Integer Quantum Hall Effect: Theoretical Approaches

Dusan Begus*
*Department of Physics, Brown University, Providence, Rhode Island, USA
(Dated: December 19, 2022)

In this article we present several successful theoretical interpretations of the integer quantum Hall effect. After a brief introduction to the Landau levels of electrons in a magnetic field, we go on to demonstrate a one-particle account of the Hall conductivity quantization. Secondly, we derive the Kubo formula for Hall conductivity and introduce the Brillouin zone Berry connections of a two-dimensional rectangular solid lattice. Finally, we derive the TKNN formula.

I. INTRODUCTION

Since von Klitzing's groundbreaking experiments in 1980, quantum Hall effects have been one of the major cornerstones of modern theoretical and experimental physics developments. In addition to the practical utility of quantum Hall effects (Kondo Effect, ATLAS-TFET, Graphene/Superconductor junctions [20]), physics of a two-dimensional electron structure under a magnetic field exhibits a number of particularly rich mathematical properties, and is hence used as a probing ground for quantum field theory, gauge theory, string theory, and many other fields of theoretical research [11, 14, 23, 26, 27]. In dealing with the Hall effect, one needs to separate the classical Hall effect [10] (discovered in 1879 by Edwin Hall), the integer quantum Hall effect [1, 3, 4, 5, 6] (discovered by von Klitzing, Dorda, Pepper [1]), and the fractional quantum Hall states [7, 11, 26, 27] (first discovered by Tsui, Stormer, Gossard [7]).
The following paper will deal with the theory of the integer quantum Hall effect. The structure of the article goes as following: In Sec. II we give a brief overview of the classical Hall effect and present von Klitzing's experimental results on a Hall sample. In Sec. III we consider the Landau levels of electrons in a magnetic field. We go on to present single-particle arguments for the formation of Hall plateaux. Aiming at more mathematical rigour, in Sec. IV we derive the Kubo formula for Hall conductivity and introduce the momentum-space Chern band numbers for a simple solid. Consequently, in Sec. V we structurize the TKNN invariants.
The primary sources for the paper were Chapter 2 in [11] and the seminal work on the topic in [3].

II. THEORETICAL SETUP

In general terms, when a two-dimensional material is exposed to a magnetic field B perpendicular to its plane, one can probe the electron structure of its very fabric by measuring the conductivity tensor of the sample. The most general conductivity relation reads [11, 15]:

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{1}$$

with E_i the electric field inside the material and J_i the electric current induced by charge carriers (electrons). The tensor σ_{ij} satisfies $\sigma_{12} = -\sigma_{21}$ because of the rotational symmetry of a 2D plane [11].
One can also write the resistivity tensor:

$$\rho_{ij} = \sigma^{-1} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \tag{2}$$

where the resistivity tensor (2) is the inverse of the conductivity tensor (1). While Edwin Hall experimentally probed (1) and (2) in [10], it is the use of the Drude model [11, 28-31] that provides us with the classical prediction for the corresponding tensors:

$$\sigma = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix} \tag{3}$$

with $\sigma_{DC} = \frac{ne^2\tau}{m}$, n being the concentration of the electrons in the sample, $\omega_B = eB/m$ the *Larmor frequency* of the electrons in the sample [28-31], τ the average scatering time of the confined electrons [13, 28-31], and e the electron charge.
Hence, one can also compute the conductivity matrix of the sample to be

$$\rho = \frac{1}{\sigma_{DC}} \begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}$$

and obtain

$$\rho_{xx} = \frac{m}{ne^2\tau} \quad \rho_{xy} = \frac{B}{ne} \tag{4}$$

The resistivities von Klitzing measured on an Si MOSFET in [1] differed peculiarly from the classical predictions in (4) (Fig. 1).

Gazing at Fig.1, we recognize that for certain periodic ranges of the value of B, ρ_{xy} takes constant values while ρ_{xx} drops to zero (except at plateaux transitions). In fact, ρ_{xy} takes the general form [1, 2, 11]:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \tag{5}$$

where $\nu \in \mathbb{N}$.
The centre of each plateux corresponds to [1,2]:

$$n = \frac{B\nu e}{2\pi\hbar} \tag{6}$$

The phenomenon described by (5-6) is reffered to as the *integer quantum Hall effect*. We ought to explain how it arises theoretically.

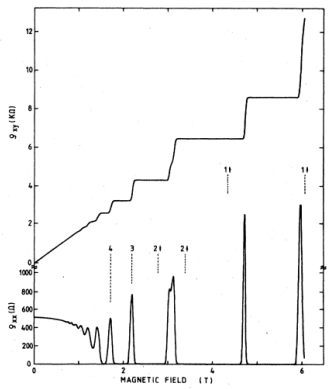

FIG. 1. Extracted from [2]: von Klitzing's experimental results: a probation of integer quantum Hall effect.

III. ONE-PARTICLE APPROACH: A CRUDE EXPLANATION OF IQHE

A. Electrons in a magnetic field

In order to understand the total conductivity of a large number of electrons in a sample ($N \sim 10^{23}$), we must first study how each of these electrons behaves in a magnetic field. We start by writing down the Lagrangian for a single electron in a magnetic field [11, 16]:

$$L = \frac{1}{2}\dot{\mathbf{x}}^2 - e\dot{\mathbf{x}} \cdot \mathbf{A} \tag{7}$$

where $\mathbf{A}$ is the electromagnetic vector potential field coupled to the electron. We note that we reasonably ignore the Coulomb interactions between electrons in this paper. While we will not offer a detailed explanation as to why electron interactions do not contribute to the explanation of IQHE, we shall note that the detailed study of the interaction dilemma can be found in [11]. On most general grounds, for a large concentration of electrons in the sample ($n \geq Be/2\pi\hbar$), by the Heisenberg uncertainty relations [22, 24], the electrons will be localized in space, and hence their momenta and energies will be large relative to the average Coulomb interaction contribution to their energy. Consequently, the inter-electronic potential contributions to the electron energies will be negligible. One can compute the canonical momentum of (7):

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}} - e\mathbf{A} \tag{8}$$

And we can thus compute the electron Hamiltonian as the Legendre transform [22, 25] of the Lagrangian

$$H = \dot{\mathbf{x}} \cdot \mathbf{p} - L = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 \tag{9}$$

Due to $\mathbf{B}$ being perpendicular to the plane xy, our gauge field $\mathbf{A}$ satisfies

$$\nabla \times \mathbf{A} = B\hat{\mathbf{z}} \tag{10}$$

We will pick the choice of *Landau gauge* [11, 24] for our field:

$$\mathbf{A} = xB\hat{\mathbf{y}} \tag{11}$$

The Hamiltonian then becomes

$$H = \frac{1}{2m}(p_x^2 + (p_y + eBx)^2)) \tag{12}$$

and with the solution ansatz

$$\psi_k(x, y) = e^{iky} f_k(x) \tag{13}$$

the energy eigenstate equation becomes the Hamiltonian for a quantum linear harmonic oscillator displaced in the x-direction:

$$H = \frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + kl_B^2)^2 \tag{14}$$

on the constant$-k$ subspace of the Hilbert space of the theory, where $l_B = \sqrt{\hbar/eB}$ is the *magnetic length* [11,17]. Consequently, the eigenvalue wavefunctions of (14) are [11]:

$$\psi_{\nu,k}(x, y) \sim e^{iky} H_\nu(x + kl_B^2) e^{-(x+kl_B^2)^2/(2l_B^2)} \tag{15}$$

where $H_\nu(x)$ is the ν-th Hermite polynomial [22]. We also have [22]:

$$E_\nu = \hbar\omega_B(\nu + \frac{1}{2}) \tag{16}$$

We can estimate the degeneracy of each energy state if we assume that the k-eigenstate is localized around $(-kl_B^2)$ [11, 25]: if we place our theory in a box of size $L_x \times L_y$, the values of k will be quantized as $k = \frac{2\pi}{L_y}$ [25], and the total degeneracy per energy value is

$$\mathcal{N} = \frac{L_y}{2\pi}\int_{-\frac{L_x}{l_B^2}}^{0} dk = \frac{L_x L_y}{2\pi l_B^2} = \frac{eBA}{2\pi\hbar} \tag{17}$$

The most important feature of (17) is the fact that the energy degeneracy depends linearly on B. This result can also be interpreted as

$$\mathcal{N} = \frac{AB}{\Phi_0} \tag{18}$$

Where $\Phi_0 = 2\pi\hbar/e$ is the *quantum of flux* [17]. It is also easy to incorporate an electric field along the x-axis into the Hamiltonian [11]:

$$H = \frac{1}{2m}(p_x^2 + (p_y^2 + eBx)^2) + eEx \tag{19}$$

whose eigenstates satisfy

$$\Psi_{\nu,k} = \psi_{\nu,k}(x + \frac{mE}{eB^2}, y) \tag{20}$$

And

$$E_{\nu,k} = \hbar\omega_B(\nu + \frac{1}{2}) - eE\left(kl_B^2 + \frac{eE}{m\omega_B^2}\right) + \frac{m}{2}\left(\frac{E}{B}\right)^2 \tag{21}$$

B. Deriving the resistivities

We will first argue that, using the Pauli exclusion principle [22], at low temperatures all electrons will fall into the least energetic possible configuration, in which no two electrons have the same wavefunction. Secondly, we will presume all of the electrons have their spins aligned with the magnetic field. Namely, this assumption follows from the fact that the magnetic fields of order $B \sim \Phi_0 n$ (which is the scale we are working in [17]), the energetic price of electrons anti-aligning their spin with the magnetic field would be much greater than the energy scales obtained through (21). More on this discussion can be found in [11], while a more scandalous investigation of the spin-approximation and its flaws can be found in [19].
Secondly, we shall try to obtain an expression for (1) when exactly ν Landau bands in (22) are filled. We will assume that a *small electric field* ($E \ll l_B B^2 e/m$) is applied to the sample, so that the general structure and $k-$*degeneracy* of Landau levels (18) is *not disturbed*, but rather probed on the account of its conductance properties. By virtue of (8), we know that the velocity vector of an electron satisfies

$$m\dot{\mathbf{x}} = \mathbf{p} + e\mathbf{A} \tag{22}$$

Thus, using the classical relation $\mathbf{I} = -e\dot{\mathbf{x}}$ we get [11]:

$$\mathbf{I} = -\frac{e}{m} \sum_{\text{filled states}} \langle\psi| - i\hbar\nabla + e\mathbf{A} \,|\psi\rangle \tag{23}$$

Working in Landau gauge with the electric field E in the x-direction, we have:

$$I_x = -\frac{e}{m} \sum_{n=1}^{\nu} \sum_{k} \langle\psi_{n,k}| - i\hbar \frac{\partial}{\partial x} |\psi_{n,k}\rangle = 0 \tag{24}$$

since the expectation value of momentum a harmonic oscillator wavefunction always vanishes [22]. On the other hand:

$$I_y = -\frac{e}{m} \sum_{n=1}^{\nu} \sum_{k} \langle\psi_{n,k}| \,\hbar k + exB \,|\psi_{n,k}\rangle \tag{25}$$

Since the second term in (25) dispenses the position expectation value of the electron [22], and we know from (20-21) that the electron Hermite polynomial wavefunction is localized around $(-\hbar k/eB - mE/eB^2)$, after some algebraic manipulations, we end up with [11]:

$$I_y = e\nu \sum_{k} \frac{E}{B} \tag{26}$$

But the sum over energy-degenerate k-states just evaluates to the number of states we previously computed in (18). Hence we conclude that

$$\boxed{I_x = 0, \quad I_y = \frac{e\nu E}{\Phi_0}} \tag{27}$$

From which we extract

$$\rho_{xx} = 0, \quad |\rho_{xy}| = \frac{2\pi\hbar}{e^2\nu} \tag{28}$$

and the corresponding value of B that fills ν Landau levels can be calculated to be:

$$\frac{N}{A} = n = \frac{B\nu}{\Phi_0} \tag{29}$$

In a neat fashion, we have recovered the information von Klitzing measured in [1,2].
While it is clear that (28-29) agrees with von Klitzing's result, we still do not have an account as for why plateaux (5-6) form. Why does ρ_{xy} remain constant over a *range* of magnetic field values?
A full explanation of (5-6) is clarified by two phenomena:

1. Disorder plays a great role in the formation of a ladder ρ_{xy} diagram. Namely, when one includes a random potential arising from the impurity scatterings of electrons [28-31] in the Hall sample, the energy dispersion relation experiences degeneracy and Landau levels (16,21) spread out (Fig. 2).

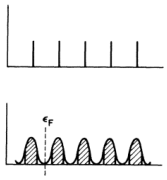

FIG. 2. Extracted from [4]: R. Laughlin's sketch of the energy degeneracy of Landau levels due to impurities. Straight lines in the upper graph correspond to exact quantized Landau levels.

   The states corresponding to the exact ν-valued Landau states are referred to as *extended states*, while the states dispersed around the mean energy are referred to as *localized states* [11, 15]. If one slowly changes the magnetic field passing thorugh the sample, instead of electrons drastically jumping from one Landau level to another, they will prefer to pass into localized states, therefore stabilizing total conductivity of the sample and rendering it constant. A less crude explanation of the *disorder argument* can be found in [4,11,17].

2. We might ask ourselves: If indeed the sample contains both localised and extended states, can we still trust (29)? Namely, our single-particle approach to understanding (5) only gave us a result when ν extended Landau levels are filled. It will turn out that, due to the nature of electromagnetic field being a gauge field [15, 24, 27], the occurrence of a phenomenon known as *spectral flow* of extended states will necessitate the resistivity values to lie exactly on the plateaux predicted by (29). More on the resistivity inertia can be found in the original work on the topic by Laughlin in [4].

IV. THE KUBO FORMULA

While the intuitive establishments presented in previous sections provide us with a hands-on motive for the formation of Fig. 1, we would still relish a rigorous, mathematical approach to the subject. In order to make advancement with the understanding of integer conductivities, we need to build a strong language of electron behaviour in 2D materials.

Let us consider a general many-electron Hamiltonian H_0 of our sample. When the material is exposed to an electromagnetic field A_μ with the choice of gauge $A_t = 0$ [24], it gains an interaction Hamiltonian [11]:

$$\Delta H = -\mathbf{J}\cdot\mathbf{A} \tag{30}$$

where $\mathbf{J}$ is the conserved current corresponding to the original Hamiltonian H_0, Ultimately, in order to compute conductivities, we will be interested in computing the full quantum expectation value of $\langle\mathbf{J}\rangle$ in the presence of $\mathbf{A}$.

By setting $A_t = 0$, from Maxwell's equations [24] we obtain $\mathbf{E} = -\partial_t\mathbf{A}$. Instead of passing a constant electric field through the plane, we employ the following trick [13]: we set $\mathbf{E}$ to be complex periodic $\mathbf{E}(t) = \mathbf{E}\exp(-i\omega t)$. Consequently, one can write

$$\mathbf{A} = \frac{\mathbf{E}}{i\omega}e^{-i\omega t} \tag{31}$$

Now we introduce the interaction vacuum state $|0(t)\rangle$ which satisfies $H_0|0(t)\rangle = 0$. All of the operators in the theory get shifted in the manner $\mathcal{O}(t) = e^{iH_0t}\mathcal{O}e^{-iH_0t}$, while the quantum states evolve like

$$|\psi(t)\rangle_I = U(t,t_0)\,|\psi(t_0)\rangle_I \tag{32}$$

$$U(t,t_0) = T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\Delta H(t')dt'\right) \tag{33}$$

with T the time-ordering operator [11,24].

We would like to calculate $\langle\mathbf{J}\rangle$, and so we write [13]:

$$\langle\mathbf{J}(t)\rangle = \langle 0(t)|\,\mathbf{J}(t)\,|0(t)\rangle = \langle 0|\,U^{-1}(t)\mathbf{J}(t)U(t)\,|0\rangle \approx \tag{34}$$

$$\approx \langle 0|\left(\mathbf{J}(t) + \frac{i}{\hbar}\int_{-\infty}^{t}dt'[\Delta H(t'),\mathbf{J}(t)]\right)|0\rangle \tag{35}$$

where we used the Baker-Campbell-Hausdorff formula [22] in the second line.

Thus, we have [13, 21]:

$$\langle J_i(t)\rangle = \frac{1}{\hbar\omega}\int_{-\infty}^{t}dt'\,\langle 0|\,[J_j(t'),J_i(t)]\,|0\rangle\,E_je^{-i\omega t'} \tag{36}$$

Using the time translation invariance of the correlator in (36) [13], we may switch $t\to t'' = t - t'$ and write

$$\langle J_i(t)\rangle = \frac{1}{\hbar\omega}\left(\int_0^\infty dt''e^{iwt''}\,\langle 0|\,[J_j(0),J_i(t'')]\,|0\rangle\right)E_je^{-i\omega t} \tag{37}$$

Hence, by the dynamical definition of conductivity tensor given in (1), we realize that

$$\sigma_{xy}(\omega) = \frac{1}{\hbar\omega}\int_0^\infty dte^{i\omega t}\,\langle 0|\,[J_y(0),J_x(t)]\,|0\rangle \tag{38}$$

We now notice that the current operator evolves as $\mathbf{J}(t) = e^{iH_0t}\mathbf{J}(0)e^{-iH_0t}$. Injecting (38) with a complete basis of eigenstates of H_0 through $\mathbf{1} = \sum|n\rangle\langle n|$, we get

$$\sigma_{xy}(\omega) = \frac{1}{\hbar\omega}\int_0^\infty dte^{i\omega t}\sum_n\{\langle 0|\,J_y\,|n\rangle\,\langle n|\,J_x\,|0\rangle\,e^{i(E_n-E_0)t/\hbar} - \\ - \langle 0|\,J_x\,|n\rangle\,\langle n|\,J_y\,|0\rangle\,e^{(E_0-E_n)t/\hbar}\}$$

And, since the first and second term $n = 0$ contributions to the previous expression cancel each other out, we also have (after integrating over time):

$$\sigma_{xy}(\omega) = -\frac{i}{\omega}\sum_{n\neq 0}\{\frac{\langle 0|\,J_y\,|n\rangle\,\langle n|\,J_x\,|0\rangle}{\hbar\omega + E_n - E_0} - \frac{\langle 0|\,J_x\,|n\rangle\,\langle n|\,J_y\,|0\rangle}{\hbar\omega + E_0 - E_n}\} \tag{39}$$

The idea now is to take the "DC"-limit of (39), since then the gauge potential profile (31) stabilizes in time and yields our theoretical setup in Sec. II:

$$\frac{1}{\hbar\omega + E_n - E_0} \approx \frac{1}{E_n - E_0} - \frac{\hbar\omega}{(E_n - E_0)^2} + \cdots \tag{40}$$

Finally, due to the rotational symmetry of the sample [11, 17], the transformation

$$x \longrightarrow y$$
$$y \longrightarrow -x$$
$$J_x \longrightarrow J_y$$
$$J_y \longrightarrow -J_x$$

should leave σ_{xy} invariant, and hence the $\sim 1/(E_n - E_0)$ contributions from the first and second term in (39) cancel each other out.

Finally, we get [11,17]:

$$\boxed{\sigma_{xy} = i\hbar\sum_{n\neq 0}\frac{\langle 0|\,J_y\,|n\rangle\,\langle n|\,J_x\,|0\rangle - \langle 0|\,J_x\,|n\rangle\,\langle n|\,J_y\,|0\rangle}{(E_n - E_0)^2}} \tag{41}$$

In literature, the expression (41) is known as the Kubo formula for Hall conductivity [11, 13, 21].

V. TKNN

A. The Berry Connection

In a general 2D solid electron structure with the rectangular first Brillouin zone [28-31], we will have

$$0 \le k_x \le \frac{\pi}{a}, \quad -0 \le k_y \le \frac{\pi}{b}. \tag{42}$$

Electrons will have Bloch wavefunctions (neglecting the band index) [28-31]:

$$\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{kx}} u_{\mathbf{k}}(\mathbf{x}) \tag{43}$$

Thus, in each energy band one can form the following vector expression

$$\mathcal{A}_i(\mathbf{k}) = -i \left\langle u_{\mathbf{k}} \right| \frac{\partial}{\partial k^i} \left| u_{\mathbf{k}} \right\rangle \tag{44}$$

usually referred to as the *Berry connection* [9]. Much like its electromagnetic analogue A_μ [24], we can use it to construct the *Berry curvature*:

$$\mathcal{F}_{xy} = \frac{\partial \mathcal{A}_x}{\partial k^y} - \frac{\partial \mathcal{A}_y}{\partial k^x} = -i \left\langle \frac{\partial u}{\partial k^y} \middle| \frac{\partial u}{\partial k^x} \right\rangle + i \left\langle \frac{\partial u}{\partial k^x} \middle| \frac{\partial u}{\partial k^y} \right\rangle \tag{45}$$

A nice property of (45) follows from integrating the Berry curvature across the entire Brillouin zone (which is topologically a torus [28-31]):

$$C = -\frac{1}{2\pi} \int_{\mathbf{T}^2} d^2k \mathcal{F}_{xy} \tag{46}$$

In 1946, S. S. Chern proved that C *must be* an integer [8, 18]. Thus, we observe a restriction of topological origin on the space of all connections (44) across the Brillouin zone.

B. Calculation of the crucial formula

In the following calculation, we will assume that the corresponding material electron interactions are negligible and that the same material is cooled to low temperatures ($k_B T \ll E_F$) [3, 11].

For any material whose energy spectrum decomposes into bands, and whose Fermi energy lies in one of the band gaps, we can write its off-diagonal conductivity through the Kubo formula (41), rendering all states beneath the Fermi energy filled and thus collectively representing the interacting vaccum $|0\rangle\,(t)$. All of the states above the Fermi energy represent possible electron excitations.

We write (α, γ are band indices):

$$\sigma_{xy} = i\hbar \sum_{E_\alpha < E_F < E_\gamma} \int_{\mathbf{T}^2} \frac{\left\langle u_{\mathbf{k}}^\alpha \right| J_y \left| u_{\mathbf{p}}^\gamma \right\rangle \left\langle u_{\mathbf{p}}^\gamma \right| J_x \left| u_{\mathbf{k}}^\alpha \right\rangle}{(E_\gamma(\mathbf{k}) - E_\alpha(\mathbf{p}))^2} - \tag{47}$$

$$- \frac{\left\langle u_{\mathbf{k}}^\alpha \right| J_x \left| u_{\mathbf{p}}^\gamma \right\rangle \left\langle u_{\mathbf{p}}^\gamma \right| J_y \left| u_{\mathbf{k}}^\alpha \right\rangle}{(E_\gamma(\mathbf{k}) - E_\alpha(\mathbf{p}))^2} \tag{48}$$

where $\mathbf{k}, \mathbf{p}$ represent the momenta which we integrate over.

Writing the eigenstate equation for (43), we realize that

$$H \left| \psi_{\mathbf{k}} \right\rangle = E_{\mathbf{k}} \left| \psi_{\mathbf{k}} \right\rangle \Rightarrow (e^{-i\mathbf{kx}} H e^{+i\mathbf{kx}}) \left| u_{\mathbf{k}} \right\rangle = E_{\mathbf{k}} \left| u_{\mathbf{k}} \right\rangle \tag{49}$$

and hence, defining $\tilde{H} = e^{-i\mathbf{kx}} H e^{+i\mathbf{k}}$ we have

$$\tilde{H}(\mathbf{k}) \left| u_{\mathbf{k}} \right\rangle = E_{\mathbf{k}} \left| u_{\mathbf{k}} \right\rangle \tag{50}$$

Using the definition of Bloch wavefunction electron current in [28], we posit that

$$\mathbf{J} = \frac{e}{\hbar} \frac{\partial \tilde{H}}{\partial \mathbf{k}} \tag{51}$$

and so, substituting (51) into (47-48), one finds that

$$\sigma_{xy} = i \frac{e^2}{\hbar} \sum_{E_\alpha < E_F < E_\gamma} \int_{\mathbf{T}^2} \frac{\left\langle u_{\mathbf{k}}^\alpha \right| \partial_y \tilde{H} \left| u_{\mathbf{p}}^\gamma \right\rangle \left\langle u_{\mathbf{p}}^\gamma \right| \partial_x \tilde{H} \left| u_{\mathbf{k}}^\alpha \right\rangle}{(E_\gamma(\mathbf{k}) - E_\alpha(\mathbf{p}))^2} - \tag{52}$$

$$- \frac{\left\langle u_{\mathbf{k}}^\alpha \right| \partial_x \tilde{H} \left| u_{\mathbf{p}}^\gamma \right\rangle \left\langle u_{\mathbf{p}}^\gamma \right| \partial_y \tilde{H} \left| u_{\mathbf{k}}^\alpha \right\rangle}{(E_\gamma(\mathbf{k}) - E_\alpha(\mathbf{p}))^2} \tag{53}$$

After using the relation

$$\left\langle u_{\mathbf{k}}^\alpha \right| \partial_i \tilde{H} \left| u_{\mathbf{p}}^\gamma \right\rangle = \left\langle u_{\mathbf{k}}^\alpha \right| \partial_i \left(\tilde{H} \left| u_{\mathbf{p}}^\gamma \right\rangle \right) - \left\langle u_{\mathbf{k}}^\alpha \right| \tilde{H} \left| \partial_i u_{\mathbf{p}}^\gamma \right\rangle =$$

$$= (E_\gamma(\mathbf{k}) - E_\alpha(\mathbf{p})) \left\langle u_{\mathbf{k}}^\alpha | \partial_i u_{\mathbf{p}}^\gamma \right\rangle = -(E_\gamma(\mathbf{k}) - E_\alpha(\mathbf{p})) \left\langle \partial_i u_{\mathbf{k}}^\alpha | u_{\mathbf{p}}^\gamma \right\rangle$$

and algebraically manipulating (52-53), we realize that

$$\sigma_{xy} = i \frac{e^2}{\hbar} \sum_{E_\alpha < E_F < E_\gamma} \int_{\mathbf{T}^2} \{ \left\langle \partial_y u_{\mathbf{k}}^\alpha | u_{\mathbf{p}}^\gamma \right\rangle \left\langle u_{\mathbf{p}}^\gamma | \partial_x u_{\mathbf{k}}^\alpha \right\rangle - \tag{54}$$

$$- \left\langle \partial_x u_{\mathbf{k}}^\alpha | u_{\mathbf{p}}^\gamma \right\rangle \left\langle u_{\mathbf{p}}^\gamma | \partial_y u_{\mathbf{k}}^\alpha \right\rangle \} \tag{55}$$

Using the identity relation on the entire Hilbert space [22]:

$$\int_{\mathbf{T}^2} \sum_\gamma \left| u_{\mathbf{k}}^\gamma \right\rangle \left\langle u_{\mathbf{k}}^\gamma \right| = \int_{\mathbf{T}^2} \left(1 - \sum_\alpha \left| u_{\mathbf{k}}^\alpha \right\rangle \left\langle u_{\mathbf{k}}^\alpha \right| \right) \tag{56}$$

one ends up with

$$\boxed{\sigma_{xy} = \frac{ie^2}{\hbar} \sum_\alpha \int_{\mathbf{T}^2} \frac{d^2k}{(2\pi)^2} \left\langle \partial_y u_{\mathbf{k}}^\alpha | \partial_x u_{\mathbf{k}}^\alpha \right\rangle - \left\langle \partial_x u_{\mathbf{k}}^\alpha | \partial_y u_{\mathbf{k}}^\alpha \right\rangle} \tag{57}$$

Comparing (57) to (45), and applying (46), we can conclude that

$$\boxed{\sigma_{xy} = -\frac{e^2}{2\pi\hbar} \sum_\alpha C_\alpha} \tag{58}$$

where the integer C_α represents the *Chern number* of a band indexed as α.

Equation (58) is known as the TKNN formula [3].

It is telling us that the Hall conductivity of a material should be interpreted as a topological invariant of the underlying solid lattice. In general terms, (58) also reproduces the result (5) and thus offers another explanation of IQHE, *if* we assume that a material exposed to a magnetic field forms electronic energy bands. A tight-binding [28-31] theoretical approach to modelling a lattice in a magnetic field can be made: it turns out that, if the flux through the lattice unit cell is a *rational multiple* of the flux quantum, electrons still form band structures. In such cases, electron momenta lie in *magnetic Brillouin zones* [12] and so TKNN works to a great precision in treating those types of integer quantum Hall effects [3, 12, 23].

[1] K. von Klitzing, G. Dorda and M. Pepper, *New method for high accuracy determination of the fine structure constant based on quantized Hall resistance* Phys. Rev. Lett. **45**, 494-497 (1980)

[2] K. von Klitzing, *The quantized Hall effect* Rev. Mod. Phys. **58**, 519-531 (1986) doi:10.1103/RevModPhys.58.519

[3] D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs, *Quantized Hall Conductance in a Two-Dimensional Periodic Potential* Phys. Rev. Lett. **49**, 405-408 (1982)

[4] R. B. Laughlin, *Quantized Hall conductivity in two-dimensions* Phys. Rev. B **23**, 5632-5733 (1981)

[5] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015-2018 (1988) doi:10.1103/PhysRevLett.61.2015

[6] D. J. Thouless (1983). *Quantization of particle transport* Phys. Rev. B. 27 (10): 6083–6087

[7] D. C. Tsui, H. L. Stormer and A. C. Gossard, *Two-dimensional magnetotransport in the extreme quantum limit* Phys. Rev. Lett. **48**, 1559-1562 (1982)

[8] Chern, Shiing-Shen (1946), *Characteristic classes of Hermitian Manifolds* Annals of Mathematics, Second Series, 47 (1): 85–121

[9] Berry, M. V. (1984). *Quantal Phase Factors Accompanying Adiabatic Changes* Proceedings of the Royal Society A. 392 (1802): 45–57.

[10] Edwin Hall (1879). *On a New Action of the Magnet on Electric Currents* American Journal of Mathematics. 2 (3): 287–92.

[11] D. Tong *Lectures on the Quantum Hall Effect* Cambridge University Accessed December 2022

[12] D. R. Hofstadter, Phys. Rev. B **14**, 2239-2249 (1976) doi:10.1103/PhysRevB.14.2239

[13] D. Tong *Lectures on Kinetic Theory* Cambridge University Accessed December 2022

[14] A. Zee, *Quantum Hall fluids* Lect. Notes Phys. **456**, 99-153 (1995) doi:10.1007/BFb0113369 [arXiv:cond-mat/9501022 [cond-mat]].

[15] A. P. Balachandran, *Gauge symmetries, topology, and quantisation* AIP Conf. Proc. **317**, 1-81 (1994) [arXiv:hep-th/9210111 [hep-th]].

[16] M. Asorey, *Topological phases of quantum theories. Chern-Simons theory* J. Geom. Phys. **11**, 63-94 (1993)

[17] Fradkin, E. (2013). *Field Theories of Condensed Matter Physics* (2nd ed.). Cambridge: Cambridge University Press.

[18] M. Nakahara, *Geometry, topology and physics* (2003)

[19] S. L. Sondhi, A. Karlhede, S. A. Kivelson and E. H. Rezayi, *Skyrmions and the crossover from the integer to fractional quantum Hall effect at small Zeeman energies* Phys. Rev. B **47**, 16419-16426 (1993)

[20] v. Klitzing, K. (1987). *Physics and Applications of the Quantum Hall Effect* In: Mendez, E.E., von Klitzing, K. (eds) Physics and Applications of Quantum Wells and Superlattices. NATO ASI Series, vol 170. Springer, Boston, MA.

[21] Kubo, Ryogo (1957). *Statistical-Mechanical Theory of Irreversible Processes. I. General Theory and Simple Applications to Magnetic and Conduction Problems* J. Phys. Soc. Jpn. 12 (6): 570–586.

[22] Griffiths, D., Schroeter, D. (2018). *Introduction to Quantum Mechanics (3rd ed.)* Cambridge: Cambridge University Press.

[23] Ezawa, Zyun F. (2013). *Quantum Hall Effects: Recent Theoretical and Experimental Developments* (3rd ed.). World Scientific.

[24] M. Srednicki, *Quantum field theory* Cambridge University Press, 2007.

[25] D. Tong *Lectures on Statistical Physics* Cambridge University, Accessed December 2022.

[26] X. G. Wen, *Topological Order in Rigid States* Int. J. Mod. Phys. B **4**, 239 (1990)

[27] F. Wilczek and A. Zee, *Appearance of Gauge Structure in Simple Dynamical Systems* Phys. Rev. Lett. **52**, 2111-2114 (1984)

[28] Simon, Steven H, *The Oxford solid state basics* Oxford Univ. Press Oxford, UK 2013

[29] E. Witten, *Three lectures on topological phases of matter* Riv. Nuovo Cim. **39**, no.7, 313-370 (2016) [arXiv:1510.07698 [cond-mat.mes-hall]].

[30] Ashcroft, N. W. and Mermin, N. D. D.*Solid State Physics* Holt-Saunders, 1976.

[31] Kittel, C. *Introduction to Solid State Physics* 2004, Wiley