Initial_Values

1. "Eigendirection" of recovered $R(t)$

We would like to derive the "eigenvector" initial condition for $R(0)$ and a small enough $\phi(0)$, such that

$$\mathcal{R}_{i,0} = \mathcal{R}_i(0) = \max(\mathcal{R}_i)$$

as in NoteForR_i.

We start with MSV ODE system:

$$\begin{cases} S(t) = G_p(\phi(t)) \\ I(t) = 1 - S(t) - R(t) \\ \dot{R}(t) = \gamma I(t) \end{cases}$$

$$\dot{\phi} = -\beta\phi_I = -\beta(\phi - \phi_S - \phi_R) = -\beta\phi + \beta\frac{G_p'(\phi)}{\delta} + \gamma(1-\phi)$$

where $\phi(t)$ is the probability that a randomly chosen edge has not yet transmitted the disease at time t.
At $t=0$, $\phi(0) \to 1$ so we take

- $\omega = 1 - \phi \Leftrightarrow \phi = 1 - \omega$
- $V(t) = 1 - S(t) = 1 - G_p(\phi(t)) = 1 - G_p(1 - \omega(t))$

Now consider the ODE for $\omega(t)$ and $R(t)$ based on previous system, we have

$$\begin{cases} \dot{\omega} = -\dot{\phi} = \beta(1-\omega) - \beta\frac{G_p'(1-\omega)}{\delta} - \gamma\omega \\ \dot{R}(t) = \gamma(1 - G_p(\phi(t)) - R(t)) \end{cases}$$

Similar with the derivation of $\mathcal{R}_{c,0}$, using first order approximation, we have:

$$\begin{aligned} G_p(\phi(t)) = G_p(1-\omega(t)) &= \sum_k p_k(1-\omega)^k \\ &= \sum_k p_k[1 - k\omega + o(\omega^2)] \\ &\approx \sum_k p_k - \omega\sum_k kp_k \\ &= 1 - \delta\omega \end{aligned}$$

and

$$
\begin{aligned}
G_p'(\phi(t)) = G_p'(1-\omega(t)) &= \sum_k kp_k(1-\omega)^{k-1} \\
&= \sum_k kp_k[1-(k-1)\omega + o(\omega^2)] \\
&\approx \sum_k kp_k - \omega \sum_k k(k-1)p_k \\
&= \delta(1 - \frac{G_p''(1)}{\delta} \times \omega)
\end{aligned}
$$

Using these approximation, the linearized ODEs near $t \to 0$ are

$$
\begin{cases}
\dot{\omega} \approx \beta(1-\omega) - \beta(1 - \frac{G_p''(1)}{\delta} \times \omega) - \gamma\omega = [\beta \times \frac{G_p''(1)}{\delta} - (\beta+\gamma)]\omega = \eta\omega \\
\dot{R}(t) \approx \gamma(\delta\omega - R(t))
\end{cases}
$$

- Note:

$$
\eta = \beta \times \frac{G_p''(1)}{\delta} - (\beta+\gamma) = (\beta+\gamma)(\mathcal{R}_{c,0} - 1)
$$

  Then this linearized system has the matrix form:

$$
\begin{bmatrix} \dot{\omega} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ \delta\gamma & -\gamma \end{bmatrix} \times \begin{bmatrix} \omega \\ R \end{bmatrix}
$$

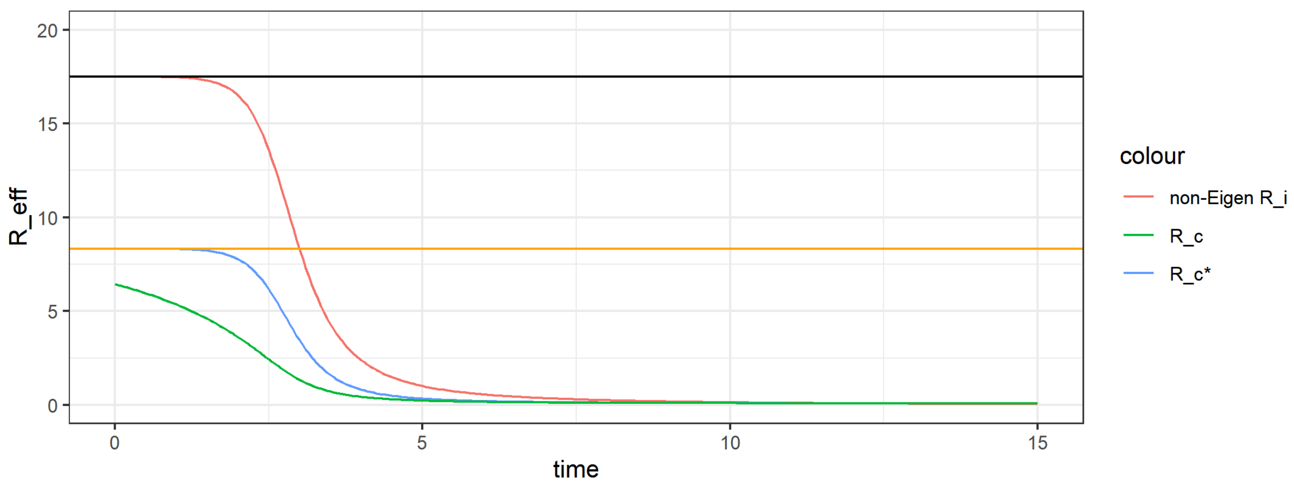
The eigenvalues are just η and $-\gamma$.

For the dominant eigenvalue η, the eigenvector satisfy:

$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \delta\gamma & -\gamma-\eta \end{bmatrix} \times \begin{bmatrix} \omega(0) \\ R(0) \end{bmatrix} \Leftrightarrow R(0) = \frac{\delta\gamma}{\gamma+\eta} \times \omega(0)
$$

This should give us a initial condition on the eigen-direction.

This estimation works well s.t. the $\mathcal{R}_i$(Red curve on top, "non-Eigen" is a typo) match with the theoretical $max(R_i)$ (black horizontal line):



2. Migrating to $p(0)$

Based on Todd's [Rc.tex](Rc.tex)/[Rc.pdf](Rc.pdf) note, we have an additional ODE for $p(t)$: the probability of a newly infected nodes infect one of its neighbour before recovery.

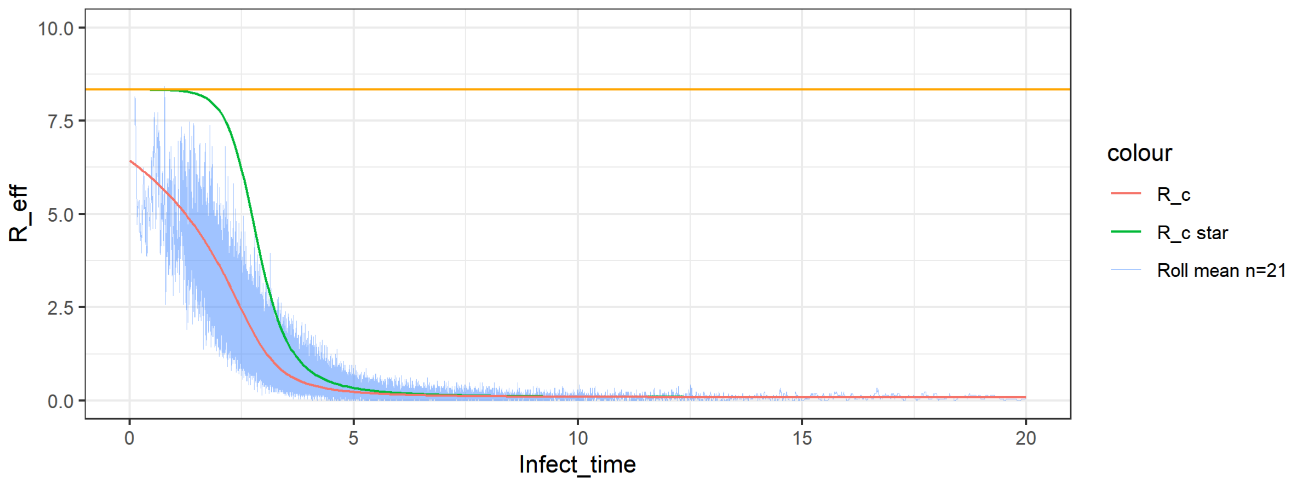
$$\frac{d}{dt}p(t) = -\beta\phi_S(t) + (\beta+\gamma)p(t) = -\beta\frac{G_p''(\phi(t))}{\delta} + (\beta+\gamma)p(t)$$

We have conclude this should be a final value problem, since when $t \to +\infty$, $\frac{d}{dt}p(\infty) \to 0$ as the system reaching its equilibrium state.

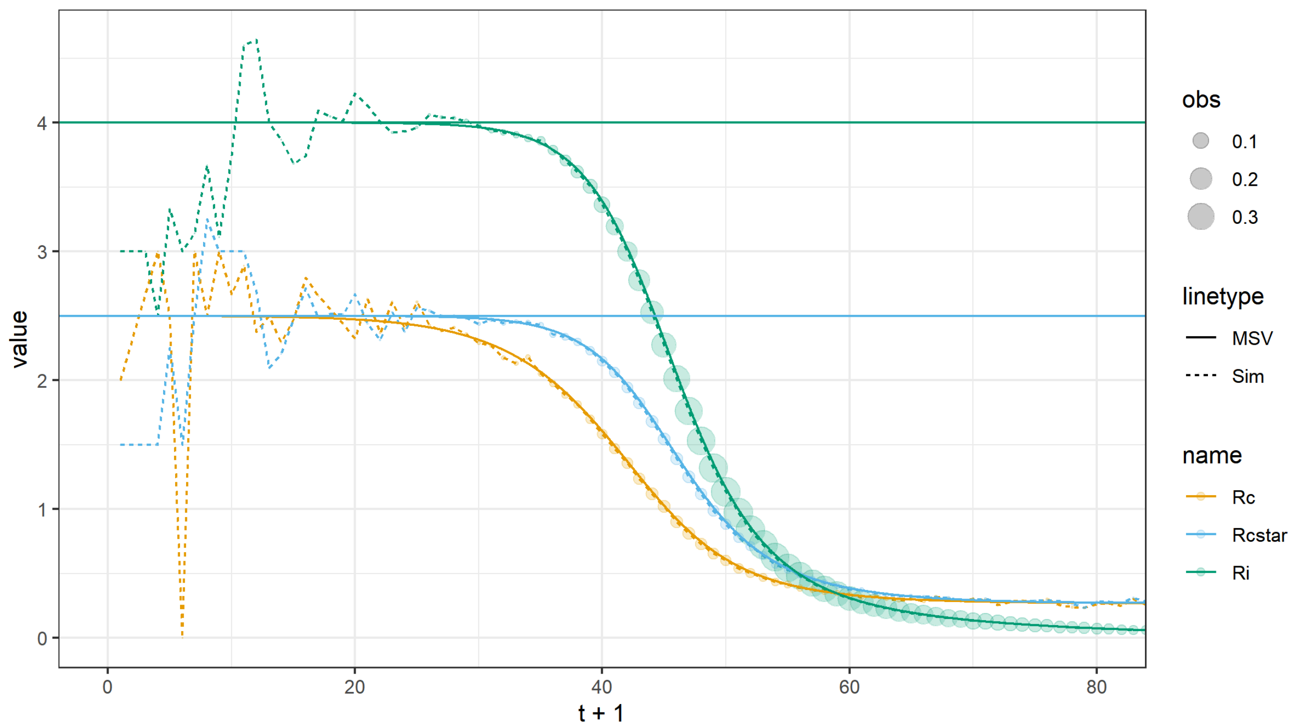
Then we could derived the $\mathcal{R}_c$ using $p(t)$:

$$\mathcal{R}_c(t) = \frac{p(t)}{\phi(t)} \times (\mathbb{E}[K_I^*] - 1) = p(t) \times \frac{G_p''(\phi(t))}{G_p'(\phi(t))}$$

For $N = 50,000, \gamma = 0.20, \beta = 0.25, I_0 = 1$: 16 simulation



For $N = 1,000,000, \gamma = \beta = 0.1, I_0 = 1$: New simulation



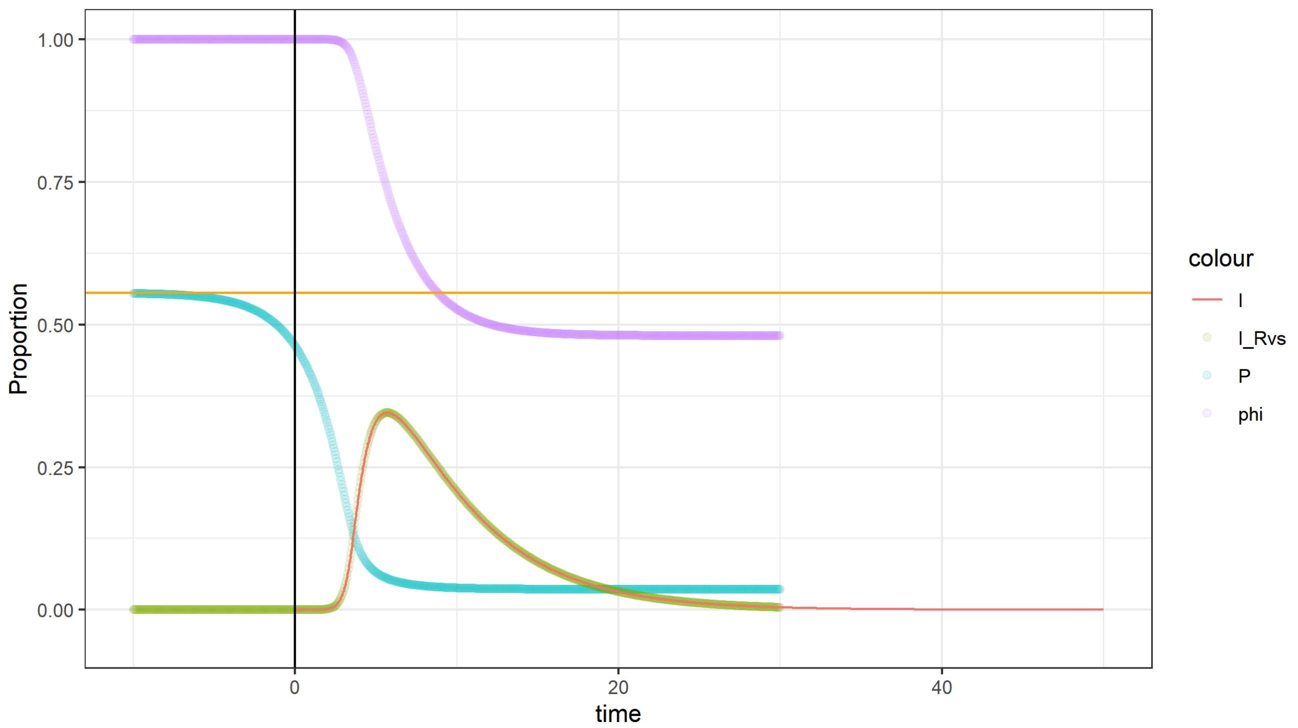
However, as observed in the old simulation, if network size N not large enough (and γ is small, i.e. infectious duration D is long), there is a good possibility that competing infection occurs, even for nodes infected at $t \to 0$ but being infectious for a long time, s.t.

$$\frac{d}{dt}p(t)|_{t=0} \neq 0 \Leftrightarrow p(0) \neq \frac{\beta}{\beta+\gamma}$$

But we could still say

$$p(-\infty) = \frac{\beta}{\beta+\gamma}$$

Currently, we reversely solve the ODEs from some t_{end} that close to the end of the outbreak for $p(t)$.
For $N = 500,000, \gamma = 0.20, \beta = 0.25, I_0 = 1$



**So for $N = 50,000$ case, even $\phi(0)$ at $t = 0$ is still very close to its initial value $\phi(-\infty) = 1$ (i.e. still governed by on its eigenvector), $p(0)$ is already away from $p(-\infty)$. Even if we do the eigenvector estimation, the magnitude of $\omega = 1 - \phi = 2 \times 10^{-5}$ could not match with $q(0) = \frac{\beta}{\beta+\gamma} - p(0) \approx 1 \times 10^{-1}$.

Thus RZ doubt if this eigenvector idea still work for NOT large enough N.** We might have to find the proper magnitude for N for this estimation idea to work, like what Todd tried to do in Rc.pdf.

As network size N increase (e.g. $N = 1 \times 10^7$), competing infection would be extremely rare at the beginning, s.t. $p(0)$ is much close to $p(-\infty)$, then this estimation would work better.

In Rc.tex/Rc.pdf, Todd tried to estimate $p(0)$ using linearization. We are also interested to know how $p(0)$ connects to N, so we have a better idea for how large the network should be so that competing infection could be rear, at least at the very beginning s.t.

$$p(0) \approx \frac{\beta}{\beta+\gamma} \Leftrightarrow \mathcal{R}_c^*(0) \approx \mathcal{R}_c(0)$$

As an alternative way, we would like to see if we can use similar "eigendirection" idea to estimate $p(0)$ like $R(0)$.

- But RZ doubt if it works, especially for larger $\rho = \frac{\beta}{\gamma}$ case where $\frac{d}{dt}p(t)|_{t=0}$ seems to be quite different with $\frac{d}{dt}p(t)|_{t=-\infty}$.
- Increasing N seems lower $\omega(0)$ and make $p(0)$ closer to $p(-\infty)$.

Consider $q(t) = \frac{\beta}{\beta+\gamma} - p(t)$, then

$$\dot{q}(t) = \frac{d}{dt}q(t) = -\frac{d}{dt}p(t) = +\beta\frac{G_p''(\phi(t))}{\delta} - (\beta+\gamma)(\frac{\beta}{\beta+\gamma} - q(t))$$

Thus as $q \to 0$ and $\omega \to 0$, the linearized ODEs as $t \to -\infty$ are

$$\begin{cases} \dot{\omega} \approx \beta(1-\omega) - \beta(1 - \frac{G_p''(1)}{\delta} \times \omega) - \gamma\omega = [\beta \times \frac{G_p''(1)}{\delta} - (\beta+\gamma)]\omega = \eta\omega \\ \dot{q} \approx +\beta(1 - \frac{G_p''(1)}{\delta}\omega) - (\beta+\gamma)(\frac{\beta}{\beta+\gamma} - q) = -\beta\frac{G_p''(1)}{\delta}\omega + (\beta+\gamma)q \end{cases}$$

- Note:

  $$\eta = \beta \times \frac{G_p''(1)}{\delta} - (\beta+\gamma) = (\beta+\gamma)(\mathcal{R}_c^*(0) - 1)$$

  Then this linearized system has the matrix form:

  $$\begin{bmatrix} \dot{\omega} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ -\beta \times \frac{G_p''(1)}{\delta} & \beta+\gamma \end{bmatrix} \times \begin{bmatrix} \omega \\ q \end{bmatrix}$$

The eigenvalues are just η and $\beta+\gamma$.

For most cases that $\mathcal{R}_c^*(0) = \frac{\beta}{\beta+\gamma}\frac{G_p''(1)}{\delta} > 2$, the dominant eigenvalue is η, the eigenvector satisfy:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\beta \times \frac{G_p''(1)}{\delta} & (\beta+\gamma) - \eta \end{bmatrix} \times \begin{bmatrix} \omega(0) \\ q(0) \end{bmatrix} \Leftrightarrow q(0) = \frac{\beta \times \frac{G_p''(1)}{\delta}}{(\beta+\gamma) - \eta} \times \omega(0) < 0$$

??? $q(0) = \frac{\beta}{\beta+\gamma} - p(0)$ should be positive. RZ verified the derivation several times (also via maple), still not be able to find the mistake. A second opinion would be appreciated