Game values and (sur)real numbers

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1 Introduction

GOALS

- Describe:
  - Combinatoric games
  - Surreal numbers
  - Where the real numbers fit in
- Stay on this side of sanity

Game theory

- Classic game theory is the theory of games with *imperfect information*
- Nash equilibria and so on

Combinatorial game theory

- The analysis of games with perfect information
- ... accidentally led to some of the most beautiful theories of analysis

Hackenbush

- On your turn, you remove one line
  - Lines no longer connected to ground are removed
- bLue lines can be removed by Left
- Red lines can be removed by Right
- greeN lines can be removed by aNyone

Domineering

- On your turn, you place a domino
- Left places verticaL dominoes
- Right places hoRizontal dominoes

Resources

- *On Numbers and Games*, Conway
- *Surreal Numbers*, Knuth
- *Winning Ways*, Berlekamp, Conway, Guy

Review

- We define the real numbers by:
  - Building the integers as nested sets
  - Building the rationals as equivalence classes of ordered pairs of integers
  - Building the reals as cuts of the rationals
- With deterministic games, we build all this at once
  - ... and much more!

2 Combinatoric games

Axiom 1

- A game is:
  - a set of options for the Left player, and a set of options for the Right player
    - $X = (X^L \mid X^R)$
  - Options are *previously defined* games
- So, what are some games?

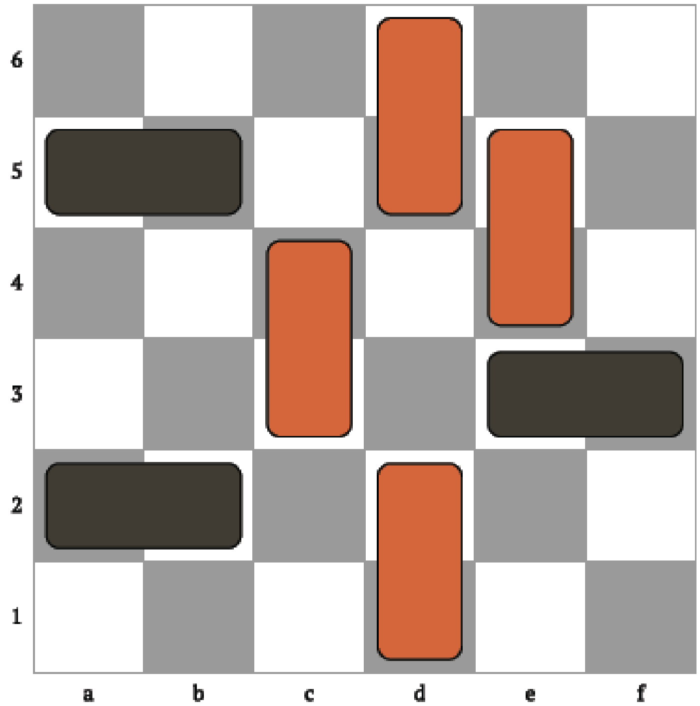
Some games

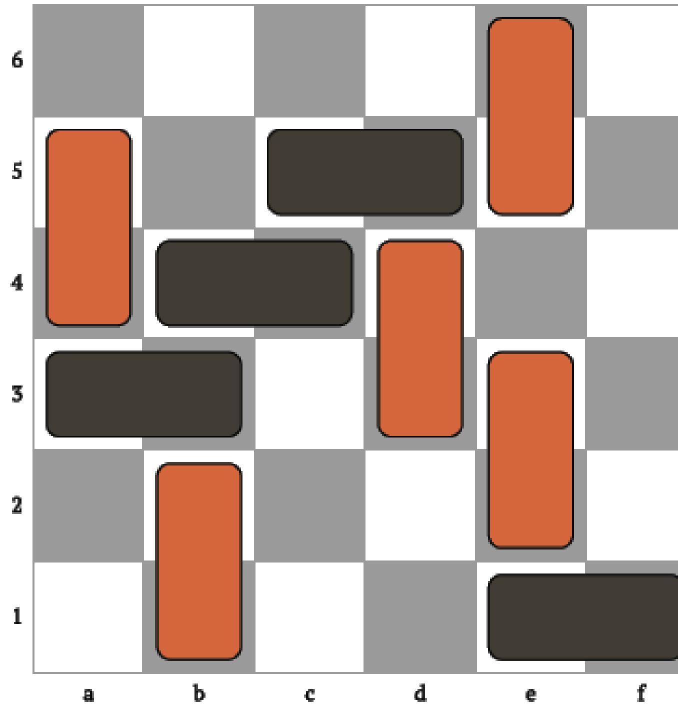
- A set of options for the Left player, and a set of options for the Right player
- $(\emptyset \mid \emptyset) = (|)$
  - 0
- $(0 \mid)$
  - 1
- $(\mid 0)$
  - -1
- $(0 \mid 0)$
  - $*$

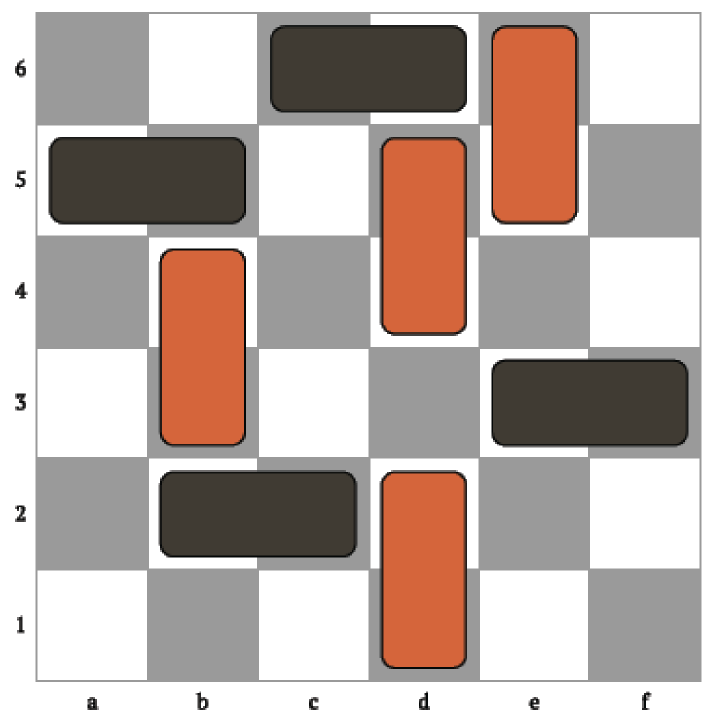
How to play a game?

- If it's your turn, you choose an option
- It's then the other player's turn in that game
- If you have no options than you lose

3 Adding games







Axiom 2

- $A + B = (A + b^L, a^L + B \mid A + b^R, a^R + B)$
- In other words, left can pick an left option from B and add it to A (and so on)
- This is perfectly well defined, and beautifully inductive
  - All games are defined in terms of previously defined games

Definition

- The **negative** of a game reverses the roles of Left and Right
- This has a nice, recursive definition
  - $A = (A^L \mid A^R)$
  - $-A \equiv (-A^R \mid -A^L)$
- Again, relying on beautiful induction

4 Ordering games

- We say that games are better if they are better for Left
- ...in the context of adding games together

Axiom 3

- Adding game A to an existing game can't hurt Left *unless*
  - Right has a good move
- $A \geq 0$ *unless*
  - Some option $a^R \leq 0$ **Def:** $-a^R \geq 0$
- In other words, this game can't hurt me unless
  - you can move to a game that can't hurt you

Partial ordering

- **Def:** $A \geq B \iff A - B \geq 0$
  - **Def:** $A + -B \geq 0$
- $A \geq B \ B \geq A \implies A = B$
- $A \geq B \ B \ngeq A \implies A > B$
- $A \ngeq B \ B \geq A \implies A < B$
- $A \ngeq B \ B \ngeq A \implies A || B$

0=0

- Any game that is “equal to” 0 can’t hurt either player
- If I add a game which the second player wins, I don’t change the outcome of any game
- Similarly, if $A = B$, then adding A has the same effect on any game as adding B
- This is not necessarily true if $A||B$

5 Values and numbers

Equivalence classes

- We **define** a game value as an *equivalence class* of games
- The rational numbers were defined for you in a similar way:
  - 1/2 is the equivalence class of ordered pairs (1, 2); (2, 4); ...

Numbers

- The values I’ve defined are a very cool group.
- But not very numerical:
  - $* + * = 0$
- Games have “numerical” value if you can count free moves, which works when moving is always bad.

Axiom 1N: what is a (surreal) number?

- Recall: a game is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L \mid x^R)$
  - Options are *previously defined* games
- A number-game is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L \mid x^R)$, s.t. no $x^L \geq x^R$
  - Options are *previously defined* number-games
- A number is a value associated with a class of number-games

Integers

- We create the natural numbers as $n + 1 = (n|)$
- Negative integers are then defined by the negation rule:
  - $-n - 1 = (| - n)$

Binary fractions

- We can create any finite binary expansion
  - $(2k + 1)/2^{n+1} = (k/2^n \mid (k + 1)/2^n)$
  - e.g., $7/16 = (3/8 \mid 1/2)$

The limit

- What happens if we take the limit of all numbers we can make in a finite number of steps?
- We can get all the reals ...
  - e.g., $1/3 = (0, 1/4, 5/16, \ldots \mid 1, 1/2, 3/8, \ldots)$
- plus some very weird stuff
  - $\omega = (0, 1, 2, \ldots \mid)$
  - $1/\omega = (0|1, 1/2, 1/4, \ldots)$

0.999...

- Is 0.999... really equal to 1?
- Depends on your definitions
- What is 0.1111... (base 2) as a game?

Ordinals

- You can take as many limits as you want, and get all of the infinite ordinals, and a wide range of infinitesimals

Axiom 1R: what is a (real) number?

- Recall: a number is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L \mid x^R)$, s.t.:
    - no $x^L \geq x^R$
  - Options are *previously defined* numbers
- A real number is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L \mid x^R)$, s.t.:
    - no $x^L \geq x^R$
    - x^L has a largest element iff x^R has a smallest element
  - Options are *previously defined* real numbers

Axiom 4 (not shown)

- You can define multiplication
  - Motivation: $(x - x^S)(y - y^S)$ has a known sign
- ... and construct division
  - Insane simultaneous induction on simpler quotients, and on the main quotient
- The surreal numbers are a *field*

Surreal arithmetic

- $\omega - 1$,
- $\omega/2$, $\sqrt{(}\omega)$
- Even crazier stuff: $\sqrt[3]{\omega - 1} - \pi/\omega$

6 Beyond numbers

Micro-infinitesimals

- If we allow values that aren't numbers, we have infinitesimals that are smaller than the smallest infinitesimal numbers

Temperature

- Cold games are games where moving makes the position worse for your side
  - Number games are games that are (recursively) cold
  - Red-blue hackenbush
- Neutral games are games where the positions are the same for left and right
  - The theory of Nim values
  - Green hackenbush
- Hot games are games where there can be a positive value to moving
  - Example: domineering

Conclusion

- We can define a bewildering array of games with a simple, recursive definition
- By defining addition, we can organize these into values, which form a group under sensible game addition
- By recursively requiring making a move to have a cost, we identify a subset that we call the surreal numbers
  - these contain the reals, the infinite ordinals and a consistent set of infinitesimals
  - These surreal numbers form a field
- There are also interesting game values that are *not* numbers
- Game values are the best thing

Beyond the conclusions

- The option framework is sort of a generalization of
  - the Cantor framework for the ordinals
    - (building up, never a right option)
  - the Dedekind framework for the reals
    - (filling in, always a right option)

Simplicity theorem (numbers)

- The value of $(x^L \mid x^R)$ is the simplest, non-prohibited value
- Prohibited: if if is larger than some x^R or less than some x^L
- Simplest: earliest created; it has no options that are not prohibited
  - ...or else those would be simpler, non-prohibited values

More simplicity

- If no non-prohibited value already exists, then the value is
  - $(x^L + x^R)/2$, if both exist
  - $x^L + 1$, if only x^L exists
  - ...

Finitude

- Any game takes a *finite* number of moves to play
  - Induction: if I have a new game, and play it, it will take one more move than the option I chose
- This number is not necessarily *bounded*.
- In particular, a number-game that does not correspond to a finite binary expansion has an unbounded number of possible moves
  - (depending on what games it is added to)