus we must have either $\det(A) = 0$ or $\det(B) = 0$ (including the possibility that both of them are zero).

20. Show that if k is a scalar and A is an $n \times n$ matrix, then $\det(kA) = k^n \det(A)$.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let $B = [ka_{ij}]$ be the matrix obtained by multiplying A by a constant k . If $k = 0$ then B has a row of zeros and hence $\det(B) = 0 = 0^n \det(A)$. Otherwise, using the definition, $\det(B) = \sum (\pm) b_{1j_1} b_{2j_2} \cdots b_{\ell j_{\ell}} \cdots b_{nj_n} = \sum (\pm) ka_{1j_1} ka_{2j_2} \cdots ka_{\ell j_{\ell}} \cdots ka_{nj_n}$ (since the entries have all been scaled by k)

Then, factoring out the k from each term and pulling it outside the sum, $\det(B) = k^n \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{\ell j_{\ell}} \cdots a_{nj_n} = k^n \det(A)$. \square

21. Prove Theorem 3.5

Theorem 3.5: If B is obtained from A by multiplying a row (column) of A by a real number k , then $\det(B) = k \det(A)$.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let $B = [b_{ij}]$ be the matrix obtained by multiplying the ℓ th row of A by a constant k . If $k = 0$ then B has a row of zeros and hence $\det(B) = 0 = 0 \det(A)$. Otherwise, using the definition, $\det(B) = \sum (\pm) b_{1j_1} b_{2j_2} \cdots b_{\ell j_{\ell}} \cdots b_{nj_n} = \sum (\pm) a_{1j_1} a_{2j_2} \cdots ka_{\ell j_{\ell}} \cdots a_{nj_n}$ (since the entries for all other rows remain the same, and the entry drawn from row ℓ has been scaled by k)

Then, factoring out the k from each term and pulling it outside the sum, $\det(B) = k \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{\ell j_{\ell}} \cdots a_{nj_n} = k \det(A)$

The column case follows by observing that a column operation on A is a row operation on A^T and using the fact that $\det(A) = \det(A^T)$. \square

22. Prove Theorem 3.8

Theorem 3.8: If A is an $n \times n$ matrix, then A is nonsingular if and only if $\det(A) \neq 0$.

Proof:

First, suppose that A is nonsingular. Then, by Theorem 2.8, $A = E_1 E_2 \cdots E_n$ where each E_i is an elementary matrix. Then, using Lemma 3.1 n times, $\det(A) = \det(E_1 E_2 \cdots E_n) = \det(E_1) \det(E_2) \cdots \det(E_n) \neq 0$ (as discussed in the sketch of the proof of Lemma 3.1 above, for each i , $\det(E_i) \neq 0$).

Conversely, suppose A is singular. Then, using Theorem 2.10, A is row equivalent to a matrix B with a row of zeros. Then, by Theorem 3.4, $\det(B) = 0$. Therefore, since $A = F_1 F_2 \cdots F_\ell B$, where each F_i is an elementary matrix, $\det(A) = \det(F_1 F_2 \cdots F_\ell B) = \det(F_1) \det(F_2) \cdots \det(F_\ell) \det(B) = 0$.

23. Let $A = \begin{bmatrix} 3 & 1 & 0 & -1 \\ 2 & 0 & -5 & 1 \\ 2 & 1 & -4 & 0 \\ 0 & -1 & 3 & 2 \end{bmatrix}$. Find:

(a) $M_{14} = \begin{bmatrix} 2 & 0 & -5 \\ 2 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix}$

(b) $M_{32} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 1 \\ 0 & 3 & 2 \end{bmatrix}$

(c) $\det(M_{32}) = \begin{vmatrix} 3 & 0 & -1 \\ 2 & -5 & 1 \\ 0 & 3 & 2 \end{vmatrix} = (-30) + (0) + (-6) - (0) - (0) - (9) = -45$

(d) $A_{32}(-1)^{3+2} \det(M_{32}) = (-1)(-45) = 45$

(e) $A_{41} = (-1)^{4+1} \det(M_{41}) = (-1) \begin{vmatrix} 1 & 0 & -1 \\ 0 & -5 & 1 \\ 1 & -4 & 0 \end{vmatrix} = (0) + (0) + (0) - (5) - (0) - (-4) = (-1)(-1) = 1$

24. Find the determinant of the matrix A from the previous problem by using the expansion of $\det(A)$ along the last row.

$$\begin{aligned} \det(A) &= (0)A_{41} + (-1)A_{42} + (3)A_{43} + (2)A_{44} = 0 + (-1) \begin{vmatrix} 3 & 0 & -1 \\ 2 & -5 & 1 \\ 2 & -4 & 0 \end{vmatrix} + (3) \begin{vmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} + (2) \begin{vmatrix} 3 & 1 & 0 \\ 2 & 0 & -5 \\ 2 & 1 & -4 \end{vmatrix} \\ &= 0 + (-1)[(0) + (0) + (8) - (10) - (0) - (-12)] + (3)[(0) + (2) + (-2) - (0) - (0) - (-3)] + (2)[(0) + (-10) + (0) - (0) - (-8) - (-15)] \\ &= 0 + (-1)(10) + (3)(-3) + (2)(13) = 0 - 10 - 9 + 26 = 7 \end{aligned}$$