Introduction to dynamical modeling

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2016 Summer Course on Mathematical Modeling and Analysis of Infectious Diseases

National Taiwan University

Goals

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  - introduce the idea of dynamical modeling

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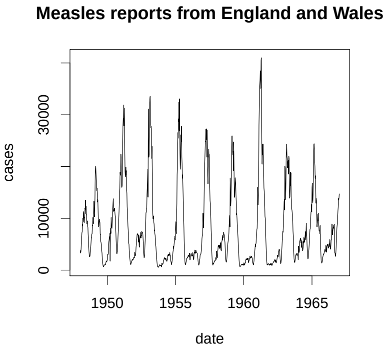
Dynamic modeling connects scales



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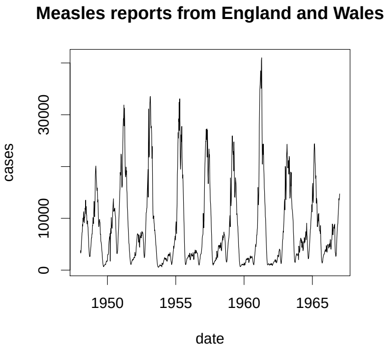

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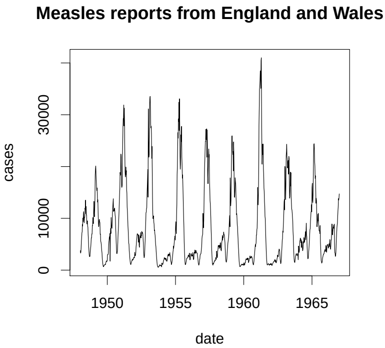

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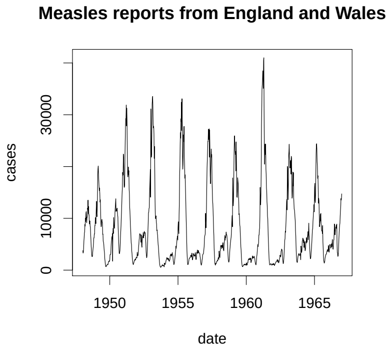

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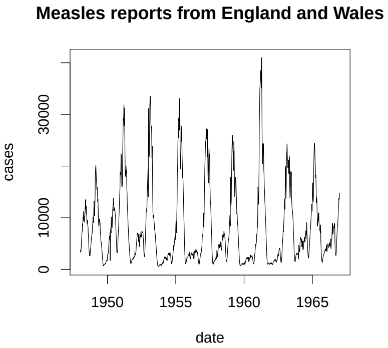

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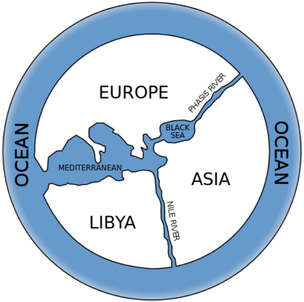

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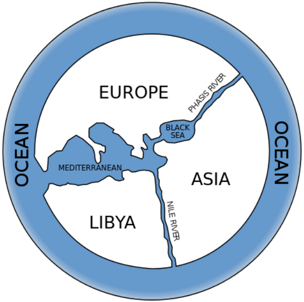

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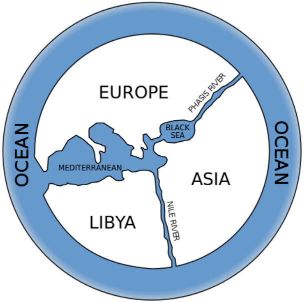

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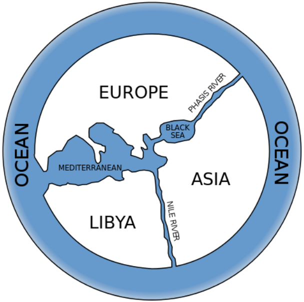

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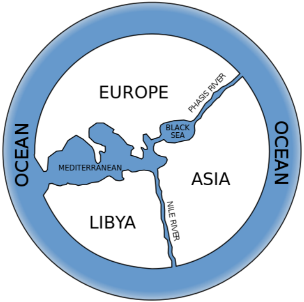

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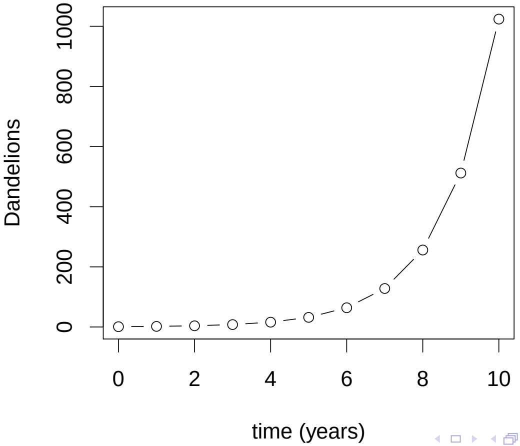

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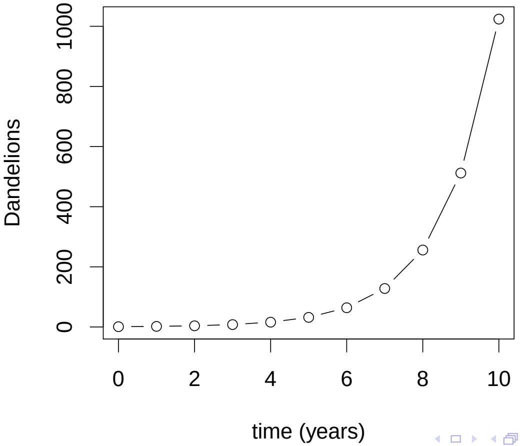

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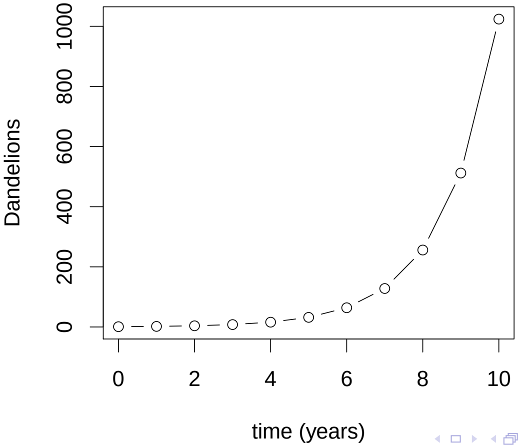

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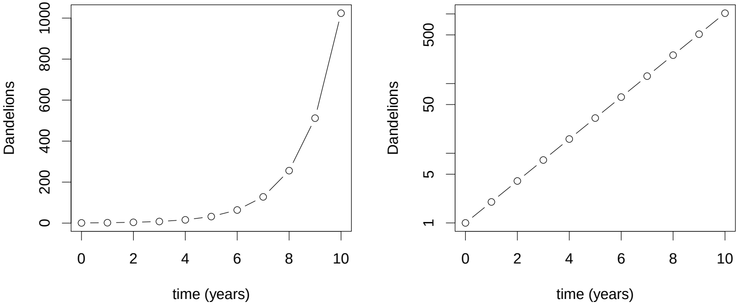

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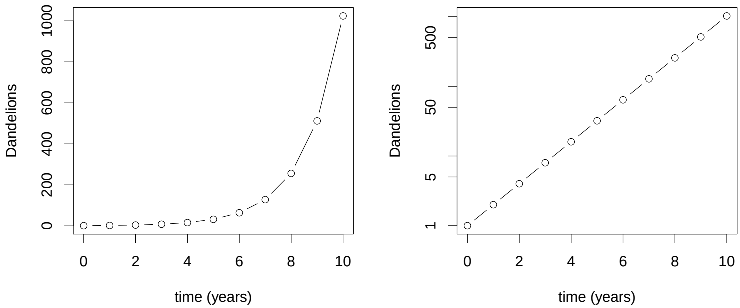

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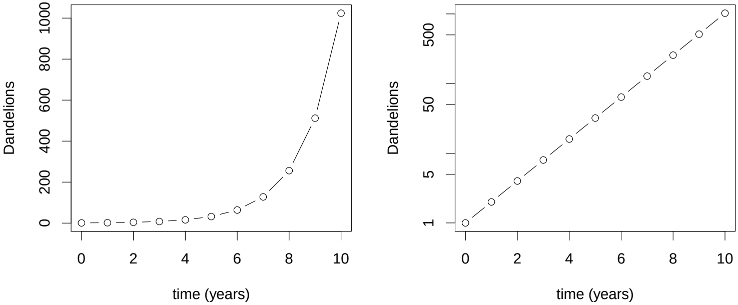

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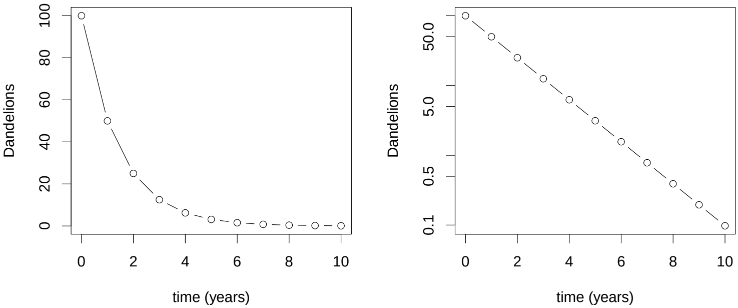

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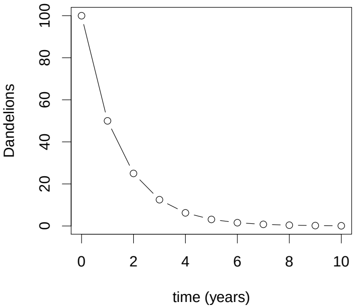

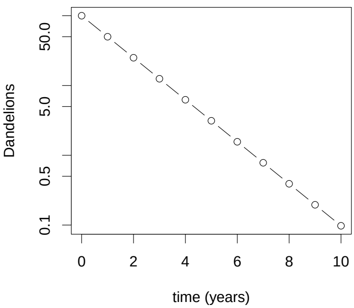

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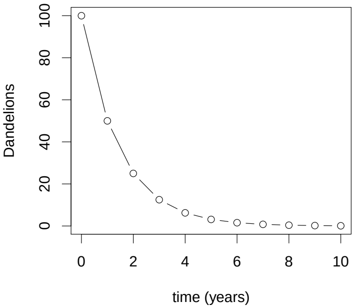

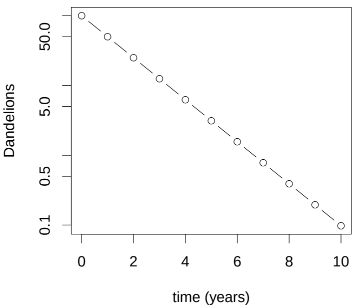

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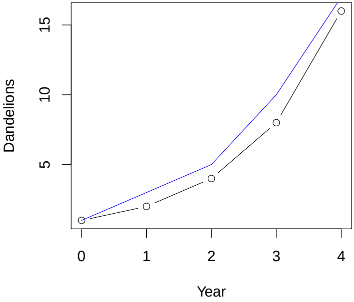

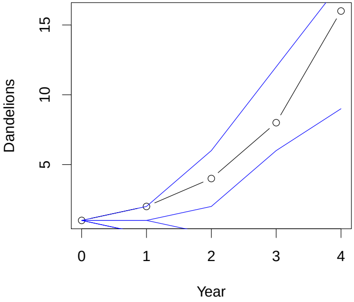

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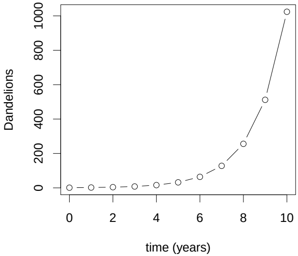

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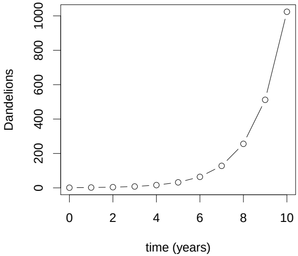

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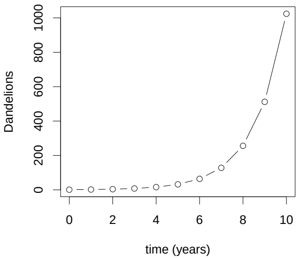

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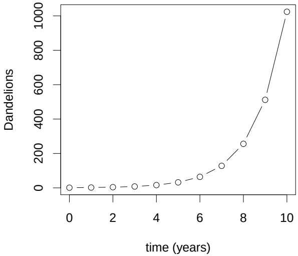

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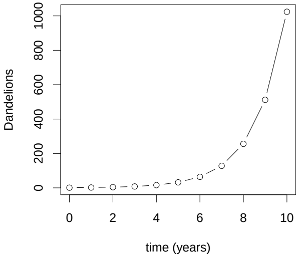

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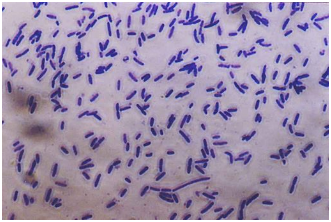

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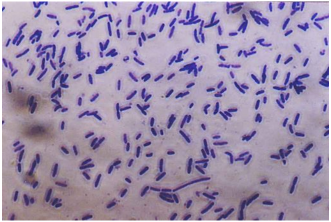

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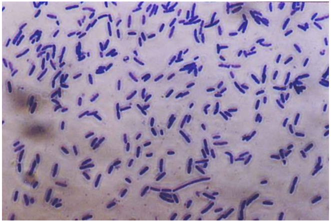

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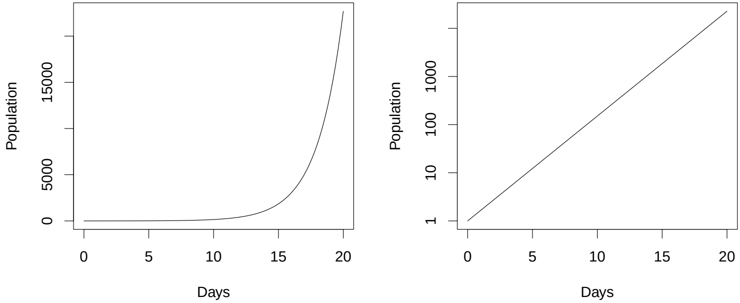

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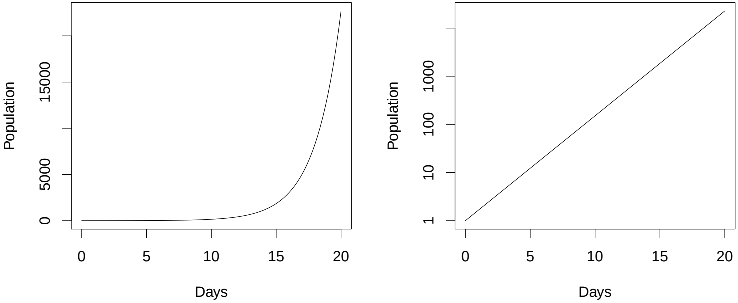

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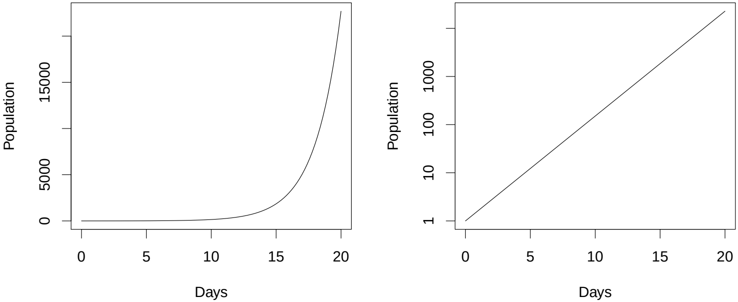

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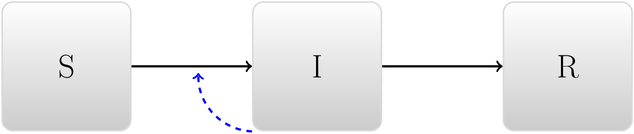

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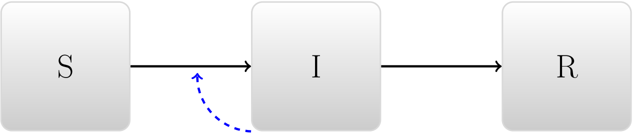

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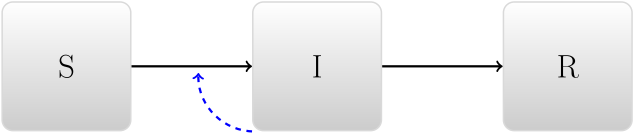

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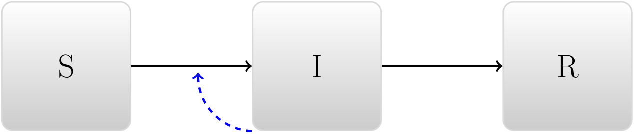

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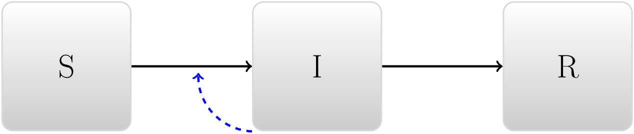

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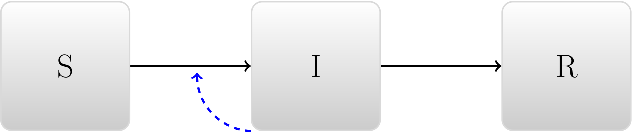

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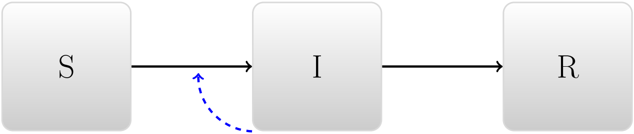

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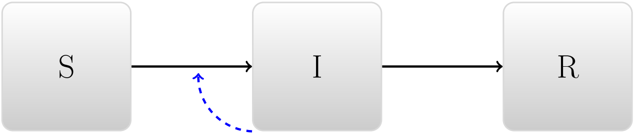

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Conceptual modeling

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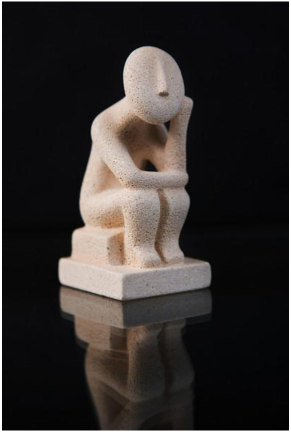

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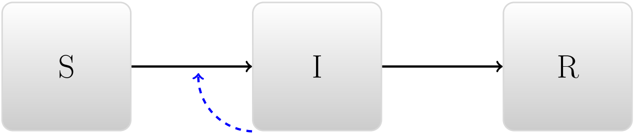

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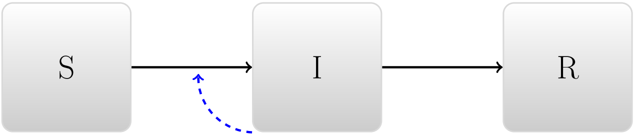

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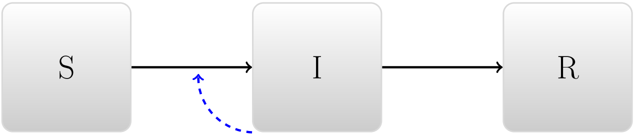
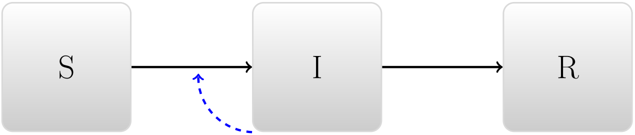
- When does disease increase, decrease?

Conceptual modeling



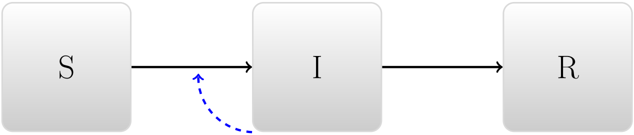
- What is the final result?
- When does disease increase, decrease?

Dynamic implementation



- Requires assumptions about time distributions

Dynamic implementation



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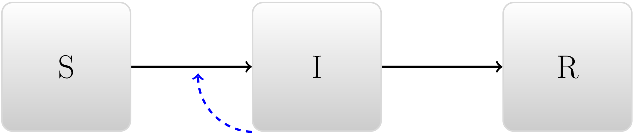

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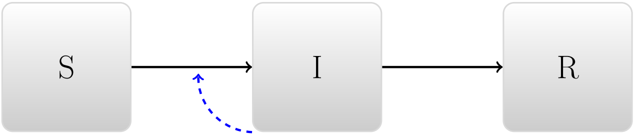

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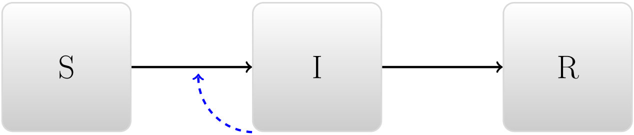

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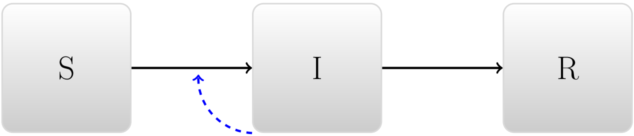

- Infectious people recover at *per capita* rate γ

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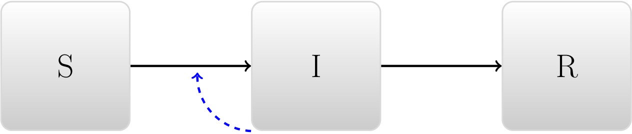

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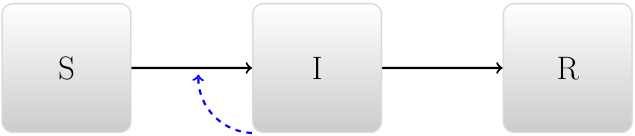

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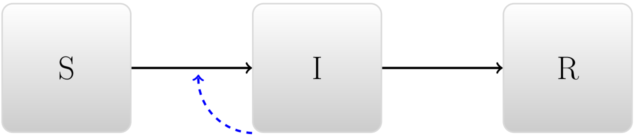

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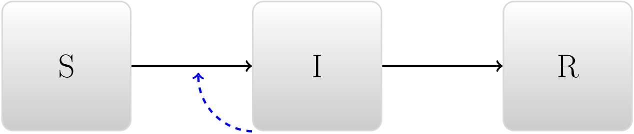
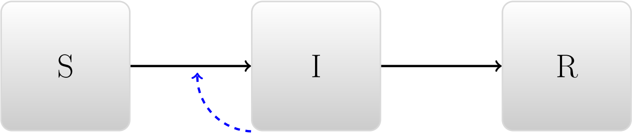


- Susceptible people get infected by:

Transmission



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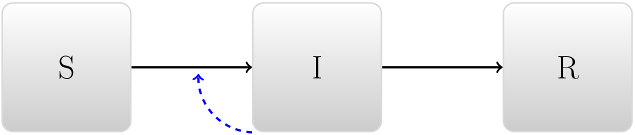

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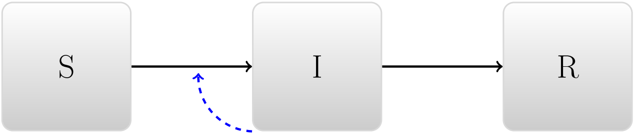

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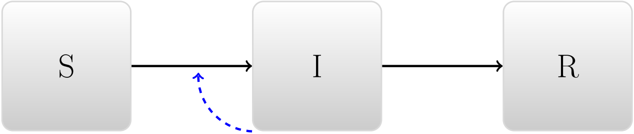

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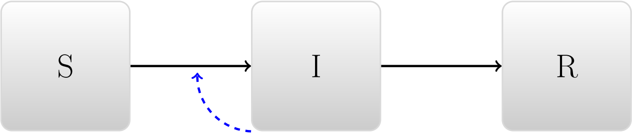

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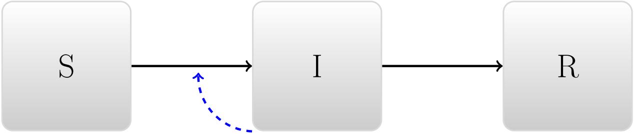

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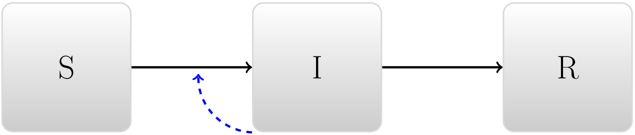

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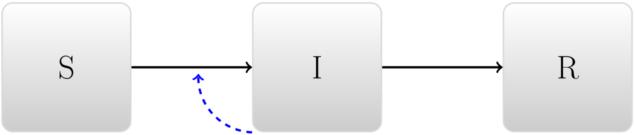

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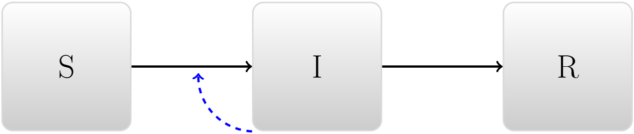

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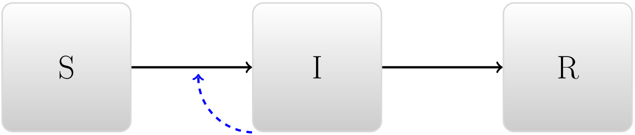

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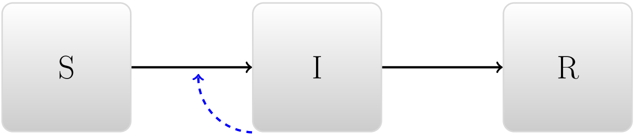

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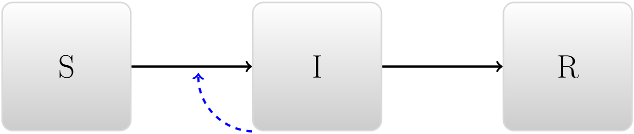

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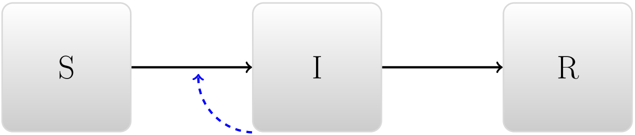

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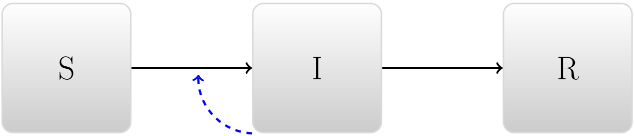
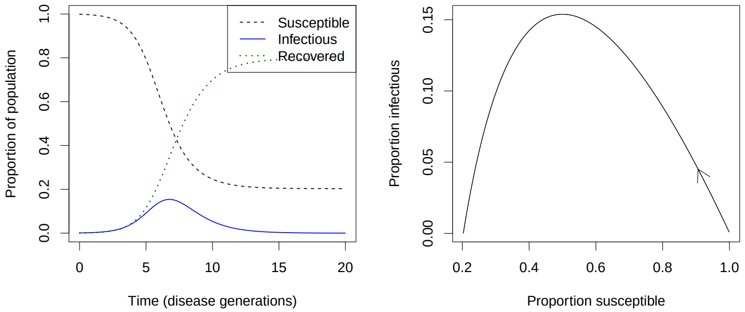
ODE implementation



$$\begin{aligned}\frac{dS}{dt} &= -\beta\frac{SI}{N}\\ \frac{dI}{dt} &= \beta\frac{SI}{N} - \gamma I\\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

ODE implementation



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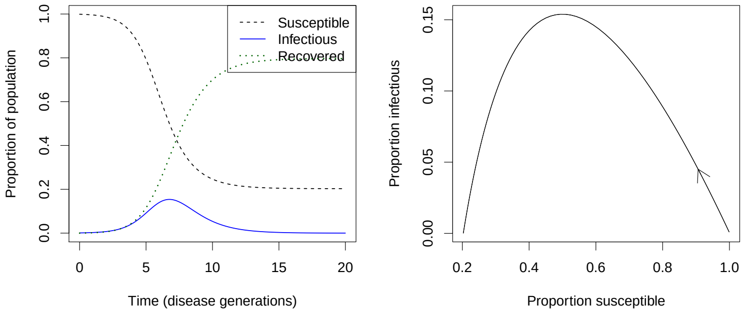
ODE implementation



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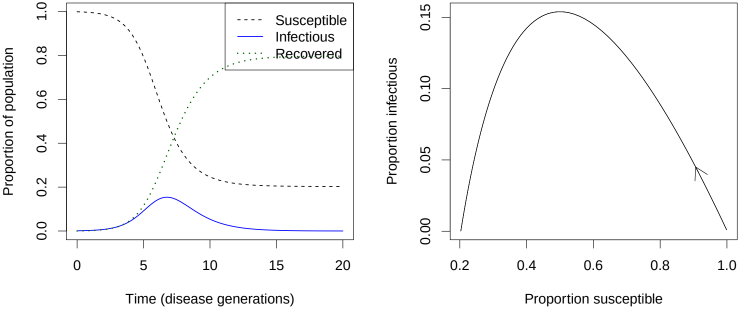

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ODE assumptions

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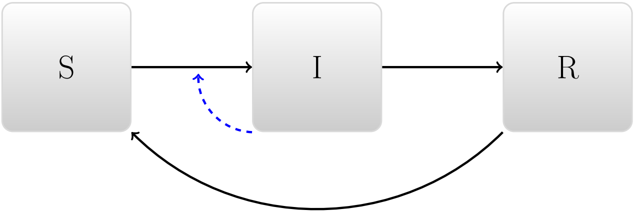
Closing the circle



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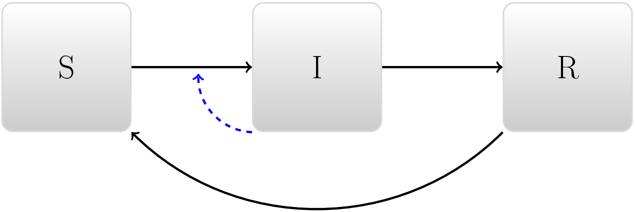

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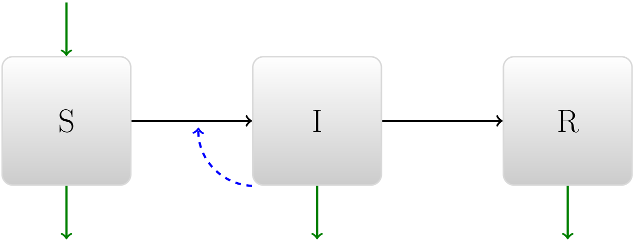

- Births and deaths

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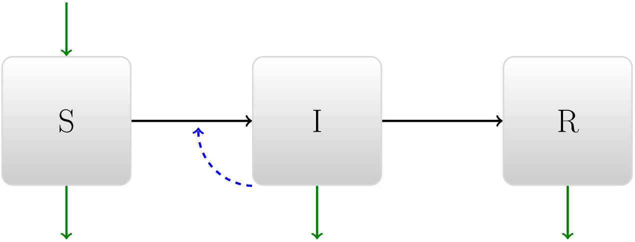

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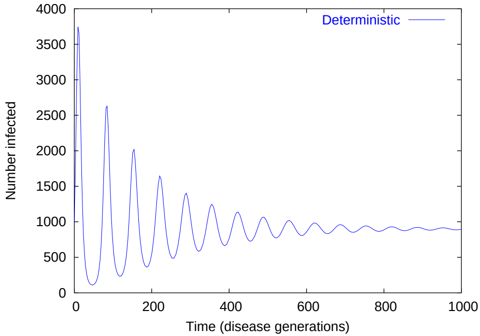
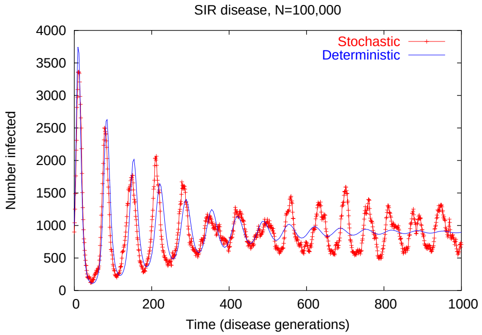
Tendency to oscillate



Modeling individuals as individuals usually requires a *stochastic* model

With individuality



Even in the simplest form, this can cause large random oscillations even in large populations

Types

- **Discrete** vs. **Continuous** time steps

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