

# A Numerical Approach to Computing $\mathcal{R}_c(t)$

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We need to compute

$$p(t) = \mathbb{P} \{ \text{an individual infected at time } t \text{ infects a neighbour} \}. \quad (1)$$

The primary will infect the secondary if it makes an infectious contact before recovering *and* the secondary is not infected at the time of contact. Using the MSV formalism, the probability that a randomly chosen neighbour is not infected at time  $t$  is  $\sigma(t) = G_q(\phi(t))$ .

One way to compute  $p(t)$  is by comparing times of random events: let  $T_r$  be the time after infection that the primary recovers,  $T_c$  be the time after infection that the primary makes its first contact with the secondary, and let  $T_n$  be the first time that the secondary neighbour has an infectious contact from one of its other neighbours:

$$\mathbb{P} \{ T_r > u \} = e^{-\gamma u} \quad (2a)$$

$$\mathbb{P} \{ T_c > u \} = e^{-\beta u} \quad (2b)$$

$$\mathbb{P} \{ T_n > t + u \} = \sigma(t + u). \quad (2c)$$

Further,

$$p(t) = \mathbb{P} \{ t + T_c < (t + T_r) \wedge T_n \}, \quad (3)$$

where  $\wedge$  denotes the minimum (this notation is commonly used in probability and stochastic processes), while, exploiting the independence of  $T_r$  and  $T_n$ ,

$$\mathbb{P} \{ (t + T_r) \wedge T_n > t + u \} = \mathbb{P} \{ T_r > u; T_n > t + u \} \quad (4a)$$

$$= \mathbb{P} \{ T_r > u \} \mathbb{P} \{ T_n > t + u \} \quad (4b)$$

$$= e^{-\gamma u} \sigma(t + u). \quad (4c)$$

Thus,

$$p(t) = \int_0^\infty \beta e^{-(\beta+\gamma)u} \sigma(t + u) du \quad (5)$$

and

$$\frac{dp}{dt} = \int_0^\infty \beta e^{-(\beta+\gamma)u} \frac{d}{dt} \sigma(t + u) du \quad (6a)$$

$$= \int_0^\infty \beta e^{-(\beta+\gamma)u} \frac{d}{du} \sigma(t + u) du \quad (6b)$$

$$= \beta e^{-(\beta+\gamma)u} \sigma(t + u) \Big|_0^\infty + \int_0^\infty \beta(\beta + \gamma) e^{-(\beta+\gamma)u} \sigma(t + u) du \quad (6c)$$

$$= (\beta + \gamma)p(t) - \beta\sigma(t). \quad (6d)$$

We can thus add one additional equation to the MSV equations to compute  $p(t)$ .

What is missing here are boundary conditions. Now, if we look at (??),  $\sigma(t+u)$  is bounded, while  $e^{-(\beta+\gamma)u}$  is integrable, so we can use Lebesgue's dominated convergence theorem to interchange integration and limits to see that

$$\lim_{t \rightarrow \infty} p(t) = \int_0^\infty \beta e^{-(\beta+\gamma)u} \lim_{t \rightarrow \infty} \sigma(t+u) du \quad (7a)$$

$$= \int_0^\infty \beta e^{-(\beta+\gamma)u} G_q(\phi(\infty)) du \quad (7b)$$

$$= \frac{\beta}{\beta + \gamma} G_q(\phi(\infty)). \quad (7c)$$

This is the final condition that allows us to compute  $p(t)$ ; one way to do so is via the *shooting method*: we look for the value  $p(0) = p_0$  so that the solution with this initial condition eventually goes to the correct final condition.

We can also approximate the initial condition by linearizing  $\ell = -\ln \phi$  about the disease-free equilibrium, taking  $\ell(0) = \ell_0$  where  $0 < \ell_0 \ll 1$ . We then have

$$-\dot{\ell} = -\beta + \beta G_q(e^{-\ell})e^\ell + \gamma(1 - e^\ell) \quad (8a)$$

$$= -\beta + \beta \frac{\sum_{d=1}^\infty dp_d e^{-(d-1)\ell}}{\sum_{d=1}^\infty dp_d} + \gamma(1 - e^\ell) \quad (8b)$$

$$= -\left( \beta \frac{\sum_{d=1}^\infty d(d-1)p_d}{\sum_{d=1}^\infty dp_d} + \gamma \right) \ell + \mathcal{O}(\ell^2) \quad (8c)$$

$$= -(\beta G'_q(1) + \gamma) \ell + \mathcal{O}(\ell^2) \quad (8d)$$

whence, keeping only the lowest order terms,

$$\ell(t) = \ell_0 e^{(\beta G'_q(1) + \gamma)t}. \quad (9)$$

Substituting  $\phi(t) = e^{-\ell(t)}$  into (??) using (??) yields

$$p(0) = \int_0^\infty \beta e^{-(\beta+\gamma)t} G_q(e^{-\ell_0 e^{(\beta G'_q(1) + \gamma)t}}) dt. \quad (10a)$$

making the change of variable  $u = \ell_0 e^{(\beta G'_q(1) + \gamma)t}$ , we have

$$= \frac{\beta}{\beta G'_q(1) + \gamma} \ell_0^{\frac{\beta+\gamma}{\beta G'_q(1) + \gamma}} \int_{\ell_0}^\infty u^{-\frac{\beta+\gamma}{\beta G'_q(1) + \gamma} - 1} G_q(e^{-u}) du. \quad (10b)$$

Now, let's make a few observations. First, since  $G_q(z)$  is a generating function, it is increasing, and has an increasing derivative. Moreover,  $G_q(0) = 0$ , as by definition, it is the p.g.f. of the degree of a vertex with at least one neighbour. Thus, using the mean value theorem, for any  $z \in (0, 1)$  there exists  $\eta \in [0, z]$  such that

$$\frac{G_q(z)}{z} = G'_q(\zeta), \quad (11)$$

so that  $G'_q(0)z \leq G_q(z) \leq G'_q(1)z$ . Thus,

$$\begin{aligned} G'_q(0) \int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} e^{-u} du \\ \leq \int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du \leq G'_q(1) \int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} e^{-u} du, \end{aligned} \quad (12)$$

and the left and right hand side limits exist as  $\ell_0 \rightarrow 0$ :

$$\int_0^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} e^{-u} du = \Gamma\left(-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}\right), \quad (13)$$

which is well posed because  $0 < \frac{\beta+\gamma}{\beta G'_q(1)+\gamma} < 1$  (recall that  $\mathcal{R}_{0,c} = \frac{\beta}{\beta+\gamma G'_q(1)} \geq 1$ ). Moreover, all integrands in (??) are positive, and are thus all the integrals are increasing functions of  $\ell_0$ , so the integral

$$\int_0^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du \quad (14)$$

exists, and is bounded and non-zero.

Next, using l'Hôpital's rule, we see that

$$\lim_{\ell_0 \rightarrow 0} \frac{\int_0^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du - \int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du}{\ell_0^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}}} \quad (15a)$$

$$= \lim_{\ell_0 \rightarrow 0} \frac{\int_0^{\ell_0} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du}{\ell_0^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}}} \quad (15b)$$

$$= \lim_{\ell_0 \rightarrow 0} \frac{\ell_0^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-\ell_0})}{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma} \ell_0^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1}} \quad (15c)$$

$$= -\frac{\beta G'_q(1) + \gamma}{\beta + \gamma}. \quad (15d)$$

Combining the above, we see that

$$\begin{aligned} \int_{\ell_0}^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du \\ = \int_0^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du - \ell_0^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}} \left( \frac{\beta G'_q(1) + \gamma}{\beta + \gamma} + o(\ell_0) \right). \end{aligned} \quad (16)$$

Substituting this into (??) gives us

$$p(0) = \frac{\beta}{\beta + \gamma} + \frac{\beta}{\beta G'_q(1) + \gamma} \ell_0^{\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}} \int_0^{\infty} u^{-\frac{\beta+\gamma}{\beta G'_q(1)+\gamma}-1} G_q(e^{-u}) du + o(\ell_0). \quad (17)$$

Now, we've already observed that  $\frac{\beta+\gamma}{\beta G'_q(1)+\gamma} = \frac{1}{\mathcal{R}_{0,c} + \frac{\gamma}{\beta+\gamma}}$  lies strictly in  $(0,1)$ , so the initial condition  $\ell_0$  is appearing in a form that is strictly larger than  $\mathcal{O}(\ell_0)$ , with the effect becoming more pronounced as  $\mathcal{R}_{0,c}$  increases.

The missing step here is to characterize  $\ell_0$ . Given that  $S(t) = G_p(\phi(t))$  we might reasonably take  $S(0) = 1 - \frac{1}{n}$  and  $\phi(0) = G_p^{-1}\left(1 - \frac{1}{n}\right)$ , whence, using the inverse function theorem,

$$\ell_0 = -\ln G_p^{-1}\left(1 - \frac{1}{n}\right) \approx -\ln\left(1 - \frac{(G_p^{-1})'(1)}{n}\right) = -\ln\left(1 - \frac{1}{G'_p(1)n}\right) \approx \frac{1}{G'_p(1)n}, \quad (18)$$

and  $p(0) - \frac{\beta}{\beta+\gamma} = \mathcal{O}\left(n^{-\frac{1}{\mathcal{R}_{0,c} + \frac{\gamma}{\beta+\gamma}}}\right)$ .