fixed point of BP algorithm = stationary point (minima) of G_{Bethe}

$\Leftrightarrow$ Equations of BP ① $\Leftrightarrow$ Equations of G_{Bethe} (stationarity conditions) ②

①: belief equations for BP

$$b_i(x_i) = \phi_i(x_i) \prod_{j \in N(i)} m_{ji}(x_i) \quad (a)$$

$$b_{ij}(x_i, x_j) = \phi_i(x_i)\phi_j(x_j)\psi_{ij}(x_i,x_j) \prod_{k \in N(i)\setminus j} m_{ki}(x_i) \prod_{l \in N(j)\setminus i} m_{lj}(x_j) \quad (b)$$

$$m_{ji}(x_i) = \sum_{x_j} \psi_{ij}(x_i,x_j)\phi_j(x_j) \prod_{k \in N(j)\setminus i} m_{kj}(x_j) \quad (c) \rightarrow \text{fixed point}$$

② Stationarity conditions for G_{Bethe}.

$$G_{Bethe} = \sum_{i,j}\sum_{x_i x_j} b_{ij}(x_i,x_j)\left[E_{ij}(x_i,x_j) + \ln b_{ij}(x_i,x_j)\right] - \sum_i (q_i - 1)\sum_{x_i} b_i(x_i)\left[E_i(x_i) + \ln b_i(x_i)\right]$$

Restrictions: $\sum_{x_i} b_i(x_i) = 1 \rightarrow \gamma_i$ $\quad \sum_{x_i,x_j} b_{ij}(x_i,x_j) = 1 \rightarrow \gamma_{ij}$ $\quad$ } Lagrange multiplier.

$\sum_{x_j} b_{ij}(x_i,x_j) = b_i(x_i) \rightarrow \lambda_{ij}(x_i)$

$$L = G_{Bethe} + \sum \gamma_i (1 - b_i(x_i)) + \sum_{i,j}\gamma_{ij}\left(1 - \sum_{x_i x_j} b_{ij}(x_i,x_j)\right) + \sum_{i,j}\sum_{x_j}\lambda_{ij}(x_i)\left(b_i(x_i) - \sum_{x_j} b_{ij}(x_i,x_j)\right)$$

$$\frac{\partial L}{\partial b_{ij}(x_i,x_j)} = 0 \Rightarrow \left(E_{ij}(x_i,x_j) + \ln b_{ij}(x_i,x_j) + 1\right) - \gamma_{ij} - \lambda_{ij}(x_i) - \lambda_{ij}(x_j) = 0 \quad (d)$$

$$\frac{\partial L}{\partial b_i(x_i)} = 0 \Rightarrow -(q_i - 1)\left(E_i(x_i) + \ln b_i(x_i) + 1\right) - \gamma_i - \sum_{j \in N(i)} \lambda_{ij}(x_i) \quad (e)$$

$\leftarrow$ "pairwise" $b_{ij}(x_i,x_j)$ ~~only defined for~~ neighboring i,j

(f) $\lambda_{ij}(x_i) = \ln \prod_{k \in N(i)\setminus j} m_{ki}(x_i)$

(f) + (a) + (b) + (c) $\Rightarrow$ (d) + (e) satisfied

(f) + (d) + (e) $\Rightarrow$ (a) + (b).

$\therefore$ (a)+(b)+(c) $\Leftrightarrow$ (d) + (e) i.e. #.