p0Note

Following Todd's outputs/Rc.pdf/Rc.tex note:

Linearizing $\ell = -log(\phi) \Leftrightarrow \phi = e^{-\ell}$ near the disease-free equilibrium, s.t. $\ell(0) = \ell_0$ and $\ell_0 \to 0^+$.
Then we have:

$$\begin{aligned}-\dot{\ell} &= \frac{d}{dt}[-log(\phi(t))] = \frac{\dot{\phi}}{\phi} = -\beta + \beta\frac{G_p'(\phi)}{\delta\phi} + \gamma(\frac{1}{\phi} - 1) \\ &= -\beta + \frac{\beta}{\delta}\left[\sum_{d=1}^{\infty} dp_d e^{-\ell(d-1)}\right] \times e^{\ell} + \gamma(e^{\ell} - 1)\end{aligned}$$

Take into $e^{\ell} = 1 + \ell + o(\ell^2)$ and $e^{-\ell} = 1 - \ell + o(\ell^2)$ near $\ell \to 0^+$, we further have:

$$\begin{aligned}-\dot{\ell} &= -\beta + \frac{\beta}{\delta}\left[\sum_{d=1}^{\infty} dp_d(1 - \ell + o(\ell^2))^{d-1}\right] \times (1 + \ell + o(\ell^2)) + \gamma\ell \\ &= -\beta + \frac{\beta}{\delta}\left[\sum_{d=1}^{\infty} dp_d - \sum_{d=1}^{\infty} d(d-1)p_d\ell + o(\ell^2)\right] \times (1 + \ell + o(\ell^2)) + \gamma\ell \\ &= -\beta + \frac{\beta}{\delta}\left[\delta - \ell G_p''(1) + o(\ell^2)\right] \times (1 + \ell + o(\ell^2)) + \gamma\ell \\ &= -\beta + \frac{\beta}{\delta}\left[\delta - \ell G_p''(1) + \ell\delta + o(\ell^2)\right] + \gamma\ell \\ &= \ell(-\frac{\beta G_p''(1)}{\delta} + \beta + \gamma) + o(\ell^2) \\ &= \ell(\beta + \gamma)(1 - \mathcal{R}_{c,0}) + o(\ell^2)\end{aligned}$$

We take only the first order term of $\dot{\ell}$ to estimate $\ell(t)$ near ℓ_0, which provides:

$$\ell(t) \approx \ell_0 e^{(\beta+\gamma)(1-\mathcal{R}_{c,0})\times t}$$

Thus further

$$\phi(t) = e^{-\ell(t)} \approx e^{-\ell_0 e^{(\beta+\gamma)(1-\mathcal{R}_{c,0})t}}$$

Since

$$p(t) = \int_0^{\infty} \beta e^{-(\beta+\gamma)u}\phi_S(t+u)du = \int_0^{\infty} \beta e^{-(\beta+\gamma)u}\frac{G_p'(\phi(t+u))}{\delta}du$$

, take into the approximation gives us:

$$p(0) = \int_0^{\infty} \beta e^{-(\beta+\gamma)u}\frac{G_p'(\phi(u))}{\delta}du \approx \int_0^{\infty} \beta e^{-(\beta+\gamma)u}\frac{G_p'(e^{-\ell_0 e^{(\beta+\gamma)(1-\mathcal{R}_{c,0})u}})}{\delta}du$$

Now change variable with

$$v = \ell_0 e^{(\beta+\gamma)(1-\mathcal{R}_{c,0})u} \Leftrightarrow u = \frac{\log(v) - \log(\ell_0)}{(\beta+\gamma)(1-\mathcal{R}_{c,0})} \Leftrightarrow du = \frac{v^{-1}}{(\beta+\gamma)(1-\mathcal{R}_{c,0})} \times dv$$

The approximation then comes to

$$p(0) \approx \frac{1}{(\beta+\gamma)(1-\mathcal{R}_{c,0})\delta}\ell_0^{-\frac{1}{\mathcal{R}_{c,0-1}}} \int_{\ell_0}^{\infty} v^{\frac{1}{\mathcal{R}_{c,0-1}}-1} G_p'(e^{-v})dv$$