

Mathematical foundations for dynamics

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of Infectious Diseases

National Taiwan University

Goals

- ▶ This lecture will explain
 - ▶ exponential growth (and decline)
 - ▶ simple qualitative methods for analyzing ODE-based dynamical systems
 - ▶ the importance of linear equations
 - ▶ some basic ideas about matrices and eigenvalues

Outline

Exponential change

Population growth example

Disease model

Matrices and eigenvalues

Modeling decline

- ▶ We have some bacteria in a tank
- ▶ They have no food, so they are simply dying at a *per capita* rate of 0.02/hr.
- ▶ If the current density is 100 bacteria/ml, what will be the density after 1 hr?
- ▶ What will be the density after 1 wk?

A simple model of population growth

- ▶ $\frac{dN}{dt} = (b - d)N$
- ▶ $\frac{dN}{dt} = rN$
- ▶ This is the only differential equation you need to solve!
- ▶ $N(t) = N(0)e^{rt} = N(0) \exp(rt)$
- ▶ Bacteria example

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A more realistic model of population growth

- ▶ Populations don't grow forever
 - ▶ or decline forever
- ▶ Probably the birth rate will decline if the population is too crowded
- ▶ Let's let the birth rate go down as population goes up:
- ▶ $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- ▶ *We don't want to solve this equation!*

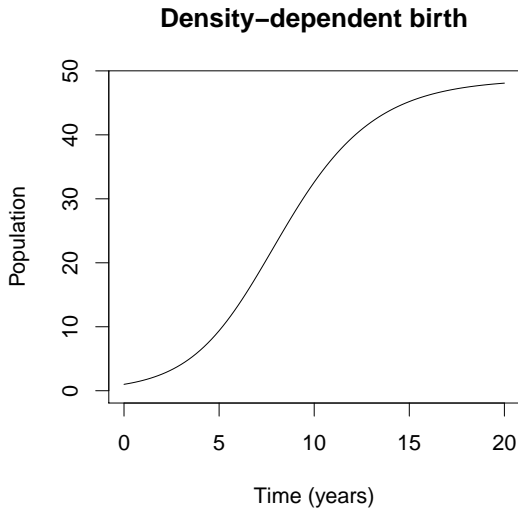
What can we do instead?

- ▶ *Computer simulations*: what will happen with particular parameters?
- ▶ *Qualitative analysis*: what can we learn in general?

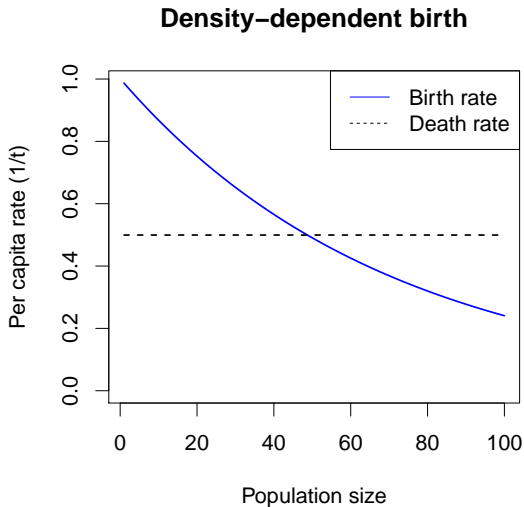
Population growth model

- ▶ Structure: $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d)N$
- ▶ Parameters?
 - ▶ b_0 : *per capita* birth rate [1/time]
 - ▶ d : *per capita* death rate [1/time]
 - ▶ N_b : Scale of population regulation [indiv]
- ▶ State variables?
 - ▶ N : Population size [indiv]

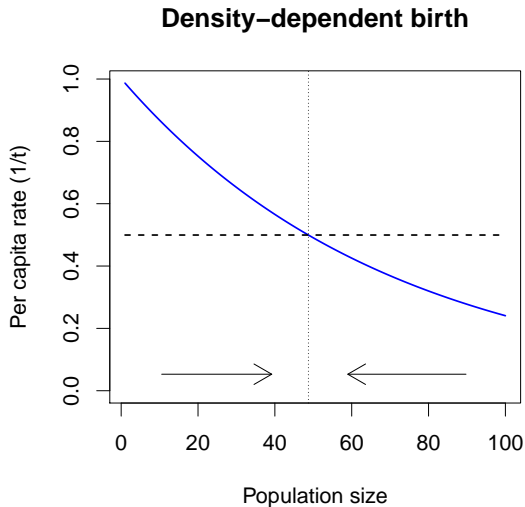
Computer simulation



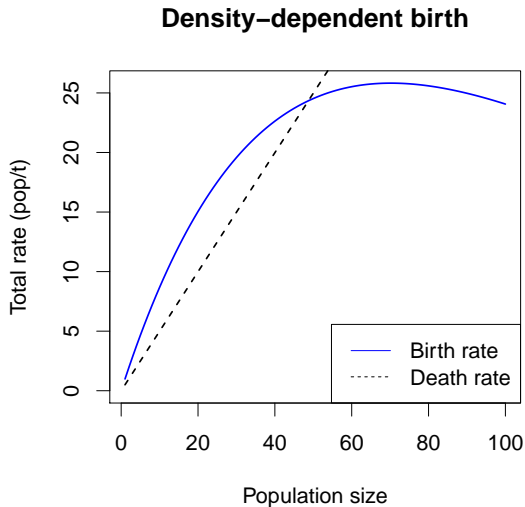
What will this model do?



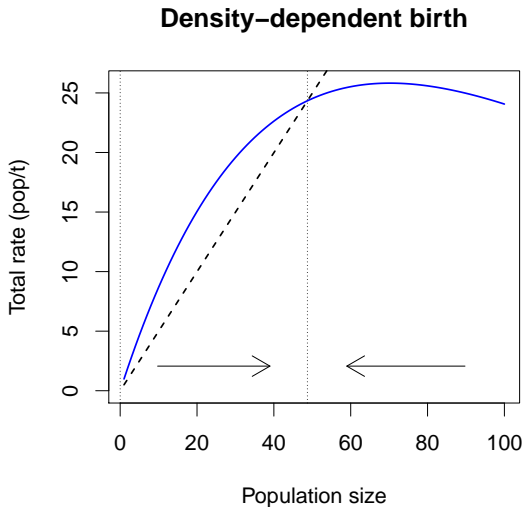
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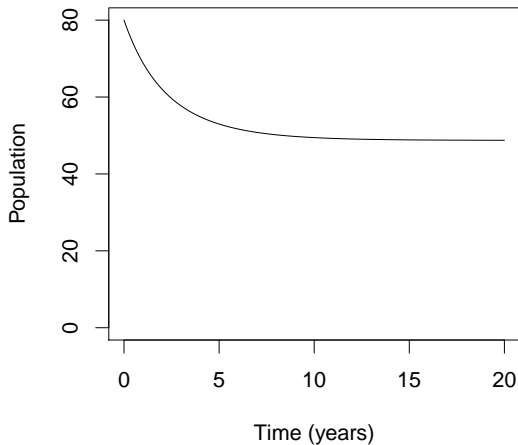


What will this model do?



Computer simulation

Density-dependent birth

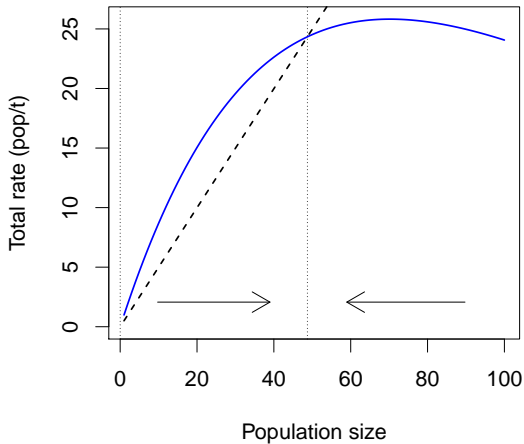


Qualitative analysis

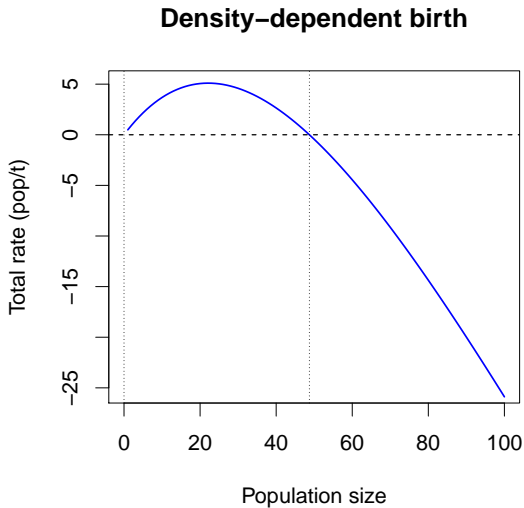
- ▶ Find *equilibria* – points where the population will not change
 - ▶ Structure: $\frac{dN}{dt} = f(N)$
 - ▶ Equilibria when $f(N) = 0$
- ▶ Analyze equilibrium *stability* – if we are *near* the equilibrium, we will move toward it or away from it?
 - ▶ How does $f(N)$ *change* near an equilibrium?

Linearization

Density-dependent birth

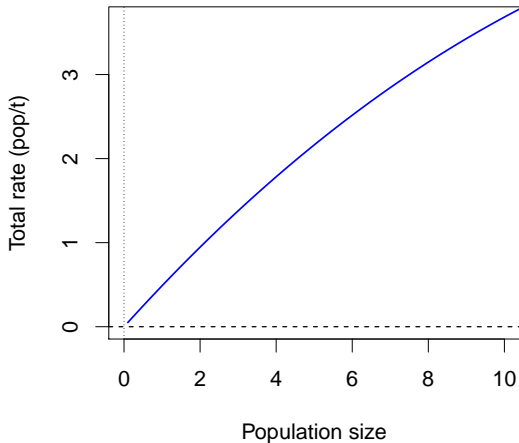


Linearization

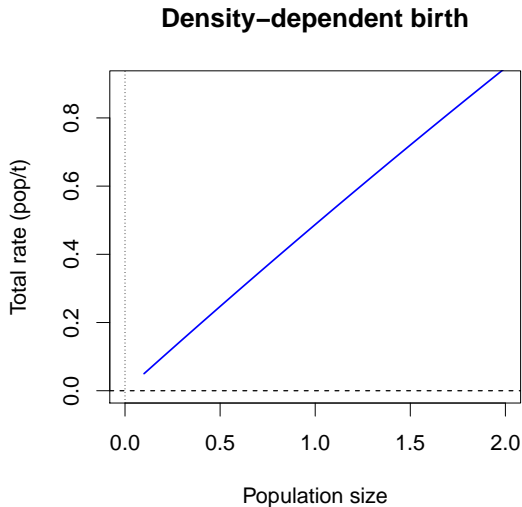


Zoom to extinction equilibrium

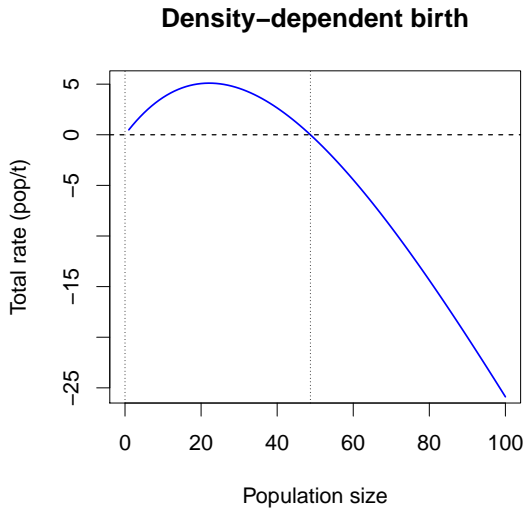
Density-dependent birth



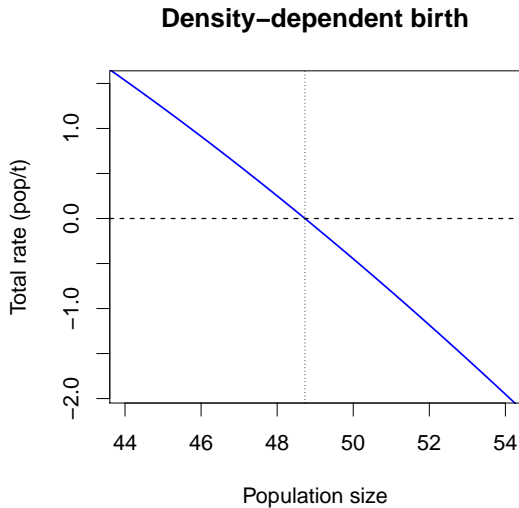
Zoom to extinction equilibrium



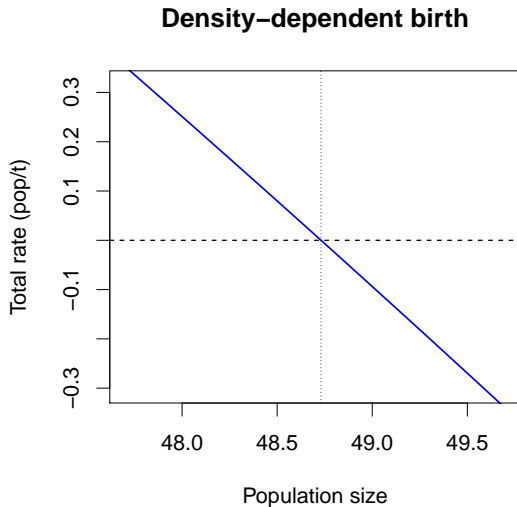
Linearization



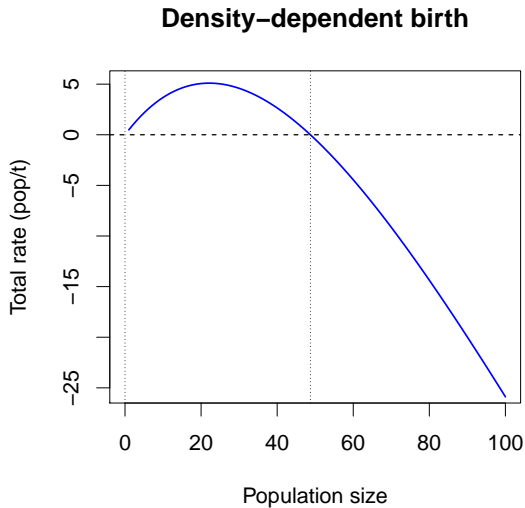
Zoom to carrying capacity



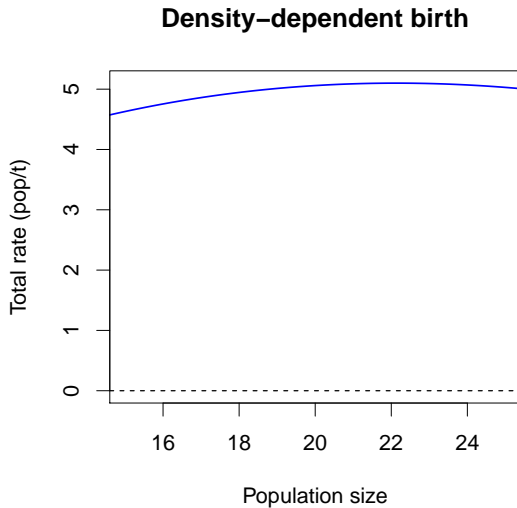
Zoom to carrying capacity



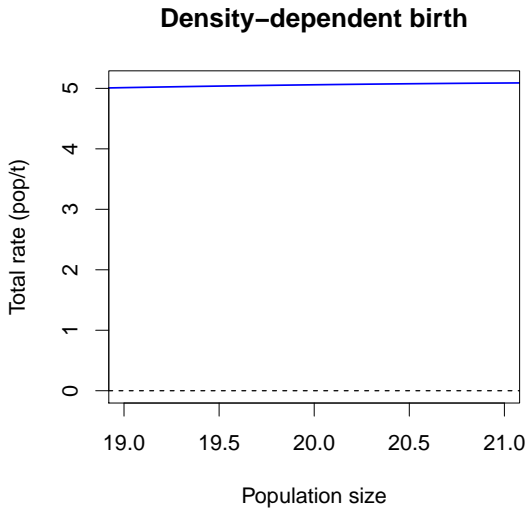
Linearization



Zoom to other point



Zoom to other point



Linearization

- ▶ Near an equilibrium, the system behaves like:

- ▶ $\frac{dx}{dt} = Jx$

- ▶ x is the distance from equilibrium

- ▶ $J = \frac{\partial f}{\partial x}$

- ▶ The solution is $x(t) = x(0) \exp(Jt)$

- ▶ Moves away exponentially if $J > 0$
 - ▶ Moves in exponentially if $J < 0$

Outline

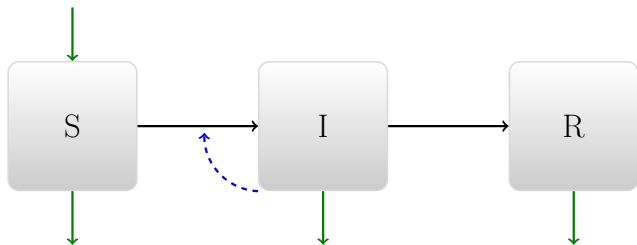
Exponential change

Population growth example

Disease model

Matrices and eigenvalues

What about our simple disease model?



$$\begin{aligned}\frac{dS}{dt} &= \mu N - \beta \frac{SI}{N} - \mu S \\ \frac{dI}{dt} &= \beta \frac{SI}{N} - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R\end{aligned}$$

Disease model

- ▶ Parameters?

- ▶ μ : Death rate [1/time]
- ▶ β : Transmission rate [1/time]
- ▶ γ : Recovery rate [1/time]
- ▶ N : Population size [indiv]

- ▶ State variables?

- ▶ S, I, R – but we are going to ignore R
 - ▶ * It does not affect S or I under our assumptions
 - ▶ * It is redundant (we know it if we know N, S and I .)

Equilibria

- ▶ $I = 0, S = N$
 - ▶ The *disease-free equilibrium* (DFE)
- ▶ $S = \gamma/\beta, I = (\text{something})$
 - ▶ The *endemic equilibrium* (EE)

Qualitative analysis



$$\begin{aligned}\frac{dS}{dt} &= f(S, I) \\ \frac{dI}{dt} &= g(S, I)\end{aligned}$$

- ▶ We still have linear equations near the equilibrium
- ▶ This is the only kind of equation we can solve
- ▶ Behaviour is determined by



$$J = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} \end{pmatrix}$$

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Rabbits

- ▶ Imagine we have a population of rabbits
 - ▶ Baby rabbits become adults after one month
 - ▶ Each pair of adult rabbits produces one pair of baby rabbits each month
 - ▶ Rabbits never die
- ▶ What happens to this population?

Matrix equations

- ▶ We describe this as equations for **A**dult and **B**aby rabbits:
 - ▶ $A' = A + B$
 - ▶ $B' = A$
- ▶ In matrix terms, we write:

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

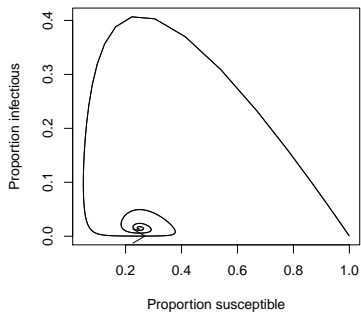
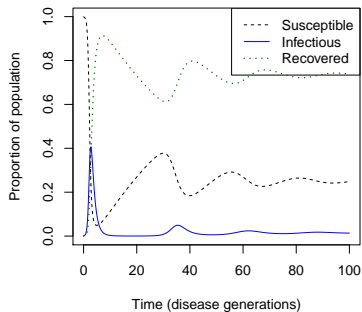
Eigenvectors and eigenvalues

- ▶ We describe matrix dynamics using eigenvectors and eigenvalues
 - ▶ An *eigenvector* is a vector which keeps its shape when multiplied by the matrix (it is just multiplied by a regular number)
 - ▶ An *eigenvalue* is the number we multiply by

Dominant values

- ▶ Usually, matrix dynamics have a single *dominant* eigenvalue (and eigenvector)
 - ▶ This is just the one that is most important for the dynamics we are studying

Disease example



Disease-free equilibrium

- ▶ Dominant eigenvalue is (usually) $\beta - \gamma$
 - ▶ Describes how fast the epidemic grows exponentially
 - ▶ Eigenvector describes relationship between *increase in I* and *decrease in S*
- ▶ Other eigenvalue describes how fast susceptibles recover to equilibrium when there is no disease

Endemic equilibrium

- ▶ There is a pair of *complex* eigenvalues
 - ▶ $a + bi$, where $i = \sqrt{-1}$
- ▶ In complex eigenvalues:
 - ▶ real part (a) describes exponential growth (or decline)
 - ▶ imaginary part (b) describes rate of oscillation