Topological Properties of the Witten Index

Dusan Begus*
*Department of Physics, Brown University, Providence, Rhode Island, USA
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In this article we give an overview of several topological properties of the Witten Index in theories of supersymmetric quantum mechanics. After a brief introduction to SUSY QM, we set up a supersymmetric theory on a real line. We then show that the path integral expression for the Witten index localizes for a generic superpotential. Secondly, we place our theory on an arbitrary compact manifold. We relate the Witten index of a supersymmetric sigma model and the Euler characteristic of the target manifold. Finally, we derive the Chern-Gauss-Bonnet theorem.

I. INTRODUCTION

A powerful tool in theoretical physics, supersymmetry represents an idea of a symmetry between bosonic and fermionic degrees of freedom in a quantum field theory. A popular consensus in the physics community is that any unified theory of fundamental interactions should have supersymmetry in its construction [11, 13, 19].
Supersymmetric Quantum Mechanics, a general class of one-dimensional supersymmetric theories, was first introduced by Witten [1,3] and Cooper and Freedman [2] in pursuit of explaining the *chierarchy problem* [13], and it was only in this context that SUSY QM was probed in its early days. As time went by, physicists and mathematicians started to understand that supersymmetric theories of quantum mechanics have a rich mathematical structure. In particular, they can be used to explain the most novel results of 20th century mathematics, such as the Hodge decomposition theorems [15], Morse theory [16], and the Atiyah-Singer index theorem [17, 18].
In our brief article, we shall try to examine the topological implications of the seminal works on SUSY QM in [1] and [3]. In Sec. II, we present the essential features of a generic one-dimensional supersymmetric theory. In Sec. III we introduce our first supersymmetric theory: we place a bosonic and a fermionic field on a real line. We explore the localization of the path integral expression for the Witten index. In Sec. IV we introduce another theory: the sigma model. This time, the target space of the theory is a manifold, and the physical predictions we make will be reflected by the topology of the image space of the theory. We go on to prove a simple relation between the Witten index of the sigma model and the Euler character of the target manifold. Finally, in Sec. V we combine all of the established results to derive the Chern-Gauss-Bonnet Theorem. Besides the papers [1] and [3], the main additional resources used in writing this paper were Chapters 1 and 3 in [5].

II. PROPERTIES OF SUPERSYMMETRIC THEORIES

The *supersymmetry algebra* is defined by

$$\{Q, Q^\dagger\} = 2H \tag{1}$$

and

$$\{Q, Q\} = 0,\ \{Q^\dagger, Q^\dagger\} = 0 \tag{2}$$

where $\{\ ,\ \}$ represents the *anti-commutator* between the operators in the algebra, and H is the *Hamiltonian* of our theory.
The first basic property of a generic supersymmetric quantum mechanical theory is that any state in its Hilbert space will have non-negative energy:

$$\langle\psi| H |\psi\rangle = \frac{1}{2}\langle\psi| (QQ^\dagger + Q^\dagger Q) |\psi\rangle = \frac{1}{2}(\left|Q^\dagger |\psi\rangle\right|^2 + |Q|\psi\rangle|^2) \geq 0 \tag{3}$$

The ground states of the theory satisfy [1]:

$$Q|\psi\rangle = Q^\dagger|\psi\rangle = 0 \tag{4}$$

Therefore, we can immediately conclude that only a certain class of quantum mechanical potentials can actually be supersymmetric (those with positive semi-definite Hamiltonians) [2].
Secondly, for energy eigenstates of H with $E \neq 0$, we also have [3, 5]:

$$\{Q, Q^\dagger\} = 2E \Rightarrow \{c, c^\dagger\} = 1 \tag{5}$$

with $c = Q/\sqrt{2E}$.
There is a unique irreducible representation of the operators $c, c^\dagger$ on a two-dimensional vector space [3,20]:

$$c|0\rangle = 0,\ |1\rangle = c^\dagger|0\rangle \tag{6}$$

with the basis vectors $\{|0\rangle, |1\rangle\}$.
Hence, one can immediately conclude that all of the energy eigenstates with non-zero energy *must come in pairs* $\{|\psi_0\rangle, |\psi_1\rangle\}$, where $|\psi_1\rangle = c^\dagger|\psi_0\rangle$.
On the other hand, due to the ambiguity in defining c for $E = 0$, the zero-energy states can (and sometimes will) have no *supersymmetric partners.*
One can define the operator [3]:

$$F = c^\dagger c \tag{7}$$

Which is popularly referred to as *the fermion number operator* [5]. It is clearly well-defined on all the energy eigenstates of the Hamiltonian with $E \neq 0$. Due to (5-6), we recognize that F acts on eigenstates $H|\psi\rangle = E|\psi\rangle$ as

$F|\psi\rangle = |\psi\rangle$ (known in literature as *fermionic states*) or $F|\psi\rangle = 0$ (known in literature as *bosonic states*). The space of fermionic states is usually labelled as $\mathcal{H}_F$, while the space of bosonic states is labelled as $\mathcal{H}_B$ [3]. Clearly we have [1]:

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F \tag{8}$$

In particular, we get [4]:

$$[F, H] = 0$$

The extension of the operator F on zero-energy eigenstates depends on the quantisation programme of the SUSY theory we work with and will be presented later in the article.
Finally, we can introduce the *Witten index* of the theory, defined by [3]:

$$\boxed{\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta H}} \tag{9}$$

where Tr represents the trace map, and β is an arbitrary real number.
We will first show that the Witten index is in fact independent of β:

$$\frac{d\mathcal{I}}{d\beta} = 0 \tag{10}$$

Proof: We notice the canonical isomorphism

$$\mathcal{H}_B|_{E>0} \cong \mathcal{H}_F|_{E>0}$$

induced by the operators $c, c^\dagger$ acting on every supersymmetric pair with $E > 0$. This implies that

$$\mathrm{Tr}|_{E>0}\,(-1)^F e^{-\beta H} = 0$$

Hence we are left with

$$\boxed{\mathcal{I} = \mathrm{dim}\mathcal{H}_{0,B} - \mathrm{dim}\mathcal{H}_{0,F}} \tag{11}$$

With $\{0, X\}$ marking a zero-energy subspace of the (fermionic, bosonic) subspace of our graded Hilbert space. Noting that (11) is independent of β, we arrive at equation (10). $\diamondsuit$

III. SUPERSYMMETRIC QUANTUM MECHANICS ON A REAL LINE

We now introduce a one-dimensional field theory given by the action [5]:

$$S = \int dt L = \int dt \left(\frac{1}{2}\dot{x}^2 + i\psi^\dagger\dot{\psi} - \frac{1}{2}h'^2 + \frac{1}{2}h''[\psi^\dagger, \psi] \right) \tag{12}$$

with x a real-valued c-number field. On the other hand, $\psi, \psi^\dagger$ are Grassmann-valued fields and satisfiy anticommutation relations $\{\psi, \psi^\dagger\} = 0$. The action (12) is invariant under the following symmetry [4-5]:

$$\delta x = \epsilon^\dagger\psi - \epsilon\psi^\dagger \tag{13}$$
$$\delta\psi = \epsilon(-i\dot{x} + h') \tag{14}$$
$$\delta\psi^\dagger = \epsilon^\dagger(i\dot{x} + h') \tag{15}$$

where ϵ a Grassmann variable.
The symmetry generating operators of (13-15) are [5]:

$$\mathcal{Q} = \int dt \left[\psi(t)\frac{\delta}{\delta x(t)} + (i\dot{x} + h')\frac{\delta}{\delta\psi^\dagger(t)} \right] \tag{16}$$
$$\mathcal{Q}^\dagger = \int dt \left[-\psi^\dagger(t)\frac{\delta}{\delta x(t)} - (i\dot{x} - h')\frac{\delta}{\delta\psi(t)} \right] \tag{17}$$

With the definition of the functional derivative given in [6].
One can show [4-5] that:

$$\mathcal{Q}S = \mathcal{Q}^\dagger S = 0 \tag{18}$$

and the conserved charges [6] of the theory are:

$$Q = (\dot{x} - ih')\psi \tag{19}$$
$$Q^\dagger = (\dot{x} + ih')\psi^\dagger \tag{20}$$

Moving on to the Hamiltonian picture, we can calculate the corresponding canonical momenta to fields in (12):

$$p(t) = \frac{\delta S}{\delta\dot{x}(t)} = \dot{x}(t), \;\; \pi(t) = \frac{\delta S}{\delta\dot{\psi}(t)} = i\psi^\dagger(t) \tag{21}$$

In the quantum picture, we impose the canonical (anti)-commutation relations between the operators in (21) in the following manner:

$$[x, p] = i, \;\; \{\psi, \psi^\dagger\} = 1 \tag{22}$$

in the Heisenberg picture [6] at a fixed time. After Legendre transforming [6] L in (12), we obtain the Hamiltonian of the theory:

$$H = \frac{1}{2}(p^2 + h'^2) - \frac{1}{2}h''(\psi^\dagger\psi - \psi\psi^\dagger)$$

By the virtue of [4], we may evaluate:

$$\{Q, Q^\dagger\} = 2H \tag{23}$$

with $Q = (\dot{x} - ih')\psi = (p - ih')\psi$, $Q^\dagger = (p + ih')\psi^\dagger$, and the theory is wonderfully *supersymmetric*.

A. The Anomalous Path Integral

The key insight into the utility of the index defined in (9) comes from the path integral evaluation of quantum amplitudes: namely, the partition function [6,22] of the theory and the Witten index of the theory can be computed in incredibly similar manners [5]:

$$\mathcal{Z} = \mathrm{Tr}e^{-\beta H} = \int_{\psi(\beta)=-\psi(0)} \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger(t)e^{-S_E[x(t),\psi(t),\psi^\dagger(t)]} \tag{24}$$

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int_{\psi(\beta)=\psi(0)} \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger(t)e^{-S_E[x(t),\psi(t),\psi^\dagger(t)]} \tag{25}$$

With the natural Wick rotation to Euclidean action [6,10] and the imposal of periodic time ($x(\beta) = x(0)$) on the theory, due to the nature of thermal field theory partition functions [6,21].

One of the more astonishing properties of the Witten Index is the fact that (9) is invariant under the rescaling of the superpotential $h(x)$ in (12). Namely, we shall prove that under the rescaling of the potential $h(x) \to \lambda h(x)$ for an arbitrary non-zero λ, we get:

$$\frac{d\mathcal{I}}{d\lambda} = 0 \tag{26}$$

Step 1: Wick rotating the action (12), we have [5]:

$$S_E = \oint d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \psi^\dagger\frac{d\psi}{d\tau} - h''\psi^\dagger\psi\right] \tag{27}$$

and so, by writing the Witten index as a function of λ through the path integral expression (25), we obtain:

$$\frac{d\mathcal{I}}{d\lambda} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger(t)(-\oint d\tau(\lambda h'^2 - h''\psi^\dagger\psi))e^{-S_E} \tag{28}$$

where the $\oint$ symbol indicates that the integral should be taken along a closed time loop.

Step 2: The λ-rescaled symmetry generator reads:

$$\mathcal{Q}^\dagger_\lambda = \int dt\left[-\psi^\dagger(\tau)\frac{\delta}{\delta x(\tau)} + (\frac{dx}{d\tau} + \lambda h')\frac{\delta}{\delta\psi(\tau)}\right] \tag{29}$$

and one can show [5] that:

$$\mathcal{Q}^\dagger_\lambda \oint d\tau h'\psi = \oint d\tau(-h''\psi^\dagger\psi + h'\frac{dx}{d\tau} + \lambda h'^2) \tag{30}$$

Step 3: After losing the total derivatives in (30), we arrive at the following result:

$$\mathcal{Q}^\dagger_\lambda \oint d\tau h'\psi = \oint d\tau(-h''\psi^\dagger\psi + \lambda h'^2) \tag{31}$$

Step 4: Since the Euclidean action (27) is invariant under the $\mathcal{Q}^\dagger_\lambda$ operator (18), we learn that:

$$\frac{d\mathcal{I}}{d\lambda} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger\mathcal{Q}^\dagger_\lambda\left(-e^{-S_E}\oint d\tau(\lambda h'^2 - h''\psi^\dagger\psi)\right) \tag{32}$$

Finally, as the integral of any $\mathcal{Q}$-exact quantity vanishes by the axioms of Berezin integration [10], we conclude that the Witten index (9) is invariant under the rescaling of the potential ,i.e. we recover the relation (26). $\Diamond$

Now we perform the following procedure:

We rescale the potential $h(x)$ so it becomes infinitely steep ($\lambda \to \infty$). Thus, only the critical points of the potential ($h'(X_i) = 0$) contribute to the Witten index path integral expression (25), while the other contributions ($h'(X_i) \neq 0$) to the path integral get exponentially suppressed [4]. Consequently, we reduce the path integral expression for the Witten index into a sum of *regular integrals*.

Let X be a critical point of the rescaled potential $h(x)$. We can Taylor expand around X:

$$h'(x) \approx h''(X)(x - X) \Rightarrow V(x) \approx \frac{1}{2}h''(X)^2(x - X)^2 \tag{33}$$

We also recall the Gaussian expressions [6]:

$$\int d^n x e^{-x^T M x} = \sqrt{\frac{1}{detM}} \tag{34}$$

$$\int d\theta d\bar{\theta} e^{\bar{\theta}M\theta} = detM \tag{35}$$

for a linear operator M on the infinite-dimenisonal space of fields $x(\tau), \psi(\tau), \psi^\dagger(\tau)$. After several algebraic manipulations of (25) using (33-35), one ends up with [5]:

$$\boxed{\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_X \frac{det(\frac{d}{d\tau} + h''(X))}{det^{\frac{1}{2}}(-\frac{d^2}{d\tau^2} + h''(X))}} \tag{36}$$

where the sum is taken over the critical points of the potential.

The calculation of the determinants in (36) should be done for a harmonic oscillator given by (33). A full such evaluation using *zeta-function regularization* can be found in Chapter 2 of [5], and it yields the following result:

$$det^{\frac{1}{2}}(-\frac{d^2}{d\tau^2} + h''(X)) = 2\sinh\left(\frac{\beta|\omega|}{2}\right) \tag{37}$$

$$det(\frac{d}{d\tau} + \omega) = 2\sinh\left(\frac{\beta\omega}{2}\right) \tag{38}$$

Thus we obtain:

$$\boxed{\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta H} = \sum_X \frac{h''(X)}{|h''(X)|} = \sum_X \mathrm{sign}(h''(X))} \tag{39}$$

The phenomenon we have just described is popularly referred to as *localization* of the index integral expression (25), since the value of the Witten index only depends on a sum of signs of $h''(X)$ over the critical points $h'(X) = 0$. Eq. (39) is not dependent on particular values of $h(x)$. Consequently, the Witten index of theories given by (12) is robust in the sense that it does not change under generic (small) perturbations of the potential $h(x)$, and is hence topologically protected. In order to probe more topological properties of the Witten index, we must turn to a theory much different than (12).

IV. THE SUPERSYMMETRIC SIGMA MODEL

We now introduce another supersymmetric theory, usually refferred to as the supersymmetric sigma model [3]:

$$L = \int dt \frac{1}{2} g_{ij}(x) + i g_{ij} \psi^{\dagger i} \nabla_t \psi^j - \frac{1}{4} R_{ijkl} \psi^i \psi^j \psi^{\dagger k} \psi^{\dagger l} \tag{40}$$

With M a compact n-dimensional manifold, g_{ij} its Riemannian metric, and R_{ijkl} its Riemann tensor [8]. All of the Latin indices in (40) run from 1 to n. The covariant derivative of the fermionic field given by

$$\nabla_t \psi^i = \frac{d\psi^i}{dt} + \Gamma^i_{jk} \frac{dx^j}{dt} \psi^k \tag{41}$$

The action obtained with (40) is invariant under the following transformations [4,5]:

$$\delta x^i = \epsilon^\dagger \psi^i - \epsilon \psi^{\dagger i} \tag{42}$$

$$\delta \psi^i = \epsilon(-i\dot{x}^i + \Gamma^i_{jk} \psi^{\dagger j} \psi^k) \tag{43}$$

$$\delta \psi^{\dagger i} = \epsilon^\dagger(+i\dot{x}^i + \Gamma^i_{jk} \psi^{\dagger j} \psi^k) \tag{44}$$

and the associated supercharges are [5]:

$$Q = g_{ij} \dot{x}^i \psi^{\dagger j}, \quad Q^\dagger = g_{ij} \dot{x}^i \psi^j \tag{45}$$

The canonical momenta of the theory are evaluated to be:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij}(\dot{x}^i + i\Gamma^i_{kl} \psi^{\dagger k} \psi^l) \tag{46}$$

$$\frac{\partial L}{\partial \dot{\psi}^i} = i g_{ij} \psi^{\dagger i} \tag{47}$$

We also impose the (anti)-commutation relations on fields:

$$[x^i, p_j] = \delta^i_j, \quad \{\psi^i, \psi^{\dagger j}\} = g^{ij} \tag{48}$$

After introducing the *mechanical momentum* [5]:

$$\pi_i = g_{ij} \dot{x}^j = p_i - i g_{il} \Gamma^l_{jk} \psi^{\dagger j} \psi^k$$

we also require:

$$[\pi_i, \psi^j] = i\Gamma^j_{ik} \psi^k, \quad \{\psi_i, \psi^{\dagger j}\} = i\Gamma^j_{ik} \psi^{\dagger k} \tag{49}$$

$$[\pi_i, \pi_j] = -R_{ijkl} \psi^{\dagger k} \psi^l \tag{50}$$

The upshot of a series of relations we introduced in (48-50) is that we can now actually make the following (highly nontrivial) isomorphism between the algebra of supersymmetric $x, \psi, \psi^\dagger$ fields and the algebra of p-forms [8] on the target manifold M [1]:

$$x^i \longleftrightarrow x^i \tag{51}$$

$$p_i \longleftrightarrow -i \frac{\partial}{\partial x^i} \tag{52}$$

$$\psi^{\dagger i} \longleftrightarrow dx^i \tag{53}$$

$$\psi^i \longleftrightarrow g^{ij} \iota_{\frac{\partial}{\partial x^j}} \tag{54}$$

$$Q = i\psi^{\dagger i} p_i \longleftrightarrow dx^i \wedge \frac{\partial}{\partial x^i} = d \tag{55}$$

$$Q^\dagger = i\psi^i p_i \longleftrightarrow g^{ij} \iota_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = d^\dagger \tag{56}$$

$$H = \frac{1}{2}\{Q, Q^\dagger\} \longleftrightarrow H = \frac{1}{2}\Delta \tag{57}$$

where

1. The map $\star : \Lambda^p M \to \Lambda^{n-p} M$ is referred to in literature as the Hodge dual [8, 12, 15].
2. $d : \Lambda^p M \to \Lambda^{p+1} M$ is the *exterior derivative* operator on $\Lambda^p M$ [8].
3. $d^\dagger = (-1)^{n(p+1)+1} \star d \star : \Lambda^p M \to \Lambda^{p-1} M$ is the corresponding *adjoint operator* to d [8].
4. *The Laplacian operator* is defined as $\Delta = (d + d^\dagger)^2$ [5, 12, 15].

Operators on the left-hand side of (51-57) act on the Hilbert space of the theory (10), while the operators on the right-hand side of (51-57) act on the direct sum of p-form spaces $\oplus_p \Lambda^p M$.
We will further posit that we can diagonalize our quantum Hamiltonian so that the energy eigenstates correspond to exactly one type of forms, i.e. energy eigenstates correspond to pure p-forms [1,3]. The aforementioned assumption follows from the fact that the fermion number operator F commutes with the Hamiltonian and:

$$F : \mathcal{H} \to \mathbb{R} \longleftrightarrow \phi : \oplus_p \Lambda^p M \to \mathbb{R} \tag{58}$$

where ϕ is the linear extension of the "degree of the form" map $\phi|_{\Lambda^p M} = p$.
Due to the nature of expression (4), we can write the following relation for the *ground states*:

$$Q |\psi\rangle = Q^\dagger |\psi\rangle = 0 \longleftrightarrow d\omega = d^\dagger \omega = 0 \tag{59}$$

Consequently, our ground states correspond to *harmonic forms* on M [15]. Hence we realize that

$$\boxed{\mathcal{H}_{\text{ground}} = \oplus_p \text{Harm}^p(M)} \tag{60}$$

where $\text{Harm}^p(M)$ is the space of harmonic p-forms on M.

A. Euler characteristic $\leftrightarrow$ Witten index

We now quote the following theorem (due to the courtesy of W.V.D. Hodge [15]):
Theorem (Hodge's Theorem).
There is an isomorphism

$$H^p(M) \cong \mathrm{Harm}^p(M) \tag{61}$$

where $H^p(M)$ is the usual de Rham cohomology group of the manifold [8, 9, 12].
The big twist in the understanding of SUSY nonlinear sigma models comes from the expression (11). Namely, the Witten index counts the weighted sum of the dimensions of the fermionic and bosonic Hilbert subspaces (9-11). The correlation between (8) and (58) associates the bosonic states with even p-forms and the fermionic states with odd p-forms [1]. We thus have [1]:

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta H} = \sum_p (-1)^p \dim\ \mathrm{Harm}^p(M) \tag{62}$$

On the other hand, we recall the result from algebraic topology [9]:

$$\chi(M) = \sum_p (-1)^p \dim\ H^p(M) \tag{63}$$

Employing (61) on (62), we conclude that:

$$\boxed{\mathcal{I} = \chi(M)} \tag{64}$$

Since the Euler characteristic of a manifold is invariant up to a *homotopy invariance* [9], we realize that the Witten index of the underlying nonlinear sigma model on the manifold does not change even if the manifold experiences drastic changes in the metric, including changes in dimensions due to retraction homotopies [9]. Accordingly, we come to understand that (40) represents a novel *topological field theory* [14].

V. THE CHERN-GAUSS-BONNET THEOREM

One can extract more information from (25, 40) than just the expression (64). Let us write down the path integral expression (25) for the Witten index in Wick-rotated time [5]:

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta H} = \int \mathcal{D}x\mathcal{D}\psi^\dagger\mathcal{D}\psi e^{-S_E[x,\psi,\psi^\dagger]} \tag{65}$$

with

$$S_E = \oint d\tau \frac{1}{2} g_{ij}(x)\dot{x}^i\ \dot{x}^j + g_{ij}\psi^{\dagger i}\nabla_\tau\psi^j + \frac{1}{4}R_{ijkl}\psi^i\psi^j\psi^{\dagger k}\psi^{\dagger l} \tag{66}$$

and

$$\nabla_\tau\psi^i = \dot{\psi}^i + \Gamma^i_{jk}\dot{x}^j\psi^k \tag{67}$$

We have already derived that the Witten index (9) is independent of β (10). Hence, we can always rescale the looped time interval of the domain of fields in (40). Thus, we argue that supersymmetric theories (1) have no *renormalization group flow* [6,10] with respect to time. Armed with the β-invariance of (9), we now employ the following *quasi-renormalisation scaling protocol* [22]:

$$\tau \longrightarrow \tau' = \frac{\tau}{\beta} \tag{68}$$

$$\psi \longrightarrow \psi' = \beta^{-\frac{1}{4}}\psi \tag{69}$$

$$\psi^\dagger \longrightarrow {\psi'}^\dagger = \beta^{-\frac{1}{4}}\psi^\dagger \tag{70}$$

to obtain the transformed Euclidean action:

$$S_E = \oint_0^1 d\tau' \frac{1}{2\beta} g_{ij}(x)\dot{x}^i\dot{x}^j + \frac{1}{\sqrt{\beta}} g_{ij}\psi^{\dagger i}\nabla_\tau\psi^i + \frac{1}{4}R_{ijkl}\psi^i\psi^j\psi^{\dagger k}\psi^{\dagger l} \tag{71}$$

When we calculate S_E in the limit $\beta \to 0$, the modes with $\dot{x}^i$, $\dot{\psi}^i$ nonzero get highly suppressed. Hence, the path integral (65) reduces to a regular integral over functionally-constant values of $x, \psi, \psi^\dagger$:

$$\mathcal{I} = \frac{1}{(2\pi)^{n/2}} \int d^n x \frac{1}{\sqrt{g}} \int d^n\psi d^n\psi^\dagger \exp(-\frac{1}{4}R_{ijkl}\psi^i\psi^j\psi^{\dagger k}\psi^{\dagger l}) \tag{72}$$

Remembering the rules of the exponential Grassmann integration (35) while using the Riemann tensor identities [8]:

$$R_{ijkl} = R_{klij}$$
$$R_{ijkl} = -R_{jikl}$$
$$R_{ijkl} = -R_{ijlk}$$

and employing a number of combinatoric tricks, we end up with two results [4-5]:
Result 1:

$$\mathcal{I} = \chi(M) = 0 \tag{73}$$

for n odd.
The result (73) is a consequence of the Poincaré duality [9].

Result 2:

$$\mathcal{I} = \frac{1}{(4\pi)^{n/2}(n/2)!} \int d^n x \frac{1}{\sqrt{g}} \epsilon^{i_1\cdots i_n}\epsilon^{j_1\cdots j_n} R_{i_1 i_2 j_1 j_2}\cdots R_{i_{n-1} i_n j_{n-1} j_n} \tag{74}$$

for n even.
The eq. (74) is known as the *Chern-Gauss-Bonnet theorem* [7,12].

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[22] The author coined the term *quasi-renormalisation* and he apologizes in advance.