

Game values and (sur)real numbers
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GOALS

- Describe:
 - Combinatoric games
 - Surreal numbers
 - Where the real numbers fit in
- Stay on this side of sanity

Game theory

- Classic game theory is the theory of games with *imperfect information*
- Why would that be?

Determinism

- Games with perfect information are *boring*
 - Mathematically, not practically
- Analyze the game tree; figure out who wins

Combinatorial game theory

- Except that deterministic games are not boring at all
- Conway decided to think about what it might mean to add two deterministic games together
- The result was the best thing

Resources

- *On Numbers and Games*, Conway
- *Surreal Numbers*, Knuth
- *Winning Ways*, Berlekamp, Conway, Guy

Review

- We define the real numbers by:
 - Building the integers as nested sets
 - Building the rationals as equivalence classes of ordered pairs of integers
 - Building the reals as cuts of the rationals
- A lot of work, also, we're left with three definitions of the number 3 (and 2 of the number $3/2$)

Axiom 1: what is a game?

- A game is: a set of options for the Left player, and a set of options for the Right player
 - $x = (x^L \mid x^R)$
 - Options are *previously defined* games
- A game *state* is a game together with a specification of whose turn it is
- Motivation: Clearly define a wide range of deterministic games
 - in a way that's going to make it easy to add and subtract them
- Bonus: Highly inductive

Um, what?

- I have just defined a bewilderingly wonderful agglomeration of objects
 - We will need to “chop” it three times to get to the real numbers
- But is it clear that I've defined any objects at all?

What are some games?

- A set of options for the Left player, and a set of options for the Right player
- $(\emptyset \mid \emptyset) = (|)$
 - 0
- $(0 \mid)$
 - 1
- $(\mid 0)$
 - -1
- $(0 \mid 0)$
 - *

How to play a game?

- If it's your turn, you choose an option
- It's then the other player's turn in that game
- If you have no options than you lose

Hackenbush

- Uses a drawing with blue, red and green lines, and a “ground”
- On your turn, you remove a line
 - Lines no longer connected to ground are removed
- blue lines can be removed by Left
- Red lines can be removed by Right
- green lines can be removed by anyone

What outcomes can a game have?

- $\mathcal{O}(0) = S$ – second player wins
- $\mathcal{O}(*) = F$ – first player wins
- $\mathcal{O}(1) = L$ – Left player wins
- $\mathcal{O}(-1) = R$ – Right player wins

Axiom 2: Adding games

- To play in the game $A + B$, you move *either* in A or in B
 - $A + B = (A + B^L, A^L + B \mid A + B^R, A^R + B)$
- This is perfectly well defined, and beautifully inductive
 - All games are defined in terms of previously defined games
- Motivation: related to thinking about certain kinds of specific games
 - Also, turns out to be super-cool

Examples

- What happens if we add games with various outcomes?
 - $S + S = S$
 - $F + F = ?$
 - $L + L = L$
 - $L + R = ?$
 - $L + F = ?$

Some games are better

- We say $A \leq B$ if B is at least as good for the Left player as A
- Motivation:
 - classify games by their potential additive effects
 - put a (partial) ordering on the games

Definition

- The **negative** of a game reverses the roles of Left and Right
- This has a nice, recursive definition
 - $A = (A^L \mid A^R)$
 - $-A \equiv (-A^R \mid -A^L)$
- We then evaluate $A : B$ by looking at the outcome of $A - B \equiv A + (-B)$

At least as good

- A is at least as good as B (for Left) if $A - B$ has no good moves (for Right)
 - This means $\mathcal{O}(A - B) =$
 - * L, or S

Mirror world

- It is sometimes useful to construct $A - B$ by imagining a mirror, and putting B on the opposite side of the mirror (Left and Right are reversed there)

Axiom 3: Partial ordering

- We say position $A - B$ is good for Left, *unless*
 - Right has a good move
- We say $A \geq B$ *unless*
 - Some $A^R \leq B$, or
 - Some $B^L \geq A$

Partial ordering

- $\mathcal{O}(A - B)$?
 - L $\implies A > B$
 - R $\implies A < B$
 - S $\implies A = B$
 - F $\implies A \sim B$

Theorem

- If $A = B$, then:
 - $\forall X, \mathcal{O}(X + A) = \mathcal{O}(X + B)$
 - $\mathcal{O}(X + A)$
 - $* = \mathcal{O}((X + A) + (B - A))$
 - $* = \mathcal{O}((X + B) + (A - A))$
 - $* = \mathcal{O}(X + B)$

Values

- We can thus define a game value as an *equivalence class* of games
 - A set of games that are linked by an equivalence relation
 - The rational numbers were defined last week in a similar way:
 - * $1/2$ is the equivalence class of ordered pairs $(1, 2); (2, 4); \dots$

Numbers

- The values I've defined are a very cool group.
- But not very numerical:
 - $* + * = 0$
- Games have “numerical” value if you can count free moves, which works when moving is always bad.

Axiom 1N: what is a (surreal) number?

- Recall: a game is: a set of options for the Left player, and a set of options for the Right player
 - $x = (x^L \mid x^R)$
 - Options are *previously defined* games
- A number is: a set of options for the Left player, and a set of options for the Right player
 - $x = (x^L \mid x^R)$, s.t. no $x^L \geq x^R$
 - Options are *previously defined* numbers

Examples

- $1 + 1 = 2$
- $(0|1)$
- $(0|2)$
- $(0|3)$

Simplicity theorem

- The value of $(x^L \mid x^R)$ is the simplest, non-prohibited value
- Prohibited: if it is larger than some x^R or less than some x^L
- Simplest: earliest created; it has no options that are not prohibited
 - ...or else those would be simpler, non-prohibited values

Integers

- We create the integers as $n + 1 = (n \mid)$

Binary fractions

- We create the (fractional) dyadic rationals as
 - $(2k + 1)/2^{n+1} = (k/2^n \mid (k + 1)/2^n)$
 - e.g., $7/16 = (3/8 \mid 1/2)$
- This is also how we *define* the dyadic rationals: integers divided by powers of two.

The limit

- What happens if we take the limit of all numbers we can make in a finite number of steps?
- We can get all the reals ...
 - e.g., $1/3 = (0, 1/4, 5/16, \dots \mid 1, 1/2, 3/8, \dots)$
- plus some very weird stuff
 - $\omega = (0, 1, 2, \dots \mid)$
 - $1/\omega = (0 \mid 1, 1/2, 1/4, \dots)$

0.999...

- Is 0.999... really equal to 1?
- Depends on your definitions
- What is 0.1111... (base 2) as a game?

Ordinals

- You can take as many limits as you want, and get all of the infinite ordinals, and a wide range of infinitesimals

Finitude

- Any game takes a *finite* number of moves to play
 - Induction: if I have a new game, and play it, it will take one more move than the option I chose
- This number is not necessarily *bounded*. Given a game that does not correspond to a dyadic number, it is possible to take more than N moves in it, $\forall N$.

Axiom 1R: what is a (real) number?

- Recall: a number is: a set of options for the Left player, and a set of options for the Right player
 - $x = (x^L \mid x^R)$, s.t.:
 - * no $x^L \geq x^R$
 - Options are *previously defined* numbers
- A real number is: a set of options for the Left player, and a set of options for the Right player
 - $x = (x^L \mid x^R)$, s.t.:
 - * no $x^L \geq x^R$
 - * x^L has a largest element iff x^R has a smallest element
 - Options are *previously defined* real numbers

Axiom 4

- You can define multiplication
 - Motivation: $(x - x^S)(y - y^S)$ has a known sign

Theorem

- You can construct division and show that the surreal numbers are a field
 - Insane induction that only a genius could come up with, seriously
 - Induction simultaneously on simpler quotients, and on the quotient itself

Surreal arithmetic

- $\omega - 1$,
- $\omega/2$, $\sqrt{\omega}$
- Even crazier stuff: $\sqrt[3]{\omega - 1} - \pi/\omega$

Micro-infinitesimals

- If we allow values that aren't numbers, we have infinitesimals that are smaller than the smallest infinitesimal numbers

Nimbers

- We can define neutral games by identifying options for Left and Right
- This is the theory of Nim values

Hot games

- Hot games are games where there can be a positive value to moving
- Example: domineering

Conclusion

- We can define a bewildering array of games with a simple, recursive definition
- By defining addition, we can chop these into values, which form a group under sensible game addition
- By recursively requiring making a move to have a cost, we can chop these further into numbers, which contain the reals, the infinite ordinals and a consistent set of infinitesimals
 - These surreal numbers form a field
- There are also interesting game values that are *not* numbers
- Game values are the best thing