Stochastic models:

Concepts and analysis

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Outline

Introduction

Describing a stochastic process

Equilibrium and quasi-equilibrium

Analytic methods

Conclusions

Modelling individual events

- Differential equations model continuous processes
- Disease spreads in the real world through discrete events
- Discrete events are fundamentally stochastic
  - Even in theory we don't know when the next event will occur, nor even what it will be

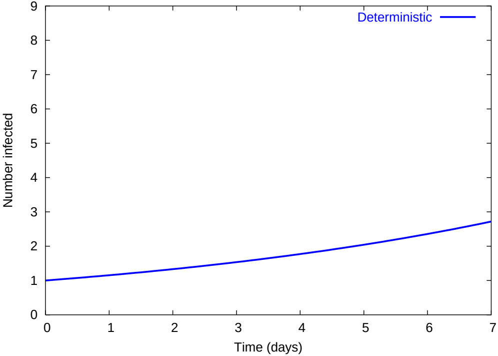
Types of stochasticity

- Demographic stochasticity is caused by the existence of individual people and discrete events
- Environmental stochasticity refers to events that affect more than one person at a time
  - Weather
  - Politics
  - Economics

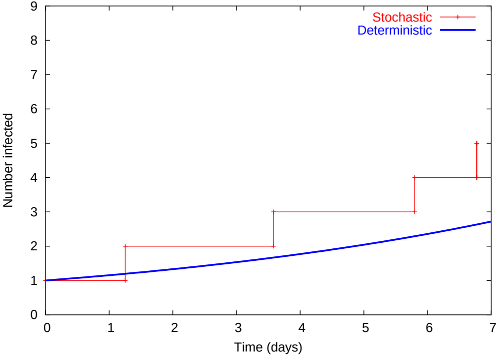
Modelling individual events

A stochastic potential-transmission event

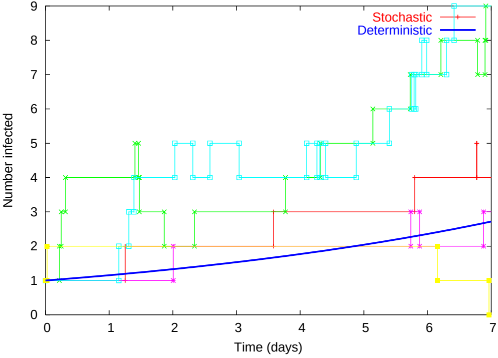
Deterministic spread



Demographic spread



Demographic spread

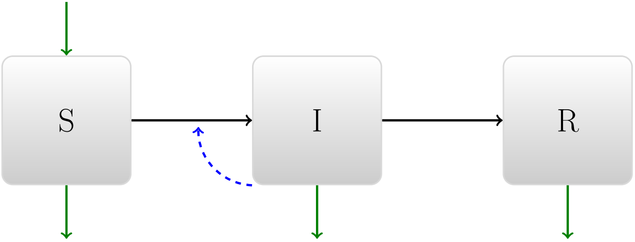


Outline

States and rates

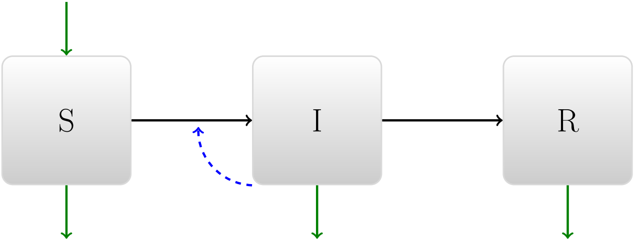
- We describe our system in terms of the *probability rates* of events happening
  - If the rate of event E is $r_E(t)$, the probability of the event occurring in the time interval $(t, t + dt)$ is $r_E(t)dt$
- If the system is *Markovian*, $r_E(t)$ depends only on the state of the system at time t
  - The Markovian assumption is convenient, but can have unwanted consequences

States and rates (Deterministic)



$$\frac{dS}{dt} = \mu\hat{N} - \beta\frac{SI}{N} - \mu S$$
$$\frac{dI}{dt} = \beta\frac{SI}{N} - \gamma I - \mu I$$

States and rates (Demographic)



Event	**transition**	**rate**	**Effect** (S, I)
Infection	$S \rightarrow I$	$\beta SI/N$	$(-1, 1)$
Recovery	$I \rightarrow R$	γI	$(0, -1)$
Rebirth	$R \rightarrow S$	$\mu(N - S - I)$	$(1, 0)$
Rebirth	$I \rightarrow S$	μI	$(1, -1)$

Analogy

- The demographic model is an exact analogue of the deterministic one
  - Conceptually
  - In the limit as $N \to \infty$

Realizations and ensembles

- How do we think about the behavior of a stochastic process?
  - A single example of how the process could go (e.g., from a stochastic simulation) is called a *realization*
  - The universe of possible realizations is called the *ensemble*.
  - The probability distribution that describes what state we expect the population to be in at time t is called the *ensemble distribution*

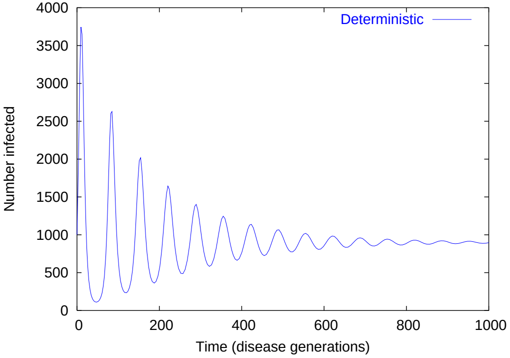
Some techniques

- Simulate one or many realizations
- Simulate the ensemble distribution
  - Requires one state variable for each possible state of the system
- Solve the ensemble distribution dynamics exactly!
  - Rarely possible
- Analytic approximations to the ensemble distribution

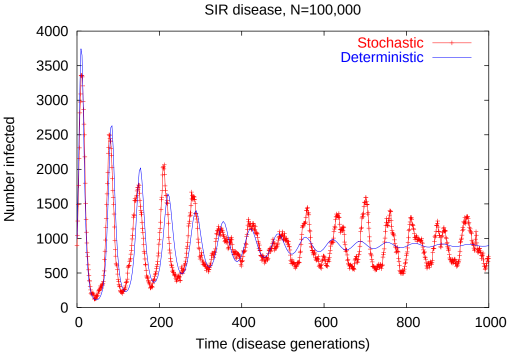
Simulating a realization

- Given a state of the system:
  - List possible events, and associated rates
  - Calculate the total rate r_T: this gives the rate at which the next event (whatever it is) will happen
    - i.e., an exponential waiting time with mean $1/r_T$
  - The probability of event E is r_E / r_T
  - Randomly select the time and nature of the next event
  - Change the system state appropriately
  - Repeat forever
    - Or until system is extinct
    - Or until you are tired

Deterministic model



Demographic model



Modelling the ensemble distribution

- We model the ensemble distribution by creating one conceptual ‘box’ for each possible state of the system, and asking what is the probability that the system is in each box.
  - This can be a lot of boxes
- The probabilities change as follows:
- $\dot{p}_S = \sum_{S'} p_{S'} r_{S' \to S} - p_S \sum_{S'} r_{S \to S'}$

Modelling the ensemble distribution

- If our system is small enough (particularly, if it has one state dimension) we might be able to simulate the ensemble distribution
- We might seek to solve these equations analytically
  - Only in special cases
- We might seek analytic approximations to increase our understanding
  - Moment approximations
  - Diffusion approximations

Questions

- What kind of questions do we want to ask with a stochastic model?
  - How does stochasticity affect disease dynamics?
    - Spatial distribution
    - Establishment
    - Persistence
  - How much variance do we expect stochasticity to cause?
  - Under what circumstances can we eliminate or eradicate a disease?

Outline

Equilibrium

- Define equilibrium as an ensemble distribution that does not change with time
- What are the equilibria of our stochastic SIR system?
  - Disease free equilibrium
  - Others?
- There is no equilibrium corresponding to the endemic equilibrium of the deterministic system!
  - As long as any populations not extinct, proportion extinct will increase.

Quasi-equilibrium

- Consider the ensemble distribution confined to the subset of states where nothing is extinct
- This system can be described as a *fully connected* (you can get anywhere from anywhere else, *open* (you can leave the set) flow.
- Linear algebra tells us that such a system will converge to a stable *relative* distribution of probabilities of being in each non-extinct box

Interpreting the quasi-equilibrium

- The quasi-equilibrium is the asymptotic distribution of system states given that nothing has gone extinct
- The eigenvalue λ_q associated with the quasi-equilibrium distribution is the rate at which the probability of non-extinction decays (exponentially)
  - The distribution of persistence times must be asymptotically exponential
  - The expected persistence time (looking forward) approaches $-1/\lambda_q$ as the system continues to persist

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Modelling the quasi-equilibrium

- We can also model the probability of being in a particular state given that extinction has not occured
  - Computationally convenient
  - Can also keep track of cumulative extinction probability
- Define $q_S = p_S/(1 - p_N)$ where N is a 'null' state that we cannot escape from.
- Use quotient rule to find dynamic equations for q_S.

The fate of infectious disease

- Fizzle
  - Disease fails to “establish”
  - We will make this precise later
- Burn-out
  - Disease goes extinct after first epidemic
- Fade-out
  - Disease goes extinct after system approaches quasi-equilibrium
  - Can take a *long* time

Outline

Linearization

- Two of the most useful tools for understanding deterministic disease models are linearizations:
  - **Disease-free equilibrium:** what factors control whether the disease can invade and persist?
  - **Endemic equilibrium:** tendency to cycle, damping or persistence of cycles
- Both of these methods have analogues in demographic models

Linear birth-death process

- We do an invasion analysis by asking how the number of infectives behaves in the limit where we assume that virtually the whole population is susceptible.
- This corresponds to a demographic model with the state determined by the number of infectious individuals I
- This system has only two events:
  - Infection at rate $R_0 I$
  - Recovery at rate I

Long-term behavior

- Unlike the finite systems discussed before, the probability of eventual extinction in this system is not one!
- Why not?
  - Probability of extinction given persistence goes to zero, as expected number of infectious individuals goes to ∞

Extinction probability

- Chains of infection are independent in this model
- We can use this fact to solve directly for the probability of extinction when starting from I infections, E_I
  - $E_I = R_0^{-I}$, when $R_0 > 1$
  - 1, otherwise
- We can define this as the 'fizzle' probability: the disease would have gone extinct even without depleting any susceptibles.

Moment calculations

- Ask: what is the expected behavior of the mean, variance, …of the ensemble?
- Define: $\mu = \sum_I I p_i$
- How does μ change through time?
  - $\dot{\mu} = \sum_I I \dot{p}_I$
  - $= \sum_I (b_I - m_I) p_I$, where $b(I) = R_0 I$ is the 'birth' rate, and $m(I) = I$ is the 'death' rate
- These equations can be solved in the linear system, or approximated (by "moment closure") for non-linear systems

Diffusion approximations

- We can approximate the discrete-valued demographic system with a real-valued system that reflects the mean *and variance* of the demographic system
  - Thus we can incorporate demographic stochasticity in a continuous system
  - An excellent approximation except when some values are very small
- In a linear (or linearized) system, we can solve the equilibrium distribution of the continuous equations, and thus approximate the quasi-equilibrium distribution
  - Disease persistence
  - Size of demographic fluctuations

Diffusion approximations

- We linearize about the endemic equilibrium, in exact analogy to Jacobian methods for stability in deterministic models
- Diffusion (and thus demographic stochasticity) is relatively unimportant when the square of the number infected is large compared to the demographic variance
  - Number infected at equilibrium: $\frac{(R_0 - 1)\rho N}{R}$
    - $\approx \rho N$
  - Demographic variance: $\approx N$
- Diffusion index $\approx \rho^2 N$. If ρ is small, demographic stochasticity can be important even for very large populations.

Outline

Conclusions

- Treating individuals as individuals can have dramatic effects on models of disease transmission
  - Acquired immunity is an important part of this phenomenon
- Stochastic models are hard, and we usually combine techniques to understand them:
  - Analytic approximation
  - Simulating ensemble distributions
  - Simulating realizations
- Remember: People are individuals $\implies$ demographic stochasticity is real!

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