Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in AI and ML

II Semester 2020-21

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Course Name	Mathematical Foundation for Data Science
Nature of Exam	Open Book
Weightage for grading	30%
Duration	120 minutes
Date of Exam	30-07-2023 AN

# Pages	3
# Questions	3

Instructions

1. All questions are compulsory
2. Questions are to be answered in the order in which they appear in this paper and in the page numbers mentioned before each of them.

Q1 Answer the following questions with justifications.

a) Given two linearly independent $D \times 1$ vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, where $D > 3$, we intend to create a $D \times 3$ matrix where each column of the matrix is a linear combination of the given vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. If possible, find vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and values for the coefficients in the linear combination such that the resulting matrix has full column rank. Otherwise, explain with a suitable mathematical argument why this is not possible. (4)

b) Consider a $n \times n$ matrix $\boldsymbol{A}$ which we can write as $\boldsymbol{A} = \boldsymbol{L}^T\boldsymbol{L}$ where $\boldsymbol{L}$ is a lower-triangular matrix. We have a function $F(\boldsymbol{x}, \boldsymbol{R}, \boldsymbol{y})$ that returns the value $\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{y}$ where $\boldsymbol{R}$ is an $n \times n$ matrix and $\boldsymbol{x}$ and $\boldsymbol{y}$ are $n \times 1$ vectors respectively. If we are given both the matrix $\boldsymbol{A}$ and $\boldsymbol{L}$, what is the smallest number of calls that need to be made to this function in terms of n in order to decide that the given matrix $\boldsymbol{A}$ is positive definite, and why? (3)

c) We are given a function $f(x) = e^{\alpha x+\beta}$. Can the series $1+3x+5x^2$ be the Taylor's polynomial to second degree that approximates this function around $x = 0$? (3)

Q2 Answer the following questions with justifications.

a) You are given a system of three linear equations as shown below:

$$\alpha x + 2y + 6z = b_1$$
$$\alpha x + \alpha y + 5z = b_2$$
$$\alpha x + \alpha y + \alpha z = b_3$$

In each of the equations, you can observe presence of an unknown constant α. Derive the three possible values of α for which the given linear system will fail to have three pivots, giving justification(s). (3)

b) A data analyst at an oil exploration company came across a very interesting $m \times m$ matrix $\mathbf{B}$ while building an automated oil exploration software. He observed that the matrix $\mathbf{B}$ matrix can be defined as (4)

$$\mathbf{B} = \sum_{i=1}^{n} \left(6^i \cdot \mathbf{p}_i \mathbf{p}_i^T \right)$$

where $\mathbf{p}_i \in \mathbb{R}^m$ and $n < m$. While examining the summation he made an important observation that the vectors $\mathbf{p_1}, \mathbf{p_2}, \ldots\ldots, \mathbf{p_n}$ have the following interesting property:

$$\langle \mathbf{p}_i, \mathbf{p}_j \rangle = 0 \text{ whenever } i \neq j$$
$$\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 1 \text{ for all } i = 1, 2, \ldots, n$$

i) Derive at least n eigenvalues of $\mathbf{B}$ to help the analyst.
ii) Derive the corresponding eigenvectors.
iii) Derive the trace of the matrix $\mathbf{B}$.
iv) Derive the determinant of the matrix $\mathbf{B}$.

c) A professor teaching Linear Algebra introduced a matrix $\mathbf{M} \in \mathbb{R}^{\mathbf{r} \times \mathbf{r}}$ having an interesting property that $\mathbf{M}^k = \mathbf{0}$ for some $k < r$. Answer the following questions with proper reasoning in each case. (3)

i) Derive the trace of this matrix $\mathbf{M}$.
ii) Derive the determinant of this matrix $\mathbf{M}$.
iii) Derive all the individual eigenvalues of this matrix $\mathbf{M}$.
iv) Is the matrix $\mathbf{M}$ invertible ? Give reason for your answer.

Q3 Answer the following questions with justifications.

a) If a data analysis led to a real matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{pmatrix}$$

with distinct a_1, a_2, a_3. Find all possible values of $\mathbf{b} \in \mathbb{R}^3$ such that the system $\mathbf{Ax} = \mathbf{b}$ is consistent where $\mathbf{x} \in \mathbb{R}^3$. (1.5)

b) Let $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ and define $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} = \mathbf{x}^T \mathbf{A} \mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$. Then,

i) prove $\langle ., . \rangle_{\mathbf{A}}$ is an inner product on $\mathbb{R}^4$ (1.5)

ii) find all possible $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$ such that $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} = 0$ where $\mathbf{x} = [1, 0, 0, 0]^T$ (1)

c) If $f : \mathbb{R}^3 \to \mathbb{R}$ is such that $f(\mathbf{x}) = f(-\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^3$ and $f(0, 0, 0) = 4$. Find linear approximation of f about $[0, 0, 0]^T$ if all first order partial dervatives of f exist. (2)

d) Let $g(x, y) = (3x + y)^2(2y + 1)$ and $h(x, y) = 6y^2(2x + y)$

i) Find $\dfrac{\partial g}{\partial x}, \dfrac{\partial g}{\partial y}, \dfrac{\partial h}{\partial x}, \dfrac{\partial h}{\partial y}$ (2)

ii) Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ as

$$f(x, y, z) = \begin{pmatrix} g(x, y) + h(y, z) \\ h(x, y) + g(y, z) \end{pmatrix}.$$

Compute $\dfrac{df}{d\mathbf{b}}(0, 1, 0)$ where $\mathbf{b} = [x, y, z]^T$. (2)