Elementary Graph Algorithms

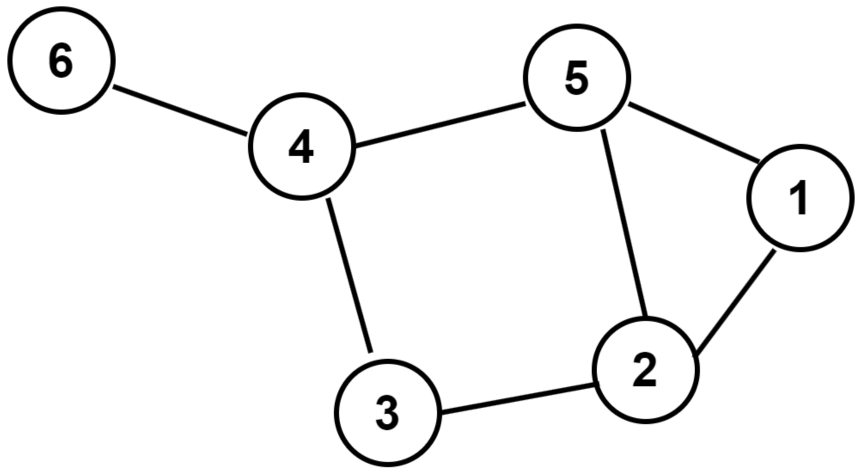
- Design and analysis of algorithms

Introduction

- Graph and set theory
- Graph and computer science

Graph and set theory

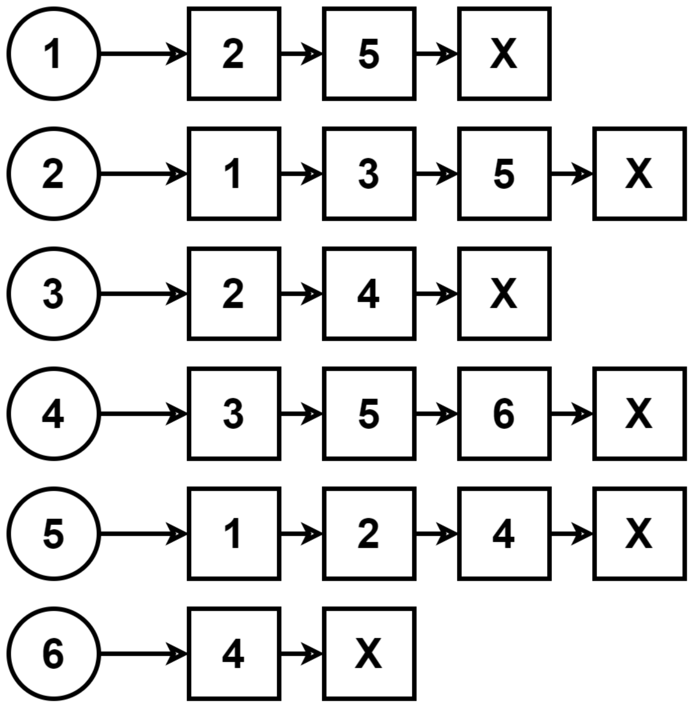
- In set theory, a graph G is defined as a set of vertices V and a relation E describing how the vertices are related to each other.
- A graph is denoted as: $G = (V, E)$
- A graph may be directed or undirected.
- A graph may also be weighted, i.e. the edges of the graph are assigned values or weights.



An undirected graph $G = (V, E)$ where $V = \{6, 4, 5, 3, 1, 2\}$ and $E = \{(6, 4), (4, 5), (4, 3), (5, 1), (5, 2), (3, 2), (2, 1)\}$

Graph and computer science

- In computer science, graph is an abstract data type which is an implementation of a graph in set theory.
- One of the ways of implementing graphs is by using linked lists.
- The circular nodes are the vertices and the square nodes are the vertices they are connected to.
- So the path from a circular node to any of the square node in the adjacent list is an edge.



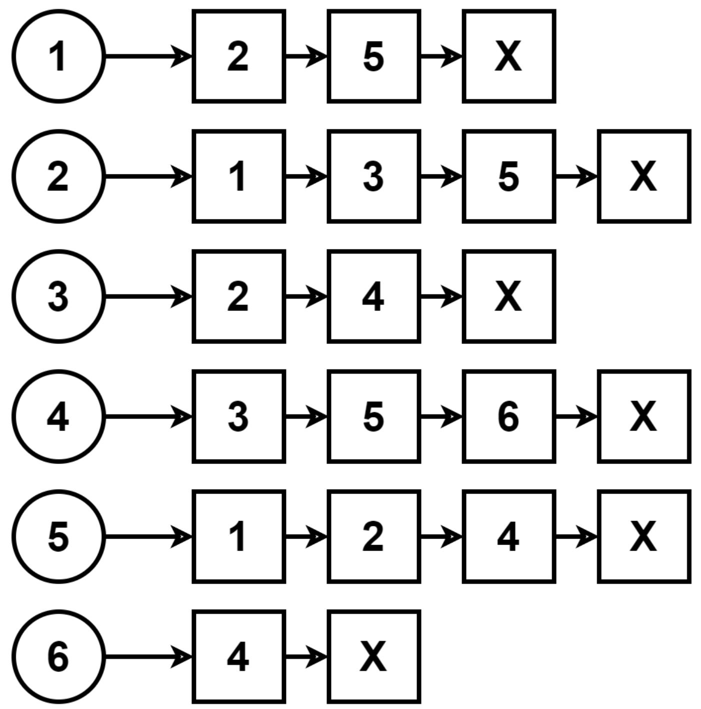
Implementation using linked lists

Representations of graph

- Adjacency list
- Adjacency list – time complexity
- Adjacency matrix
- Adjacency matrix – time complexity

Adjacency list

- In this method, linked lists are used to store the adjacent vertices.
- A graph consists of a list of vertices. Each vertex in the list, in turn, has a pointer to a list of vertices directly connected to it.
- So every vertex in the graph has an edge to each vertex in its adjacency list.
- To represent a weighted graph, an additional attribute of weight can be added to each vertex in the adjacency list.



Implementation using adjacency list

Adjacency list – time complexity

Operation	Time complexity
Add vertex	$O(1)$
Add edge	$O(1)$
Remove vertex	$O(\|E\|)$
Remove edge	$O(\|V\|)$
Adjacency	$O(\|V\|)$

Adjacency matrix

- In this method, a matrix is used to represent if two vertices are connected.
- A graph contains a list of vertices and a $n \times n$ matrix such that n is the number of vertices in the graph.
- If there is an edge between the x^{th} and y^{th} vertex, then $M_{xy} = 1$. Else $M_{xy} = 0$.
- To represent a weighted graph, weights can be stored at the row and column corresponding to the edge in the adjacency matrix.

	1	2	3	4	5	6
1	0	1	0	0	1	0
2	1	0	1	0	1	0
3	0	1	0	1	0	0
4	0	0	1	0	1	1
5	1	1	0	1	0	0
6	0	0	0	1	0	0

Implementation using adjacency matrix

Adjacency matrix – time complexity

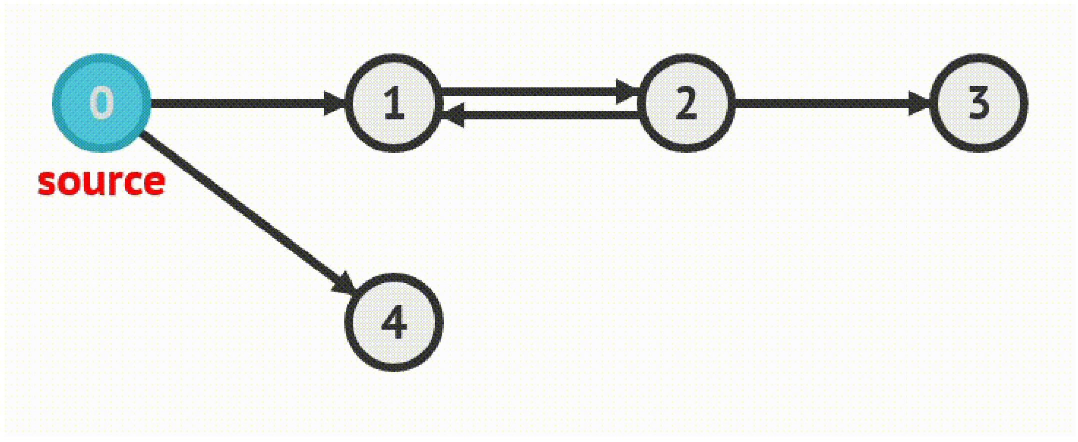
Operation	Time complexity
Add vertex	$O(\|V\|^2)$
Add edge	$O(1)$
Remove vertex	$O(\|V\|^2)$
Remove edge	$O(1)$
Adjacency	$O(1)$

Breadth first search

- What is BFS?
- Attributes
- Algorithm
- Analysis
- Properties
- Applications

What is BFS?

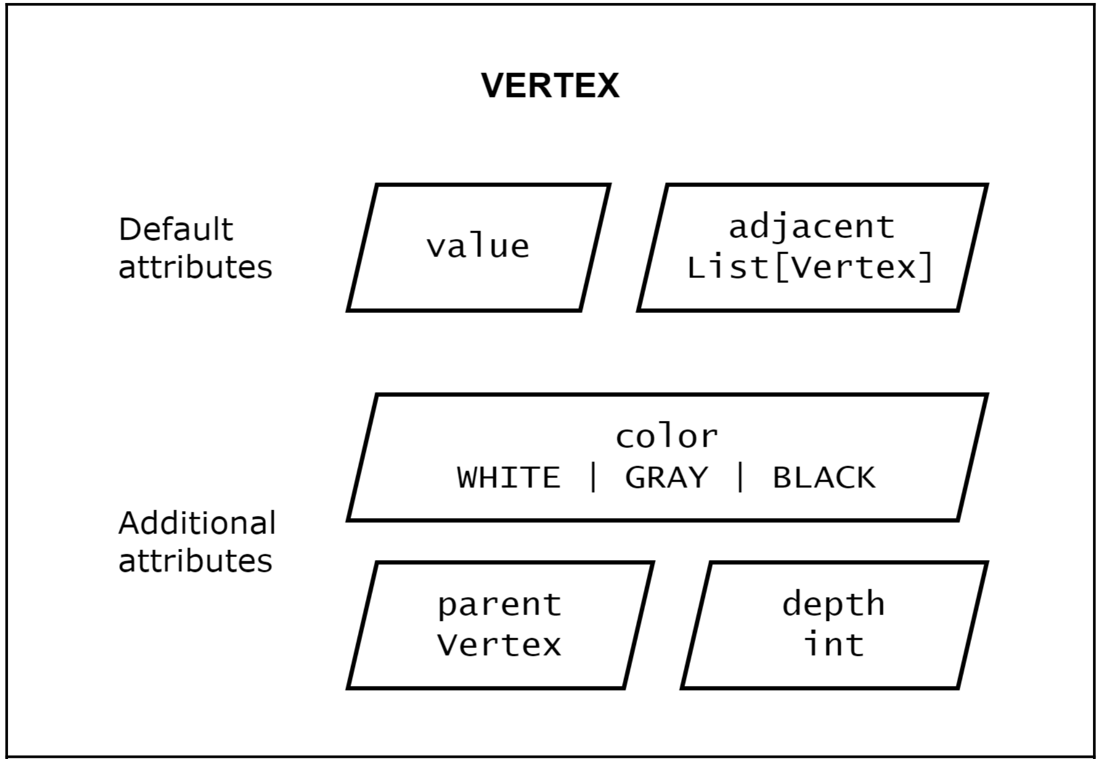
- During breadth first search, a *source* vertex is selected from where traversal will start.
- The algorithm systematically traverses all the adjacent vertices of *source*.
- Then for all the discovered vertices, it then discovers the next set of adjacent vertices.
- This goes on until all the vertices connected to *source* have been discovered.



BFS traversal

Attributes

- For BFS traversal, some additional attributes are added to each vertex.
- *color*: $WHITE$ means that the vertex is yet to be reached, $GRAY$ means that the vertex has been reached but not yet traversed, and $BLACK$ means that the vertex has been traversed.
- *depth*: Distance of the vertex from the source vertex.
- *parent*: Parent of the vertex in the breadth first tree.



BFS vertex attributes

Algorithm

```
function BFS(graph, source):
    # whiten all vertices
    for v in graph.vertices:
        v.color = WHITE
        v.parent = NULL
    # traverse source
    source.color = GRAY
    source.depth = 0
    source.parent = NULL
    enqueue(source)
```

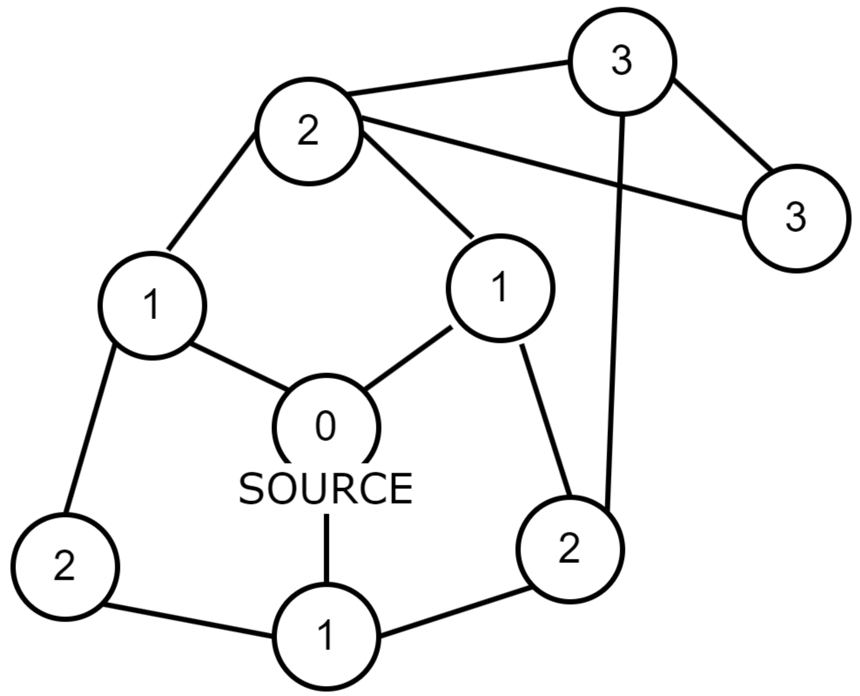
```
    # keep dequeuing
    while not is_queue_empty():
        curr = dequeue()
        # traverse nodes adjacent to source
        for v in curr.adjacent:
            if v.color == WHITE:
                v.color = GRAY
                v.depth = curr.depth + 1
                v.parent = curr
                enqueue(v)
```

Analysis

- Whitening all vertices takes $O(V)$ time.
- Traversing source vertex takes $O(1)$ time.
- Since a vertex is enqueued only once, the queue can be dequeued V times.
- The sum of all adjacent vertices of every vertex is E.
- Time complexity of BFS is $O(V + E)$.

Properties

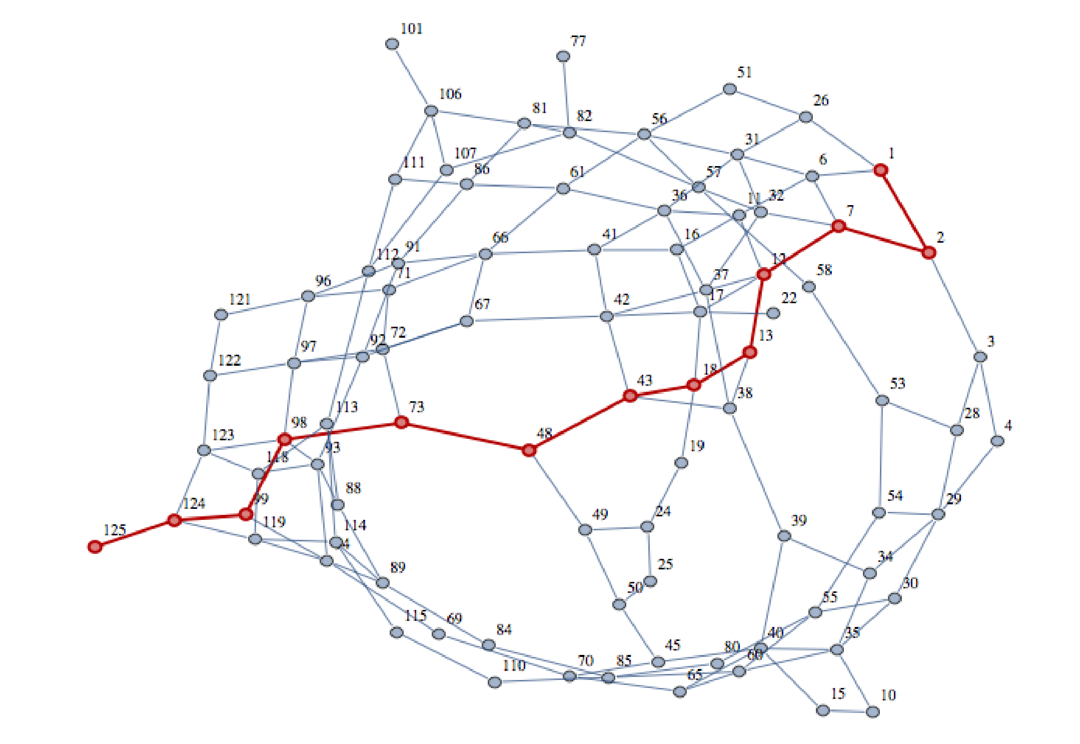
- Breadth first search creates a breadth first tree where the parent of v is $v.parent$
- $source$ is the root of the tree and $source.parent$ is $NULL$.
- For any vertex v, $v.depth$ is the shortest distance from $source$ to v.



Shortest path from source

Applications

- Prim's algorithm uses BFS procedure to find minimal spanning tree in weighted graphs.
- Dijkstra's shortest path algorithm also uses BFS procedure for path finding.
- Alternatively, it can also be used to find solution to mazes.
- BFS traversal is also used to analyze relationships and connections, especially in social networking.



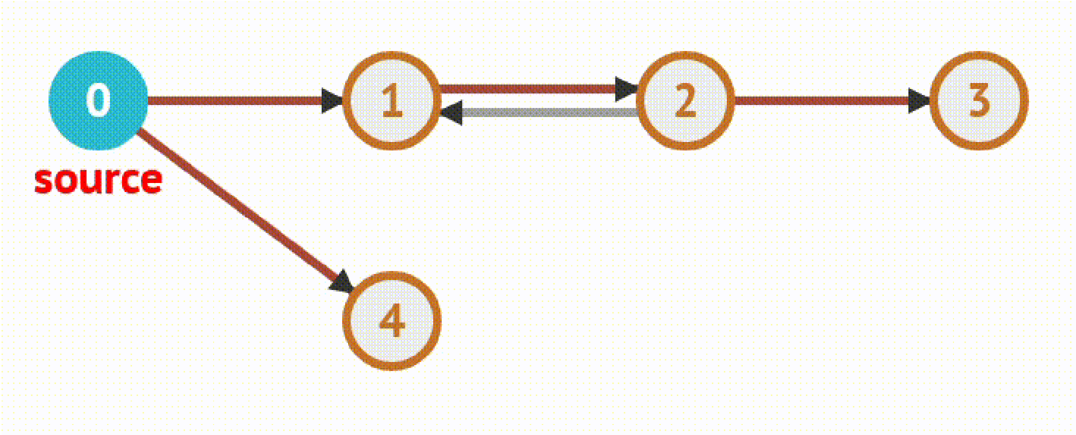
Path finding

Depth first search

- What is DFS?
- Attributes
- Algorithm
- Analysis
- Properties
- Types of edges
- White-path theorem
- Applications

What is DFS?

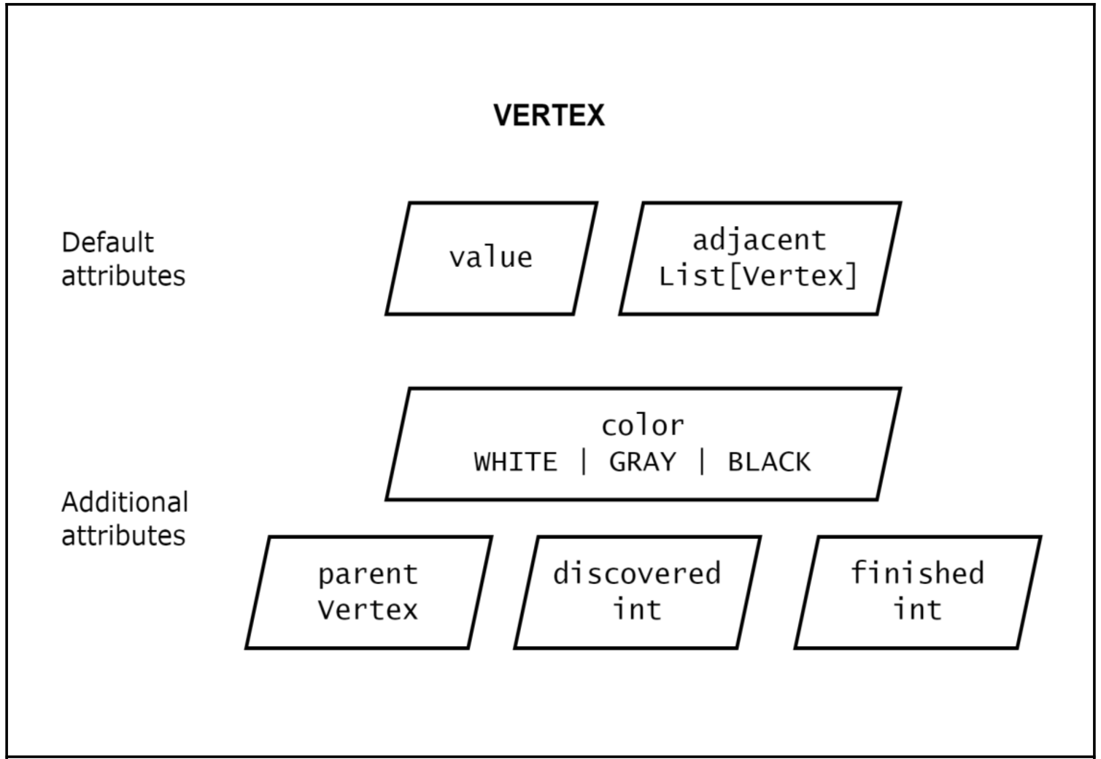
- The aim of depth first search is to traverse *deeper* vertices first whenever possible.
- The algorithm starts with a vertex, keeps discovering *deeper* vertices until there are no more undiscovered vertices.
- Then the same process is repeated for every other undiscovered vertices.



DFS traversal

Attributes

- DFS traversal also defines some more attributes for vertices.
- $color$: $WHITE$, $GRAY$ or $BLACK$.
- $discovered$: Timestamp when vertex was discovered.
- $finished$: Timestamp when the traversal of the vertex finished.
- $parent$: Parent of the vertex in the breadth first tree.



DFS vertex attributes

Algorithm

```
function DFS(graph):
    # whiten all vertices
    for v in graph.vertices:
        v.color = WHITE
        v.parent = NULL
    # reset timestamp
    timestamp = 0
    # visit every vertex
    for v in graph.vertices:
        if v.color == WHITE:
            DFS_VISIT(graph, v)
```

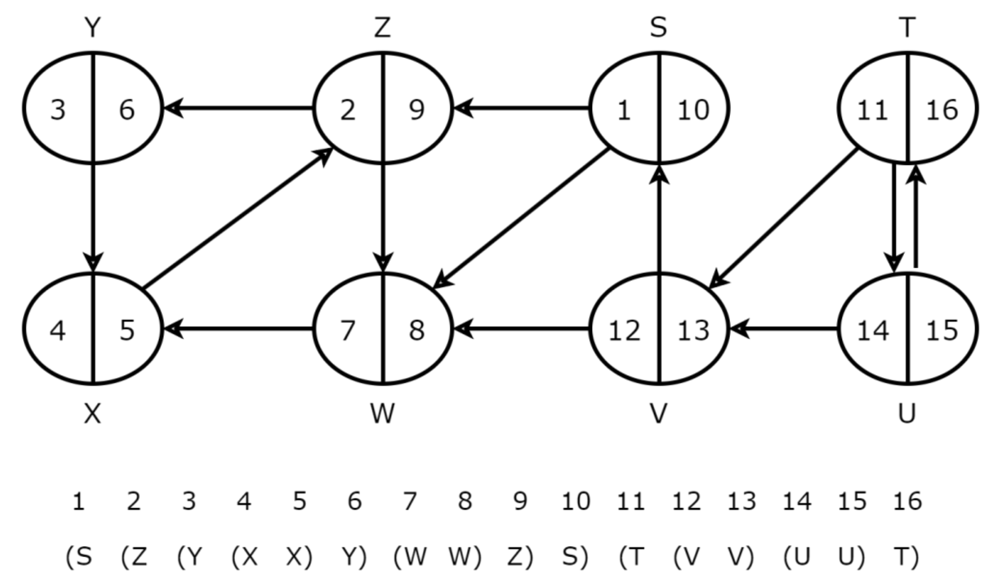
```
function DFS_VISIT(vertex):
    timestamp += 1
    vertex.discovered = timestamp
    vertex.color = GRAY
    for v in vertex.adjacent:
        if v.color == WHITE:
            v.parent = vertex
            DFS_VISIT(v)
    timestamp += 1
    vertex.finished = timestamp
    vertex.color = BLACK
```

Analysis

- Whitening all vertices takes $O(V)$ time.
- Since a vertex is pushed only once, the stack can be popped V times, i.e. DFS_VISIT will be called V times.
- The sum of all adjacent vertices of every vertex is E, i.e. during DFS_VISIT, adjacent vertices will be checked E times.
- Time complexity of DFS is $O(V + E)$.

Properties

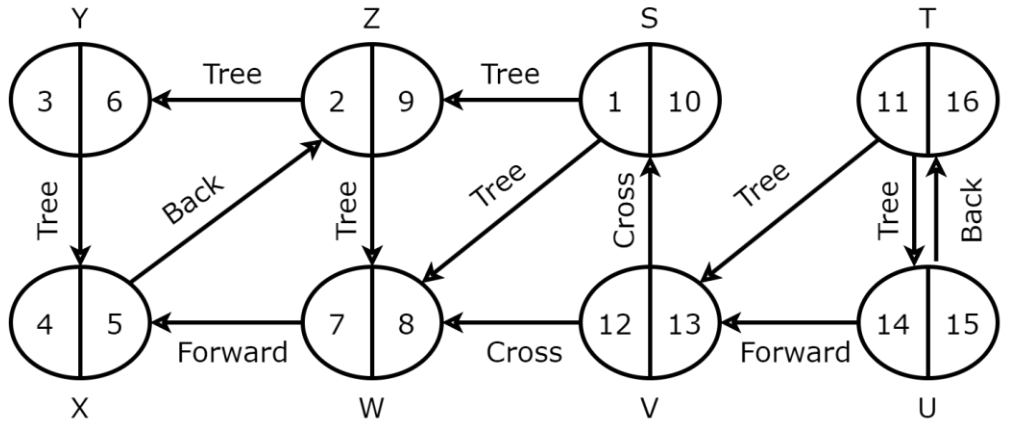
- Depth first search creates multiple trees or a depth first forest where the parent of v is $v.parent$
- Each disconnected vertex has a corresponding tree, the root of the trees are vertices whose $v.parent$ is $NULL$.
- DFS discovery and finish times have parenthesis structure.



Parenthesis structure

Types of edges

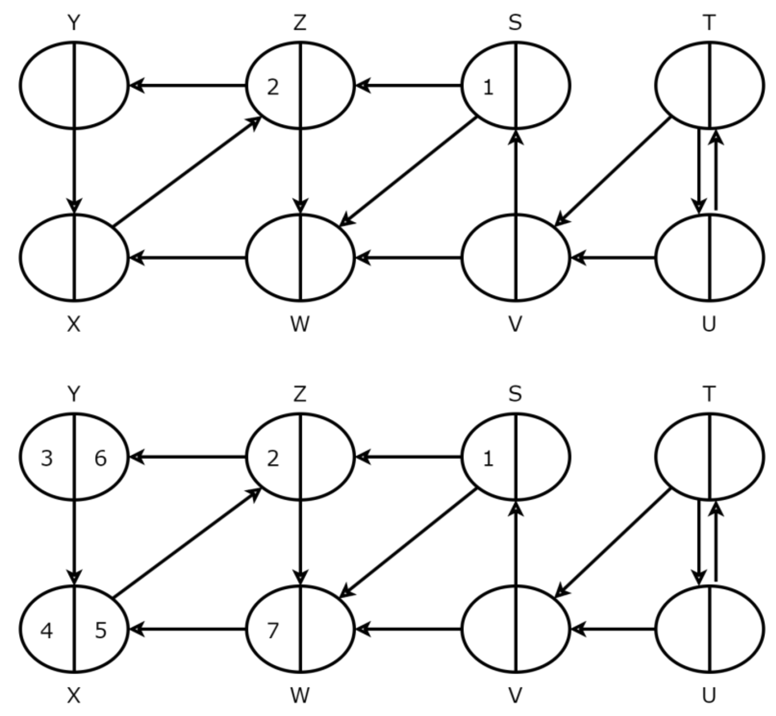
- Tree edges: $u \to v$ if v is the child of u, i.e. when u is discovered, v is $WHITE$.
- Back edges: $u \to v$ if v is an ancestor of u, i.e. when u is discovered, v is $GRAY$.
- Forward edges: $u \to v$ if v is a descendent of u in the same tree, i.e. when u is discovered, v is $BLACK$ and in the same tree.
- Cross edges: $u \to v$ if v and u are in different trees, i.e. when u is discovered, v is BLACK but in a different tree.



Types of edges

White-path theorem

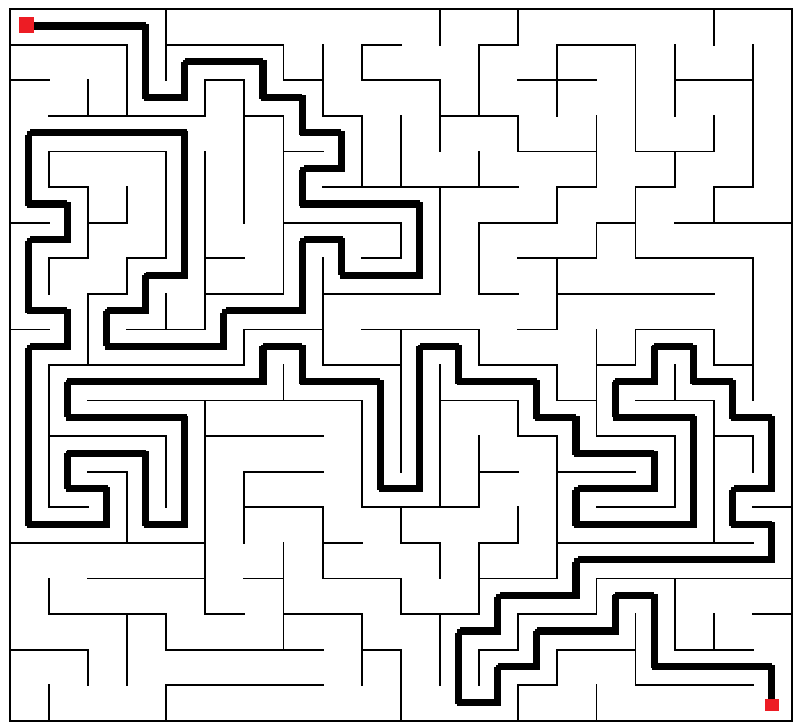
- White-path theorem states that in a depth first forest, vertex v is a descendent of vertex u if and only if at $u.discovered$, i.e. when u was discovered, there is a path from u to v consisting entirely of white vertices.
- In the depth first forest of the graph in the example, z is an ancestor of vertices x.
- w is not an ancestor of z even if there exists a path from w to z because it does not consist of all white vertices.



Discovery time of z and w respectively

Applications

- DFS provides key information about the structure of the graph, e.g. if the graph is cyclic in nature.
- DFS can also be used to identify connected components in a graph.
- DFS is used in topological sort and topological sort is used to resolve dependencies in complex environments.
- It can also be used to solve mazes.



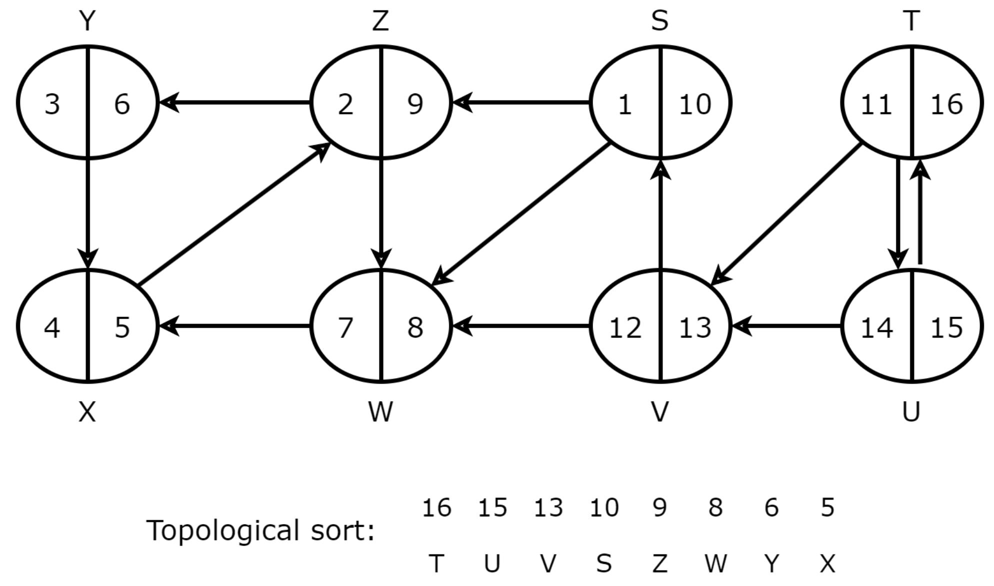
Maze solving

Topological sort

- What is topological sort?
- Algorithm
- Analysis
- Applications

What is topological sort?

- Topological sort is a way to order the vertices of a directed acyclic graph such that for any edge $a \rightarrow b$, a always appears before b in the ordering.
- A directed acyclic graph or DAG is a graph with directed edges and no loops.
- Topological sort is only possible for DAGs because no linear ordering is possible for graphs that contain a loop.



Topological sort

Algorithm

```
function TOPOLOGICAL_SORT(graph):
    stack = []
    timestamp = 0
    # whiten all vertices
    for v in graph.vertices:
        v.color = WHITE
    # visit every vertex
    for v in graph.vertices:
        if v.color == WHITE:
            TOPOLOGICAL_VISIT(v, stack)
```

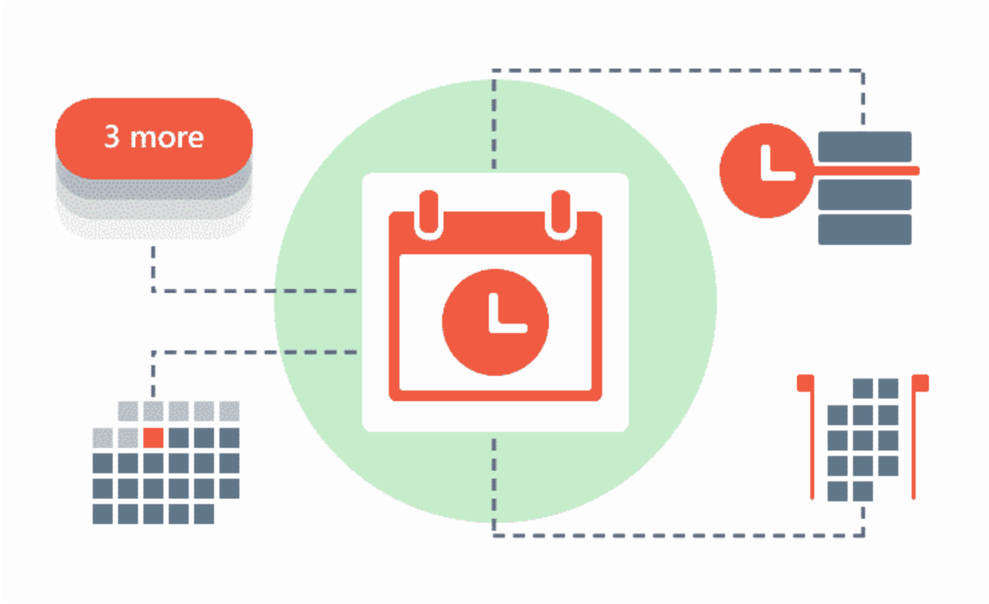
```
function TOPOLOGICAL_VISIT(vertex, stack):
    # increase timestamp
    timestamp += 1
    vertex.color = GRAY
    for v in vertex.adjacent:
        if v.color == WHITE:
            TOPOLOGICAL_VISIT(v)
    timestamp += 1
    vertex.finished = timestamp
    stack.push_front(vertex)
    vertex.color = BLACK
```

Analysis

- Topological sort either uses a slightly modified depth first search algorithm or just uses DFS algorithm and then sorts the vertices in decreasing order of finish timestamps.
- Thus the time complexity of topological sort is the same as that of DFS.
- Time complexity of topological sort: $O(V + E)$

Applications

- Software compilation: Compilers use topological sort to find the order in which to compile source files according to their dependencies.
- Task scheduling: Similarly, CPU schedulers also use this sorting technique to find the order of tasks according to their dependencies.
- Project planning: It can be also useful while planning a complex projects.



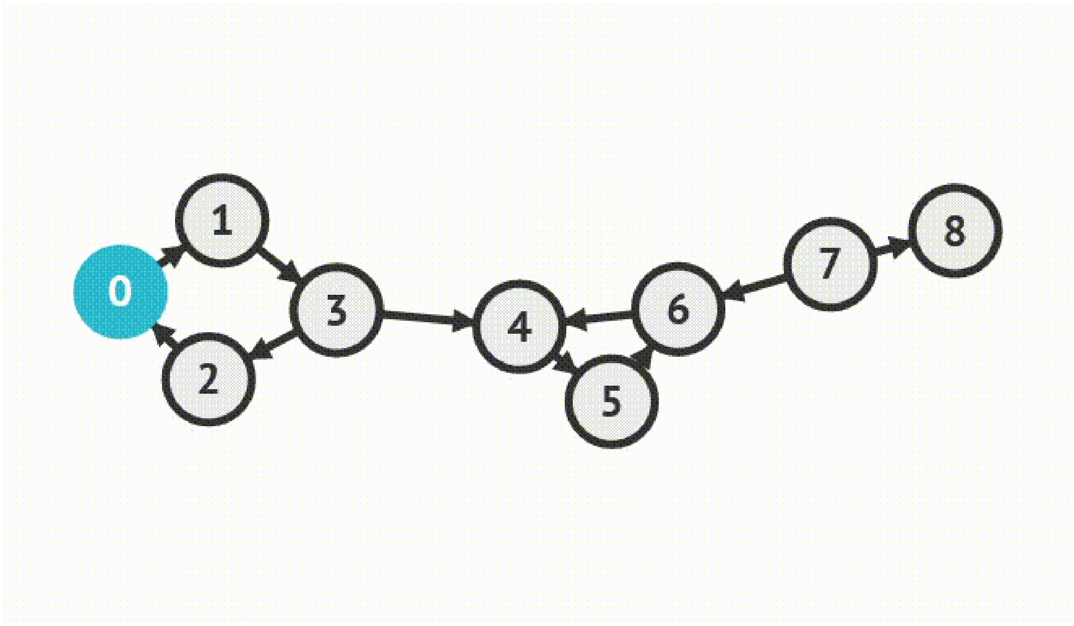
Scheduling

Strongly connected components

- What is a strongly connected component?
- Algorithm
- Analysis

What is strongly connected component?

- A strongly connected component or SCC of a directed graph is a maximal set of vertices where each vertex is connected to every other vertex.
- SCCs can be used to divide a complex graph into multiple sub-graphs which can be then worked upon individually.
- After the individual operations are complete, they can be then merged together.



Kosaraju's algorithm

Algorithm

```
function STRONGLY_CONNECTED_COMPONENTS(graph):
    DFS(graph)
    TOPOLOGICAL_SORT(graph)
    transpose = TRANSPOSE(graph)
    # each tree in the forest is a strongly
    # connected component
    DFS(transpose)
    forest = transpose.forest
    return TRANSPOSE(forest)
```

```
function TRANSPOSE(graph):
    transpose = Graph()
    # reverse every edge direction
    for v in graph.vertices:
        for u in v.adjacent:
            graph.vertex[u] = v
    return transpose
```

Analysis

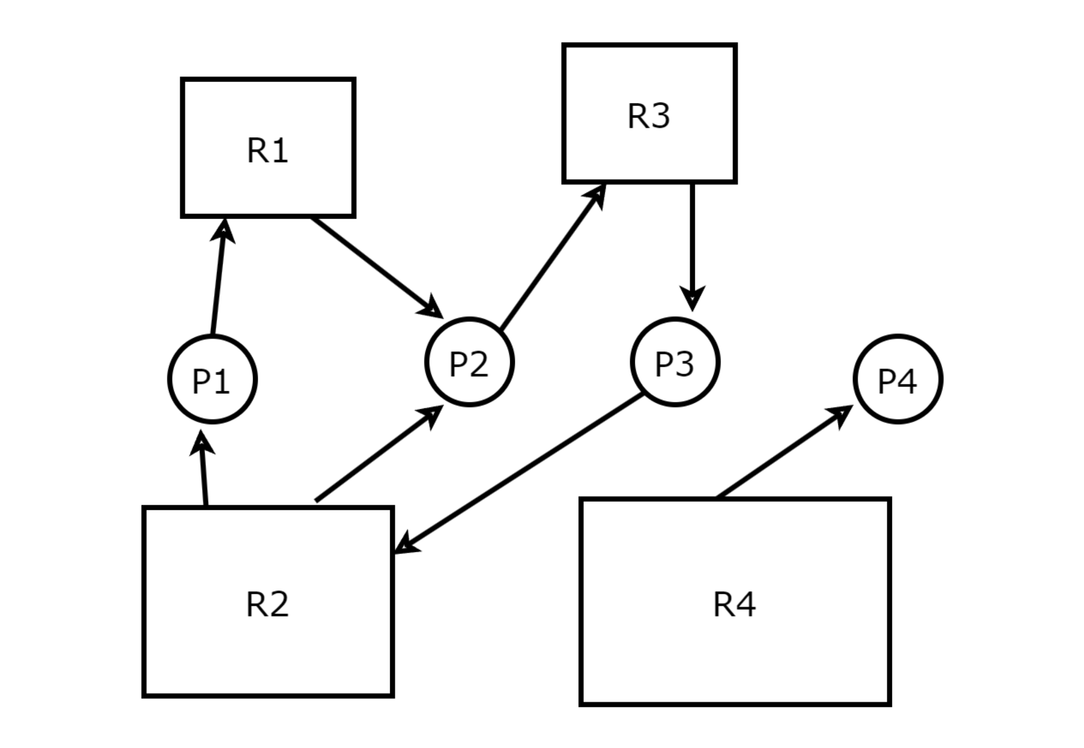
- Time complexity of DFS is $O(V + E)$
- During transpose, outer for loop runs V times and inner for loop cumulatively runs E times. Thus time complexity of transpose is also $O(V + E)$
- DFS on transpose graph is again $O(V + E)$
- Time complexity of finding strongly connected components is $O(V + E)$

Conclusion

- Applications of graph: I
- Applications of graph: II

Applications of graph: I

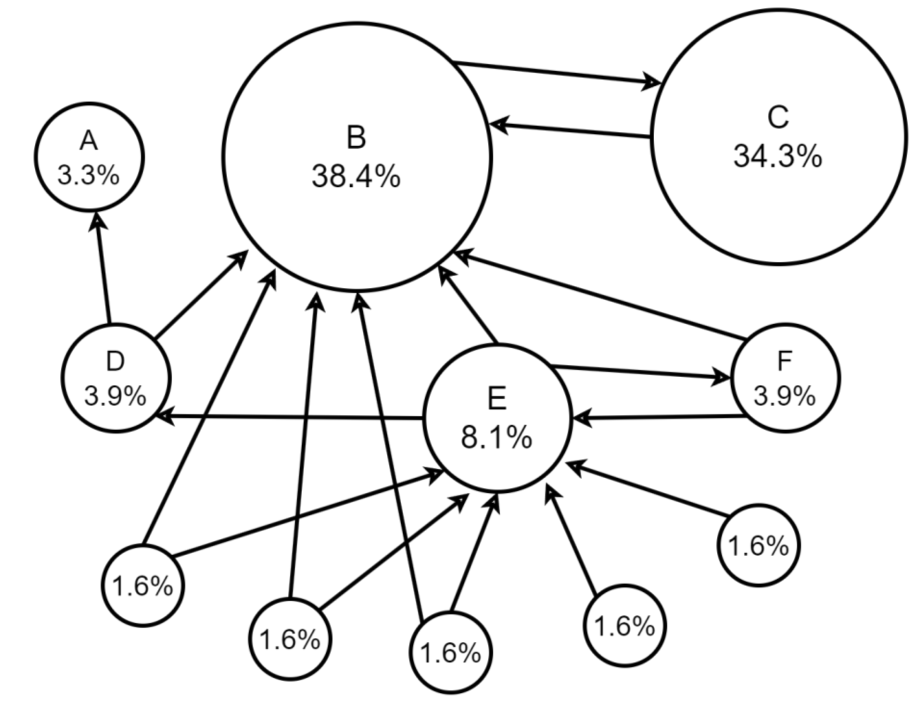
- Maps: Locations in a map are vertices and the paths connecting them are edges.
- Social media: All the users are vertices and if two users are friends there is an edge between them.
- Resource allocation graphs: In operating systems, processes and resources are vertices and an edge represents if an user is using a resource.



Resource allocation graph

Applications of graph: II

- Web: All the pages are vertices and if two pages are linked an edge is created between the two pages.
- Google uses graph algorithms to find the popularity of webpages. The popularity of a page is decided on the basis of the number and popularity of the pages which have links to it.
- Graphs are also used to solve many real life problems, like path finding and scheduling.



Google's Page Ranking algorithm

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