Introduction Statistical Methods- Session-6

1. **Bernoulli Distribution**:

- **Definition:** It represents the outcome of a single experiment where there are only two possible outcomes: success (1) and failure (0).
- **Example:** Tossing a coin:
  - Probability of heads (p) = 0.5.
  - Probability of tails ($1 - p$) = 0.5.
- **Formula:** $P(X = x) = p^x(1-p)^{1-x}$, where $x = 0 \text{ or } 1$.

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2. **Binomial Distribution**:

- **Definition:** It is the sum of multiple independent Bernoulli trials.
- **Example:** Tossing a coin 10 times and counting the number of heads.
  - Probability of success (p) = 0.5.
  - Number of trials (n) = 10.
- **Formula:** $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$, where k is the number of successes.

3. **Poisson Distribution**:

- **Definition:** It represents the number of events occurring in a fixed interval of time or space, assuming these events happen at a constant rate independently.
- **Example:** The number of emails you receive in an hour.
  - Average rate (λ) = 5 emails per hour.
- **Formula:** $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$.

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4. **Normal Distribution**:

- **Definition:** It is a continuous distribution shaped like a bell curve, defined by its mean (μ) and standard deviation (σ).
- **Example:** Heights of people in a population.
  - Mean height (μ) = 170 cm.
  - Standard deviation (σ) = 10 cm.
- **Formula:** $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

5. **t-Distribution**:

- **Definition:** It is used for estimating population parameters when the sample size is small, and the population standard deviation is unknown.
- **Example:** Calculating the confidence interval for the mean of a sample of 10 students' test scores.
- **Formula:** Similar to the normal distribution but with heavier tails.

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6. **F-Distribution**:

- **Definition:** It is used to compare the variances of two populations and is commonly applied in ANOVA tests.
- **Example:** Testing if two manufacturing processes have the same variance.
- **Formula:** Based on the ratio of two chi-square distributed variables divided by their degrees of freedom.

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7. **Chi-Square Distribution**:

- **Definition:** It is used in hypothesis testing, particularly for categorical data (e.g., goodness-of-fit or independence tests).
- **Example:** Testing whether a die is fair (each face has equal probability).
- **Formula:** $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$, where O_i is observed frequency, and E_i is expected frequency.

Example 1

In a particular restaurant, an average of 3 out of every 5 customers ask for water with their meal. A random sample of 10 customers is selected. Find the probability that

(i) Exactly 6 ask for water with their meal,

(ii) Less than 9 ask for water with their meal.

(iii) No one asks for water with their meal?

(iv) At most 2 ask for water with their meal?

(v) At least 3 ask for water with their meal?

Key Details:

- **Number of trials** (n): 10 customers
- **Probability of success** (p): 3/5 = 0.6 (customer asks for water)
- **Probability of failure** ($1 - p$): 1-0.6 = 0.4 (customer does not ask for water)
- The **binomial formula** for $P(X = k)$ is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where:

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

(i) Probability that exactly 6 customers ask for water ($P(X = 6)$):

Step 1: Plug in $n = 10$, $k = 6$, and $p = 0.6$:

$$P(X = 6) = \binom{10}{6}(0.6)^6(0.4)^4$$

Step 2: Calculate the binomial coefficient $\binom{10}{6}$:

$$\binom{10}{6} = \frac{10!}{6!(10-6)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

Step 3: Calculate $(0.6)^6$:

$$(0.6)^6 = 0.046656$$

Step 4: Calculate $(0.4)^4$:

$$(0.4)^4 = 0.0256$$

Step 5: Multiply all terms together:

$$P(X = 6) = 210 \cdot 0.046656 \cdot 0.0256 = 0.2519$$

Thus, the probability that exactly 6 customers ask for water is approximately **0.2519**.

(ii) Probability that fewer than 9 customers ask for water ($P(X < 9)$):

Step 1: Use the complement rule:

$$P(X < 9) = 1 - P(X \geq 9)$$

This means:

$$P(X < 9) = 1 - (P(X = 9) + P(X = 10))$$

Step 2: Calculate $P(X = 9)$:

$$P(X = 9) = \binom{10}{9}(0.6)^9(0.4)^1$$

- $\binom{10}{9} = 10$,
- $(0.6)^9 = 0.010077696$,
- $(0.4)^1 = 0.4$,

$$P(X = 9) = 10 \cdot 0.010077696 \cdot 0.4 = 0.0403$$

Step 3: Calculate $P(X = 10)$:

$$P(X = 10) = \binom{10}{10}(0.6)^{10}(0.4)^{0}$$

- $\binom{10}{10} = 1$,
- $(0.6)^{10} = 0.0060466176$,

$$P(X = 10) = 1 \cdot 0.0060466176 \cdot 1 = 0.0060$$

Step 4: Add $P(X = 9)$ and $P(X = 10)$:

$$P(X \geq 9) = P(X = 9) + P(X = 10) = 0.0403 + 0.0060 = 0.0463$$

Step 5: Subtract from 1:

$$P(X < 9) = 1 - 0.0463 = 0.9537$$

Thus, the probability that fewer than 9 customers ask for water is approximately **0.9537**.

(iii) Probability that no one asks for water ($P(X = 0)$):

Step 1: Use the formula:

$$P(X = 0) = \binom{10}{0}(0.6)^{0}(0.4)^{10}$$

Step 2: Compute each term:

- $\binom{10}{0} = 1$,
- $(0.6)^{0} = 1$,
- $(0.4)^{10} = 0.0001048576$,

$$P(X = 0) = 1 \cdot 1 \cdot 0.0001048576 = 0.0001$$

Thus, the probability that no one asks for water is approximately **0.0001**.

(iv) Probability that at most 2 customers ask for water ($P(X \leq 2)$):

Step 1: Sum $P(X = 0)$, $P(X = 1)$, and $P(X = 2)$:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

Step 2: Use previous results for $P(X = 0)$:

$$P(X = 0) = 0.0001$$

Step 3: Calculate $P(X = 1)$:

$$P(X = 1) = \binom{10}{1}(0.6)^1(0.4)^9$$

- $\binom{10}{1} = 10$,
- $(0.6)^1 = 0.6$,
- $(0.4)^9 = 0.000262144$,

$$P(X = 1) = 10 \cdot 0.6 \cdot 0.000262144 = 0.0016$$

Step 4: Calculate $P(X = 2)$:

$$P(X = 2) = \binom{10}{2}(0.6)^2(0.4)^8$$

- $\binom{10}{2} = 45$,
- $(0.6)^2 = 0.36$,
- $(0.4)^8 = 0.00065536$,

$$P(X = 2) = 45 \cdot 0.36 \cdot 0.00065536 = 0.0106$$

Step 5: Add them all:

$$P(X \leq 2) = 0.0001 + 0.0016 + 0.0106 = 0.0123$$

Thus, the probability that at most 2 customers ask for water is approximately **0.0123**.

(v) Probability that at least 3 customers ask for water ($P(X \geq 3)$):

Step 1: Use the complement rule:

$$P(X \geq 3) = 1 - P(X \leq 2)$$

Step 2: Substitute $P(X \leq 2)$:

$$P(X \geq 3) = 1 - 0.0123 = 0.9877$$

Thus, the probability that at least 3 customers ask for water is approximately **0.9877**.

Example 2:

Ten coins are thrown simultaneously. what is the probability of getting 3 or fewer heads?

Step 1: Identify the problem and variables

This is a **Binomial Probability problem** because:

1. You are performing multiple independent trials (tossing 10 coins).
2. Each trial has two possible outcomes: **head** or **tail**.
3. The probability of getting a head ($p = 0.5$) remains constant across all trials.

Key variables:

- $n = 10$: Total number of coin tosses.
- $p = 0.5$: Probability of getting heads on a single toss.
- $q = 1 - p = 0.5$: Probability of getting tails.
- X: Number of heads we are interested in.

We want to calculate $P(X \leq 3)$, i.e., the probability of getting **3 or fewer heads**:

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

Step 2: Binomial Probability Formula

The probability for any specific number of heads ($X = k$) is given by:

$$P(X = k) = \binom{n}{k} p^k q^{n-k}$$

Where:

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.
  - This counts the number of ways k heads can occur in n trials.
- p^k: Probability of getting k heads.
- q^{n-k}: Probability of getting $(n - k)$ tails.

We need to calculate this formula for $k = 0, 1, 2, 3$ and then add them.

Step 3: Calculate $P(X = 0)$

Substitute $n = 10, k = 0, p = 0.5, q = 0.5$ into the formula:

$$P(X = 0) = \binom{10}{0} (0.5)^0 (0.5)^{10}$$

Step-by-step:

1. $\binom{10}{0} = \frac{10!}{0!(10-0)!} = 1$: There is only one way to get 0 heads.
2. $(0.5)^0 = 1$: Any number to the power 0 is 1.
3. $(0.5)^{10} = 0.0009765625$: Multiply 0.5 by itself 10 times.

Final probability:

$$P(X = 0) = 1 \cdot 1 \cdot 0.0009765625 = 0.0009765625$$

Step 4: Calculate $P(X = 1)$

Substitute $n = 10, k = 1, p = 0.5, q = 0.5$:

$$P(X = 1) = \binom{10}{1} (0.5)^1 (0.5)^9$$

Step-by-step:

1. $\binom{10}{1} = \frac{10!}{1!(10-1)!} = \frac{10}{1} = 10$: There are 10 ways to get 1 head in 10 tosses.
2. $(0.5)^1 = 0.5$: This is the probability of getting 1 head.
3. $(0.5)^9 = 0.001953125$: Multiply 0.5 by itself 9 times.

Final probability:

$$P(X = 1) = 10 \cdot 0.5 \cdot 0.001953125 = 0.009765625$$

Step 5: Calculate $P(X = 2)$

Substitute $n = 10, k = 2, p = 0.5, q = 0.5$:

$$P(X = 2) = \binom{10}{2}(0.5)^2(0.5)^8$$

Step-by-step:

1. $\binom{10}{2} = \frac{10!}{2!(10-2)!} = \frac{10 \cdot 9}{2 \cdot 1} = 45$: There are 45 ways to get 2 heads in 10 tosses.
2. $(0.5)^2 = 0.25$: This is the probability of getting 2 heads.
3. $(0.5)^8 = 0.00390625$: Multiply 0.5 by itself 8 times.

Final probability:

$$P(X = 2) = 45 \cdot 0.25 \cdot 0.00390625 = 0.0439453125$$

Step 6: Calculate $P(X = 3)$

Substitute $n = 10, k = 3, p = 0.5, q = 0.5$:

$$P(X = 3) = \binom{10}{3}(0.5)^3(0.5)^7$$

Step-by-step:

1. $\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$: There are 120 ways to get 3 heads in 10 tosses.
2. $(0.5)^3 = 0.125$: This is the probability of getting 3 heads.
3. $(0.5)^7 = 0.0078125$: Multiply 0.5 by itself 7 times.

Final probability:

$$P(X = 3) = 120 \cdot 0.125 \cdot 0.0078125 = 0.1171875$$

Step 7: Add the probabilities

Now, sum up all the probabilities for $X = 0, 1, 2, 3$:

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

Substitute the values:

$$P(X \leq 3) = 0.0009765625 + 0.009765625 + 0.0439453125 + 0.1171875$$

Perform the addition:

$$P(X \leq 3) = 0.171875$$

Example 3:

A pair of dice is rolled 5 times. If getting the product of 6 is considered as a success. Find the probability of getting at least 4 successes.

Step 1: Understand the Problem

We are rolling **two dice 5 times**. We define **success** as when the product of the numbers on the two dice equals 6. The question asks for the probability of getting **at least 4 successes** in these 5 rolls.

What does "at least 4 successes" mean?

It means either:

- Exactly 4 successes (in 5 trials), or
- Exactly 5 successes (in 5 trials).

So we want to calculate:

$$P(X \geq 4) = P(X = 4) + P(X = 5),$$

where X is the number of successes.

Step 2: Calculate the Probability of Success (p) in One Trial

What is success in one trial?

A trial involves rolling two dice. Success occurs if the product of the numbers on the dice equals 6.

Total Outcomes:

- Rolling two dice gives $6 \times 6 = 36$ possible outcomes.

Favorable Outcomes (Product = 6):

We list the dice rolls where the product equals 6:

1. $(1, 6)$
2. $(2, 3)$
3. $(3, 2)$
4. $(6, 1)$

So there are 4 favorable outcomes.

Probability of success (p):

The probability of success in one trial is:

$$p = \frac{\text{Number of favorable outcomes}}{\text{Total outcomes}} = \frac{4}{36} = \frac{1}{9}.$$

Probability of failure ($1 - p$):

If success happens with probability $p = \frac{1}{9}$, then failure happens with probability:

$$1 - p = 1 - \frac{1}{9} = \frac{8}{9}.$$

Step 3: Use the Binomial Distribution

The number of successes X in $n = 5$ trials follows a **Binomial Distribution.** The probability of $X = k$ successes is given by:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where:

- $n = 5$: Number of trials,
- k: Number of successes,
- $p = \frac{1}{9}$: Probability of success in one trial,
- $1 - p = \frac{8}{9}$: Probability of failure.

Step 4: Calculate $P(X \geq 4)$

We need to calculate:

$$P(X \geq 4) = P(X = 4) + P(X = 5).$$

Step 4.1: Calculate $P(X = 4)$

Substitute $n = 5, k = 4, p = \frac{1}{9}, (1-p) = \frac{8}{9}$ into the Binomial Formula:

$$P(X = 4) = \binom{5}{4} \left(\frac{1}{9}\right)^4 \left(\frac{8}{9}\right)^1.$$

Step-by-step:

1. **Calculate the binomial coefficient ($\binom{5}{4}$):**

$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = \frac{5 \cdot 4!}{4! \cdot 1} = 5.$$

2. **Calculate $\left(\frac{1}{9}\right)^4$:**

$$\left(\frac{1}{9}\right)^4 = \frac{1}{9 \cdot 9 \cdot 9 \cdot 9} = \frac{1}{6561}.$$

3. **Calculate $\left(\frac{8}{9}\right)^1$:**

$$\left(\frac{8}{9}\right)^1 = \frac{8}{9}.$$

4. **Multiply them together:**

$$P(X = 4) = 5 \cdot \frac{1}{6561} \cdot \frac{8}{9} = \frac{40}{59049}.$$

Step 4.2: Calculate $P(X = 5)$

Substitute $n = 5, k = 5, p = \frac{1}{9}, (1 - p) = \frac{8}{9}$ into the Binomial Formula:

$$P(X = 5) = \binom{5}{5} \left(\frac{1}{9}\right)^5 \left(\frac{8}{9}\right)^0.$$

Step-by-step:

1. **Calculate the binomial coefficient ($\binom{5}{5}$):**

$$\binom{5}{5} = \frac{5!}{5!(5-5)!} = 1.$$

2. **Calculate $\left(\frac{1}{9}\right)^5$:**

$$\left(\frac{1}{9}\right)^5 = \frac{1}{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9} = \frac{1}{59049}.$$

3. **Calculate $\left(\frac{8}{9}\right)^0$:**

$$\left(\frac{8}{9}\right)^0 = 1.$$

4. **Multiply them together:**

$$P(X = 5) = 1 \cdot \frac{1}{59049} \cdot 1 = \frac{1}{59049}.$$

Step 4.3: Add $P(X = 4)$ and $P(X = 5)$

Finally, add the probabilities:

$$P(X \geq 4) = P(X = 4) + P(X = 5) = \frac{40}{59049} + \frac{1}{59049}.$$

Step-by-step:

1. Add the numerators:

$$40 + 1 = 41.$$

2. Keep the denominator the same:

$$P(X \geq 4) = \frac{41}{59049}.$$

Step 5: Final Answer

The probability of getting at least 4 successes is:

$$P(X \geq 4) = \frac{41}{59049} \approx 0.000694 \,(\text{approximately } 0.0694\%).$$

Example 4:

Each of five questions on a multiple-choice examination has four choices, only one of which is correct. The student is attempting to guess the answers. The random variable X is the number of questions answered correctly. What is the probability that the student will get

a) exactly three correct answers?

b) at most three correct answers?

c) at least one correct answer?

d) Find Mean and variance.

Problem Context

We have:

1. **5 multiple-choice questions.**
2. Each question has **4 options**, **1 of which is correct.**
3. The student is guessing the answers, which means:
   - Probability of a correct answer (p) = $1/4 = 0.25$,
   - Probability of an incorrect answer ($1-p$) = $3/4 = 0.75$.
4. The problem uses the **binomial distribution**, where:
   - $n = 5$ (number of trials),
   - X is the number of correct answers,
   - Binomial probability formula is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where $\binom{n}{k}$ is the **binomial coefficient** and equals:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Part (a): Exactly 3 Correct Answers

We want to calculate $P(X=3)$, where $k=3$:

$$P(X=3)=\binom{5}{3}(0.25)^3(0.75)^2.$$

Step-by-Step Explanation:

1. **Binomial Coefficient:**
   - $\binom{5}{3}=\frac{5!}{3!(5-3)!}=\frac{5\cdot4\cdot3!}{3!\cdot2\cdot1}=\frac{20}{2}=10.$
2. **Probability of 3 Correct Answers (0.25^3):**
   - $0.25^3=0.25\cdot0.25\cdot0.25=0.015625.$
3. **Probability of 2 Incorrect Answers (0.75^2):**
   - $0.75^2=0.75\cdot0.75=0.5625.$
4. **Combine the Results:**
   - $P(X=3)=10\cdot0.015625\cdot0.5625,$
   - $P(X=3)=0.08789.$

Part (b): At Most 3 Correct Answers

At most 3 correct answers means:

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P($$

We already know $P(X = 3)$, so let's calculate the others.

1. $P(X = 0)$:

$$P(X = 0) = \binom{5}{0}(0.25)^0(0.75)^5.$$

- $\binom{5}{0} = 1$,
- $(0.25)^0 = 1$,
- $(0.75)^5 = 0.75 \cdot 0.75 \cdot 0.75 \cdot 0.75 \cdot 0.75 = 0.2373$,
- $P(X = 0) = 1 \cdot 1 \cdot 0.2373 = 0.2373$.

2. $P(X = 1)$:

$$P(X = 1) = \binom{5}{1}(0.25)^1(0.75)^4.$$

- $\binom{5}{1} = 5$,
- $0.25^1 = 0.25$,
- $(0.75)^4 = 0.3164$,
- $P(X = 1) = 5 \cdot 0.25 \cdot 0.3164 = 0.3955$.

3. $P(X = 2)$:

$$P(X = 2) = \binom{5}{2}(0.25)^2(0.75)^3.$$

- $\binom{5}{2} = 10$,
- $(0.25)^2 = 0.0625$,
- $(0.75)^3 = 0.4219$,
- $P(X = 2) = 10 \cdot 0.0625 \cdot 0.4219 = 0.2637$.

4. **Combine All Probabilities:**

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3),$$

$$P(X \leq 3) = 0.2373 + 0.3955 + 0.2637 + 0.08789 = 0.9844.$$

Part (c): At Least 1 Correct Answer

At least 1 correct means:

$$P(X \geq 1) = 1 - P(X = 0).$$

From part (b), $P(X = 0) = 0.2373$:

$$P(X \geq 1) = 1 - 0.2373 = 0.7627.$$

Part (d): Mean and Variance

For a binomial distribution:

$$\mu = n \cdot p, \quad \sigma^2 = n \cdot p \cdot (1 - p).$$

1. **Mean (μ):**

$$\mu = 5 \cdot 0.25 = 1.25.$$

2. **Variance (σ^2):**

$$\sigma^2 = 5 \cdot 0.25 \cdot 0.75 = 0.9375.$$

Example 5:

- Find the binomial distribution if the mean is 6 and variance is 4.

Step 1: Understand the Binomial Distribution

The binomial distribution models the number of successes in n trials, where the probability of success in each trial is p. Two important characteristics of the binomial distribution are:

1. **Mean (μ):** This tells us the expected number of successes in n trials. The formula is:

$$\mu = n \cdot p,$$

   where n is the total number of trials, and p is the probability of success for each trial.

2. **Variance (σ^2):** This measures the spread of the distribution and is given by:

$$\sigma^2 = n \cdot p \cdot (1 - p).$$

In this problem, we are **given**:

$$\mu = 6 \quad \text{and} \quad \sigma^2 = 4.$$

We need to find n (number of trials) and p (probability of success).

Step 2: Write the Formulas and Equations

We'll use the two given formulas:

1. Mean:

$$n \cdot p = 6 \tag{1}$$

   This comes from the formula for mean ($\mu = n \cdot p$).

2. Variance:

$$n \cdot p \cdot (1 - p) = 4 \tag{2}$$

   This comes from the formula for variance ($\sigma^2 = n \cdot p \cdot (1 - p)$).

These two equations will allow us to solve for n and p.

Step 3: Express p in Terms of n

From Equation (1):

$$p = \frac{6}{n}.$$

This expresses the probability p in terms of n, which we will use in the variance formula.

Step 4: Substitute $p = \frac{6}{n}$ into the Variance Formula

Now substitute $p = \frac{6}{n}$ into Equation (2):

$$n \cdot \frac{6}{n} \cdot \left(1 - \frac{6}{n}\right) = 4.$$

1. The first term $n \cdot \frac{6}{n}$ simplifies to 6:

$$6 \cdot \left(1 - \frac{6}{n}\right) = 4.$$

2. Expand the term inside the parentheses:

$$6 \cdot \left(1 - \frac{6}{n}\right) = 6 - \frac{36}{n}.$$

So the equation becomes:

$$6 - \frac{36}{n} = 4.$$

Step 5: Solve for n

Rearrange the equation:

$$\frac{36}{n} = 2.$$

Solve for n:

$$n = \frac{36}{2} = 18.$$

Thus, the total number of trials is $n = 18$.

Step 6: Solve for p

Now that we know $n = 18$, substitute it back into Equation (1):

$$n \cdot p = 6.$$

Substitute $n = 18$:

$$18 \cdot p = 6.$$

Solve for p:

$$p = \frac{6}{18} = \frac{1}{3}.$$

Thus, the probability of success in each trial is $p = \frac{1}{3}$.

Step 7: Verify the Variance

To ensure our values for n and p are correct, substitute them into the variance formula:

$$\sigma^2 = n \cdot p \cdot (1 - p).$$

Substitute $n = 18$ and $p = \frac{1}{3}$:

$$\sigma^2 = 18 \cdot \frac{1}{3} \cdot \left(1 - \frac{1}{3}\right).$$

1. Simplify $18 \cdot \frac{1}{3}$:

$$18 \cdot \frac{1}{3} = 6.$$

2. Simplify $1 - \frac{1}{3}$:

$$1 - \frac{1}{3} = \frac{2}{3}.$$

3. Multiply everything together:

$$\sigma^2 = 6 \cdot \frac{2}{3} = 4.$$

The variance matches the given value, so our solution is correct.

Final Answer

The binomial distribution has:

$$n = 18 \quad \text{and} \quad p = \frac{1}{3}.$$

The binomial distribution is:

$$X \sim \text{Binomial}(n = 18, p = \frac{1}{3}).$$

Example 6:

In a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.

Step 1: Recognize the Distribution

We are told that the mean number of defective parts in a sample of 20 is $\lambda = 2$. Since we are dealing with the **number of occurrences of defects** in a fixed number of samples (discrete outcomes), the **Poisson distribution** is appropriate.

The Poisson probability formula is:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Here:

- X is the number of defective parts,
- λ is the average number of defects in a sample,
- e is a constant ($e \approx 2.718$),
- $x!$ is the factorial of x, e.g., $3! = 3 \times 2 \times 1 = 6$.

Step 2: Interpret the Question

We need to find how many out of **1000 samples** would contain **at least 3 defective parts**. Mathematically, this means finding $P(X \geq 3)$.

The complement rule makes this easier:

$$P(X \geq 3) = 1 - P(X \leq 2)$$

Where $P(X \leq 2)$ is the probability of having **0, 1, or 2 defective parts**.

Step 3: Calculate $P(X \le 2)$

To calculate $P(X \le 2)$, we need to sum the probabilities of $X = 0$, $X = 1$, and $X =$ compute each individually using the Poisson formula.

Case 1: $P(X = 0)$

Substitute $x = 0$ and $\lambda = 2$ into the formula:

$$P(X = 0) = \frac{2^0 e^{-2}}{0!}$$

- $2^0 = 1,$
- $0! = 1,$
- $e^{-2} \approx 0.1353.$

So:

$$P(X = 0) = \frac{1 \cdot 0.1353}{1} = 0.1353$$

Case 2: $P(X = 1)$

Substitute $x = 1$ and $\lambda = 2$ into the formula:

$$P(X = 1) = \frac{2^1 e^{-2}}{1!}$$

- $2^1 = 2,$
- $1! = 1,$
- $e^{-2} \approx 0.1353.$

So:

$$P(X = 1) = \frac{2 \cdot 0.1353}{1} = 0.2707$$

Case 3: $P(X = 2)$

Substitute $x = 2$ and $\lambda = 2$ into the formula:

$$P(X = 2) = \frac{2^2 e^{-2}}{2!}$$

- $2^2 = 4,$
- $2! = 2,$
- $e^{-2} \approx 0.1353.$

So:

$$P(X = 2) = \frac{4 \cdot 0.1353}{2} = 0.2707$$

Sum Up $P(X \leq 2)$

Now add these probabilities:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$P(X \leq 2) = 0.1353 + 0.2707 + 0.2707 = 0.6767$$

Step 4: Calculate $P(X \geq 3)$

Using the complement rule:

$$P(X \geq 3) = 1 - P(X \leq 2)$$

$$P(X \geq 3) = 1 - 0.6767 = 0.3233$$

Step 5: Scale to 1000 Samples

The probability $P(X \geq 3)$ represents the fraction of samples with at least 3 defective parts. To find the expected number of such samples out of 1000:

$$\text{Expected number} = P(X \geq 3) \times 1000$$

$$\text{Expected number} = 0.3233 \times 1000 = 323.3$$

Final Answer

The expected number of samples containing at least 3 defective parts is:

$$\boxed{323 \text{ samples.}}$$

Example 6:

Telephone calls enter a college switchboard on the average of two every three minutes. What is the probability of 5 or more calls arriving in a 9-minute period?

Step 1: Identify the Distribution Type

This problem involves events (telephone calls) happening over time. Since:

- The calls occur randomly and independently,
- The average number of calls in a given time period is constant,

We use the **Poisson distribution.**

Step 2: Formula for Poisson Distribution

The formula is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where:

- $P(X = k)$: Probability of exactly k events in a time interval.
- λ: Expected number of events in the given interval (average rate of occurrence).
- k: The specific number of events for which we calculate the probability.
- e: A mathematical constant (approximately 2.718).

Step 3: Problem Details

1. The average number of calls is **2 calls every 3 minutes.**
2. The time interval of interest is **9 minutes.**
3. The question asks for the probability of **5 or more calls.**

Step 4: Compute λ

The **expected number of events (calls)** in 9 minutes is given by:

$$\lambda = (\text{Average calls in 3 minutes}) \times (\text{Number of 3-minute intervals in 9 minutes})$$

$$\lambda = 2 \times 3 = 6$$

So, in a 9-minute period, the expected number of calls is $\lambda = 6$.

Step 5: What We Need to Find

We are tasked with finding $P(X \geq 5)$, the probability of 5 or more calls. Using the complement rule:

$$P(X \geq 5) = 1 - P(X \leq 4)$$

So, instead of calculating probabilities for 5, 6, 7,... (which is tedious), we calculate $P(X \leq 4)$ and subtract it from 1.

Step 6: Compute $P(X \leq 4)$

To compute $P(X \leq 4)$, we calculate the probabilities of $X = 0, 1, 2, 3,$ and 4 using the Poisson formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Let's compute them step-by-step with $\lambda = 6$:

Step 6.1: Calculate $P(X = 0)$

$$P(X = 0) = \frac{6^0 e^{-6}}{0!} = \frac{1 \cdot e^{-6}}{1} = e^{-6}$$

Step 6.2: Calculate $P(X = 1)$

$$P(X = 1) = \frac{6^1 e^{-6}}{1!} = \frac{6 \cdot e^{-6}}{1} = 6e^{-6}$$

Step 6.3: Calculate $P(X = 2)$

$$P(X = 2) = \frac{6^2 e^{-6}}{2!} = \frac{36 \cdot e^{-6}}{2} = 18e^{-6}$$

Step 6.4: Calculate $P(X = 3)$

$$P(X = 3) = \frac{6^3 e^{-6}}{3!} = \frac{216 \cdot e^{-6}}{6} = 36e^{-6}$$

Step 6.5: Calculate $P(X = 4)$

$$P(X = 4) = \frac{6^4 e^{-6}}{4!} = \frac{1296 \cdot e^{-6}}{24} = 54e^{-6}$$

Step 7: Sum $P(X \leq 4)$

Now, sum up all the probabilities:

$$P(X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

Substitute the values:

$$P(X \leq 4) = e^{-6}(1 + 6 + 18 + 36 + 54)$$
$$P(X \leq 4) = e^{-6} \cdot 115$$

Step 8: Approximate $P(X \leq 4)$

From a table or calculator, $e^{-6} \approx 0.002478$:

$$P(X \leq 4) = 0.002478 \cdot 115 \approx 0.28497$$

Step 9: Compute $P(X \geq 5)$

Now, use the complement rule:

$$P(X \geq 5) = 1 - P(X \leq 4)$$
$$P(X \geq 5) = 1 - 0.28497 \approx 0.715$$

Final Answer

The probability of getting **5 or more calls** in 9 minutes is approximately:

P(X \geq 5) \approx 0.715 \, \text{(or 71.5%)}.

Example 7:

If 3% of electronic units manufactured by a company are defective. Find the probability that in a sample of 200 units, less than 2 bulbs are defective.

Step 1: Understand the problem

- A company manufactures electronic units, and 3% of the units are defective. This means the probability of a single unit being defective is $p = 0.03$.
- A sample of 200 units ($n = 200$) is randomly chosen.
- We need to calculate the probability that **less than 2 units are defective** in this sample. In mathematical terms:

$$P(X < 2) = P(X = 0) + P(X = 1)$$

Step 2: Why Poisson approximation is used?

The problem involves a **binomial distribution** since each unit has a probability $p = 0.03$ of being defective, and the number of defective units in 200 trials is:

$$X \sim \text{Binomial}(n = 200, p = 0.03)$$

For the binomial distribution, the formula for probability is:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

However:

- $n = 200$ is large,
- $p = 0.03$ is small.

In such cases, the **Poisson approximation** simplifies the calculations:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where:

- $\lambda = n \cdot p$.

Step 3: Compute λ

For the Poisson approximation, the parameter λ is the expected number of defective units in the sample:

$$\lambda = n \cdot p$$

Substitute $n = 200$ and $p = 0.03$:

$$\lambda = 200 \cdot 0.03 = 6$$

So, the problem now reduces to finding probabilities using a Poisson distribution with:

$$X \sim \text{Poisson}(\lambda = 6)$$

Step 4: Break down $P(X < 2)$

We are asked to find the probability of **less than 2 defective units**. This means:

$$P(X < 2) = P(X = 0) + P(X = 1)$$

Step 5: Use the Poisson formula

For the Poisson distribution, the probability mass function is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Step 5.1: Calculate $P(X = 0)$

Substitute $k = 0$ and $\lambda = 6$ into the formula:

$$P(X = 0) = \frac{6^0 e^{-6}}{0!}$$

- $6^0 = 1$,
- $0! = 1$.

So:

$$P(X = 0) = e^{-6}$$

Step 5.2: Calculate $P(X = 1)$

Substitute $k = 1$ and $\lambda = 6$ into the formula:

$$P(X = 1) = \frac{6^1 e^{-6}}{1!}$$

- $6^1 = 6$,
- $1! = 1$.

So:

$$P(X = 1) = 6e^{-6}$$

Step 6: Add the probabilities

Now, sum $P(X=0)$ and $P(X=1)$:

$$P(X<2)=P(X=0)+P(X=1)$$

Substitute the values:

$$P(X<2)=e^{-6}+6e^{-6}$$

Factor out e^{-6}:

$$P(X<2)=e^{-6}(1+6)$$

Step 7: Approximate e^{-6}

From a calculator or a table, the value of e^{-6} (the exponential function) is approximately:

$$e^{-6}\approx 0.002478$$

Substitute this into the formula:

$$P(X<2)=0.002478\cdot(1+6)$$

$$P(X<2)=0.002478\cdot 7$$

$$P(X<2)\approx 0.017346$$

Step 8: Final Answer

The probability that less than 2 units are defective is:

P(X < 2) \approx 0.0173 \, \text{(or 1.73%)}.

Example 8:

The number of customers arriving at a cafeteria at an average rate of 0.3 per minute.

(a) Find the probability that exactly 2 customers arrive in a 10-minute span.

(b) Find the probability that 2 or more customers arrive in a 10-minute span.

(c) Find the probability that exactly one customer arrives in a 5-minute span and one customer arrives in the next 5-minute span.

Problem Recap

We are dealing with a **Poisson distribution problem**, where customer arrivals follow a Poisson process. The **average rate** (λ) is **0.3 customers per minute.**

For a time span of t minutes, the mean number of arrivals (λ_t) is:

$$\lambda_t = \lambda \cdot t$$

Part (a): Probability that exactly 2 customers arrive in a 10-minute span

1. **Determine the time span and mean (λ_t):**
   - The time span is 10 minutes.
   - $\lambda_t = 0.3 \cdot 10 = 3.$

   This means, on average, we expect **3 customers** to arrive in 10 minutes.
2. **Poisson Probability Formula:** The probability of exactly k arrivals is given by:

$$P(X = k) = \frac{\lambda_t^k e^{-\lambda_t}}{k!}$$

3. **Plug in values for $k = 2$:**
   - $\lambda_t = 3$ (mean number of arrivals in 10 minutes),
   - $k = 2$ (exactly 2 customers).

   Substituting these values:

$$P(X = 2) = \frac{3^2 e^{-3}}{2!}$$

4. **Simplify the terms:**
   - $3^2 = 9,$
   - $2! = 2,$
   - Approximation for e^{-3}: $e^{-3} \approx 0.0498.$

   Substituting:

$$P(X = 2) = \frac{9 \cdot 0.0498}{2} = \frac{0.4482}{2} = 0.2241$$

5. **Conclusion:** The probability of exactly 2 customers arriving in 10 minutes is:
   P(X = 2) \approx 0.2241 \, \text{(or 22.41%)}.

Part (b): Probability that 2 or more customers arrive in a 10-minute span

1. **Break down the problem:** "2 or more customers" means:

$$P(X \geq 2) = 1 - P(X < 2)$$

   Where:

$$P(X < 2) = P(X = 0) + P(X = 1)$$

2. **Find $P(X = 0)$:** Using the Poisson formula:

$$P(X = 0) = \frac{3^0 e^{-3}}{0!} = \frac{1 \cdot e^{-3}}{1} = e^{-3}$$

   Approximation for e^{-3}:

$$P(X = 0) \approx 0.0498$$

3. **Find $P(X = 1)$:** Using the Poisson formula:

$$P(X = 1) = \frac{3^1 e^{-3}}{1!} = \frac{3 \cdot e^{-3}}{1} = 3 \cdot e^{-3}$$

   Substituting $e^{-3} \approx 0.0498$:

$$P(X = 1) \approx 3 \cdot 0.0498 = 0.1494$$

4. **Find $P(X < 2)$:** Add the probabilities:

$$P(X < 2) = P(X = 0) + P(X = 1) = 0.0498 + 0.1494 = 0.1992$$

5. **Find $P(X \geq 2)$:** Use the complement rule:

$$P(X \geq 2) = 1 - P(X < 2) = 1 - 0.1992 = 0.8008$$

6. **Conclusion:** The probability of 2 or more customers arriving in 10 minutes is:
   P(X \geq 2) \approx 0.8008 \, \text{(or 80.08%)}.

Part (c): Probability that exactly one customer arrives in each 5-minute span

1. **Understand the situation:** The 10-minute span is divided into **two independent 5-minute intervals.** The mean number of arrivals for each 5-minute span is:

$$\lambda_t = 0.3 \cdot 5 = 1.5$$

The probability of **exactly 1 customer** in each span is independent.

2. **Find $P(X = 1)$ for a single span:** Using the Poisson formula:

$$P(X = 1) = \frac{\lambda_t^1 e^{-\lambda_t}}{1!} = \frac{1.5^1 e^{-1.5}}{1} = 1.5 \cdot e^{-1.5}$$

Approximation for $e^{-1.5}$: $e^{-1.5} \approx 0.2231$.

Substituting:

$$P(X = 1) \approx 1.5 \cdot 0.2231 = 0.3347$$

3. **Combine the probabilities for both spans:** Since the events are independent:

$$P(1 \text{ customer in each span}) = P(X_1 = 1) \cdot P(X_2 = 1)$$

Substitute:

$$P(1 \text{ customer in each span}) = 0.3347 \cdot 0.3347 = 0.1120$$

4. **Conclusion:** The probability of exactly one customer arriving in each 5-minute span is:
P(\text{1 customer in each span}) \approx 0.1120 \, \text{(or 11.20%)}.

Summary of Results

1. (a) P(X = 2) \approx 0.2241 \, \text{(22.41%)},
2. (b) P(X \geq 2) \approx 0.8008 \, \text{(80.08%)},
3. (c) P(\text{1 customer in each span}) \approx 0.1120 \, \text{(11.20%)}.

Example 9:

It is known that 5% of the books bound at a certain bindery have defective bindings. Find the probability that 2 of 100 books bound by this bindery will have defective bindings using

(a)The formula for the binomial distribution;

(b)The Poisson approximation to the binomial distribution

Problem Information:

- **Defective probability per unit** (p): 0.05 (5% defective bindings).
- **Sample size** (n): 100 books.
- **We need** $P(X = 2)$: The probability that 2 books are defective out of 100 books.

(a) Using the Binomial Distribution Formula:

The **binomial distribution formula** is:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Where:

- $n = 100$: Total number of trials (books).
- $k = 2$: Number of successes (defective books).
- $p = 0.05$: Probability of success (defective bindings).
- $1 - p = 0.95$: Probability of failure (not defective).

Step 1: Compute $\binom{100}{2}$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

For $n = 100$ and $k = 2$:

$$\binom{100}{2} = \frac{100 \cdot 99}{2!} = \frac{100 \cdot 99}{2} = 4950$$

Step 2: Compute $p^k = (0.05)^2$:

$$p^k = (0.05)^2 = 0.05 \cdot 0.05 = 0.0025$$

Step 3: Compute $(1-p)^{n-k} = (0.95)^{98}$:

Using a scientific calculator or approximation:

$$0.95^{98} \approx 0.0059$$

Step 4: Combine all components:

Now substitute everything into the formula:

$$P(X=2) = \binom{100}{2} p^k (1-p)^{n-k}$$

$$P(X=2) = 4950 \cdot 0.0025 \cdot 0.0059$$

Multiply step-by-step:

1. $4950 \cdot 0.0025 = 12.375$,
2. $12.375 \cdot 0.0059 \approx 0.081$.

So:

$$P(X=2) \approx 0.081$$

(b) Using Poisson Approximation:

For the **Poisson approximation**, the formula is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where:

- $\lambda = n \cdot p = 100 \cdot 0.05 = 5,$
- $k = 2,$
- $e \approx 2.718.$

Step 1: Substitute values into the formula:

$$P(X = 2) = \frac{5^2 e^{-5}}{2!}$$

Step 2: Simplify the numerator 5^2:

$$5^2 = 25$$

Step 3: Compute $2!$:

$$2! = 2 \cdot 1 = 2$$

Step 4: Compute e^{-5}:

Using a calculator:

$$e^{-5} = \frac{1}{e^5} \approx \frac{1}{148.413} \approx 0.0067$$

Step 5: Combine all terms:

Now substitute everything into the formula:

$$P(X = 2) = \frac{25 \cdot 0.0067}{2}$$

Multiply step-by-step:

1. $25 \cdot 0.0067 = 0.1675,$
2. $\frac{0.1675}{2} = 0.084.$

So:

$$P(X = 2) \approx 0.084$$

Final Results:

- **Binomial distribution:** $P(X = 2) = 0.081,$
- **Poisson approximation:** $P(X = 2) = 0.084.$

Example 10:

Given that a switch board of a consultant's office receives on the average 0.6 calls per minute, find the probabilities that

(a) in a given minute there will be at least 1 call;

(b) in a 4-minute interval there will be at least 3 calls.

Given Data:

1. The average number of calls per minute is $\lambda = 0.6$.
2. The Poisson formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

(a) Probability that in a given minute there will be at least 1 call ($P(X \geq 1)$):

The probability of at least 1 call is calculated as:

$$P(X \geq 1) = 1 - P(X = 0)$$

Step 1: Find $P(X = 0)$:

Substituting into the Poisson formula:

$$P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!}$$

- Here, $\lambda = 0.6$, $k = 0$, and $0! = 1$.
- $\lambda^0 = (0.6)^0 = 1$.
- $e^{-\lambda} = e^{-0.6}$.

Calculation:

$$P(X = 0) = \frac{1 \cdot e^{-0.6}}{1} = e^{-0.6}$$

Using a calculator:

$$e^{-0.6} \approx 0.5488$$

So:

$$P(X = 0) \approx 0.5488$$

Step 2: Calculate $P(X \geq 1)$:

Using the formula:

$$P(X \geq 1) = 1 - P(X = 0)$$

Substitute $P(X = 0) \approx 0.5488$:

$$P(X \geq 1) = 1 - 0.5488 = 0.4512$$

(b) Probability that in a 4-minute interval there will be at least 3 calls ($P(X \geq 3)$):

For a 4-minute interval:

$$\lambda = 4 \cdot 0.6 = 2.4$$

The probability of at least 3 calls is:

$$P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$$

Step 1: Calculate $P(X = 0)$:

Using the Poisson formula:

$$P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!}$$

- Here, $\lambda = 2.4$, $k = 0$, and $0! = 1$.
- $\lambda^0 = (2.4)^0 = 1$.
- $e^{-\lambda} = e^{-2.4}$.

Calculation:

$$P(X = 0) = \frac{1 \cdot e^{-2.4}}{1} = e^{-2.4}$$

Using a calculator:

$$e^{-2.4} \approx 0.0907$$

So:

$$P(X = 0) \approx 0.0907$$

Step 2: Calculate $P(X = 1)$:

Using the Poisson formula:

$$P(X=1) = \frac{\lambda^1 e^{-\lambda}}{1!}$$

- Here, $\lambda = 2.4$, $k = 1$, and $1! = 1$.
- $\lambda^1 = (2.4)^1 = 2.4$.
- $e^{-\lambda} = e^{-2.4} \approx 0.0907$.

Calculation:

$$P(X=1) = \frac{2.4 \cdot 0.0907}{1} = 2.4 \cdot 0.0907 \approx 0.2177$$

Step 3: Calculate $P(X = 2)$:

Using the Poisson formula:

$$P(X=2) = \frac{\lambda^2 e^{-\lambda}}{2!}$$

- Here, $\lambda = 2.4$, $k = 2$, and $2! = 2$.
- $\lambda^2 = (2.4)^2 = 5.76$.
- $e^{-\lambda} = e^{-2.4} \approx 0.0907$.

Calculation:

$$P(X=2) = \frac{5.76 \cdot 0.0907}{2} = \frac{0.5224}{2} \approx 0.2612$$

Step 4: Calculate $P(X \geq 3)$:

Using the formula:

$$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$

Substitute the values:

$$P(X \geq 3) = 1 - 0.0907 - 0.2177 - 0.2612$$

Simplify:

$$P(X \geq 3) = 1 - 0.5696 = 0.4304$$

Final Answer (b):

$$P(X \geq 3) \approx 0.4304$$

Poisson Distribution: Applications

The **Poisson distribution** is used to model the probability of a given number of events occurring in a fixed interval of time or space, assuming the events happen independently of each other and at a constant average rate.

Common Applications of Poisson Distribution

1. **Number of Calls at a Call Center**
   - Example: The average number of calls received at a call center is 5 per hour. The Poisson distribution can be used to find the probability of receiving exactly 7 calls in an hour.
2. **Machine Failures**
   - Example: A machine breaks down on average 2 times per month. The Poisson distribution can help estimate the probability of the machine breaking down 4 times in a given month.
3. **Accidents**
   - Example: The average number of car accidents occurring at an intersection is 3 per week. The Poisson distribution can calculate the probability of observing exactly 5 accidents in a week.
4. **Customers in a Store**
   - Example: A store observes 10 customers arriving per hour on average. The Poisson distribution can determine the probability of exactly 15 customers arriving in a given hour.
5. **Biological Applications**
   - Example: Modeling the number of mutations in a given length of DNA sequence.

Example to Illustrate

Scenario

A library receives an average of 3 requests for a specific book every day. Use the Poisson distribution to:

1. Find the probability of receiving exactly 5 requests in a day.
2. Find the probability of receiving fewer than 2 requests in a day.

Step 1: Identify Parameters

- The average number of events (λ) = 3 (requests per day).
- The Poisson formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Step 2: Solve for (1) Exactly 5 Requests

Here, $k = 5$ and $\lambda = 3$.

Substitute into the formula:

$$P(X = 5) = \frac{3^5 e^{-3}}{5!}$$

- Calculate $3^5 = 243$.
- Calculate $e^{-3} \approx 0.0498$.
- Calculate $5! = 120$.

$$P(X = 5) = \frac{243 \cdot 0.0498}{120} \approx \frac{12.1}{120} \approx 0.101$$

So, the probability of exactly 5 requests is approximately **0.101**.

Step 3: Solve for (2) Fewer than 2 Requests

This means $P(X < 2) = P(X = 0) + P(X = 1)$.

- For $P(X = 0)$:

$$P(X = 0) = \frac{3^0 e^{-3}}{0!} = e^{-3} \approx 0.0498$$

- For $P(X = 1)$:

$$P(X = 1) = \frac{3^1 e^{-3}}{1!} = \frac{3 \cdot 0.0498}{1} \approx 0.1494$$

Add them together:

$$P(X < 2) = 0.0498 + 0.1494 = 0.1992$$

So, the probability of fewer than 2 requests is approximately **0.199**.

Key Insights

- The Poisson distribution is ideal for modeling rare events.
- Its simplicity lies in having only one parameter (λ), the average rate of occurrence.
- The events must occur independently.

Problem 11:

If a random variable has the standard normal distribution, find the probability that it will take on a value

a)less than 1.75 = P(Z < 1.75) = F(1.75) = 0.9599

b)less than -1.25 = P(Z < -1.25) = F(-1.25) = 1 –F(1.25) = 0.1056

c)greater than 2.06 = P(Z >2.06) = 1 – F(2.06) = 0.0197

d)greater than -1.82 = P(Z > -1.82) = 1- F(-1.82) =1-0.0344=0.9656

P(Z > -1.82) = 1 –F(-1.82) = 1 –(1 – F(1.82) = F(1.82) = 0.9656

Key Formula and Table Use

1. The **standard normal distribution table (Z-table)** provides cumulative probabilities $F(z)$ for $P(Z < z)$.
2. **Basic Relationships:**
   - $P(Z < z) = F(z)$: Probability of being less than z.
   - $P(Z > z) = 1 - F(z)$: Probability of being greater than z.
   - For negative z, use symmetry: $F(-z) = 1 - F(z)$.

---

a) Less than 1.75: $P(Z < 1.75)$

Here, we are asked for the cumulative probability $F(1.75)$:

$$P(Z < 1.75) = F(1.75)$$

From the Z-table:

- Look for $Z = 1.75$, the value is **0.9599**.

Thus:

$$P(Z < 1.75) = 0.9599$$

Interpretation: The probability of the standard normal variable Z being less than 1.75 is 0.9599 or **95.99%**.

b) Less than -1.25: $P(Z < -1.25)$

Using symmetry of the standard normal distribution:

$$F(-z) = 1 - F(z)$$

Thus:

$$P(Z < -1.25) = F(-1.25) = 1 - F(1.25)$$

From the Z-table:

- Look for $Z = 1.25$, the value is **0.8944**.

Calculate:

$$P(Z < -1.25) = 1 - 0.8944 = 0.1056$$

Interpretation: The probability of Z being less than -1.25 is 0.1056 or **10.56%**.

---

c) Greater than 2.06: $P(Z > 2.06)$

Using the complement rule:

$$P(Z > z) = 1 - F(z)$$

For $Z = 2.06$:

- From the Z-table, $F(2.06) = 0.9803$.

Calculate:

$$P(Z > 2.06) = 1 - 0.9803 = 0.0197$$

Interpretation: The probability of Z being greater than 2.06 is 0.0197 or **1.97%**.

d) Greater than -1.82: $P(Z > -1.82)$

We use the complement and symmetry properties:

$$P(Z > -1.82) = 1 - F(-1.82)$$

First, find $F(-1.82)$ using symmetry:

$$F(-1.82) = 1 - F(1.82)$$

For $Z = 1.82$:

- From the Z-table, $F(1.82) = 0.9656$.

Thus:

$$F(-1.82) = 1 - 0.9656 = 0.0344$$

Now calculate:

$$P(Z > -1.82) = 1 - F(-1.82) = 1 - 0.0344 = 0.9656$$

Interpretation: The probability of Z being greater than -1.82 is 0.9656 or **96.56%**.

Summary of Results

- a) $P(Z < 1.75) = 0.9599$
- b) $P(Z < -1.25) = 0.1056$
- c) $P(Z > 2.06) = 0.0197$
- d) $P(Z > -1.82) = 0.9656$

Problem 12:

Scores on a particular test are normally distributed with a mean of 30 and standard deviation of 4. What is the probability of anyone scoring less than 40?

Step 1: Understanding the Problem

We are given:

- μ (mean) = 30,
- σ (standard deviation) = 4,
- X (value of interest) = 40.

We need to find $P(X < 40)$, the probability of scoring less than 40, where scores are normally distributed.

Step 2: Convert $X = 40$ to a Standard Z-score

The Z-score formula is:

$$Z = \frac{X - \mu}{\sigma}$$

Substitute the given values:

- $X = 40$,
- $\mu = 30$,
- $\sigma = 4$.

Perform the calculation:

$$Z = \frac{40 - 30}{4} = \frac{10}{4} = 2.5$$

So, the Z-score for $X = 40$ is $Z = 2.5$. This means that a score of 40 is **2.5 standard deviations above the mean.**

Step 3: Use the Z-table to Find $P(Z < 2.5)$

What the Z-table Represents

- The Z-table provides cumulative probabilities for standard normal distribution values, i.e., probabilities of Z being less than a certain value.
- For $Z = 2.5$, the Z-table gives $F(2.5) = 0.9938$, which means:

$$P(Z < 2.5) = 0.9938$$

Why This is Useful

The Z-table tells us that the probability of Z being less than 2.5 is 99.38%.

Step 4: Interpret the Result

Since $Z = 2.5$ corresponds to $X = 40$, the probability $P(X < 40)$ is the same as $P(Z < 2.5)$.

Thus:

$$P(X < 40) = P(Z < 2.5) = 0.9938$$

This means there is a 99.38% chance that someone scores less than 40 on this test.

Expanded Explanation for Each Step

1. Z-Score Calculation

- **Why Z-score is Needed:** Z-scores standardize different normal distributions, so we can use a universal table (the Z-table).
- **How Calculations Work:** Subtract the mean (μ) from X to find how far X is from the center of the distribution. Divide by the standard deviation (σ) to scale this distance in terms of standard deviations.

2. Look Up Z in the Table

- The Z-table is a cumulative probability table that gives the total area under the curve to the left of a given Z-score.
- For $Z = 2.5$, we find 0.9938 in the table. This means 99.38% of the data lies to the left of $Z = 2.5$.

3. Interpretation

- The cumulative probability 0.9938 means $P(X < 40) = 0.9938$. In other words, about 99.38% of scores are below 40.

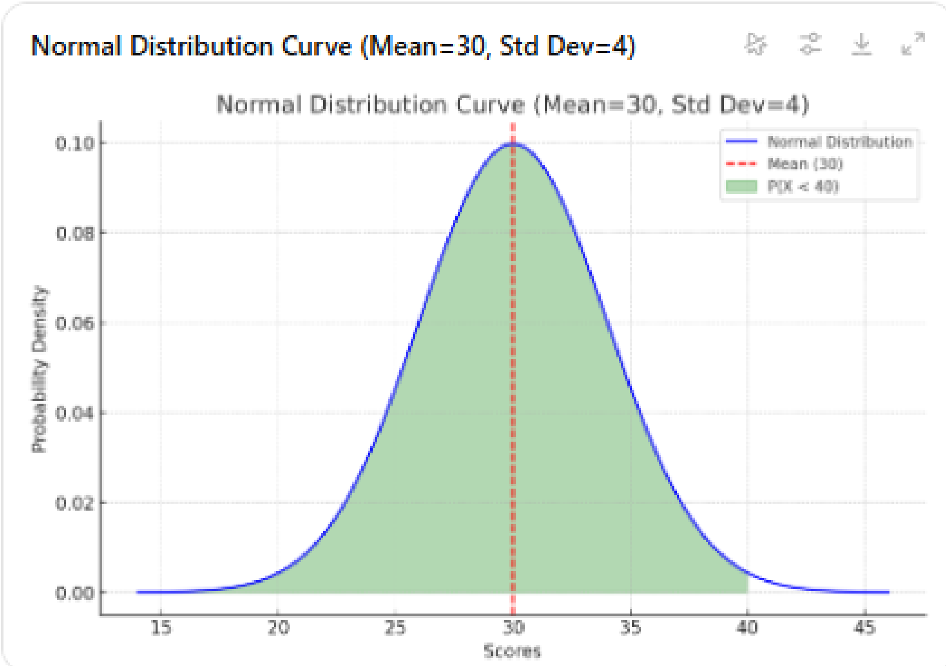
Visual Aid (Optional)

If you plot the normal distribution curve:

- Mark $X = 30$ as the center (mean).
- Mark $X = 40$ at $Z = 2.5$, 2.5 standard deviations above the mean.
- The shaded area to the left of $Z = 2.5$ (up to $X = 40$) represents $P(X < 40) = 0.9938$.

Final Answer

$$P(X < 40) = 0.9938$$



Here is the normal distribution curve for scores with a mean of 30 and a standard deviation of 4. The shaded green area represents the probability of scoring less than 40.

The red dashed line shows the mean, and the curve demonstrates how scores are distributed around the mean. The probability of scoring less than 40 corresponds to the cumulative area under the curve to the left of $x = 40$. [>-]

Problem 13:

Scores on a particular test are normally distributed with a mean of 30 and standard deviation of 4. What is the probability of anyone scoring greater than 38?

Step 1: Define the problem

We are looking for $P(X > 38)$, the probability that a score is greater than 38.

Step 2: Convert $X = 38$ to the standard normal variable Z

The standard normal variable Z is calculated using the formula:

$$Z = \frac{X - \mu}{\sigma}$$

Here:

- $X = 38$ (the given value for which we're calculating the probability),
- $\mu = 30$ (the mean of the distribution),
- $\sigma = 4$ (the standard deviation).

Substitute the values:

$$Z = \frac{38 - 30}{4} = \frac{8}{4} = 2$$

Thus, $Z = 2$.

Step 3: Find the cumulative probability for $Z = 2$

Using the standard normal distribution table, find $P(Z < 2)$. From the table:

$$P(Z < 2) = 0.9772$$

This means the probability of a score being less than 38 is 0.9772.

Step 4: Calculate $P(Z > 2)$

The total probability under the curve is 1. Therefore:

$$P(Z > 2) = 1 - P(Z < 2)$$

Substitute $P(Z < 2) = 0.9772$:

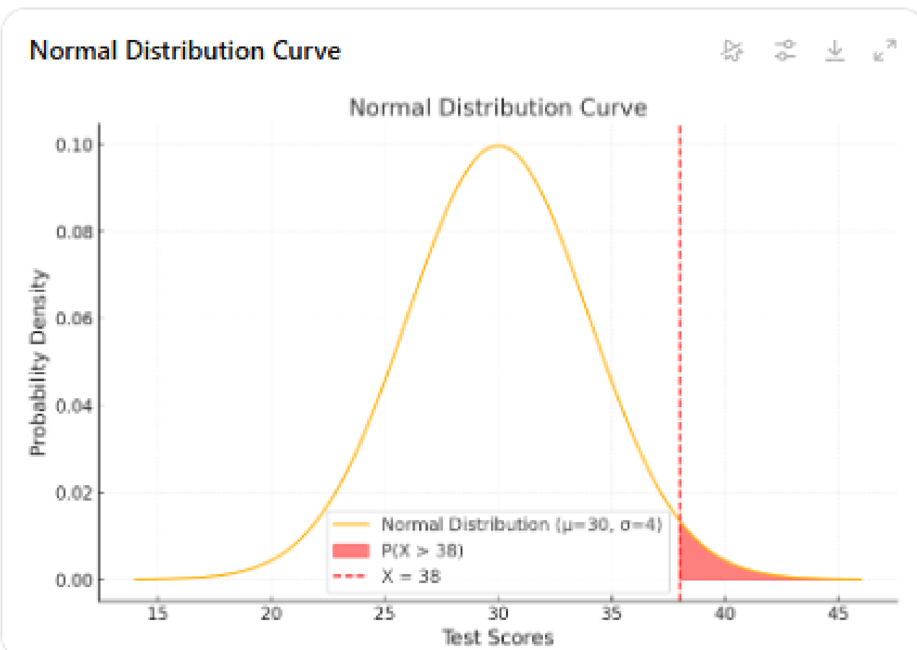
$$P(Z > 2) = 1 - 0.9772 = 0.0228$$

Step 5: Interpret the result

The probability of scoring greater than 38 is 0.0228, or 2.28%.

Final Answer:

The probability of scoring greater than 38 is **2.28%**.



Here is the normal distribution curve for test scores with a mean of 30 and a standard deviation of 4. The shaded red region represents the probability $P(X > 38)$, and the dashed red line indicates the threshold score of 38. [>-]

Problem 14:

Problem Restatement:

We are given:

- Mean (μ) = 75
- Standard deviation (σ) = 4
- Test scores follow a **normal distribution**. We need to find $P(70 \leq X \leq 80)$, i.e., the probability of a student scoring between 70 and 80.

Step 1: Standardize the Scores (Find Z-scores)

To use the standard normal table (Z-table), we need to convert the test scores into **Z-scores** using the formula:

$$Z = \frac{X - \mu}{\sigma}$$

1. For $X = 70$:

$$Z_1 = \frac{70 - 75}{4} = \frac{-5}{4} = -1.25$$

2. For $X = 80$:

$$Z_2 = \frac{80 - 75}{4} = \frac{5}{4} = 1.25$$

The Z-scores represent how many standard deviations the values (70 and 80) are away from the mean.

Step 2: Use the Standard Normal Table (Z-table)

The Z-table gives the **cumulative probability** $P(Z < z)$, which is the probability that the random variable Z is less than the given Z-value.

1. Find $P(Z < -1.25)$: From the Z-table, $P(Z < -1.25) = 0.1056$.
2. Find $P(Z < 1.25)$: From the Z-table, $P(Z < 1.25) = 0.8944$.

Step 3: Calculate the Probability Between 70 and 80

The probability of a score between 70 and 80 is the difference between the cumulative probabilities for Z_2 and Z_1:

$$P(70 \leq X \leq 80) = P(Z < 1.25) - P(Z < -1.25)$$

Substitute the values:

$$P(70 \leq X \leq 80) = 0.8944 - 0.1056 = 0.7888$$

Thus, the probability is **0.7888** or **78.88%**.

Step 4: Verify the Results

1. Check that $Z_1 = -1.25$ and $Z_2 = 1.25$ are correctly computed.
2. Use a Z-table or statistical tool to confirm $P(Z < -1.25) = 0.1056$ and $P(Z < 1.25) = 0.8944$.
3. Confirm subtraction is correct: $0.8944 - 0.1056 = 0.7888$.

Step 5: Visualize the Normal Distribution Curve

1. The normal curve has a peak at $\mu = 75$.
2. Shade the region between $X = 70$ and $X = 80$ (equivalent to $Z = -1.25$ to $Z = 1.25$).
3. The shaded area under the curve represents the probability 0.7888, as calculated.

Step 6: Interpretation

- **Key Insight**: About 78.88% of students will score between 70 and 80 on the test.
- This makes sense because 70 and 80 are close to the mean (75), so the probability of being within this range is high.

Problem 15:

If X is a normal variate with mean 30 and SD 5. Find the probabilities that

(i) 26≤ X ≤ 40 (ii) X > 45

Problem Restatement

We are solving a problem involving a **normal distribution** with:

- Mean (μ) = 30
- Standard Deviation (σ) = 5

The two parts to solve are:

1. $P(26 \le X \le 40)$
2. $P(X > 45)$

Part 1: $P(26 \le X \le 40)$

Step 1: Standardize the Values (Find Z-scores)

We first convert $X = 26$ and $X = 40$ to Z-scores using the formula:

$$Z = \frac{X - \mu}{\sigma}$$

1. For $X = 26$:

$$Z_1 = \frac{26 - 30}{5} = \frac{-4}{5} = -0.8$$

2. For $X = 40$:

$$Z_2 = \frac{40 - 30}{5} = \frac{10}{5} = 2$$

Step 2: Use the Standard Normal Table (Z-table)

The Z-table gives the **cumulative probability** $P(Z < z)$, which is the probability that the random variable Z is less than the given Z-value.

1. From the Z-table:
   - $P(Z < -0.8) = 0.2119$
   - $P(Z < 2) = 0.9772$

Step 3: Calculate the Probability

The probability $P(26 \leq X \leq 40)$ is the difference between $P(Z < 2)$ and $P(Z < -0.8)$:

$$P(26 \leq X \leq 40) = P(Z < 2) - P(Z < -0.8)$$

Substitute the values:

$$P(26 \leq X \leq 40) = 0.9772 - 0.2119 = 0.7653$$

Thus, the probability is **0.7653** or **76.53%**.

Part 2: $P(X > 45)$

Step 1: Standardize the Value (Find Z-score)

For $X = 45$:

$$Z = \frac{45 - 30}{5} = \frac{15}{5} = 3$$

Step 2: Use the Z-table

From the Z-table:

- $P(Z < 3) = 0.9987$

Step 3: Calculate the Probability

The probability $P(X > 45)$ is the complement of $P(Z < 3)$:

$$P(X > 45) = 1 - P(Z < 3)$$

Substitute the value:

$$P(X > 45) = 1 - 0.9987 = 0.0013$$

Thus, the probability is **0.0013** or **0.13%**.

Final Results

1. $P(26 \leq X \leq 40) = 0.7653$ (76.53%)
2. $P(X > 45) = 0.0013$ (0.13%)

Problem 16:

Butterfly-style valves used in heating and ventilating industries have a high flow coefficient. Flow coefficient can be modeled by a normal distribution with mean 496 C_v and standard deviation 25C_v. Find the probability that a valve will have a

flow coefficient of

a) atleast 450C_v

b) between 445.5 and 522C_v

Part (a): Probability that a valve will have a flow coefficient of at least 450Cv

Step 1: Identify the parameters

We are given:

- The mean (μ) = 496Cv
- The standard deviation (σ) = 25Cv
- We need to calculate $P(X \geq 450)$.

Step 2: Standardize the variable

To use the standard normal distribution, we convert the raw score ($X = 450$) into a **Z-score** using the formula:

$$Z = \frac{X - \mu}{\sigma}$$

Substitute the values:

$$Z = \frac{450 - 496}{25}$$

$$Z = \frac{-46}{25} = -1.84$$

Step 3: Interpret the Z-score

The Z-score of -1.84 means that the value $450Cv$ is **1.84 standard deviations below the mean.**

Step 4: Use the Z-table

The Z-table gives us the cumulative probability up to the Z-score (i.e., $P(Z \leq z)$).

From the Z-table:

$$P(Z \leq -1.84) = 0.0329$$

This means the probability of Z being less than -1.84 is **0.0329**.

Step 5: Calculate $P(Z \geq -1.84)$

Since the total probability is 1 (the entire area under the curve):

$$P(Z \geq -1.84) = 1 - P(Z \leq -1.84)$$

$$P(Z \geq -1.84) = 1 - 0.0329 = 0.9671$$

Thus:

$$P(X \geq 450) = 0.9671$$

Conclusion:

The probability of a valve having a flow coefficient of **at least 450Cv** is:

$$0.9671 \text{ or } 96.71\%.$$

Part (b): Probability that a valve has a flow coefficient between 445.5Cv and 522Cv

Step 1: Identify the bounds

We are calculating $P(445.5 \leq X \leq 522)$. This involves two steps:

1. Calculate $P(X \leq 522)$
2. Calculate $P(X \leq 445.5)$
3. Subtract the smaller cumulative probability from the larger one.

Step 2: Standardize the bounds

1. For $X = 445.5$:

$$Z = \frac{X - \mu}{\sigma}$$

$$Z = \frac{445.5 - 496}{25} = \frac{-50.5}{25} = -2.02$$

2. For $X = 522$:

$$Z = \frac{522 - 496}{25} = \frac{26}{25} = 1.04$$

Step 3: Use the Z-table

1. From the Z-table, the cumulative probability for $Z = -2.02$:

$$P(Z \leq -2.02) = 0.0217$$

2. For $Z = 1.04$:

$$P(Z \leq 1.04) = 0.8508$$

Step 4: Subtract cumulative probabilities

The probability between $Z = -2.02$ and $Z = 1.04$ is:

$$P(-2.02 \leq Z \leq 1.04) = P(Z \leq 1.04) - P(Z \leq -2.02)$$

$$P(-2.02 \leq Z \leq 1.04) = 0.8508 - 0.0217 = 0.8291$$

Thus:

$$P(445.5 \leq X \leq 522) = 0.8291$$

Conclusion:

The probability of a valve having a flow coefficient **between 445.5Cv and 522Cv** is:

$$0.8291 \text{ or } 82.91\%.$$

Summary

1. $P(X \geq 450) = 0.9671$ (96.71%)
2. $P(445.5 \leq X \leq 522) = 0.8291$ (82.91%)

Problem 17:

Decide whether the normal distribution to approximate *x* may be used in the following examples:

1.In a survey of 8 to 18-year-old heavy media users in the United States,47% said they get fair or poor grades (C's or below). You randomly select forty-five 8 to 18-year-old heavy media users in the United States and ask them whether they get fair or poor grades.

2.Thirty-six percent of people in the United States own a dog. You randomly select 25 people in the United States and ask them if they own a dog.

To decide whether the normal distribution can be used to approximate the distribution of X, the number of successes (i.e., heavy media users who get fair or poor grades), we need to check if the **conditions for normal approximation to the binomial distribution** are satisfied.

Step 1: Identify the problem parameters

The problem involves a **binomial distribution**:

- $n = 45$ (sample size)
- $p = 0.47$ (proportion of successes, or probability of a user reporting fair or poor grades)
- $q = 1 - p = 0.53$ (proportion of failures, or probability of a user reporting better grades)

We need to check if the normal approximation can be applied to approximate X, where X is the count of successes (fair or poor grades).

Step 2: Conditions for normal approximation

The normal distribution can be used to approximate a binomial distribution if:

$$np \geq 5 \quad \text{and} \quad nq \geq 5$$

Step 2.1: Check np

$$np = 45 \times 0.47 = 21.15$$

Since $np = 21.15 \geq 5$, the first condition is satisfied.

Step 2.2: Check nq

$$nq = 45 \times 0.53 = 23.85$$

Since $nq = 23.85 \geq 5$, the second condition is satisfied.

Step 3: Conclusion

Since both $np \geq 5$ and $nq \geq 5$, the normal distribution **can be used** to approximate the binomial distribution of X.

This problem involves determining whether the **normal approximation to the binom** can be applied to approximate the distribution of X, where X represents the numb owners in a sample of 25 people.

Step 1: Identify the parameters

The problem involves a **binomial distribution**:

- $n = 25$ (sample size)
- $p = 0.36$ (probability of owning a dog)
- $q = 1 - p = 0.64$ (probability of not owning a dog)

Step 2: Conditions for normal approximation

The normal distribution can be used to approximate a binomial distribution if:

$$np \geq 5 \quad \text{and} \quad nq \geq 5$$

Step 2.1: Check np

$$np = 25 \times 0.36 = 9$$

Since $np = 9 \geq 5$, the first condition is satisfied.

Step 2.2: Check nq

$$nq = 25 \times 0.64 = 16$$

Since $nq = 16 \geq 5$, the second condition is satisfied.

Problem 18:

Thirty-one percent of the seniors in a certain high school plan to attend college. If 50 students are randomly selected, find the probability that less than 14 students plan to attend college by normal approximation.

To solve this problem using the **normal approximation to the binomial distribution**, follow thes steps:

Step 1: Identify the binomial distribution parameters

The problem involves a **binomial distribution**:

- $n = 50$ (sample size),
- $p = 0.31$ (probability a senior plans to attend college),
- $q = 1 - p = 0.69$ (probability a senior does NOT plan to attend college).

We want to find $P(X < 14)$, where X is the number of students planning to attend college.

Step 2: Conditions for normal approximation

The normal distribution can approximate the binomial distribution if:

$$np \geq 5 \quad \text{and} \quad nq \geq 5$$

Check np:

$$np = 50 \times 0.31 = 15.5 \quad (\text{satisfies } np \geq 5)$$

Check nq:

$$nq = 50 \times 0.69 = 34.5 \quad (\text{satisfies } nq \geq 5)$$

Since both conditions are met, we can use the normal approximation.

Step 3: Find the mean and standard deviation

The mean (μ) and standard deviation (σ) for a binomial distribution are:

$$\mu = np, \quad \sigma = \sqrt{npq}$$

Calculate μ:

$$\mu = 50 \times 0.31 = 15.5$$

Calculate σ:

$$\sigma = \sqrt{50 \times 0.31 \times 0.69} = \sqrt{10.695} \approx 3.27$$

Step 4: Apply the continuity correction

Since we want $P(X < 14)$, the continuity correction adjusts this to:

$$P(X < 14) \approx P(Y < 13.5),$$

where Y is the approximating normal variable.

Step 5: Standardize the value

The standard normal variable Z is calculated as:

$$Z = \frac{Y - \mu}{\sigma}$$

Calculate Z for $Y = 13.5$:

$$Z = \frac{13.5 - 15.5}{3.27} = \frac{-2}{3.27} \approx -0.61$$

Step 6: Find the probability using the standard normal table

From the standard normal table, the cumulative probability for $Z = -0.61$ is:

$$P(Z < -0.61) \approx 0.2709$$

Step 7: Final answer

$$P(X < 14) \approx 0.2709$$

Summary of Steps:

1. Verified normal approximation conditions ($np \geq 5, nq \geq 5$).
2. Found $\mu = 15.5$ and $\sigma = 3.27$.
3. Applied continuity correction: $P(X < 14) \to P(Y < 13.5)$.
4. Standardized $Z = \frac{Y-\mu}{\sigma}$.
5. Used the standard normal table to find $P(Z < -0.61) = 0.2709$.

Problem 19:

If 62% of all clouds seeded with silver iodide show spectacular growth, What is the probability that among 40 clouds seeded with silver iodide at most 20 will show spectacular growth by normal approximation?

To solve this problem using **normal approximation to the binomial distribution**, follow these steps:

Step 1: Identify the binomial distribution parameters

The problem involves a **binomial distribution**:

- $n = 40$ (sample size),
- $p = 0.62$ (probability a cloud shows spectacular growth),
- $q = 1 - p = 0.38$ (probability a cloud does NOT show spectacular growth).

We want to find $P(X \leq 20)$, where X is the number of clouds showing spectacular growth.

Step 2: Conditions for normal approximation

The normal distribution can approximate the binomial distribution if:

$$np \geq 5 \quad \text{and} \quad nq \geq 5$$

Check np:

$$np = 40 \times 0.62 = 24.8 \quad (\text{satisfies } np \geq 5)$$

Check nq:

$$nq = 40 \times 0.38 = 15.2 \quad (\text{satisfies } nq \geq 5)$$

Since both conditions are satisfied, we can use the normal approximation.

Step 3: Find the mean and standard deviation

The mean (μ) and standard deviation (σ) for a binomial distribution are:

$$\mu = np, \quad \sigma = \sqrt{npq}$$

Calculate μ:

$$\mu = 40 \times 0.62 = 24.8$$

Calculate σ:

$$\sigma = \sqrt{40 \times 0.62 \times 0.38} = \sqrt{9.424} \approx 3.07$$

Step 4: Apply the continuity correction

Since we want $P(X \leq 20)$, the continuity correction adjusts this to:

$$P(X \leq 20) \approx P(Y \leq 20.5),$$

where Y is the approximating normal variable.

Step 5: Standardize the value

The standard normal variable Z is calculated as:

$$Z = \frac{Y - \mu}{\sigma}$$

Calculate Z for $Y = 20.5$:

$$Z = \frac{20.5 - 24.8}{3.07} = \frac{-4.3}{3.07} \approx -1.4$$

Step 6: Find the probability using the standard normal table

From the standard normal table, the cumulative probability for $Z = -1.4$ is:

$$P(Z \leq -1.4) \approx 0.0808$$

Step 7: Final answer

$$P(X \leq 20) \approx 0.0808$$

Summary of Steps:

1. Verified normal approximation conditions ($np \geq 5$, $nq \geq 5$).
2. Found $\mu = 24.8$ and $\sigma = 3.07$.
3. Applied continuity correction: $P(X \leq 20) \rightarrow P(Y \leq 20.5)$.
4. Standardized $Z = \frac{Y-\mu}{\sigma}$.
5. Used the standard normal table to find $P(Z \leq -1.4) = 0.0808$.

Practice Problems

In a factory, 8% of all machines break down at least once a year. Use the Poisson approximation to the binomial distribution to determine the probabilities that among 25 machines (randomly chosen in the factory):

(a) 5 will break down at least once a year;

(b) at least 4 will break down once a year:

(c) anywhere from 3 to 8, inclusive, will break down at least once a year.

Step 1: Define the parameters of the binomial distribution

The problem involves a binomial distribution:

- $n = 25$ (sample size),
- $p = 0.08$ (probability that a machine breaks down at least once a year),
- $q = 1 - p = 0.92$.

Using the Poisson approximation to the binomial distribution is valid when:

$$n \cdot p \leq 10.$$

Check $n \cdot p$:

$$n \cdot p = 25 \times 0.08 = 2 \quad \text{(valid for Poisson approximation).}$$

Step 2: Use the Poisson approximation

For the Poisson approximation:

- The mean $\lambda = n \cdot p$,
- $\lambda = 25 \times 0.08 = 2$.

The Poisson probability mass function (PMF) is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

where k is the number of machines breaking down.

(b) Probability that **at least 4** machines break down

"At least 4" means $P(X \geq 4)$. Using the complement rule:

$$P(X \geq 4) = 1 - P(X \leq 3).$$

Compute $P(X \leq 3)$:

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3).$$

Use the PMF for each term:

1. $P(X = 0) = \frac{2^0 e^{-2}}{0!} = e^{-2} \approx 0.1353,$
2. $P(X = 1) = \frac{2^1 e^{-2}}{1!} = \frac{2 \cdot 0.1353}{1} \approx 0.2706,$
3. $P(X = 2) = \frac{2^2 e^{-2}}{2!} = \frac{4 \cdot 0.1353}{2} \approx 0.2706,$
4. $P(X = 3) = \frac{2^3 e^{-2}}{3!} = \frac{8 \cdot 0.1353}{6} \approx 0.1804.$

Add these probabilities:

$$P(X \leq 3) \approx 0.1353 + 0.2706 + 0.2706 + 0.1804 \approx 0.8569.$$

Now find $P(X \geq 4)$:

$$P(X \geq 4) = 1 - P(X \leq 3) \approx 1 - 0.8569 = 0.1431.$$

Thus:

$$P(X \geq 4) \approx 0.1431.$$

(c) Probability that **anywhere from 3 to 8 machines**, inclusive, will break down

This means $P(3 \leq X \leq 8)$, which can be expressed as:

$P(3 \leq X \leq 8) = P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)$.

We already know $P(X = 3) \approx 0.1804$ and $P(X = 5) \approx 0.036$. Let's calculate $P(X = 4)$, $P(X = 6)$, $P(X = 7)$, and $P(X = 8)$.

$P(X = 4)$:

$$P(X = 4) = \frac{2^4 e^{-2}}{4!} = \frac{16 \cdot 0.1353}{24} \approx 0.0902.$$

$P(X = 6)$:

$$P(X = 6) = \frac{2^6 e^{-2}}{6!} = \frac{64 \cdot 0.1353}{720} \approx 0.0120.$$

$P(X = 7)$:

$$P(X = 7) = \frac{2^7 e^{-2}}{7!} = \frac{128 \cdot 0.1353}{5040} \approx 0.0034.$$

$P(X = 8)$:

$$P(X = 8) = \frac{2^8 e^{-2}}{8!} = \frac{256 \cdot 0.1353}{40320} \approx 0.0009.$$

Now sum all probabilities from $X = 3$ to $X = 8$:

$$P(3 \leq X \leq 8) = 0.1804 + 0.0902 + 0.036 + 0.0120 + 0.0034 + 0.0009 \approx 0.3229.$$

Final Answers:

(a) $P(X = 5) \approx 0.036$,

(b) $P(X \geq 4) \approx 0.1431$,

(c) $P(3 \leq X \leq 8) \approx 0.3229$.

Practice Problems

The average number of trucks arriving on any one day at a truck depot in a certain city is known to be 12. what is the probability that on a given day fewer than nine trucks will arrive at this depot?

To calculate the probability that fewer than 9 trucks will arrive on a given day when the average number of trucks arriving per day is 12 ($\lambda = 12$), we use the **Poisson distribution**. Here's the detailed explanation step-by-step.

Step 1: Define the problem

The Poisson distribution formula is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where:

- $\lambda = 12$ (average number of trucks arriving per day),
- k is the specific number of trucks,
- e is approximately 2.71828.

We are tasked with finding $P(X < 9)$, which means:

$$P(X < 9) = P(X = 0) + P(X = 1) + P(X = 2) + \cdots + P(X = 8)$$

Step 2: Calculate individual probabilities for $k = 0, 1, 2, \dots, 8$

(i) $P(X = 0)$:

$$P(X = 0) = \frac{12^0 e^{-12}}{0!} = e^{-12}$$

Using a calculator:

$$e^{-12} \approx 6.1442 \times 10^{-6}.$$

Thus:

$$P(X = 0) \approx 0.000006.$$

(ii) $P(X = 1)$:

$$P(X = 1) = \frac{12^1 e^{-12}}{1!} = 12 \cdot e^{-12}.$$

$$P(X = 1) = 12 \cdot 0.000006 \approx 0.000074.$$

(iii) $P(X = 2)$:

$$P(X = 2) = \frac{12^2 e^{-12}}{2!} = \frac{144 \cdot e^{-12}}{2}.$$

$$P(X = 2) = \frac{144 \cdot 0.000006}{2} \approx 0.00044.$$

(iv) $P(X = 3)$**:**

$$P(X = 3) = \frac{12^3 e^{-12}}{3!} = \frac{1728 \cdot e^{-12}}{6}.$$

$$P(X = 3) = \frac{1728 \cdot 0.000006}{6} \approx 0.0021.$$

(v) $P(X = 4)$**:**

$$P(X = 4) = \frac{12^4 e^{-12}}{4!} = \frac{20736 \cdot e^{-12}}{24}.$$

$$P(X = 4) = \frac{20736 \cdot 0.000006}{24} \approx 0.0052.$$

(vi) $P(X = 5)$**:**

$$P(X = 5) = \frac{12^5 e^{-12}}{5!}.$$

$$P(X = 5) = \frac{248832 \cdot e^{-12}}{120}.$$

$$P(X = 5) = \frac{248832 \cdot 0.000006}{120} \approx 0.0124.$$

(vii) $P(X=6)$**:**

$$P(X=6)=\frac{12^6e^{-12}}{6!}.$$

$$P(X=6)=\frac{2985984\cdot e^{-12}}{720}.$$

$$P(X=6)=\frac{2985984\cdot 0.000006}{720}\approx 0.0248.$$

(viii) $P(X=7)$**:**

$$P(X=7)=\frac{12^7e^{-12}}{7!}.$$

$$P(X=7)=\frac{35831808\cdot e^{-12}}{5040}.$$

$$P(X=7)=\frac{35831808\cdot 0.000006}{5040}\approx 0.0426.$$

(ix) $P(X=8)$**:**

$$P(X=8)=\frac{12^8e^{-12}}{8!}.$$

$$P(X=8)=\frac{429981696\cdot e^{-12}}{40320}.$$

$$P(X=8)=\frac{429981696\cdot 0.000006}{40320}\approx 0.0639.$$

Step 3: Add probabilities

Now, add all the probabilities calculated above:

$$P(X<9)=P(X=0)+P(X=1)+\cdots+P(X=8)$$

$$P(X<9)\approx 0.000006+0.000074+0.00044+0.0021+0.0052+0.0124+0.0248+0.0426+0.0639.$$

$$P(X<9)\approx 0.1515.$$

Final Answer:

The probability that fewer than 9 trucks will arrive on a given day is approximately:

$$\boxed{0.1515 \text{ or } 15.15\%}.$$

Practice Problems

A certain kind of sheet metal has on the average, five defects per 10 square feet. If we assume a Poisson distribution, what is the probability that a 15-square-foot sheet of the metal will have at least six defects?

Given Data:

- The average number of defects per 10 square feet is $\lambda = 5$.
- We are looking at a 15-square-foot sheet. Therefore, the expected number of defects for this area is:

$$\lambda = \frac{15 \times 5}{10} = 7.5$$

- The problem requires us to find the probability that the number of defects X is at least 6:

$$P(X \geq 6) = 1 - P(X < 6)$$

Step 1: Write $P(X < 6)$ in terms of the Poisson probability formula

The Poisson probability mass function (PMF) is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Here, $\lambda = 7.5$. For $P(X < 6)$, we calculate:

$$P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$

Step 2: Calculate $P(X = k)$ for $k = 0, 1, 2, 3, 4, 5$

Substitute $\lambda = 7.5$ into the Poisson formula:

1. $P(X = 0) = \frac{7.5^0 e^{-7.5}}{0!} = e^{-7.5} \approx 0.000553$
2. $P(X = 1) = \frac{7.5^1 e^{-7.5}}{1!} = 7.5e^{-7.5} \approx 0.004145$
3. $P(X = 2) = \frac{7.5^2 e^{-7.5}}{2!} = \frac{56.25e^{-7.5}}{2} \approx 0.015545$
4. $P(X = 3) = \frac{7.5^3 e^{-7.5}}{3!} = \frac{421.875e^{-7.5}}{6} \approx 0.038863$
5. $P(X = 4) = \frac{7.5^4 e^{-7.5}}{4!} = \frac{3164.0625e^{-7.5}}{24} \approx 0.072421$
6. $P(X = 5) = \frac{7.5^5 e^{-7.5}}{5!} = \frac{23730.46875e^{-7.5}}{120} \approx 0.108632$

Step 3: Add probabilities for $P(X < 6)$

$$P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$

$$P(X < 6) = 0.000553 + 0.004145 + 0.015545 + 0.038863 + 0.072421 + 0.108632 \approx 0.240159$$

Step 4: Calculate $P(X \geq 6)$

Using the complement rule:

$$P(X \geq 6) = 1 - P(X < 6)$$

$$P(X \geq 6) = 1 - 0.240159 \approx 0.759841$$

Final Answer:

The probability that at least six defects occur is approximately:

P(X \geq 6) \approx 0.7598 \text{ (or 75.98%)}

Practice Problems

What is the probability that a standard normal variate Z will be

(i) greater than 1.09

(ii) less than –1.65

(iii) lying between –1.00 and 1.96

(iv) lying between 1.25 and 2.75

(i) $P(Z > 1.09)$

The standard normal variate Z has a mean of 0 and a standard deviation of 1.

1. From the standard normal table, find $P(Z < 1.09)$. For $Z = 1.09$:

$$P(Z < 1.09) = 0.8621$$

2. The complement rule states:

$$P(Z > 1.09) = 1 - P(Z < 1.09)$$

   Substituting the value:

$$P(Z > 1.09) = 1 - 0.8621 = 0.1379$$

Answer for (i): $P(Z > 1.09) = 0.1379$ or 13.79%.

(ii) $P(Z < -1.65)$

1. Using the symmetry of the normal distribution, $P(Z < -1.65) = P(Z > 1.65)$.
2. From the standard normal table, $P(Z < 1.65) = 0.9505$. Thus:

$$P(Z > 1.65) = 1 - P(Z < 1.65)$$

   Substituting the value:

$$P(Z > 1.65) = 1 - 0.9505 = 0.0495$$

Answer for (ii): $P(Z < -1.65) = 0.0495$ or 4.95%.

(iii) $P(-1.00 < Z < 1.96)$

1. Use the property of cumulative probabilities:
$$P(-1.00 < Z < 1.96) = P(Z < 1.96) - P(Z < -1.00)$$
2. From the standard normal table:
   - $P(Z < 1.96) = 0.9750$
   - $P(Z < -1.00) = 1 - P(Z < 1.00) = 1 - 0.8413 = 0.1587$
3. Subtract the probabilities:
$$P(-1.00 < Z < 1.96) = 0.9750 - 0.1587 = 0.8163$$

Answer for (iii): $P(-1.00 < Z < 1.96) = 0.8163$ or 81.63%.

(iv) $P(1.25 < Z < 2.75)$

1. Use the property of cumulative probabilities:
$$P(1.25 < Z < 2.75) = P(Z < 2.75) - P(Z < 1.25)$$
2. From the standard normal table:
   - $P(Z < 2.75) = 0.9970$
   - $P(Z < 1.25) = 0.8944$
3. Subtract the probabilities:
$$P(1.25 < Z < 2.75) = 0.9970 - 0.8944 = 0.1026$$

Answer for (iv): $P(1.25 < Z < 2.75) = 0.1026$ or 10.26%.

Final Answers:

1. $P(Z > 1.09) = 0.1379$
2. $P(Z < -1.65) = 0.0495$
3. $P(-1.00 < Z < 1.96) = 0.8163$
4. $P(1.25 < Z < 2.75) = 0.1026$

Practice Problems

The length of life of an instrument produced by a machine has a normal distribution with a mean of 12 months and a standard deviation of 2 months. Find the probability that an instrument produced by this machine will last
a) less than 7 months.
b) between 7 and 12 months.

We are solving this problem based on the **normal distribution** with:

- Mean $\mu = 12$ months
- Standard deviation $\sigma = 2$ months.

We'll standardize the values using the **Z-score formula:**

$$Z = \frac{X - \mu}{\sigma}$$

(a) Probability that the instrument will last less than 7 months, $P(X < 7)$

1. Compute the Z-score for $X = 7$:

$$Z = \frac{7 - 12}{2} = \frac{-5}{2} = -2.5$$

2. Using the standard normal distribution table, find $P(Z < -2.5)$:
   - From the table, $P(Z < -2.5) = 0.0062$.

Thus, the probability is:

$$P(X < 7) = 0.0062$$

Answer for (a): The probability is $P(X < 7) = 0.0062$ or 0.62%.

(b) Probability that the instrument will last between 7 and 12 months, $P(7 < X < 12)$

1. Compute the Z-score for $X = 7$ (already computed in part (a)):

$$Z = -2.5$$

2. Compute the Z-score for $X = 12$:

$$Z = \frac{12 - 12}{2} = 0$$

3. Find the cumulative probabilities from the standard normal distribution table:
   - $P(Z < -2.5) = 0.0062$ (from part (a)).
   - $P(Z < 0) = 0.5000$.
4. Subtract the probabilities to find $P(7 < X < 12)$:

$$P(7 < X < 12) = P(Z < 0) - P(Z < -2.5)$$

   Substituting the values:

$$P(7 < X < 12) = 0.5000 - 0.0062 = 0.4938$$

Answer for (b): The probability is $P(7 < X < 12) = 0.4938$ or 49.38%.

Final Answers:

1. $P(X < 7) = 0.0062$ or 0.62%
2. $P(7 < X < 12) = 0.4938$ or 49.38%

Practice Problems

A computer crashes on average once every 4 months;
a) What is the probability that it will not crash in a period of 4 months?
b) What is the probability that it will crash once in a period of 4 months?
c) What is the probability that it will crash twice in a period of 4 months?
d) What is the probability that it will crash three times in a period of 4 months?

Given Data:

- Average crashes in a 4-month period (λ) = 1 crash (since it crashes once every 4 months).

The formula for the Poisson distribution is:

$$P(X = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

where:

- X is the number of events (crashes) in the given period.
- λ is the average number of events in the given period.
- k is the specific number of events we are calculating the probability for.
- $e \approx 2.718$.

(a) Probability that it will **not crash** in 4 months ($P(X = 0)$):

Using $\lambda = 1$ and $k = 0$:

$$P(X = 0) = \frac{e^{-1} \cdot 1^0}{0!} = \frac{e^{-1} \cdot 1}{1} = e^{-1}$$

From a calculator, $e^{-1} \approx 0.3679$.

Answer for (a): $P(X = 0) = 0.3679$ or 36.79%.

(b) Probability that it will **crash once** in 4 months ($P(X = 1)$):

Using $\lambda = 1$ and $k = 1$:

$$P(X = 1) = \frac{e^{-1} \cdot 1^1}{1!} = \frac{e^{-1} \cdot 1}{1} = e^{-1}$$

From a calculator, $e^{-1} \approx 0.3679$.

Answer for (b): $P(X = 1) = 0.3679$ or 36.79%.

(c) Probability that it will **crash twice** in 4 months ($P(X = 2)$):

Using $\lambda = 1$ and $k = 2$:

$$P(X = 2) = \frac{e^{-1} \cdot 1^2}{2!} = \frac{e^{-1} \cdot 1}{2} = \frac{e^{-1}}{2}$$

From a calculator:

$$P(X = 2) = \frac{0.3679}{2} = 0.1839$$

Answer for (c): $P(X = 2) = 0.1839$ or 18.39%.

(d) Probability that it will **crash three times** in 4 months ($P(X=3)$):

Using $\lambda = 1$ and $k = 3$:

$$P(X=3) = \frac{e^{-1} \cdot 1^3}{3!} = \frac{e^{-1} \cdot 1}{6} = \frac{e^{-1}}{6}$$

From a calculator:

$$P(X=3) = \frac{0.3679}{6} = 0.0613$$

Answer for (d): $P(X=3) = 0.0613$ or 6.13%.

Final Answers:

1. $P(X=0) = 0.3679$ or 36.79%
2. $P(X=1) = 0.3679$ or 36.79%
3. $P(X=2) = 0.1839$ or 18.39%
4. $P(X=3) = 0.0613$ or 6.13%

Practice Problems

To solve this problem, we will use the properties of the **binomial distribution**.

Step 1: Understanding the given data

The formula for the **mean** of a binomial distribution is:

$$\mu = n \cdot p$$

where:

- n = number of observations (trials),
- p = probability of success.

The formula for the **variance** of a binomial distribution is:

$$\text{Variance} = \sigma^2 = n \cdot p \cdot (1-p)$$

The **standard deviation** is the square root of the variance:

$$\text{Standard Deviation} = \sigma = \sqrt{n \cdot p \cdot (1-p)}$$

Step 2: Using the given information

From the question:

- $\mu = 20$ (mean),
- $n = 30$ (number of observations).

We need to find:

1. p (probability of success),
2. Variance (σ^2),
3. Standard deviation (σ).

Step 3: Find p

From the formula for mean:

$$\mu = n \cdot p$$

Substitute the known values:

$$20 = 30 \cdot p$$

Solve for p:

$$p = \frac{20}{30} = 0.6667$$

Step 4: Find the variance (σ^2)

The formula for variance is:

$$\text{Variance} = n \cdot p \cdot (1 - p)$$

Substitute the values:

$$\text{Variance} = 30 \cdot 0.6667 \cdot (1 - 0.6667)$$

Simplify:

$$\text{Variance} = 30 \cdot 0.6667 \cdot 0.3333$$

$$\text{Variance} \approx 30 \cdot 0.2222 = 6.6667$$

So, the variance is:

$$\text{Variance} \approx 6.67$$

Step 5: Find the standard deviation (σ)

The formula for standard deviation is:

$$\text{Standard Deviation} = \sqrt{\text{Variance}}$$

Substitute the value of the variance:

$$\text{Standard Deviation} = \sqrt{6.6667} \approx 2.58$$

Final Answers:

1. Variance (σ^2) = 6.67,
2. Standard Deviation (σ) = 2.58.

Practice Problems

The probability of a boy hitting a target is 0.2. How Many times must he fire so that the probability of his hitting the target at least once is greater than 2/3.

We are solving for the minimum number of trials n such that the probability of hitting the target at least once is greater than $\frac{2}{3}$. Let's work step by step:

Step 1: Understand the problem

The boy has a probability $p = 0.2$ of hitting the target in one shot. The probability of **not hitting the target** in one shot is:

$$q = 1 - p = 1 - 0.2 = 0.8$$

If the boy fires n times, the probability of missing the target in all n attempts is:

$$P(\text{missing all targets}) = q^n = (0.8)^n$$

The probability of hitting the target at least once is:

$$P(\text{at least one hit}) = 1 - P(\text{missing all targets}) = 1 - (0.8)^n$$

We need this probability to be greater than $\frac{2}{3}$, so:

$$1 - (0.8)^n > \frac{2}{3}$$

Step 2: Set up the inequality

Rearranging the inequality:

$$(0.8)^n < 1 - \frac{2}{3}$$

Simplify the right-hand side:

$$(0.8)^n < \frac{1}{3}$$

Step 3: Solve for n

Take the natural logarithm (ln) of both sides to solve for n:

$$\ln((0.8)^n) < \ln\left(\frac{1}{3}\right)$$

Using the logarithmic rule $\ln(a^b) = b\ln(a)$, this becomes:

$$n\ln(0.8) < \ln\left(\frac{1}{3}\right)$$

Since $\ln(0.8)$ is negative, dividing both sides by $\ln(0.8)$ reverses the inequality:

$$n > \frac{\ln\left(\frac{1}{3}\right)}{\ln(0.8)}$$

Step 4: Calculate the values

1. Compute $\ln\left(\frac{1}{3}\right)$:

$$\ln\left(\frac{1}{3}\right) = \ln(1) - \ln(3) = 0 - 1.0986 = -1.0986$$

2. Compute $\ln(0.8)$:

$$\ln(0.8) = -0.2231$$

3. Substitute into the inequality:

$$n > \frac{-1.0986}{-0.2231}$$

4. Simplify:

$$n > 4.923$$

Step 5: Interpret the result

Since n must be an integer, round up to the nearest whole number:

$$n = 5$$

Final Answer:

The boy must fire at least **5 times** so that the probability of hitting the target at least once is greater than $\frac{2}{3}$.

Verification:

1. Probability of hitting at least once with $n = 5$:

$$P(\text{at least one hit}) = 1 - (0.8)^5 = 1 - 0.32768 = 0.67232$$

This is greater than $\frac{2}{3}$ (0.6667).

2. For $n = 4$:

$$P(\text{at least one hit}) = 1 - (0.8)^4 = 1 - 0.4096 = 0.5904$$

This is less than $\frac{2}{3}$.

Thus, $n = 5$ is correct.

Practice Problems

A box of candies has many different colors in it. There is a 15% chance of getting a pink candy. What is the probability that exactly 4 candies in a box are pink out of 10?

Hint: n = 10, p=0.15, q=0.85, x=4

We are solving this problem using the **binomial probability formula**, as it involves a fixed number of trials ($n = 10$), two possible outcomes (getting a pink candy or not), and the probability of success ($p = 0.15$) remains constant for each trial.

Step 1: Write the Binomial Formula

The binomial probability formula is:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Where:

- $P(X = x)$ is the probability of x successes (pink candies) in n trials (candies).
- $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the binomial coefficient.
- p is the probability of success (getting a pink candy), $p = 0.15$.
- $q = 1 - p = 0.85$ is the probability of failure (not getting a pink candy).

We need to calculate the probability of getting exactly 4 pink candies ($x = 4$) out of 10 trials ($n = 10$).

Step 2: Plug in the values

$$P(X = 4) = \binom{10}{4} (0.15)^4 (0.85)^6$$

Step 3: Calculate the Binomial Coefficient ($\binom{10}{4}$)

The binomial coefficient is:

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4! \cdot 6!}$$

1. Expand the factorials:

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

2. Simplify:

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{(4 \cdot 3 \cdot 2 \cdot 1) \cdot 6!}$$

Cancel $6!$ from the numerator and denominator:

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{5040}{24} = 210$$

Step 4: Calculate the probability terms

1. $(0.15)^4$:

$$(0.15)^4 = 0.15 \cdot 0.15 \cdot 0.15 \cdot 0.15 = 0.00050625$$

2. $(0.85)^6$:

$$(0.85)^6 = 0.85 \cdot 0.85 \cdot 0.85 \cdot 0.85 \cdot 0.85 \cdot 0.85 = 0.418211943$$

Step 5: Combine all terms

Substitute into the formula:

$$P(X = 4) = 210 \cdot 0.00050625 \cdot 0.418211943$$

First, multiply $210 \cdot 0.00050625$:

$$210 \cdot 0.00050625 = 0.1063125$$

Then, multiply the result by 0.418211943:

$$0.1063125 \cdot 0.418211943 = 0.044446$$

Final Answer:

The probability of getting exactly 4 pink candies out of 10 is approximately:

$$P(X = 4) \approx 0.0444 \text{ or } 4.44\%.$$