R_c-SignIssue

This is a translated version based on Todd's notes.
We are able to describe the result with a short ODE and being able to solve it numerically together with ODE system of MSV framework.

We would like to compute:

$$p(t) = \mathbb{P}\{\text{the focal infected node infected at time } t \text{ infect random one of its neighbour}\}$$

- As in MSV framework, we assume infection of neighbors are independent, thus the number of infected neighbors follows binomial distribution. Follow the idea of Zhao2 result/derivation:
  - We count the condition that the neighbor must be susceptible $\phi_S(t)$ in $p(t)$ s.t.

$$\mu(t) = p(t)$$

  - Furthermore, correspond to $\mathcal{R}_c^*$, we have

$$\mathcal{R}_c(t) = \mu(t) \times (\mathbb{E}[K_I^*] - 1) = p(t) \times \phi(t) \frac{G_p''(\phi(t))}{G_p'(\phi(t))}$$

- Initially at $t = 0$, there should be no competing infection among neighbors, all neighbors of focal infected node can only be infected by the focal node.
  - This leads to

$$p(0) = \tau = \frac{\beta}{\beta + \gamma}$$

    as initial value of $p(t)$.
  - Furthermore, this agree with the initial value for $\mathcal{R}_c$, s.t.

$$\mathcal{R}_c(0) = \mathcal{R}_c^*(0) = \mathcal{R}_{c,0} = \frac{\beta}{\beta + \gamma} \times \frac{G_p''(1)}{\delta}$$

To find $p(t)$ in the stochastic process, we observe that the focal infected node will infect its neighbor:

- if the focal node makes an infectious contact with the neighbor under rate β before the focal node's recovery under rate γ
- $\mathbf{AND}$ the neighbor is not infected (by the neighbor's neighbor other than the focal) at the time of contact by focal.
  With the MSV framework, the probability that a randomly chosen neighbor of the focal node is not infected at time t is

$$\phi_S(t) = G_q(\phi(t)) = \frac{G_p'(\phi(t))}{\delta}$$

  In the random events, lets set the following random variables for time:
- T_r: the time after infection t that the focal infected node recovers. Based on the recovery rate γ and exponential distribution, we have

$$\mathbb{P}(T_r > s) = e^{-\gamma s}$$

- T_c: the time after infection t that the focal node makes it infectious contact with the neighbor through the edge connecting them. Based on the infection rate β and exponential distribution, we have

$$\mathbb{P}(T_c > s) = e^{-\beta s}$$

- T_n: the first time that the neighbor has an infectious contact from one of its other neighbors than the focal node. We further have

$$\mathbb{P}(T_n > t + s) = \phi_S(t + s)$$

With these RVs, we can interpret $p(t)$ in the following probability:

$$p(t) = \mathbb{P}\{t + T_c < (t + T_r) \wedge T_n\}$$

where $t + T_c < (t + T_r) \wedge T_n \ \Leftrightarrow \min((t + T_r), T_n)$.

Since T_r and T_n are independent, we can further derive:

$$\begin{aligned}\mathbb{P}\{(t + T_r) \wedge T_n > t + u\} &= \mathbb{P}\{T_r > u\}\mathbb{P}\{T_n > t + u\} \\ &= e^{-\gamma u}\phi_S(t + u)\end{aligned}$$

Then we can rewrite the expression for $p(t)$ based on law of total probability:

$$\begin{aligned}p(t) &= \mathbb{P}\{t + T_c < (t + T_r) \wedge T_n\} \\ &= \int_0^\infty \mathbb{P}\{t + u < (t + T_r) \wedge T_n | t + T_c = t + u\} \times \mathbb{P}\{t + T_c = t + u\} du \\ &= \int_0^\infty \mathbb{P}\{t + u < (t + T_r) \wedge T_n | T_c = u\} \times \mathbb{P}\{T_c = u\} du \\ &= \int_0^\infty \mathbb{P}\{t + u < (t + T_r) \wedge T_n\} \times \mathbb{P}\{T_c = u\} du \quad \text{as } T_n, T_c, T_r \text{ are independent} \\ &= \int_0^\infty [e^{-\gamma u}\phi_S(t + u)] \times [\beta e^{-\beta u}] du \\ &= \int_0^\infty \beta e^{-(\beta+\gamma)u}\phi_S(t + u) du\end{aligned}$$

and we have

$$\begin{aligned}\frac{d}{dt}p(t) &= \frac{d}{dt}\int_0^\infty \beta e^{-(\beta+\gamma)u}\phi_S(t + u) du \\ &= \int_0^\infty \beta e^{-(\beta+\gamma)u} \times [\frac{d}{dt}\phi_S(t + u)] du \\ &= \int_0^\infty \beta e^{-(\beta+\gamma)u} \times [\dot{\phi}_S(t + u)] du \\ &= [\beta e^{-(\beta+\gamma)u}\phi_S(t + u)]\Big|_{u=0}^{\infty} - \int_0^\infty -(\beta + \gamma)\beta e^{-(\beta+\gamma)u}\phi_S(t + u) du \quad \text{(I.B.P.)} \\ &= -\beta\phi_S(t) + (\beta + \gamma)\int_0^\infty \beta e^{-(\beta+\gamma)u}\phi_S(t + u) du \\ &= -\beta\phi_S(t) + (\beta + \gamma)p(t)\end{aligned}$$

as a ODE of $p(t)$.

At $t = 0$, we expect to have $\phi_S(0) = 1$ and $p(0) = \frac{\beta}{\beta+\gamma}$, so take these initial value into the ODE gives us

$$\frac{d}{dt}p(t) = -\beta + (\beta + \gamma) \times \frac{\beta}{\beta + \gamma} = 0$$

which agree with our expectation.

Sign Problem

This notes fixed some typo in Todd's original notes and the final ODE agree with the original one. I review the derivation myself twice and it seems correct.

But we expect $p(t)$ be a probability and monotonically decreasing from its initial value since the following two factor:

- ϕ_S is decreasing as the infection spread out.
- We have more competing infection happens, i.e. it is more likely to have $T_n < t + T_c$
  However, in numeric solving, this ODE for $p(t)$ leads to increasing $p(t)$ and getting larger than 1.

See R_c-SignIssue.R for numeric solutions `CM_P`, the ODE is implemented at line `#107`.

Just an observation: if alter the all signs in the ODE, the results of $p(t)$ seems to be much more reasonable.
So my conjecture now would be that we mess up the sign in the probability arguments somewhere, perhaps in the integration expression of $p(t)$.
But I am having difficulty find the problem, and a second opinion would be really appreciated.