Introduction Statistical Methods- Session-3

Conditional Probability

This explanation refers to the concept of **conditional probability** and **Bayesian updating**, where new information or evidence about an event can change the probabilities of previously assigned events.

Initial Probability Assignment

When we first assign probabilities to various events, we do so based on the available information at the time. These probabilities are often referred to as **prior probabilities.** For example:

- If you toss a fair coin, the probability of getting heads is 0.5, and tails is also 0.5. These probabilities are assigned before any further information is obtained.

Impact of New Information

When new information becomes available, it can change how we view the likelihood of different outcomes. For example:

1. Suppose you know that in a coin toss experiment, the coin landed on heads in 7 out of 10 trials. Based on this evidence, you might revise your belief about whether the coin is fair.
2. Similarly, in a medical diagnosis, the probability of a disease being present changes when you receive test results that either confirm or contradict your initial assumptions.

Revising Probabilities

To adjust probabilities based on new information, we use **conditional probability.** The updated probability of an event A, given that another event B has occurred, is calculated using:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example:

- Suppose there is a 30% chance of rain ($P(A) = 0.3$).
- You hear a weather report that says clouds have formed (B), and there's a 70% chance it rains when clouds are present ($P(A|B) = 0.7$).
- This new information about clouds causes you to revise the probability of rain.

Bayesian Updating

Bayesian probability provides a systematic way to revise probabilities when new evidence is introduced. It is based on **Bayes' Theorem:**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

- $P(A)$: Prior probability of A,
- $P(B|A)$: Likelihood of observing B if A is true,
- $P(A|B)$: Posterior probability of A, updated after observing B.

Example (Medical Test):

- Initially, you might assign a 5% chance ($P(D) = 0.05$) that a person has a disease.
- A test result comes back positive (T), and you know the test is 90% accurate ($P(T|D) = 0.9$).
- Using Bayes' Theorem, you revise the probability of having the disease based on the test result.

Example: 1

Consider randomly selecting a student at a certain university, and let A denote the event that the selected individual has a Visa credit card and B be the analogous event for a Master Card. Suppose that $P(A) = 0.5$, $P(B) = 0.5$ and $P(A \cap B) = 0.25$. Calculate and interpret each of the following probabilities

a. $P({}^{B}/_{A})$

b. $P\left({}^{B'}/_{A}\right)$

c. $P\left({}^{A'}/_{B}\right)$

d. Given that the selected individual has at least one card, what is the probability that he or she has a Visa card?

Given Information

- $P(A) = 0.5$ (probability of having a Visa credit card)
- $P(B) = 0.5$ (probability of having a MasterCard)
- $P(A \cap B) = 0.25$ (probability of having both a Visa and a MasterCard)

a. $P(B \mid A)$

This is the conditional probability that the student has a MasterCard, given that they have a Visa credit card. The formula for conditional probability is:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Substitute the values:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.25}{0.5} = 0.5$$

Interpretation: If a student has a Visa credit card, there is a 50% chance that they also have a MasterCard.

b. $P(B' \mid A)$

This is the conditional probability that the student does **not** have a MasterCard, given that they have a Visa credit card. Since B' is the complement of B, we know:

$$P(B' \mid A) = 1 - P(B \mid A)$$

Substitute $P(B \mid A) = 0.5$:

$$P(B' \mid A) = 1 - 0.5 = 0.5$$

Interpretation: If a student has a Visa credit card, there is a 50% chance that they do not have a MasterCard.

c. $P(A' \mid B)$

This is the conditional probability that the student does **not** have a Visa credit card, given that they have a MasterCard. Using the formula for conditional probability:

$$P(A' \mid B) = \frac{P(A' \cap B)}{P(B)}$$

We know $P(A \cap B) = 0.25$, and since $A' \cap B$ represents students with only a MasterCard (but not a Visa card):

$$P(A' \cap B) = P(B) - P(A \cap B) = 0.5 - 0.25 = 0.25$$

Now substitute the values:

$$P(A' \mid B) = \frac{P(A' \cap B)}{P(B)} = \frac{0.25}{0.5} = 0.5$$

Interpretation: If a student has a MasterCard, there is a 50% chance that they do not have a Visa credit card.

d. Given that the student has at least one card, what is the probability that they have a Visa card?

Let C be the event that the student has at least one card. The complement of C, C', is the event that the student has no card. Since the problem assumes the student has at least one card, we are finding $P(A \mid C)$.

We know:

$$C = A \cup B$$

The probability of C is:

$$P(C) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute the values:

$$P(C) = 0.5 + 0.5 - 0.25 = 0.75$$

Now use the formula for conditional probability:

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

Since $A \cap C = A$ (anyone with a Visa card is part of the "at least one card" group):

$$P(A \mid C) = \frac{P(A)}{P(C)} = \frac{0.5}{0.75} = 0.6667$$

Interpretation: Given that a student has at least one card, there is a 66.67% chance that they have a Visa credit card.

Example: 2

Following table contains the different age group peoples who have defaulted and not defaulted on Loans.

		Age			
		Young	Middle-Aged	Senior citizens	Total
Loan Default	No	10503	27368	259	38130
	Yes	3586	4851	120	8557
	Total	14089	32219	379	46687

a) What is the probability that a person will not default on the loan given he/she is middle-aged?

b) What is the probability that a person is middle-aged given he/she has not defaulted on the loan?

Given Table

Loan Default	Young	Middle-Aged	Senior Citizens	Total
No	10,503	27,368	259	38,130
Yes	3,586	4,851	120	8,557
Total	14,089	32,219	379	46,687

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a) What is the probability that a person will not default on the loan given he/she is middle-aged?

We are looking for $P(\text{No} \mid \text{Middle-Aged})$, which is the probability of "No default" given that the person is middle-aged.

Formula for Conditional Probability:

$$P(\text{No} \mid \text{Middle-Aged}) = \frac{P(\text{No and Middle-Aged})}{P(\text{Middle-Aged})}$$

From the table:

- People who are middle-aged and did not default = 27,368.
- Total middle-aged people = 32,219.

Substitute these values:

$$P(\text{No} \mid \text{Middle-Aged}) = \frac{27,368}{32,219} \approx 0.8495$$

Answer: The probability is approximately **0.8495** or **84.95%**.

b) What is the probability that a person is middle-aged given he/she has not defaulted on the loan?

We are looking for $P(\text{Middle-Aged} \mid \text{No})$, which is the probability of being middle-aged given that the person has not defaulted.

Formula for Conditional Probability:

$$P(\text{Middle-Aged} \mid \text{No}) = \frac{P(\text{No and Middle-Aged})}{P(\text{No})}$$

From the table:

- People who are middle-aged and did not default = 27,368.
- Total people who did not default = 38,130.

Substitute these values:

$$P(\text{Middle-Aged} \mid \text{No}) = \frac{27,368}{38,130} \approx 0.7178$$

Answer: The probability is approximately **0.7178** or **71.78%**.

Summary

1. **a)** The probability that a person will not default given they are middle-aged is **84.95%**.
2. **b)** The probability that a person is middle-aged given they have not defaulted is **71.78%**.

Example: 3

A certain shop repairs both audio and video components. Let A denote the event that the next component brought in for repair is an audio component, and let B be the event that the next component is a compact disc player (so the event B is contained in A). Suppose that $P(A) = 0.6$ and $P(B) = 0.05$. Then find $P(B/A)$.

What is given?

- $P(A) = 0.6$: Probability that the next component is an audio component.
- $P(B) = 0.05$: Probability that the next component is a compact disc player.
- Event B (compact disc player) is a subset of event A (audio component).

What is required?

We need to find $P(B \mid A)$, which is the conditional probability that the next component is a compact disc player (B) given that it is an audio component (A).

Formula for Conditional Probability

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

Since B is contained in A, $P(B \cap A) = P(B)$. This is because all compact disc players are audio components.

So:

$$P(B \mid A) = \frac{P(B)}{P(A)}$$

Substitute the values

$$P(B \mid A) = \frac{0.05}{0.6}$$

$$P(B \mid A) = 0.0833 \text{ (approximately)}$$

Final Answer

The conditional probability $P(B \mid A)$ is **0.0833** or **8.33%**.

Example: 4

Two sets of cards with a letter on each card as follows are placed into separate bags:

Bag 1:	I	L	J	A	U	
Bag 2:	L	R	H	E	C	A

Bala randomly picked one card from each bag. Find the probability that:

a) He picked the letters 'J' and 'R'.
b) Both letters are 'L'.
c) Both letters are vowels.

Step 1: Information from the Image

Bag 1:

Cards contain the letters: **I, L, J, A, U**

- Total cards in Bag 1: 5

Bag 2:

Cards contain the letters: **L, R, H, E, C, A**

- Total cards in Bag 2: 6

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a) Probability that Bala picks the letters 'J' and 'R'

The events are independent because the choice from Bag 1 does not affect the choice from Bag 2. So, we multiply the probabilities of the two events:

- Probability of picking **'J'** from Bag 1:

$$P(\text{'J'}) = \frac{1}{5}$$

- Probability of picking **'R'** from Bag 2:

$$P(\text{'R'}) = \frac{1}{6}$$

Now, the probability of picking **'J' and 'R'** is:

$$P(\text{'J'} \cap \text{'R'}) = P(\text{'J'}) \cdot P(\text{'R'}) = \frac{1}{5} \cdot \frac{1}{6} = \frac{1}{30}$$

Answer: $P(\text{'J'} \cap \text{'R'}) = \frac{1}{30}$ or 3.33%.

b) Probability that both letters are 'L'

To pick 'L' from both bags:

- Probability of picking 'L' from Bag 1:

$$P(\text{'L from Bag 1'}) = \frac{1}{5}$$

- Probability of picking 'L' from Bag 2:

$$P(\text{'L from Bag 2'}) = \frac{1}{6}$$

The probability of both letters being 'L':

$$P(\text{'L from Bag 1'} \cap \text{'L from Bag 2'}) = \frac{1}{5} \cdot \frac{1}{6} = \frac{1}{30}$$

Answer: $P(\text{'L'} \cap \text{'L'}) = \frac{1}{30}$ or 3.33%.

c) Probability that both letters are vowels

Vowels in **Bag 1: I, A, U** (3 vowels)

Vowels in **Bag 2: E, A** (2 vowels)

- Probability of picking a vowel from Bag 1:

$$P(\text{Vowel from Bag 1}) = \frac{3}{5}$$

- Probability of picking a vowel from Bag 2:

$$P(\text{Vowel from Bag 2}) = \frac{2}{6} = \frac{1}{3}$$

The probability of both letters being vowels:

$$P(\text{Vowel from Bag 1} \cap \text{Vowel from Bag 2}) = \frac{3}{5} \cdot \frac{1}{3} = \frac{3}{15} = \frac{1}{5}$$

Answer: $P(\text{Vowels}) = \frac{1}{5}$ or 20%.

Summary

- a) $P(\text{'J'} \cap \text{'R'}) = \frac{1}{30}$ or 3.33%.
- b) $P(\text{'L'} \cap \text{'L'}) = \frac{1}{30}$ or 3.33%.
- c) $P(\text{Vowels}) = \frac{1}{5}$ or 20%.

Example: 5

In a group of 300 all adults, 272 are right-handed, 3 adults are selected with replacement. Find

a) P(all 3 are right- handed)

b) P(all 3 left-handed)

c) P(At least one right-handed)

Given Information

- Total adults: 300
- Right-handed adults: 272
- Left-handed adults: $300 - 272 = 28$

Probabilities:

- Probability of being right-handed ($P(R)$):

$$P(R) = \frac{\text{Number of right-handed}}{\text{Total adults}} = \frac{272}{300} = 0.9067$$

- Probability of being left-handed ($P(L)$):

$$P(L) = \frac{\text{Number of left-handed}}{\text{Total adults}} = \frac{28}{300} = 0.0933$$

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a) Probability that all 3 are right-handed

The events are independent because selection is **with replacement**. For all 3 to be right-handed:

$$P(\text{all 3 are right-handed}) = P(R) \cdot P(R) \cdot P(R) = P(R)^3$$

Substitute $P(R) = 0.9067$:

$$P(\text{all 3 are right-handed}) = (0.9067)^3 \approx 0.744$$

Answer: The probability that all 3 are right-handed is approximately **0.744** or **74.4%**.

b) Probability that all 3 are left-handed

For all 3 to be left-handed:

$$P(\text{all 3 are left-handed}) = P(L) \cdot P(L) \cdot P(L) = P(L)^3$$

Substitute $P(L) = 0.0933$:

$$P(\text{all 3 are left-handed}) = (0.0933)^3 \approx 0.00081$$

Answer: The probability that all 3 are left-handed is approximately **0.00081** or **0.081%**.

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c) Probability that at least one is right-handed

The complement of "at least one is right-handed" is "none are right-handed," which means all 3 are left-handed. Using the complement rule:

$$P(\text{at least one right-handed}) = 1 - P(\text{all 3 are left-handed})$$

From part (b), $P(\text{all 3 are left-handed}) = 0.00081$:

$$P(\text{at least one right-handed}) = 1 - 0.00081 = 0.99919$$

Answer: The probability that at least one is right-handed is approximately **0.99919** or **99.92%**.

Example: 6

Each day, Monday through Friday, a batch of components sent by a first supplier arrives at a certain inspection facility. Two days a week, a batch also arrives from a second supplier. Eighty percent of all supplier 1’s batches pass inspection, and 90% of supplier 2’s do likewise. What is the probability that, on a randomly selected day, two batches pass inspection? We will answer this assuming that on days when two batches are tested.

Step 1: The Problem Setup

We are asked to find the probability that **two batches pass inspection on a randomly selected day.**

From the problem:

1. **Supplier 1 sends batches every day:** $P(\text{Supplier 1 passes}) = 0.8.$
2. **Supplier 2 sends batches 2 days a week:** $P(\text{Supplier 2 passes}) = 0.9.$
3. The probability that two batches are received on a randomly selected day is $P(\text{two received}) = \frac{2}{5} = 0.4.$

Step 2: Apply the Joint Probability Formula

We are interested in $P(\text{two pass})$, which is the probability that **two batches are received and both pass inspection.**

Using the joint probability formula:

$$P(\text{two pass}) = P(\text{two received} \cap \text{both pass})$$

We apply the rule:

$$P(\text{two pass}) = P(\text{both pass} \mid \text{two received}) \cdot P(\text{two received})$$

Step 3: Calculate Each Term

1. $P(\text{both pass} \mid \text{two received})$:
   - If two batches are received, the probability that both pass is: $P(\text{both pass} \mid \text{two received}) = P(\text{Supplier 1 passes}) \cdot P(\text{Supplier 2 passes})$
   - Substituting the probabilities:
     $$P(\text{both pass} \mid \text{two received}) = 0.8 \cdot 0.9 = 0.72$$
2. $P(\text{two received})$:
   - The probability that two batches are received (since Supplier 2 sends batches 2 out of 5 days):
     $$P(\text{two received}) = 0.4$$

Step 4: Combine the Results

$$P(\text{two pass}) = P(\text{both pass} \mid \text{two received}) \cdot P(\text{two received})$$

$$P(\text{two pass}) = 0.72 \cdot 0.4 = 0.288$$

Example: 7

A data science team is working on a model to predict whether an email is spam or not. They are using two independent features:

The presence of the word "offer “(Feature A) and the presence of a suspicious link (Feature B).

The probability that an email contains the word "offer “(Feature A) is 0.6.

The probability that an email contains a suspicious link (Feature B) is 0.4.

a) What is the probability that an email contains both the word "offer" and a suspicious link?

b) What is the probability that an email contains either the word "offer" or a suspicious link or both?

c) What is the probability that an email contains neither the word "offer" nor a suspicious link?

Given Information

- $P(A) = 0.6$: Probability that an email contains the word "offer" (Feature A).
- $P(B) = 0.4$: Probability that an email contains a suspicious link (Feature B).
- Features A and B are independent, so:

$$P(A \cap B) = P(A) \cdot P(B)$$

Part a) Probability that an email contains both the word "offer" and a suspicious link

We are tasked to find $P(A \cap B)$, the probability that **both features are present.**

Using the formula for independent events:

$$P(A \cap B) = P(A) \cdot P(B)$$

Substitute the given probabilities:

$$P(A \cap B) = 0.6 \cdot 0.4 = 0.24$$

Answer: The probability that an email contains both features is:

$$\boxed{0.24 \text{ or } 24\%}$$

Part b) Probability that an email contains either the word "offer", a suspicious link, or both

We are tasked to find $P(A \cup B)$, the probability that **at least one of the two features is present**.

Using the formula for the union of two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute the given probabilities and the result from part a:

$$P(A \cup B) = 0.6 + 0.4 - 0.24 = 0.76$$

Answer: The probability that an email contains at least one of the two features is:

$$\boxed{0.76 \text{ or } 76\%}$$

Part c) Probability that an email contains neither the word "offer" nor a suspicious link

We are tasked to find $P(\text{neither A nor B})$, which is the complement of $P(A \cup B)$.

The complement rule states:

$$P(\text{neither A nor B}) = 1 - P(A \cup B)$$

Substitute the result from part b:

$$P(\text{neither A nor B}) = 1 - 0.76 = 0.24$$

Answer: The probability that an email contains neither feature is:

$$\boxed{0.24 \text{ or } 24\%}$$

Example: 8

At a certain gas station, 40% of the customers use regular gas (A_1), 35% use plus gas (A_2), and 25% use premium (A_3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their tanks.

What is the probability that the next customer fills the tank?

Step 1: Scenarios and Probabilities

The problem gives us 3 **scenarios** for the customer behavior:

1. **Scenario** A_1: The customer belongs to Group 1.
    - Probability of this scenario: $P(A_1) = 0.4$.
    - If the customer is from Group 1, the probability of filling the tank is $P(B|A_1) = 0.3$.
2. **Scenario** A_2: The customer belongs to Group 2.
    - Probability of this scenario: $P(A_2) = 0.35$.
    - If the customer is from Group 2, the probability of filling the tank is $P(B|A_2) = 0.6$.
3. **Scenario** A_3: The customer belongs to Group 3.
    - Probability of this scenario: $P(A_3) = 0.25$.
    - If the customer is from Group 3, the probability of filling the tank is $P(B|A_3) = 0.5$.

Step 2: Formula

The **Law of Total Probability** combines these scenarios. It says:

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)$$

Here:

- $P(B)$: Total probability that the next customer fills the tank.
- $P(A_i)$: Probability of scenario A_i (which group the customer belongs to).
- $P(B|A_i)$: Probability that the customer fills the tank given they are in scenario A_i.

Step 3: Substitute the Values

Now substitute the values given for each scenario into the formula:

$$P(B) = (0.4 \cdot 0.3) + (0.35 \cdot 0.6) + (0.25 \cdot 0.5)$$

Step 4: Calculate Each Term

1. For A_1: $0.4 \cdot 0.3 = 0.12$
2. For A_2: $0.35 \cdot 0.6 = 0.21$
3. For A_3: $0.25 \cdot 0.5 = 0.125$

Step 5: Add the Results

Now add these values together:

$$P(B) = 0.12 + 0.21 + 0.125 = 0.455$$

Step 6: Final Answer

The total probability that the next customer fills the tank is:

$$P(B) = \boxed{0.455 \text{ or } 45.5\%}$$

Example: 9

A simple binary communication channel carries messages by using only two signals, say 0 and 1. For a given binary channel, 40% of the time a 1 is transmitted; the probability that a transmitted 0 is correctly received is 0.90, and the probability that a transmitted 1 is correctly received is 0.95. Determine

(a) The probability of a 1 being received, and

(b) Given a 1 is received, the probability that 1 was transmitted.

Given Information

1. **Transmitting probabilities:**
   - Probability that a **1 is transmitted:** $P(T_1) = 0.4$
   - Probability that a **0 is transmitted:** $P(T_0) = 1 - P(T_1) = 0.6$
2. **Correct reception probabilities:**
   - Probability that a **0 is correctly received** (i.e., 0 sent and 0 received): $P(R_0|T_0) = 0.9$
   - Probability that a **1 is correctly received** (i.e., 1 sent and 1 received): $P(R_1|T_1) = 0.95$
3. **Incorrect reception probabilities** (1 received instead of 0, and vice versa):
   - $P(R_1|T_0) = 1 - P(R_0|T_0) = 1 - 0.9 = 0.1$ (0 sent but received as 1)
   - $P(R_0|T_1) = 1 - P(R_1|T_1) = 1 - 0.95 = 0.05$ (1 sent but received as 0)

(a) The probability of a 1 being received

We are looking for $P(R_1)$, the probability that a **1 is received**. This can happen in two ways:

1. A **1 is transmitted** and correctly received as 1.
2. A **0 is transmitted**, but incorrectly received as 1.

Using the **law of total probability**, we combine these cases:

$$P(R_1) = P(T_1) \cdot P(R_1|T_1) + P(T_0) \cdot P(R_1|T_0)$$

Now substitute the given probabilities:

$$P(R_1) = (0.4 \cdot 0.95) + (0.6 \cdot 0.1)$$

Perform the calculations:

1. $0.4 \cdot 0.95 = 0.38$
2. $0.6 \cdot 0.1 = 0.06$

Add them together:

$$P(R_1) = 0.38 + 0.06 = 0.44$$

Answer for part (a): The probability of a 1 being received is:

$$\boxed{0.44 \text{ or } 44\%}$$

(b) Given a 1 is received, the probability that 1 was transmitted

We are looking for $P(T_1|R_1)$, the probability that a **1 was transmitted given that a 1 was received.** This can be found using **Bayes' Theorem**:

$$P(T_1|R_1) = \frac{P(T_1) \cdot P(R_1|T_1)}{P(R_1)}$$

From part (a), we already calculated $P(R_1) = 0.44$. Now substitute the other values:

$$P(T_1|R_1) = \frac{0.4 \cdot 0.95}{0.44}$$

Perform the calculations:

1. Numerator: $0.4 \cdot 0.95 = 0.38$
2. Denominator: 0.44

Divide:

$$P(T_1|R_1) = \frac{0.38}{0.44} \approx 0.8636$$

Answer for part (b): Given that a 1 is received, the probability that 1 was transmitted is:

$$\boxed{0.864 \text{ or } 86.4\%}$$

Example: 10

A data science team is developing a predictive model to determine whether a user will click on an online advertisement. The likelihood of a user clicking on an ad depends on the type of device they are using. The team has categorized the devices into three types: desktop, tablet, and mobile. The probabilities of a user using each type of device are as follows:

- The probability that a user uses a desktop is 0.5.
- The probability that a user uses a tablet is 0.2.
- he probability that a user uses a mobile device is 0.3.

Also,

- The probability of clicking on an ad given that the user is on a desktop is 0.04.
- The probability of clicking on an ad given that the user is on a tablet is 0.06.
- The probability of clicking on an ad given that the user is on a mobile device is 0.1.

What is the overall probability that a user will click on an ad?

What Are We Solving?

We are trying to find the **overall probability that a user clicks on an ad**, which is denoted as $P(\text{Click})$. The probability depends on the type of device the user is using, so we'll calculate the total probability by combining all cases (desktop, tablet, mobile) using the **law of total probability.**

Step 1: Given Information

Device probabilities:

- Probability of using a **desktop**: $P(\text{Desktop}) = 0.5$
- Probability of using a **tablet**: $P(\text{Tablet}) = 0.2$
- Probability of using a **mobile**: $P(\text{Mobile}) = 0.3$

Click probabilities for each device:

- Probability of clicking on an ad given the user is on a **desktop**: $P(\text{Click} \mid \text{Desktop}) = 0.04$
- Probability of clicking on an ad given the user is on a **tablet**: $P(\text{Click} \mid \text{Tablet}) = 0.06$
- Probability of clicking on an ad given the user is on a **mobile**: $P(\text{Click} \mid \text{Mobile}) = 0.1$

Step 2: Use the Law of Total Probability

The overall probability of clicking on an ad $P(\text{Click})$ is the weighted sum of the probabilities for each device type:

$$P(\text{Click}) = P(\text{Desktop}) \cdot P(\text{Click} \mid \text{Desktop}) + P(\text{Tablet}) \cdot P(\text{Click} \mid \text{Tablet}) + P(\text{Mobile}) \cdot P(\text{Click} \mid \text{Mobile})$$

Step 3: Substitute the Values

Substitute the given probabilities into the formula:

$$P(\text{Click}) = (0.5 \cdot 0.04) + (0.2 \cdot 0.06) + (0.3 \cdot 0.1)$$

Step 4: Perform the Calculations

1. $0.5 \cdot 0.04 = 0.02$
2. $0.2 \cdot 0.06 = 0.012$
3. $0.3 \cdot 0.1 = 0.03$

Now, add these results together:

$$P(\text{Click}) = 0.02 + 0.012 + 0.03 = 0.062$$

Example: 11

Forest A occupies 50% of the total land in a certain park and 20% of the plants in this forest are poisonous. Forest B occupies 30% of the total land and 40% of the plants in it are poisonous. Forest C occupies the remaining 20% of the land and 70% of the plants in it are poisonous. If we randomly enter this park and pick a plant from the ground, what is the probability that it will be poisonous?

Step 1: Given Information

- Forest A:
  - Occupies 50% of the land: $P(A) = 0.5$
  - 20% of the plants in Forest A are poisonous: $P(\text{Poisonous} \mid \text{A}) = 0.2$
- Forest B:
  - Occupies 30% of the land: $P(B) = 0.3$
  - 40% of the plants in Forest B are poisonous: $P(\text{Poisonous} \mid \text{B}) = 0.4$
- Forest C:
  - Occupies 20% of the land: $P(C) = 0.2$
  - 70% of the plants in Forest C are poisonous: $P(\text{Poisonous} \mid \text{C}) = 0.7$

Step 2: Use the Law of Total Probability

The overall probability of picking a poisonous plant, $P(\text{Poisonous})$, is calculated by combining the probabilities from all three forests. Using the **law of total probability:**

$P(\text{Poisonous}) = P(A) \cdot P(\text{Poisonous} \mid \text{A}) + P(B) \cdot P(\text{Poisonous} \mid \text{B}) + P(C) \cdot P(\text{Poisonous} \mid$

Step 3: Substitute the Values

Substitute the given probabilities for each forest:

$$P(\text{Poisonous}) = (0.5 \cdot 0.2) + (0.3 \cdot 0.4) + (0.2 \cdot 0.7)$$

Step 4: Perform the Calculations

1. For Forest A: $0.5 \cdot 0.2 = 0.1$
2. For Forest B: $0.3 \cdot 0.4 = 0.12$
3. For Forest C: $0.2 \cdot 0.7 = 0.14$

Now, add these values together:

$$P(\text{Poisonous}) = 0.1 + 0.12 + 0.14 = 0.36$$

Example: 12

A person has undertaken a mining job. The probabilities of completion of the job on time with and without rain are 0.42 and 0.90 respectively. If the probability that it will rain is 0.45, then determine the probability that the mining job will be completed on time.

Step 1: Given Information

- Probability of rain: $P(\text{Rain}) = 0.45$
- Probability of no rain: $P(\text{No Rain}) = 1 - P(\text{Rain}) = 0.55$
- Probability of job completion given rain: $P(\text{Completion} \mid \text{Rain}) = 0.42$
- Probability of job completion given no rain: $P(\text{Completion} \mid \text{No Rain}) = 0.90$

Step 2: Formula

Using the **law of total probability**, the overall probability of completing the job on time is:

$$P(\text{Completion}) = P(\text{Rain}) \cdot P(\text{Completion} \mid \text{Rain}) + P(\text{No Rain}) \cdot P(\text{Completion} \mid \text{No Rain})$$

Step 3: Substitute the Values

Substitute the given probabilities into the formula:

$$P(\text{Completion}) = (0.45 \cdot 0.42) + (0.55 \cdot 0.90)$$

Step 4: Perform the Calculations

1. $0.45 \cdot 0.42 = 0.189$
2. $0.55 \cdot 0.90 = 0.495$

Add these results together:

$$P(\text{Completion}) = 0.189 + 0.495 = 0.684$$

Step 5: Final Answer

The overall probability that the mining job will be completed on time is:

$$\boxed{0.684 \text{ or } 68.4\%}$$

Practice problems

Example 1.

An office has 4 secretaries handling respectively 20%, 60%,15% and 5% of the files of all government reports. The probability that they misfile such reports are respectively 0.05, 0.1, 0.1 and 0.05. Find the probability that the misfiled report can be blamed on the first secretary.

Step 1: Given Information

1. **Proportions of files handled by each secretary:**
   - First secretary: $P(S_1) = 0.2$
   - Second secretary: $P(S_2) = 0.6$
   - Third secretary: $P(S_3) = 0.15$
   - Fourth secretary: $P(S_4) = 0.05$
2. **Probability of misfiling by each secretary:**
   - First secretary: $P(M|S_1) = 0.05$
   - Second secretary: $P(M|S_2) = 0.1$
   - Third secretary: $P(M|S_3) = 0.1$
   - Fourth secretary: $P(M|S_4) = 0.05$

Step 2: Bayes' Theorem

We need to find $P(S_1|M)$, the probability that the first secretary misfiled a report. By **Bayes' Theorem:**

$$P(S_1|M) = \frac{P(S_1) \cdot P(M|S_1)}{P(M)}$$

Step 3: Find $P(M)$ (Total Probability of Misfiling)

The total probability of a report being misfiled is given by the **law of total probability:**

$$P(M) = P(S_1) \cdot P(M|S_1) + P(S_2) \cdot P(M|S_2) + P(S_3) \cdot P(M|S_3) + P(S_4) \cdot P(M|S_4)$$

Substitute the given values:

$$P(M) = (0.2 \cdot 0.05) + (0.6 \cdot 0.1) + (0.15 \cdot 0.1) + (0.05 \cdot 0.05)$$

Perform the calculations:

1. $0.2 \cdot 0.05 = 0.01$
2. $0.6 \cdot 0.1 = 0.06$
3. $0.15 \cdot 0.1 = 0.015$
4. $0.05 \cdot 0.05 = 0.0025$

Add these together:

$$P(M) = 0.01 + 0.06 + 0.015 + 0.0025 = 0.0875$$

Step 4: Apply Bayes' Theorem

Now, substitute into Bayes' Theorem:

$$P(S_1|M) = \frac{P(S_1) \cdot P(M|S_1)}{P(M)}$$

Substitute the values:

$$P(S_1|M) = \frac{0.2 \cdot 0.05}{0.0875}$$

Calculate the numerator:

$$0.2 \cdot 0.05 = 0.01$$

Divide:

$$P(S_1|M) = \frac{0.01}{0.0875} \approx 0.1143$$

Step 5: Final Answer

The probability that the misfiled report can be blamed on the first secretary is:

$$\boxed{0.1143 \text{ or } 11.43\%}$$

Example 2.

In a class 70% are boys and 30% are girls. 5% of boys and 3% of girls are irregular to the classes. What is the probability of a student selected at random is irregular to the classes and what is the probability that the irregular student is a girl?

Step 1: Given Information

- Student proportions:
  - $P(\text{Boy}) = 70\% = 0.7$
  - $P(\text{Girl}) = 30\% = 0.3$
- Irregular students:
  - $P(\text{Irregular} \mid \text{Boy}) = 5\% = 0.05$
  - $P(\text{Irregular} \mid \text{Girl}) = 3\% = 0.03$

Step 2: Probability of a Student Being Irregular

The total probability that a student is irregular ($P(\text{Irregular})$) can be calculated using the **law of total probability**:

$$P(\text{Irregular}) = P(\text{Boy}) \cdot P(\text{Irregular} \mid \text{Boy}) + P(\text{Girl}) \cdot P(\text{Irregular} \mid \text{Girl})$$

Substitute the given probabilities:

$$P(\text{Irregular}) = (0.7 \cdot 0.05) + (0.3 \cdot 0.03)$$

Perform the calculations:

1. $0.7 \cdot 0.05 = 0.035$
2. $0.3 \cdot 0.03 = 0.009$

Add these together:

$$P(\text{Irregular}) = 0.035 + 0.009 = 0.044$$

So, the probability that a randomly selected student is irregular is:

$$\boxed{0.044 \text{ or } 4.4\%}$$

Step 3: Probability That an Irregular Student Is a Girl

We need to find $P(\text{Girl} \mid \text{Irregular})$, the probability that an irregular student is a girl. Use **Bayes' Theorem**:

$$P(\text{Girl} \mid \text{Irregular}) = \frac{P(\text{Girl}) \cdot P(\text{Irregular} \mid \text{Girl})}{P(\text{Irregular})}$$

Substitute the known values:

$$P(\text{Girl} \mid \text{Irregular}) = \frac{0.3 \cdot 0.03}{0.044}$$

Perform the calculations:

1. Numerator: $0.3 \cdot 0.03 = 0.009$
2. Denominator: 0.044

Divide:

$$P(\text{Girl} \mid \text{Irregular}) = \frac{0.009}{0.044} \approx 0.2045$$

So, the probability that an irregular student is a girl is:

$$\boxed{0.2045 \text{ or } 20.45\%}$$

Example 3.

Three machines A, B and C produce respectively 60%, 30% and 10% of the total number of items of a factory. The percentage of defective outputs of these machines are 2%, 3% and 4%. An item is selected at random and is found to be defective.

(i) Find the probability that the item was not defective and produced by machine C?

(ii) (ii) What is the probability that the item was produced by machine C or B?

Step 1: Given Information

1. **Production percentages** (probabilities for machines):
    - $P(A) = 60\% = 0.6$
    - $P(B) = 30\% = 0.3$
    - $P(C) = 10\% = 0.1$
2. **Defective percentages** (conditional probabilities):
    - $P(\text{Defective} \mid \text{A}) = 2\% = 0.02$
    - $P(\text{Defective} \mid \text{B}) = 3\% = 0.03$
    - $P(\text{Defective} \mid \text{C}) = 4\% = 0.04$
3. **Probability of an item being **not defective** is:
    - $P(\text{Not Defective} \mid \text{A}) = 1 - 0.02 = 0.98$
    - $P(\text{Not Defective} \mid \text{B}) = 1 - 0.03 = 0.97$
    - $P(\text{Not Defective} \mid \text{C}) = 1 - 0.04 = 0.96$

Step 2: Use the Definition of Probability

The probability of an item being **not defective and produced by machine C** is given by:

$$P(\text{Not Defective and Machine C}) = P(C) \cdot P(\text{Not Defective} \mid \text{C})$$

Here:

- $P(C)$: Probability the item was produced by machine C.
- $P(\text{Not Defective} \mid \text{C})$: Probability the item is not defective given it was produced by machine C.

Step 3: Substitute the Values

Substitute the known values:

$$P(\text{Not Defective and Machine C}) = 0.1 \cdot 0.96$$

Step 4: Perform the Calculation

Multiply:

$$P(\text{Not Defective and Machine C}) = 0.096$$

Step 5: Final Answer

The probability that the item was **not defective and produced by machine C** is:

$$\boxed{0.096 \text{ or } 9.6\%}$$

Step 1: Given Information

From the problem:

- Probability of an item being produced by Machine C: $P(C) = 0.1$
- Probability of an item being produced by Machine B: $P(B) = 0.3$

Step 2: Substitute the Values

$$P(\text{C or B}) = P(C) + P(B) = 0.1 + 0.3$$

Step 3: Perform the Calculation

$$P(\text{C or B}) = 0.4$$

Step 4: Final Answer

The probability that the item was produced by Machine C or Machine B is:

$$\boxed{0.4 \text{ or } 40\%}$$

Example: 4

A card is randomly drawn from an incomplete deck of cards from which the ace of diamonds is missing.

1. What is the probability that the card is "clubs"?
2. What is the probability that the card is a "queen"?
3. Are the events "clubs" and "queen" independent

Step 1: Understand the Deck

1. A standard deck of cards has 52 cards, but the **ace of diamonds** is missing.
2. This means the deck now contains $52 - 1 = 51$ cards.

- **Number of clubs:** There are still 13 clubs (since none are removed).
- **Number of queens:** There are still 4 queens (one in each suit).

Part 1: Probability that the card is a "club"

The probability of drawing a "club" is the ratio of the number of clubs to the total number of cards:

$$P(\text{Club}) = \frac{\text{Number of clubs}}{\text{Total number of cards}}$$

Substitute the values:

$$P(\text{Club}) = \frac{13}{51}$$

So, the probability is:

$$\boxed{\frac{13}{51}}$$

Part 2: Probability that the card is a "queen"

The probability of drawing a "queen" is the ratio of the number of queens to the total number of cards:

$$P(\text{Queen}) = \frac{\text{Number of queens}}{\text{Total number of cards}}$$

Substitute the values:

$$P(\text{Queen}) = \frac{4}{51}$$

So, the probability is:

$$\boxed{\frac{4}{51}}$$

Part 3: Are the events "clubs" and "queen" independent?

Step 1: Understand Independence

Two events A and B are independent if:

$$P(A \cap B) = P(A) \cdot P(B)$$

Here:

- A: The card is a club ($P(\text{Club}) = \frac{13}{51}$).
- B: The card is a queen ($P(\text{Queen}) = \frac{4}{51}$).

Step 2: Find $P(\text{Club} \cap \text{Queen})$

The event $P(\text{Club} \cap \text{Queen})$ occurs if the card is both a club and a queen. There is **only one queen of clubs** in the deck.

So:

$$P(\text{Club} \cap \text{Queen}) = \frac{1}{51}$$

Step 3: Check Independence

Now, calculate $P(\text{Club}) \cdot P(\text{Queen})$:

$$P(\text{Club}) \cdot P(\text{Queen}) = \frac{13}{51} \cdot \frac{4}{51} = \frac{52}{2601}$$

Compare $P(\text{Club} \cap \text{Queen})$ with $P(\text{Club}) \cdot P(\text{Queen})$:

- $P(\text{Club} \cap \text{Queen}) = \frac{1}{51} = \frac{51}{2601}$.
- $P(\text{Club}) \cdot P(\text{Queen}) = \frac{52}{2601}$.

Since $P(\text{Club} \cap \text{Queen}) \neq P(\text{Club}) \cdot P(\text{Queen})$, the events **"clubs" and "queen" are not independent.**

Example: 5

Three companies produce the same tool and supply it to the market. Company A produces 30% of the tools for the market and the remaining 70% of the tools are produced in companies' B and C. 98% of the tools produced in company A, 95% of the tools produced in company B and 97% of the tools produced in company C are not defective. What percent of tools should be produced by company's B and C so that a tool picked at random in the market will have a probability of being non defective equal to 96%?

Step 1: Define the Problem

Let:

- $P(A) = 30\% = 0.3$: Company A produces 30% of the tools.
- $P(B)$: The percentage of tools produced by company B (to be determined).
- $P(C)$: The percentage of tools produced by company C (to be determined).
- Tools must sum to 100%:

$$P(B) + P(C) = 1 - P(A) = 70\% = 0.7$$

The probability of selecting a **non-defective tool** in the market is given as **96%**:

$$P(\text{Non-Defective}) = 96\% = 0.96$$

Step 2: Probability of Selecting a Non-Defective Tool

The probability of a randomly selected tool being non-defective is determined by the weighted sum of the non-defective probabilities for each company:

$$P(\text{Non-Defective}) = P(A) \cdot P(\text{Non-Defective} \mid \text{A}) + P(B) \cdot P(\text{Non-Defective} \mid \text{B}) + P(C) \cdot P(\text{Non-Defective} \mid \text{C})$$

From the problem:

- $P(\text{Non-Defective} \mid \text{A}) = 98\% = 0.98$
- $P(\text{Non-Defective} \mid \text{B}) = 95\% = 0.95$
- $P(\text{Non-Defective} \mid \text{C}) = 97\% = 0.97$

Substitute the known values:

$$0.96 = (0.3 \cdot 0.98) + (P(B) \cdot 0.95) + (P(C) \cdot 0.97)$$

Step 3: Simplify the Equation

1. Calculate the contribution of company A:

$$0.3 \cdot 0.98 = 0.294$$

2. Substitute $P(C) = 0.7 - P(B)$ (since $P(B) + P(C) = 0.7$):

$$0.96 = 0.294 + (P(B) \cdot 0.95) + ((0.7 - P(B)) \cdot 0.97)$$

3. Expand the equation:

$$0.96 = 0.294 + (P(B) \cdot 0.95) + (0.7 \cdot 0.97) - (P(B) \cdot 0.97)$$

4. Simplify:

$$0.96 = 0.294 + 0.679 - (P(B) \cdot (0.97 - 0.95))$$

$$0.96 = 0.294 + 0.679 - (P(B) \cdot 0.02)$$

5. Combine constants:

$$0.96 = 0.973 - (P(B) \cdot 0.02)$$

6. Rearrange for $P(B)$:

$$P(B) \cdot 0.02 = 0.973 - 0.96 = 0.013$$

$$P(B) = \frac{0.013}{0.02} = 0.65$$

Step 4: Find $P(C)$

Since $P(B) + P(C) = 0.7$:

$$P(C) = 0.7 - P(B) = 0.7 - 0.65 = 0.05$$

Step 5: Final Answer

- Company B should produce $\boxed{65\%}$ of the tools.
- Company C should produce $\boxed{5\%}$ of the tools.

Example: 6

A sample of 500 respondents was selected in a large metropolitan area to study consumer behavior, with the following results:

ENJOYS SHOPPING FOR CLOTHING	GENDER		
	Male	Female	Total
Yes	136	224	360
No	104	36	140
Total	240	260	500

a. Suppose the respondent chosen is a female. What is the probability that she does not enjoy shopping for clothing?

b. Suppose the respondent chosen enjoys shopping for clothing. What is the probability that the individual is a male?

Understanding the Data

The table shows the number of respondents based on gender (male/female) and whether they enjoy shopping for clothing ("Yes" or "No"). Here's the summary:

- Total respondents = **500**
- **Females:**
  - Enjoy shopping: 224
  - Do not enjoy shopping: 36
  - Total females: 260
- **Males:**
  - Enjoy shopping: 136
  - Do not enjoy shopping: 104
  - Total males: 240

Part (a): Suppose the respondent is female. What is the probability that she does not enjoy shopping for clothing?

This is a **conditional probability** question:

$$P(\text{Does not enjoy shopping} \mid \text{Female}) = \frac{\text{Number of females who do not enjoy shopping}}{\text{Total number of females}}$$

From the table:

- Number of females who do not enjoy shopping = 36
- Total number of females = 260

Substitute the values:

$$P(\text{Does not enjoy shopping} \mid \text{Female}) = \frac{36}{260}$$

Simplify:

$$P(\text{Does not enjoy shopping} \mid \text{Female}) = 0.1385$$

Answer for Part (a): The probability is:

$$\boxed{0.1385 \text{ or } 13.85\%}$$

Part (b): Suppose the respondent enjoys shopping for clothing. What is the probability that the individual is male?

This is another **conditional probability** question:

$$P(\text{Male} \mid \text{Enjoys shopping}) = \frac{\text{Number of males who enjoy shopping}}{\text{Total number of people who enjoy shopping}}$$

From the table:

- Number of males who enjoy shopping = 136
- Total number of people who enjoy shopping = 360

Substitute the values:

$$P(\text{Male} \mid \text{Enjoys shopping}) = \frac{136}{360}$$

Simplify:

$$P(\text{Male} \mid \text{Enjoys shopping}) = 0.3778$$

Answer for Part (b): The probability is:

$$\boxed{0.3778 \text{ or } 37.78\%}$$

Example: 7

A standard deck of cards is being used to play a game. There are four suits (hearts, diamonds, clubs, and spades), each having 13 faces (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, and king), making a total of 52 cards. This complete deck is thoroughly mixed, and you will receive the first 2 cards from the deck without replacement.

a. What is the probability that both cards are queens?

b. What is the probability that the first card is a 10 and the second card is a 5 or 6?

c. If you were sampling with replacement, what would be the answer in (a)?

Part (a): Probability that both cards are queens (without replacement)

Step 1: Understand the problem

- There are **4 queens** in the deck.
- The deck has **52 cards.**
- We are drawing **2 cards without replacement** (the first card affects the total for the second card).

Combination Formula

The combination formula is used to calculate the number of ways to choose r items from n items without regard to order:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Here:

- n: Total number of items.
- r: Number of items to choose.

Part (a): Probability that both cards are queens (without replacement)

Step 1: Total possible outcomes

When we draw 2 cards from a deck of 52, the total number of possible ways to choose 2 cards is:

$$\binom{52}{2} = \frac{52!}{2!(52-2)!} = \frac{52 \cdot 51}{2 \cdot 1} = 1326$$

So, there are **1326 possible outcomes**.

Step 2: Favorable outcomes

There are 4 queens in the deck. The number of ways to choose 2 queens from 4 queens is:

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

So, there are **6 favorable outcomes**.

Step 3: Probability formula

The probability is the ratio of favorable outcomes to total outcomes:

$$P(\text{Both cards are queens}) = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{\binom{4}{2}}{\binom{52}{2}}$$

Substitute the values:

$$P(\text{Both cards are queens}) = \frac{6}{1326} = \frac{1}{221} \approx 0.00452$$

Answer for (a):

$$\boxed{\frac{1}{221} \text{ or } 0.00452\ (0.452\%)}$$

Part (b): Probability that the first card is a 10 and the second card is a 5 or 6

For this part, combinations are used to break down the events step by step.

Step 1: Total possible outcomes

The total number of ways to draw 2 cards from 52 cards is:

$$\binom{52}{2} = 1326$$

Step 2: Favorable outcomes

We want:

1. The first card to be a **10**.
2. The second card to be either a **5** or **6**.

Let's calculate the number of favorable outcomes.

1. **Choose the first card (a 10):** There are **4 tens** in the deck, so there are $\binom{4}{1} = 4$ ways to choose the first card.
2. **Choose the second card (a 5 or 6):** After the first card is chosen, there are $52 - 1 = 51$ cards remaining. Among these, there are 8 cards that are either a 5 or a 6 (4 fives + 4 sixes):

$$\binom{8}{1} = 8$$

So, the total number of favorable outcomes is:

$$4 \cdot 8 = 32$$

Part (c): Probability that both cards are queens (with replacement)

When sampling with replacement, the total number of cards remains 52 after each draw. Therefore, the order of selection does not matter.

Step 1: Probability of drawing a queen

Since there are 4 queens in a deck of 52 cards, the probability of drawing a queen on any single draw is:

$$P(\text{Queen on any draw}) = \frac{4}{52} = \frac{1}{13}$$

Step 2: Multiply probabilities

Since the cards are replaced after each draw, the draws are **independent**, and the probability of drawing a queen remains the same for both draws:

$$P(\text{Both cards are queens with replacement}) = P(\text{Queen on first draw}) \cdot P(\text{Queen on second draw})$$

Substitute values:

$$P(\text{Both cards are queens with replacement}) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

Answer for (c):

$$\boxed{\frac{1}{169} \text{ or } 0.00592\ (0.592\%)}$$

Example: 8

A box of nine gloves contains two left-handed gloves and seven right-handed gloves.

a. If two gloves are randomly selected from the box without replacement, what is the probability that both gloves selected will be right-handed?

b. If two gloves are randomly selected from the box without replacement, what is the probability there will be one right-handed glove, and one left-handed glove selected?

c. If three gloves are selected with replacement, what is the probability that all three will be left-handed?

d. If you were sampling with replacement, what would be the answers to (a) and (b)?

Given Information

- Total gloves in the box: 9 gloves.
- Number of left-handed gloves: 2.
- Number of right-handed gloves: 7.
- We calculate probabilities for different scenarios, with or without replacement.

Part (a): Probability that both selected gloves are right-handed (without replacement)

Step 1: Understand the problem

We are selecting 2 **gloves without replacement**, and we want the probability that both gloves are **right-handed**.

Step 2: Total possible outcomes

The total number of ways to choose 2 gloves from 9 is:

$$\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9 \cdot 8}{2} = 36$$

Step 3: Favorable outcomes

The number of ways to choose 2 right-handed gloves from 7 is:

$$\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$$

Step 4: Probability formula

The probability is:

$$P(\text{Both right-handed}) = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{\binom{7}{2}}{\binom{9}{2}}$$

Substitute the values:

$$P(\text{Both right-handed}) = \frac{21}{36} = \frac{7}{12}$$

Answer for (a):

$$\boxed{\frac{7}{12} \text{ or } 0.5833 \ (58.33\%)}$$

Part (b): Probability of selecting one right-handed glove and one left-handed glove (without replacement)

Step 1: Favorable outcomes

To select one right-handed glove and one left-handed glove:

1. Choose 1 right-handed glove from 7:

$$\binom{7}{1} = 7$$

2. Choose 1 left-handed glove from 2:

$$\binom{2}{1} = 2$$

The total number of favorable outcomes is:

$$\binom{7}{1} \cdot \binom{2}{1} = 7 \cdot 2 = 14$$

Step 2: Total outcomes

The total number of ways to select 2 gloves from 9 is:

$$\binom{9}{2} = 36$$

Step 3: Probability formula

The probability is:

$$P(\text{One right-handed and one left-handed}) = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{14}{36} = \frac{7}{18}$$

Answer for (b):

$$\boxed{\frac{7}{18} \text{ or } 0.3889\ (38.89\%)}$$

Part (c): Probability that all 3 selected gloves are left-handed (with replacement)

Step 1: Probability of selecting a left-handed glove on one draw

Since there are 2 left-handed gloves out of 9:

$$P(\text{Left-handed glove}) = \frac{2}{9}$$

Step 2: Probability of selecting 3 left-handed gloves (with replacement)

With replacement, the draws are independent. Multiply the probabilities for 3 draws:

$$P(\text{All 3 left-handed}) = P(\text{Left-handed}) \cdot P(\text{Left-handed}) \cdot P(\text{Left-handed})$$

Substitute the value:

$$P(\text{All 3 left-handed}) = \left(\frac{2}{9}\right)^3 = \frac{2 \cdot 2 \cdot 2}{9 \cdot 9 \cdot 9} = \frac{8}{729}$$

Answer for (c):

$$\boxed{\frac{8}{729} \text{ or } 0.01097\ (1.097\%)}$$

Part (d): Sampling with replacement for (a) and (b)

For (a): Both gloves are right-handed (with replacement)

With replacement, the probability of selecting a right-handed glove on any draw is:

$$P(\text{Right-handed}) = \frac{7}{9}$$

The draws are independent, so:

$$P(\text{Both right-handed}) = P(\text{Right-handed}) \cdot P(\text{Right-handed})$$

Substitute:

$$P(\text{Both right-handed}) = \frac{7}{9} \cdot \frac{7}{9} = \frac{49}{81}$$

Answer for (a) with replacement:

$$\boxed{\frac{49}{81} \text{ or } 0.6049\ (60.49\%)}$$

For (b): One right-handed and one left-handed glove (with replacement)

With replacement, the probability of selecting a right-handed glove is $\frac{7}{9}$, and the probability of selecting a left-handed glove is $\frac{2}{9}$.

There are **two possible orders** for selecting one right-handed and one left-handed glove:

1. Right-handed first, then left-handed.
2. Left-handed first, then right-handed.

The total probability is:

$$P(\text{One right-handed and one left-handed}) = P(\text{Right then Left}) + P(\text{Left then Right})$$

Calculate each:

$$P(\text{Right then Left}) = P(\text{Right-handed}) \cdot P(\text{Left-handed}) = \frac{7}{9} \cdot \frac{2}{9} = \frac{14}{81}$$

$$P(\text{Left then Right}) = P(\text{Left-handed}) \cdot P(\text{Right-handed}) = \frac{2}{9} \cdot \frac{7}{9} = \frac{14}{81}$$

Add them together:

$$P(\text{One right-handed and one left-handed}) = \frac{14}{81} + \frac{14}{81} = \frac{28}{81}$$

Answer for (b) with replacement:

$$\boxed{\frac{28}{81} \text{ or } 0.3457\ (34.57\%)}$$

Example:9

In a jar with 5 red, 6 blue and 2 white marbles. Two marbles are selected, find the probability that both are red if:

a) If two marbles are selected with replacement.

b) If two marbles are selected without replacement.

Given Information

- Total marbles in the jar = $5 + 6 + 2 = 13$ marbles.
- Number of red marbles = 5.
- We are selecting **2 marbles**.

Part (a): Probability that both marbles are red (with replacement)

Step 1: Key Concept

When selecting **with replacement**, the first marble is put back into the jar before drawing the second marble. This makes the draws **independent**, meaning the probability of each draw remains the same.

Step 2: Probability of drawing a red marble on one draw

The probability of drawing a red marble on any draw is:

$$P(\text{Red}) = \frac{\text{Number of red marbles}}{\text{Total number of marbles}} = \frac{5}{13}$$

Step 3: Probability of both draws being red

Since the draws are independent, the probability of both marbles being red is:

$$P(\text{Both red}) = P(\text{Red on first draw}) \cdot P(\text{Red on second draw})$$

Substitute the values:

$$P(\text{Both red}) = \frac{5}{13} \cdot \frac{5}{13} = \frac{25}{169}$$

Answer for (a):

$$\boxed{\frac{25}{169} \text{ or } 0.148\ (14.8\%)}$$

Part (b): Probability that both marbles are red (without replacement)

Step 1: Key Concept

When selecting **without replacement**, the first marble is not put back into the jar. This makes the draws **dependent**, meaning the probability changes after the first draw.

Step 2: Probability of drawing a red marble on the first draw

The probability of drawing a red marble on the first draw is:

$$P(\text{Red on first draw}) = \frac{5}{13}$$

Step 3: Probability of drawing a red marble on the second draw

After one red marble is removed, there are 4 red marbles left, and the total number of marbles is reduced to 12. So:

$$P(\text{Red on second draw} \mid \text{Red on first draw}) = \frac{4}{12} = \frac{1}{3}$$

Step 4: Probability of both draws being red

Since the events are dependent, the probability is:

$$P(\text{Both red}) = P(\text{Red on first draw}) \cdot P(\text{Red on second draw} \mid \text{Red on first draw})$$

Substitute the values:

$$P(\text{Both red}) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156} = \frac{5}{39}$$

Answer for (b):

$$\boxed{\frac{5}{39} \text{ or } 0.128\ (12.8\%)}$$

Example: 10

In batch of 6400 light bulbs, 80 are defective. If 12 light bulbs are selected from the batch without replacements, find probability that all are good.

Given Information

- Total light bulbs in the batch = 6400.
- Number of defective bulbs = 80.
- Number of good bulbs = $6400 - 80 = 6320$.
- Number of bulbs selected = 12.
- The bulbs are selected **without replacement.**

We want to find the **probability that all 12 selected bulbs are good.**

Step 1: Key Formula

When selecting bulbs without replacement, the probability of all 12 being good is calculated as:

$$P(\text{All are good}) = \frac{\binom{\text{Good bulbs}}{\text{Selected}}}{\binom{\text{Total bulbs}}{\text{Selected}}}$$

Here:

- $\binom{n}{r}$ is the combination formula, which gives the number of ways to select r items from n items:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Step 2: Apply the Formula

1. **Total possible outcomes:** The total number of ways to select 12 bulbs from 6400 is:

$$\binom{6400}{12} = \frac{6400!}{12!(6400-12)!}$$

2. **Favorable outcomes:** The number of ways to select 12 good bulbs from the 6320 good bulbs is:

$$\binom{6320}{12} = \frac{6320!}{12!(6320-12)!}$$

Step 3: Simplify Using Probability

Rather than directly using factorials (which are very large numbers), we calculate the probability step by step for each selection, as the bulbs are picked **without replacement**.

First Selection

The probability that the first bulb is good:

$$P(\text{First is good}) = \frac{\text{Good bulbs}}{\text{Total bulbs}} = \frac{6320}{6400}$$

Second Selection

After selecting one good bulb, there are 6319 good bulbs left and 6399 total bulbs remaining:

$$P(\text{Second is good}) = \frac{6319}{6399}$$

Subsequent Selections

Continue this process for all 12 selections:

$$P(\text{All 12 are good}) = \frac{6320}{6400} \cdot \frac{6319}{6399} \cdot \frac{6318}{6398} \cdot \dots \cdot \frac{6309}{6389}$$

Step 4: Write the General Probability Formula

In general, the probability is:

$$P(\text{All 12 are good}) = \prod_{i=0}^{11} \frac{6320-i}{6400-i}$$

Step 5: Approximation

For large populations, probabilities involving small samples can often be approximated. Since 12 is much smaller than 6400, the probability can be simplified:

$$P(\text{All are good}) \approx \left(\frac{6320}{6400}\right)^{12}$$

Step 6: Calculation

1. Calculate the fraction:

$$\frac{6320}{6400} = 0.9875$$

2. Raise to the power of 12:

$$P(\text{All are good}) \approx (0.9875)^{12} \approx 0.868$$

Final Answer

The probability that all 12 selected bulbs are good is approximately:

$$\boxed{0.868 \text{ or } 86.8\%}$$

Example: 11

In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.

- 45 flights in the survey departed from airport A; 9 of those flights departed late.
- 55 flights in the survey departed from airport B; 11 flights departed late.
- Are the events "depart from airport A" and "departed late" independent?

Let's determine whether the events "**depart from airport A**" and "**departed late**" are independent, step by step.

Step 1: Understand Independence

Two events A and B are independent if:

$$P(A \cap B) = P(A) \cdot P(B)$$

Here:

- A: The flight departed from airport A.
- B: The flight departed late.

If this condition holds, the events are independent. Otherwise, they are dependent.

Step 2: Gather the Given Information

- Total flights: 100
- Flights from **airport** A: 45 ($P(A) = \frac{45}{100} = 0.45$).
- Flights that departed late: 20 ($P(B) = \frac{20}{100} = 0.20$).
- Flights from **airport** A that departed late: 9 ($P(A \cap B) = \frac{9}{100} = 0.09$).

Step 3: Check for Independence

To check if the events are independent, calculate:

$$P(A) \cdot P(B)$$

1. Substitute $P(A)$ and $P(B)$:

$$P(A) \cdot P(B) = 0.45 \cdot 0.20 = 0.09$$

2. Compare $P(A) \cdot P(B)$ with $P(A \cap B)$:

   - $P(A \cap B) = 0.09$
   - $P(A) \cdot P(B) = 0.09$

Since:

$$P(A \cap B) = P(A) \cdot P(B)$$

The events **"depart from airport A"** and **"departed late"** are **independent.**

Final Answer

The events **"depart from airport A"** and **"departed late"** are:

$$\boxed{\text{Independent.}}$$

Example: 12

If A and B are two events with P(A)= 1/3, P(B)= 1/2 and Find

$$P\left(\frac{A}{B}\right), P\left(\frac{B}{A}\right), \mathrm{P}\left(\frac{\overline{\mathrm{A}}}{\overline{B}}\right), \mathrm{P}\left(\frac{\overline{\mathrm{B}}}{\overline{A}}\right), \mathrm{P}\left(\frac{\mathrm{A}}{\overline{B}}\right)$$

Given Information

1. $P(A) = \frac{1}{3}$
2. $P(B) = \frac{1}{2}$

We need to calculate:

1. $P(A \mid B)$
2. $P(B \mid A)$
3. $P(A^c \mid B)$
4. $P(B^c \mid A)$
5. $P(A^c \mid B^c)$

Step 1: Key Formula

The conditional probability formula is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Similarly:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

For complements (A^c and B^c):

$$P(A^c \mid B) = 1 - P(A \mid B)$$

$$P(B^c \mid A) = 1 - P(B \mid A)$$

$$P(A^c \mid B^c) = \frac{P(A^c \cap B^c)}{P(B^c)}$$

Step 2: Assumptions

If not explicitly stated, we assume $P(A \cap B)$ to be given or derived. Let's assume $P(A \cap B) = \frac{1}{6}$ (a reasonable assumption, as it satisfies $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$).

Step 3: Calculate Each Probability

1. $P(A \mid B)$**:**

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Substitute values:

$$P(A \mid B) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

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2. $P(B \mid A)$**:**

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Substitute values:

$$P(B \mid A) = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

3. $P(A^c \mid B)$**:**

$$P(A^c \mid B) = 1 - P(A \mid B)$$

Substitute $P(A \mid B) = \frac{1}{3}$:

$$P(A^c \mid B) = 1 - \frac{1}{3} = \frac{2}{3}$$

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4. $P(B^c \mid A)$**:**

$$P(B^c \mid A) = 1 - P(B \mid A)$$

Substitute $P(B \mid A) = \frac{1}{2}$:

$$P(B^c \mid A) = 1 - \frac{1}{2} = \frac{1}{2}$$

5. $P(A^c \mid B^c)$**:**

First, calculate $P(B^c) = 1 - P(B)$:

$$P(B^c) = 1 - \frac{1}{2} = \frac{1}{2}$$

Now calculate $P(A^c \cap B^c)$:

$$P(A^c \cap B^c) = 1 - P(A \cup B)$$

Using the formula for union:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute values:

$$P(A \cup B) = \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{6} + \frac{3}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

Thus:

$$P(A^c \cap B^c) = 1 - \frac{2}{3} = \frac{1}{3}$$

Finally, calculate $P(A^c \mid B^c)$:

$$P(A^c \mid B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Final Answers

1. $P(A \mid B) = \boxed{\frac{1}{3}}$
2. $P(B \mid A) = \boxed{\frac{1}{2}}$
3. $P(A^c \mid B) = \boxed{\frac{2}{3}}$
4. $P(B^c \mid A) = \boxed{\frac{1}{2}}$
5. $P(A^c \mid B^c) = \boxed{\frac{2}{3}}$

Example: 13

Three students A,B,C write an examination. Their chances of passing are $\frac{1}{4}, \frac{1}{2}$ and $\frac{1}{3}$ respectively. Find the probability that

(i) all of them pass,

(ii) at least one of them passes,

(iii) at least two of them pass.

Given Information

The probabilities of students A, B, and C passing the exam are:

- $P(A) = \frac{1}{4}$
- $P(B) = \frac{1}{2}$
- $P(C) = \frac{1}{3}$

The complementary probabilities (failing) are:

- $P(A') = 1 - P(A) = \frac{3}{4}$
- $P(B') = 1 - P(B) = \frac{1}{2}$
- $P(C') = 1 - P(C) = \frac{2}{3}$

Part (i): Probability that all of them pass

For all three students to pass, all events must happen simultaneously. Since they are independent, we multiply the probabilities:

$$P(\text{All pass}) = P(A) \cdot P(B) \cdot P(C)$$

Substitute the values:

$$P(\text{All pass}) = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3}$$

Simplify:

$$P(\text{All pass}) = \frac{1}{24}$$

Answer for (i):

$$\boxed{\frac{1}{24} \text{ or } 0.0417\ (4.17\%)}$$

Part (ii): Probability that at least one of them passes

The complement of "at least one of them passes" is "none of them pass." First, calculate the probability that none pass, and then subtract it from 1:

$$P(\text{At least one passes}) = 1 - P(\text{None pass})$$

Step 1: Calculate $P(\text{None pass})$**:**

$$P(\text{None pass}) = P(A') \cdot P(B') \cdot P(C')$$

Substitute the values:

$$P(\text{None pass}) = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{2}{3}$$

Simplify:

$$P(\text{None pass}) = \frac{3 \cdot 1 \cdot 2}{4 \cdot 2 \cdot 3} = \frac{6}{24} = \frac{1}{4}$$

Step 2: Calculate $P(\text{At least one passes})$**:**

$$P(\text{At least one passes}) = 1 - P(\text{None pass}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Answer for (ii):

$$\boxed{\frac{3}{4} \text{ or } 0.75\ (75\%)}$$

Part (iii): Probability that at least two of them pass

To calculate this, use the complement rule and subtract the probabilities of cases where fewer than 2 students pass:

$$P(\text{At least two pass}) = 1 - P(\text{None pass}) - P(\text{Exactly one passes})$$

Step 1: Calculate $P(\text{Exactly one passes})$**:**

"Exactly one passes" means one student passes, and the other two fail. There are three cases (A passes, B passes, or C passes):

1. **Case 1: Only A passes:**

$$P(\text{Only A passes}) = P(A) \cdot P(B') \cdot P(C') = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{12}$$

2. **Case 2: Only B passes:**

$$P(\text{Only B passes}) = P(A') \cdot P(B) \cdot P(C') = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{12}$$

3. **Case 3: Only C passes:**

$$P(\text{Only C passes}) = P(A') \cdot P(B') \cdot P(C) = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{8}$$

Add these probabilities:

$$P(\text{Exactly one passes}) = \frac{1}{12} + \frac{1}{12} + \frac{1}{8} = \frac{2}{12} + \frac{3}{24} = \frac{5}{24}$$

Step 2: Calculate $P(\text{At least two pass})$**:**

$$P(\text{At least two pass}) = 1 - P(\text{None pass}) - P(\text{Exactly one passes})$$

Substitute values:

$$P(\text{At least two pass}) = 1 - \frac{1}{4} - \frac{5}{24}$$

Simplify:

$$P(\text{At least two pass}) = \frac{24}{24} - \frac{6}{24} - \frac{5}{24} = \frac{13}{24}$$

Answer for (iii):

$$\boxed{\frac{13}{24} \text{ or } 0.5417\ (54.17\%)}$$

Final Summary of Answers

1. **All of them pass:** $\boxed{\frac{1}{24} \text{ or } 4.17\%}$
2. **At least one passes:** $\boxed{\frac{3}{4} \text{ or } 75\%}$
3. **At least two pass:** $\boxed{\frac{13}{24} \text{ or } 54.17\%}$