Mathematical foundations for dynamics

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Goals

- This lecture will explain
  - exponential growth (and decline)
  - simple qualitative methods for analyzing ODE-based dynamical systems
  - the importance of linear equations
  - some basic ideas about matrices and eigenvalues

1 Exponential change

Modeling decline

- We have some bacteria in a tank
- They have no food, so they are simply dying at a *per capita* rate of 0.02/hr.
- If the current density is 100 bacteria/ml, what will be the density after 1 hr?
- What will be the density after 1 wk?

A simple model of population growth

- $\frac{dN}{dt} = (b - d)N$
- $\frac{dN}{dt} = rN$
- This is the only differential equation you need to solve!
- $N(t) = N(0)e^{rt} = N(0)\exp(rt)$
- Bacteria example

2 Population growth example

A more realistic model of population growth

- Populations don't grow forever
  - or decline forever
- Probably the birth rate will decline if the population is too crowded
- Let's let the birth rate go down as population goes up:
- $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- *We don't want to solve this equation!*

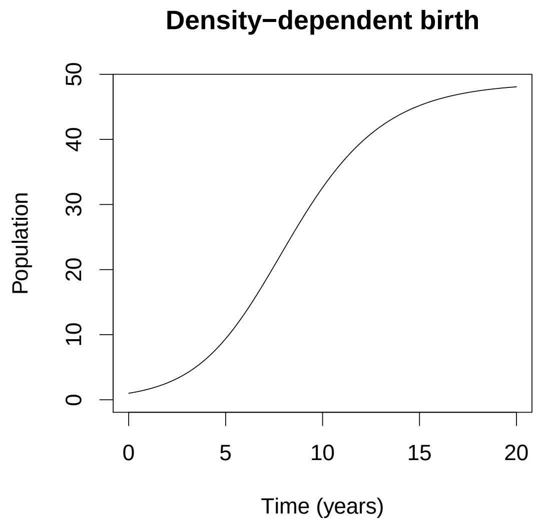
What can we do instead?

- *Computer simulations*: what will happen with particular parameters?
- *Qualitative analysis*: what can we learn in general?

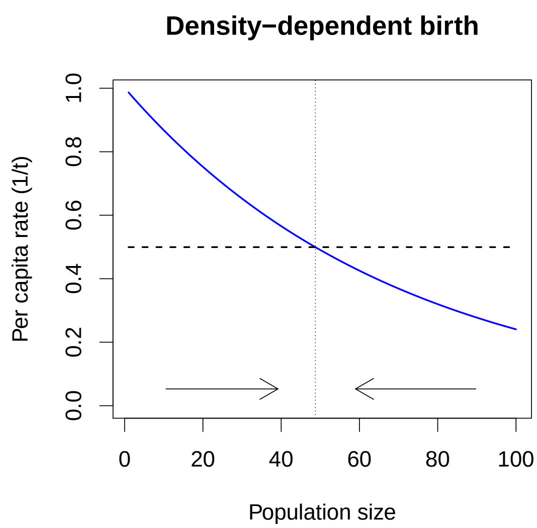
Population growth model

- Structure: $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- Parameters?
  - b_0: *per capita* birth rate [1/time]
  - d: *per capita* death rate [1/time]
  - N_b: Scale of population regulation [indiv]
- State variables?
  - N: Population size [indiv]

Computer simulation



What will this model do?



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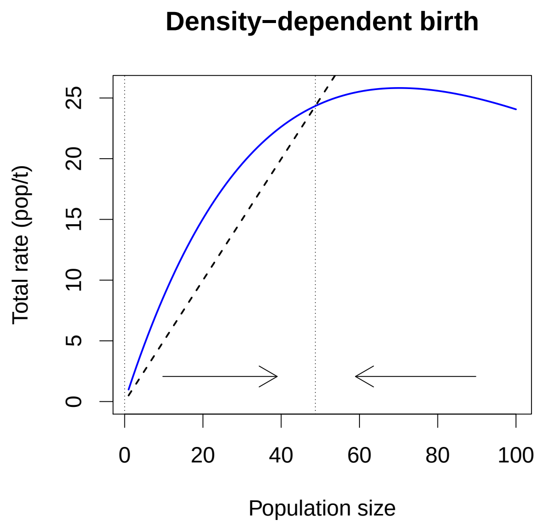
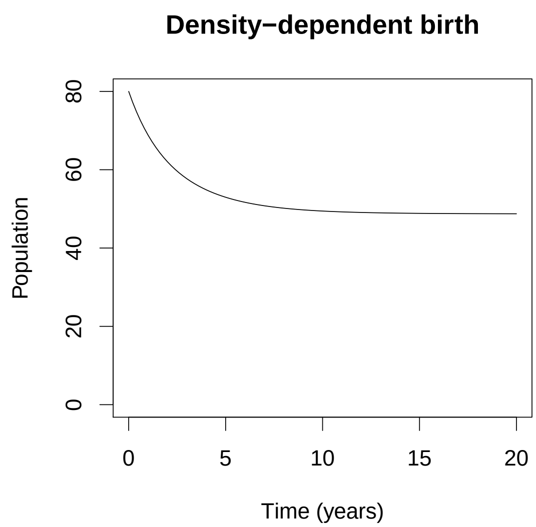


Computer simulation

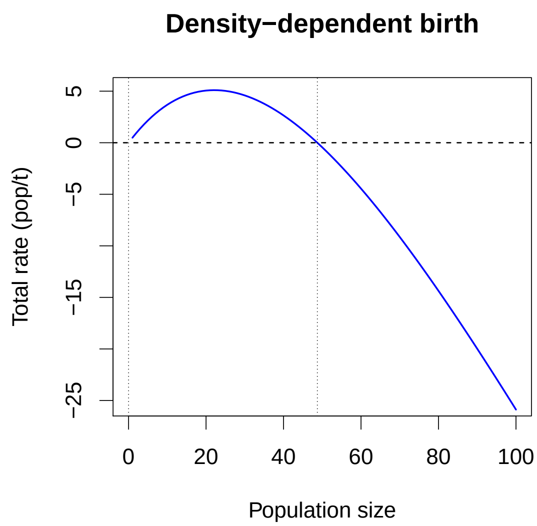


Qualitative analysis

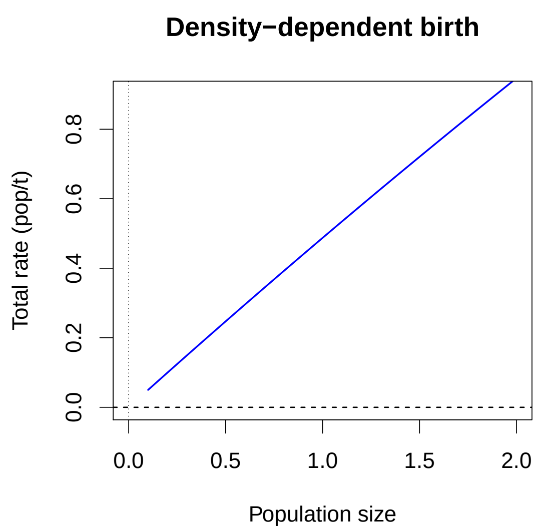
- Find *equilibria* – points where the population will not change
  - Structure: $\dfrac{dN}{dt} = f(N)$
  - Equilibria when $f(N) = 0$

- Analyze equilibrium *stability* – if we are *near* the equilibrium, we will move toward it or away from it?
    - How does $f(N)$ *change* near an equilibrium?

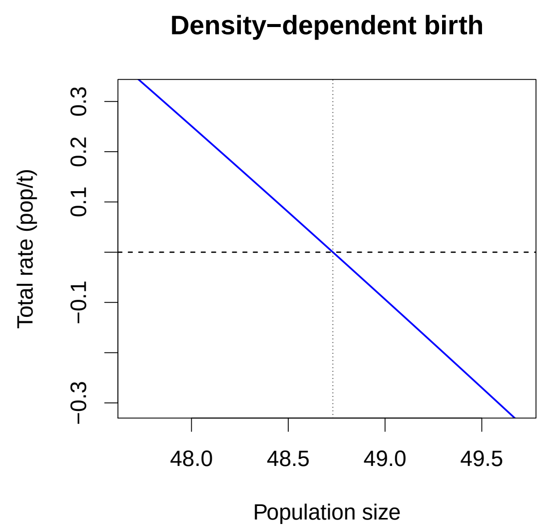
Linearization



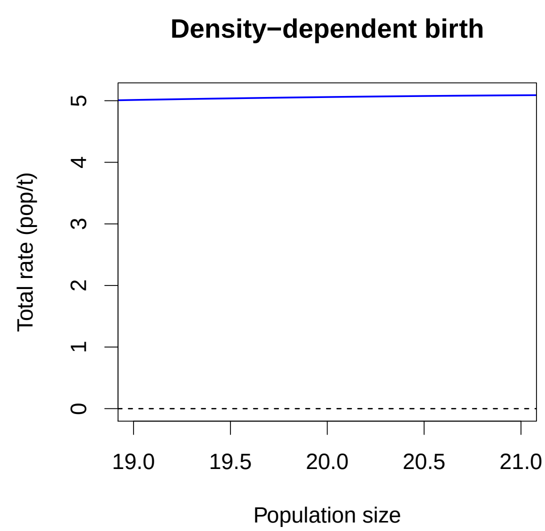
Zoom to extinction equilibrium



Zoom to carrying capacity



Zoom to other point



Linearization

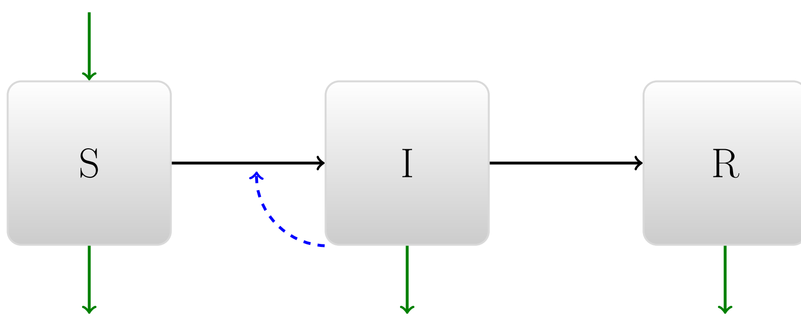
- Near an equilibrium, the system behaves like:
- $\frac{dx}{dt} = Jx$

- x is the distance from equilibrium
- $J = \frac{\partial f}{\partial x}$

- The solution is $x(t) = x(0)\exp(Jt)$
  - Moves away exponentially if $J > 0$
  - Moves in exponentially if $J < 0$

3 Disease model

What about our simple disease model?



$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \beta\frac{SI}{N} - \mu S \\
\frac{dI}{dt} &= \beta\frac{SI}{N} - \gamma I - \mu R \\
\frac{dR}{dt} &= \gamma I - \mu R
\end{aligned}$$

Disease model

- Parameters?
  - μ: Death rate [1/time]

- β: Transmission rate [1/time]
- γ: Recovery rate [1/time]
- N: Population size [indiv]

- State variables?
  - S, I, R – but we are going to ignore R
    - *
    - *

Equilibria

- $I = 0$, $S = N$
  - The *disease-free equilibrium* (DFE)
- $S = \gamma/\beta$, $I =$ (something)
  - The *endemic equilibrium* (EE)

Qualitative analysis

-
$$
\begin{aligned}
\frac{dS}{dt} &= f(S, I) \\
\frac{dI}{dt} &= g(S, I)
\end{aligned}
$$

- We still have linear equations near the equilibrium
- This is the only kind of equation we can solve
- Behaviour is determined by
-
$$
J = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} \end{pmatrix}
$$

4 Matrices and eigenvalues

Rabbits

- Imagine we have a population of rabbits
  - Baby rabbits become adults after one month
  - Each pair of adult rabbits produces one pair of baby rabbits each month
  - Rabbits never die
- What happens to this population?

Matrix equations

- We describe this as equations for **A**dult and **B**aby rabbits:
  - $A' = A + B$
  - $B' = A$
- In matrix terms, we write:
  -
$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

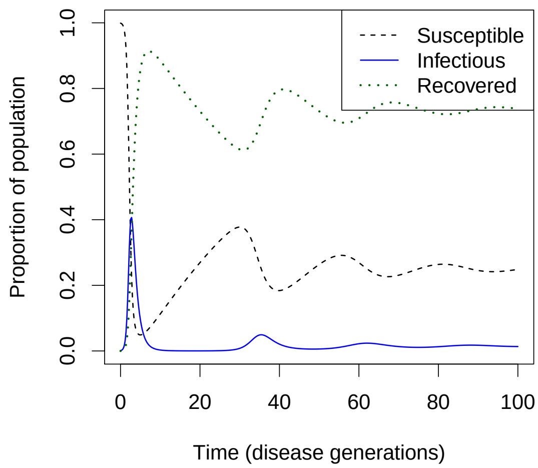
Eigenvectors and eigenvalues

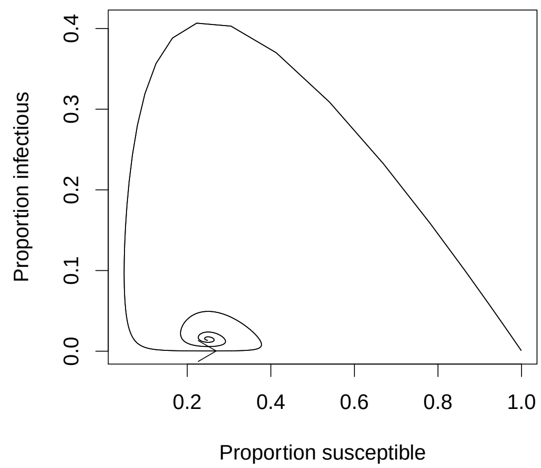
- We describe matrix dynamics using eigenvectors and eigenvalues
  - An *eigenvector* is a vector which keeps its shape when multiplied by the matrix (it is just multiplied by a regular number)
  - An *eigenvalue* is the number we multiply by

Dominant values

- Usually, matrix dynamics have a single *dominant* eigenvalue (and eigenvector)
  - This is just the one that is most important for the dynamics we are studying

Disease example





Disease-free equilibrium

- Dominant eigenvalue is (usually) $\beta - \gamma$
  - Describes how fast the epidemic grows exponentially
  - Eigenvector describes relationship between *increase in I* and *decrease in S*
- Other eigenvalue describes how fast susceptibles recover to equilibrium when there is no disease

Endemic equilibrium

- There is a pair of *complex* eigenvalues
  - $a + bi$, where $i = \sqrt{-1}$
- In complex eigenvalues:
  - real part (a) describes exponential growth (or decline)
  - imaginary part (b) describes rate of oscillation