Initial_Values

We would like to derive the "eigenvector" initial condition for $R(0)$ and a small enough $\phi(0)$, such that

$$\mathcal{R}_{i,0} = \mathcal{R}_i(0) = \max(\mathcal{R}_i)$$

as in NoteForR_i.

We start with MSV ODE system:

$$\begin{cases} S(t) = G_p(\phi(t)) \\ I(t) = 1 - S(t) - R(t) \\ \dot{R}(t) = \gamma I(t) \end{cases}$$

$$\dot{\phi} = -\beta\phi_I = -\beta(\phi - \phi_S - \phi_R) = -\beta\phi + \beta\frac{G_p'(\phi)}{\delta} + \gamma(1-\phi)$$

where $\phi(t)$ is the probability that a randomly chosen edge has not yet transmitted the disease at time t.
At $t = 0$, $\phi(0) \to 1$ so we take

- $\omega = 1 - \phi \Leftrightarrow \phi = 1 - \omega$
- $V(t) = 1 - S(t) = 1 - G_p(\phi(t)) = 1 - G_p(1 - \omega(t))$

Now consider the ODE for $\omega(t)$ and $R(t)$ based on previous system, we have

$$\begin{cases} \dot{\omega} = -\dot{\phi} = \beta(1-\omega) - \beta\frac{G_p'(1-\omega)}{\delta} - \gamma\omega \\ \dot{R}(t) = \gamma(1 - G_p(\phi(t)) - R(t)) \end{cases}$$

Similar with the derivation of $\mathcal{R}_{c,0}$, using first order approximation, we have:

$$\begin{aligned} G_p(\phi(t)) = G_p(1-\omega(t)) &= \sum_k p_k(1-\omega)^k \\ &= \sum_k p_k[1 - k\omega + o(\omega^2)] \\ &\approx \sum_k p_k - \omega\sum_k kp_k \\ &= 1 - \delta\omega \end{aligned}$$

and

$$\begin{aligned} G_p'(\phi(t)) = G_p'(1-\omega(t)) &= \sum_k kp_k(1-\omega)^{k-1} \\ &= \sum_k kp_k[1 - (k-1)\omega + o(\omega^2)] \\ &\approx \sum_k kp_k - \omega\sum_k k(k-1)p_k \\ &= \delta(1 - \frac{G_p''(1)}{\delta} \times \omega) \end{aligned}$$

Using these approximation, the linearized ODE near $t \to 0$ is

$$\begin{cases} \dot{\omega} \approx \beta(1-\omega) - \beta(1 - \frac{G_p''(1)}{\delta} \times \omega) - \gamma\omega = [\beta \times \frac{G_p''(1)}{\delta} - (\beta+\gamma)]\omega = \eta\omega \\ \dot{R}(t) \approx \gamma(\delta\omega - R(t)) \end{cases}$$

- Note:

$$\eta = \beta \times \frac{G_p''(1)}{\delta} - (\beta+\gamma) = (\beta+\gamma)(\mathcal{R}_{c,0} - 1)$$

  Then this linearized system has the matrix form:

$$\begin{bmatrix} \dot{\omega} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ \delta\gamma & -\gamma \end{bmatrix} \times \begin{bmatrix} \omega \\ R \end{bmatrix}$$

The eigenvalues are just η and $-\gamma$.

For the dominant eigenvalueη, the eigenvector satisfy:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \delta\gamma & -\gamma-\eta \end{bmatrix} \times \begin{bmatrix} \omega(0) \\ R(0) \end{bmatrix} \Leftrightarrow R(0) = \frac{\delta\gamma}{\gamma+\eta} \times \omega(0)$$

This should give us a initial condition on the eigen-direction.