Introduction to dynamical modeling

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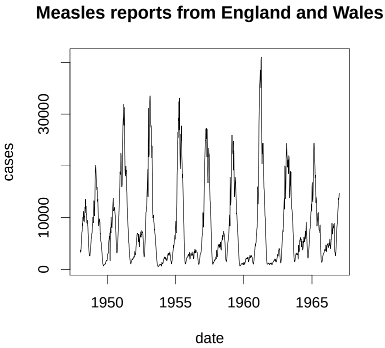
2016 Summer Course on Mathematical Modeling and Analysis of Infectious Diseases

National Taiwan University

Goals

- This lecture will:
  - introduce the idea of dynamical modeling
  - give simple examples of population modeling and disease modeling
  - discuss different types of model approaches

Dynamic modeling connects scales



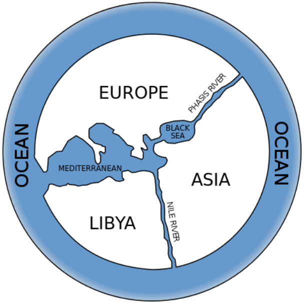
- Start with rules about how things change in short time steps
  - Usually based on *individuals*
- Calculate results over longer time periods
  - Usually about *populations*

Example: Dandelions

- Start with one dandelion; it produces 100 seeds, of which only 2% survive to reproduce.
- Seeds that survive to reproduce will produce 100 seeds after 1 year next year.
- How many dandelions after 3 years?

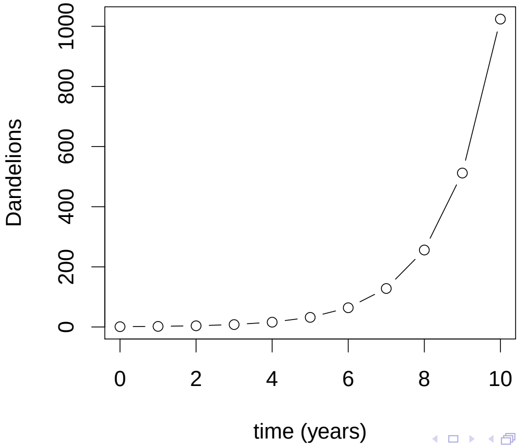
Model worlds

- A dynamic model is based on a model world
- The model world has *enough* assumptions to allow us to calculate dynamics
- The model world is *simpler* than the real world
- Essentially, all models are wrong, but some are useful. – Box and Draper (1987), *Empirical Model Building . . .*



Model result

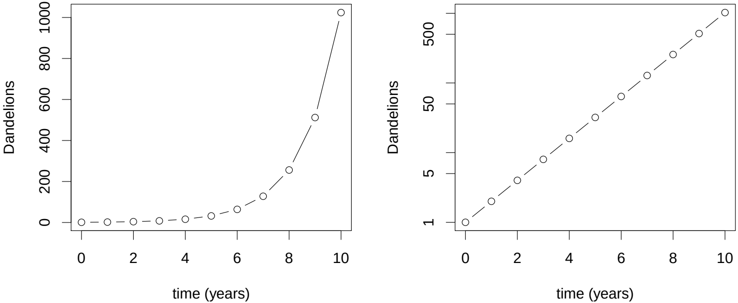
- If each individual is reproducing independently at each time step, the population changes *exponentially*
  - it is *multiplied* by the same amount in each step.



Scales

- The difference between 1 and 10 is the same as the difference between 10 and what?
  - *additive* difference:
    - * 19
  - *multiplicative* difference:
    - * 100

Scales

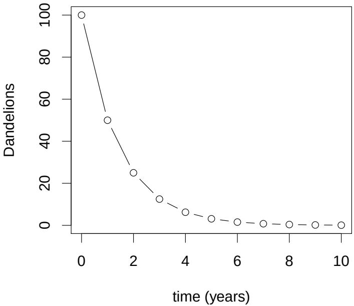


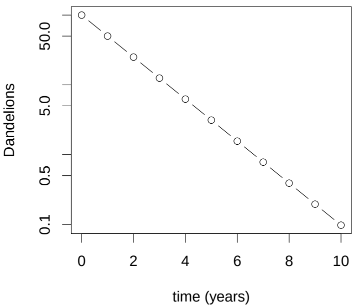
- Linear scale: 1 to 10 = 10 to 19
- Log scale: 1 to 10 = 10 to 100

Exponential change

- We can have exponential *growth* (population goes up)
- or exponential *decline* (population goes down)
- What if we spray the dandelions, so that each seed only has 0.5% chance of survival?

Exponential decline



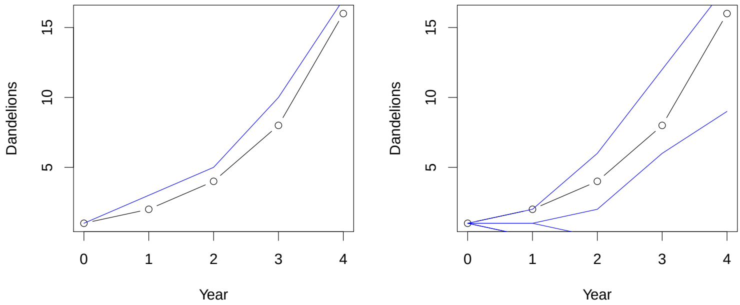


- Linear scale: 1 to 10 = 10 to 19
- Log scale: 1 to 10 = 10 to 100

Randomness

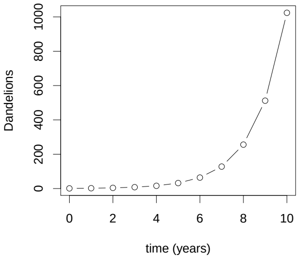
- Do our rules tell us exactly what is going to happen?
- If we have 1 dandelion this year, do we expect exactly two dandelions next year?
  - Do we expect exactly 1/2 of a dandelion?
- **Deterministic models**: rules describe exactly what will happen
- **Stochastic models**: rules describe a range of things that *might* happen

Stochastic model



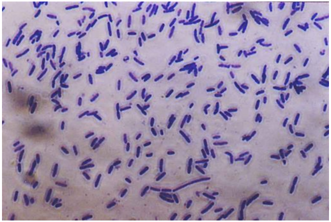
Time steps

- Dynamic models can use
  - *discrete time*: we model the population at specific time points
  - *continuous time*: we model time smoothly
- Which kind of model is the dandelion model?



Bacteria

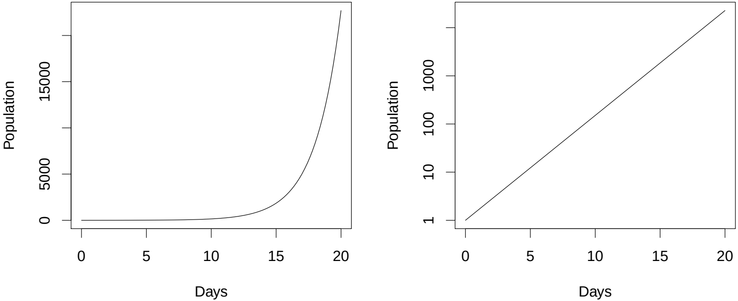
- Imagine we have some bacteria in a tank
- They are continuously dividing, and continuously dying



Bacteria

- Model world
  - The bacteria:
    - die at a constant *per capita* rate
    - divide at a constant *per capita* rate
- Model
  - $\dfrac{dN}{dt} = (b - d)N$

Model result



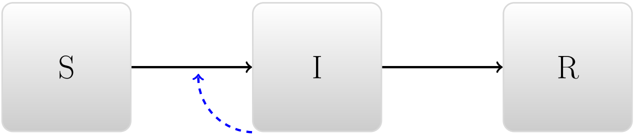
- Population grows *exponentially*
- As long as $b > d$

Describing a model

- **Model structure:** what are the rules that our model is following?
  - Each individual is dividing and dying independently at a fixed rate
- **Parameters:** what quantities (with units) determine how the rules are working
  - Birth rate is 0.04/day
- **State variables:** what changing quantities are we modeling?
  - The number of bacteria

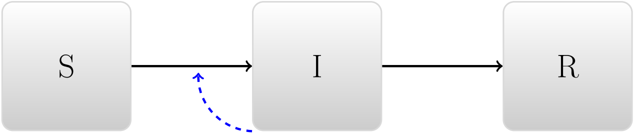
Simple models of disease spread

- Divide people into categories:



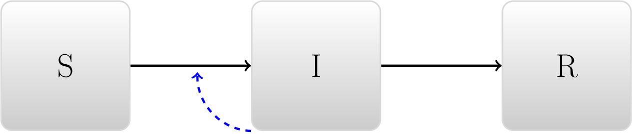
- Susceptible: can be infected
- Infectious: can infect others
- Recovered: cannot be infected

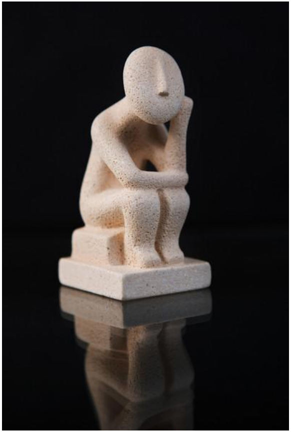
What determines transition rates?



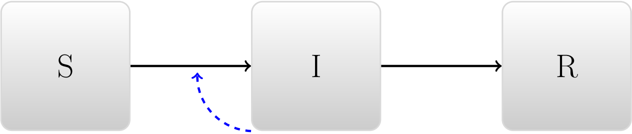
- People get better independently
- People get infected by infectious people

Conceptual modeling



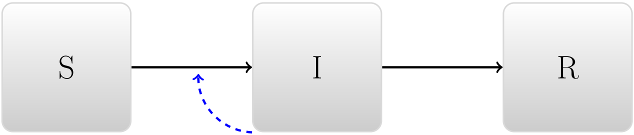


Conceptual modeling



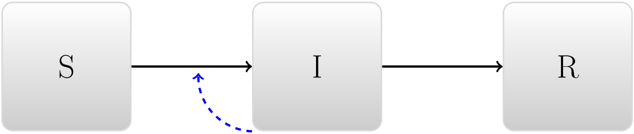
- What is the final result?
- When does disease increase, decrease?

Dynamic implementation



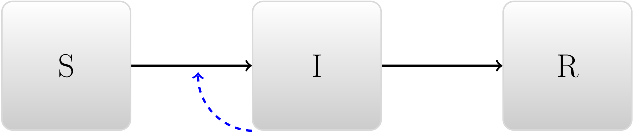
- Requires assumptions about time distributions
- The *conceptually simplest* implementation uses **Ordinary Differential Equations** (ODEs)
  - Other options may be more realistic
  - Or simpler in practice

Recovery



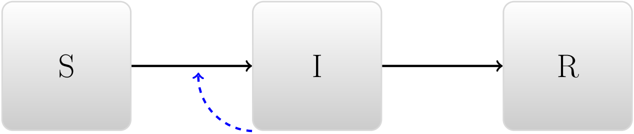
- Infectious people recover at *per capita* rate γ
  - Total recovery rate is γI
  - Mean time infectious is $D = 1/\gamma$

Transmission



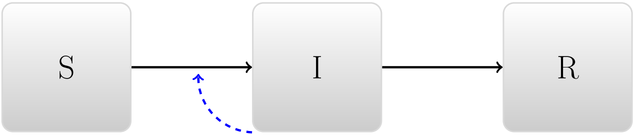
- Susceptible people get infected by:
  - Going around and contacting people (rate c)
  - Some of these people are infectious (proportion I/N)
  - Some of these contacts are effective (proportion p)
- Per capita rate of becoming infected is $cpI/N \equiv \beta I/N$
- Population-level transmission rate is $\mathcal{T} = \beta SI/N$

Another perspective on transmission



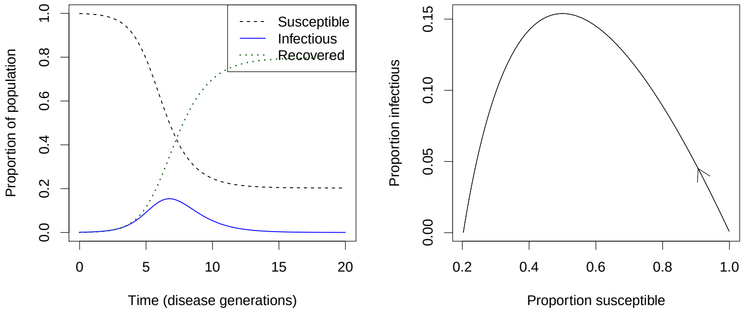
- Infectious people infect others by:
  - Going around and contacting people (rate c)
  - Some of these people are susceptible (proportion S/N)
  - Some of these contacts are effective (proportion p)
- Per capita rate of infecting others is $cpS/N \equiv \beta S/N$
- Population-level transmission rate is $\mathcal{T} = \beta SI/N$

ODE implementation



$$
\begin{aligned}
\frac{dS}{dt} &= -\beta\frac{SI}{N} \\
\frac{dI}{dt} &= \beta\frac{SI}{N} - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned}
$$

ODE implementation

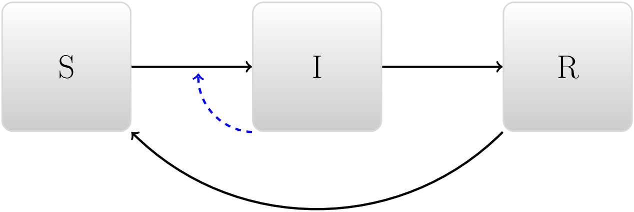


- Not everyone will get infected
- Disease starts to decline when number of susceptibles is small

ODE assumptions

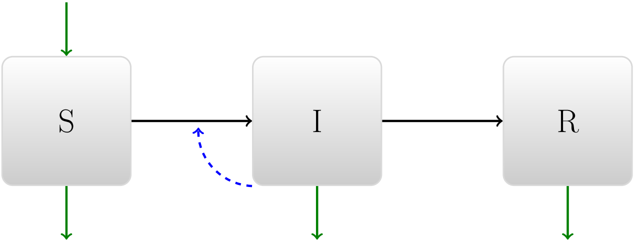
- Lots and lots of people
- Perfectly mixed

Closing the circle



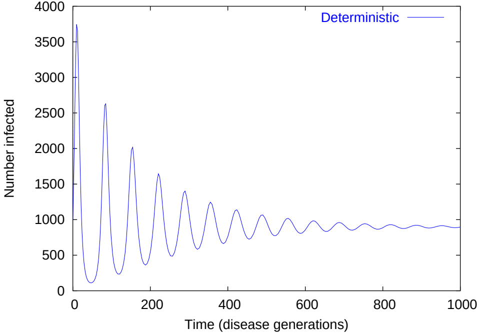
- Loss of immunity

Closing the circle



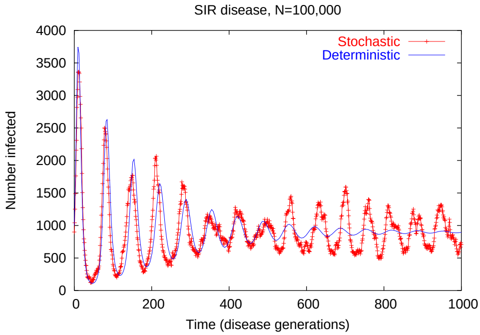
- Births and deaths

Tendency to oscillate



Modeling individuals as individuals usually requires a *stochastic* model

With individuality



Even in the simplest form, this can cause large random oscillations even in large populations

Types

- **Discrete** vs. **Continuous** time steps
- **Deterministic** vs. **Stochastic** dynamics
  - Stochastic models may have **Discrete individuals**

Summary

- Dynamics are an essential tool to link scales
- Very simple models can provide useful insights
- More complex models can provide more detail, but also require more assumptions, and more choices