WEBINAR 1

Q.1.

Consider an inner product space with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y}$ defined with the help of matrix $\mathbf{A}$ defined below.

$$\mathbf{A} = \begin{bmatrix} 5.5 & -1.5 \\ -1.5 & 5.5 \end{bmatrix}$$

Consider two vectors $\mathbf{a} = [1 \quad 5]^T$ and $\mathbf{b} = [2 \quad 7]^T$ in the inner product space.

(a) Find the distance $d(\mathbf{a}, \mathbf{b}) = \|a - b\|$ between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space where $\|\cdot\|$ is the norm induced by the inner product.

(2 marks)

(b) Find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space.

(2 marks)

Q.2.

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(A) Using elementary row operations, write the matrix in its row echelon form

(2 marks)

(B) Let $\mathbf{V}$ be a vector subspace spanned by the columns of matrix $\mathbf{A}$. Find the basis and dimension of $\mathbf{V}$.

(2 marks)

(C) Let $\mathbf{V}$ be a vector subspace spanned by vectors $\mathbf{x}$, such that $\mathbf{A}\mathbf{x} = 0$. Find the basis and dimension of $\mathbf{V}$.

(2 marks)

(D) Give the set of linearly independent rows of $\mathbf{A}$. What is the number of vectors in this set?

(2 marks)

Q.3.

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix},$$

a professor asks two of his best students enrolled in Linear Algebra class to find the maximum value of $\mathbf{x^T A x}$, subject to the fact that $\|\mathbf{x}\|^2 = 1$, where $\|\|$ is the Euclidean norm. Given that the students have not studied Calculus earlier, the first student says that this is impossible, whereas the second one is optimistic in estimating the value. Who is correct and why? Give adequate justifications.

HINT: Find a symmetric matrix $\mathbf{B}$ such that $\mathbf{x^T A x} = \mathbf{x^T B x}$

(4 marks)

Q.4.

(A) Suppose $\boldsymbol{A} = [\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4]$ where $\boldsymbol{C}_m \in \mathbb{R}^4, m = 1, 2, 3, 4$ are columns of $\boldsymbol{A}$. It is known that $\text{rank}(\boldsymbol{A}) = 2$ and $\boldsymbol{C}_2 = 3\boldsymbol{C}_1$ and $\boldsymbol{C}_4 = 2\boldsymbol{C}_1 + 3\boldsymbol{C}_3$. If a particular solution of $\boldsymbol{Ax} = \boldsymbol{b}$ is $[1, 0, 1, 0]^T$, then

(i) Find the general solution of $\boldsymbol{Ax} = \boldsymbol{b}$. [2 Marks]

(ii) Find $\boldsymbol{b}$ if $\text{RREF}(\boldsymbol{A}) = \boldsymbol{A}$. [2 Marks]

(B) Consider $M = \{\boldsymbol{A} \in \mathbb{R}^{2\times 2} \mid \boldsymbol{A} = -\boldsymbol{A}^T\}$.

(i) Prove that M is subspace of vector space V, where V is set of all 2×2 real matrices. [1.5 Marks]

(ii) Prove or disprove $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for M defined in (i).

[2.5 Marks]

(iii) Prove or disprove that $\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 3 & 1 \end{bmatrix} \right\}$ is a linearly independent set. [2 Marks]

Q.5.

(B) A function $f : \mathbb{R}^3 \Rightarrow \mathbb{R}^3$ is given below:

$$\mathbf{f([x_1, x_2, x_3]^T) = [2 * x_1 + x_2, 3 * x_1 + 7 * x_2 + 3 * x_3, 3 * x_1 + x_2 + x_3]^T}$$

(A) Using elementary row operations, write the matrix in its row echelon form

(B) Let **V** be a vector subspace spanned by the columns of matrix **A**. Give a set of vectors that form the basis for the vector space **V**. What is the number of vectors in this basis set?

(1 marks)

(C) Let **V** be a vector subspace spanned by vectors **x**, such that $\mathbf{Ax} = 0$. Give a set of vectors that form the basis for the vector space **V**. What is the number of vectors in this basis set?

(1 marks)

(D) Give the set of linearly independent rows of **A**. What is the number of vectors in this set?

(1 marks)