# A Numerical Approach to Computing $\mathcal{R}_{\mathrm{c}}(t)$

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We need to compute

$$p(t) = \mathbb{P}\left\{\text{an individual infected at time } t \text{ infects a neighbour}\right\}. \tag{1}$$

The primary will infect the secondary if it makes an infectious contact before recovering *and* the secondary is not infected at the time of contact. Using the MSV formalism, the probability that a randomly chosen neighbour is not infected at time $t$ is $\sigma(t) = G_q(\phi(t))$.

One way to compute $p(t)$ is by comparing times of random events: let $T_{\mathrm{r}}$ be the time after infection that the primary recovers, $T_{\mathrm{c}}$ be the time after infection that the primary makes it's first contact with the secondary, and let $T_{\mathrm{n}}$ be the first time that the secondary neighbour has an infectious contact from one of its other neighbours:

$$\mathbb{P}\left\{T_{\mathrm{r}} > u\right\} = e^{-\gamma u} \tag{2a}$$

$$\mathbb{P}\left\{T_{\mathrm{c}} > u\right\} = e^{-\beta u} \tag{2b}$$

$$\mathbb{P}\left\{T_{\mathrm{n}} > t + u\right\} = \sigma(t+u). \tag{2c}$$

Further,

$$p(t) = \mathbb{P}\left\{t + T_{\mathrm{c}} < (t + T_{\mathrm{r}}) \wedge T_{\mathrm{n}}\right\}, \tag{3}$$

where $\wedge$ denotes the minimum (this notation is commonly used in probability and stochastic processes), while, exploiting the independence of $T_{\mathrm{r}}$ and $T_{\mathrm{n}}$,

$$\mathbb{P}\left\{(t + T_{\mathrm{r}}) \wedge T_{\mathrm{n}} > t + u\right\} = \mathbb{P}\left\{T_{\mathrm{r}} > u; T_{\mathrm{n}} > t + u\right\} \tag{4a}$$

$$= \mathbb{P}\left\{T_{\mathrm{r}} > u\right\} \mathbb{P}\left\{T_{\mathrm{n}} > t + u\right\} \tag{4b}$$

$$= e^{\gamma u} \sigma(t+u). \tag{4c}$$

Thus,

$$p(t) = \int_0^\infty \beta e^{-(\beta+\gamma)u} \sigma(t+u) \, \mathrm{d}u \tag{5}$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \int_0^\infty \beta e^{-(\beta+\gamma)u} \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t+u)\,\mathrm{d}u \tag{6a}$$

$$= \int_0^\infty \beta e^{-(\beta+\gamma)u} \frac{\mathrm{d}}{\mathrm{d}u}\sigma(t+u)\,\mathrm{d}u \tag{6b}$$

$$= \beta e^{-(\beta+\gamma)u}\sigma(t+u)\Big|_0^\infty + \int_0^\infty \beta(\beta+\gamma) e^{-(\beta+\gamma)u} \frac{\mathrm{d}}{\mathrm{d}u}\sigma(t+u)\,\mathrm{d}u \tag{6c}$$

$$= (\beta+\gamma)p(t) - \beta\sigma(t). \tag{6d}$$

We can thus add one additional equation to the MSV equations to compute $p(t)$.