Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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Chapter 1

Introduction

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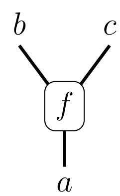
Chapter 2

Background

2.1 Drawing Monoidal Categories

A monoidal category $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold[1]. We assume that monoidal categories are strict, since all monoidal categories are monoidally equivalent to a strict one[2].

The morphisms of $\mathcal{C}$ can be drawn as string diagrams, where the morphism maps from the bottom to the top. Functions that make up the morphism are drawn as tokens or boxes. For example



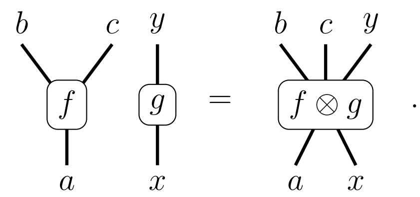
depicts a morphism $f : a \to b \otimes c$. For identity morphisms we drop the box and only draw a vertical line, so id_a is the diagram

.

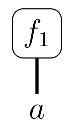
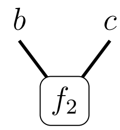
The tensor product of morphisms is the horizontal concatenation of diagrams, such that strings from separate functions don't interact. For example, given $g : x \to y$, the tensor product $f \otimes g : a \otimes x \to b \otimes c \otimes y$ is drawn as

[1] For more details see [Eti+15].

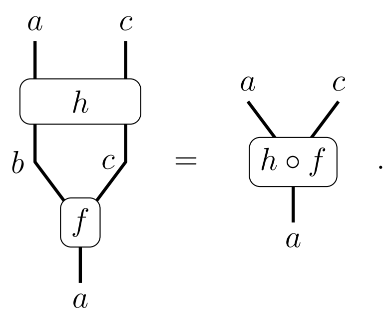
[2] See [ML98, VII.2] or [Eti+15, Thm 2.8.5]



By convention, $\mathbb{1}$ is blank and unlabelled, and strings that would join to $\mathbb{1}$ are blank. Particularly, $\mathrm{id}_{\mathbb{1}}$ is an empty diagram, and we have diagrams such as

 $: a \to \mathbb{1}$ and  $: \mathbb{1} \to b \otimes c.$

The compositions of morphisms is the vertical stacking of diagrams where domains and codomains match. For example, the composition $h \circ f : a \to b \otimes c \to a \otimes c$ of $f : a \to b \otimes c$ with $h : b \otimes c \to a \otimes c$ has the diagram



Before looking at our main example of a diagrammatic monoidal category, we first define some terminology.

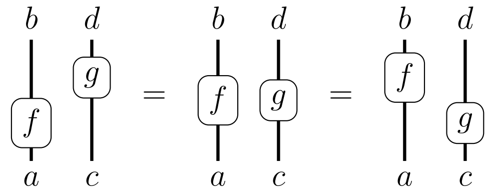
Definition 2.1.1. For a commutative ring R, an *R-linear category* is a category enriched over the category of R-modules. That is, for objects a, b, the set of morphisms $\mathrm{Hom}(a, b)$ is an R-module and the composition of morphisms is R-bilinear.

Example 2.1.2. Let $\Bbbk$ be a field. The category of vector spaces over $\Bbbk$, $\mathbf{Vect}_{\Bbbk}$, is a $\Bbbk$-linear category. This makes sense by the classical theory of linear algebra.

For a strict R-linear monoidal category $\mathcal{C}$, the bifunctoriality of $- \otimes -$ implies the following *interchange law*. For morphisms $f : a \to b$ and $g : c \to d$, $(\mathrm{id}_b \otimes g) \circ (f \otimes \mathrm{id}_c) = f \otimes g = (f \otimes \mathrm{id}_d) \circ (\mathrm{id}_a \otimes g)$. In other words the following diagram commutes.

$$\begin{tikzcd}
a \otimes c \arrow[r, "f\otimes \mathrm{id}_c"] \arrow[d, "\mathrm{id}_a \otimes g"'] \arrow[dr, "f\otimes g"] & b \otimes c \arrow[d, "\mathrm{id}_b \otimes g"] \\
a \otimes d \arrow[r, "f\otimes \mathrm{id}_d"] & b \otimes d
\end{tikzcd}$$

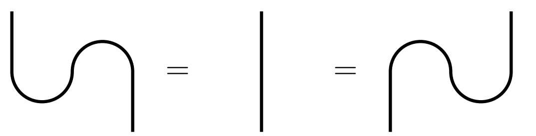
Written with string diagrams, this is



which holds up to deformation of the diagram.

Definition 2.1.3. A monoidal category $\mathcal{C}$ is *generated* by finite set S_o of objects and S_m of morphisms, when all non-unit objects are a finite tensor of objects in S_o and all non-identity morphisms are a finite combination of tensors and compositions of morphisms in S_m.

Example 2.1.4. Our first example of a diagrammatic monoidal category is the *Temperley-Lieb category.* The Temperley-Lieb category $\mathcal{TL}$ is a strict R-linear monoidal category whose objects are generated by the vertical line I and morphisms generated by the cup $\cup : \mathbb{1} \to \mathsf{I} \otimes \mathsf{I}$ and cap $\cap : \mathsf{I} \otimes \mathsf{I} \to \mathbb{1}$, with relations



Mention that composition and tensor product is as explained above
Some example
Mention bubbles and specialisation to some $\delta \in R$
Mention that these are crossingless matchings
Comment on isotopy

Chapter 3

One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

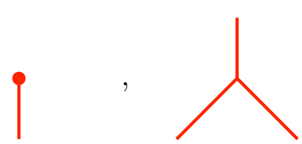
The first one-colour diagrammatic we explore is the *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = e \rangle$. At the end of this section, we see that this diagrammatic category is equivalent to the category of Soergel Bimodules under additive Karoubian closure.

Remark 3.1.1. All diagrammatics below and in Chapter 4 can be defined in the language of planar algebras, without the additional structure of categories, e.g. in [Jon21]. Nevertheless, we define them in the context of categories as we will see them as diagrammatic versions of important categories in representation theory.

Definition 3.1.2. The *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ is a $\mathbb{C}$-linear monoidal category with the following presentation.

The objects are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is concatenation. The empty tensor product, i.e. the monoidal identity, will be denoted $\varnothing =: s^0$.

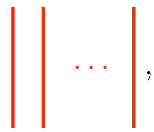
The morphisms are generated, up to isotopy, by univalent and trivalent vertices



(3.1.3)

that are maps $s \to \varnothing$ and $ss \to s$ respectively. Note that we put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). The free $\mathbb{C}$-module structure on each morphism space $\mathrm{Hom}(s^n, s^m)$ produces $\mathbb{C}$-linear combinations of such diagrams. Something about composition/tensor and addition commuting Then, composition or tensors with the zero morphism 0 result in 0. To abuse notation, the empty diagram

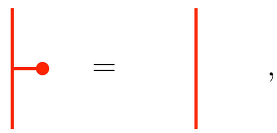
$\varnothing \to \varnothing$ will be denoted $\varnothing$. The identity morphism in $\mathrm{Hom}(s^n, s^n)$ is the diagram consisting of n (red) vertical lines



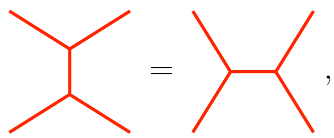
(3.1.4)

which we may identify with s^n.

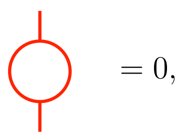
Such diagrams are subject to the following local relations



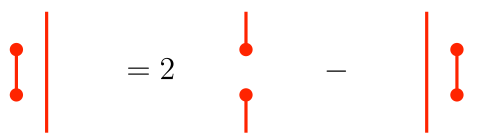
(3.1.5a)



(3.1.5b)



(3.1.5c)

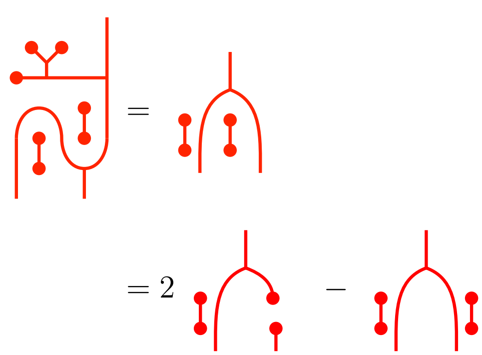


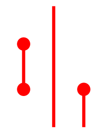
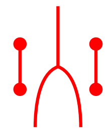
(3.1.5d)

Remark 3.1.6. The object s is a Frobenius object in $\mathcal{H}(S_2)$. The generators (3.1.3) and their horizontal reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.5a) and (3.1.5b).

Put a definition of frob object in intro

Example 3.1.7. Using the relations in (3.1.5) we can simplify the morphism in $\mathrm{Hom}(ss, s)$,



$= 2$  $-$  .

Add example of using frob associativity

The morphism space $\mathrm{Hom}(s^n, s^m)$ has a left (or right) $\mathbb{C}[$$]$-basis called the *double leaves* basis, as described in [EW16]. To define this basis, we must first define morphisms known as *light leaves*.

To make use of the group structure of S_2, we need to translate between words in $\mathcal{H}(S_2)$ and elements in S_2. Let $\phi : (\mathrm{ob}(\mathcal{H}(S_2)), \otimes) \to (S_2, *)$ be the monoid homomorphism[1] mapping $s \mapsto s$ and $\varnothing \mapsto 1$, and $\psi : S_2 \to \mathrm{ob}(\mathcal{H}(S_2))$ be the function that maps $s \mapsto s$ and $1 \mapsto \varnothing$. Should this be a definition? The maps ϕ allows words $w = s^n$ to be seen as elements of S_2, and ψ allows $1, s \in S_2$ to be seen as the objects $\varnothing, s \in \mathcal{H}(S_2)$. Clearly, $\phi\psi$ is the identity map on S_2, and the map $\psi\phi : \mathcal{H}(S_2) \to \mathcal{H}(S_2)$ takes objects to one of $\varnothing$ or s in $\mathcal{H}(S_2)$ by considering them as elements in S_2.

Definition 3.1.8. (Subexpression for S_2) Given a word $w = s^n$, a *subexpression* e is a binary string of length n. We can *apply* a subexpression to produce an object $w(e) \in \mathcal{H}(S_2)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $0 \le i \le n$, write $w(e, i)$ for the resultant object of the first i terms in e applied to the first i terms in w. Particularly $w(e, 0) = \varnothing$ and $w(e, n) = w(e)$.

For example, 0000, 0110 and 1011 are subexpressions of $s^4 = ssss$. Applying the third subexpression gives $ssss(1011) = s\varnothing ss = sss$ and $ssss(1011, 3) = sss(101) = s\varnothing s = \varnothing$, by strictness of the monoidal category. Here, each term of the subexpression is a decision to include or exclude the corresponding s in the word, where excluding an s amounts to tensoring with $\varnothing$.

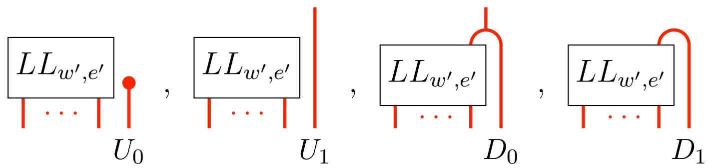
For a word w and subexpression e, we label each term by U_0, U_1, D_0 or D_1. The i-th term is labelled U_* if $\phi(w(e, i-1)) = 1 \in S_2$, and labelled D_* if $\phi(w(e, i-1)) = s \in S_2$. The label's subscript is the corresponding term in e.

Example 3.1.9. For the object $w = ssss$ and subexpression $e = 0101$, we find the labels as recorded in the following table.

Term i	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
$w(e,i)$	$\varnothing$	$\varnothing s = s$	$\varnothing s \varnothing = s$	$\varnothing s \varnothing s = ss$
Labels	U_0	U_0U_1	$U_0U_1D_0$	$U_0U_1D_0D_1$

Definition 3.1.10. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, the light leaf $LL_{w's,e'i}$ is one of

[1] A map that preserves the monoidal product and identity element.

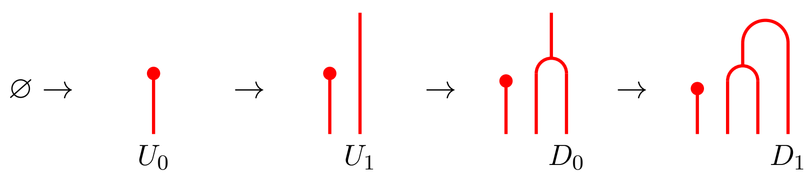


(3.1.11)

corresponding to the next label, where w' and e' are appropriate subwords[2] of w and e respectively.

Here, the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ is $\varnothing$, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. This implies that the recursive definition is consistent.

Example 3.1.12. Following from Example 3.1.9 for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.

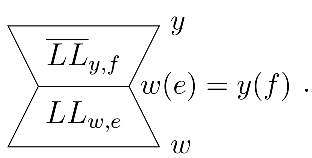


Definition 3.1.13. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y is a composition

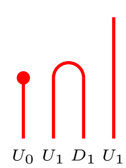
$$\mathbb{LL}_{f,e} \coloneqq \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

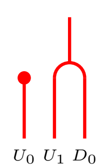
for subexpressions e of w and f of y such that $w(e) = f(y)$.

Visually this looks like a morphism from w to y factoring through $w(e) = y(f) \in \{\varnothing, s\}$,



Example 3.1.14. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

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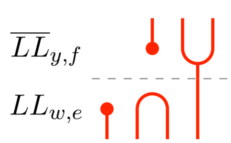
Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$, factoring through s, is



[2] A word with some letters removed.

Note that these double leaves have no floating diagrams such as . In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{C}[\,]$-module, where the (left) -action is left concatenation by . Since we can move barbells to the right via. the relation (3.1.5d) and double leaves cut down the middle are double leaves factoring through $\varnothing$, we can equivalently act by $\mathbb{C}[\,]$ on the right. This leads us to the following theorem.

Theorem 3.1.15 (Elias-Williamson [EW16, Theorem 1.2])**.** *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{LL}(w, y)$ be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $w(e) = y(f)$. Then $\mathbb{LL}(w, y)$ is a basis for* $\mathrm{Hom}(w, y)$ *as a left (or right) $\mathbb{C}[\,]$-module.*

A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.16. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky's construction in [Lib08].

The morphisms in this category can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1. The degree of a general diagram is the sum of the degrees of the generators that appear in it.

How is grading on objects defined? Do we just grade them all to be 0 (like identity map)?

Say more about how this puts a grading on the category.

Put example

The double leaves bases allow us to show that the Karoubi envelope of $\mathcal{H}(S_2)$ is equivalent to the category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$ over S_2 as monoidal categories.

Theorem 3.1.17 (Elias-Williamson [EW16, Theorem 6.30])**.** *The category* $\mathrm{Kar}_{\oplus}(\mathcal{H}(S_2))$ *and the category of Soergel Bimodules* $\mathbb{S}\mathrm{Bim}$ *over S_2 are equivalent as graded $\mathbb{C}$-linear monoidal categories.*

The proof in [EW16] gives an equivalence of graded $\mathbb{C}$-linear monoidal categories $\mathcal{H}(S_2) \cong \mathbb{BS}\mathrm{Bim}$ where $\mathbb{BS}\mathrm{Bim}$ is the category of Bott-Samelson bimodules over S_2. This was done by comparing the graded dimensions of morphism spaces using double leaves bases. Since $\mathrm{Kar}_{\oplus}(\mathbb{BS}\mathrm{Bim}) \cong \mathbb{S}\mathrm{Bim}$ and Karoubi envelope preserves equivalences, we obtain $\mathrm{Kar}_{\oplus}(\mathcal{H}(S_2)) \cong \mathbb{S}\mathrm{Bim}$.

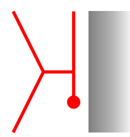
3.2 Diagrammatic $\mathcal{O}_0(\mathfrak{sl}_2)$

A little bit about category $\mathcal{O}$, and our example of $\mathfrak{sl}_2$

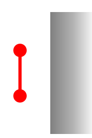
With the diagrammatic category $\mathcal{H}(S_2)$, we can describe diagrammatics for the category $\mathcal{O}_0$ for the Lie algebra $\mathfrak{sl}_2$. In particular, we define module category $\mathcal{DO}_0(\mathfrak{sl}_2)$ with a left-action of $\mathcal{H}(S_2)$. At the end, we give a description of $\mathcal{O}_0(\mathfrak{sl}_2)$ and a proof for a type of equivalence of these categories.

Remark 3.2.1. The following is actually only a diagrammatic description for the projective objects $\mathrm{proj}(\mathcal{O}_0)$ of $\mathcal{O}_0$ and not $\mathcal{O}_0$ itself. We can pass from $\mathrm{proj}(\mathcal{O}_0)$ to $\mathcal{O}_0$ by observing that $K^b(\mathrm{proj}(\mathcal{O}_0))$ is equivalent to $D^b(\mathcal{O}_0)$ as graded $\mathbb{Z}$-linear monoidal triangulated categories. This is a standard trick in the field, see for example the introduction of [RW18][3]. However for our purposes it is not important to understand how this works.

Definition 3.2.2. Let $\mathcal{DO}_0(\mathfrak{sl}_2)$ be the $\mathbb{C}$-linear (Define this in background) left $\mathcal{H}(S_2)$-module category with elements generated (Define what this means.) by the monoidal identity $\varnothing$ of $\mathcal{H}(S_2)$ and morphisms generated by the empty diagram $\varnothing$, where $\mathcal{H}(S_2)$ acts on the left by left concatenation for both objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0,1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0,1] \times \mathbb{R}$. For example a morphism may be

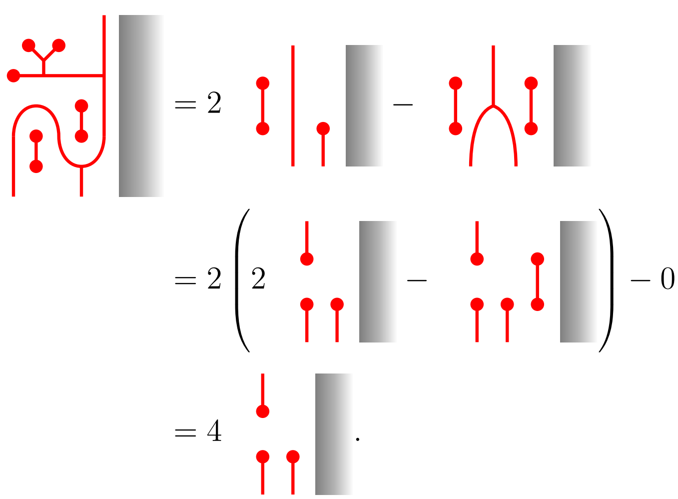
 .

We impose the relation that diagrams are related to the wall by

 $= 0.$ (3.2.3)

In this section we may write $\mathcal{DO}_0$ for this category. Talk about the $\mathbb{C}$-linear structure and how that works.

Example 3.2.4. Using the new relation (3.2.3), we can further simplify the morphism in Example (3.1.7) by



[3] A self-contained summary of how diagrammatic categories can be related to abelian categories.

The objects of this category are identical to objects in $\mathcal{H}(S_2)$ and the morphisms are the same modulo the wall relation (3.2.3). A natural question to ask is whether double leaves still form bases for the morphism spaces here. Notice that double leaves appear in $\mathcal{DO}_0$ by acting on $\varnothing$ by double leaves in $\mathcal{H}(S_2)$. All morphisms in $\mathcal{DO}_0$ are morphisms in $\mathcal{H}(S_2)$ so they can be written as $\mathbb{C}[\,]$-linear combinations of double leaves, though some have collapsed to 0. Thus double leaves span the morphism spaces of $\mathcal{DO}_0$ as (left) $\mathbb{C}[\,]$-modules. However they may not be linearly independent as neither left nor right modules. For example, any pair of double leaves that factor through $\varnothing$ become 0 when multiplied by on either side (by shifting the barbell to the right). Although double leaves are not always a basis for its respective morphism space as $\mathbb{C}[\,]$-modules, it turns out they are a basis over $\mathbb{C}$.

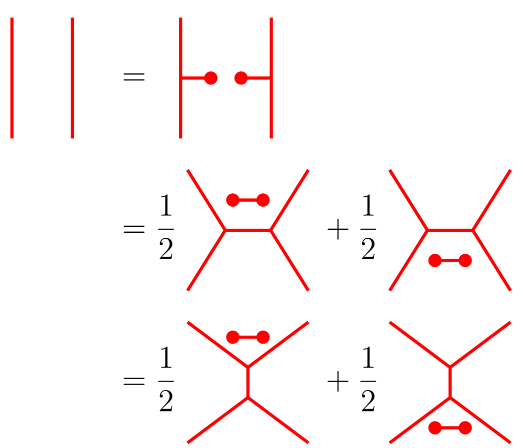
Lemma 3.2.5. *Let $\pi : \mathrm{mor}(\mathcal{H}(S_2)) \to \mathrm{mor}(\mathcal{DO}_0)$ be the projection map which takes a morphism f to the result of its action on $\varnothing$. Then the image $\pi(\mathbb{LL}(w,y))$ is a basis for $\mathrm{Hom}_{\mathcal{DO}_0}(w,y)$ as a $\mathbb{C}$-module.*

Proof. We consider morphisms $\mathrm{Hom}(w,y)$ in $\mathcal{DO}_0$ for fixed objects w, y, and write $\mathbb{LL} := \pi(\mathbb{LL}(w,y))$ for the set of double leaves in $\mathcal{DO}_0$. Any diagram in $\mathcal{DO}_0$ can be written as a $\mathbb{C}$-linear combination of morphisms without floating diagrams by simplifying them to barbells, pulling them to the right and using (3.2.3). The diagrams $\mathbb{LL}$ span morphisms without floating components as a $\mathbb{C}$-module. This can be observed by writing morphisms as a linear combination of double leaves, by (3.1.15) with the right action, then applying π and (3.2.3). Since the barbell-wall relation (3.2.3) has no effect on $\mathbb{C}$-linear combinations of $\mathbb{LL}$, it follows from linear independence over $\mathbb{C}[\,]$ that they are linearly independent over $\mathbb{C}$ in $\mathcal{DO}_0$. Check the proof. □

Maybe put this next bit in section 3.1
Say more about what this is, and why we say it here

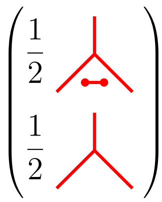
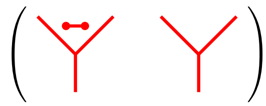
Lemma 3.2.6. *In the additive closure of $\mathcal{H}(S_2)$ we have an isomorphism $s \otimes s \cong s \oplus s$.*

Proof. In $\mathcal{H}(S_2)$ we have the relation



(3.2.7)

This implies we have maps

 $: ss \to s \oplus s$ and  $: s \oplus s \to ss.$

It follows from (3.1.5d), (3.1.5c) and the above calculation (3.2.7), that these maps are inverses. Maybe put the inverse calculation here. □

Before giving the main theorem, (Reword this, this may be wrong) we provide a useful description of $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$. This can be found in general in [Hum08, Sections 3.8–3.10], or in [Maz09, Section 5.2] for the specific case of $\mathfrak{sl}_2$. The main category of interest is $\mathcal{O}$, of modules over semisimple Lie algebras satisfying certain finiteness conditions. The category $\mathcal{O}$ is a direct sum of subcategories, and in the case of $\mathfrak{sl}_2$, all non-trivial summands in this direct sum are equivalent to $\mathcal{O}_0$ as Check: abelian categories. The category $\mathrm{proj}(\mathcal{O}_0)$ is a full subcategory of $\mathcal{O}_0$ containing only projective modules, which is in particular additive and contains all direct summands.

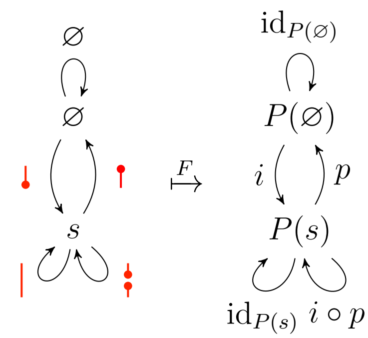
(Reword this) The following result is essentially due to Soergel [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4] (see also [Soe98]) but was not originally formulated as such. The key arguments are in [Soe90] so we attribute this theorem to Soergel.

Theorem 3.2.8 (Soergel, [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4])**.** *The diagrammatic category* $\mathrm{Kar}_{\oplus}(\mathcal{DO}_0(\mathfrak{sl}_2))$ *and* $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$ *are equivalent as* $\mathbb{C}$*-linear* $\mathcal{H}(S_2)$*-module categories.*

Check all of this & Put precise references
Clean up the differences between proj $\mathcal{O}_0$, $\mathcal{O}_0$, $\mathcal{O}$.
Maybe write description as a soergel module outside the proof

Proof. As a shorthand, we write $\mathrm{proj}(\mathcal{O}_0)$ for $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$. The work of Soergel in [Soe90, Section 2.4] shows that $\mathrm{proj}(\mathcal{O}_0)$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_\varnothing, \Theta_s \in \mathrm{End}(\mathcal{O})$ (corresponding to elements in S_2). Explains what this means, how its related to the $\mathcal{H}(S_2)$ module category From classical results e.g. [Maz09, Proposition 5.84 and Lemma 5.87], it follows that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$. Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure, which become the desired properties in $\mathrm{proj}(\mathcal{O}_0)$ when applied to the appropriate module. Additionally, [Soe90, Section 2.4] shows that there is a relation in $\mathcal{O}_0$ analogous to the barbell-wall relation (3.2.3), and [Maz09, Proposition 5.90] shows that there is a natural isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ which is analogous to the isomorphism given by Lemma 3.2.6.

Define the functor $F : \mathcal{DO}_0 \to \mathrm{proj}(\mathcal{O}_0)$ that sends the empty object $\varnothing$ to the trivial module $P(\varnothing)$, and the Soergel module action corresponding to s to the translation functor Θ_s. Then the object s maps to $\Theta_s(P(\varnothing)) =: P(s)$, and s^3 maps to $\Theta_s^3(P(\varnothing)) = \Theta_s\Theta_s\Theta_s(P(\varnothing))$. In order for F to be functorial, it must map identity diagrams $s^n \to s^n$ to $\mathrm{id}_{\Theta_s^n(P(\varnothing))}$. For non-identity maps, we let $F(\ ) = i$ be the inclusion $P(\varnothing) \to P(s)$ and $F(\ ) = p$ be the projection $P(s) \to P(\varnothing)$. Should we talk about the Soergel module action of a diagram The mapping of F is depicted by the following diagram.



Note that the projection and inclusion maps are exactly the unit and counit of Θ_s evaluated at $P(\varnothing)$. This completely determines the image of F by linearity, additivity and the isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ What are the details here?. Applying the natural transformations in the Frobenius object structure for Θ_s to $P(\varnothing)$ give corresponding maps for $P(\varnothing), P(s)$ and $\Theta_s^2(P(\varnothing))$. Along with the analogous barbell-wall relation in $\mathrm{proj}(\mathcal{O}_0)$, we have that F is well defined. Note that by construction F preserves $\mathbb{C}$-linearity and the Soregel module structure in [Soe90].

Now we show that F is fully faithful. It follows from $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ and the description of $P(\varnothing)$ and $P(s)$ in [Maz09, Section 5.2] that the image of and generate all morphisms of the form $\Theta_s^n(P(\varnothing)) \to \Theta_s^m(P(\varnothing))$. Hence F is full. For the faithfulness of F, it suffices Why? to match the dimensions of $\mathbb{C}$-bases for morphism spaces involving $P(\varnothing)$ and $P(s)$. By Lemma 3.2.5, $\mathrm{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_\varnothing\}$, $\mathrm{Hom}(s, \varnothing)$ has a basis $\{\ \}$, $\mathrm{Hom}(\varnothing, s)$ has a basis $\{\ \}$, and $\mathrm{Hom}(s, s)$ has a basis $\{\mathrm{id}_s, \ \circ\ \}$. The dimensions coincide with the corresponding morphism spaces in $\mathrm{proj}(\mathcal{O}_0)$ Ref?. Therefore F is fully faithful.

Since objects in $\mathrm{proj}(\mathcal{O}_0)$ are direct sums and direct summands of the elements $\Theta_s^n(P(\varnothing))$ for non-negative integers n, the additive Karoubi envelope induces the equivalence $\mathrm{Kar}_\oplus(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ as $\mathbb{C}$-linear left $\mathcal{H}(S_2)$-module Should this be $\mathrm{Kar}_\oplus(\mathcal{H}(S_2))$ categories. □

Remark 3.2.9. The category $\mathcal{DO}_0$ is graded by the same grading as $\mathcal{H}(S_2)$ in Section 3.1. The equivalence $\mathrm{Kar}_\oplus(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ includes a grading of $\mathrm{proj}(\mathcal{O}_0)$ and hence a grading of $\mathcal{O}$, which would otherwise be ungraded. Check

Chapter 4

Two-colour Diagrammatics

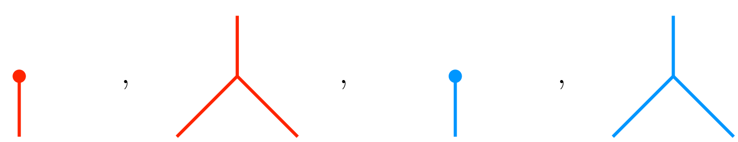
The previous chapter had its focus on the symmetric group generated by one element S_2, which brought about one-colour diagrammatics. We shift our attention to a more complex example by adding an extra generator, that is, another colour. In particular, we consider the case for the affine symmetric group on two elements $\widetilde{S}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$.

4.1 Two-colour Diagrammatic Hecke Category

Corresponding to $\widetilde{S}_2$, we define the two-colour (diagrammatic) Hecke category $\mathcal{H}(\widetilde{S}_2)$. This is a (strict) $\mathbb{C}$-linear monoidal category given by the following isotopy presentation.

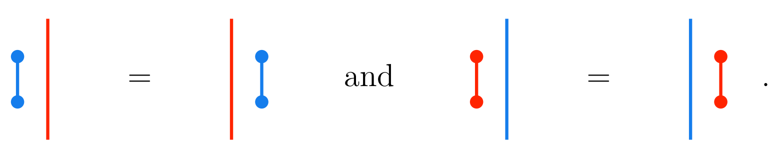
Objects in $\mathcal{H}(\widetilde{S}_2)$ are generated by formal tensor products of the non-identity elements $s, t \in \widetilde{S}_2$. As before, we write objects as words such as $sstttst =: s^2t^3st$ where the tensor product is concatenation, and associate the colour red to s and blue to t. The empty word is the monoidal identity, which we write as $\varnothing$.

The morphisms are generated by the univalent and trivalent vertices



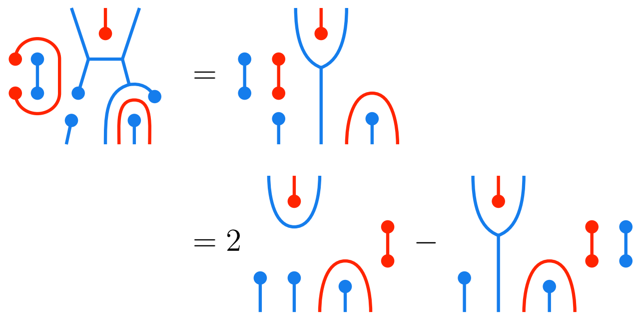
(4.1.1)

that are maps $s \to \varnothing$, $ss \to s$, $t \to \varnothing$ and $tt \to t$ respectively. As in the one-colour case, tensor product is horizontal concatenation, composition is appropriate vertical stacking, and we denote the empty diagram $\varnothing \to \varnothing$ by $\varnothing$. For each colour, these diagrams have the one-colour relations given by (3.1.5). Since we have two colours now, we also need to describe how the colours interact. This is given by the *two-colour* relations



(4.1.2)

Example 4.1.3. The following morphism in Hom$(ttsts, tst)$ can be simplified using the one-colour and two-colour relations.



Remark 4.1.4. Notice that the red and blue lines never cross as no generators that allow crossings. This is a consequence of working over affine S_2 in which the generators s and t have no relation. Mention example of crossing and S_3.

Definition 4.1.5. For a group with a presentation in terms of generators and relations, the *length* of a product of generators is the number of generators in the product. We say that a product of generators is *reduced* if it's length cannot be shortened with relations.

In $\widetilde{S}_2$ products can be shortened by the relation $s^2 = t^2 = 1$. For instance, $sttsts$ is not reduced because it is equal to ts which is reduced. Notice that for $\widetilde{S}_2$ each element can be written uniquely as a reduced product of generators. This is true since otherwise we have two distinct reduced products for the same element in $\widetilde{S}_2$ so they must be related by $s^2 = t^2$. This means they can be reduced further by $s^2 = t^2 = 1$, which contradicts minimality of their length.

Notice that there is a notational similarity between products in the group and words in $\mathcal{H}(\widetilde{S}_2)$. This motivates the following definitions. Let $\phi : (\mathrm{ob}(\mathcal{H}(\widetilde{S}_2)), \otimes) \to (\widetilde{S}_2, *)$ be the monoid homomorphism mapping $\varnothing \mapsto 1$, $s \mapsto s$ and $t \mapsto t$. Also define $\psi : \widetilde{S}_2 \to \mathrm{ob}(\mathcal{H}(\widetilde{S}_2))$ to be the function mapping elements $x \in \widetilde{S}_2$ to the tensor product of s and t in $\mathcal{H}(\widetilde{S}_2)$ corresponding to the reduced product of x in $\widetilde{S}_2$. This is well defined because reduced products are unique and two different reduced products cannot equal the same element of $\widetilde{S}_2$. The composition $\psi\phi : \mathcal{H}(\widetilde{S}_2) \to \mathcal{H}(\widetilde{S}_2)$ maps words w to the tensor of s and t corresponding to the reduced product of $\phi(w)$, and $\phi\psi$ is the identity map on $\widetilde{S}_2$.

The following definition is a more general version of Definition 3.1.8.

Definition 4.1.6 (Subexpression)**.** Given a word w of length n, a *subexpression* e is a binary string of length n. A subexpression can be *applied* to produce an word $w(e)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $1 \leq i \leq n$, we write $w(e, i)$ for the result of the first i terms of e applied to the first i terms in w. Particularly $w(e, 0) = \varnothing$ and $w(e, n) = w(e)$.

For example, in $\mathcal{H}(\widetilde{S}_2)$, if $w = sttts$ and $e = 11001$ then $w(e) = st\varnothing\varnothing s = sts$ and $w(e, 3) = sts(110) = st\varnothing = st$ in $\mathcal{H}(\widetilde{S}_2)$.

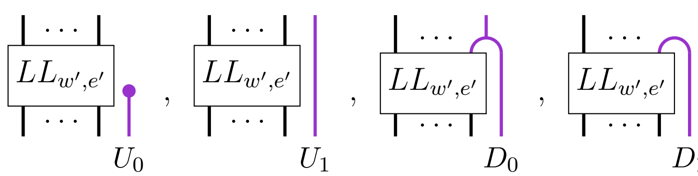
Let the *length* of a word be the number of generators in its tensor product. As before, given an object w and a subexpression e of w, we label each of the n terms by one of U_0, U_1, D_0, D_1. Let $i \geq 0$, and write x for the i-th term of w. We label the i-th term U_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. In other words we write U_* if the next term of w will make $\psi\phi$ applied to the partially evaluated subexpression longer, regardless of the i-term of e. We label D_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. The label's subscript is the i-th term of e. Note that this construction is well defined because $\psi\phi(w(e, i-1) \otimes x) = \psi(\phi(w(e, i-1)) * \phi(x)) = \psi(\phi(w(e, i-1)) * x)$ is always either longer or shorter, since the last element of the reduced product is either the same as x or different. When they are the same, the word is shorter via $s^2 = t^2 = 1$, and when they are different it is longer as no relations can make it shorter.

Remark 4.1.7. This description of the labels (via. reduced products) is more akin to the definition for general Coxeter groups than in Section 3.1.

Example 4.1.8. Consider the word $w = sttts$ and subexpression $e = 11001$. The labels can be constructed as in the following table.

Term i	1	2	3	4	5
Partial w	s	st	stt	$sttt$	$sttts$
Partial e	1	11	110	1100	11001
$w(e,i)$	s	st	$st\varnothing = st$	$st\varnothing\varnothing = st$	$st\varnothing\varnothing s = sts$
Labels	U_1	U_1U_1	$U_1U_1D_0$	$U_1U_1D_0D_0$	$U_1U_1D_0D_0U_1$

Definition 4.1.9. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and a subexpression e is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given appropriate subwords w' and e' of w and e respectively, and if the next terms are x in w and i in e, the light leaf $LL_{w'x,e'i}$ is one of



(4.1.10)

corresponding to the next label. The purple strand represents either red or blue corresponding to whether the next term x is s or t, respectively.

Notice that the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ does not end with the colour corresponding to x, and if the next label is D_* the codomain of $LL_{w',e'}$ ends with a strand with the colour corresponding to x. This implies the recursive definition in the diagram above is consistent. Note that in the case of D_*, one of the black strands in the domain of $LL_{w',e'}$ must have the colour of x in order for the colour to appear in its codomain.

4.2 Diagrammatic Tilt(SL(2))

Blah

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