Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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Background

Notation: we write 1 for the neutral element of a group.

2.1 Coxeter Groups

Definition 2.1.1. A *Coxeter system* (W, S) is a group W and a finite subset $S = \{s_1, ..., s_n\} \subset W$ under the following conditions. For any $s, t \in S$, $(st)^{m_{st}} = 1$ where $m_{st} \in \mathbb{Z}_{>0} \cup \{\infty\}$ such that $m_{st} = 1$ if $s = t$, and $m_{st} = m_{ts} \in \{2, 3, ...\} \cup \{\infty\}$ if $s \neq t$. In other words, $W = \langle s \in S \mid (st)^{m_{st}} = 1 \rangle$ with generator S. We call W a *Coxeter group*.

The value $m_{st} = \infty$ indicates there are no relations of the form $(st)^m = 1$ for any $m \in \mathbb{Z}_{>0}$. We often call relations of the form $s^2 = 1$ *quadratic relations*. The quadratic relations on the generators of the Coxeter group imply that $s^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1} = ts$. Moreover, if $s \neq t$ and $m_{st} < \infty$, then we can use the quadratic relations to write $(st)^{m_{st}} = 1$ equivalently as

$$\underbrace{sts...}_{m_{st}} = \underbrace{tst...}_{m_{st}},$$

which we call *braid relations*. Coxeter systems are closely related to reflections, so we often call elements of S *simple reflections*, and elements in W that are conjugates to elements in S *reflections*.

Example 2.1.2. The permutation group of n elements S_n is a Coxeter group generated by the set of transpositions $S = \{(i, i+1) \in S_n : 1 \leq i \leq n-1\}$. Let $s_i := (i, i+1)$. We know from algebra that S generates S_n, so let us check the relations.

- For any i, $s_i^2 = (i, i+1)(i, i+1) = 1$.
- For $i > j+1$, the transpositions $(i, i+1)$ and $(j, j+1)$ are disjoint so $(s_i s_j)^2 = (i, i+1)(j, j+1)(i, i+1)(j, j+1) = 1$.
- For $i = j+1$, $(s_i s_j)^3 = ((i, i+1)(j, j+1))^3 = (i, i+1, i+2)^3 = 1$.

These are sometimes called the Coxeter system of type A_{n-1}, for $n \geq 2$.

An easy case is the Coxeter group $W \simeq S_3$ with generators $S = \{s, t\}$ where s, t correspond to transpositions (12) and (23) respectively. By the quadratic and braid relations, we find that the elements of W are exactly $1, s, t, st, ts, sts = tst$. We will frequently revisit this example.

Definition 2.1.3. Let $w \in W$. As S generates W, we can write $w = s_1 s_2 ... s_k$ for some $s_1, ..., s_k \in S$. We say the sequence $(s_1, ..., s_k)$ is an *expression* for w of *length* k. Given the relations in the definition, $w \in W$ is not uniquely expressed as such a sequence, so we write $\underline{w}$ to denote a choice of expression $(s_1, ..., s_k)$ for w.

Definition 2.1.4. Let $w \in W$. For any expression $\underline{w} = (s_1, ..., s_k)$, we say the *length of* $\underline{w}$ is k, and write $\ell(\underline{w}) = k$. The *length of* w, written $\ell(w)$ is the smallest integer k such that w admits an expression of length k. We say an expression $\underline{w}$ is *reduced* if $\ell(\underline{w}) = \ell(w)$.

Note that $\ell(w) = 0$ if and only if $w = 1$.

The following are useful results regarding reduced expressions.

Theorem 2.1.5 (Exchange condition)**.** *Let $\underline{w} = (s_1, ..., s_k)$ be a reduced expression for $w \in W$, and let $t \in S$. If $\ell(wt) < \ell(w)$, then there exists an integer $i \in \{1, 2, ..., k\}$ such that $wt = s_1 ... \hat{s}_i ... s_k$, i.e. with s_i omitted.*

Corollary 2.1.6 (Deletion Condition)**.** *Let $\underline{w} = (s_1, ..., s_k)$ be an expression for $w \in W$ where $\ell(w) < k$, i.e. not a reduced expression. Then there exists some $i < j$ such that $w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_k$.*

In other words, if an expression is not reduced, two elements in the expression may be cancelled to result in a shorter expression.

Theorem 2.1.7 (Matsumoto, 1964)**.** *Any two reduced expressions for $w \in W$ are related by braid relations.*

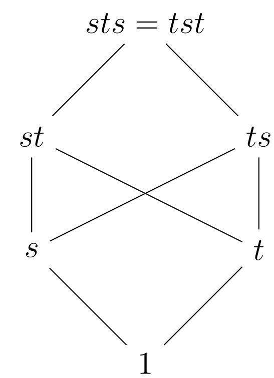
We can also define a partial order on W.

Definition 2.1.8 (Bruhat Order)**.** Let T be the set of elements of W that are conjugate to elements in S. Define a partial order $\leq$ on W such that for $x, y \in W$, $x \leq y$ if and only if there exists a chain $x = x_0, x_1, ..., x_m = y$ of elements in W such that $\ell(x_i) < \ell(x_{i+1})$ and $x_i^{-1} x_{i+1} \in T$ for each $i = 0, 1, ..., m-1$.

That is x_{i+1} is x_i multiplied on the right with a conjugate of an element in S such that its length is longer than x_i. Note that we can equivalently multiply on the left because for any $t \in T$ we can write $xt = (xtx^{-1})x$ where $xtx^{-1} \in T$.

Equivalently let $y = s_1 ... s_k$ be a reduced expression, and we can define $x \leq y$ to be if and only if there exists a reduced expression $x = s_{i_1} ... s_{i_\ell}$ such that $1 \leq i_1 < ... < i_\ell \leq k$. In other words, x is y after removing some terms from a reduced expression (we say x is a *subexpression* of y).

Example 2.1.9. The Hasse diagram for the Bruhat order on S_3 is as follows (using the labelling of elements from Example 2.1.2).



Definition 2.1.10. Given a Coxeter system (W, S), define a representation V of W as follows. Let V be a vector space over $\mathbb{R}$ generated by the basis $\{\alpha_s : s \in S\}$. Equip V with a symmetric bilinear form $(-, -)$ defined by

$$(\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}}.$$

If $m_{st} = \infty$ we define $\pi/m_{st} = 0$. Define the W-action on V such that for $s \in S$ and $v \in V$,

$$s \cdot v = v - 2(v, \alpha_s)\alpha_s.$$

We call this the *geometric representation* of the Coxeter system.

This is defined for both finite and infinite Coxeter groups.

Proposition 2.1.11. *For any Coxeter system, the geometric representation is faithful.*

In this paper, we will work with this representation of the Coxeter group.

2.2 Hecke Algebra

Let $\mathbb{Z}[v, v^{-1}]$ be the set of integer Laurent polynomials, for an indeterminate v.

Definition 2.2.1. The *Hecke algebra* $\mathcal{H}$ for a Coxeter system (W, S) is the unital associative algebra over $\mathbb{Z}[v, v^{-1}]$ generated by $\{\delta_s : s \in S\}$ with the following relations.

- $\delta_s^2 = (v^{-1} - v)\delta_s + 1$, for any $s \in S$.
- $\underbrace{\delta_s\delta_t\delta_s...}_{m_{st}} = \underbrace{\delta_t\delta_s\delta_t...}_{m_{st}}$, for any $s, t \in S$ where $m_{st} < \infty$.

Recall that an algebra over a commutative ring R is an R-module with an R-bilinear multiplication operation. A unital associative algebra over R is then an algebra over R for which multiplication is associative and has a multiplicative identity.

Similarly to Coxeter groups, we call the first relations *quadratic relations* and the second *braid relations.*

Note that the quadratic relation is equivalent to $(\delta_s - v^{-1})(\delta_s + v) = 0$.

For $w \in W$ with reduced expression $w = s_1 s_2 ... s_k$, define the element $\delta_w = \delta_{s_1}\delta_{s_2}...\delta_{s_k}$ of $\mathcal{H}$. Since $\mathcal{H}$ has braid relations identical to W, Matsumoto's theorem (Theorem 2.1.7) implies that this is independent of the choice of reduced expression. Note that we set $\delta_1 = 1$

Theorem 2.2.2. *The Hecke algebra is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{\delta_w : w \in W\}$.*

Definition 2.2.3. We call $\{\delta_w : w \in W\}$ the *standard basis* of $\mathcal{H}$.

Proposition 2.2.4. *The following multiplication formulae hold in $\mathcal{H}$. For $w \in W$ and $s \in S$,*

$$\delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } ws > w, \\ (v^{-1} - v)\delta_w + \delta_{ws} & \text{if } ws < w, \end{cases}$$

and

$$\delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } ws > w, \\ (v^{-1} - v)\delta_w + \delta_{sw} & \text{if } ws < w. \end{cases}$$

Proposition 2.2.5. *For any simple reflection $s \in S$,*

$$\delta_s^{-1} = \delta_s + (v - v^{-1}).$$

This follows from the quadratic relation in $\mathcal{H}$.

Proposition 2.2.6. *Since the generators $\{\delta_s : s \in S\}$ of $\mathcal{H}$ are invertible, δ_w is invertible for every $w \in W$. Moreover,*

$$\delta_{w^{-1}}^{-1} = \delta_w + \sum_{x<w} a_x \delta_x$$

for some $a_x \in \mathbb{Z}[v, v^{-1}]$.

There is another basis known as the Kazhdan-Lusztig basis.

Definition 2.2.7. The *Kazhdan-Lusztig involution* or *bar involution* is a $\mathbb{Z}$-linear involution $\mathcal{H} \to \mathcal{H}, h \mapsto \overline{h}$ defined on generators $\overline{v} = v^{-1}$ and $\overline{\delta_s} = \delta_s^{-1}$ for $s \in S$, such that it distributes across products as a ring automorphism.

Definition 2.2.8. The *Kazhdan-Lusztig basis* for $\mathcal{H}$ is the set $\{b_w : w \in W\} \subseteq \mathcal{H}$ such that for any $w \in W$,

- b_x is self-dual, i.e. $\overline{b_x} = b_x$, and

- b_x has the form

$$b_x = \delta_x + \sum_{y<x} h_{y,x}\delta_y$$

  for some $h_{y,x} \in v\mathbb{Z}[v]$, where $<$ is the Bruhat order.

The coefficients $h_{y,x} \in v\mathbb{Z}[v]$ are called *Kazhdan-Lusztig polynomials*.

Additionally, we set $h_{x,x} = 1$ and $h_{y,x} = 0$ if $y \not\leq x$ in the Bruhat order. The second condition is sometimes called the *degree bound* condition.

Lemma 2.2.9. *The Kazhdan-Lusztig basis is unique.*

Furthermore, the corresponding Kazhdan-Lusztig basis element for $s \in S$ is $b_s = \delta_s + v$.

Definition 2.2.10. The *Kazhdan-Lusztig anti-involution* $\omega : \mathcal{H} \to \mathcal{H}$ is an involution defined similarly to the Kazhdan-Lusztig involution, but distributes across products as a ring *anti*-automorphism. That is for $a, b \in \mathcal{H}$, $\omega(ab) = \omega(b)\omega(a)$.

Definition 2.2.11. The *standard trace* $\epsilon : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ is a $\mathbb{Z}[v, v^{-1}]$-linear map which extracts the coefficient of δ_{id} for elements written in the standard basis.

Definition 2.2.12. The *standard form* $(-,-) : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ is a sesquilinear form (with respect to either involution restricted to $\mathbb{Z}[v, v^{-1}]$) such that $(a, b) := \epsilon(\omega(a)b)$ for $a, b \in \mathcal{H}$.

Here, sesquilinear means that the form is linear in the second variable and in the first variable, $(fa, b) = \overline{f}(a, b)$ for $a, b \in \mathcal{H}$ and $f \in \mathbb{Z}[v, v^{-1}]$. Note the restricted involution inverts each v extending linearly to $\mathbb{Z}[v, v^{-1}]$. The bar involution and anti-involution restricted to $\mathbb{Z}[v, v^{-1}]$ are the same, as this ring is commutative.

Theorem 2.2.13. *The Kazhdan-Lusztig basis is asymptotically orthonormal. That is for $x, y \in W$,*

$$(b_x, b_y) = \begin{cases} 1 + v\mathbb{Z}[v] & \textit{if } x = y, \\ v\mathbb{Z}[v] & \textit{otherwise.} \end{cases}$$

2.3 Soergel Bimodules

Definition 2.3.1. A *$\mathbb{Z}$-graded ring R* is a ring with a decomposition

$$R = \bigoplus_{i\in\mathbb{Z}} R^i$$

into a direct sum of additive subgroups $R_i \subseteq R$ such that $R^i R^j \subseteq R^{i+j}$.

Gradings are defined in this way to generalise a notion of 'degree', where the degree of a product is the sum of their degrees. This definition can naturally be extended to $\mathbb{Z}$-graded modules over some a $\mathbb{Z}$-graded ring.

Definition 2.3.2. Let R be a $\mathbb{Z}$-graded ring. A *$\mathbb{Z}$-graded R-module* M is a module over R with a decomposition

$$M = \bigoplus_{i\in\mathbb{Z}} M^i$$

into a direct sum of additive subgroups $M^i \subseteq M$ such that $R^iM^j \subseteq M^{i+j}$. We call the M^i *graded pieces* of M, and the elements of M^i *homogeneous of degree i*.

For the remainder of this paper we will only be working with $\mathbb{Z}$-graded objects, so we just say *graded*.

Example 2.3.3. For any ring R, the *trivial grading* of R is the decomposition where $R^0 = R$ and $R^i = 0$ for all $i \neq 0$.

Example 2.3.4. Let F be a field with the trivial grading. The vector space of real polynomials (in one or several variables) over F has a natural grading where the n-graded piece is the subspace generated by degree n monomials. For example, the $\mathbb{R}$-vector space $\mathbb{R}[x]$ has a decomposition

$$\mathbb{R}[x] = V^0 \oplus V^1 \oplus V^2 \oplus \dots$$

where V^i is the subspace spanned by $\{x^i\}$. This example is a $\mathbb{Z}$-grading where the n-graded piece is 0 for $n < 0$.

Gradings for other algebraic objects, such as algebras and bimodules, can be similarly defined. The following definitions are for general graded objects.

Definition 2.3.5. Let M and N be graded objects. For $i \in \mathbb{Z}$, define $M(i)$ to be the graded object with graded pieces $M(i)^j := M^{i+j}$. We say this is obtained by a *shift in grading* of M.

If we visualise the graded pieces horizontally in ascending order of degree, the grading of $M(i)$ is the grading of M shifted to the left by i places. Particularly, if $x \in M^n$ is homogeneous of degree n in M, then it is homogeneous of degree $n - i$ in $M(i)$.

Degree	-2	-1	0	1	2
M	M^{-2}	M^{-1}	M^0	M^1	M^2
$M(1)$	M^{-1}	M^0	M^1	M^2	M^3
$M(2)$	M^0	M^1	M^2	M^3	M^4
$M(-1)$	M^{-3}	M^{-2}	M^{-1}	M^0	M^1
$M(i)$	M^{-2+i}	M^{-1+i}	M^i	M^{1+i}	M^{2+i}

Definition 2.3.6. Let M and N be graded objects. A morphism $f : M \to N$ is *homogeneous of degree k* if $f(M^i) \subseteq N^{i+k}$ for all $i \in \mathbb{Z}$. Typically we assume morphisms between graded objects are homogeneous of degree 0, and call them *graded morphisms*. A *graded isomorphism* is a graded morphism with a graded (two-sided) inverse. We say that M and N are *isomorphic up to shift* if there is a graded isomorphism $M \simeq N(i)$ for some $i \in \mathbb{Z}$. The *graded morphism space* (or *graded Hom space*) between M and N is

$$\mathrm{Hom}^\bullet(M, N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(M, N(i)).$$

Notice for any morphism $M \to N$ of degree k, there is a morphism $M \to N(k)$ of degree 0 that contains the same information.

Definition 2.3.7. Let M be a graded object in an additive category (i.e. direct sums are defined on M), and let $p = \sum_i p_i v^i \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ be a Laurent polynomial with positive integer coefficients. Define

$$M^{\oplus p} := \bigoplus_{i \in \mathbb{Z}} M(i)^{\oplus p_i}$$

where $M^{\oplus k} := \bigoplus_{j=1}^{k} M$ for $k \in \mathbb{Z}_{\geq 0}$.

Definition 2.3.8. Let R be a graded ring and M a graded R-module. A *graded submodule* of M is a submodule $N \subseteq M$ with the induced grading $N^i = N \cap M^i$ for all $i \in \mathbb{Z}$. A *graded direct summand* of M is a graded module N such that $M \simeq N \oplus N'$ as graded modules for some graded submodule $N' \subseteq M$. We say that M is *graded free* if it has an R-basis of homogeneous elements of M. If this basis is finite, then there is a unique $p \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ such that $M \simeq R^{\oplus p}$, and we call p the *graded rank* of M.

For our purposes, fix a Coxeter system (W, S) and consider it's geometric representation V. Let $R = \mathrm{Sym}(V) \simeq \mathbb{R}[\alpha_s : s \in S]$ be the symmetric algebra of V, which we will think of as the real polynomial ring generated, as a ring, by the basis of V. We can think of R as a graded algebra[1], such that $V \subseteq R$ is homogeneous of degree 2, i.e. $\deg \alpha_s = 2$ and the 'monomials' that are products of i basis elements are degree $2i$.

There is a natural action of W on R, induced by its action on V, that for any $w \in W$,

$$w \cdot \prod_{s \in S} \alpha_s^{k_s} = \prod_{s \in S} (w \cdot \alpha_s)^{k_s}$$

where $k_s \in \mathbb{Z}_{\geq 0}$, extending linearly to R.

[1] A graded module that is also a graded ring.