Mathematical foundations for dynamics

Jonathan Dushoff, McMaster University
`http://lalashan.mcmaster.ca/DushoffLab`

2016 Summer Course on Mathematical Modeling and Analysis of Infectious Diseases

National Taiwan University

Goals

- This lecture will explain

Goals

- This lecture will explain
  - exponential growth (and decline)

Goals

- This lecture will explain
  - exponential growth (and decline)
  - simple qualitative methods for analyzing ODE-based dynamical systems

Goals

- This lecture will explain
  - exponential growth (and decline)
  - simple qualitative methods for analyzing ODE-based dynamical systems
  - the importance of linear equations

Goals

- This lecture will explain
  - exponential growth (and decline)
  - simple qualitative methods for analyzing ODE-based dynamical systems
  - the importance of linear equations
  - some basic ideas about matrices and eigenvalues

Goals

- This lecture will explain
  - exponential growth (and decline)
  - simple qualitative methods for analyzing ODE-based dynamical systems
  - the importance of linear equations
  - some basic ideas about matrices and eigenvalues

Outline

Modeling decline

- We have some bacteria in a tank

Modeling decline

- We have some bacteria in a tank
- They have no food, so they are simply dying at a *per capita* rate of 0.02/hr.

Modeling decline

- We have some bacteria in a tank
- They have no food, so they are simply dying at a *per capita* rate of 0.02/hr.
- If the current density is 100 bacteria/ml, what will be the density after 1 hr?

Modeling decline

- We have some bacteria in a tank
- They have no food, so they are simply dying at a *per capita* rate of 0.02/hr.
- If the current density is 100 bacteria/ml, what will be the density after 1 hr?
- What will be the density after 1 wk?

Modeling decline

- We have some bacteria in a tank
- They have no food, so they are simply dying at a *per capita* rate of 0.02/hr.
- If the current density is 100 bacteria/ml, what will be the density after 1 hr?
- What will be the density after 1 wk?

A simple model of population growth

- $\dfrac{dN}{dt} = (b - d)N$

A simple model of population growth

- $\frac{dN}{dt} = (b - d)N$
- $\frac{dN}{dt} = rN$

A simple model of population growth

- $\frac{dN}{dt} = (b - d)N$
- $\frac{dN}{dt} = rN$
- This is the only differential equation you need to solve!

A simple model of population growth

- $\dfrac{dN}{dt} = (b - d)N$
- $\dfrac{dN}{dt} = rN$
- This is the only differential equation you need to solve!
- $N(t) = N(0)e^{rt} = N(0)\exp(rt)$

A simple model of population growth

- $\dfrac{dN}{dt} = (b - d)N$
- $\dfrac{dN}{dt} = rN$
- This is the only differential equation you need to solve!
- $N(t) = N(0)e^{rt} = N(0)\exp(rt)$
- Bacteria example

A simple model of population growth

- $\frac{dN}{dt} = (b - d)N$
- $\frac{dN}{dt} = rN$
- This is the only differential equation you need to solve!
- $N(t) = N(0)e^{rt} = N(0)\exp(rt)$
- Bacteria example

Outline

A more realistic model of population growth

- Populations don't grow forever

A more realistic model of population growth

- Populations don't grow forever
  - or decline forever

A more realistic model of population growth

- Populations don't grow forever
  - or decline forever
- Probably the birth rate will decline if the population is too crowded

A more realistic model of population growth

- Populations don't grow forever
  - or decline forever
- Probably the birth rate will decline if the population is too crowded
- Let's let the birth rate go down as population goes up:

A more realistic model of population growth

- Populations don't grow forever
  - or decline forever
- Probably the birth rate will decline if the population is too crowded
- Let's let the birth rate go down as population goes up:
- $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$

A more realistic model of population growth

- Populations don't grow forever
  - or decline forever
- Probably the birth rate will decline if the population is too crowded
- Let's let the birth rate go down as population goes up:
- $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- *We don't want to solve this equation!*

A more realistic model of population growth

- Populations don't grow forever
  - or decline forever
- Probably the birth rate will decline if the population is too crowded
- Let's let the birth rate go down as population goes up:
- $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- *We don't want to solve this equation!*

What can we do instead?

- *Computer simulations*: what will happen with particular parameters?

What can we do instead?

- *Computer simulations*: what will happen with particular parameters?
- *Qualitative analysis*: what can we learn in general?

What can we do instead?

- *Computer simulations*: what will happen with particular parameters?
- *Qualitative analysis*: what can we learn in general?

Population growth model

- Structure: $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$

Population growth model

- Structure: $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?

Population growth model

- Structure: $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?
  - b_0: *per capita* birth rate [1/time]

Population growth model

- Structure: $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?
  - b_0: *per capita* birth rate [1/time]
  - d: *per capita* death rate [1/time]

Population growth model

- Structure: $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?
  - b_0: *per capita* birth rate [1/time]
  - d: *per capita* death rate [1/time]
  - N_b: Scale of population regulation [indiv]

Population growth model

- Structure: $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?
  - b_0: *per capita* birth rate [1/time]
  - d: *per capita* death rate [1/time]
  - N_b: Scale of population regulation [indiv]
- State variables?

Population growth model

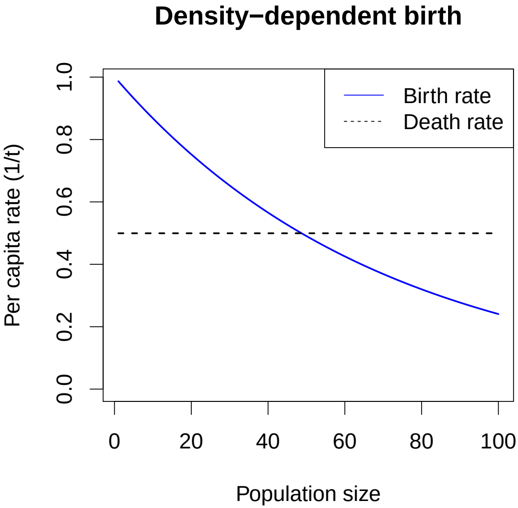
- Structure: $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?
  - b_0: *per capita* birth rate [1/time]
  - d: *per capita* death rate [1/time]
  - N_b: Scale of population regulation [indiv]
- State variables?
  - N: Population size [indiv]

Population growth model

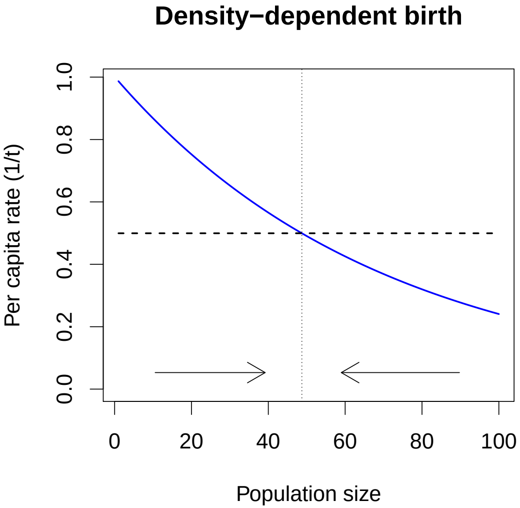
- Structure: $\dfrac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?
  - b_0: *per capita* birth rate [1/time]
  - d: *per capita* death rate [1/time]
  - N_b: Scale of population regulation [indiv]
- State variables?
  - N: Population size [indiv]

Computer simulation

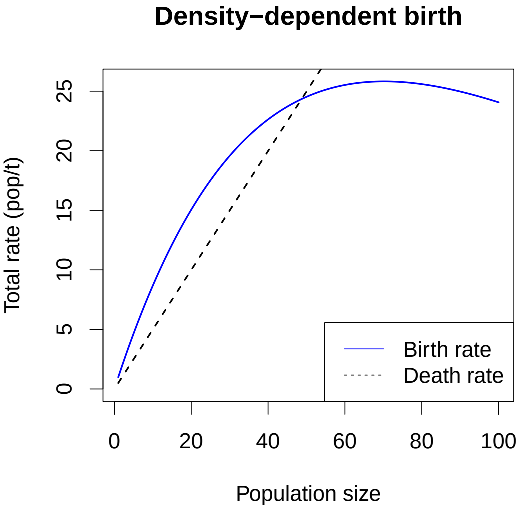
What will this model do?



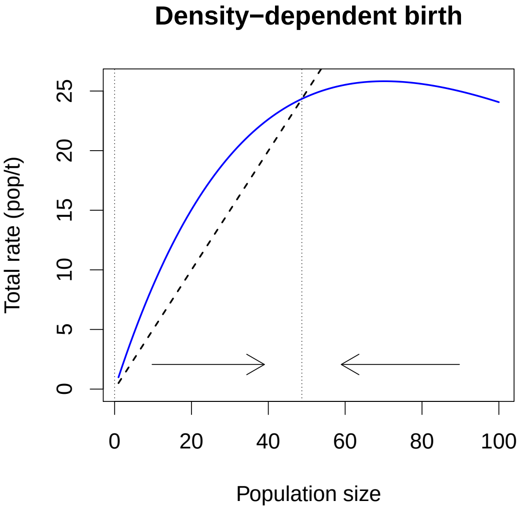
What will this model do?



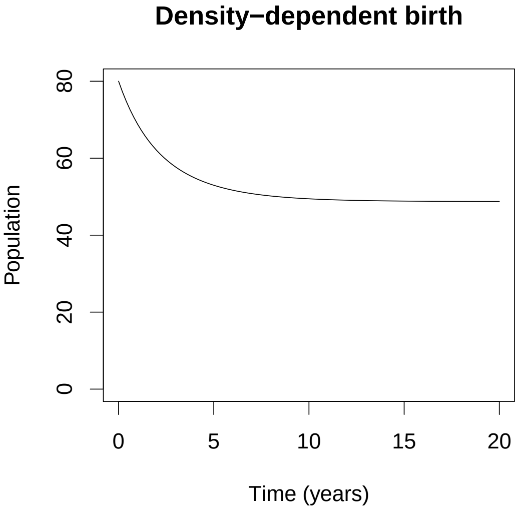
What will this model do?



What will this model do?



Computer simulation



Qualitative analysis

- Find *equilibria* – points where the population will not change

Qualitative analysis

- Find *equilibria* – points where the population will not change
  - Structure: $\dfrac{dN}{dt} = f(N)$

Qualitative analysis

- Find *equilibria* – points where the population will not change
  - Structure: $\dfrac{dN}{dt} = f(N)$
  - Equilibria when $f(N) = 0$

Qualitative analysis

- Find *equilibria* – points where the population will not change
  - Structure: $\dfrac{dN}{dt} = f(N)$
  - Equilibria when $f(N) = 0$
- Analyze equilibrium *stability* – if we are *near* the equilibrium, we will move toward it or away from it?

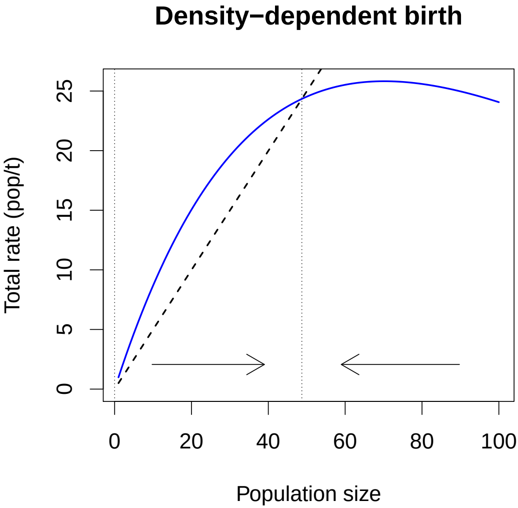
Qualitative analysis

- Find *equilibria* – points where the population will not change
  - Structure: $\frac{dN}{dt} = f(N)$
  - Equilibria when $f(N) = 0$
- Analyze equilibrium *stability* – if we are *near* the equilibrium, we will move toward it or away from it?
  - How does $f(N)$ *change* near an equilibrium?

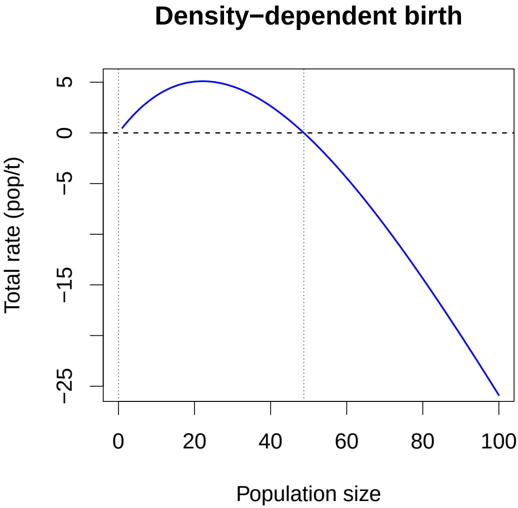
Qualitative analysis

- Find *equilibria* – points where the population will not change
  - Structure: $\dfrac{dN}{dt} = f(N)$
  - Equilibria when $f(N) = 0$
- Analyze equilibrium *stability* – if we are *near* the equilibrium, we will move toward it or away from it?
  - How does $f(N)$ *change* near an equilibrium?

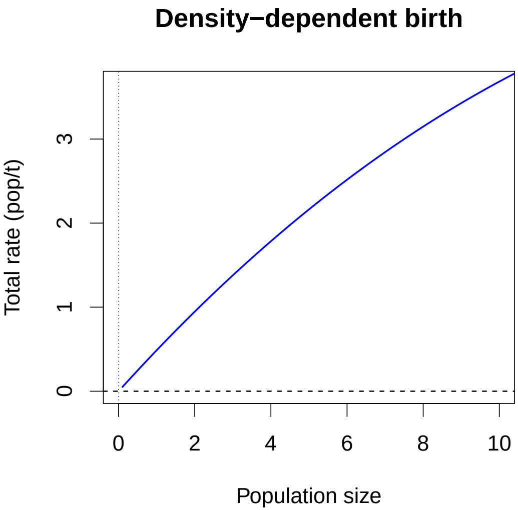
Linearization



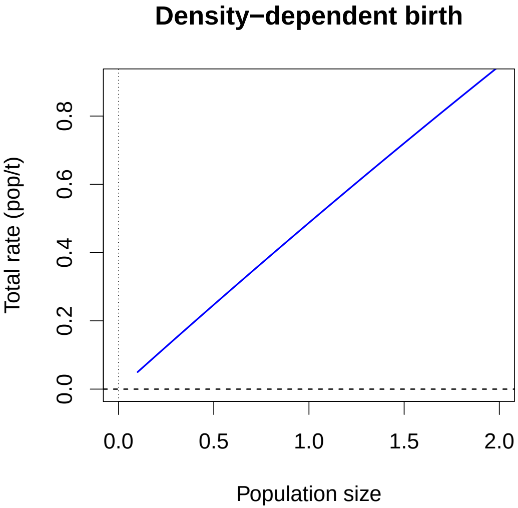
Linearization



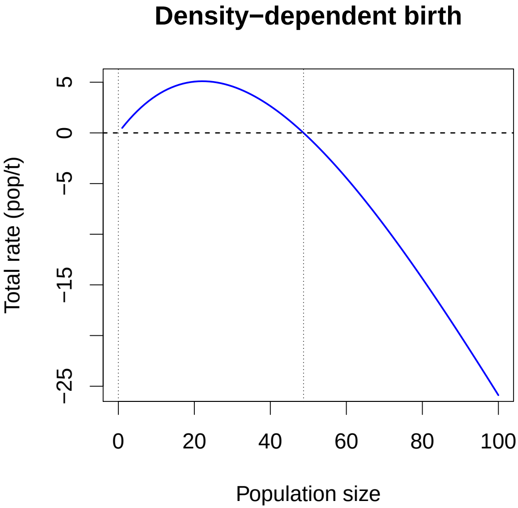
Zoom to extinction equilibrium



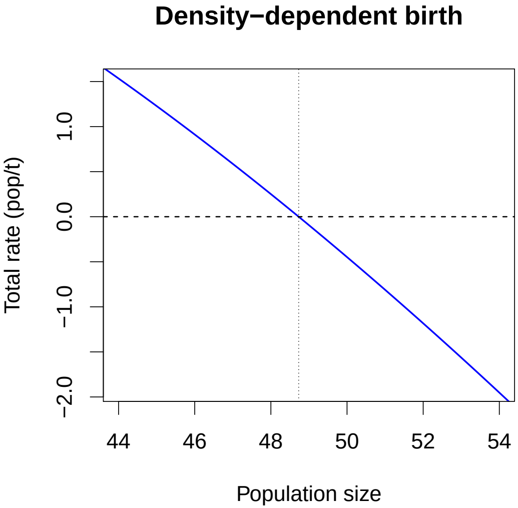
Zoom to extinction equilibrium



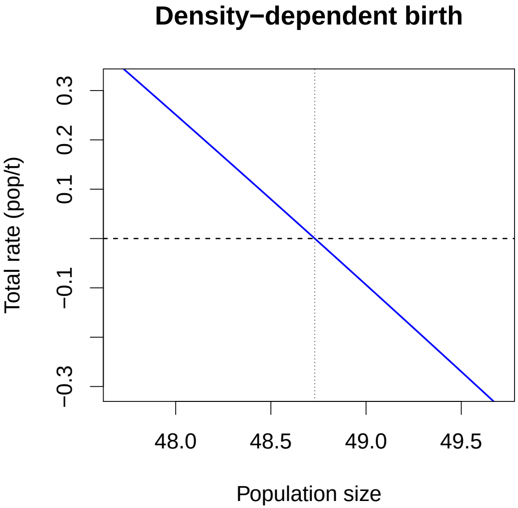
Linearization



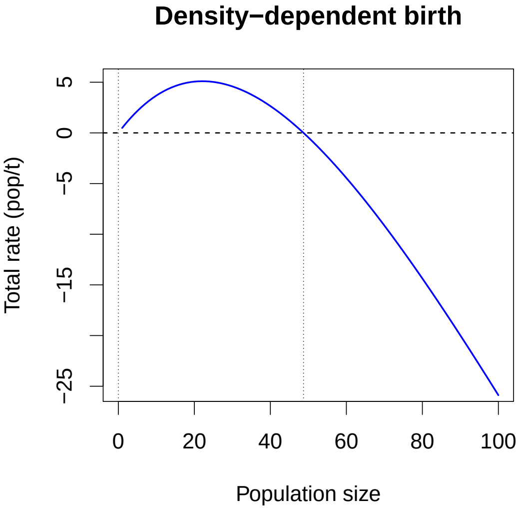
Zoom to carrying capacity



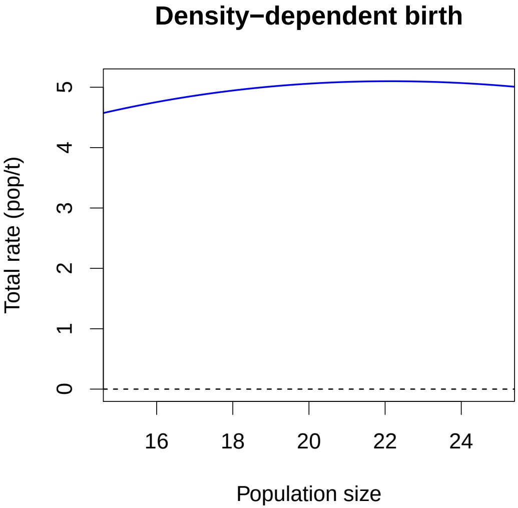
Zoom to carrying capacity



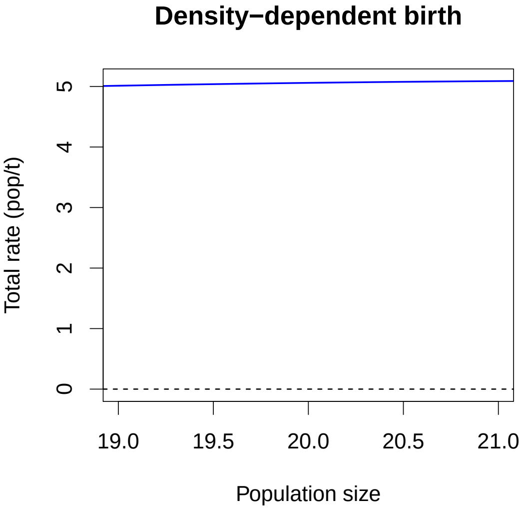
Linearization



Zoom to other point



Zoom to other point



Linearization

- Near an equilibrium, the system behaves like:

Linearization

- Near an equilibrium, the system behaves like:
- $\frac{dx}{dt} = Jx$

Linearization

- Near an equilibrium, the system behaves like:
- $\dfrac{dx}{dt} = Jx$
    - x is the distance from equilibrium

Linearization

- Near an equilibrium, the system behaves like:
- $\frac{dx}{dt} = Jx$
  - x is the distance from equilibrium
  - $J = \frac{\partial f}{\partial x}$

Linearization

- Near an equilibrium, the system behaves like:
- $\dfrac{dx}{dt} = Jx$
  - x is the distance from equilibrium
  - $J = \dfrac{\partial f}{\partial x}$
- The solution is $x(t) = x(0)\exp(Jt)$

Linearization

- Near an equilibrium, the system behaves like:
- $\dfrac{dx}{dt} = Jx$
  - x is the distance from equilibrium
  - $J = \dfrac{\partial f}{\partial x}$
- The solution is $x(t) = x(0)\exp(Jt)$
  - Moves away exponentially if $J > 0$

Linearization

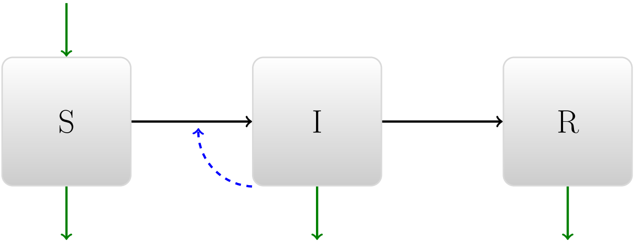
- Near an equilibrium, the system behaves like:
- $\dfrac{dx}{dt} = Jx$
  - x is the distance from equilibrium
  - $J = \dfrac{\partial f}{\partial x}$
- The solution is $x(t) = x(0)\exp(Jt)$
  - Moves away exponentially if $J > 0$
  - Moves in exponentially if $J < 0$

Linearization

- Near an equilibrium, the system behaves like:
- $\dfrac{dx}{dt} = Jx$
  - x is the distance from equilibrium
  - $J = \dfrac{\partial f}{\partial x}$
- The solution is $x(t) = x(0)\exp(Jt)$
  - Moves away exponentially if $J > 0$
  - Moves in exponentially if $J < 0$

Outline

What about our simple disease model?



$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \beta \frac{SI}{N} - \mu S \\
\frac{dI}{dt} &= \beta \frac{SI}{N} - \gamma I - \mu R \\
\frac{dR}{dt} &= \gamma I - \mu R
\end{aligned}$$

Disease model

- Parameters?

Disease model

- Parameters?
  - μ: Death rate [1/time]

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]
- State variables?

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]
- State variables?
  - S, I, R – but we are going to ignore R

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]
- State variables?
  - S, I, R – but we are going to ignore R
    - *

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]
- State variables?
  - S, I, R – but we are going to ignore R
    - * It does not affect S or I under our assumptions

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]
- State variables?
  - S, I, R – but we are going to ignore R
    - * It does not affect S or I under our assumptions
    - *

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]
- State variables?
  - S, I, R – but we are going to ignore R
    - * It does not affect S or I under our assumptions
    - * It is redundant (we know it if we know N, S and I.

Disease model

- Parameters?
  - μ: Death rate [1/time]
  - β: Transmission rate [1/time]
  - γ: Recovery rate [1/time]
  - N: Population size [indiv]
- State variables?
  - S, I, R – but we are going to ignore R
    - * It does not affect S or I under our assumptions
    - * It is redundant (we know it if we know N, S and I.

Equilibria

- $I = 0$, $S = N$

Equilibria

- $I = 0$, $S = N$
  - The *disease-free equilibrium* (DFE)

Equilibria

- $I = 0$, $S = N$
  - The *disease-free equilibrium* (DFE)
- $S = \gamma/\beta$, $I =$ (something)

Equilibria

- $I = 0$, $S = N$
  - The *disease-free equilibrium* (DFE)
- $S = \gamma/\beta$, $I =$ (something)
  - The *endemic equilibrium* (EE)

Equilibria

- $I = 0$, $S = N$
  - The *disease-free equilibrium* (DFE)
- $S = \gamma/\beta$, $I =$ (something)
  - The *endemic equilibrium* (EE)

Qualitative analysis

- $$\begin{aligned}\frac{dS}{dt} &= f(S, I)\\ \frac{dI}{dt} &= g(S, I)\end{aligned}$$

Qualitative analysis

- $$\begin{aligned}\frac{dS}{dt} &= f(S, I)\\ \frac{dI}{dt} &= g(S, I)\end{aligned}$$
- We still have linear equations near the equilibrium

Qualitative analysis

- $$\begin{aligned} \frac{dS}{dt} &= f(S, I) \\ \frac{dI}{dt} &= g(S, I) \end{aligned}$$
- We still have linear equations near the equilibrium
- This is the only kind of equation we can solve

Qualitative analysis

- $$\begin{aligned}\frac{dS}{dt} &= f(S, I) \\ \frac{dI}{dt} &= g(S, I)\end{aligned}$$
- We still have linear equations near the equilibrium
- This is the only kind of equation we can solve
- Behaviour is determined by

Qualitative analysis

- $$\begin{aligned}\frac{dS}{dt} &= f(S, I) \\ \frac{dI}{dt} &= g(S, I)\end{aligned}$$
- We still have linear equations near the equilibrium
- This is the only kind of equation we can solve
- Behaviour is determined by
- $$J = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} \end{pmatrix}$$

Qualitative analysis

- $$\begin{aligned} \frac{dS}{dt} &= f(S, I) \\ \frac{dI}{dt} &= g(S, I) \end{aligned}$$
- We still have linear equations near the equilibrium
- This is the only kind of equation we can solve
- Behaviour is determined by
- $$J = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} \end{pmatrix}$$

Outline

Rabbits

- Imagine we have a population of rabbits

Rabbits

- Imagine we have a population of rabbits
  - Baby rabbits become adults after one month

Rabbits

- Imagine we have a population of rabbits
  - Baby rabbits become adults after one month
  - Each pair of adult rabbits produces one pair of baby rabbits each month

Rabbits

- Imagine we have a population of rabbits
  - Baby rabbits become adults after one month
  - Each pair of adult rabbits produces one pair of baby rabbits each month
  - Rabbits never die

Rabbits

- Imagine we have a population of rabbits
  - Baby rabbits become adults after one month
  - Each pair of adult rabbits produces one pair of baby rabbits each month
  - Rabbits never die
- What happens to this population?

Rabbits

- Imagine we have a population of rabbits
  - Baby rabbits become adults after one month
  - Each pair of adult rabbits produces one pair of baby rabbits each month
  - Rabbits never die
- What happens to this population?

Matrix equations

- We describe this as equations for **A**dult and **B**aby rabbits:

Matrix equations

- We describe this as equations for **A**dult and **B**aby rabbits:
  - $A' = A + B$

Matrix equations

- We describe this as equations for **A**dult and **B**aby rabbits:
  - $A' = A + B$
  - $B' = A$

Matrix equations

- We describe this as equations for **A**dult and **B**aby rabbits:
  - $A' = A + B$
  - $B' = A$
- In matrix terms, we write:

Matrix equations

- We describe this as equations for **A**dult and **B**aby rabbits:
  - $A' = A + B$
  - $B' = A$
- In matrix terms, we write:
  - $$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix}$$

Matrix equations

- We describe this as equations for **A**dult and **B**aby rabbits:
  - $A' = A + B$
  - $B' = A$
- In matrix terms, we write:
  - $$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix}$$

Eigenvectors and eigenvalues

- We describe matrix dynamics using eigenvectors and eigenvalues

Eigenvectors and eigenvalues

- We describe matrix dynamics using eigenvectors and eigenvalues
  - An *eigenvector* is a vector which keeps its shape when multiplied by the matrix (it is just multiplied by a regular number)

Eigenvectors and eigenvalues

- We describe matrix dynamics using eigenvectors and eigenvalues
  - An *eigenvector* is a vector which keeps its shape when multiplied by the matrix (it is just multiplied by a regular number)
  - An *eigenvalue* is the number we multiply by

Eigenvectors and eigenvalues

- We describe matrix dynamics using eigenvectors and eigenvalues
  - An *eigenvector* is a vector which keeps its shape when multiplied by the matrix (it is just multiplied by a regular number)
  - An *eigenvalue* is the number we multiply by

Dominant values

- Usually, matrix dynamics have a single *dominant* eigenvalue (and eigenvector)

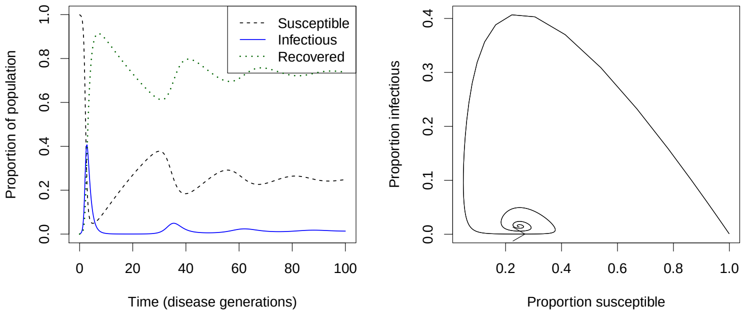
Dominant values

- Usually, matrix dynamics have a single *dominant* eigenvalue (and eigenvector)
  - This is just the one that is most important for the dynamics we are studying

Dominant values

- Usually, matrix dynamics have a single *dominant* eigenvalue (and eigenvector)
    - This is just the one that is most important for the dynamics we are studying

Disease example



Disease-free equilibrium

- Dominant eigenvalue is (usually) $\beta - \gamma$

Disease-free equilibrium

- Dominant eigenvalue is (usually) $\beta - \gamma$
  - Describes how fast the epidemic grows exponentially

Disease-free equilibrium

- Dominant eigenvalue is (usually) $\beta - \gamma$
  - Describes how fast the epidemic grows exponentially
  - Eigenvector describes relationship between *increase in I* and *decrease in S*

Disease-free equilibrium

- Dominant eigenvalue is (usually) $\beta - \gamma$
  - Describes how fast the epidemic grows exponentially
  - Eigenvector describes relationship between *increase in I* and *decrease in S*
- Other eigenvalue describes how fast susceptibles recover to equilibrium when there is no disease

Disease-free equilibrium

- Dominant eigenvalue is (usually) $\beta - \gamma$
  - Describes how fast the epidemic grows exponentially
  - Eigenvector describes relationship between *increase in I* and *decrease in S*
- Other eigenvalue describes how fast susceptibles recover to equilibrium when there is no disease

Endemic equilibrium

- There is a pair of *complex* eigenvalues

Endemic equilibrium

- There is a pair of *complex* eigenvalues
  - $a + bi$, where $i = \sqrt{-1}$

Endemic equilibrium

- There is a pair of *complex* eigenvalues
  - $a + bi$, where $i = \sqrt{-1}$
- In complex eigenvalues:

Endemic equilibrium

- There is a pair of *complex* eigenvalues
  - $a + bi$, where $i = \sqrt{-1}$
- In complex eigenvalues:
  - real part (a) describes exponential growth (or decline)

Endemic equilibrium

- There is a pair of *complex* eigenvalues
  - $a + bi$, where $i = \sqrt{-1}$
- In complex eigenvalues:
  - real part (a) describes exponential growth (or decline)
  - imaginary part (b) describes rate of oscillation

Endemic equilibrium

- There is a pair of *complex* eigenvalues
  - $a + bi$, where $i = \sqrt{-1}$
- In complex eigenvalues:
  - real part (a) describes exponential growth (or decline)
  - imaginary part (b) describes rate of oscillation