# Computing Intersections of Integral Cones (Version 1)

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## 1 Cones

Let $A$ be a finite rank free abelian group (e.g., the Grothendieck group of projective modules over an Artinian ring), and let $C \subset A$ be a subset closed under addition and scalar multiplication by elements of $\mathbb{Z}^{\geq 0}$ (e.g., the set of elements of a Grothendieck group which are the class of some module). We say that $C$ is a **cone**.

We say that $C$ is **full** if, for any choice of $x \in A$ and $n \in \mathbb{Z}^{>0}$, we have $x \in C$ if and only if $nx \in C$. In other words, $C$ is precisely the intersection of $\mathbb{Q} \otimes_{\mathbb{Z}} C$ with the lattice $A \hookrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q}$.

## 2 Grothendieck Groups of Symmetric groups and related Hecke algebras

Here I'll describe the main abstract theorem which motivates me to do these computations.

Fix $p$ a prime number, our characteristic of interest, and $n$ corresponding to $\mathfrak{S}_n$, the symmetric group on $n$ letters. We consider various values of $e = p^i$ at once. For each $e$, let $\zeta_e$ denote a primitive $e^{\text{th}}$ root of unity.

Recall the **Grothendieck group** $K_0(A\text{-proj})$ for $A$ an Artinian ring is the free abelian group generated by indecomposable projective $A$-modules. It also has a more abstract characterization utilizing all finitely generated projective modules modulo relations induced by exact sequences. The Grothendieck group contains a cone of $\mathbb{Z}^{\geq 0}$-linear combinations of indecomposable projective modules, aka the cone of modules.

The algebras of interest are the group algebras $\mathbb{Q}\mathfrak{S}_n$ and $\mathbb{F}_p\mathfrak{S}_n$, along with the **Hecke algebras** $FH_q(S_n)$ for $F$ a Cyclotomic field and $q \in F$, given by generators $T_1, \ldots, T_{n-1}$ and relations

$$(T_i - q)(T_i + 1) = 0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

and

$$T_i T_j = T_j T_i \text{ when } |i - j| > 1.$$

**Theorem 2.1** *There are canonical embeddings (coming from the $e$ parts of analogues of the Brauer-Nesbitt CDE triangle)*

$$\begin{tikzcd}
 & K_0\left(\mathbb{Q}(\zeta_p)H_{\zeta_p}(S_n)\text{-proj}\right) \arrow[dr, hook] & \\
K_0(\mathbb{F}_pS_n\text{-proj}) \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] \arrow[ddr, hook] & K_0\left(\mathbb{Q}(\zeta_{p^2})H_{\zeta_{p^2}}(S_n)\text{-proj}\right) \arrow[r, hook] & K_0(\mathbb{Q}S_n\text{-mod}) \\
 & K_0\left(\mathbb{Q}(\zeta_{p^3})H_{\zeta_{p^3}}(S_n)\text{-proj}\right) \arrow[ur, hook] & \\
 & \vdots \arrow[uur, hook] &
\end{tikzcd}$$

*Each embedding gives an embedding of cones of modules.*

Since the representation theory of $\mathbb{Q}\mathfrak{S}_n$ is well-understood in terms of Young Diagrams, $K_0(\mathbb{Q}S_n\text{-mod})$ is easily described as a free abelian group on partitions of $n$. Thus in theory these embeddings give the Grothendieck groups and their cones very explicit descriptions. In fact, we have ready-made combinatorial descriptions for the embeddings of $K_0\left(\mathbb{Q}(\zeta_e)H_{\zeta_e}(S_n)\text{-proj}\right)$ into $K_0(\mathbb{Q}S_n\text{-mod})$ for each $e$, including their cones of modules.

Moreover, for $e = p$ the embedding

$$K_0(\mathbb{F}_pS_n\text{-proj}) \hookrightarrow K_0\left(\mathbb{Q}(\zeta_p)H_{\zeta_p}(S_n)\text{-proj}\right)$$

defines an isomorphism of abelian groups. It does *not*, however, define an isomorphism of cones in general. The other embeddings can be seen to provide obstructions to an equality of cones here. Thus, the cone of modules inside $K_0(\mathbb{F}_pS_n\text{-proj})$ is contained in an intersection of cones of the corresponding hecke algebras. Our goal here is to compute those intersections.

For technical reasons, knowing the cone of modules is equivalent to the problem of knowing the **decomposition numbers** of $\mathbb{F}_p\mathfrak{S}_n$, which is the major unsolved problem of modular representation theory of symmetric groups. In fact, since the decomposition matrices are unitriangular, it is easy to show that all of these cones are full, and therefore the intersection of cones is full.

# 3 Computing Intersections of Cones

Here I describe how I convert the question of intersecting cones into an algorithm. The basic idea is pretty simple: if $C \subset A$ is a cone generated by a subset of a basis for $A$ (as is the case for the Hecke algebra projective Grothendieck groups, due to the triangular structure of their decomposition matrices), then $C$ is characterized by its dual

$$C^* := \{f \in A^* \mid f(x) \geq 0 \forall x \in C\},$$

i.e., $C$ is its own double dual. Taking an intersection of these cones then corresponds to taking a sum of their duals. Since taking a sum of cones given by generators corresponds simply to extending the list of generators, it is easy to compute a sum of cones. Then we throw out the decomposables in our lists of generators and dualize again.

With that in mind, we establish notation:

Fix $p$ a prime number, our characteristic of interest, and $n$ corresponding to $\mathfrak{S}_n$. We consider various values of $e = p^i$ at once. For each $e$, let $\zeta_e$ denote a primitive $e^{\text{th}}$ root of unity. Let $k$ denote the number of irreducible modules over $\mathbb{Q}\mathfrak{S}_n$ and, for each $e$, $\ell_e$ denote the number of projective indecomposable modules for $\mathbb{Q}(\zeta_e)H_{\zeta_e}(S_n)$, i.e., $k$ is the number of partitions of $n$ and $\ell_e$ is the number of $e$-regular partitions of $n$.

Let $P_e$ denote the $k \times \ell_e$ decomposition matrix of $\mathbb{Q}\mathfrak{S}_n$ over $\mathbb{Q}(\zeta_e)H_{\zeta_e}(S_n)$, which we conceptualize as a list of columns representing indecomposable modules for $\mathbb{Q}(\zeta_e)H_{\zeta_e}(S_n)$ under the embeddings of section 2. We let $B_e$ denote an arbitrary choice of a $k \times (k - \ell_e)$ matrix which extends $P_e$ to a basis, i.e., so that the block matrix

$$\left( \begin{array}{c|c} P_e & B_e \end{array} \right)$$

is an invertible integer $k \times k$ matrix. One obvious way to do this is to have the columns of $B_e$ consist of the $e$-singular partitions of $n$.

The cone of projective modules under this picture is the $\mathbb{Z}^{\geq 0}$-span of the columns of $P_e$. The dual to this cone can be computed by taking the inverse of the block matrix,

$$\left( \frac{Q_e}{B'_e} \right) := \left( \begin{array}{c|c} P_e & B_e \end{array} \right)^{-1}$$

where we take $Q_e$ to be a $\ell_e \times k$ matrix. The dual cone is then the $\mathbb{Z}^{\geq 0}$-span of the rows of $Q_e$ taken together with the $\mathbb{Z}$-span of the rows of $B'_e$.

Due to the nature of the embeddings of Theorem 2.1, one can see that the rows of $B'_e$ must be inside the $\mathbb{Z}$-span of the rows of $B'_p$. Therefore when we wish to dualize again, we can simply copy down $B'_p$ for the bottom half.

**This next part is the naïve method:**

And now we have one more trick: in order to get a canonical set of generators for the "pointed part" of the sum of dual cones, we project the rows of all $Q_e$ onto the subspace spanned by the rows of $Q_p$, along the subspace spanned by the rows of $B'_p$. The projection matrix corresponding to this operation comes from a change of basis of a standard projection matrix, and we've already committed to computing the inverse of the change-of-basis matrices, so it is given by the formula

$$\pi := \left( \begin{array}{c|c} P_p & B_p \end{array} \right) \left( \begin{array}{c|c} I_{\ell_p} & 0 \\ \hline 0 & 0 \end{array} \right) \left( \frac{Q_p}{B'_p} \right).$$

Thus a full list of generators for the dual cone is given by the rows of $Q_e\pi$ for each $e = p^i$, along with $\pm$ the rows of $B'_p$. As a sanity check, we should probably make sure $B'_e\pi = 0$ within our algorithm, as we expect to be the case.

Then we throw out the extraneous generators: first eliminate duplicates, and then search for decomposables. Once we have the list of indecomposable generators, we can then dualize again to get the intersection of cones.

It should be noted that there are two issues with this idea:

(1) There is not a clear algorithm for throwing out decomposables of a cone given by generators if we don't have a terminating procedure. We need to contain the sum of dual cones inside a half-plane in order to apply an iterative procedure.

(2) If the indecomposable elements of the sum of cones do not extend the rows of $B'_p$ to a basis, then I don't know how to compute their dual.

**More recent method:**

Once we have $Q_e$, the way to project the rows of $Q_e$ onto the subspace generated by the rows of $Q_p$ is to right-multiply by $P_p$. This brings us into the lowest dimensional vector space $K_0\left(\mathbb{Q}(\zeta_p)H_{\zeta_p}(S_n)\text{-proj}\right)$, which helps to optimize computations quite dramatically. Together with the observation that the cones are full, and therefore we don't need to dogmatically operate over $\mathbb{Z}$ in order to know what the $\mathbb{Z}$-lattice cones are, and this becomes a literal linear programming problem, which sage can solve beautifully.