Linear Algebra Practice Problems

Math 240 — Calculus III

Summer 2015, Session II

1. Determine whether the given set is a vector space. If not, give at least one axiom that is not satisfied. Unless otherwise stated, assume that vector addition and scalar multiplication are the ordinary operations defined on the set.

   (a) The set of vectors $\{(a, b) \in \mathbb{R}^2 : b = 3a + 1\}$
   Answer: This is not a vector space. It does not contain the zero vector, and is not closed under either addition or scalar multiplication.

   (b) The set of vectors $\{(a, b) \in \mathbb{R}^2\}$ with scalar multiplication defined by $k(a, b) = (ka, b)$
   Answer: This is not a vector space. The scalar multiplication defined above does not distribute over the usual addition of vectors.

   $$(r + s)(a, b) = ((r + s)a, b) = (ra + sa, b)$$
   $$\text{but } r(a, b) + s(a, b) = (ra, b) + (sa, b) = (ra + sa, 2b)$$

   (c) The set of vectors $\{(a, b) \in \mathbb{R}^2\}$ with scalar multiplication defined by $k(a, b) = (ka, 0)$
   Answer: This is not a vector space. It does not obey the identity property of scalar multiplication because
   $$1(a, b) = (1a, 0) = (a, 0) \neq (a, b).$$

   (d) The set of real numbers, with addition defined by $\mathbf{x} + \mathbf{y} = x - y$
   Answer: This is not a vector space. This method of vector addition is neither associative nor commutative.

   (e) The set $\mathbb{R}^3$, with the vector addition operation $\oplus$ defined by

   $$(a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1 + 5, a_2 + b_2 - 7, a_3 + b_3 + 1)$$

   and scalar multiplication $\odot$ defined by

   $$c \odot (a_1, a_2, a_3) = (ca_1 + 5(c - 1), ca_2 - 7(c - 1), ca_3 + c - 1).$$

   Answer: This is a vector space. The zero vector is $(-5, 7, -1)$.

2. Determine whether or not the given set is a subspace of the indicated vector space.

   (a) $\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$
   Answer: This is not a subspace of $\mathbb{R}^3$. It does not contain the zero vector $\mathbf{0} = (0, 0, 0)$ and it is not closed under either addition or scalar multiplication.

   (b) All polynomials in P_2 that are divisible by $x - 2$
   Answer: This is a subspace of P_2.

(c) $\{f \in C^0[a,b] : \int_a^b f(x)\,dx = 0\}$
Remember that $C^0[a,b]$ is the vector space of continuous, real-valued functions defined on the closed interval $[a,b]$ with $a < b$.
Answer: This is a subspace of $C^0[a,b]$. It is the kernel of the linear transformation $T : C^0[a,b] \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\,dx$.

3. If $A = \begin{bmatrix} 1 & 4 \\ 5 & 10 \\ 8 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 6 & -3 \\ 1 & -3 & 2 \end{bmatrix}$, determine (a) AB and (b) BA.

Answer: (a) $AB = \begin{bmatrix} 0 & -6 & 5 \\ -10 & 0 & 5 \\ -20 & 12 & 0 \end{bmatrix}$, (b) $BA = \begin{bmatrix} 2 & 8 \\ 2 & -2 \end{bmatrix}$

4. Use either Gaussian elimination or Gauss-Jordan elimination to solve the given system or show that no solution exists.

(a)
$$\begin{aligned} x_1 - x_2 - x_3 &= -3 \\ 2x_1 + 3x_2 + 5x_3 &= 7 \\ x_1 - 2x_2 + 3x_3 &= -11 \end{aligned}$$

Answer: $x_1 = 0,\ x_2 = 4,\ x_3 = -1$

(b)
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Answer: $x_1 = t,\ x_2 = -t,\ x_3 = 0$ for any $t \in \mathbb{R}$

(c)
$$\begin{aligned} x_1 - x_2 - x_3 &= 8 \\ x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

Answer: The system is inconsistent; there is no solution.

(d)
$$\begin{aligned} x_1 + 2x_2 \phantom{{}+ x_3} + x_4 &= 0 \\ 4x_1 + 9x_2 + x_3 + 12x_4 &= 0 \\ 3x_1 + 9x_2 + 6x_3 + 21x_4 &= 0 \\ x_1 + 3x_2 + x_3 + 9x_4 &= 0 \end{aligned}$$

Answer: $x_1 = 19t,\ x_2 = -10t,\ x_3 = 2t,\ x_4 = t$ for any $t \in \mathbb{R}$

5. Determine the rank of the given matrix.

(a) $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$
Answer: 2

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{bmatrix}$
Answer: 3

(c) $\begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & \frac{2}{3} & 3 & \frac{1}{3} \\ 6 & 6 & 6 & 12 & 0 \end{bmatrix}$
Answer: 3

6. Determine whether the given set of vectors in $\mathbb{R}^n$ is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (1,0,1)$, $\mathbf{v}_3 = (1,-1,5)$
Answer: These vectors are linearly independent.

(b) $\mathbf{v}_1 = (2,6,3)$, $\mathbf{v}_2 = (1,-1,4)$, $\mathbf{v}_3 = (3,2,1)$, $\mathbf{v}_4 = (2,5,4)$
Answer: These vectors are linearly dependent. They are vectors in $\mathbb{R}^3$, which is a 3-dimensional vector space. Any set of more than 3 vectors in $\mathbb{R}^3$ is linearly dependent.

(c) $\mathbf{v}_1 = (1,-1,3,-1)$, $\mathbf{v}_2 = (1,-1,4,2)$, $\mathbf{v}_3 = (1,-1,5,7)$.
Answer: These vectors are linearly independent.

(d) $\mathbf{v}_1 = (2,1,1,5)$, $\mathbf{v}_2 = (2,2,1,1)$, $\mathbf{v}_3 = (3,-1,6,1)$, $\mathbf{v}_4 = (1,1,1,-1)$
Answer: These vectors are linearly independent.

7. Determine a basis for the subspace of $\mathbb{R}^n$ spanned by the given set of vectors.

(a) $\{(1,3,3),\ (-3,-9,-9),\ (1,5,-1),\ (2,7,4),\ (1,4,1)\}$
Answer: $\{(1,3,3),\ (1,5,-1)\}$ (answers may vary)

(b) $\{(1,1,-1,2),\ (2,1,3,-4),\ (1,2,-6,10)\}$
Answer: $\{(1,1,-1,2),\ (2,1,3,-4)\}$ (answers may vary)

8. Evaluate the determinant of the given matrix.

(a) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{bmatrix}$
Answer: -48

(b) $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 7 & 1 \\ 2 & 6 & 4 \end{bmatrix}$
Answer: 62

(c) $\begin{bmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$
Answer: 0

(d) $\begin{bmatrix} -2 & -1 & 4 \\ -3 & 6 & 1 \\ -3 & 4 & 8 \end{bmatrix}$
Answer: -85

(e) $\begin{bmatrix} 6 & 1 & 8 & 10 \\ 0 & \frac{2}{3} & 7 & 2 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -5 \end{bmatrix}$
Answer: 80

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 5 & 8 & 20 \end{bmatrix}$
Answer: 16

9. Find the values of λ that satisfy the equation

$$\begin{vmatrix} -3-\lambda & 10 \\ 2 & 5-\lambda \end{vmatrix} = 0.$$

Answer: $\lambda = -5, 7$

10. If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5$, evaluate the determinant of the matrix

$$\begin{bmatrix} 2a_1 & a_2 & a_3 \\ 6b_1 & 3b_2 & 3b_3 \\ 2c_1 & c_2 & c_3 \end{bmatrix}.$$

Answer: 30

11. Recall that a square matrix A is said to be **skew-symmetric** if $A^T = -A$. If A is a 5×5 skew-symmetric matrix, show that $\det(A) = 0$.

Answer: We know that $\det\left(A^T\right) = \det(A)$. Also, if A is 5×5, then $\det(-A) = (-1)^5 \det(A)$. Putting these together with the information $A^T = -A$, we get

$$\det(A) = \det\left(A^T\right) = \det(-A) = (-1)^5 \det(A) = -\det(A).$$

The only number that is equal to its negative is 0.

12. Determine whether the given matrix is singular or nonsingular. If it is nonsingular, find its inverse.

(a) $\begin{bmatrix} 6 & 0 \\ -3 & 2 \end{bmatrix}$

Answer: $\begin{bmatrix} 1/6 & 0 \\ 1/4 & 1/2 \end{bmatrix}$

(b) $\begin{bmatrix} -2\pi & -\pi \\ -\pi & \pi \end{bmatrix}$

Answer: $\dfrac{-1}{3\pi} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ -1 & 5 & 1 \end{bmatrix}$

Answer:
$\begin{bmatrix} 7/15 & -13/30 & -8/15 \\ 1/15 & -2/15 & 1/15 \\ 2/15 & 7/30 & 2/15 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Answer:
$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix}$

Answer:
$\dfrac{1}{3} \begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -1 & 1 \\ -1 & 5 & 0 \\ 0 & 6 & -1 \end{bmatrix}$

Answer: This matrix is singular.

13. Use an inverse matrix to solve the linear system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 2x_2 + 2x_3 &= -3 \\ 3x_1 - x_2 + 5x_3 &= 7 \end{aligned}$$

Answer: $x_1 = 21,\ x_2 = 1,\ x_3 = -11$

14. Write the linear system

$$\begin{aligned} 7x_1 - 2x_2 &= b_1, \\ 3x_1 - 2x_2 &= b_2, \end{aligned}$$

in the form $A\mathbf{x} = \mathbf{b}$. Use $\mathbf{x} = A^{-1}\mathbf{b}$ to solve the system for each $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 10 \\ 50 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 0 \\ -20 \end{bmatrix}.$$

Answer: $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 7 & -2 \\ 3 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

$$A^{-1} = \begin{bmatrix} 1/4 & -1/4 \\ 3/8 & -7/8 \end{bmatrix},\quad A^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -13/8 \end{bmatrix},\quad A^{-1}\begin{bmatrix} 10 \\ 50 \end{bmatrix} = \begin{bmatrix} -10 \\ -40 \end{bmatrix},\quad A^{-1}\begin{bmatrix} 0 \\ -20 \end{bmatrix} = \begin{bmatrix} 5 \\ 35/2 \end{bmatrix}$$

15. Determine the matrix representation for the given linear transformation T relative to the ordered bases B and C.

(a) $T : M_2(\mathbb{R}) \to P_3$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-d) + 3bx^2 + (c-a)x^3$ with

i. $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $C = \{1, x, x^2, x^3\}$,

Answer: $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

ii. $B = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ and $C = \{x, 1, x^3, x^2\}$.

Answer: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(b) $T : V \to V$, where $V = \operatorname{span}\{e^{2x}, e^{-3x}\}$, given by $T(f) = f'$ with

i. $B = C = \{e^{2x}, e^{-3x}\}$,

Answer: $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

ii. $B = \{e^{2x} - 3e^{-3x}, 2e^{-3x}\}$ and $C = \{e^{2x} + e^{-3x}, -e^{2x}\}$.

Answer: $\begin{bmatrix} 9 & -6 \\ 7 & -6 \end{bmatrix}$

16. Determine which of the indicated column vectors are eigenvectors of the given matrix A. Give the corresponding eigenvalue for each one that is.

(a) $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$, $\mathbf{v}_1 = (5, -2),\ \mathbf{v}_2 = (2, 5),\ \mathbf{v}_3 = (-2, 5)$

Answer: $\mathbf{v}_3$ is an eigenvector for the eigenvalue $\lambda = -1$.

(b) $A = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$, $\mathbf{v}_1 = (1, 2 - \sqrt{2}),\ \mathbf{v}_2 = (2 + \sqrt{2}, 2),\ \mathbf{v}_3 = (\sqrt{2}, -\sqrt{2})$

Answer: $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the eigenvalue $\lambda = \sqrt{2}$.

(c) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$, $\mathbf{v}_1 = (0, 0),\ \mathbf{v}_2 = (2 + 2i, -1),\ \mathbf{v}_3 = (2 + 2i, 1)$

Answer: $\mathbf{v}_2$ is an eigenvector for the eigenvalue $\lambda = 2i$.

Note: The zero vector is not allowed as an eigenvector.

(d) $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$, $\mathbf{v}_1 = (-1, 4, 3),\ \mathbf{v}_2 = (1, 4, 3),\ \mathbf{v}_3 = (3, 1, 4)$

Answer: $\mathbf{v}_2$ is an eigenvector for the eigenvalue $\lambda = 3$.

17. Find the eigenvalues and eigenvectors of the given matrix.

(a) $\begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix}$

Answer: $\lambda_1 = 1,\ \mathbf{v}_1 = (1,1),\ \lambda_2 = 6,\ \mathbf{v}_2 = (2,7)$

(b) $\begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix}$

Answer: $\lambda_1 = 3i,\ \mathbf{v}_1 = (2, 1+3i),\ \lambda_2 = -3i,\ \mathbf{v}_2 = (2, 1-3i)$

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Answer: $\lambda_1 = 1+i,\ \mathbf{v}_1 = (i,1),\ \lambda_2 = 1-i,\ \mathbf{v}_2 = (-i,1)$

(d) $\begin{bmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{bmatrix}$

Answer: $\lambda_1 = 4,\ \mathbf{v}_1 = (1,1,1),\ \lambda_2 = -4,\ \mathbf{v}_2 = (1,9,1),\ \lambda_3 = 0,\ \mathbf{v}_3 = (9,45,25)$

(e) $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

Answer: $\lambda_1 = -1,\ \mathbf{v}_1 = (1,-1,1),\ \lambda_2 = i,\ \mathbf{v}_2 = (i,1,1),\ \lambda_3 = -i,\ \mathbf{v}_3 = (-i,1,1)$

(f) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{bmatrix}$

Answer: $\lambda_1 = 1,\ \mathbf{v}_1 = (1,0,0),\ \lambda_2 = 5,\ \mathbf{v}_2 = (1,2,0),\ \lambda_3 = -7,\ \mathbf{v}_3 = (1,2,-4)$

18. Determine whether the given matrix A is diagonalizable. If so, find the matrix P that diagonalizes A and the diagonal matrix D such that $D = P^{-1}AP$.

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

Answer: This matrix has one eigenvalue with algebraic multiplicity 2, but only one linearly independent eigenvector. It is defective and therefore not diagonalizable.

(b) $\begin{bmatrix} -9 & 13 \\ -2 & 6 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 13 \\ 1 & 2 \end{bmatrix},\ D = \begin{bmatrix} 4 & 0 \\ 0 & -7 \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix},\ D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

(d) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(e) $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: This matrix has two eigenvalues with algebraic multiplicities of 1 and 2, respectively. The latter has only one linearly independent eigenvector, hence the matrix is defective and not diagonalizable.

(f) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & -\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

19. Use diagonalization to solve the given system of differential equations.

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} 3c^1 e^{7t} + 2c_2 e^{-4t} \\ c_1 e^{7t} - 3c_2 e^{-4t} \end{bmatrix}$$

(b) $\mathbf{x}' = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{-4t} \\ c_1 e^{2t} - c_2 e^{-4t} \\ c_1 e^{2t} + c_3 e^{6t} \end{bmatrix}$$

(c) $\mathbf{x}' = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} c_1 e^{2\sqrt{2}t} + c_2 e^{-2\sqrt{2}t} + c_3 \\ \sqrt{2} c_1 e^{2\sqrt{2}t} - \sqrt{2} c_2 e^{-2\sqrt{2}t} \\ c_1 e^{2\sqrt{2}t} + c_2 e^{-2\sqrt{2}t} - c_3 \end{bmatrix}$$

MATH 2210Q MIDTERM EXAM I PRACTICE PROBLEMS

Date and place: September 30, 2020, 5:00–8:00pm.
Material: Sections 1.1–1.5, 1.7–1.9, 2.1–2.3 Lectures 1–12 HW1-4, and the practice exam and additional practice problems below.
Policy: No calculators will be allowed at the exam.
Format: The actual exam will have the same format as the practice problems. The content will be reasonably similar to the practice exam and the additional practice problems.

Practice Exam:

True/False Questions

________ (1) If an $m \times n$ matrix A has m pivots after row reduction, then for each $\mathbf{b} \in \mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

________ (2) If a system of linear equations $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then the system of linear equations $A\mathbf{x} = \mathbf{c}$ for any other vector $\mathbf{c}$ has either no solution or infinitely many solutions.

________ (3) For three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, if $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent; then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

________ (4) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linearly dependent, then $\mathbf{u}$ is a linearly combination of $\mathbf{v}$ and $\mathbf{w}$.

________ (5) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and assume that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set in $\mathbb{R}^n$. Then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set in $\mathbb{R}^m$.

________ (6) For a matrix A, any linear combination of its columns can be written as $A\mathbf{x}$ for suitable $\mathbf{x}$.

________ (7) Let A be a matrix A. Suppose that $\mathbf{x} \mapsto A\mathbf{x}$ defines a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, then A^T can define a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$ by setting $\mathbf{x} \mapsto A^T\mathbf{x}$.

________ (8) The linear transformation $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 + 2x_2 \\ x_1 + x_2 \\ x_1 \end{pmatrix}$ is one-to-one.

________ (9) If the linear equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the columns of A are linearly independent.

Problem I Consider the following system of linear equations

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 + x_2 + 2x_3 = 4 \\ x_1 + ax_2 + 2x_3 = b \end{cases}$$

(1) Write the augmented matrix for the system.
(2) For which a, does the system have a free variable?
(3) When $a = 1$, for which b, is the system consistent? When the system is consistent, give the solution of the system.

Problem II Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

(1) For which values of a and b, is the vector $\mathbf{w} = \begin{pmatrix} 2 \\ a \\ b \end{pmatrix}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

(2) When $a = 5$ and $b = 8$, write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Are there more than one ways to write $\mathbf{w}$ as a linear combination?

(3) Consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. Is T one-to-one; is T onto?

Problem III

(1) Let T be the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$, given by first reflecting about line $x_1 = x_2$, and then inserting the plane $\mathbb{R}^2$ naturally as the x_1x_2-plane of $\mathbb{R}^3$. Compute the standard matrix for T (denoted by A).

(2) Are the columns of A linearly independent?

(3) For which value b is the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ b \end{pmatrix}$ in the range of T? When it is in the range of T, find a vector $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$.

Problem IV

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and $\mathbf{u} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$.

Determine if the following expression makes sense. If it does, compute its value.

(1) $B^T + \mathbf{u}$.

(2) ACB^T.

(3) $(A\mathbf{u})^T C$.

Problem V

Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & 5 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix}.$$

(1) Write down the associated homogeneous equation.

(2) Observe that the middle column is the sum of the other two columns. What does this fact tell you about the solution to the homogeneous solution?

(3) Observe that $\mathbf{b}$ is sum of all the columns of A. Which particular solution does this fact give you?

(4) Purely based on the information of (2) and (3), what can you say about the general solution to the linear system $A\mathbf{x} = \mathbf{b}$?

Additional practice problems

Problem 1 Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$. Write the solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ in the form of $\mathbf{x} = \mathbf{p} + t\mathbf{v}$.

Problem 2 (1) Let T be the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, given by first rotating clockwise $\pi/4$, and then projecting to the x-axis. Compute the standard matrix for T (denoted by A).

(2) Using the geometric description of T, explain whether it is true that the system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

(3) Is the linear transformation T one-to-one?

Problem 3 Consider the following three linear transformations:

- $R : \mathbb{R}^2 \to \mathbb{R}^2$ is given by rotating about the origin of radian $\pi/2$ counterclockwise.
- $S : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection about x_1-axis.
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection about the line $x_1 = x_2$.

Write the standard matrices of R, S, and T. Prove that $R = T \circ S$ as linear transformations.

Solution to the Practice Exam:

True/False Questions

(1) If an $m \times n$ matrix A has m pivots after row reduction, then for each $\mathbf{b} \in \mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

True Since there is a pivot in each row, there is no room for the system to be inconsistent. The augmented system would look like

$$\begin{pmatrix} \blacksquare & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * \end{pmatrix}$$

(2) If a system of linear equations $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then the system of linear equations $A\mathbf{x} = \mathbf{c}$ for any other vector $\mathbf{c}$ has either no solution or infinitely many solutions.

True The condition that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions implies that the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solution $\mathbf{x}_h$. Therefore, for a vector $\mathbf{c}$, if $A\mathbf{x} = \mathbf{c}$ has one solution, say $\mathbf{p}$, the general solution will be $\mathbf{x} = \mathbf{p} + \mathbf{x}_h$. Since there are infinitely many $\mathbf{x}_h$, so there are infinitely many solutions to $A\mathbf{x} = \mathbf{c}$ in this case.

(3) For three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, if $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent; then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

False For example, in $\mathbb{R}^2$, $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. These three vectors are not multiples of each other. Thus $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ are linearly independent. Yet, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ has three vectors in $\mathbb{R}^2$, it must be linearly dependent.

(4) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linearly dependent, then $\mathbf{u}$ is a linearly combination of $\mathbf{v}$ and $\mathbf{w}$.

False It is true that one of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a linearly combination of others, but it need not be $\mathbf{v}$. For example, $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ contains the zero vector, the set is linearly dependent. But $\mathbf{u}$ is clearly not a linear combination of $\mathbf{v}, \mathbf{w}$.

(5) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and assume that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set in $\mathbb{R}^n$. Then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set in $\mathbb{R}^m$.

True If $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ is a linear relation. Then $\mathbf{0} = T(\mathbf{0}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3)$. So $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent.

(6) For a matrix A, any linear combination of its columns can be written as $A\mathbf{x}$ for suitable $\mathbf{x}$.

True A linear combination of columns of A with weights $c_1, \ldots, c_n$, is $A\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

(7) Let A be a matrix A. Suppose that $\mathbf{x} \mapsto A\mathbf{x}$ defines a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, then A^T can define a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$ by setting $\mathbf{x} \mapsto A^T\mathbf{x}$.

True Since $\mathbf{x} \mapsto A\mathbf{x}$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, then A is an $m \times n$ matrix. Then A^T is an $n \times m$ matrix, so $\mathbf{x} \mapsto A^T\mathbf{x}$ is a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

(8) The linear transformation $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 + 2x_2 \\ x_1 + x_2 \\ x_1 \end{pmatrix}$ is one-to-one.

True The standard matrix for T is $\begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$. Since the columns of A are not multiple of each other and thus linearly independent, T is one-to-one.

(9) If the linear equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the columns of A are linearly independent.

True Since $A\mathbf{x} = \mathbf{b}$ has a unique solution, the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution. Thus the columns of A are linearly independent.

Problem I Consider the following system of linear equations

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 + x_2 + 2x_3 = 4 \\ x_1 + ax_2 + 2x_3 = b \end{cases}$$

(1) Write the augmented matrix for the system.
(2) For which a, does the system have a free variable?
(3) When $a = 1$, for which b, is the system consistent? When the system is consistent, give the solution of the system.

Solution (1) The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 1 & a & 2 & b \end{pmatrix}$$

(2) We perform a row reduction

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 1 & a & 2 & b \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & a-1 & 1 & b-3 \end{pmatrix}$$

If the system has a free variable, we need $a = 1$.

(3) When $a = 1$, we continue the row operation as follows.

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & b-3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & b-4 \end{pmatrix}$$

The system is consistent when $b = 4$.

When $b = 4$, we have

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution is

$$\begin{cases} x_1 = 2 - x_2 \\ x_2 \text{ is free} \\ x_3 = 1. \end{cases}$$

Problem II Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}3\\4\\5\end{pmatrix}.$$

(1) For which values of a and b, is the vector $\mathbf{w} = \begin{pmatrix}2\\a\\b\end{pmatrix}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

(2) When $a = 5$ and $b = 8$, write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Are there more than one ways to write $\mathbf{w}$ as a linear combination?

(3) Consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$. Is T one-to-one; is T onto?

Solution (1) We try to solve the linear system

$$\begin{pmatrix}1&1&3&2\\1&2&4&a\\1&3&5&b\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&1&3&2\\0&1&1&a-2\\0&2&2&b-2\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&1&3&2\\0&1&1&a-2\\0&0&0&b-2-2(a-2)\end{pmatrix}$$

The system is consistent if and only if $b-2-2(a-2) = b-2a+2 = 0$. So when $b-2a+2=0$, $\mathbf{w}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(2) When $a = 5$ and $b = 8$, we continue the system above.

$$\begin{pmatrix}1&1&3&2\\0&1&1&3\\0&0&0&0\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&0&2&-1\\0&1&1&3\\0&0&0&0\end{pmatrix}$$

Then solution is

$$\begin{cases} x_1 = -1 - 2x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free.} \end{cases}$$

For example, when $x_3 = 0$, $x_1 = -1$ and $x_2 = 3$, we can write

$$\mathbf{w} = -\mathbf{v}_1 + 3\mathbf{v}_2 + 0\mathbf{v}_3.$$

If you prefer, you can take x_3 to be any number, say $x_3 = 2$, then $x_1 = -5$ and $x_2 = 1$ and

$$\mathbf{w} = -5\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3.$$

Since one can choose x_3 to be any number we want, there are infinitely many linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ to represent $\mathbf{w}$.

(3) By the discussion in (1), T is not onto (because if $b-2a+2 \neq 0$, $\mathbf{w} = \begin{pmatrix}2\\a\\b\end{pmatrix}$ is not in the range of T). T is not one-to-one because we have seen in (2) that the system has a free variable.

Problem III

(1) Let T be the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$, given by first reflecting about line $x_1 = x_2$, and then inserting the plane $\mathbb{R}^2$ naturally as the x_1x_2-plane of $\mathbb{R}^3$. Compute the standard matrix for T (denoted by A).

(2) Are the columns of A linearly independent?

(3) For which value b is the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ b \end{pmatrix}$ in the range of T? When it is in the range of T, find a vector $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$.

Solution (1) We compute the images of the standard basis

$$T : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad T : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

So the standard matrix of T is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(2) The columns of A are linearly independent as they are not multiple of each other.

(3) We solve the following system

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & b \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & b \end{pmatrix}.$$

When $b = 0$, the system is consistent and $\mathbf{b}$ is in the range. So when $b = 0$, we get $x_1 = 2$ and $x_2 = 1$. So for $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we get $T(\mathbf{x}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

Problem IV

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and $\mathbf{u} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$.

Determine if the following expression makes sense. If it does, compute its value.

(1) $B^T + \mathbf{u}$.

(2) ACB^T.

(3) $(A\mathbf{u})^T C$.

Solution (1)

$$B^T + \mathbf{u} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 8 \end{pmatrix}.$$

(2)

$$ACB^T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

We see that $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ does not make sense. So the total product does not exist.

(3)

$$(A\mathbf{u})^T C = \left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right)^T \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \left(\begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 \\ 5 + 6 + 7 \end{pmatrix} \right)^T \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 18 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 92 & 148 \end{pmatrix}$$

Problem V

Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & 5 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix}.$$

(1) Write down the associated homogeneous equation.

(2) Observe that the middle column is the sum of the other two columns. What does this fact tell you about the solution to the homogeneous solution?

(3) Observe that $\mathbf{b}$ is sum of all the columns of A. Which particular solution does this fact give you?

(4) Purely based on the information of (2) and (3), what can you say about the general solution to the linear system $A\mathbf{x} = \mathbf{b}$?

Solution (1) The associated homogeneous equation is $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(2) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ denote the columns of A. As we observed that $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$. So $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. This means that $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a homogeneous solution.

(3) As pointed out, $\mathbf{b} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Then $A\mathbf{x} = \mathbf{b}$ has a particular solution $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(4) From (2) and (3), we see that

$$\mathbf{x} = \mathbf{p} + t\mathbf{x}_h = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

is a general solution to $A\mathbf{x} = \mathbf{b}$.

Solutions to Additional practice problems

Problem 1 Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$. Write the solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ in the form of $\mathbf{x} = \mathbf{p} + t\mathbf{v}$.

Solution We solve

$$\begin{pmatrix} 1 & 1 & 3 & 1 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We get

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 - 2x_3 \\ 2 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

Problem 2 (1) Let T be the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, given by first rotating clockwise $\pi/4$, and then projecting to the x-axis. Compute the standard matrix for T (denoted by A).

(2) Using the geometric description of T, explain whether it is true that the system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

(3) Is the linear transformation T one-to-one?

Solution (1) We compute the image of T under the described linear transformation.

$$T: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

So the standard matrix of T is

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 \end{pmatrix}$$

(2) Note that the last step is the projection to the x-axis. So the range of T is contained in the x-axis and thus T cannot be onto. Thus $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$.

(3) The linear transformation T is not one-to-one because the columns of A are linearly dependent.

Problem 3 Consider the following three linear transformations:

- $R : \mathbb{R}^2 \to \mathbb{R}^2$ is given by rotating about the origin of radian $\pi/2$ counterclockwise.
- $S : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection about x_1-axis.
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection about the line $x_1 = x_2$.

Write the standard matrices of R, S, and T. Prove that $R = T \circ S$ as linear transformations.

Solution

$$R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad R\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathrm{Mat}(R) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \mathrm{Mat}(S) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathrm{Mat}(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We check that

$$\mathrm{Mat}(T)\mathrm{Mat}(S) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathrm{Mat}(R)$$

So it follows that $R = T \circ S$ as linear transformations.

MATH 2210Q MIDTERM EXAM II PRACTICE PROBLEMS

Date and place: Thursday, November 6^{th}, 2020.
Material: Sections 3.1-3.2-3.3-4.1–4.7. Lecture 13–24, HW5-6-7, and the practice exam and additional practice problems below.
Policy: No calculators will be allowed at the exam.
Format: The actual exam will have the same format as the practice problems. The content will be reasonably similar to the practice exam and the additional practice problems.

Practice Exam:

True/False Questions

________ (1) If A and B are $n \times n$ matrices, then $(A-B)(A+B) = A^2 - B^2$.

________ (2) If A is 3×3 matrix and the equation $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has a unique solution, then A is invertible.

________ (3) If two rows of a 3×3-matrix A are the same, then $\det A = 0$.

________ (4) If B is formed by replacing Row 1 of A by the sum of Row 1 and Row 3 of A, then $\det B = \det A$.

________ (5) If B is formed by replacing one row of A by a linear combination of the other rows of A, then $\det B = \det A$.

________ (6) $\det(-A) = -\det(A)$

________ (7) If V is a p-dimensional vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set spanning V. Then S cannot be linearly dependent.

________ (8) If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that $H = \mathrm{Col}(A)$.

________ (9) If A is a $m \times n$ matrix and the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then $\mathrm{rank}(A) = m$.

________ (10) A change-of-coordinates matrix is always invertible.

Short Problems

(1) Give the inverse of the following matrix.

$$\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}.$$

(2) Find vectors that span the following subspace

$$H = \left\{ \begin{pmatrix} a+3b+c \\ 3b+c \\ -a+c \\ a+b \end{pmatrix} ;\ \text{for } a,b,c \in \mathbb{R} \right\}.$$

(3) If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is a matrix with $\det A = 5$. What is the determinant of $\begin{pmatrix} 9a & 6b & 3c \\ 6d & 4e & 2f \\ 3g & 2h & i \end{pmatrix}$?

Problem I For the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ -3 & 1 & 3 \\ 2 & -3 & 1 \end{pmatrix}$,

(1) Compute its determinant.
(2) Determine $\det((\frac{1}{2}A)^{-1})$.
(3) Consider the parallelepiped $\mathbf{P}$ with one vertex at the origin, and adjacent vertices

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix},$$

and consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathbf{x} \mapsto A\mathbf{x}$. What is the volume of $T(\mathbf{P})$?

Problem II Consider $\mathbf{P}_2 = \{\text{Polynomials of degree } \leq 2\}$ with basis

$$\mathcal{B} = \{1 + t,\ t^2,\ 2 + t + t^2\}.$$

(1) If a polynomial has $\mathcal{B}$-coordinate $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. What is this polynomial?

(2) Find the $\mathcal{B}$-coordinate of the polynomial $2 + 3t + t^2$.

Problem III

Consider the matrix $A = \begin{pmatrix} 1 & 3 & -1 & 1 & -2 \\ 2 & 6 & -2 & 3 & -6 \\ -1 & -3 & 1 & 5 & -10 \end{pmatrix}$.

(1)If $Range A)$ is a subspace of $\mathbb{R}^k$, what is k?
(2) If $\ker(A)$ is a subspace of $\mathbb{R}^\ell$, what is ℓ?
(3) Find a basis of $\ker(A)$, and $Range(A)$.

(4) Determine if the vector $\mathbf{u} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ is in $\ker(A)$.

(5) Determine if the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$ is in $Range(A)$.

Problem IV Consider the following vectors in $\mathbb{R}^2$.

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{c}_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases of $\mathbb{R}^2$.

(1) What is the $\mathcal{B}$-coordinate of the vector $\mathbf{v} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$?

(2) Which vector in $\mathbb{R}^2$ has $\mathcal{B}$-coordinate $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

(3) What is the change-of-coordinate matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{B}$?
(4) What is the change-of-coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$?

Additional practice problems

Problem 1 Compute the determinant and the inverse of the following matrix.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 3 \\ 2 & -3 & 4 \end{pmatrix}$$

Problem 2 Compute the following determinant

$$\begin{vmatrix} 1 & 3 & 2 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 7 & 3 & 0 \\ 0 & 2 & -3 & 2 \end{vmatrix}$$

Problem 3 Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 2 & 0 & 2 & 3 \\ 3 & 0 & 3 & 3 \\ 4 & 0 & 4 & 3 \end{pmatrix}$$

and $\dim\ker(A)$.

Solution to the Practice Exam:

True/False Questions

(1) If A and B are $n \times n$ matrices, then $(A-B)(A+B) = A^2 - B^2$.

False. Note that $(A-B)(A+B) = A(A+B) - B(A+B) = A^2 + AB - BA - B^2$. In general, $AB \neq BA$, so $(A+B)(A-B)$ need not be equal to $A^2 - B^2$.

(2) If A is 3×3 matrix and the equation $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has a unique solution, then A is invertible.

True. By the property of homogeneous/non-homogeneous equations, if $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has a unique solution, then $A\mathbf{x} = \mathbf{0}$ has a unique solution. Therefore A is invertible.

(3) If two rows of a 3×3-matrix A are the same, then $\det A = 0$.

True.

(4) If B is formed by replacing Row 1 of A by the sum of Row 1 and Row 3 of A, then $\det B = \det A$.

True.

(5) If B is formed by replacing one row of A by a linear combination of other rows of A, then $\det B = \det A$.

False. Such formed matrix B must have linearly dependent rows; and therefore, we always have $\det B = 0$.

Maybe it is good to give an example to illustrate the difference between (4) and (5): For (4), we are aiming to explain:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+2d+g & b+2e+h & c+2f+i \\ d & e & f \\ g & h & i \end{vmatrix}.$$

For (5), we are aiming to explain:

$$\begin{vmatrix} 2d+g & 2e+h & 2f+i \\ d & e & f \\ g & h & i \end{vmatrix} = 0.$$

(6) $\det(-A) = -\det(A)$

False. If A is $n \times n$, then $\det(-A) = (-1)^n \cdot \det(A)$.

(7) If V is a p-dimensional vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set spanning V. Then S cannot be linearly dependent.

True. If the number of vectors of a spanning set is equal to the dimension, the set is linearly independent.

(8) If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that $H = \mathrm{Col}(A)$.

True. Since H is a subspace of $\mathbb{R}^3$, then H has a spanning set with ≤ 3 vectors. Putting these spanning vectors as columns of A and maybe adding zero columns at the end, gives a 3×3 matrix. Clearly $H = \mathrm{Col}(A)$.

(9) If A is a $m \times n$ matrix and the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then $\mathrm{rank}(A) = m$.

True. If $\mathbf{x} \mapsto A\mathbf{x}$ is onto, the column space of A is exactly $\mathbb{R}^m$; and so $\mathrm{rank}(A) = \dim \mathrm{Col}(A) = m$.

(10) A change-of-coordinates matrix is always invertible.

True.

Short Problems

(1) Give the inverse of the following matrix.

$$\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}.$$

Solution We have

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{3\cdot 4 - 7\cdot 2}\begin{pmatrix} 4 & -7 \\ -2 & 3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 & -7 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 7/2 \\ 1 & -3/2 \end{pmatrix}.$$

(2) Find vectors that span the following subspace

$$H = \left\{ \begin{pmatrix} a+3b+c \\ 3b+c \\ -a+c \\ a+b \end{pmatrix} ;\ \text{for } a,b,c \in \mathbb{R} \right\}.$$

Solution We have

$$H = \left\{ \begin{pmatrix} a+3b+c \\ 3b+c \\ -a+c \\ a+b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 3 \\ 3 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} ;\ \text{for } a,b,c \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Remark: If I ask for a basis of H, we need to make a computation to see these three vectors are linearly dependent.

(3) If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is a matrix with $\det A = 5$. What is the determinant of $\begin{pmatrix} 9a & 6b & 3c \\ 6d & 4e & 2f \\ 3g & 2h & i \end{pmatrix}$?

Solution We solve this by the following list of equalities.

$$\begin{vmatrix} 9a & 6b & 3c \\ 6d & 4e & 2f \\ 3g & 2h & i \end{vmatrix} = 3\cdot 2\cdot \begin{vmatrix} 3a & 2b & c \\ 3d & 2e & f \\ 3g & 2h & i \end{vmatrix} = 3\cdot 2\cdot 3\cdot 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3\cdot 2\cdot 3\cdot 2\cdot 5 = 180.$$

Here, for the first equality, we take out a common factor of 3 from the first row (making $9a$ to $3a$, $6b$ to $2b$, and $3c$ to c), and take out a common factor of 2 from the second row. For the second equality, we take out a common factor of 3 from the first column and a common factor of 2 from the second column.

Problem I For the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ -3 & 1 & 3 \\ 2 & -3 & 1 \end{pmatrix}$,

(1) Compute its determinant.

(2) Determine $\det((\frac{1}{2}A)^{-1})$.

(3) Consider the parallelepiped $\mathbf{P}$ with one vertex at the origin, and adjacent vertices

$$\begin{pmatrix}3\\0\\0\end{pmatrix}, \begin{pmatrix}8\\2\\0\end{pmatrix}, \begin{pmatrix}4\\9\\-3\end{pmatrix},$$

and consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathbf{x} \mapsto A\mathbf{x}$. What is the volume of $T(\mathbf{P})$? the system consistent? When the system is consistent, give the solution of the system.

Solution (1) We compute the determinant as follows

$$\begin{vmatrix} 2 & 0 & 4 \\ -3 & 1 & 3 \\ 2 & -3 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 0 & 2 \\ -3 & 1 & 3 \\ 2 & -3 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 9 \\ 0 & -3 & -3 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 9 \\ 0 & 0 & 24 \end{vmatrix} = 2 \cdot 24 = 48.$$

(2) We have

$$\det((\tfrac{1}{2}A)^{-1}) = \left(\det(\tfrac{1}{2}A)\right)^{-1} = \left((\tfrac{1}{2})^3 \cdot 48\right)^{-1} = \frac{1}{6}.$$

Here the first equality follows from the fact that $\det(B^{-1}) = \det(B)^{-1}$ for a matrix B. For the second equality, we note that $\frac{1}{2}A$ means to multiply every entry of A by $\frac{1}{2}$. This is equivalent to multiply the first row by $\frac{1}{2}$, and then multiply the second row by $\frac{1}{2}$, and finally multiply the last row by $\frac{1}{2}$. So going from $\det A$ to $\det(\frac{1}{2}A)$, we need to multiply the determinant by $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$, one factor for each row. The cubic power corresponds to the matrix having size 3×3.

(3) The volume of the parallelepiped $\mathbf{P}$ is

$$\begin{vmatrix} 3 & 8 & 4 \\ 0 & 2 & 9 \\ 0 & 0 & -3 \end{vmatrix} = \left|3 \cdot 2 \cdot (-3)\right| = 18.$$

The linear transformation T will scale volume by $\det(A)$. So the volume of $T(\mathbf{P})$ is

$$48 \cdot 18 = 864.$$

(The actual exam will have smaller numbers in the computation.)

Problem II Consider $\mathbf{P}_2 = \{\text{Polynomails of degree } \leq 2\}$ with basis

$$\mathcal{B} = \{1 + t,\ t^2,\ 2 + t + t^2\}.$$

(1) If a polynomial has $\mathcal{B}$-coordinate $\begin{pmatrix}1\\2\\3\end{pmatrix}$. What is this polynomial?

(2) Find the $\mathcal{B}$-coordinate of the polynomial $2 + 3t + t^2$.

Solution (1) The polynomial is

$$1 \cdot (1 + t) + 2 \cdot t^2 + 3 \cdot (2 + t + t^2) = 7 + 4t + 5t^2.$$

(2) We write

$$2 + 3t + t^2 = c_1 \cdot (1 + t) + c_2 \cdot t^2 + c_3 \cdot (2 + t + t^2)$$

Comparing coefficients, we get

$$\begin{cases} 2 = c_1 + 2c_3 \\ 3 = c_1 + c_3 \\ 1 = c_2 + c_3. \end{cases}$$

$$\begin{pmatrix}1 & 0 & 2 & 2\\ 1 & 0 & 1 & 3\\ 0 & 1 & 1 & 1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 0 & -1 & 1\\ 0 & 1 & 1 & 1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & -1 & 1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & 0 & 4\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1\end{pmatrix}$$

So we have

$$[2+3t+t^2]_{\mathcal{B}} = \begin{pmatrix}4\\ 2\\ -1\end{pmatrix}.$$

Problem III

Consider the matrix $A = \begin{pmatrix}1 & 3 & -1 & 1 & -2\\ 2 & 6 & -2 & 3 & -6\\ -1 & -3 & 1 & 5 & -10\end{pmatrix}$.

(1) If $Range(A)$ is a subspace of $\mathbb{R}^k$, what is k?
(2) If $\ker(A)$ is a subspace of $\mathbb{R}^\ell$, what is ℓ?
(3) Find a basis of $\ker(A)$, and $Range(A)$.
(4) Determine if the vector $\mathbf{u} = \begin{pmatrix}3\\ -1\\ 0\\ 2\\ 1\end{pmatrix}$ is in $\ker(A)$.
(5) Determine if the vector $\mathbf{v} = \begin{pmatrix}3\\ 5\\ 3\end{pmatrix}$ is in $Range(A)$.

Solution (1) $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^3$; so $k=3$.
(2) $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^5$; so $\ell = 5$.
(3) We perform a row reduction process on A:

$$\begin{pmatrix}1 & 3 & -1 & 1 & -2\\ 2 & 6 & -2 & 3 & -6\\ -1 & -3 & 1 & 5 & -10\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 3 & -1 & 1 & -2\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 6 & -12\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 3 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

The columns 1, 4 are pivot columns, and columns 2, 3 and 5 correspond to free variables.

A basis of $\mathrm{Row}(A)$ is given by

$$\begin{pmatrix}1 & 3 & -1 & 0 & 0\end{pmatrix},\ \begin{pmatrix}0 & 0 & 0 & 1 & -2\end{pmatrix}.$$

A basis of $\mathrm{Col}(A)$ is given by

$$\begin{pmatrix}1\\ 2\\ -1\end{pmatrix},\ \begin{pmatrix}1\\ 3\\ 5\end{pmatrix}.$$

For $\mathrm{Nul}(A)$, we solve $A\mathbf{x} = \mathbf{0}$ to get

$$\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix} = \begin{pmatrix}-3x_2 + x_3\\ x_2\\ x_3\\ 2x_5\\ x_5\end{pmatrix} = x_2\begin{pmatrix}-3\\ 1\\ 0\\ 0\\ 0\end{pmatrix} + x_3\begin{pmatrix}1\\ 0\\ 1\\ 0\\ 0\end{pmatrix} + x_5\begin{pmatrix}0\\ 0\\ 0\\ 2\\ 1\end{pmatrix}$$

So a basis of $\mathrm{Nul}(A)$ is given by

$$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

(4) For this, we compute

$$\begin{pmatrix} 1 & 3 & -1 & 1 & -2 \\ 2 & 6 & -2 & 3 & -6 \\ -1 & -3 & 1 & 5 & -10 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-3+2-2 \\ 6-6+6-6 \\ -3+3+10-10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So $\mathbf{u} \in \mathrm{Nul}(A)$.

(5) For this, we solve

$$\begin{pmatrix} 1 & 3 & -1 & 1 & -2 & 3 \\ 2 & 6 & -2 & 3 & -6 & 5 \\ -1 & -3 & 1 & 5 & -10 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & -1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 6 & -12 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & -1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}$$

So the system is inconsistent, and therefore, $\mathbf{v}$ does not belong to $\mathrm{Col}(A)$.

Problem IV Consider the following vectors in $\mathbb{R}^2$.

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{c}_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases of $\mathbb{R}^2$.

(1) What is the $\mathcal{B}$-coordinate of the vector $\mathbf{v} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$?

(2) Which vector in $\mathbb{R}^2$ has $\mathcal{B}$-coordinate $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

(3) What is the change-of-coordinate matrix $P_{\mathcal{B} \leftarrow \mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{B}$?

(4) What is the change-of-coordinate matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$?

Solution

(1) For this, we solve

$$\begin{pmatrix} 1 & 3 & 7 \\ 5 & 1 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 7 \\ 0 & -14 & -28 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

So $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(2) The vector is

$$\begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+9 \\ 10+3 \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}.$$

(3) For this, we solve

$$\begin{pmatrix} 1 & 3 & 4 & 5 \\ 5 & 1 & 6 & 11 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -14 & -14 & -14 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

So we get $P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

(4) The change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ can be computed by taking the inverse:

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \frac{1}{1\cdot 1 - 1\cdot 2}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Solutions to Additional practice problems

Problem 1 Compute the inverse and the determinant of the following matrix.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 3 \\ 2 & -3 & 4 \end{pmatrix}.$$

Solution For determinant, we compute

$$\begin{vmatrix} 1 & 0 & -2 \\ -3 & 1 & 3 \\ 2 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & -3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{vmatrix} = 1 \cdot 1 \cdot (-1) = -1.$$

We compute A^{-1} as follows.

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 3 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & -1 & 7 & 3 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -7 & -3 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -13 & -6 & -2 \\ 0 & 1 & 0 & -18 & -8 & -3 \\ 0 & 0 & 1 & -7 & -3 & -1 \end{pmatrix}$$

So we have

$$A^{-1} = \begin{pmatrix} -13 & -6 & -2 \\ -18 & -8 & -3 \\ -7 & -3 & -1 \end{pmatrix}.$$

Problem 2 Compute the following determinant

$$\begin{vmatrix} 1 & 3 & 2 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 7 & 3 & 0 \\ 0 & 2 & -3 & 2 \end{vmatrix}$$

Solution We could compute by expansion by rows and columns:

$$\begin{vmatrix} 1 & 3 & 2 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 7 & 3 & 0 \\ 0 & 2 & -3 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 0 & 0 \\ 7 & 3 & 0 \\ 2 & -3 & 2 \end{vmatrix} - 0 \cdot ? + 0 \cdot ? - 0 \cdot ?.$$

Now this becomes a lower triangular matrix; so we get

$$1 \cdot 4 \cdot 3 \cdot 2 = 24.$$

Problem 3 Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 2 & 0 & 2 & 3 \\ 3 & 0 & 3 & 3 \\ 4 & 0 & 4 & 3 \end{pmatrix}$$

and $\dim \mathrm{Nul}(A)$.

Solution We first compute the column space of A:

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} \right\}$$

Here the second equality, we removed the zero vector and the duplicated vector.

Now the two vectors are not multiple of each other; so they are linearly independent and form a basis of $\mathrm{Col}(A)$.

It them follows that $\mathrm{rank}(A) = \dim \mathrm{Col}(A) = 2$, and therefore $\dim \mathrm{Nul}(A) = 4 - \dim \mathrm{rank}(A) = 2$.