

The SIR Model Family

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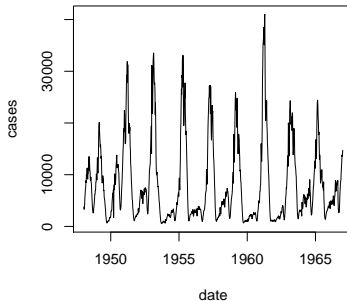
DAIDD 2016

<http://www.ici3d.org/daidd/>

Dynamic modeling connects scales



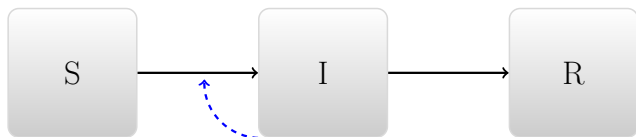
Measles reports from England and Wales



- ▶ Start with rules about how things change in short time steps
 - ▶ Usually based on *individuals*

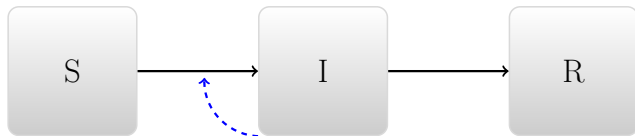
Compartmental models

Divide people into categories:



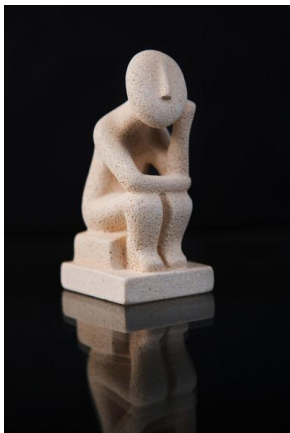
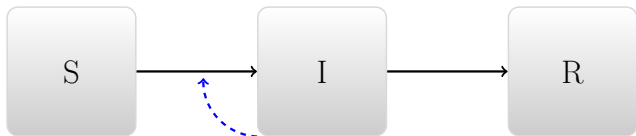
- Susceptible \rightarrow Infectious \rightarrow Recovered

What determines transition rates?

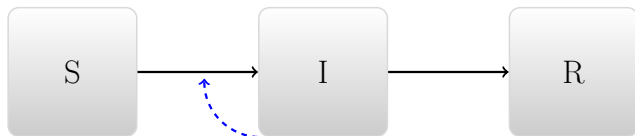


- ▶ People get better independently
- ▶ People get infected by infectious people

Conceptual modeling

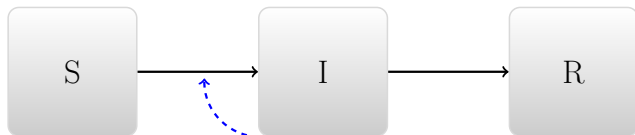


Conceptual modeling



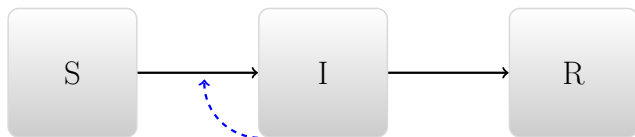
- ▶ What is the final result?
- ▶ When does disease increase, decrease?

Dynamic implementation



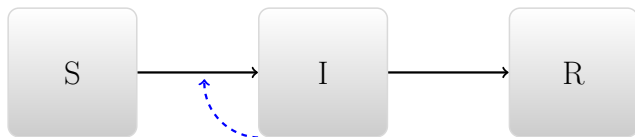
- ▶ Requires assumptions about recovery and transmission
- ▶ The *conceptually simplest* implementation uses Ordinary Differential Equations (ODEs)
 - ▶ Other options may be more realistic
 - ▶ Or simpler in practice

Recovery



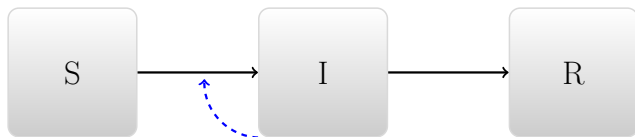
- ▶ Infectious people recover at *per capita* rate γ
 - ▶ Total recovery rate is γI
 - ▶ Mean time infectious is $D = 1/\gamma$

Transmission



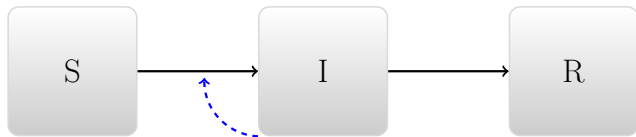
- ▶ Susceptible people get infected by:
 - ▶ Going around and contacting people (rate c)
 - ▶ Some of these people are infectious (proportion I/N)
 - ▶ Some of these contacts are effective (proportion p)
- ▶ Per capita rate of becoming infected is $cpI/N \equiv \beta I/N$
- ▶ Population-level transmission rate is $\mathcal{T} = \beta SI/N$

Another perspective on transmission



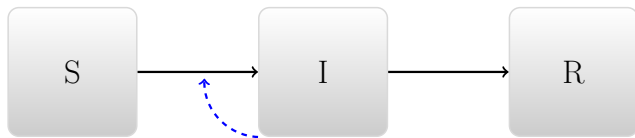
- ▶ Infectious people infect others by:
 - ▶ Going around and contacting people (rate c)
 - ▶ Some of these people are susceptible (proportion S/N)
 - ▶ Some of these contacts are effective (proportion p)
- ▶ Per capita rate of infecting others is $cpS/N \equiv \beta S/N$
- ▶ Population-level transmission rate is $\mathcal{T} = \beta SI/N$

ODE implementation



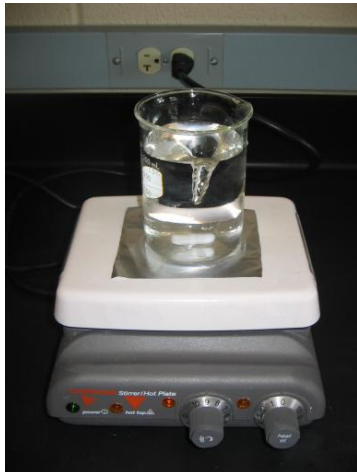
$$\begin{aligned}\frac{dS}{dt} &= -\beta \frac{SI}{N} \\ \frac{dI}{dt} &= \beta \frac{SI}{N} - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

Spreadsheet implementation



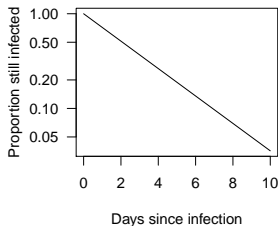
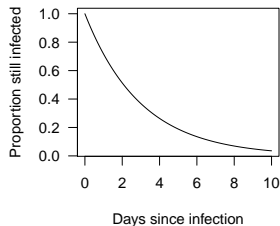
<http://tinyurl.com/SIR-DAIDD-2016>

ODE assumptions



- ▶ Lots and lots of people
- ▶ Perfectly mixed

ODE assumptions



- ▶ Waiting times are exponentially distributed
- ▶ Rarely realistic

Scripts vs. spreadsheets

	Susceptibles	infectious	Removed	Total
	people	people	people	people
0	999		1	0
1	998.6004	1.1996		0.2
2	998.121231584064	1.438848415936		0.43992
3	997.546773522873	1.72553679393953	0.7276896831872	
4	996.858252058318	2.06895089970737	1.07279704197511	
5	996.033271747327	2.48014103075661	1.48658722191658	
6	995.045150553223	2.97223401870901	1.9826154280679	
7	993.862147734573	3.58079003361749	2.5770622318097	
8	992.446573962396	4.26420579907117	3.2892202385332	
9	990.753775388012	5.10416321364044	4.14206139834744	
10	988.730987798368	6.1061181605567	5.16289404107552	
11	986.316064502168	7.29981782464567	6.38411767318686	
12	983.436093466813	8.71982529507146	7.844081238116	
13	980.005937097253	10.4060166056164	9.58804629713026	

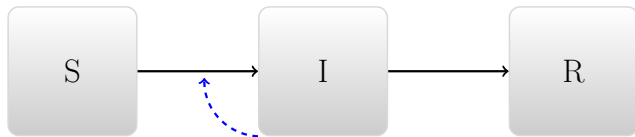
```
return(as.list(  
  Sdot = - beta*S*I/N,  
  Idot = S*I/N - I/D,  
  Rdot = I/D  
))
```

More about transmission



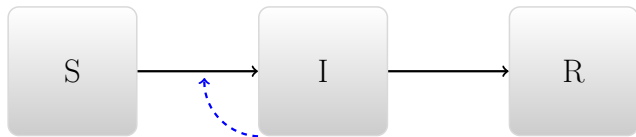
- ▶ $\beta = pc$
 - ▶ What is a contact?
 - ▶ What is the probability of transmission?
- ▶ Sometimes this decomposition is clear
- ▶ But usually it's not

Population sizes



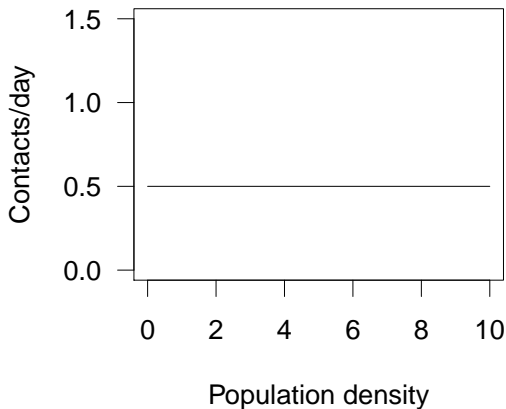
$$\begin{aligned}\frac{dS}{dt} &= -\beta \frac{SI}{N} \\ \frac{dI}{dt} &= \beta \frac{SI}{N} - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

Population sizes



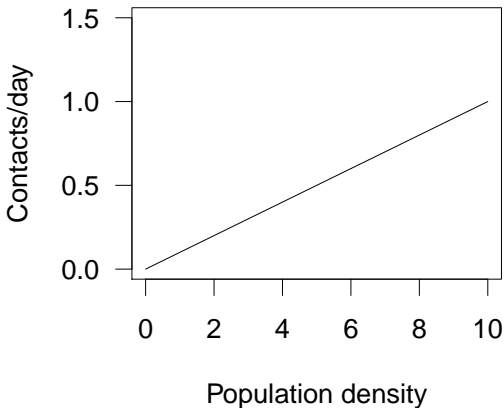
$$\begin{aligned}\frac{dS}{dt} &= -\beta(N)\frac{SI}{N} \\ \frac{dI}{dt} &= \beta(N)\frac{SI}{N} - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

Standard incidence



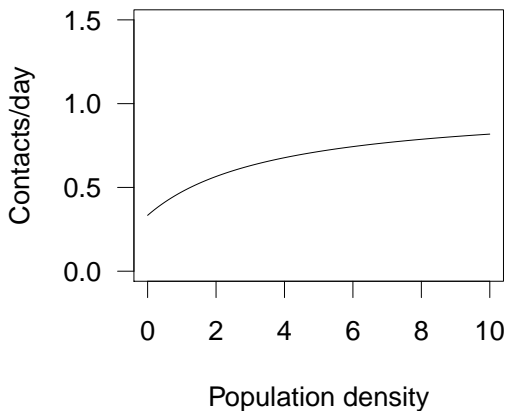
- ▶ $\beta(N) = \beta_0$
- ▶ $\mathcal{T} = \frac{\beta_0 SI}{N}$
- ▶ Also known as *frequency-dependent* transmission

Mass action



- ▶ $\beta(N) = \beta_1 N$
- ▶ $\mathcal{T} = \beta_1 SI$
- ▶ Also known as *density-dependent* transmission

General

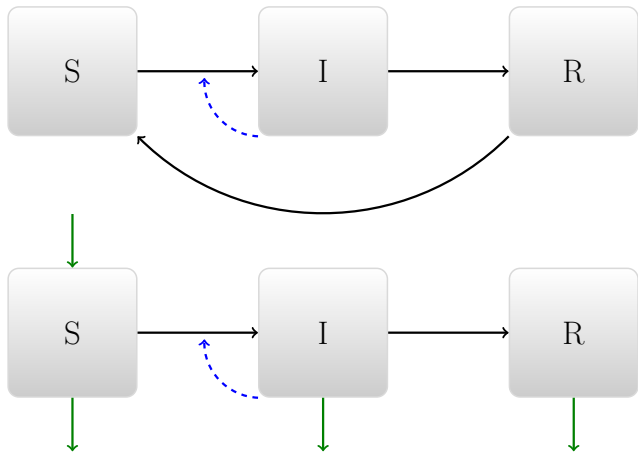


- ▶ May not go to zero when N does
- ▶ May not go to ∞ when N does

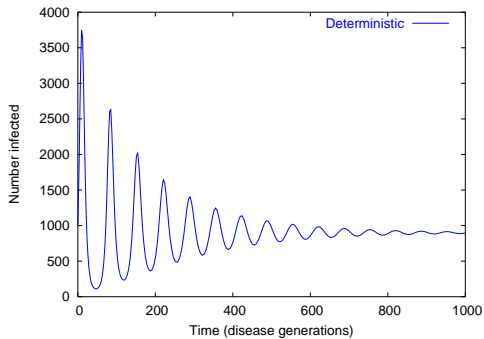
Digression – units

- ▶ $\mathcal{T} = \beta SI/N : [\text{ppl}/\text{time}]$
- ▶ $\beta : [1/\text{time}]$
 - ▶ $\beta/\gamma = \beta D : [1]$
 - ▶ Standard incidence, $\beta_0 : [1/\text{time}]$
 - ▶ Mass-action incidence, $\beta_1 : [1/(\text{people} \cdot \text{time})]$

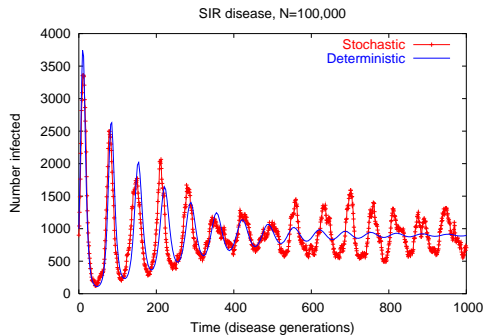
Closing the circle



Tendency to oscillate



With individuality



Summary

- ▶ Dynamics are an essential tool to link scales
- ▶ Very simple models can provide useful insights
- ▶ More complex models can provide more detail, but also require more assumptions, and more choices

Conclusions from simple models

- ▶ There is a link between individual-level processes and population-level outcomes
- ▶ The reproductive number (number of cases per case) is a key quantity
 - ▶ Disease increases when $\mathcal{R} > 1$
 - ▶ Decreases when $\mathcal{R} < 1$
- ▶ Oscillation
 - ▶ If susceptibles are replenished, diseases have a tendency to oscillate
 - ▶ These oscillations tend to be damped (get smaller through time)
- ▶ These conclusions from **simple** models help *guide* our understanding of more complicated models