

# Mathematical foundations for dynamics

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2016 Summer Course on Mathematical Modeling and Analysis  
of Infectious Diseases

National Taiwan University

# Goals

- ▶ This lecture will explain
  - ▶ exponential growth (and decline)
  - ▶ simple qualitative methods for analyzing ODE-based dynamical systems
  - ▶ the importance of linear equations
  - ▶ some basic ideas about matrices and eigenvalues

# Outline

Exponential change

Population growth example

Disease model

Matrices and eigenvalues

# Modeling decline

- ▶ We have some bacteria in a tank
- ▶ They have no food, so they are simply dying at a *per capita* rate of 0.02/hr.
- ▶ If the current density is 100 bacteria/ml, what will be the density after 1 hr?
- ▶ What will be the density after 1 wk?

# A simple model of population growth

- ▶  $\frac{dN}{dt} = (b - d)N$
- ▶  $\frac{dN}{dt} = rN$
- ▶ This is the only differential equation you need to solve!
- ▶  $N(t) = N(0)e^{rt} = N(0) \exp(rt)$
- ▶ Bacteria example

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Exponential change

Population growth example

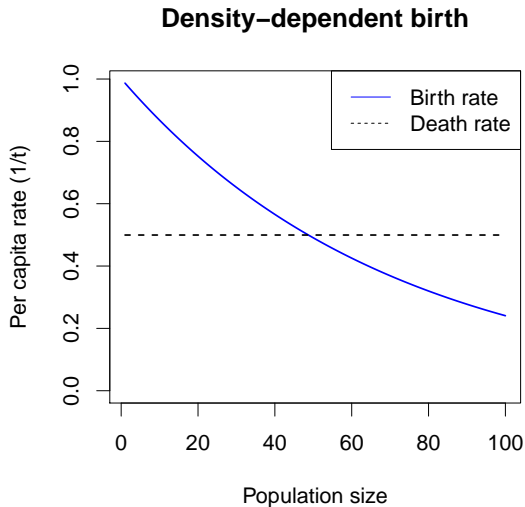
Disease model

Matrices and eigenvalues

# A more realistic model of population growth

- ▶ Populations don't grow forever
  - ▶ or decline forever
- ▶ Probably the birth rate will decline if the population is too crowded
- ▶ Let's let the birth rate go down as population goes up:
- ▶  $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$

# A model of population growth





# A model of population growth

- ▶  $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- ▶ *We don't want to solve this equation!*

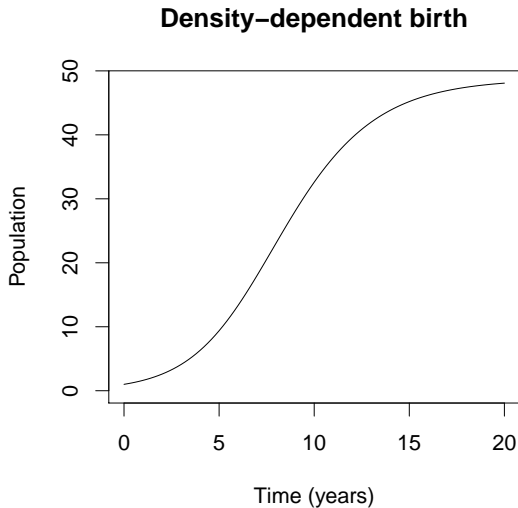
# What can we do instead?

- ▶ *Computer simulations*: what will happen with particular parameters?
- ▶ *Qualitative analysis*: what can we learn in general?

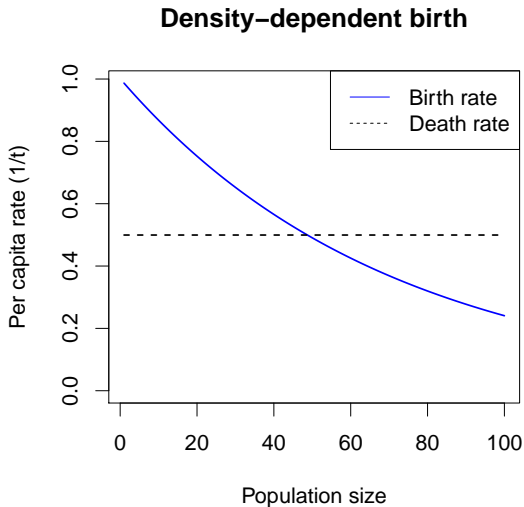
# Population growth model

- ▶ Structure:  $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- ▶ Parameters?
  - ▶  $b_0$ : *per capita* birth rate [1/time]
  - ▶  $d$ : *per capita* death rate [1/time]
  - ▶  $N_b$ : Scale of population regulation [indiv]
- ▶ State variables?
  - ▶  $N$ : Population size [indiv]

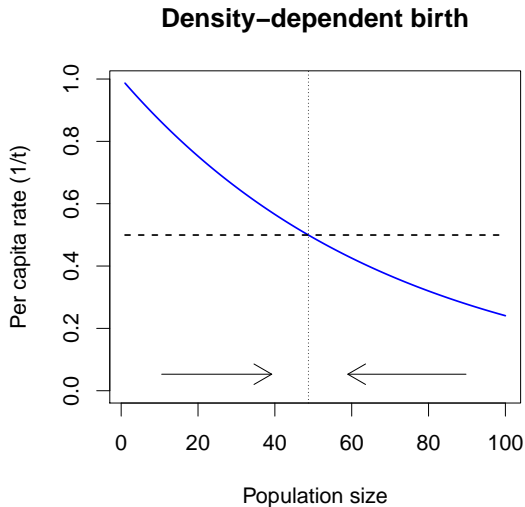
# Computer simulation



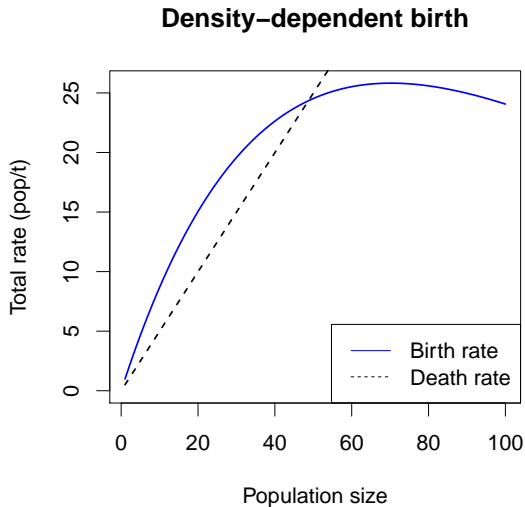
# What will this model do?



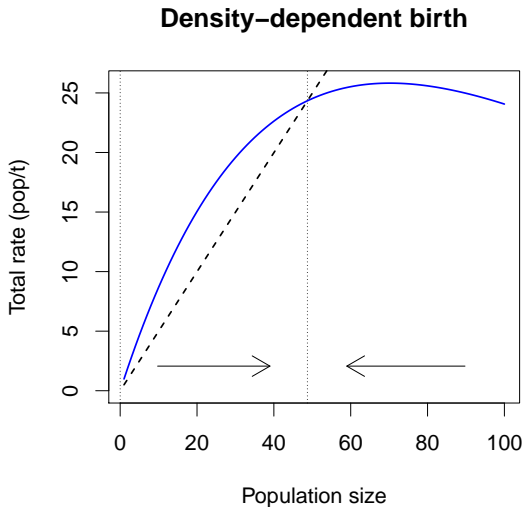
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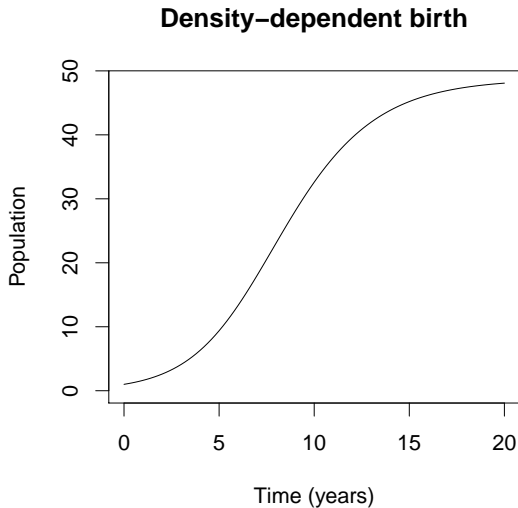


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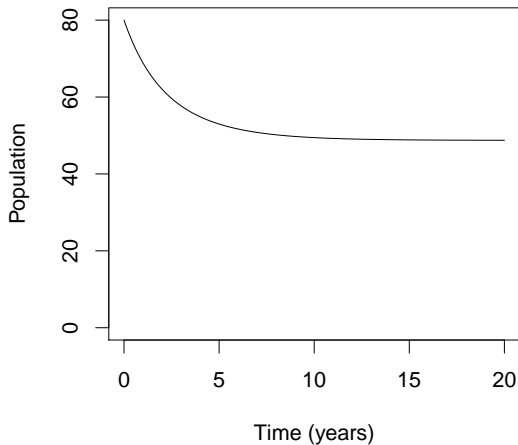


## Computer simulation



# Computer simulation

## Density-dependent birth

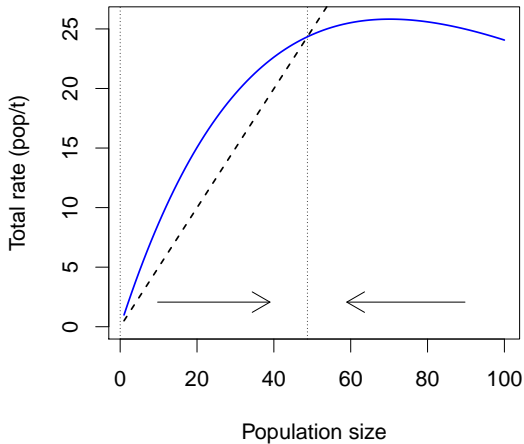


# Qualitative analysis

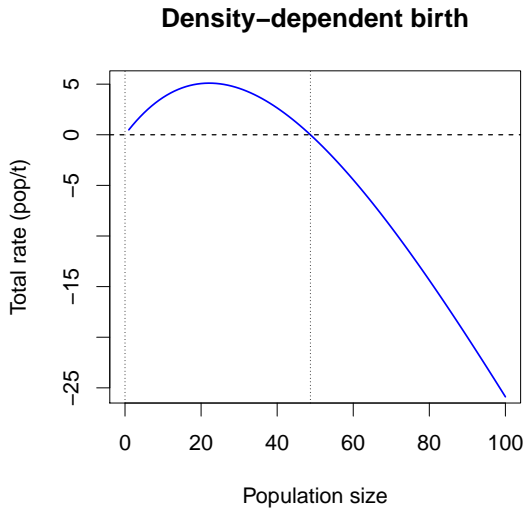
- ▶ Find *equilibria* – points where the population will not change
  - ▶ Structure:  $\frac{dN}{dt} = f(N)$
  - ▶ Equilibria when  $f(N) = 0$
- ▶ Analyze equilibrium *stability* – if we are *near* the equilibrium, we will move toward it or away from it?
  - ▶ How does  $f(N)$  *change* near an equilibrium?

# Linearization

## Density-dependent birth

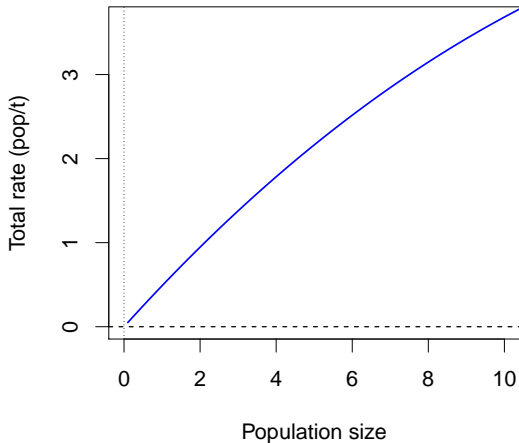


# Linearization

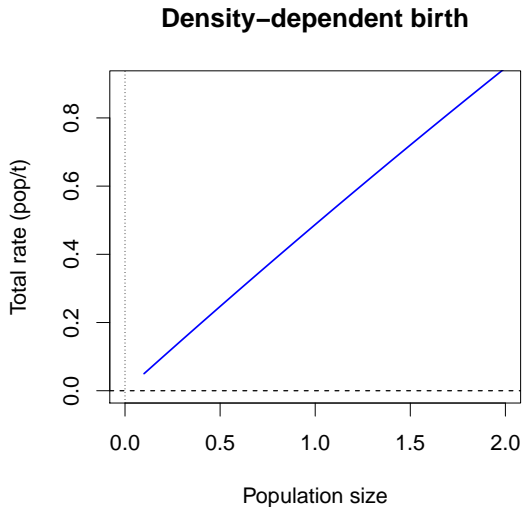


# Zoom to extinction equilibrium

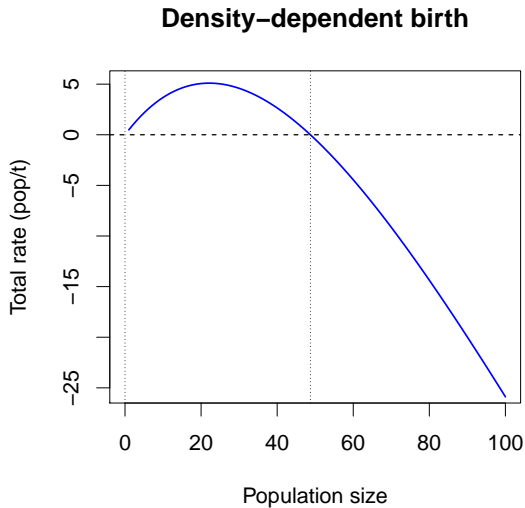
## Density-dependent birth



# Zoom to extinction equilibrium

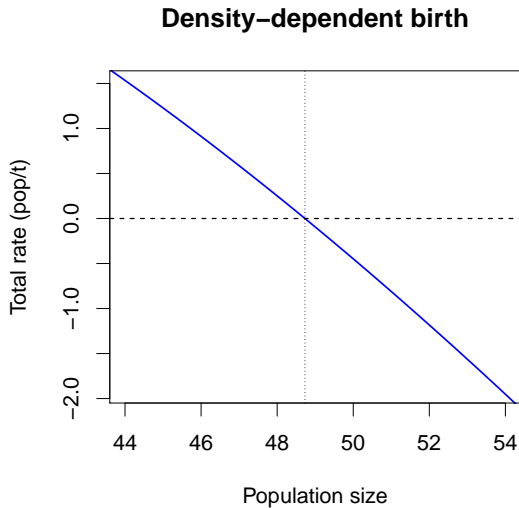


# Linearization

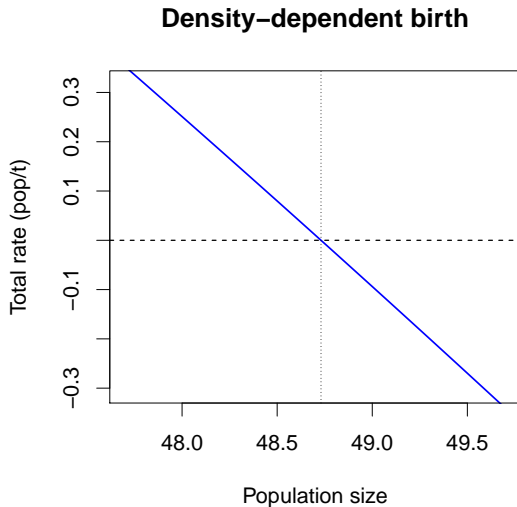




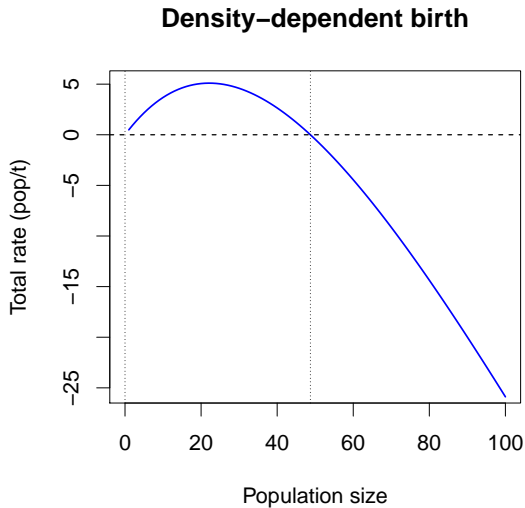
# Zoom to carrying capacity



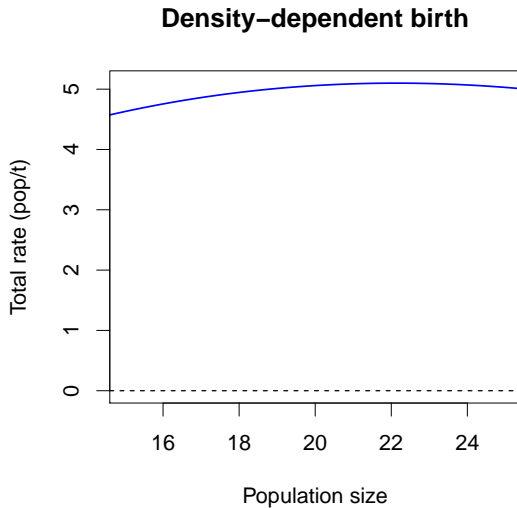
# Zoom to carrying capacity



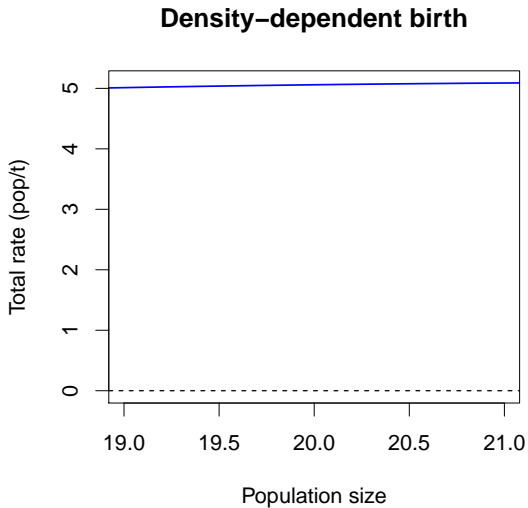
# Linearization



## Zoom to other point



## Zoom to other point



# Linearization

- ▶ Near an equilibrium, the system behaves like:

- ▶  $\frac{dx}{dt} = Jx$

- ▶  $x$  is the distance from equilibrium

- ▶  $J = \frac{\partial f}{\partial x}$

- ▶ The solution is  $x(t) = x(0) \exp(Jt)$

- ▶ Moves away exponentially if  $J > 0$
  - ▶ Moves in exponentially if  $J < 0$

# Outline

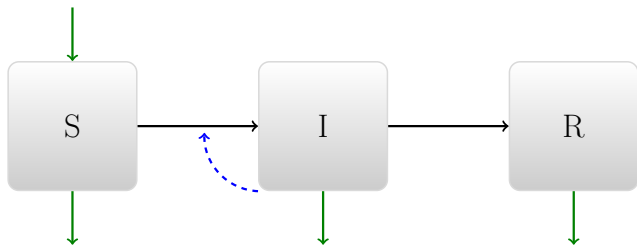
Exponential change

Population growth example

Disease model

Matrices and eigenvalues

## What about our simple disease model?



$$\begin{aligned}\frac{dS}{dt} &= \mu N - \beta \frac{SI}{N} - \mu S \\ \frac{dI}{dt} &= \beta \frac{SI}{N} - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R\end{aligned}$$



# Disease model

- ▶ Parameters?

- ▶  $\mu$ : Death rate [1/time]
- ▶  $\beta$ : Transmission rate [1/time]
- ▶  $\gamma$ : Recovery rate [1/time]
- ▶  $N$ : Population size [indiv]

- ▶ State variables?

- ▶  $S, I, R$  – but we are going to ignore  $R$ 
  - ▶ \* It does not affect  $S$  or  $I$  under our assumptions
  - ▶ \* It is redundant (we know it if we know  $N, S$  and  $I$ .)

# Equilibria

- ▶  $I = 0, S = N$ 
  - ▶ The *disease-free equilibrium* (DFE)
- ▶  $S = \gamma/\beta, I = (\text{something})$ 
  - ▶ The *endemic equilibrium* (EE)

# Qualitative analysis



$$\begin{aligned}\frac{dS}{dt} &= f(S, I) \\ \frac{dI}{dt} &= g(S, I)\end{aligned}$$

- ▶ We still have linear equations near the equilibrium
- ▶ This is the only kind of equation we can solve
- ▶ Behaviour is determined by



$$J = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} \end{pmatrix}$$

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# Rabbits

- ▶ Imagine we have a population of rabbits
  - ▶ Baby rabbits become adults after one month
  - ▶ Each pair of adult rabbits produces one pair of baby rabbits each month
  - ▶ Rabbits never die
- ▶ What happens to this population?

# Matrix equations

- ▶ We describe this as equations for **A**dult and **B**aby rabbits:
  - ▶  $A' = A + B$
  - ▶  $B' = A$
- ▶ In matrix terms, we write:

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

# Eigenvectors and eigenvalues

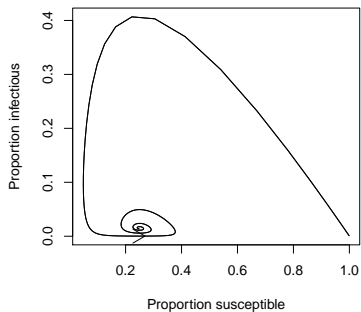
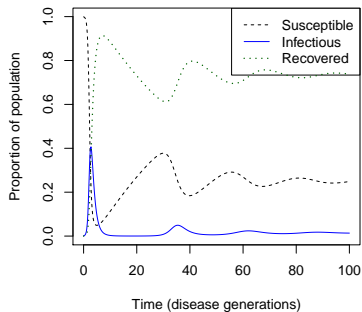
- ▶ We describe matrix dynamics using eigenvectors and eigenvalues
  - ▶ An *eigenvector* is a vector which keeps its shape when multiplied by the matrix (it is just multiplied by a regular number)
  - ▶ An *eigenvalue* is the number we multiply by

# Dominant values

- ▶ Usually, matrix dynamics have a single *dominant* eigenvalue (and eigenvector)
  - ▶ This is just the one that is most important for the dynamics we are studying



# Disease example



# Disease-free equilibrium

- ▶ Dominant eigenvalue is (usually)  $\beta - \gamma$ 
  - ▶ Describes how fast the epidemic grows exponentially
  - ▶ Eigenvector describes relationship between *increase in I* and *decrease in S*
- ▶ Other eigenvalue describes how fast susceptibles recover to equilibrium when there is no disease

# Endemic equilibrium

- ▶ There is a pair of *complex* eigenvalues
  - ▶  $a + bi$ , where  $i = \sqrt{-1}$
- ▶ In complex eigenvalues:
  - ▶ real part ( $a$ ) describes exponential growth (or decline)
  - ▶ imaginary part ( $b$ ) describes rate of oscillation