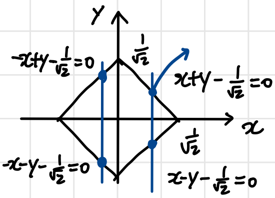
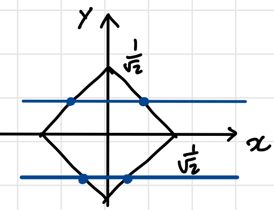
#1) $P(x,y) = \begin{cases} 1 & \left(|x|+|y| \le \frac{1}{\sqrt{2}}\right) \\ 0 & (\text{others}) \end{cases}$



$$P_X(x) = \int_{-\infty}^{\infty} p(x,y)\,dy = \begin{cases} \displaystyle\int_{x-\frac{1}{\sqrt{2}}}^{-x+\frac{1}{\sqrt{2}}} 1\,dy = -2x+\sqrt{2} & \left(0 \le x \le \frac{1}{\sqrt{2}}\right) \\ \displaystyle\int_{-x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} 1\,dy = 2x+\sqrt{2} & \left(-\frac{1}{\sqrt{2}} \le x \le 0\right) \end{cases}$$



$$P_Y(y) = \int_{-\infty}^{\infty} p(x,y)\,dx = \begin{cases} \displaystyle\int_{y-\frac{1}{\sqrt{2}}}^{-y+\frac{1}{\sqrt{2}}} 1\,dx = -2y+\sqrt{2} & \left(0 \le y \le \frac{1}{\sqrt{2}}\right) \\ \displaystyle\int_{-y-\frac{1}{\sqrt{2}}}^{y+\frac{1}{\sqrt{2}}} 1\,dx = 2y+\sqrt{2} & \left(-\frac{1}{\sqrt{2}} \le y \le 0\right) \end{cases}$$

$0 \le x \le \frac{1}{\sqrt{2}},\ 0 \le y \le \frac{1}{\sqrt{2}}$ 일 때 $P_X(x)P_Y(y) = (-2x+\sqrt{2})(-2y+\sqrt{2})$

$\ne P_{X,Y}(x,y)$

이므로 독립이 아니다!

$$E(X) = \int_0^{\frac{1}{\sqrt{2}}} -2x^2+\sqrt{2}x \, dx + \int_{-\frac{1}{\sqrt{2}}}^{0} (2x^2+\sqrt{2}x)\, dx$$

$$= \left[-\frac{2}{3}x^3+\frac{\sqrt{2}}{2}x^2\right]_0^{\frac{1}{\sqrt{2}}} + \left[\frac{2}{3}x^3+\frac{\sqrt{2}}{2}x^2\right]_{-\frac{1}{\sqrt{2}}}^{0}$$

$$= \left(-\frac{1}{3\sqrt{2}}+\frac{\sqrt{2}}{4}\right) + \left(\frac{1}{3\sqrt{2}}-\frac{\sqrt{2}}{4}\right) = 0$$

$$E(Y) = \int_0^{\frac{1}{\sqrt{2}}} (-2y^2+\sqrt{2}y)\, dy + \int_{-\frac{1}{\sqrt{2}}}^{0} (2y^2+\sqrt{2}y)\, dy$$

$$= 0$$

$$\mathrm{COV}(X, Y) = E((X-0)(Y-0))$$

$$= E(XY)$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \int_{x-\frac{1}{\sqrt{2}}}^{-x+\frac{1}{\sqrt{2}}} xy \cdot 1 \, dy dx + \int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} xy \cdot 1 \, dy dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x \left[\frac{1}{2}y^2\right]_{x-\frac{1}{\sqrt{2}}}^{-x+\frac{1}{\sqrt{2}}} dx + \int_{-\frac{1}{\sqrt{2}}}^{0} x \left[\frac{1}{2}y^2\right]_{-x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} dx$$

$$= 0$$

공분산이 0이어도, 즉 두 확률변수가 상관없어도 독립은 아닐수 있다

#2. $d \times d$ 행렬 A, 확률변수 $X=(X_1, X_2, \cdots, X_d)^T$ 에 대해

#2.1 $E(AX) = AE(X)$

sol) 나의 풀이

$$E(AX) = E\left(\begin{pmatrix} \sum_{j=1}^{d} a_{1j} X_j \\ \sum_{j=1}^{d} a_{2j} X_j \\ \vdots \\ \sum_{j=1}^{d} a_{dj} X_j \end{pmatrix}\right) = \begin{pmatrix} E\left(\sum_{j=1}^{d} a_{1j} X_j\right) \\ E\left(\sum_{j=1}^{d} a_{2j} X_j\right) \\ \vdots \\ E\left(\sum_{j=1}^{d} a_{dj} X_j\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{d} a_{1j} E(X_j) \\ \sum_{j=1}^{d} a_{2j} E(X_j) \\ \vdots \\ \sum_{j=1}^{d} a_{dj} E(X_j) \end{pmatrix} \xrightarrow{\text{전개}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ & \vdots & & \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{pmatrix} \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_d) \end{pmatrix} = AE(X)$$

#2.1 $E(AX) = AE(X)$

sol2) 교수님 언급하신 풀이

$$E(AX) = E\begin{pmatrix}\begin{pmatrix}\sum_{j=1}^{d} a_{1j}X_j \\ \sum_{j=1}^{d} a_{2j}X_j \\ \vdots \\ \sum_{j=1}^{d} a_{dj}X_j\end{pmatrix}\end{pmatrix} = \begin{pmatrix} E(\sum_{j=1}^{d} a_{1j}X_j) \\ E(\sum_{j=1}^{d} a_{2j}X_j) \\ \vdots \\ E(\sum_{j=1}^{d} a_{dj}X_j)\end{pmatrix}$$

$$= \begin{pmatrix} \int\cdots\int \sum_{j=1}^{d} a_{1j}x_j\, p(x_1, x_2, \cdots, x_d)\, dx_1 \cdots dx_d \\ \int\cdots\int \sum_{j=1}^{d} a_{2j}x_j\, p(x_1, x_2, \cdots, x_d)\, dx_1 \cdots dx_d \\ \vdots \\ \int\cdots\int \sum_{j=1}^{d} a_{dj}x_j\, p(x_1, x_2, \cdots, x_d)\, dx_1 \cdots dx_d \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{d} a_{1j} \int\cdots\int x_j\, p(x_1, x_2, \cdots, x_d)\, dx_1 \cdots dx_n \\ \sum_{j=1}^{d} a_{2j} \int\cdots\int x_j\, p(x_1, x_2, \cdots, x_d)\, dx_1 \cdots dx_n \\ \\ \sum_{j=1}^{d} a_{dj} \int\cdots\int x_j\, p(x_1, x_2, \cdots, x_d)\, dx_1 \cdots dx_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{dj} \end{pmatrix} \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_d) \end{pmatrix} = AE(X)$$

cf)

$$E_{X_1}(x_1) = \int\cdots\int x_1\, p(x_1, x_2, \cdots, x_d)\, dx_1 \cdots dx_d$$

$$= \int x_1 \underbrace{\int\cdots\int p(x_1, x_2, \cdots, x_d)\, dx_2 \cdots dx_d}_{p_{X_1}(x_1)}\, dx_1$$

$$= \int x_1\, p_{X_1}(x_1)\, dx_1$$

#2.2 $V(AX) = AV(X)A^T$

$$V(AX) = E\left(\left(AX - E(AX)\right)\left(AX - E(AX)\right)^T\right)$$

$$= E\left(\left(AX - AE(X)\right)\left(AX - AE(X)\right)^T\right)$$

$$= E\left(A\left(X - E(X)\right)\left(X - E(X)\right)^T A^T\right)$$

$$= A\,E\left(\left(X - E(X)\right)\left(X - E(X)\right)^T\right) A^T$$

$$= AV(X)A^T$$