A Numerical Approach to Computing $\mathcal{R}_{\mathrm{c}}(t)$

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October 6, 2025

We need to compute

$$p(t) = \mathbb{P}\left\{\text{an individual infected at time } t \text{ infects a neighbour}\right\}. \tag{1}$$

The primary will infect the secondary if it makes an infectious contact before recovering *and* the secondary is not infected at the time of contact. Using the MSV formalism, the probability that a randomly chosen neighbour is not infected at time t is $\sigma(t) = G_q(\phi(t))$.

One way to compute $p(t)$ is by comparing times of random events: let T_{r} be the time after infection that the primary recovers, T_{c} be the time after infection that the primary makes it's first contact with the secondary, and let T_{n} be the first time that the secondary neighbour has an infectious contact from one of its other neighbours:

$$\mathbb{P}\left\{T_{\mathrm{r}} > u\right\} = e^{-\gamma u} \tag{2a}$$

$$\mathbb{P}\left\{T_{\mathrm{c}} > u\right\} = e^{-\beta u} \tag{2b}$$

$$\mathbb{P}\left\{T_{\mathrm{n}} > t + u\right\} = \sigma(t+u). \tag{2c}$$

Further,

$$p(t) = \mathbb{P}\left\{t + T_{\mathrm{c}} < (t + T_{\mathrm{r}}) \wedge T_{\mathrm{n}}\right\}, \tag{3}$$

where $\wedge$ denotes the minimum (this notation is commonly used in probability and stochastic processes), while, exploiting the independence of T_{r} and T_{n},

$$\begin{aligned}
\mathbb{P}\left\{(t + T_{\mathrm{r}}) \wedge T_{\mathrm{n}} > t + u\right\} &= \mathbb{P}\left\{T_{\mathrm{r}} > u; T_{\mathrm{n}} > t + u\right\} && (4a)\\
&= \mathbb{P}\left\{T_{\mathrm{r}} > u\right\}\mathbb{P}\left\{T_{\mathrm{n}} > t + u\right\} && (4b)\\
&= e^{\gamma u}\sigma(t+u). && (4c)
\end{aligned}$$

Thus,

$$p(t) = \int_0^\infty \beta e^{-(\beta+\gamma)u}\sigma(t+u)\,\mathrm{d}u \tag{5}$$

and

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= \int_0^\infty \beta e^{-(\beta+\gamma)u}\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t+u)\,\mathrm{d}u && (6a)\\
&= \int_0^\infty \beta e^{-(\beta+\gamma)u}\frac{\mathrm{d}}{\mathrm{d}u}\sigma(t+u)\,\mathrm{d}u && (6b)\\
&= \beta e^{-(\beta+\gamma)u}\sigma(t+u)\Big|_0^\infty + \int_0^\infty \beta(\beta+\gamma)e^{-(\beta+\gamma)u}\sigma(t+u)\,\mathrm{d}u && (6c)\\
&= (\beta+\gamma)p(t) - \beta\sigma(t). && (6d)
\end{aligned}$$

We can thus add one additional equation to the MSV equations to compute $p(t)$.

What is missing here are boundary conditions. Now, if we look at (5), $\sigma(t+u)$ is bounded, while $e^{-(\beta+\gamma)u}$ is integrable, so we can use Lebesgue's dominated convergence theorem to interchange integration and limits to see that

$$\lim_{t\to\infty} p(t) = \int_0^\infty \beta e^{-(\beta+\gamma)u} \lim_{t\to\infty} \sigma(t+u)\,\mathrm{d}u \tag{7a}$$

$$= \int_0^\infty \beta e^{-(\beta+\gamma)u} G_q(\phi(\infty))\,\mathrm{d}u \tag{7b}$$

$$= \frac{\beta}{\beta+\gamma} G_q(\phi(\infty)). \tag{7c}$$

This is the final condition that allows us to compute $p(t)$; one way to do so is via the *shooting method*: we look for the value $p(0) = p_0$ so that the solution with this initial condition eventually goes to the correct final condition.

We can also approximate the initial condition by linearizing $\ell = -\ln\phi$ about the disease-free equilibrium, taking $\ell(0) = \ell_0$ where and $0 < \ell_0 \ll 1$. We then have

$$-\dot{\ell} = -\beta + \beta G_q(e^{-\ell})e^{\ell} + \gamma(1-e^{\ell}) \tag{8a}$$

$$= -\beta + \beta \frac{\sum_{d=1}^\infty d p_d e^{-(d-1)\ell}}{\sum_{d=1}^\infty d p_d} + \gamma(1-e^{\ell}) \tag{8b}$$

$$= -\left(\beta \frac{\sum_{d=1}^\infty d(d-1) p_d}{\sum_{d=1}^\infty d p_d} + \gamma\right)\ell + \mathcal{O}(\ell^2) \tag{8c}$$

$$= -(\beta G_q'(1) + \gamma)\ell + \mathcal{O}(\ell^2) \tag{8d}$$

whence, keeping only the lowest order terms,

$$\ell(t) = \ell_0 e^{(\beta G_q'(1)+\gamma)t}. \tag{9}$$

Substituting $\phi(t) = e^{-\ell(t)}$ into (5) using (9) yields

$$p(0) = \int_0^\infty \beta e^{-(\beta+\gamma)t} G_q\big(e^{-\ell_0 e^{(\beta G_q'(1)+\gamma)t}}\big)\,\mathrm{d}t\,. \tag{10a}$$

making the change of variable $u = \ell_0 e^{(\beta G_q'(1)+\gamma)t}$, we have

$$= \frac{\beta}{\beta G_q'(1)+\gamma} \ell_0^{\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}} \int_{\ell_0}^\infty u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u. \tag{10b}$$

Now, let's make a few observations. First, since $G_q(z)$ is a generating function, it is increasing, and has an increasing derivative. Moreover, $G_q(0) = 0$, as by definition, it is the p.g.f. of the degree of a vertex with at least one neighbour. Thus, using the mean value theorem, for any $z \in (0,1)$ there exists $\eta \in [0,z]$ such that

$$\frac{G_q(z)}{z} = G_q'(\zeta), \tag{11}$$

so that $G_q'(0)z \le G_q(z) \le G_q'(1)z$. Thus,

$$G_q'(0)\int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} e^{-u}, \mathrm{d}u \\ \le \int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u \le G_q'(1)\int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} e^{-u}\,\mathrm{d}u, \quad (12)$$

and the left and right hand side limits exist as $\ell_0 \to 0$:

$$\int_{0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} e^{-u}, \mathrm{d}u = \Gamma\Big(-\tfrac{\beta+\gamma}{\beta G_q'(1)+\gamma}\Big), \quad (13)$$

which is well posed because $0 < \frac{\beta+\gamma}{\beta G_q'(1)+\gamma} < 1$ (recall that $\mathcal{R}_{0,\mathrm{c}} = \frac{\beta}{\beta+\gamma G_q'(1)} \ge 1$). Moreover, all integrands in (12) are positive, and are thus all the integrals are increasing functions of ℓ_0, so the integral

$$\int_{0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u \quad (14)$$

exists, and is bounded and non-zero.

Next, using l'Hôpital's rule, we see that

$$\lim_{\ell_0\to 0} \frac{\int_0^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u - \int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u}{\ell_0^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}}} \quad (15a)$$

$$= \lim_{\ell_0\to 0} \frac{\int_0^{\ell_0} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u}{\ell_0^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}}} \quad (15b)$$

$$= \lim_{\ell_0\to 0} \frac{\ell_0^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-\ell_0})\,\mathrm{d}u}{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}\ell_0^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1}} \quad (15c)$$

$$= -\frac{\beta G_q'(1)+\gamma}{\beta+\gamma}. \quad (15d)$$

Combining the above, we see that

$$\int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u \\ = \int_{0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u - \ell_0^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}} \left(\frac{\beta G_q'(1)+\gamma}{\beta+\gamma} + o(\ell_0)\right). \quad (16)$$

Substituting this into (10b) gives us

$$p(0) = \frac{\beta}{\beta+\gamma} + \frac{\beta}{\beta G_q'(1)+\gamma}\ell_0^{\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}} \int_{0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G_q'(1)+\gamma}-1} G_q(e^{-u})\,\mathrm{d}u + o(\ell_0). \quad (17)$$

Now, we've already observed that $\frac{\beta+\gamma}{\beta G_q'(1)+\gamma} = \frac{1}{\mathcal{R}_{0,\mathrm{c}}+\frac{\gamma}{\beta+\gamma}}$ lies strictly in $(0,1)$, so the initial condition ℓ_0 is appearing in a form that is strictly larger than $\mathcal{O}(\ell_0)$, with the effect becoming more pronounced as $\mathcal{R}_{0,\mathrm{c}}$ increases.

The missing step here is to characterize ℓ_0. Given that $S(t) = G_p(\phi(t))$ we might reasonably take $S(0) = 1 - \frac{1}{n}$ and $\phi(0) = G_p^{-1}\left(1 - \frac{1}{n}\right)$, whence, using the inverse function theorem,

$$\ell_0 = -\ln G_p^{-1}\left(1 - \tfrac{1}{n}\right) \approx -\ln\left(1 - \tfrac{(G_p^{-1})'(1)}{n}\right) = -\ln\left(1 - \tfrac{1}{G_p'(1)n}\right) \approx \frac{1}{G_p'(1)n}, \tag{18}$$

and $p(0) - \frac{\beta}{\beta+\gamma} = \mathcal{O}\left(n^{-\frac{1}{\mathcal{R}_{0,\mathrm{c}}+\frac{\gamma}{\beta+\gamma}}}\right)$.