BITS Pilani
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Mathematical Foundations

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Webinar#2

Agenda

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Problems on

- **Orthogonal Matrices**
- **Gram-Schmidt orthogonalization**
- **Characteristic polynomial**
- **Eigen values and Eigen vectors**
- **Spectral theorem**
- **Singular Value Decomposition**

Some Important Definitions

A square matrix $'A'$ is said to be orthogonal if its columns are orthonormal

(Orthonormal columns means dot product of any two columns is zero and each column is of length 1)

Equivalently, we can say $AA^T = A^T A = I$

$$A^{-1} = A^T$$

Remark: Rows of the orthogonal matrices are also orthonormal

Example: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix for all value of θ.

Gram Schmidt Process

Example: Convert the given basis vectors into orthonormal basis of $\mathbb{R}^3$. Given basis vectors are $a = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Solution:

Let A be a 3×3 matrix containing a, b and c as its columns. Then

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A^T\mathrm{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Now,

$$[A^T A | A^T] = \left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 0 & -1 & 1 \\ 1 & 2 & -1 & 1 & 0 & 1 \\ -1 & -1 & 2 & -1 & 1 & 0 \end{array}\right)$$

Example continued

Now, apply Gaussian elimination on $[A^T A|A^T]$, we have

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 0 & -1 & 1 \\ 1 & 2 & -1 & 1 & 0 & 1 \\ -1 & -1 & 2 & -1 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 + \frac{R_1}{2}]{R_2 \leftarrow R_2 - \frac{R_1}{2}} \left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 0 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{-1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{-1}{2} & \frac{3}{2} & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 + \frac{R_2}{3}]{} \left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 0 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{-1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} & \frac{-2}{3} & \frac{2}{3} & \frac{2}{3} \end{array}\right)$$

Example continued

On performing Gaussian elimination on $[A^T A|A^T]$,

we end up with $\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 0 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{-1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} & \frac{-2}{3} & \frac{2}{3} & \frac{2}{3} \end{array}\right)$

Note that the rows on the right hand side of the above matrix are **mutually orthogonal** and the squares of the lengths of these rows appear on the diagonal of the left side of the matrix.

They form a basis for $\mathbb{R}^3$. We can normalize the vectors to get an **orthonormal basis**.

Gram Schmidt Process

- Given a set of basis vectors for a vector space, can we convert the given basis into an orthogonal basis? Yes, we shall use Gaussian elimination to construct such a basis.
- Let us start with an example: Consider $\mathbb{R}^2$ and two basis vectors $\boldsymbol{v}_1 = (3, 1)^T$ and $\boldsymbol{v}_2 = (2, 2)^T$. Put these vectors into columns of a matrix $\boldsymbol{A}$ such that $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.
- The next step is to perform Gaussian elimination on the following augmented matrix: $[\boldsymbol{A}^T\boldsymbol{A}|\boldsymbol{A}^T] = \left[\begin{array}{cc|cc} 10 & 8 & 3 & 1 \\ 8 & 8 & 2 & 2 \end{array}\right]$
- On performing Gaussian elimination of this augmented matrix we end up with $\left[\begin{array}{cc|cc} 1 & 0.8 & 0.3 & 0.1 \\ 0 & 1 & -0.25 & 0.75 \end{array}\right]$

Gram Schmidt Process

- Note that after the completion of Gaussian elimination the two rows on the right hand side are orthogonal. They form a basis for R^2. We can normalize the vectors to get an orthonormal basis. Let us justify this technique.

- First we see that when the $m \times n$ matrix $\boldsymbol{A}$ has full column rank, then the matrix $\boldsymbol{A}^T\boldsymbol{A}$ is positive definite. To see this note that any solution $\boldsymbol{x}$ to $\boldsymbol{Ax} = \mathbf{0}$ is also a solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \mathbf{0}$ and vice-versa. Why is this the case?
- When $\boldsymbol{A}$ has linearly independent columns, there are no non-trivial solutions to $\boldsymbol{Ax} = \mathbf{0}$. Thus the fact that there are no non-trivial solutions to $\boldsymbol{Ax} = \mathbf{0}$ means that $\forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \neq \mathbf{0}, \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{Ax} > \mathbf{0}$. Note that since $\boldsymbol{A}^T\boldsymbol{A}$ is positive-definite, Gaussian elimination can be carried out on $\boldsymbol{A}^T\boldsymbol{A}$ without row exchanges.

Elementary transformations

- One of the steps of Gaussian elimination is the the subtraction of a multiple of a given row from a row below it. This step can be achieved by pre-multiplication of the given matrix by an elementary matrix. An elementary matrix is like an identity matrix except that one of the entries below the diagonal is allowed to be non-zero.
- To show how the process of elimination works using an elementary matrix consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and assume that we want to subtract two times the first row from the second row.

Elementary transformations

This can be accomplished by the following elementary matrix

$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so that the product

$$\begin{aligned} EA &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 2a_{11} & a_{22} - 2a_{12} & a_{23} - 2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned}$$

Product of elementary transformations

- A series of Gaussian elimination steps can be represented as a product of elementary transformations acting on $\boldsymbol{A}$: $\boldsymbol{E}_m \boldsymbol{E}_{m-1} \ldots \boldsymbol{E}_1 \boldsymbol{A}$.
- The product of lower triangular matrices can be seen to be lower triangular, and the inverse of a lower triangular matrix can also be seen as a lower triangular matrix.
- Thus the action of Gaussian elimination operations can be seen in the following terms $\boldsymbol{L}^{-1}\boldsymbol{A} = \boldsymbol{U}$ where the product of the elementary transformations is represented as the inverse of a lower triangular matrix for notational convenience, and the right hand side U is an upper triangular matrix. Thus we have $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$.

Final argument

- Returning to our problem we are performing Gaussian elimination on the matrix $\boldsymbol{A}^T\boldsymbol{A}$ where $\boldsymbol{A}$ contains the basis vectors as its columns. Upon Gaussian elimination on the augmented matrix we reduce $[\boldsymbol{A}^T\boldsymbol{A}|\boldsymbol{A}^T]$ to get $[\boldsymbol{U}|\boldsymbol{L}^{-1}\boldsymbol{A}^T$ where $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{L}\,\boldsymbol{U}$.
- Now we shall show that $\boldsymbol{Q}^T = \boldsymbol{L}^{-1}\boldsymbol{A}^T$ is an orthogonal matrix whose rows are orthogonal.
- Consider $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{L}^{-1}\boldsymbol{A}^T\boldsymbol{A}(\boldsymbol{L}^{-1})^T = \boldsymbol{U}(\boldsymbol{L}^{-1})^T =$ some upper triangular matrix
- But $\boldsymbol{Q}^T\boldsymbol{Q}$ is a symmetric matrix and can only be upper triangular if it is diagonal. Therefore $\boldsymbol{Q}$ is an orthogonal matrix whose columns are orthogonal. They can be normalized to obtain an orthonormal basis.

Characteristics Polynomial

Characteristics polynomial : For $\lambda \in \mathbb{R}$ and Matrix $A \in \mathbb{R}^{n \times n}$ we can define a polynomial $\rho_A(\lambda) = \det(A - \lambda \mathrm{I})$.

This polynomial can be written as:

$\rho_A(\lambda) = c_0 + c_1\lambda + \ldots + c_{n-1}\lambda^{n-1} + (-1)^n c_n \lambda^n$

where $c_{0,}\, c_1, \ldots c_{n\cdot} \quad \in \mathbb{R}$.

Example: Let the Matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

The Characteristics polynomial is $\det(A - \lambda \mathrm{I}) = (1 - \lambda)^2 - 1$.

Eigenvalues and eigenvectors

Let $A \in R$ be $n \times n$ square matrix.

$\lambda \in R$ is called eigenvalue of A and $x \in R^n \setminus \{0\}$ is the corresponding eigenvector of λ if

$$Ax = \lambda x$$

This equation is called the eigenvalue equation.

Spectral Theorem

Theorem: If A is n x n symmetric matrix then there exists an orthonormal basis of the corresponding vector space V consisting of the eigenvectors of A, and each eigenvalue is real.

Singular Value Decomposition

Singular values:
The square root of eigen values of a symmetric matrix A^TA or AA^T are called singular values of the matrix A.

Singular Vectors:
The eigen vectors of A^TA corresponding to singular values of A are called singular vectors.
The singular vectors of A^TA are called right singular vectors and singular vectors of AA^T are called left singular vectors.

Definition of SVD:
Let A be any matrix $m \times n$ then this matrix can be decomposed in to product of three matrices given by $A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V^T_{n\times n}$ and it is called singular value decomposition of A.
Where U and V are called orthogonal matrices or unitary matrices and Σ is called singular matrix.

Working Rule:

Step-1: Compute A^TA or AA^T .

Step-2: Find the eigen values of A^TA or AA^T .

Step-3: Compute the square root of non-zero eigen values of above step called singular values of A.

Step-4: Construct a singular matrix Σ of order same as A having singular values on the diagonal in decreasing order.

Step-5: Find the eigen vector of A^TA or AA^T and construct a orthogonal matrix U or V consisting of orthonormal eigen vectors of A^TA.

Step-6: Construct the vectors $\sigma_i u_i = Av_i$ and the orthogonal matrix U.

Step-7: Finally $A = U\Sigma V^T$ gives the singular value decomposition of matrix A.

Example

Find Singular value decomposition of the matrix $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

Solution:

Required solution is $A_{3\times 2} = U_{3\times 3}\Sigma_{3\times 2}V_{2\times 2}^T$

Step-1: Compute $A^T A$

$$\Rightarrow A^T A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} * \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

Step-2: Find the eigen values of $A^T A$: The characteristic equation(CE) of matrix $A^T A$.

$\Rightarrow \lambda^2 - Trace(A^T A)\lambda + |A^T A| = 0$

$Trace = 90,\ |A^T A| = 0$

$\Rightarrow CE : \lambda^2 - 90\lambda + 0 = 0$

$\Rightarrow \lambda(\lambda - 90) = 0$

$\Rightarrow \lambda_1 = 90, \lambda_2 = 0$ are called eigen values of matrix $A^T A$.

Step-3: Next find the singular values of A: $\sigma_1 = \sqrt{\lambda_1} = \sqrt{90}$ and $\sigma_2 = 0$.

Step-4: Construct the singular matrix of size same as A and with its diagonal elements being singular values in decreasing order and all other entries are zeros(equivalent diagonal matrix)

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Step-5: Find the eigen vectors of A^TA called right singular vector of A with respect to each eigen value;

For $\lambda = 90$

$$\Rightarrow A^TA - 90I = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} - 90 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix} \Rightarrow y = k \text{ is a free variable.}$$

Hence from first row we get $-9x - 27y = 0 \Rightarrow x = -3y = -3k$

Therefore eigen vectors w.r.t $\lambda = 90$ is

$$X_1 = k \begin{bmatrix} -3 \\ 1 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

For $\lambda = 0$

$$\Rightarrow A^T A - 0I = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix} \Rightarrow y = k \text{ is a free variable.}$$

Hence from first row we get $81x - 27y = 0 \Rightarrow x = \frac{1}{3}y = \frac{1}{3}k$

Therefore eigen vectors w.r.t $\lambda = 0$ is

$$X_2 = \frac{k}{3} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Step-6: Write the orthogonal matrix or right singular vectors matrix $V = [\hat{v}_1 || \hat{v}_2]$

$$V = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Step-7: Find the left eigen vectors or left singular vectors $\hat{u}_i$ using

$$\hat{u}_i = \frac{Av_i}{\sigma_i}$$

$i = 1$

$$u_1 = \frac{Av_i}{\sigma_i} = \frac{1}{\sqrt{90}} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} * \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{10}{\sqrt{900}} \\ -\frac{20}{\sqrt{900}} \\ -\frac{20}{\sqrt{900}} \end{bmatrix}$$

Since the second eigen value is zero $lambda_2 = \sigma_2 = 0$, to find the other two vector let us construct the orthogonal vectors using orthogonality conditions

Let the next vector be $w = (x, y, z)$ then $w \cdot u_1 = 0$

$\Rightarrow 10x - 20y - 20z = 0$

$\Rightarrow y = k_1$ and $z = k_2$ are free variables hence

$x = 2y + 2z = 2k_1 + 2k_2$

$$\Rightarrow w = k_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence the vectors orthogonal to u_1 are

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\text{Let } u_2 = w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \hat{u}_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$u_3 = w_2 - \frac{\langle w_2, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

$$\Rightarrow u_3 = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} \Rightarrow \hat{u}_3 = \frac{u_3}{\|u_3\|} = \begin{bmatrix} \frac{2}{\sqrt{45}} \\ -\frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{bmatrix}$$

Hence the orthogonal matrix U or matrix of left singular vectors is given by

$$U = \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}$$

Step-8:

Hence the singular value decomposition of given matrix A is

$$A = U\Sigma V^T$$

$$= \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Problem 1

Q Answer the following questions with justifications.

(A) Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix},$$

a professor asks two of his best students enrolled in Linear Algebra class to find the maximum value of $\mathbf{x^T A x}$, subject to the fact that $\|\mathbf{x}\|^2 = 1$, where $\|\|$ is the Euclidean norm. Given that the students have not studied Calculus earlier, the first student says that this is impossible, whereas the second one is optimistic in estimating the value. Who is correct and why? Give adequate justifications.
HINT: Find a symmetric matrix $\mathbf{B}$ such that $\mathbf{x^T A x} = \mathbf{x^T B x}$

(4 marks)

(B) Is $\lambda = 4$ an eigenvalue of

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}?$$

If yes, find the corresponding eigenvector.

(3 Marks)

NOTE: 1 additional mark if you can do it **without explicitly** finding the eigenvalues and checking if 4 is one of the eigenvalues.

Solution

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Answers

(A) expanding

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and rearranging terms gives us the off diagonal terms to be $(5+3)/2 = 4$ for the symmetric matrix. Thus,

$$\mathrm{B} = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$$

Let λ_1, λ_2 be eigenvalues of $\mathbf{B}$ with $\mathbf{e_1}$, $\mathbf{e_2}$ being the corresponding eigenvectors. Any $\mathbf{x}$ in $\mathbb{R}^2$ can be written as:

$$\mathbf{x} = p * \mathbf{e_1} + q * \mathbf{e_2}, p^2 + q^2 = 1$$

Now,

$$\mathbf{x}^{\mathbf{T}}\mathrm{B}\mathbf{x} = \begin{bmatrix} p\mathbf{e_1} & q\mathbf{e_2} \end{bmatrix} \begin{bmatrix} p\mathbf{e_1}\lambda_1 \\ \mathbf{e_2}\lambda_2 \end{bmatrix} = p^2\lambda_1 + q^2\lambda_2 \leq \max(\lambda_1, \lambda_2)$$

$\lambda_{max} = \frac{3+\sqrt{65}}{2}$

Solution

(B) If $\lambda = 4$ is an eigenvalue, then the eigenvector is in the null space of $\mathbf{A} - \lambda\mathbf{I_3}$. For this, the null space should have dimension > 0. Which can be verified as:

$$\left[\begin{array}{ccc|c} 3-4 & 0 & -1 & 0 \\ 2 & 3-4 & 1 & 0 \\ -3 & 4 & 5-4 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 * -1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2 * R_1, R_3 \leftarrow R_3 + 3 * R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Corresponding eigenvector is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} t$$

Problem 2

Q Answer the following questions with justifications.

(1) Given the characteristic equation of a matrix A, can we compute the characteristic equation of cA where c is a non-zero scalar without knowing the entries of A? If so, show how to do it using detailed calculations. Otherwise explain why it is not possible. Clearly state all your assumptions

Solution:

(1) The characteristic equation of the matrix cA is $det(cA - \lambda I) = 0$. We can rewrite this as $det(c(A - \frac{\lambda}{c}I)) = 0$ which can then be written as $c^n det(A - \frac{\lambda}{c}I) = 0$ or $det(A - \frac{\lambda}{c}I) = 0$. Let $\mu = \frac{\lambda}{c}$, and assume that the characteristic equation $det(A - \mu I) = 0$ can be written in terms of the polynomial $\mu^n + a_{n-1}\mu^{n-1} + \ldots a_1\mu + a_0 = 0$. Substituting $\lambda/c = \mu$ in this polynomial we get $(\frac{\lambda}{c})^n + a_{n-1}(\frac{\lambda}{c})^{n-1} + \ldots a_1(\frac{\lambda}{c}) + a_0 = 0$. Multiplying throught by c^n we finally get $\lambda^n + ca_{n-1}\lambda^{n-1} + \ldots a_1c^{n-1}\lambda + a_0c^n = 0$.

Thus we see that the characteristic equation of cA can be obtained by taking the coefficient a_k of the kth term of the characteristic equation of A and multiplying it by c^{n-k}. Thus there is no need to look at the entries of the matrix A.

Problem 3

Q Answer the following

(A) Given that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & b \end{bmatrix}, a, b \in \mathbb{R}$$

has 2 and -1 as its eigenvalues with algebraic multiplicity of 2 and 1 respectively. Further, the eigenvector corresponding to eigenvalue -1 is

$$\mathbf{e_3} = \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

(a) Find the value of b

(1 mark)

(b) Write all the eigenvalues and their geometric multiplicity

(1 mark)

(c) By observation, and using the properties of a symmetric matrix, find the other two eigenvectors ($\mathbf{e_1}$ and $\mathbf{e_2}$) of $\mathbf{A}$. (2 mark)

(d) Write the spectral decomposition of the matrix $\mathbf{A}$

(1 mark)

(e) find the value of a

(1 mark)

Problem 3 – Solution

Answers

(A) (a) The eigenvalues of **A** are 2,2,-1. Hence, 1+2+b = 2+2-1. Thus, b = 0

(b) since A is symmetric, eigenvalue 2 has GM 2 and eigenvalue -1 has GM 1

(c) The eigenvector corresponding to eigenvalue -1 is given

$$\mathbf{e_3} = \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

The other two eigenvectors correspond to eigenvalue 2 and both should lie in a plane orthogonal to $\mathbf{e_3}$ Thus, $\mathbf{e_1}$ and $\mathbf{e_2}$ are:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

Problem 3 - Solution

(d) The spectral decomposition of $\mathbf{A}$ is:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & b \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

(e) multiplying only the relevant terms:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & b \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & 0 \\ & \frac{2}{\sqrt{3}} \\ & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

$\mathrm{a} = \sqrt{2}$

Problem 3 B

(B) Given that the Singular Value Decomposition of $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & e_{22} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & \lambda_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

(a) Find the value of λ_2

(1 mark)

(b) Find the value of e_{22}

Problem 3 B solution

(B) (a)

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{2}} & & \\ \frac{\lambda_2}{\sqrt{18}} & -\frac{\lambda_2}{\sqrt{18}} & \frac{4\lambda_2}{\sqrt{18}} \end{bmatrix}$$

$$\frac{5}{2} + \frac{\lambda_2}{\sqrt{2}\sqrt{18}} = 3$$

$$\lambda_2 = 3$$

(b) $e_{22} = \frac{1}{\sqrt{2}}$

Problem 4

Q Answer the following for the given matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(A) Obtain the left-singular vectors of $\mathbf{A}$.

(3 marks)

(B) Obtain the right-singular vectors of $\mathbf{A}$.

(3 marks)

(C) Obtain the singular value matrix $\boldsymbol{\Sigma}$. What is the spectral norm of $\mathbf{A}$?

(2 marks)

Problem 4 solution

A Answers

(A)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A^T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{AA^T} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

eigenvalues and eigenvectors of AA^T :

$$\lambda_1 = \sigma_1^2 = 6, u_1 = \begin{bmatrix} 5 & 2 & 1 \end{bmatrix}^T, \|u_1\| = \sqrt{30}$$

$$\lambda_2 = \sigma_2^2 = 1, u_2 = \begin{bmatrix} 0 & \frac{-1}{2} & 1 \end{bmatrix}^T, \|u_2\| = \frac{\sqrt{5}}{2}$$

$$\lambda_3 = \sigma_3^2 = 0, u_3 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}^T, \|u_3\| = \sqrt{6}$$

Singular value matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix}$$

Left singular vectors

$$\mathbf{U} = \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} \end{bmatrix}$$

solution

(B) Right Singular vectors:

$$v_1 = \frac{1}{\sigma_1}\mathbf{A}^T u_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}2 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\2\\1\end{bmatrix}\frac{1}{\sqrt{30}} = \begin{bmatrix}2\\1\end{bmatrix}\frac{6}{\sqrt{6}\sqrt{30}}$$

$$v_2 = \frac{1}{\sigma_2}\mathbf{A}^T u_2 = \begin{bmatrix}2 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\\frac{1}{2}\\1\end{bmatrix}\frac{2}{\sqrt{5}} = \begin{bmatrix}-1\\2\end{bmatrix}\frac{1}{\sqrt{5}}$$

(C) Singular value matrix:

$$\mathbf{\Sigma} = \begin{bmatrix}\sqrt{6} & 0\\0 & \sqrt{1}\\0 & 0\end{bmatrix}$$

Spectral norm $= \sqrt{6}$ Marking Scheme: 1 mark for matrix, 1 mark for spectral norm

THANK YOU