

Introduction to dynamical modeling

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2016 Summer Course on Mathematical Modeling and Analysis
of Infectious Diseases

National Taiwan University

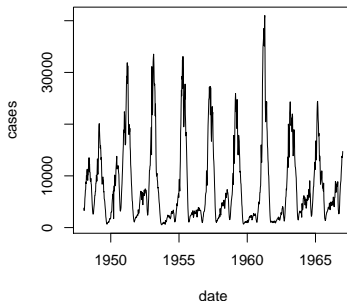
Goals

- ▶ This lecture will:
 - ▶ introduce the idea of dynamical modeling
 - ▶ give simple examples of population modeling and disease modeling
 - ▶ discuss different types of model approaches

Dynamic modeling connects scales



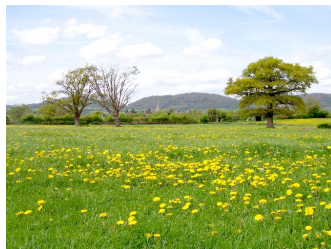
Measles reports from England and Wales



- ▶ Start with rules about how things change in short time steps
 - ▶ Usually based on *individuals*
- ▶ Calculate results over longer time periods
 - ▶ Usually about *populations*

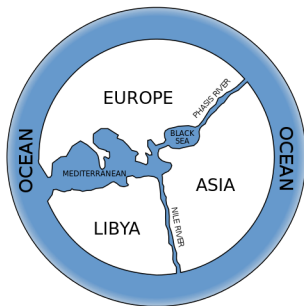
Example: Dandelions

- ▶ Start with one dandelion; it produces 100 seeds, of which only 2% survive to reproduce.
- ▶ Seeds that survive to reproduce will produce 100 seeds after 1 year next year.
- ▶ How many dandelions after 3 years?



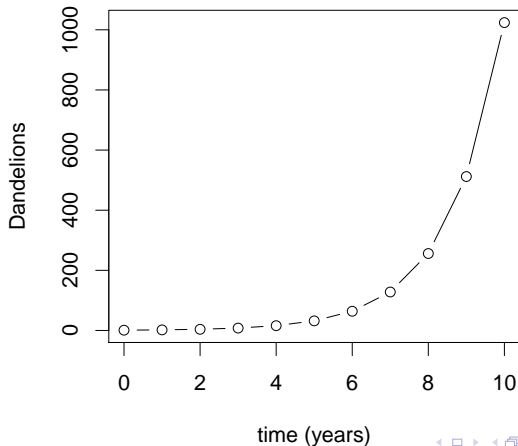
Model worlds

- ▶ A dynamic model is based on a model world
- ▶ The model world has *enough* assumptions to allow us to calculate dynamics
- ▶ The model world is *simpler* than the real world
- ▶ Essentially, all models are wrong, but some are useful. – Box and Draper (1987), *Empirical Model Building ...*



Model result

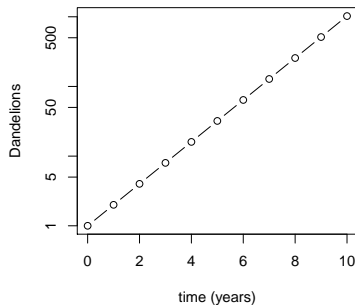
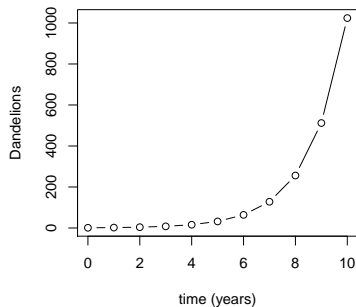
- ▶ If each individual is reproducing independently at each time step, the population changes *exponentially*
 - ▶ it is *multiplied* by the same amount in each step.



Scales

- ▶ The difference between 1 and 10 is the same as the difference between 10 and what?
 - ▶ *additive* difference:
 - ▶ * 19
 - ▶ *multiplicative* difference:
 - ▶ * 100

Scales

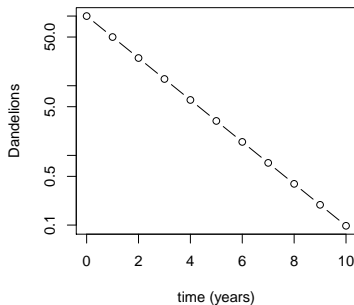
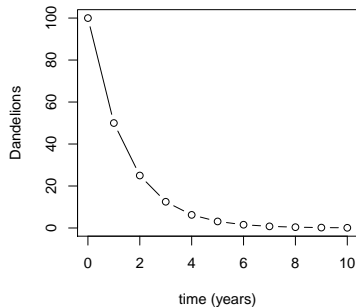


- ▶ Linear scale: 1 to 10 = 10 to 19
- ▶ Log scale: 1 to 10 = 10 to 100

Exponential change

- ▶ We can have exponential *growth* (population goes up)
- ▶ or exponential *decline* (population goes down)
- ▶ What if we spray the dandelions, so that each seed only has 0.5% chance of survival?

Exponential decline

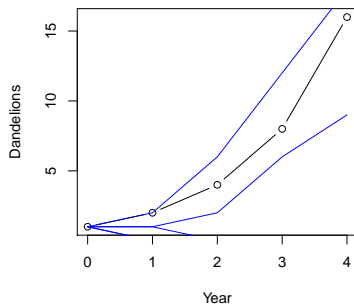
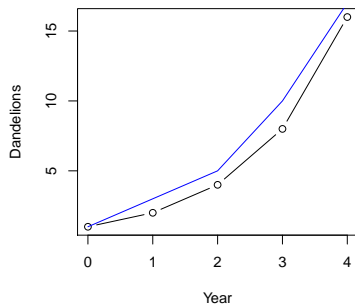


- ▶ Linear scale: 1 to 10 = 10 to 19
- ▶ Log scale: 1 to 10 = 10 to 100

Randomness

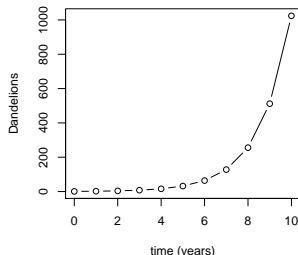
- ▶ Do our rules tell us exactly what is going to happen?
- ▶ If we have 1 dandelion this year, do we expect exactly two dandelions next year?
 - ▶ Do we expect exactly $1/2$ of a dandelion?
- ▶ **Deterministic models:** rules describe exactly what will happen
- ▶ **Stochastic models:** rules describe a range of things that *might* happen

Stochastic model



Time steps

- ▶ Dynamic models can use
 - ▶ *discrete time*: we model the population at specific time points
 - ▶ *continuous time*: we model time smoothly
- ▶ Which kind of model is the dandelion model?



Bacteria

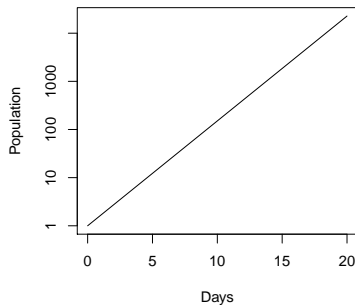
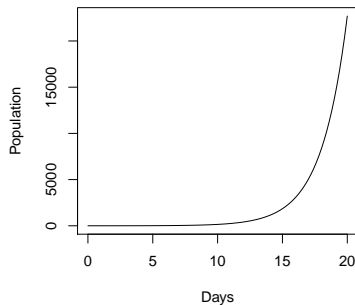
- ▶ Imagine we have some bacteria in a tank
- ▶ They are continuously dividing, and continuously dying



Bacteria

- ▶ Model world
 - ▶ The bacteria:
 - ▶ die at a constant *per capita* rate
 - ▶ divide at a constant *per capita* rate
- ▶ Model
 - ▶ $\frac{dN}{dt} = (b - d)N$

Model result



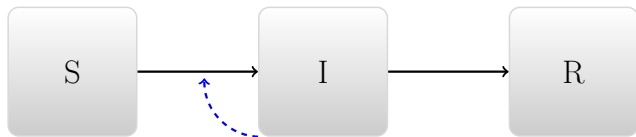
- ▶ Population grows *exponentially*
- ▶ As long as $b > d$

Describing a model

- ▶ **Model structure:** what are the rules that our model is following?
 - ▶ Each individual is dividing and dying independently at a fixed rate
- ▶ **Parameters:** what quantities (with units) determine how the rules are working
 - ▶ Birth rate is 0.04/day
- ▶ **State variables:** what changing quantities are we modeling?
 - ▶ The number of bacteria

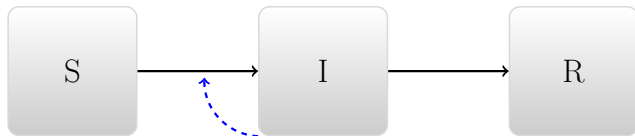
Simple models of disease spread

- ▶ Divide people into categories:



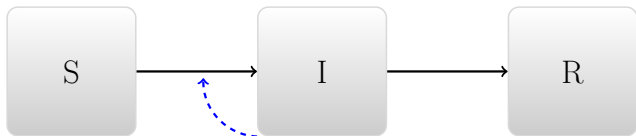
- ▶ Susceptible: can be infected
- ▶ Infectious: can infect others
- ▶ Recovered: cannot be infected

What determines transition rates?

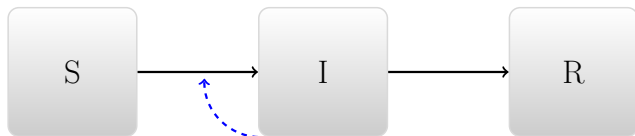


- ▶ People get better independently
- ▶ People get infected by infectious people

Conceptual modeling

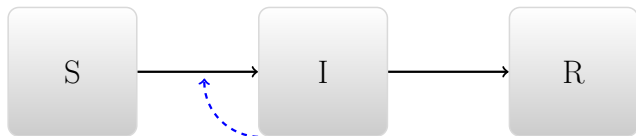


Conceptual modeling



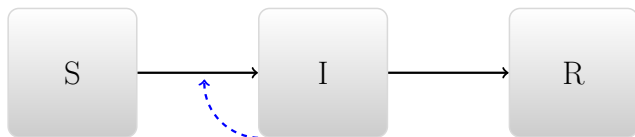
- ▶ What is the final result?
- ▶ When does disease increase, decrease?

Dynamic implementation



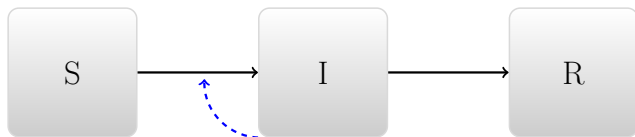
- ▶ Requires assumptions about time distributions
- ▶ The *conceptually simplest* implementation uses **Ordinary Differential Equations** (ODEs)
 - ▶ Other options may be more realistic
 - ▶ Or simpler in practice

Recovery



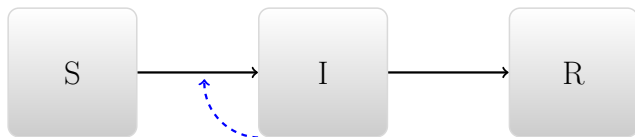
- ▶ Infectious people recover at *per capita* rate γ
 - ▶ Total recovery rate is γI
 - ▶ Mean time infectious is $D = 1/\gamma$

Transmission



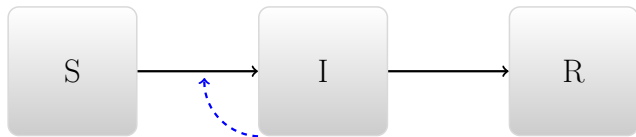
- ▶ Susceptible people get infected by:
 - ▶ Going around and contacting people (rate c)
 - ▶ Some of these people are infectious (proportion I/N)
 - ▶ Some of these contacts are effective (proportion p)
- ▶ Per capita rate of becoming infected is $cpI/N \equiv \beta I/N$
- ▶ Population-level transmission rate is $\mathcal{T} = \beta SI/N$

Another perspective on transmission



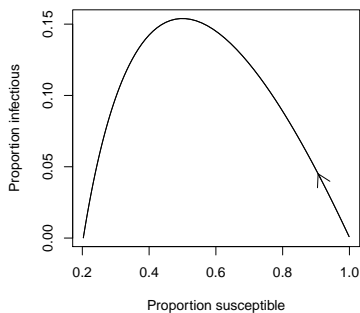
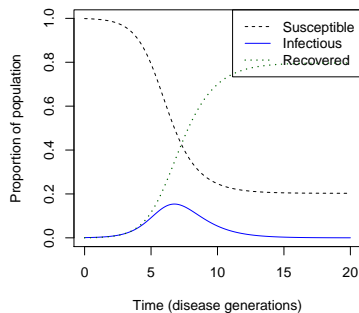
- ▶ Infectious people infect others by:
 - ▶ Going around and contacting people (rate c)
 - ▶ Some of these people are susceptible (proportion S/N)
 - ▶ Some of these contacts are effective (proportion p)
- ▶ Per capita rate of infecting others is $cpS/N \equiv \beta S/N$
- ▶ Population-level transmission rate is $\mathcal{T} = \beta SI/N$

ODE implementation



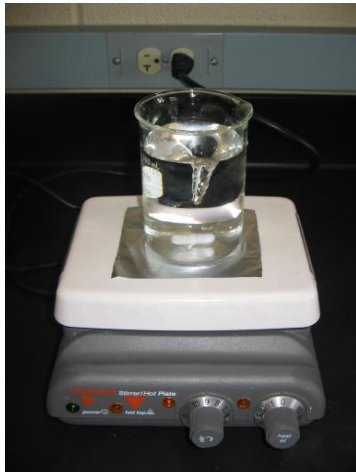
$$\begin{aligned}\frac{dS}{dt} &= -\beta \frac{SI}{N} \\ \frac{dI}{dt} &= \beta \frac{SI}{N} - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

ODE implementation



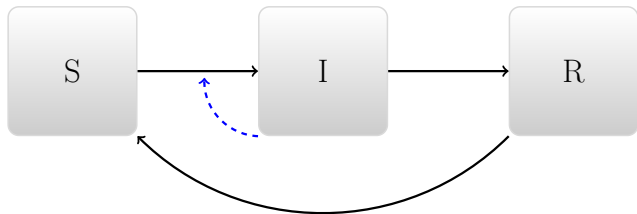
- ▶ Not everyone will get infected
- ▶ Disease starts to decline when number of susceptibles is small

ODE assumptions



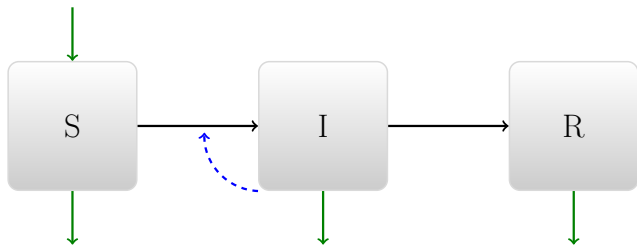
- ▶ Lots and lots of people
- ▶ Perfectly mixed

Closing the circle



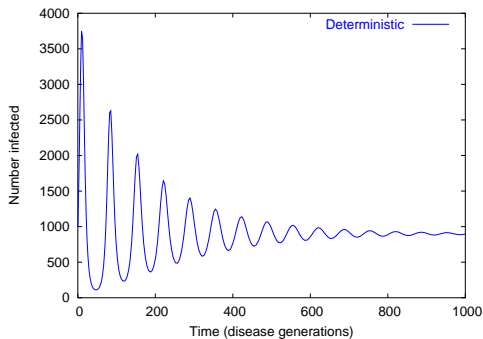
- Loss of immunity

Closing the circle



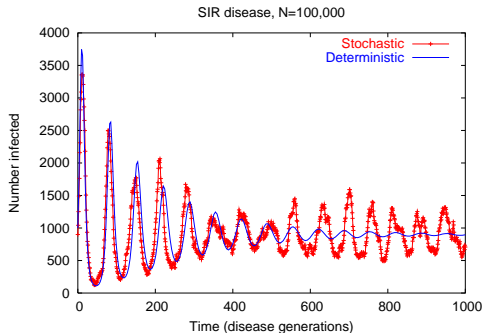
- Births and deaths

Tendency to oscillate



Modeling individuals as individuals usually requires a *stochastic* model

With individuality



Even in the simplest form, this can cause large random oscillations even in large populations

Types

- ▶ **Discrete** vs. **Continuous** time steps
- ▶ **Deterministic** vs. **Stochastic** dynamics
 - ▶ Stochastic models may have **Discrete individuals**

Summary

- ▶ Dynamics are an essential tool to link scales
- ▶ Very simple models can provide useful insights
- ▶ More complex models can provide more detail, but also require more assumptions, and more choices