Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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Introduction

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Background

Notation: we write 1 for the neutral element of a group.

2.1 Coxeter Groups

Definition 2.1.1. A *Coxeter system* (W, S) is a group W and a finite subset $S = \{s_1, ..., s_n\} \subset W$ under the following conditions. For any $s, t \in S$, $(st)^{m_{st}} = 1$ where $m_{st} \in \mathbb{Z}_{>0} \cup \{\infty\}$ such that $m_{st} = 1$ if $s = t$, and $m_{st} = m_{ts} \in \{2, 3, ...\} \cup \{\infty\}$ if $s \neq t$. In other words, $W = \langle s \in S \mid (st)^{m_{st}} = 1 \rangle$ with generator S. We call W a *Coxeter group*.

The value $m_{st} = \infty$ indicates there are no relations of the form $(st)^m = 1$ for any $m \in \mathbb{Z}_{>0}$. We often call relations of the form $s^2 = 1$ *quadratic relations*. The quadratic relations on the generators of the Coxeter group imply that $s^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1} = ts$. Moreover, if $s \neq t$ and $m_{st} < \infty$, then we can use the quadratic relations to write $(st)^{m_{st}} = 1$ equivalently as

$$\underbrace{sts...}_{m_{st}} = \underbrace{tst...}_{m_{st}},$$

which we call *braid relations*. Coxeter systems are closely related to reflections, so we often call elements of S *simple reflections*, and elements in W that are conjugates to elements in S *reflections*.

Example 2.1.2. The permutation group of n elements S_n is a Coxeter group generated by the set of transpositions $S = \{(i, i + 1) \in S_n : 1 \leq i \leq n - 1\}$. Let $s_i := (i, i + 1)$. We know from algebra that S generates S_n, so let us check the relations.

- For any i, $s_i^2 = (i, i + 1)(i, i + 1) = 1$.
- For $i > j + 1$, the transpositions $(i, i + 1)$ and $(j, j + 1)$ are disjoint so $(s_i s_j)^2 = (i, i + 1)(j, j + 1)(i, i + 1)(j, j + 1) = 1$.
- For $i = j + 1$, $(s_i s_j)^3 = ((i, i + 1)(j, j + 1))^3 = (i, i + 1, i + 2)^3 = 1$.

These are sometimes called the Coxeter system of type A_{n-1}, for $n \geq 2$.

An easy case is the Coxeter group $W \simeq S_3$ with generators $S = \{s, t\}$ where s, t correspond to transpositions (12) and (23) respectively. By the quadratic and braid relations, we find that the elements of W are exactly $1, s, t, st, ts, sts = tst$. We will frequently revisit this example.

Definition 2.1.3. Let $w \in W$. As S generates W, we can write $w = s_1 s_2 ... s_k$ for some $s_1, ..., s_k \in S$. We say the sequence $(s_1, ..., s_k)$ is an *expression* for w of *length* k. Given the relations in the definition, $w \in W$ is not uniquely expressed as such a sequence, so we write $\underline{w}$ to denote a choice of expression $(s_1, ..., s_k)$ for w.

Definition 2.1.4. Let $w \in W$. For any expression $\underline{w} = (s_1, ..., s_k)$, we say the *length of* $\underline{w}$ is k, and write $\ell(\underline{w}) = k$. The *length of* w, written $\ell(w)$ is the smallest integer k such that w admits an expression of length k. We say an expression $\underline{w}$ is *reduced* if $\ell(\underline{w}) = \ell(w)$.

Note that $\ell(w) = 0$ if and only if $w = 1$.

The following are results regarding reduced expressions.

Theorem 2.1.5 (Exchange condition)**.** *Let $\underline{w} = (s_1, ..., s_k)$ be a reduced expression for $w \in W$, and let $t \in S$. If $\ell(wt) < \ell(w)$, then there exists an integer $i \in \{1, 2, ..., k\}$ such that $wt = s_1 ... \hat{s}_i ... s_k$, i.e. with s_i omitted.*

Corollary 2.1.6 (Deletion Condition)**.** *Let $\underline{w} = (s_1, ..., s_k)$ be an expression for $w \in W$ where $\ell(w) < k$, i.e. not a reduced expression. Then there exists some $i < j$ such that $w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_k$.*

In other words, if an expression is not reduced, two elements in the expression may be cancelled to result in a shorter expression.

Theorem 2.1.7 (Matsumoto, 1964)**.** *Any two reduced expressions for $w \in W$ are related by braid relations.*

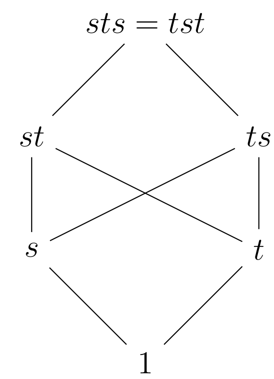
We can also define a partial order on W.

Definition 2.1.8 (Bruhat Order)**.** Let T be the set of elements of W that are conjugate to elements in S. Define a partial order $\leq$ on W such that for $x, y \in W$, $x \leq y$ if and only if there exists a chain $x = x_0, x_1, ..., x_m = y$ of elements in W such that $\ell(x_i) < \ell(x_{i+1})$ and $x_i^{-1} x_{i+1} \in T$ for each $i = 0, 1, ..., m-1$.

That is x_{i+1} is x_i multiplied on the right with a conjugate of an element in S such that its length is longer than x_i. Note that we can equivalently multiply on the left because for any $t \in T$ we can write $xt = (xtx^{-1})x$ where $xtx^{-1} \in T$.

Equivalently let $y = s_1 ... s_k$ be a reduced expression, and we can define $x \leq y$ to be if and only if there exists a reduced expression $x = s_{i_1} ... s_{i_\ell}$ such that $1 \leq i_1 < ... < i_\ell \leq k$. In other words, x is y after removing some terms from a reduced expression (we say x is a *subexpression* of y).

Example 2.1.9. The Hasse diagram for the Bruhat order on S_3 is as follows (using the labelling of elements from Example 2.1.2).



We will work with the geometric representation of Coxeter systems.

Definition 2.1.10. Given a Coxeter system (W, S), define a representation V of W as follows. Let V be a vector space over $\mathbb{R}$ with basis elements $\{\alpha_s : s \in S\}$. Equip V with a symmetric bilinear form $(-, -)$ defined by

$$(\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}}.$$

If $m_{st} = \infty$ we define $\pi/m_{st} = 0$. Define the W-action on V such that for $s \in S$ and $v \in V$,

$$s \cdot v = v - 2(v, \alpha_s)\alpha_s.$$

We call this the *geometric representation* of the Coxeter system.

Note that this is defined for both finite and infinite Coxeter groups.

Proposition 2.1.11. *For any Coxeter system, the geometric representation is faithful.*

2.2 Hecke Algebra

Let $\mathbb{Z}[v, v^{-1}]$ be the set of integer Laurent polynomials, for an indeterminate v.

Definition 2.2.1. The *Hecke algebra* $\mathcal{H}$ for a Coxeter system (W, S) is the unital associative algebra over $\mathbb{Z}[v, v^{-1}]$ generated by $\{\delta_s : s \in S\}$ with the following relations.

- $\delta_s^2 = (v^{-1} - v)\delta_s + 1$, for any $s \in S$.
- $\underbrace{\delta_s\delta_t\delta_s\ldots}_{m_{st}} = \underbrace{\delta_t\delta_s\delta_t\ldots}_{m_{st}}$, for any $s, t \in S$ where $m_{st} < \infty$.

Recall that an algebra over a commutative ring R is an R-module with an R-bilinear multiplication operation. A unital associative algebra over R is then an algebra over R for which multiplication is associative and has a multiplicative identity.

Similarly to Coxeter groups, we call the first relations *quadratic relations* and the second *braid relations*.

Note that the quadratic relation is equivalent to $(\delta_s - v^{-1})(\delta_s + v) = 0$.
For $w \in W$ with reduced expression $w = s_1 s_2 ... s_k$, define the element $\delta_w = \delta_{s_1}\delta_{s_2}...\delta_{s_k}$ of $\mathcal{H}$. Since $\mathcal{H}$ has braid relations identical to W, Matsumoto's theorem (Theorem 2.1.7) implies that this is independent of the choice of reduced expression. Note that we set $\delta_1 = 1$

Theorem 2.2.2. *The Hecke algebra is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{\delta_w : w \in W\}$.*

Definition 2.2.3. We call $\{\delta_w : w \in W\}$ the *standard basis* of $\mathcal{H}$.

Proposition 2.2.4. *The following multiplication formulae hold in $\mathcal{H}$. For $w \in W$ and $s \in S$,*

$$\delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } ws > w, \\ (v^{-1} - v)\delta_w + \delta_{ws} & \text{if } ws < w, \end{cases}$$

and

$$\delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } ws > w, \\ (v^{-1} - v)\delta_w + \delta_{sw} & \text{if } ws < w. \end{cases}$$

Proposition 2.2.5. *For any simple reflection $s \in S$,*

$$\delta_s^{-1} = \delta_s + (v - v^{-1}).$$

This follows from the quadratic relation in $\mathcal{H}$.

Proposition 2.2.6. *Since the generators $\{\delta_s : s \in S\}$ of $\mathcal{H}$ are invertible, δ_w is invertible for every $w \in W$. Moreover,*

$$\delta_{w^{-1}}^{-1} = \delta_w + \sum_{x<w} a_x \delta_x$$

for some $a_x \in \mathbb{Z}[v, v^{-1}]$.

Kazhdan-Lusztig Basis

There is another basis known as the Kazhdan-Lusztig basis.

Definition 2.2.7. The *Kazhdan-Lusztig involution* or *bar involution* is a $\mathbb{Z}$-linear involution $\mathcal{H} \to \mathcal{H}, h \mapsto \overline{h}$ defined on generators $\overline{v} = v^{-1}$ and $\overline{\delta_s} = \delta_s^{-1}$ for $s \in S$, such that it distributes across products as a ring automorphism.

Definition 2.2.8. The *Kazhdan-Lusztig basis* for $\mathcal{H}$ is the set $\{b_w : w \in W\} \subseteq \mathcal{H}$ such that for any $w \in W$,

- b_x is self-dual, i.e. $\overline{b_x} = b_x$, and
- b_x has the form
$$b_x = \delta_x + \sum_{y<x} h_{y,x}\delta_y$$
for some $h_{y,x} \in v\mathbb{Z}[v]$, where $<$ is the Bruhat order.

The coefficients $h_{y,x} \in v\mathbb{Z}[v]$ are called *Kazhdan-Lusztig polynomials*.

Additionally, we set $h_{x,x} = 1$ and $h_{y,x} = 0$ if $y \not\leq x$ in the Bruhat order. The second condition is sometimes called the *degree bound* condition.

Lemma 2.2.9. *The Kazhdan-Lusztig basis is unique.*

Standard Form

Construction of Kazhdan-Lusztig basis

Category Theory

Soergel Bimodules