

Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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Acknowledgements

I like to acknowledge ... blah blah blah

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Chapter 1

Introduction

Visual interpretations of functions simplify calculations and often provides insights into the mathematical objects they encode. [More detail about what this is](#) This general philosophy takes form in various settings. A simple example are string diagrams for permutations. A permutation can be drawn as strings between two copies of a set determining how the objects are permuted. Compositions of these permutations is the operation of joining corresponding strings start to end in order to create a larger string diagram representing their product.

A significant example are *planar algebras* in the work of Jones. These are certain algebras of planar diagrams that describe operators. His study of the Temperley-Lieb planar algebra lead to the discovery of an important invariant in knot theory in the 1983, which we know now as the Jones polynomial. For this and surrounding works he received a Fields medal. This technology of planar algebras have been since used to study subfactors in functional analysis [\[Jon21\]](#)¹ and have consequences in for example statistical mechanics and mathematical physics.

In representation theory, our main motivational example is given by the proof of the Kazhdan–Lusztig conjecture through the diagrammatics of Soergel bimodules. The conjecture relates Kazhdan–Lusztig polynomials, arising from the Weyl group associated with a Lie algebra, to Jordan–Hölder multiplicities of particular representations of Lie algebras called Verma modules. Proofs were discovered independently by Beilinson–Bernstein and Brylinski–Kashiwara in 1981 but by geometric methods, which was unsatisfying to many. Around this time, Soergel was working toward an algebraic proof by Soergel bimodules, however he hit a technical road block. In 2010’s, Elias and Williamson [Ref?](#) developed planar diagrams for morphisms on Soergel bimodules and were able to overcome the technical point where Soergel got stuck, to prove the conjecture diagrammatically. The diagrams can greatly simplify algebraic calculations and the diagrammatic category can be considered independently from Soergel bimodules. We explore this diagrammatics for S_2 in [Section 3.1](#).

One of the advantages of the diagrammatic Soergel bimodules is that it can be defined over \mathbb{Z} and extended to fields of characteristic p where classical Soergel bimodules are

¹Originally from 1999, and was recently published.

ill-behaved. Characters in the category of tilting modules (certain representations of a Lie algebra) can be calculated via Kazhdan–Lusztig polynomial in characteristic zero, however these polynomials were unknown in characteristic p . Riche and Williamson in [RW18] were able to construct these characteristic p Kazhdan–Lusztig polynomials by considering diagrammatic Soergel bimodules in characteristic p .

In this paper we give an introduction to drawing morphisms in monoidal categories, put more here and define some mechanisms to form an additive and idempotent complete category. In Chapter 3 we define diagrammatic Soergel bimodules associated with the symmetric group S_2 , construct a basis for its morphism spaces and state the theorem for its equivalence to the category of algebraic Soergel bimodules. We use this diagrammatic category to construct a diagrammatic module category with an extra relation, then prove its equivalence to the category of projective objects in the principle block of the category \mathcal{O} . In Chapter 4 we consider the affine symmetric group \tilde{S}_2 to define the diagrammatic Soergel bimodules associated it, construct a basis for its morphism spaces and state the theorem for its equivalence to the category of algebraic Soergel bimodules. The extra generator in \tilde{S}_2 compared with S_2 provides some additional complexity to the structure of the category. We then form a module category with two extra relations and provide a proof of its equivalence to the category of tilting modules for \mathfrak{sl}_2 . In the last chapter we discuss the consequences of diagrammatics in relation to Chapter 3 and Chapter 4, mention generalisations of the results and further areas of interest.

The contents of this thesis are for honours students and future readers that are interested in this topic. The reader is assumed to have some familiarity with undergraduate algebra (such as groups, rings, algebras and fields), basic ideas in representation theory, basic category theory and monoidal categories.

Talk about we dont need to know about category \mathcal{O} and Tilt. One of the advantages of diagrammatics is that we don't need to understand these complex categories in representation theory to study them. For this reason, we will defer the details in the proofs involving them to other sources.

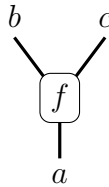
Chapter 2

Background

2.1 Drawing Monoidal Categories

A *monoidal category* \mathcal{C} is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold¹. The bifunctor \otimes is called the *tensor* or *monoidal product*. A monoidal category is *strict* if $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ and $A = \mathbb{1} \otimes A = A \otimes \mathbb{1}$ for objects and similarly for morphisms. In this paper, we will assume that monoidal categories are strict, since all monoidal categories are monoidally equivalent to a strict one².

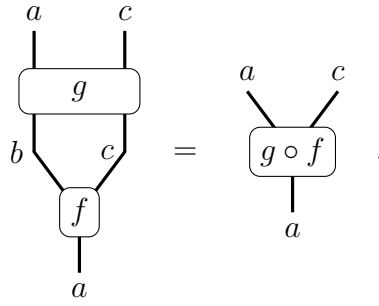
The morphisms of \mathcal{C} can be drawn as string diagrams, where the morphism maps from the bottom to the top. Functions that make up the morphism are drawn as tokens or boxes. **Say something about this being in a planar strip, with domain and codomain as bottom and top.** For example



depicts a morphism $f : a \rightarrow b \otimes c$. Notice here that tensor products are displayed with its factors laid out horizontally. The compositions of morphisms is the vertical stacking of diagrams whenever labels on domains and codomains match. For example, the composition $g \circ f : a \rightarrow b \otimes c \rightarrow a \otimes c$ of $f : a \rightarrow b \otimes c$ with $g : b \otimes c \rightarrow a \otimes c$ has the diagram

¹For more details see [Eti+15].

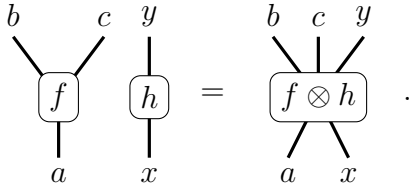
²See [ML98, VII.2] or [Eti+15, Thm 2.8.5]



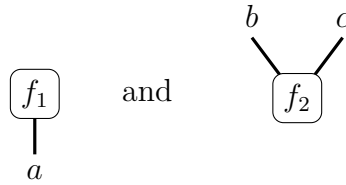
For identity morphisms we just draw a vertical line, so id_a is the diagram



This is sensible as composing a function with identities should not change the function, and this is clearly evident with diagrams. The tensor product of morphisms is the horizontal concatenation of diagrams, such that strings from separate functions don't interact. For example, given $h : x \rightarrow y$, the tensor product $f \otimes h : a \otimes x \rightarrow b \otimes c \otimes y$ is drawn as

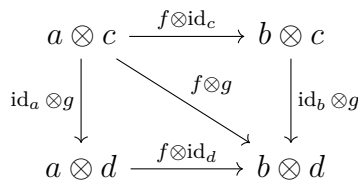


We let the monoidal unit $\mathbb{1}$ be blank and unlabelled, and strings that would join to $\mathbb{1}$ are blank. Particularly, $\text{id}_{\mathbb{1}}$ is an empty diagram. It makes sense to display $\mathbb{1}$ in this way since tensoring with $\mathbb{1}$ does nothing to objects and tensoring with $\text{id}_{\mathbb{1}}$ does nothing to morphisms in a strict monoidal category. Then we have diagrams such as



for morphisms $f_1 : a \rightarrow \mathbb{1}$ and $f_2 : \mathbb{1} \rightarrow b \otimes c$.

For a monoidal category \mathcal{C} , the bifactoriality of $- \otimes -$ implies the following *interchange law*. For morphisms $f : a \rightarrow b$ and $g : c \rightarrow d$, $(\text{id}_b \otimes g) \circ (f \otimes \text{id}_c) = f \otimes g = (f \otimes \text{id}_d) \circ (\text{id}_a \otimes g)$. In other words the following diagram commutes.



Written with string diagrams, this is

$$\begin{array}{c} b \\ | \\ \boxed{f} \\ | \\ a \end{array} \quad \begin{array}{c} d \\ | \\ \boxed{g} \\ | \\ c \end{array} = \begin{array}{c} b \\ | \\ \boxed{f} \\ | \\ a \end{array} \quad \begin{array}{c} d \\ | \\ \boxed{g} \\ | \\ c \end{array} = \begin{array}{c} b \\ | \\ \boxed{f} \\ | \\ a \end{array} \quad \begin{array}{c} d \\ | \\ \boxed{g} \\ | \\ c \end{array}$$

which holds up to deformation of the diagram.

Before looking at our main example of a diagrammatic monoidal category, we first define some terminology.

Definition 2.1.1. For a commutative ring R , an R -linear category is a category enriched over the category of R -modules. That is, for objects a, b , the set of morphisms $\text{Hom}(a, b)$ is an R -module and the composition of morphisms is R -bilinear. A R -linear monoidal category is a category that is both monoidal and R -linear such that the monoidal product on morphisms is R -bilinear.

Talk about composition with 0 is 0 Talk about tensor with 0 is 0 (is this even right?)

Example 2.1.2. Let \mathbb{k} be a field. The category of vector spaces over \mathbb{k} , $\mathbf{Vect}_{\mathbb{k}}$, is a \mathbb{k} -linear monoidal category. This follows by the classical theory of linear algebra.

Definition 2.1.3. A monoidal category \mathcal{C} is *generated* by finite set S_o of objects and S_m of morphisms, when all non-unit objects are a finite tensor of objects in S_o and all non-identity morphisms are a finite combination of tensors and compositions of morphisms in S_m .

Example 2.1.4. Our first example of a diagrammatic monoidal category is the *Temperley-Lieb category*. The Temperley-Lieb category \mathcal{TL} is a strict R -linear monoidal category whose objects are generated by the vertical line $\mathbb{1}$ and morphisms generated by the cup $\cup : \mathbb{1} \rightarrow \mathbb{1} \otimes \mathbb{1}$ and cap $\cap : \mathbb{1} \otimes \mathbb{1} \rightarrow \mathbb{1}$, with relations

$$\begin{array}{c} \cup \\ \cap \end{array} = \mathbb{1} = \begin{array}{c} \cap \\ \cup \end{array}.$$

Mention that composition and tensor product is as explained above

Some example

Mention bubbles and specialisation to some $\delta \in R$

Mention that these are crossingless matchings

Comment on isotopy

2.2 Frobenius Objects

Something something about Many relations in categorical structures can be written in diagrammatic terms - adjunctions, monoid

Something about isotopy

Let \mathcal{C} be a (strict) monoidal category. We can define the following objects.

Definition 2.2.1. A *monoid object* in \mathcal{C} is a triple (M, μ, η) for an object $M \in \mathcal{C}$, a *multiplication* map $\mu : M \otimes M \rightarrow M$ and a *unit* map $\eta : \mathbb{1} \rightarrow M$, such that

$$\begin{array}{ccccc}
 & & M \otimes M \otimes M & & \\
 & \swarrow \mu \otimes \text{id}_M & & \searrow \text{id}_M \otimes \mu & \\
 M \otimes M & & & & M \otimes M \\
 & \searrow \mu & & \swarrow \mu & \\
 & & M & &
 \end{array}$$

and

$$\begin{array}{ccccc}
 \mathbb{1} \otimes M & \xrightarrow{\eta \otimes \text{id}_M} & M \otimes M & \xleftarrow{\text{id}_M \otimes \eta} & M \otimes \mathbb{1} \\
 & \searrow \text{id}_M & \downarrow \mu & \swarrow \text{id}_M & \\
 & & M & &
 \end{array}$$

commute. The first diagram is the *associativity* relation $\mu \circ (\mu \otimes \text{id}_M) = \mu \circ (\text{id}_M \otimes \mu)$ and the second diagram is the *unit* relation $\text{id}_M = \mu \circ (\eta \otimes \text{id}_M) = \mu \circ (\text{id}_M \otimes \eta)$.

Dually, a *comonoid object* in \mathcal{C} is a triple (M, δ, ϵ) for an object $M \in \mathcal{C}$, a *comultiplication* map $\delta : M \rightarrow M \otimes M$ and a *counit* map $\epsilon : M \rightarrow \mathbb{1}$, satisfying the *coassociativity* relation

$$\begin{array}{ccccc}
 & & M \otimes M \otimes M & & \\
 & \swarrow \delta \otimes \text{id}_M & & \searrow \text{id}_M \otimes \delta & \\
 M \otimes M & & & & M \otimes M \\
 & \swarrow \delta & & \searrow \delta & \\
 & & M & &
 \end{array}$$

and *counit* relation

$$\begin{array}{ccccc}
 \mathbb{1} \otimes M & \xleftarrow{\epsilon \otimes \text{id}_M} & M \otimes M & \xrightarrow{\text{id}_M \otimes \epsilon} & M \otimes \mathbb{1} \\
 & \swarrow \text{id}_M & \uparrow \delta & \searrow \text{id}_M & \\
 & & M & &
 \end{array} .$$

Monoid objects generalise monoids, i.e. sets with an identity equipped with an associative binary operation.

Definition 2.2.2. A *Frobenius object* in \mathcal{C} is a quintuple $(A, \mu, \eta, \delta, \epsilon)$ such that (A, μ, η) is a monoid object, (A, δ, ϵ) is a comonoid object, and the maps satisfy the *Frobenius relations*

$$\begin{array}{ccccc}
 & & A \otimes A & & \\
 & \delta \otimes \text{id}_A \swarrow & \downarrow \mu & \searrow \text{id}_A \otimes \delta & \\
 A \otimes A \otimes A & & A & & A \otimes A \otimes A \\
 & \text{id}_A \otimes \mu \searrow & \downarrow \delta & \swarrow \mu \otimes \text{id}_A & \\
 & & A \otimes A & &
 \end{array} ,$$

that is $(\text{id}_A \otimes \mu) \circ (\delta \otimes \text{id}_A) = \delta \circ \mu = (\mu \otimes \text{id}_A) \circ (\text{id}_A \otimes \delta)$.

The maps and relations for a Frobenius object $(A, \mu, \eta, \delta, \epsilon)$ have a nice description with the diagrams given in [Section 2.1](#). The structure maps are drawn as

$$\begin{array}{c} A \\ | \\ \boxed{\mu} \\ / \quad \backslash \\ A \quad A \end{array} , \quad \begin{array}{c} A \\ | \\ \boxed{\eta} \end{array} , \quad \begin{array}{c} A \quad A \\ \backslash \quad / \\ \boxed{\delta} \\ | \\ A \end{array} , \quad \begin{array}{c} \boxed{\epsilon} \\ | \\ A \end{array} .$$

For the rest of this section, we only work with the Frobenius object A and $\mathbb{1}$. We can stop putting the label A by identifying A with the identity strand $\mathbb{1} = \text{id}_A$. Diagrammatically, the associativity relation $\mu \circ (\mu \otimes \text{id}_M) = \mu \circ (\text{id}_M \otimes \mu)$ is

$$\begin{array}{c} \text{---} \\ | \\ \boxed{\mu} \\ / \quad \backslash \\ \boxed{\mu} \quad \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{\mu} \\ \backslash \quad / \\ \text{---} \quad \boxed{\mu} \end{array} ,$$

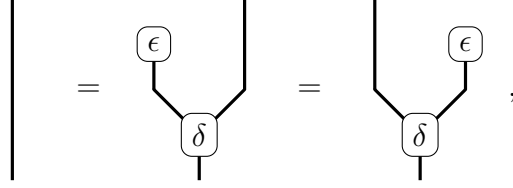
the coassociativity relation $(\delta \otimes \text{id}_A) \circ \delta = (\text{id}_A \otimes \delta) \circ \delta$ is

$$\begin{array}{c} \text{---} \quad \text{---} \\ \backslash \quad / \\ \boxed{\delta} \\ | \\ \boxed{\delta} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \quad \text{---} \\ \backslash \quad / \\ \boxed{\delta} \\ \backslash \quad / \\ \text{---} \quad \boxed{\delta} \\ | \\ \text{---} \end{array} ,$$

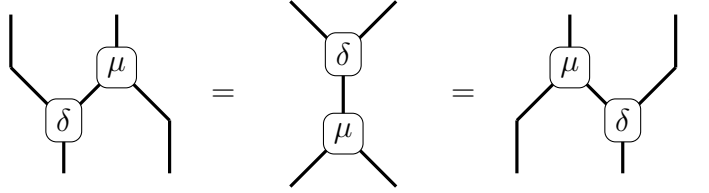
the unit relation $\text{id}_A = \mu \circ (\eta \otimes \text{id}_A) = \mu \circ (\text{id}_A \otimes \eta)$ is

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{\mu} \\ / \quad \backslash \\ \boxed{\eta} \quad \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{\mu} \\ \backslash \quad / \\ \text{---} \quad \boxed{\eta} \end{array} ,$$

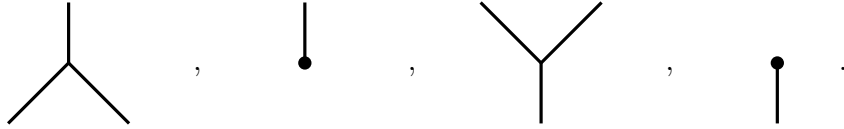
the counit relation $\text{id}_A = (\epsilon \otimes \text{id}_A) \circ \delta = (\text{id}_A \otimes \epsilon) \circ \delta$ is



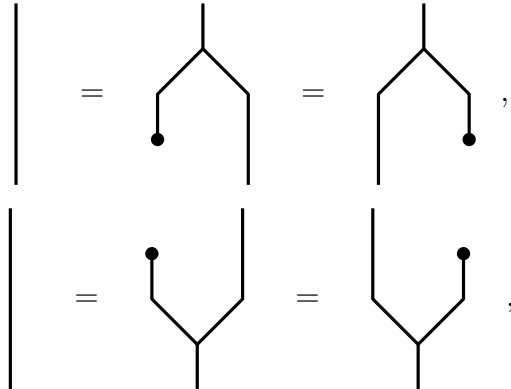
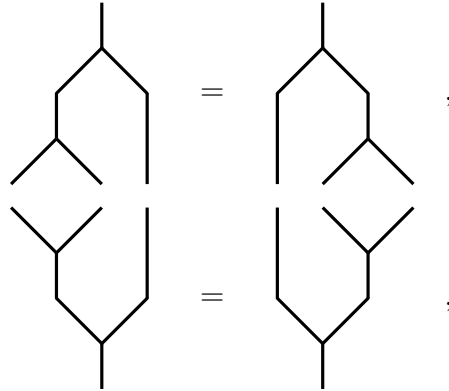
and the Frobenius relation $(\text{id}_A \otimes \mu) \circ (\delta \otimes \text{id}_A) = \delta \circ \mu = (\mu \otimes \text{id}_A) \circ (\text{id}_A \otimes \delta)$ is



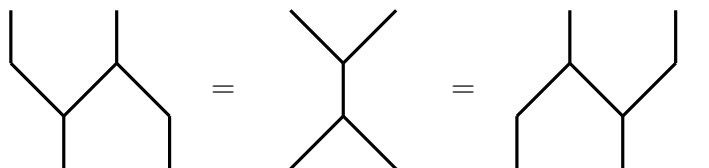
To simplify the diagrams, we stop labelling the functions and draw the structure maps as



So the relations become



and



Talk about isotopy

Maybe something about the (diagrammatic?) category Frob , capturing the data of a Frobenius object

2.3 Module Categories

Chapter 3

One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = e \rangle$. At the end of this section, we see that this diagrammatic category is equivalent to the category of Soergel Bimodules under additive Karoubian closure.

Remark 3.1.1. All diagrammatics below and in [Chapter 4](#) can be defined in the language of planar algebras, without mentioning (monoidal) categories, e.g. in [\[Jon21\]](#). Nevertheless, we define them in the context of categories as we will see them as diagrammatic versions of important categories in representation theory.

What do we do about \mathbb{C} ? Do the theorems (at the end) apply over \mathbb{Z} or \mathbb{C} or both? If we define over \mathbb{Z} , how do we use it over \mathbb{C} for the next section?

Definition 3.1.2. The *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ is a \mathbb{Z} -linear monoidal category with the following presentation.

The objects are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. $s, ssss =: s^4, sssssss =: s^7$, where the tensor product is concatenation. The empty tensor product, i.e. the monoidal identity, will be denoted $\emptyset =: s^0$.

The morphisms are generated, up to isotopy, by univalent and trivalent vertices

$$\begin{array}{c} \bullet \\ | \end{array}, \quad \begin{array}{c} | \\ \diagup \quad \diagdown \end{array}, \quad (G1)$$

that are maps $s \rightarrow \emptyset$ and $ss \rightarrow s$ respectively, and their vertical reflections. We put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). The free \mathbb{Z} -module structure on each morphism space $\text{Hom}(s^n, s^m)$ produces \mathbb{Z} -linear combinations

of such diagrams. **Something about composition/tensor and addition commuting** Then, composition or tensors with the zero morphism 0 result in 0. To abuse notation, the empty diagram $\emptyset \rightarrow \emptyset$ will be denoted \emptyset . We may sometimes identify the identity morphism id_s^n (consisting of n red strings) with s^n .

Such diagrams are subject to the following local relations

$$\begin{array}{c} \text{---} \bullet \\ | \end{array} = \begin{array}{c} | \end{array}, \quad (\text{R1a})$$

$$\begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array}, \quad (\text{R1b})$$

$$\begin{array}{c} | \\ \bigcirc \end{array} = 0, \quad (\text{R1c})$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} | \\ \bullet \\ | \\ \bullet \end{array}. \quad (\text{R1d})$$

Note that the local relations apply to any subdiagram.

Remark 3.1.3. The object s is a Frobenius object in $\mathcal{H}(S_2)$. The generators (G1) and their vertical reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (R1a) and (R1b).

Put a definition of frob object in intro Talk about how the underlying thing that inspired these diagrams is frobenius object s . Isotopy is natural for frob objects. Talk about isotopy

Example 3.1.4. Using isotopy and the relations in (R1) we can simplify the morphism in $\text{Hom}(ss, s)$,

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ | \end{array} = \begin{array}{c} | \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ | \end{array} \\ \\ = 2 \begin{array}{c} | \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ | \end{array} - \begin{array}{c} | \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ | \end{array}$$

$$= 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} | \\ \cup \\ | \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}.$$

Add example of using frob associativity

Proposition 3.1.5. *All diagrams with floating diagrams, i.e. diagrams not connected to the domain or codomain by a red strand, are linear combination of diagrams where all floating diagrams are barbells.*

Proof. By isotopy and (R1a), floating diagrams can be drawn as barbells with ‘bubbles’ and possibly floating subdiagrams inside each bubble. For example,



The Frobenius relation (R1b) allows us to ‘straighten out’ the bubbles to a chain of individual bubbles. The diagram above becomes



For a floating diagram without floating subdiagrams, it is either 0 by (R1c), or \bullet which can be removed from any diagrams containing it via (R1d). Repeating this process produces a linear combination of diagrams where all floating diagrams are barbells. \square

The morphism space $\text{Hom}(s^n, s^m)$ has a left (or right) $\mathbb{Z}[\textcolor{red}{!}]$ -basis called the *double leaves* basis, as described in [EW16]. To define this basis, we must first define morphisms known as *light leaves*.

To make use of the group structure of S_2 , we need to translate between words in $\mathcal{H}(S_2)$ and elements in S_2 . Let $\phi : (\text{ob}(\mathcal{H}(S_2)), \otimes) \rightarrow (S_2, *)$ be the monoid homomorphism¹ mapping $s \mapsto s$ and $\emptyset \mapsto 1$, and $\psi : S_2 \rightarrow \text{ob}(\mathcal{H}(S_2))$ be the function that maps $s \mapsto s$ and $1 \mapsto \emptyset$. **Should this be a definition?** The maps ϕ allows words $w = s^n$ to be seen as elements of S_2 , and ψ allows $1, s \in S_2$ to be seen as the objects $\emptyset, s \in \mathcal{H}(S_2)$. Clearly, $\phi\psi$ is the identity map on S_2 , and the map $\psi\phi : \mathcal{H}(S_2) \rightarrow \mathcal{H}(S_2)$ takes objects to one of \emptyset or s in $\mathcal{H}(S_2)$ by considering them as elements in S_2 .

Definition 3.1.6. (Subexpression for S_2) Given a word $w = s^n$, a *subexpression* e is a binary string of length n . We can *apply* a subexpression to produce an object $w(e) \in \mathcal{H}(S_2)$, which is w where terms corresponding to 0 in e are replaced with \emptyset . For $0 \leq i \leq n$, write $w(e, i)$ for the resultant object of the first i terms in e applied to the first i terms in w . Particularly $w(e, 0) = \emptyset$ and $w(e, n) = w(e)$.

¹A map that preserves the monoidal product and identity element.

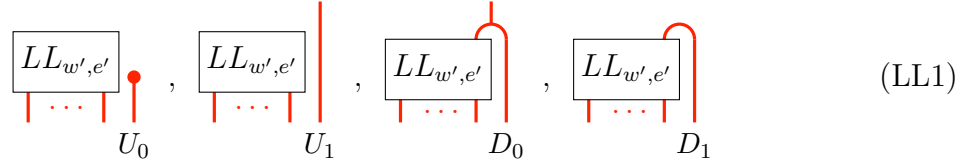
For example, 0000, 0110 and 1011 are subexpressions of $s^4 = ssss$. Applying the third subexpression gives $ssss(1011) = s\emptyset ss = sss$ and $ssss(1011, 3) = sss(101) = s\emptyset s = \emptyset$, by strictness of the monoidal category. Here, each term of the subexpression is a decision to include or exclude the corresponding s in the word, where excluding an s amounts to tensoring with \emptyset .

For a word w and subexpression e , we label each term by U_0, U_1, D_0 or D_1 . The i -th term is labelled U_* if $\phi(w(e, i-1)) = 1 \in S_2$, and labelled D_* if $\phi(w(e, i-1)) = s \in S_2$. The label's subscript is the corresponding term in e .

Example 3.1.7. For the object $w = ssss$ and subexpression $e = 0101$, we find the labels as recorded in the following table.

Term i	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
$w(e, i)$	\emptyset	$\emptyset s = s$	$\emptyset s \emptyset = s$	$\emptyset s \emptyset s = ss$
Labels	U_0	$U_0 U_1$	$U_0 U_1 D_0$	$U_0 U_1 D_0 D_1$

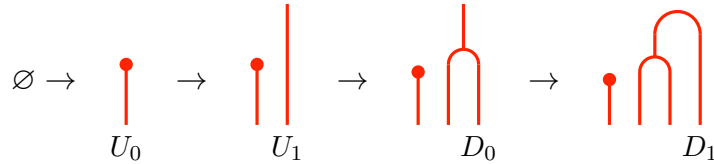
Definition 3.1.8. The *light leaf* $LL_{w,e} \in \text{Hom}(w, \psi\phi(w(e)))$ for a word w and subexpression e , is defined iteratively as follows. Let $LL_{\emptyset, \emptyset} = \emptyset$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, the light leaf $LL_{w',e'i}$ is one of



corresponding to the next label, where w' and e' are appropriate subwords² of w and e respectively.

Here, the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ is \emptyset , and when the next label is D_* the codomain of $LL_{w',e'}$ is s . This implies that the recursive definition is consistent.

Example 3.1.9. Following from [Example 3.1.7](#) for $w = ssss$ and $e = 0101$, we have labels $U_0 U_1 D_0 D_1$ so the light leaf $LL_{w,e}$ is built as follows.



Definition 3.1.10. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y in $\mathcal{H}(S_2)$ is a composition

$$\mathbb{L}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \rightarrow y$$

²A word with some letters removed.

for subexpressions e of w and f of y such that $\psi\phi(w(e)) = \psi\phi(f(y))$.

Visually these are diagrams from w to y factoring through $\psi\phi(w(e)) = \psi\phi(f(y)) \in \{\emptyset, s\}$,

$$\begin{array}{c} y \\ \overline{LL}_{y,f} \\ \overline{LL}_{w,e} \\ w \end{array} \psi\phi(w(e)) = \psi\phi(f(y)) \ .$$

Example 3.1.11. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w , and $f = 010$ be a subexpression of y . The corresponding light leaves are

$$LL_{w,e} = \begin{array}{c} \bullet \\ | \\ U_0 \end{array} \begin{array}{c} \cup \\ | \\ U_1 \end{array} \begin{array}{c} | \\ | \\ D_1 \end{array} \begin{array}{c} | \\ | \\ U_1 \end{array} \quad \text{and} \quad LL_{y,f} = \begin{array}{c} \bullet \\ | \\ U_0 \end{array} \begin{array}{c} \cup \\ | \\ U_1 \end{array} \begin{array}{c} | \\ | \\ D_0 \end{array} .$$

Then the double leaf $\mathbb{L}_{f,e} = \overline{L}\overline{L}_{y,f} \circ LL_{w,e} : ssss \rightarrow sss$, factoring through s , is

The diagram shows two horizontal lines. The top line is labeled $\overline{LL}_{y,f}$ and the bottom line is labeled $LL_{w,e}$. A dashed horizontal line is positioned between them. Three vertical red lines are shown. The leftmost line has a red dot on the $LL_{w,e}$ line and extends upwards. The middle line has a red dot on the $\overline{LL}_{y,f}$ line and extends upwards. The rightmost line has a red dot on the $\overline{LL}_{y,f}$ line and extends upwards, with a small red dot on the $LL_{w,e}$ line.

Note that these double leaves have no floating diagrams, which are $\textcircled{\bullet}$ by [Proposition 3.1.5](#). In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{Z}[\textcircled{\bullet}]$ -module, where the (left) $\textcircled{\bullet}$ -action is left concatenation by $\textcircled{\bullet}$. Since we can move barbells to the right via the relation [\(R1d\)](#) and double leaves cut down the middle are double leaves factoring through \emptyset , we can equivalently act by $\mathbb{Z}[\textcircled{\bullet}]$ on the right. This leads us to the following theorem.

Theorem 3.1.12 (Elias-Williamson [EW16, Theorem 1.2]). *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{L}\mathbb{L}(w, y)$ be the collection of double leaves $\mathbb{L}\mathbb{L}_{f,e}$ for subexpressions e of w and f of y , such that $\psi\phi(w(e)) = \psi\phi(y(f))$. Then $\mathbb{L}\mathbb{L}(w, y)$ is a basis for $\text{Hom}(w, y)$ as a left (or right) $\mathbb{Z}[\bullet]$ -module.*

A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.13. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky’s construction in [Lib08].

The morphisms in this category can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1 . The degree of a diagram is the sum of the degrees of the generators that appear in it. This makes $\mathcal{H}(S_2)$ a \mathbb{Z} -graded category.

Maybe mention what a grading is.

Put graded category definition in background

Put example

The category $\mathcal{H}(S_2)$ (under the additive Karoubi Envelope) is a diagrammatic version of the category of Soergel bimodules $\mathbb{S}\text{Bim}$ for S_2 . However $\mathbb{S}\text{Bim}$ is not well behaved with morphisms over \mathbb{Z} , so we must first alter the morphism spaces in $\mathcal{H}(S_2)$ to be over \mathbb{C} instead³. Formally we merely tensor the morphism spaces on the left by the \mathbb{C} - \mathbb{Z} -bimodule \mathbb{C} , where the right action is induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$. We write $\mathcal{H}_{\mathbb{C}}(S_2)$ for this \mathbb{C} -linear monoidal category. This process is quite simple and does little to the category itself. In particular, double leaves in $\mathcal{H}_{\mathbb{C}}(S_2)$ remain as $\mathbb{C}[\textcolor{red}{\bullet}]$ -bases⁴ for the morphisms.

Theorem 3.1.14 (Elias-Williamson [EW16, Theorem 6.30]). *The diagrammatic category $\text{Kar}^{\oplus}(\mathcal{H}_{\mathbb{C}}(S_2))$ and the category of Soergel Bimodules $\mathbb{S}\text{Bim}$ over S_2 are equivalent as graded \mathbb{C} -linear monoidal categories.*

The proof in [EW16] gives an equivalence of graded \mathbb{C} -linear monoidal categories $\mathcal{H}_{\mathbb{C}}(S_2) \cong \mathbb{B}\text{SBim}$ where $\mathbb{B}\text{SBim}$ is the category of Bott-Samelson bimodules over S_2 . Since the additive Karoubi envelope preserves equivalences, $\text{Kar}^{\oplus}(\mathbb{B}\text{SBim}) \cong \mathbb{S}\text{Bim}$ implies $\text{Kar}^{\oplus}(\mathcal{H}_{\mathbb{C}}(S_2)) \cong \mathbb{S}\text{Bim}$.

3.2 Diagrammatic $\mathcal{O}(\mathfrak{sl}_2)$

A little bit about category \mathcal{O} , and our example of \mathfrak{sl}_2

For this section, our category of interest is \mathcal{O} for the semisimple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. A description of the category \mathcal{O} can be found in general in [Hum08, Sections 3.8–3.10] or in [Maz09, Section 5.2] for the case of $\mathfrak{sl}_2(\mathbb{C})$, however we will only give a brief overview. The category \mathcal{O} is a category of certain modules (or representations) over a semisimple Lie algebra. It is a direct sum of subcategories, where, in the case of \mathfrak{sl}_2 over \mathbb{C} , the non-trivial summands are equivalent as abelian categories to the subcategory \mathcal{O}_0 . Within this, we look to the full subcategory $\text{proj}(\mathcal{O}_0)$ of projective modules in \mathcal{O}_0 , which, in particular, is additive and contains all direct summands.

In [Soe90, Section 2.4], Soergel shows that the category \mathcal{O} , and hence the subcategory $\text{proj}(\mathcal{O}_0)$, is a Soergel module category, i.e. it has an action of the monoidal category $\mathbb{S}\text{Bim}$. By the equivalence in Theorem 3.1.14 we will view $\text{proj}(\mathcal{O})$ as a $\mathcal{H}_{\mathbb{C}}(S_2)$ -module category, extending via the additive Karoubi envelope. Since $\mathcal{H}_{\mathbb{C}}(S_2)$ is diagrammatic, this action allows us to describe $\text{proj}(\mathcal{O}_0)$ (thus essentially \mathcal{O}_0 and \mathcal{O}) diagrammatically.

Remark 3.2.1. We can pass from $\text{proj}(\mathcal{O}_0)$ to \mathcal{O}_0 by observing that $K^b(\text{proj}(\mathcal{O}_0))$ is equivalent to $D^b(\mathcal{O}_0)$ as graded \mathbb{Z} -linear Should this be \mathbb{C} ? monoidal triangulated categories. This is a standard trick in the field, for example see the introduction of [RW18]⁵. However for our purposes it is not important to understand how this works.

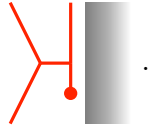
³The equivalence actually holds in more generality, but we choose \mathbb{C} because it is easy to work with.

⁴It is not hard to check that double leaves tensored with $1 \in \mathbb{C}$ on the left form a basis.

⁵A self-contained summary of how diagrammatic categories can be related to abelian categories.

Although we need to work over \mathbb{C} for $\text{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$, the diagrammatic category can be defined more simply, that is over \mathbb{Z} .

Definition 3.2.2. Let $\mathcal{DO}_0 := \mathcal{DO}_0(\mathfrak{sl}_2)$ be the \mathbb{Z} -linear (Define this in background) left $\mathcal{H}(S_2)$ -module category with elements generated (Define what this means.) by the monoidal identity \emptyset of $\mathcal{H}(S_2)$ and morphisms generated by the empty diagram \emptyset . The action of $\mathcal{H}(S_2)$ on the left is left concatenation for both objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbell on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0, 1] \times \mathbb{R}_{<0}$ instead of in the double-sided strip $[0, 1] \times \mathbb{R}$. For example a morphism may be



We impose the local relation that diagrams are related to the wall by

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \text{grey wall} \end{array} = 0. \quad (\text{W1})$$

Note that this local relations applies to any subdiagram involving the wall.

The objects of this category are identical to objects in $\mathcal{H}(S_2)$ and the morphisms are the same modulo the wall relation (W1). We can further simplify Example 3.1.4 from the previous section.

Example 3.2.3. Using the new relation (W1), we can further simplify the morphism in Example 3.1.4 by

$$\begin{aligned} & \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \text{grey wall} \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \text{grey wall} \end{array} - \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \text{grey wall} \end{array} \\ & = 2 \left(2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \text{grey wall} \end{array} - \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \text{grey wall} \end{array} \right) - 0 \\ & = 4 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \text{grey wall} \end{array}. \end{aligned}$$

A natural question to ask is whether double leaves still form bases for the morphism spaces here. Notice that double leaves appear in \mathcal{DO}_0 by acting on \emptyset by double leaves in

$\mathcal{H}(S_2)$. All morphisms in \mathcal{DO}_0 are morphisms in $\mathcal{H}(S_2)$ so they can be written as $\mathbb{Z}[\text{!}]$ -linear combinations of double leaves, though some have collapsed to 0. Thus double leaves span the morphism spaces of \mathcal{DO}_0 as (left) $\mathbb{Z}[\text{!}]$ -modules. However they may not be linearly independent as neither left nor right modules. For example, any pair of double leaves that factor through \emptyset become 0 when multiplied by ! on either side (by translating the barbell to the right). Although double leaves are not always a basis for its respective morphism space as $\mathbb{Z}[\text{!}]$ -modules, it turns out they are a basis over \mathbb{Z} .

Lemma 3.2.4. *Let $\pi : \text{mor}(\mathcal{H}(S_2)) \rightarrow \text{mor}(\mathcal{DO}_0)$ be the projection map which takes a morphism to the result of its action on the empty diagram \emptyset . Then the image $\pi(\mathbb{LL}(w, y))$ is a basis for $\text{Hom}_{\mathcal{DO}_0}(w, y)$ as a \mathbb{Z} -module.*

Proof. We consider morphisms $\text{Hom}(w, y)$ in \mathcal{DO}_0 for fixed objects w, y , and write $\mathbb{LL} := \pi(\mathbb{LL}(w, y))$ for the set of double leaves in \mathcal{DO}_0 . Any diagram in \mathcal{DO}_0 can be written as a \mathbb{Z} -linear combination of morphisms without floating diagrams, by simplifying them to barbells, pulling them to the right and killing them with (W1). We can write each of these as a $\mathbb{Z}[\text{!}]$ -linear combination of double leaves by (3.1.12) with the right action, and reduce it to a \mathbb{Z} -linear combination by (W1). This implies that \mathbb{LL} spans $\text{Hom}(w, y)$ as a \mathbb{Z} -module. Since the barbell-wall relation (W1) has no effect on \mathbb{Z} -linear combinations of \mathbb{LL} , it follows from linear independence over $\mathbb{Z}[\text{!}]$ that they are linearly independent over \mathbb{Z} in \mathcal{DO}_0 . *Check the proof.* \square

Our goal is to prove that this diagrammatic category is equivalent to $\text{proj}(\mathcal{O}_0)$. To that end, we will shift our focus from \mathbb{Z} to \mathbb{C} for the remainder of this section. From now on we write \mathcal{DO}_0 for the \mathbb{C} -linear $\mathcal{H}_{\mathbb{C}}(S_2)$ -module category obtained by replacing \mathbb{Z} with \mathbb{C} and $\mathcal{H}(S_2)$ with $\mathcal{H}_{\mathbb{C}}(S_2)$ in Definition 3.2.2. The above discussion and Lemma 3.2.4 still apply to \mathcal{DO}_0 over \mathbb{C} .

Maybe put this next bit in section 3.1

Say more about what this is, and why we say it here

The next result allows us to reduce all morphisms to a matrix of diagrams only involving \emptyset and s .

Lemma 3.2.5. *In the additive closure of $\mathcal{H}_{\mathbb{C}}(S_2)$ we have an explicit isomorphisms $ss \cong s \oplus s$, as detailed in the proof. Particularly, these are isomorphisms in the additive closure of \mathcal{DO}_0 .*

Proof. In $\mathcal{H}_{\mathbb{C}}(S_2)$ we have the relation

$$\begin{aligned}
 \begin{array}{|c|} \hline \\ \hline \end{array} &= \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \\
 &= \frac{1}{2} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} + \frac{1}{2} \begin{array}{|c|} \hline \bullet \\ \hline \end{array}
 \end{aligned}$$

$$= \frac{1}{2} \begin{array}{c} \bullet \bullet \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} + \frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \bullet \end{array} . \quad (3.2.6)$$

This implies that in $\mathcal{H}_{\mathbb{C}}(S_2)^{\oplus}$, we have maps

$$\begin{pmatrix} \frac{1}{2} \begin{array}{c} \bullet \bullet \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \\ \frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \end{pmatrix} : ss \rightarrow s \oplus s \text{ and } \begin{pmatrix} \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \\ \begin{array}{c} \bullet \bullet \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \end{pmatrix} : s \oplus s \rightarrow ss.$$

It follows from (R1d), (R1c) and the calculation (3.2.6), that these maps are inverses. Maybe put the inverse calculation here. \square

Be clear that I don't understand category \mathcal{O} very well.

As a shorthand, we write $\text{proj}(\mathcal{O}_0)$ for $\text{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$. The work of Soergel in [Soe90, Section 2.4] shows that $\text{proj}(\mathcal{O}_0)$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_{\varnothing}, \Theta_s \in \text{End}(\mathcal{O})$ (corresponding to elements in S_2). Explains what this means, how its related to the $\mathcal{H}(S_2)$ module category We construct a functor that maps faithfully into a full subcategory of $\text{proj}(\mathcal{O}_0)$, which will become the whole of $\text{proj}(\mathcal{O}_0)$ under the additive Karoubi envelope. This is the same strategy as in the proof for Theorem 3.1.14.

Definition 3.2.7. Let $F : \mathcal{DO}_0^{\oplus} \rightarrow \text{proj}(\mathcal{O}_0)$ be the \mathbb{C} -linear $\mathcal{H}_{\mathbb{C}}(S_2)$ -module functor that sends the empty object \varnothing to the trivial module $P(\varnothing)$, and the Soergel module action corresponding to s to the translation functor Θ_s . Then the object s maps to $\Theta_s(P(\varnothing)) =: P(s)$, and for example s^3 maps to the composition of three Θ_s on $P(\varnothing)$, $\Theta_s^3(P(\varnothing)) = \Theta_s \Theta_s \Theta_s(P(\varnothing))$. In order for F to be functorial, it must map identity diagrams $s^n \rightarrow s^n$ to $\text{id}_{\Theta_s^n(P(\varnothing))}$. On non-identity maps, we let $F(\uparrow)$ be the inclusion $i : P(\varnothing) \rightarrow P(s)$ and $F(\downarrow)$ be the projection $p : P(s) \rightarrow P(\varnothing)$. We then extend F by composition, additivity and linearity. The mapping of F is depicted by the following picture. Need to talk about compositions, why is F well defined if its generated by compositions of these? Are there any clashes? – Actually preserves compositions by construction

$$\begin{array}{ccc} \begin{array}{c} \varnothing \\ \downarrow \\ \varnothing \\ \downarrow \left(\begin{array}{c} \uparrow \quad \downarrow \end{array} \right) \uparrow \\ s \\ \downarrow \left(\begin{array}{c} \uparrow \quad \downarrow \end{array} \right) \uparrow \end{array} & \xrightarrow{F} & \begin{array}{c} \text{id}_{P(\varnothing)} \\ \downarrow \\ P(\varnothing) \\ \downarrow \left(\begin{array}{c} i \quad p \end{array} \right) \\ P(s) \\ \downarrow \left(\begin{array}{c} \text{id}_{P(s)} \quad i \circ p \end{array} \right) \end{array} \end{array} \quad (3.2.8)$$

Actually refer to the picture

Lemma 3.2.9. *The functor F is well defined.*

Proof. From [Maz09, Proposition 5.90], there is a natural isomorphism $\Theta_s \Theta_s \cong \Theta_s \oplus \Theta_s$ analogous to the isomorphism $ss \cong s \oplus s$ given in the proof of Lemma 3.2.5. We consider the additive closure \mathcal{DO}_0^\oplus , which does not cause problems since we will eventually take this anyway **Maybe just do additive karoubi closure in the first place**. Given a morphism in \mathcal{DO}_0 from s^n to s^m , repeated precomposition and postcomposition with $ss \rightarrow s \oplus s$ and $ss \oplus s \rightarrow s$ from Lemma 3.2.5 results in an isomorphic matrix of diagrams with domain and codomain in $\{\emptyset, s\}$. By Lemma 3.2.4 over \mathbb{C} , $\text{Hom}(\emptyset, \emptyset)$ has a basis $\{\emptyset = \text{id}_\emptyset\}$, $\text{Hom}(s, \emptyset)$ has a basis $\{\downarrow\}$, $\text{Hom}(\emptyset, s)$ has a basis $\{\uparrow\}$, and $\text{Hom}(s, s)$ has a basis $\{\text{id}_s, \uparrow \circ \downarrow\}$. Therefore, extending by linearity, the picture above completely describes the image of F .

Next we check that all the relations are preserved. From classical results e.g. [Maz09, Proposition 5.84 and Lemma 5.87], it follows that Θ_s is a Frobenius object in the category of endofunctors of \mathcal{O} . Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure. Applying these to $P(\emptyset)$ result in the same relations in $\text{proj}(\mathcal{O}_0)$ for $P(\emptyset)$, $P(s)$ and $\Theta_s^2(P(\emptyset))$. Note that the projection and inclusion maps above are exactly the unit and counit of Θ_s evaluated at $P(\emptyset)$, and the trivalent vertices provided by projecting the isomorphisms in Lemma 3.2.5 map exactly to the multiplication and comultiplication maps. Hence the Frobenius relations (R1a) and (R1b) are satisfied. **Ref for needle and barbell?** Furthermore, in [Soe90, Section 2.4] we see that $p \circ i = 0$ in $\text{proj}(\mathcal{O}_0)$ which is analogous⁶ to the barbell-wall relation (W1). Hence all the relations in \mathcal{DO}_0 are preserved by F . By construction, F preserves \mathbb{C} -linear combinations and the Soergel module structure in [Soe90], so F is well defined as a functor between \mathbb{C} -linear $\mathcal{H}(S_2)$ -module categories. \square

Theorem 3.2.10 (Soergel, [Soe90, Endomorphismensatz 7, Struktursatz 9 and Section 2.4]). *The diagrammatic category $\text{Kar}^\oplus(\mathcal{DO}_0(\mathfrak{sl}_2))$ and $\text{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$ are equivalent as \mathbb{C} -linear $\mathcal{H}(S_2)$ -module categories.*

Check all of this & Put precise references

Maybe write description as a soergel module outside the proof

Proof. First we show that F is full and faithful. It follows from Lemma 3.2.5 and the description of $P(\emptyset)$ and $P(s)$ in [Maz09, Section 5.2] that the image of \downarrow and \uparrow generate all morphisms of the form $\Theta_s^n(P(\emptyset)) \rightarrow \Theta_s^m(P(\emptyset))$. Hence F is full. Now the mapping of F on all morphism spaces are determined by those depicted in the above picture. So, for faithfulness, it suffices to compare the \mathbb{C} -dimensions of morphism spaces between objects shown in the picture. As mentioned above, the double leaves basis are precisely the diagrams depicted in the image. The bases for the corresponding morphism spaces in $\text{proj}(\mathcal{O}_0)$ are also those in the image **Ref? - that these are actually the bases of the hom spaces**, so the dimensions of Hom spaces coincide. Therefore F is fully faithful.

⁶This relation extends to the analogue of the local barbell-wall relation, as all ‘barbell on the right’ morphisms in $\text{proj}(\mathcal{O}_0)$ are linear combinations of applications of Θ_s to $p \circ i$, which is 0.

All objects in $\text{proj}(\mathcal{O}_0)$ appear as direct sums and direct summands of the elements $\Theta_s^n(P(\emptyset))$ for non-negative integers n . Therefore the additive Karoubi envelope induces an equivalence $\text{Kar}^\oplus(\mathcal{DO}_0) \cong \text{proj}(\mathcal{O}_0)$ as \mathbb{C} -linear left $\mathcal{H}(S_2)$ -module categories. \square

This result is essentially due to Soergel [Soe90, Endomorphismensatz 7, Struktursatz 9 and Section 2.4] (see also [Soe98]) but this was not its original formulation. Nevertheless we attribute this theorem to Soergel.

Maybe talk about Soergel modules and $\mathcal{H}(S_2)$ -modules vs $\text{Kar}^\oplus(\mathcal{H}(S_2))$ -modules

Remark 3.2.11. The morphisms spaces in \mathcal{DO}_0 are graded by the same grading as $\mathcal{H}(S_2)$ in Section 3.1. The equivalence $\text{Kar}^\oplus(\mathcal{DO}_0) \cong \text{proj}(\mathcal{O}_0)$ includes a grading of morphisms in $\text{proj}(\mathcal{O}_0)$ **Check!** and hence a grading morphisms of \mathcal{O} , which is otherwise ungraded.

Some more consequences

Chapter 4

Two-colour Diagrammatics

The previous chapter had its focus on the symmetric group generated by one element S_2 , which brought about one-colour diagrammatics. We now explore a more complex example by adding an extra generator, that is, another colour. In particular, we consider the affine symmetric group on two elements¹ $\tilde{S}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$. [Refine this](#)

4.1 Two-colour Diagrammatic Hecke Category

The definition is similar to the one-colour case, so we will be brief.

Definition 4.1.1. The *two-colour (diagrammatic) Hecke category* $\mathcal{H}(\tilde{S}_2)$ is a (strict) \mathbb{Z} -linear monoidal category given by the following presentation.

Objects in $\mathcal{H}(\tilde{S}_2)$ are generated by formal tensor products of the non-identity elements $s, t \in \tilde{S}_2$. As before, we write objects as words such as $sstttst =: s^2t^3st$ where the tensor product is concatenation, and associate the colour [red](#) to s and [blue](#) to t . The empty word is the monoidal identity, which we write as \emptyset .

The morphisms are generated by the univalent and trivalent vertices

$$\begin{array}{c} \bullet \\ | \\ \hline \end{array} \quad , \quad \begin{array}{c} | \\ \diagup \quad \diagdown \\ \hline \end{array} \quad , \quad \begin{array}{c} \bullet \\ | \\ \hline \end{array} \quad , \quad \begin{array}{c} | \\ \diagup \quad \diagdown \\ \hline \end{array} \quad (G2)$$

that are maps $s \rightarrow \emptyset$, $ss \rightarrow s$, $t \rightarrow \emptyset$ and $tt \rightarrow t$ respectively, and their vertical reflections. As in the one-colour case, tensor product is horizontal concatenation, composition is appropriate vertical stacking, and we denote the empty diagram $\emptyset \rightarrow \emptyset$ by \emptyset . For each colour, these diagrams have the one-colour relations given by [\(R1\)](#). As we have another colour, we need to describe how different colours interact. This is given by the

¹Also known as the infinite dihedral group.

two-colour relations

$$\begin{array}{c} \text{blue dot} \\ | \\ \text{blue dot} \end{array} \left| \begin{array}{c} \text{red line} \end{array} \right. = \left| \begin{array}{c} \text{blue dot} \\ | \\ \text{blue dot} \end{array} \right| + \left| \begin{array}{c} \text{red dot} \\ | \\ \text{red dot} \end{array} \right| - \left| \begin{array}{c} \text{red dot} \\ | \\ \text{red dot} \end{array} \right| \quad (\text{R2})$$

$$= \left| \begin{array}{c} \text{blue dot} \\ | \\ \text{blue dot} \end{array} \right| + 2 \left| \begin{array}{c} \text{red dot} \\ | \\ \text{red dot} \end{array} \right| - 2 \left| \begin{array}{c} \text{red dot} \\ | \\ \text{red dot} \end{array} \right| \quad (4.1.2)$$

and with red and blue swapped.

Example 4.1.3. Using the one-colour and two-colour relations on the following morphism in $\text{Hom}(ttsts, tst)$ we have

$$\begin{aligned} \text{Diagram 1} &= \text{Diagram 2} \\ &= 2 \left(\text{Diagram 3} - \text{Diagram 4} \right) \\ &= 2 \left(\text{Diagram 5} - \text{Diagram 6} - 2 \text{Diagram 7} + 2 \text{Diagram 8} \right) \\ &= \left(\text{Diagram 9} \otimes (2 \text{Diagram 10} + 2 \text{Diagram 11}) - \text{Diagram 12} \otimes (2 \text{Diagram 13} + 2 \text{Diagram 14}) \right). \end{aligned}$$

Talk about this containing $\mathcal{H}(S_2)$

Remark 4.1.4. Notice that the red and blue lines never cross as no generators that allow crossings. This is a consequence of working over affine S_2 in which the generators s and t have no relation. Mention example of crossing and S_3 .

In this two-colour case, [Proposition 3.1.5](#) holds by replacing (R1d) with (R2) in the proof. This handles the new possibility of floating subdiagrams with alternating colours.

Definition 4.1.5. For a group with a presentation in terms of generators and relations, the *length* of a product of generators is the number of generators in the product. We say that a product of generators is *reduced* if it's length cannot be shortened with relations.

In \tilde{S}_2 products can be shortened by the relation $s^2 = t^2 = 1$. For instance, $sttsts$ is not reduced because it is equal to ts which is reduced. Notice that for \tilde{S}_2 each element

can be written uniquely as a reduced product of generators. This is true since otherwise we have two distinct reduced products for the same element in \tilde{S}_2 so they must be related by $s^2 = t^2$. This means they can be reduced further by $s^2 = t^2 = 1$, which contradicts minimality of their length. Note that the reduced products in \tilde{S}_2 are either the identity or alternating products of s and t .

Notice that there is a notational similarity between products in the group and words in $\mathcal{H}(\tilde{S}_2)$. This motivates the following definitions. Let $\phi : (\text{ob}(\mathcal{H}(\tilde{S}_2)), \otimes) \rightarrow (\tilde{S}_2, *)$ be the monoid homomorphism mapping $\emptyset \mapsto 1$, $s \mapsto s$ and $t \mapsto t$. Also define the function $\psi : \tilde{S}_2 \rightarrow \text{ob}(\mathcal{H}(\tilde{S}_2))$ to map elements $x \in \tilde{S}_2$ to the tensor product of s and t in $\mathcal{H}(\tilde{S}_2)$ corresponding to the reduced product of x in \tilde{S}_2 . This is well defined because reduced products are unique and two different reduced products cannot equal the same element of \tilde{S}_2 . Note that the image $\psi(\tilde{S}_2)$ is the set containing \emptyset and words of alternating s and t . The composition $\psi\phi : \mathcal{H}(\tilde{S}_2) \rightarrow \mathcal{H}(\tilde{S}_2)$ maps words w to the tensor of s and t corresponding to the reduced product of $\phi(w)$, and $\phi\psi$ is the identity map on \tilde{S}_2 .

The following definition is a more general version of [Definition 3.1.6](#).

Definition 4.1.6 (Subexpression). Given a word w of length n , a *subexpression* e is a binary string of length n . A subexpression can be *applied* to produce an word $w(e)$, which is w where terms corresponding to 0 in e are replaced with \emptyset . For $1 \leq i \leq n$, we write $w(e, i)$ for the result of the first i terms of e applied to the first i terms in w . Particularly $w(e, 0) = \emptyset$ and $w(e, n) = w(e)$.

For example, in $\mathcal{H}(\tilde{S}_2)$, if $w = sttts$ and $e = 11001$ then $w(e) = st\emptyset\emptyset s = sts$ and $w(e, 3) = sts(110) = st\emptyset = st$ in $\mathcal{H}(\tilde{S}_2)$.

Let the *length* of a word be the number of generators in its tensor product. As before, given an object w and a subexpression e of w , we label each of the n terms by one of U_0, U_1, D_0, D_1 . Let $i \geq 0$, and write x for the i -th term of w . We label the i -th term U_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. In other words we write U_* if the next term of w will make $\psi\phi$ applied to the partially evaluated subexpression longer, regardless of the i -term of e . We label D_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. The label's subscript is the i -th term of e . Note that this construction is well defined because $\psi\phi(w(e, i-1) \otimes x) = \psi(\phi(w(e, i-1)) * \phi(x)) = \psi(\phi(w(e, i-1)) * x)$ is always either longer or shorter, since the last element of the reduced product is either the same as x or different. When they are the same, the word is shorter via $s^2 = t^2 = 1$, and when they are different it is longer as no relations can make it shorter.

Remark 4.1.7. This description of the labels (via. reduced products) is more akin to the definition in general for Coxeter groups, than in [Section 3.1](#). [Mention the words coxeter group in intro](#)

Example 4.1.8. Consider the word $w = sttst$ and subexpression $e = 10011$. The labels can be constructed as in the following table.

Example 4.1.12. Let $w = sst$ with the subexpression $e = 101$ and $y = tstst$ with the subexpression $f = 01001$. The corresponding light leaves are

$$LL_{w,e} = \begin{array}{c} \text{red arc} \\ \text{blue line} \\ U_1 \quad D_0 \quad U_1 \end{array} \quad \text{and} \quad LL_{y,f} = \begin{array}{c} \text{blue dots} \\ \text{red arc} \\ \text{blue line} \\ U_0 \quad U_1 \quad U_0 \quad D_0 \quad U_1 \end{array} .$$

Then the double leaf $\mathbb{L}L_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : sst \rightarrow tstst$, factoring through st , is

$$\begin{array}{c} \overline{LL}_{y,f} \\ \text{blue dots} \\ \text{red arc} \\ \text{blue line} \\ LL_{w,e} \\ \text{red arc} \\ \text{blue line} \end{array} .$$

As with the one-colour case, the set of double leaves $\mathbb{L}L(w, y)$ from words w to y in $\mathcal{H}(\tilde{S}_2)$ form a basis for $\text{Hom}(w, y)$ over $\mathbb{Z}[\text{red}, \text{blue}]$. The Hom spaces are graded such that the univalent vertices have degree 1 and trivalent vertices have degree -1 for either colour.

Let $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$ be the \mathbb{C} -linear monoidal category obtained by extending the scalars of morphisms spaces in $\mathcal{H}(\tilde{S}_2)$ from \mathbb{Z} with \mathbb{C} . All the results above also hold for $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$. Additionally, a result similar to [Theorem 3.1.14](#) states that $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$ is equivalent to SBim over \tilde{S}_2 as graded [is this graded?](#) \mathbb{C} -linear monoidal categories.

Remark 4.1.13. The construction of the diagrammatic Hecke category, light leaves, [Theorem 3.1.12](#) and [Theorem 3.1.14](#) all generalise to general Coxeter groups. The details can be found in [\[EW16\]](#).

4.2 Diagrammatic $\text{Tilt}(\mathfrak{sl}_2)$

[Something something about Tilt](#)

[Something something about extending \$\mathcal{H}\(\tilde{S}_2\)\$ from \$\mathbb{Z}\$ to \$\mathbb{C}\$.](#)

Although need to work over \mathbb{C} for $\text{Tilt}(\mathfrak{sl}_2)$, the diagrammatic category below can be defined more simply over \mathbb{Z} .

Definition 4.2.1. Let $\mathcal{DT} := \mathcal{DT}(\mathfrak{sl}_2)$ be the \mathbb{Z} -linear left $\mathcal{H}(\tilde{S}_2)$ -module category with elements generated by the monoidal identity \emptyset of $\mathcal{H}(\tilde{S}_2)$, and morphisms generated by the empty diagram \emptyset . The action of $\mathcal{H}(\tilde{S}_2)$ on the left is left concatenation for objects and morphisms. The relations on diagrams in $\mathcal{DT}(\mathfrak{sl}_2)$ are inherited from those in $\mathcal{H}(\tilde{S}_2)$. Additionally, we imagine a wall on the right of diagrams and impose the local wall-annihilation relations

$$\begin{array}{c} \text{red dot} \\ \text{red line} \end{array} \text{ wall} = \begin{array}{c} \text{blue line} \end{array} \text{ wall} = 0. \quad (\text{W2})$$

In other words, if a red barbell or blue string can come close to the wall without anything in between, then the diagram is 0. Note that local relations in (W2) involve the wall.

Example 4.2.2. The morphism in Example 4.1.3 collapses to 0 because all the diagrams have either blue or barbell on the right.

TODO: Another example clarifying 'blue on the right'

Using (W2) we can extend (R2).

Proposition 4.2.3. *In the following diagrams, the domain and codomain alternate colours and to the left of the ellipsis, pink and purple represent different colours in {red, blue}. For integers $k \geq 1$*

$$\begin{array}{c} \text{pink barbell} \quad \underbrace{\text{purple} \quad \text{purple} \quad \dots}_{k} \quad \text{red} \quad \text{grey wall} = -2 \quad \underbrace{\text{purple} \quad \text{purple} \quad \dots}_{k} \quad \text{red} \quad \text{grey wall} \end{array} \quad (4.2.3a)$$

$$\begin{array}{c} \text{purple barbell} \quad \underbrace{\text{purple} \quad \text{purple} \quad \dots}_{k} \quad \text{red} \quad \text{grey wall} = 2 \quad \underbrace{\text{purple} \quad \text{purple} \quad \dots}_{k} \quad \text{red} \quad \text{grey wall} \end{array} \quad (4.2.3b)$$

and for $k \geq 3$

$$\begin{array}{c} \text{pink} \quad \text{purple} \quad \underbrace{\text{purple} \quad \text{purple} \quad \dots}_{k} \quad \text{red} \quad \text{grey wall} = 0. \end{array} \quad (4.2.3c)$$

Proof. For $k \in \{1, 2\}$, we check the second two relations by hand. For $k = 1$, pulling the barbell through the line using (R1d) and (R2), then applying (W2) gives us

$$\begin{array}{c} \text{blue barbell} \quad \text{red} \quad \text{grey wall} = \text{red} \quad \text{blue barbell} \quad \text{grey wall} + 2 \quad \text{red} \quad \text{red barbell} \quad \text{grey wall} - 2 \quad \text{red} \quad \text{red barbell} \quad \text{grey wall} = -2 \quad \text{red} \quad \text{red barbell} \quad \text{grey wall} \end{array}$$

and

$$\begin{array}{c} \text{red barbell} \quad \text{red} \quad \text{grey wall} = 2 \quad \text{red} \quad \text{red barbell} \quad \text{grey wall} - \text{red} \quad \text{red barbell} \quad \text{grey wall} = 2 \quad \text{red} \quad \text{red barbell} \quad \text{grey wall} . \end{array}$$

By a similar proof, using the $k = 1$ relations locally, we have for $k = 2$,

$$\begin{aligned}
& \text{Diagram 1} = \text{Diagram 2} + 2 \text{Diagram 3} - 2 \text{Diagram 4} \\
& \stackrel{(k=1)}{=} 2 \text{Diagram 5} + 2(-2) \text{Diagram 6} - 2 \text{Diagram 7} = -2 \text{Diagram 8}
\end{aligned}$$

and

$$\begin{aligned}
& \text{Diagram 1} = 2 \text{ Diagram 2} - \text{Diagram 3} \\
& \stackrel{(k=1)}{=} 2 \text{ Diagram 4} - (-2) \text{ Diagram 5} = 2 \text{ Diagram 6}
\end{aligned}$$

Now we proceed by induction on k . For $k = 3$ we first show (4.2.3c). By a similar argument to (3.2.6) we have

$$\begin{aligned}
 & \left(\text{Diagram 1} \right) = \left(\text{Diagram 2} \right) = \frac{1}{2} \left(\text{Diagram 3} \right) + \frac{1}{2} \left(\text{Diagram 4} \right) \\
 & \stackrel{(k=1)}{=} \frac{2}{2} \left(\text{Diagram 5} \right) + \frac{2}{2} \left(\text{Diagram 6} \right) = 0
 \end{aligned}$$

since the wall is accessible by the blue dot. Then

$$\begin{aligned}
& \text{Diagram 1} = \text{Diagram 2} + 2 \text{Diagram 3} - 2 \text{Diagram 4} \\
& \stackrel{(k=2)}{=} 2 \text{Diagram 5} + 2(-2) \text{Diagram 6} - 2 \text{Diagram 7} \\
& = -2 \text{Diagram 8}
\end{aligned}$$

and

$$\begin{aligned}
&= 2 \left(\text{diagram with two red vertical lines and a red dot on the left line} \right) - \left(\text{diagram with two red vertical lines and a red dot on the right line} \right) \\
&\stackrel{(k=2)}{=} 2 \left(\text{diagram with two red vertical lines and a red dot on the left line} \right) - (-2) \left(\text{diagram with two blue vertical lines and a blue dot on the right line} \right) \\
&= 2 \left(\text{diagram with two red vertical lines and a red dot on the left line} \right).
\end{aligned}$$

Let $k \geq 4$ and assume the relations hold for diagrams with $k-1, k-2, \dots, 1$. Again, the argument to (3.2.6) implies

$$\begin{aligned}
& \underbrace{\left| \begin{array}{c} \text{Diagram with } k \text{ vertical lines and dots} \end{array} \right\rangle} = \frac{1}{2} \left| \begin{array}{c} \text{Diagram with } k \text{ vertical lines and dots} \end{array} \right\rangle + \frac{1}{2} \left| \begin{array}{c} \text{Diagram with } k \text{ vertical lines and dots} \end{array} \right\rangle \\
& \stackrel{\text{ind.}}{=} \frac{2}{2} \left| \begin{array}{c} \text{Diagram with } k \text{ vertical lines and dots} \end{array} \right\rangle + \frac{2}{2} \left| \begin{array}{c} \text{Diagram with } k \text{ vertical lines and dots} \end{array} \right\rangle \\
& = \left| \begin{array}{c} \text{Diagram with } k \text{ vertical lines and dots} \end{array} \right\rangle + \left| \begin{array}{c} \text{Diagram with } k \text{ vertical lines and dots} \end{array} \right\rangle \\
& \stackrel{\text{ind.}}{=} 0
\end{aligned}$$

where the rightmost pink string is the right red string when $k = 4$. Furthermore, we have

$$\begin{aligned}
& \underbrace{\left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \dots \left| \begin{array}{c} \text{red} \end{array} \right|}_{k} = \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \dots \left| \begin{array}{c} \text{red} \end{array} \right| + 2 \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \dots \left| \begin{array}{c} \text{red} \end{array} \right| - 2 \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \dots \left| \begin{array}{c} \text{red} \end{array} \right| \\
& \stackrel{\text{ind.}}{=} 2 \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \dots \left| \begin{array}{c} \text{red} \end{array} \right| + 2(-2) \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \dots \left| \begin{array}{c} \text{red} \end{array} \right| - 2 \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \left| \begin{array}{c} \text{purple} \\ \text{purple} \end{array} \right| \dots \left| \begin{array}{c} \text{red} \end{array} \right|
\end{aligned}$$

$$= -2 \begin{array}{c} \text{purple dot} \\ | \\ \text{purple dot} \end{array} \begin{array}{c} | \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ | \end{array}$$

and

$$\begin{aligned} \begin{array}{c} \text{purple dot} \\ | \\ \text{purple dot} \end{array} \underbrace{\begin{array}{c} | \\ | \\ | \end{array}}_k \cdots \begin{array}{c} | \\ | \\ | \end{array} &= 2 \begin{array}{c} \text{purple dot} \\ | \\ \text{purple dot} \end{array} \begin{array}{c} | \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ | \end{array} - \begin{array}{c} \text{purple dot} \\ | \\ \text{purple dot} \end{array} \begin{array}{c} | \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ | \end{array} \\ &\stackrel{\text{ind.}}{=} 2 \begin{array}{c} \text{purple dot} \\ | \\ \text{purple dot} \end{array} \begin{array}{c} | \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ | \end{array} - (-2) \begin{array}{c} \text{purple dot} \\ | \\ \text{purple dot} \end{array} \begin{array}{c} | \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ | \end{array} \\ &= 2 \begin{array}{c} \text{purple dot} \\ | \\ \text{purple dot} \end{array} \begin{array}{c} | \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ | \end{array}. \end{aligned}$$

□

The objects of this category are identical to objects in $\mathcal{H}(\tilde{S}_2)$ and the morphisms are the same modulo the wall relations (W2). Naturally, we wonder whether double leaves form bases for the morphism spaces in \mathcal{DT} . It is easy to see that double leaves appear in \mathcal{DT} by acting on \emptyset by double leaves in $\mathcal{H}(\tilde{S}_2)$. All morphisms in \mathcal{DT} are morphisms in $\mathcal{H}(\tilde{S}_2)$ so they can be written as $\mathbb{Z}[\text{red dot}, \text{blue dot}]$ -linear combinations of double leaves, though some of these leaves have collapsed to 0. This makes it clear that double leaves span the morphism spaces of \mathcal{DT} as (left) $\mathbb{Z}[\text{red dot}, \text{blue dot}]$ -modules. However they may not be linearly independent as neither left nor right modules as with the one-colour case. Although double leaves are not always a basis for its respective morphism space as $\mathbb{Z}[\text{red dot}, \text{blue dot}]$ -modules, it turns out a subset of them are a basis over \mathbb{Z} .

Lemma 4.2.6. *Let $\pi : \text{mor}(\mathcal{H}(\tilde{S}_2)) \rightarrow \text{mor}(\mathcal{DT})$ be the projection map which takes a morphism to the result of its action on the empty diagram \emptyset . Then the image $\pi(\mathbb{LL}(w, y))$ without zero morphisms is a basis for $\text{Hom}_{\mathcal{DT}}(w, y)$ as a \mathbb{C} -module.*

Proof. Consider morphisms $\text{Hom}(w, y)$ in \mathcal{DT} for fixed objects w, y , and write $\mathbb{LL} := \pi(\mathbb{LL}(w, y))$ for the set of double leaves in \mathcal{DT} . Any diagram in \mathcal{DT} can be written as a \mathbb{C} -linear combination of morphisms without floating diagrams by pulling floating diagrams to the right with (R1d) and (R2) then applying the wall relation (W2). We can write each of these as a $\mathbb{Z}[\text{red dot}, \text{blue dot}]$ -linear combination of double leaves with a right action, and reduce it to a \mathbb{Z} -linear combination by (W2). This implies that \mathbb{LL} spans $\text{Hom}(w, y)$ as a \mathbb{Z} -module. Now \mathbb{LL} may not be linearly independent because the two-colour wall relation (W2) reduces all diagrams factoring through a word ending with t

to 0. The set of light leaves after removing morphisms killed by (W2), i.e. $\mathbb{LL} \setminus \{0\}$, still spans $\text{Hom}(w, y)$ by the argument above. This set is linearly independent since, by construction, (W2) has no effect on \mathbb{Z} -linear combinations of $\mathbb{LL} \setminus \{0\}$. Then it follows from linear independence over $\mathbb{Z}[\textcolor{red}{\downarrow}, \textcolor{blue}{\downarrow}]$ that this set is linearly independent over \mathbb{Z} in \mathcal{DT} . \square

We aim to show that the additive Karoubi envelope of this diagrammatic category is equivalent to $\text{Tilt}(\mathfrak{sl}_2)$. From now on, we write \mathcal{DT} for the \mathbb{C} -linear $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$ -module category obtained by extending scalars from \mathbb{Z} with \mathbb{C} . All the above discussion and results still apply to \mathcal{DT} over \mathbb{C} . For brevity we also write \mathcal{T} for $\text{Tilt}(\mathfrak{sl}_2)$.

Since $\mathcal{H}_{\mathbb{C}}(S_2)$ appears inside $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$ for each colour, Lemma 3.2.5 provides explicit isomorphisms $ss \cong s \oplus s$ and $tt \cong t \oplus t$ in the additive closure of $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$.

Definition 4.2.7. Let $F : \mathcal{DT}^{\oplus} \rightarrow \text{Tilt}(\mathfrak{sl}_2)$ to be the additive \mathbb{C} -linear $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$ -module functor defined as follows. Map the empty word \emptyset to the trivial module $T(\emptyset)$ **Is that right to say?** Given a general word $s_n \dots s_1$ in \mathcal{DT} , for $s_i \in \{s, t\}$, map $F(s_n \dots s_1) = \Theta_{s_n} \dots \Theta_{s_1} T(\emptyset)$ where Θ_s, Θ_t are translation functors associated to generators of \tilde{S}_2 .

On morphisms, we define F recursively. Note that we only have red strands on the right since otherwise (W2) reduces it to 0. For $k \geq 0$, define

$$\text{id}_k^d := \underbrace{\begin{array}{c} | \\ | \\ \vdots \\ | \end{array}}_k \quad , \quad i_k^d := \begin{array}{c} \bullet \\ | \\ \vdots \\ | \end{array} \underbrace{\begin{array}{c} | \\ | \\ \vdots \\ | \end{array}}_k \quad , \quad p_k^d := \begin{array}{c} \bullet \\ | \\ \vdots \\ | \end{array} \underbrace{\begin{array}{c} | \\ | \\ \vdots \\ | \end{array}}_k$$

where colours alternate and a red strand on the right when $k \neq 0$. Similarly we define $\text{id}_k : \Theta_x \dots \Theta_s(T(\emptyset)) \rightarrow \Theta_x \dots \Theta_s(T(\emptyset))$, $i_k : \Theta_x \dots \Theta_s(T(\emptyset)) \rightarrow \Theta_y \Theta_x \dots \Theta_s(T(\emptyset))$ and $p_k : \Theta_y \Theta_x \dots \Theta_s(T(\emptyset)) \rightarrow \Theta_x \dots \Theta_s(T(\emptyset))$ to be the identity, inclusion and projection maps in \mathcal{T} , where the subscripts alternate s, t and $\Theta_x \dots \Theta_s$ is a composition of k translation functors. Further we write $\tilde{p}_k := \frac{1}{(-2)^{k+1}} p_k$. Let $F(\text{id}_k^d) = \text{id}_k$. On the generators (G2) of \mathcal{DT} , map

$$\begin{array}{l} \begin{array}{c} | \\ \vdots \\ | \end{array} \boxed{\text{id}_k^d} \begin{array}{c} | \\ | \\ \vdots \\ | \end{array} \xrightarrow{F} \begin{cases} \text{id}_{k+1}, & \text{if } k \text{ even,} \\ \begin{pmatrix} \text{id}_k & 0 \\ 0 & \text{id}_k \end{pmatrix}, & \text{if } k \text{ odd,} \end{cases} \\ \begin{array}{c} \bullet \\ | \\ \vdots \\ | \end{array} \boxed{\text{id}_k^d} \begin{array}{c} | \\ | \\ \vdots \\ | \end{array} \xrightarrow{F} \begin{cases} \tilde{p}_k, & \text{if } k \text{ even,} \\ \begin{pmatrix} i_{k-1} \circ \tilde{p}_{k-1} \\ \text{id}_k \end{pmatrix}, & \text{if } k \text{ odd,} \end{cases} \\ \begin{array}{c} \bullet \\ | \\ \vdots \\ | \end{array} \boxed{\text{id}_k^d} \begin{array}{c} | \\ | \\ \vdots \\ | \end{array} \xrightarrow{F} \begin{cases} i_k, & \text{if } k \text{ even,} \\ \begin{pmatrix} \text{id}_k & i_{k-1} \circ \tilde{p}_{k-1} \end{pmatrix}, & \text{if } k \text{ odd,} \end{cases} \end{array}$$

$$\begin{array}{c}
\text{Red fork} \quad \left(\begin{array}{c} \cdots \\ \text{id}_k^d \\ \cdots \end{array} \right) \quad \text{Grey wall} \quad \xrightarrow{F} \begin{cases} \begin{pmatrix} 0 & \text{id}_{k+1} \end{pmatrix}, & \text{if } k \text{ even,} \\ \begin{pmatrix} 0 & 0 & \text{id}_k & 0 \\ 0 & 0 & 0 & \text{id}_k \end{pmatrix}, & \text{if } k \text{ odd,} \end{cases} \\
\\
\text{Blue fork} \quad \left(\begin{array}{c} \cdots \\ \text{id}_k^d \\ \cdots \end{array} \right) \quad \text{Grey wall} \quad \xrightarrow{F} \begin{cases} \begin{pmatrix} \text{id}_{k+1} \\ 0 \end{pmatrix}, & \text{if } k \text{ even,} \\ \begin{pmatrix} \text{id}_k & 0 \\ 0 & \text{id}_k \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } k \text{ odd,} \end{cases}
\end{array}$$

where each entry in the matrix are matrices themselves. **CHECK THE SCALING!!!!!!!**
For **blue** generators, the definition is the same with the words even and odd swapped.
Putting a **red** (resp. **blue**) identity strands on the left of a diagram is applying Θ_s (resp. Θ_t) to the output morphism. Pictorially, for a morphism f in \mathcal{DT} ,

$$\left(\begin{array}{c} \cdots \\ f \\ \cdots \end{array} \right) \quad \text{Grey wall} \quad \xrightarrow{F} \Theta_s F(f).$$

We extend the functor by composition, additivity and linearity.

The mappings that don't involve matrices are summarised in the picture below.

$$\begin{array}{ccc}
\begin{array}{c} \emptyset \\ \downarrow \\ \emptyset \\ \downarrow \\ \text{Red strand} \downarrow \text{Red strand} \\ \text{Red strand} \downarrow \text{Red strand} \\ \text{Blue strand} \downarrow \text{Blue strand} \\ \text{Blue strand} \downarrow \text{Blue strand} \\ \text{Red strand} \downarrow \text{Red strand} \\ \text{Red strand} \downarrow \text{Red strand} \\ \vdots \end{array} & \xrightarrow{F} & \begin{array}{c} \text{id}_0 \\ \downarrow \\ T(\emptyset) \\ \downarrow \\ i_0 \left(\begin{array}{c} \text{Red strand} \downarrow \text{Red strand} \end{array} \right) p_0 \\ \downarrow \\ \text{id}_1 \left(\begin{array}{c} \text{Blue strand} \downarrow \text{Blue strand} \end{array} \right) \Theta_s T(\emptyset) \text{id}_1 \circ p_0 \\ \downarrow \\ i_1 \left(\begin{array}{c} \text{Red strand} \downarrow \text{Red strand} \end{array} \right) \Theta_t \Theta_s T(\emptyset) \text{id}_1 \circ p_1 \\ \downarrow \\ \text{id}_2 \left(\begin{array}{c} \text{Blue strand} \downarrow \text{Blue strand} \end{array} \right) \Theta_s \Theta_t \Theta_s T(\emptyset) \text{id}_2 \circ p_2 \\ \downarrow \\ i_2 \left(\begin{array}{c} \text{Red strand} \downarrow \text{Red strand} \end{array} \right) \Theta_s \Theta_t \Theta_s T(\emptyset) \text{id}_2 \circ p_2 \\ \downarrow \\ \vdots \end{array} \end{array} \quad (4.2.8)$$

The right wall on each diagram is not shown to reduce clutter.

The definition on generators is a consequence of the isomorphism $\Theta_s \Theta_s \cong \Theta_s \oplus \Theta_s$ analogous to $ss \cong s \oplus s$ (and respectively for t) from **Lemma 3.2.5**.

Remark 4.2.9. The action of an arbitrary morphism of $\mathcal{H}(\tilde{S}_2)$ on the left of a morphism in \mathcal{DT} is sent to the Godement product² of the natural transformations underlying the image of morphisms under F . Taking the Godement product of natural transformations $\Theta_x \dots \Theta_s \rightarrow \Theta_y \dots \Theta_s$, when viewed as diagrams in \mathcal{DT} , is just a left tensor of the corresponding diagrams. Visually, the construction looks like putting identity morphisms on the left of one morphism on the right of the other, so that the codomains align, and then composing them. In \mathcal{T} , this is the Kronecker product of matrices.

Lemma 4.2.10. *The functor F is well defined as a functor between additive \mathbb{C} -linear $\mathcal{H}(S_2)$ -module categories.*

Proof. By Remark 4.2.9, the definition of F defines an action of every morphism in \mathcal{DT}^\oplus . It remains to check that all relations are preserved. It follows from [AT17, Proposition 2.35] Check ref translation functors Θ_s, Θ_t are Frobenius objects in the category of endofunctors of \mathcal{T} and there are unit, counit, multiplication and comultiplication natural transformations and corresponding relations from the Frobenius object structure. Applying these to $T(\emptyset)$ result in Frobenius object relations in \mathcal{T} for $T(\emptyset), \Theta_s T(\emptyset)$ and $\Theta_s^2(T(\emptyset))$. Note that \downarrow and \uparrow map to i_0 and \tilde{p}_0 Is the scaling here right? which are exactly the unit and counit of Θ_s evaluated at $T(\emptyset)$ (up to scaling), and the trivalent vertices defined with id_0^d are mapped exactly to the multiplication and comultiplication maps. The isomorphism Lemma 3.2.5 we use to reduce domain and codomain has an analogue $\Theta_s \circ \Theta_s \cong \Theta_s \oplus \Theta_s$ as in [AT17, Corollary 2.35(a)], and similarly for t . Furthermore, in [AT17, Section 2.5] we see that $p_0 \circ i_0 = 0$, $p_{k+1} \circ i_{k+1} = i_k \circ p_k$ that are analogous to the relations in Proposition 4.2.3, up to an adjusting scalar given in the definition. From [AT17, Corollary 2.35] the translation functors satisfy properties analogous to the two-colour wall relations (W2). The proof in [AT17, Lemma 4.26] show that the remaining relations (R1c), (R1d), (R2) and (W2) all hold in \mathcal{T} . Hence all the relations in \mathcal{DT} are preserved by F . By construction, F preserves direct sums, \mathbb{C} -linear combinations and the Soergel module structure, so F is well defined as a functor between additive \mathbb{C} -linear $\mathcal{H}(S_2)$ -module categories.

Talk about how extending by composition makes sense. □

Say something here?

The following result states that \mathcal{DT} is indeed a diagrammatic incarnation of $\text{Tilt}(\mathfrak{sl}_2)$.

Be clear that I don't understand Tilt very well.

Theorem 4.2.11 (???). *The diagrammatic category $\text{Kar}^\oplus(\mathcal{DT}(\mathfrak{sl}_2))$ and $\text{Tilt}(\mathfrak{sl}_2)$ are equivalent as additive Karoubian \mathbb{C} -linear $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$ -module categories.*

Proof. Since \mathcal{T} is additive and Karoubian, our functor F extends uniquely to a functor $F' : \text{Kar}^\oplus(\mathcal{DT}) \rightarrow \mathcal{T}$. By the argument in [AT17, Theorem 4.27], every element in \mathcal{T} is isomorphic to F' applied to a direct sum of images of Jones–Wenzl projectors, so F' is essentially surjective. Particularly, this shows that the images of JW-projectors map exactly to the indecomposable ‘leading’ tilting modules.

²The horizontal composition of natural transformations.

By Lemma 3.2.5 and (W2), we just consider words with alternating generators and ending with s . Write $T(...ts)$ for the leading indecomposable summand of $...\Theta_t\Theta_s(T(\emptyset))$ in \mathcal{T} , and write $b_{...ts}$ for the image of $\text{JW}_{...ts}$. By [Eli16, Section 5.4.2], Jones–Wenzl projectors are primitive idempotents and their images are all the indecomposables in \mathcal{DT} , and as mentioned above they map to the leading indecomposables in \mathcal{T} . For full and faithfulness, it is sufficient to check that the dimensions of the morphism spaces between indecomposables $\text{Hom}_{\mathcal{DT}}(b_{x...ts}, b_{y...ts})$ and $\text{Hom}_{\mathcal{T}}(T(x...ts), T(y...ts))$ coincide. On the diagrammatic side, a morphism $b_{x...ts} \rightarrow b_{y...ts}$ is given by $\text{JW}_{y...ts} f \text{JW}_{x...ts}$ where $f : x...ts \rightarrow y...ts$. Now consider f as a linear combination of double leaves. For $\text{JW}_{y...ts} f \text{JW}_{x...ts}$, all double leaves with pitchforks are killed This is from [AT17, Lemma 4.14], put in sec4.1. Since the domain and codomain alternate colours, the remaining diagrams are a tensor and composition of \downarrow , \uparrow and identity strands. Notice that we have the relation

$$\begin{aligned}
\begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \text{---} \Big| &= \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \text{---} \Big| = \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \text{---} \Big| + \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \text{---} \Big| \\
&= \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \text{---} \Big| + \frac{2}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \text{---} \Big| = \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \text{---} \Big|.
\end{aligned}$$

Proceeding inductively on the number of identity strands on the right, we have

$$\begin{aligned}
\begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big| &= \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big| = \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big| + \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big| \\
&= \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big| + \frac{2}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big| \\
&= \frac{1}{2} \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big| + \begin{array}{c} \downarrow \\ \uparrow \end{array} \Big| \cdots \Big|
\end{aligned}$$

where the third equality follows from Proposition 4.2.3. By induction the second term is a linear combination of diagrams with pitchforks, hence this diagram is a linear combinations of diagrams with pitchforks. Particularly, these are killed by Jones–Wenzl projectors. The same holds for the vertically reflected diagram. Along with Proposition 4.2.3, we conclude that the only double leaves we should consider are id_k , i_k^d , p_k^d and their composition $i_k^d \circ p_k^d$. This is informally summarised by the diagram below (similar

to (4.2.8)).

$$\begin{array}{ccc}
\begin{array}{c}
\emptyset \\
\downarrow \\
\emptyset \\
\downarrow \left(\begin{array}{c} \text{red dot} \end{array} \right) \left(\begin{array}{c} \text{red dot} \end{array} \right) \\
| \quad \left(\begin{array}{c} \text{red dot} \end{array} \right) \left(\begin{array}{c} \text{red dot} \end{array} \right) \quad | \\
\downarrow \left(\begin{array}{c} \text{blue dot} \end{array} \right) \left(\begin{array}{c} \text{blue dot} \end{array} \right) \\
|| \quad \left(\begin{array}{c} \text{blue dot} \end{array} \right) \left(\begin{array}{c} \text{blue dot} \end{array} \right) \quad || \\
\downarrow \left(\begin{array}{c} \text{red dot} \end{array} \right) \left(\begin{array}{c} \text{red dot} \end{array} \right) \\
||| \quad \left(\begin{array}{c} \text{red dot} \end{array} \right) \left(\begin{array}{c} \text{red dot} \end{array} \right) \quad ||| \\
\downarrow \left(\begin{array}{c} \text{blue dot} \end{array} \right) \left(\begin{array}{c} \text{blue dot} \end{array} \right) \\
\vdots
\end{array}
& \xrightarrow{F'} &
\begin{array}{c}
\text{id}_0 \\
\downarrow \\
T(\emptyset) \\
\downarrow \left(\begin{array}{c} \text{red dot} \end{array} \right) p_0 \\
\text{id}_1 \left(\begin{array}{c} \text{red dot} \end{array} \right) T(s) \left(\begin{array}{c} \text{red dot} \end{array} \right) i_0 \circ p_0 \\
\downarrow \left(\begin{array}{c} \text{blue dot} \end{array} \right) p_1 \\
\text{id}_2 \left(\begin{array}{c} \text{blue dot} \end{array} \right) T(ts) \left(\begin{array}{c} \text{blue dot} \end{array} \right) i_1 \circ p_1 \\
\downarrow \left(\begin{array}{c} \text{red dot} \end{array} \right) p_2 \\
\text{id}_3 \left(\begin{array}{c} \text{red dot} \end{array} \right) T(sts) \left(\begin{array}{c} \text{red dot} \end{array} \right) i_2 \circ p_2 \\
\downarrow \left(\begin{array}{c} \text{blue dot} \end{array} \right) p_3 \\
\vdots
\end{array}
\end{array} \tag{4.2.12}$$

Although not drawn, all the diagrams are flanked by Jones–Wenzl projectors, and the matching morphisms in \mathcal{T} are pre and post-composed with the idempotents corresponding to the appropriate JW-projectors. Putting JW-projectors above and below any of these diagrams clearly do not result in zero. Moreover, in the endomorphism space of each non-trivial indecomposable, the morphisms id_k^d and $i_{k-1}^d \circ p_{k-1}^d$, with JW-projectors before and after, can easily be checked to be linearly independent. Hence the bases for the spaces can be read off the picture (4.2.12). In \mathcal{T} , the analogous bases for the morphism spaces in [AT17, Corollary 2.3.1] have matching dimensions, hence F' is fully faithful. Therefore the categories $\text{Kar}^\oplus(\mathcal{DT})$ and $\text{Tilt}(\mathfrak{sl}_2)$ are equivalent as additive Karoubian \mathbb{C} -linear $\mathcal{H}_{\mathbb{C}}(\tilde{S}_2)$ -module categories. \square

Comment on grading?

Comment on consequences

Say similaritied with $\text{proj}(\mathcal{O})$

Bibliography

- [AT17] Henning Haahr Andersen and Daniel Tubbenhauer. “Diagram Categories for U_q -Tilting Modules at Roots of Unity”. In: *Transformation Groups* 22 (2017), pp. 29–89. DOI: [10.1007/s00031-016-9363-z](https://doi.org/10.1007/s00031-016-9363-z).
- [Eli16] Ben Elias. “The two-color Soergel calculus”. In: *Compositio Mathematica* 152.2 (2016), pp. 327–398. DOI: [10.1112/S0010437X15007587](https://doi.org/10.1112/S0010437X15007587).
- [EW16] Ben Elias and Geordie Williamson. “Soergel Calculus”. In: *Representation Theory of the American Mathematical Society* 20 (Oct. 2016). DOI: [10.1090/ert/481](https://doi.org/10.1090/ert/481).
- [Eti+15] Pavel Etingof et al. *Tensor Categories*. Vol. 205. Mathematical Surveys and Monographs. American Mathematical Society, 2015. DOI: <http://dx.doi.org/10.1090/surv/205>.
- [Hum08] James E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category \mathcal{O}* . Vol. 94. Graduate Studies in Mathematics. American Mathematical Society, 2008. DOI: <http://dx.doi.org/10.1090/gsm/094>.
- [Jon21] Vaughan F. R. Jones. “Planar algebras”. In: *New Zealand Journal of Mathematics* 52 (2021), pp. 1–107. DOI: [10.53733/172](https://doi.org/10.53733/172). URL: <https://nzjmath.org/index.php/NZJMATH/article/view/172>.
- [Lib08] Nicolas Libedinsky. “Sur la catégorie des bimodules de Soergel”. In: *Journal of Algebra* 320.7 (2008). (French), pp. 2675–2694. ISSN: 0021-8693. DOI: <https://doi.org/10.1016/j.jalgebra.2008.05.027>.
- [ML98] Saunders Mac Lane. *Categories for the Working Mathematician*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. Springer, 1998. DOI: <https://doi.org/10.1007/978-1-4757-4721-8>.
- [Maz09] Volodymyr Mazorchuk. *Lectures on $\mathfrak{sl}_2(\mathbb{C})$ -Modules*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. IMPERIAL COLLEGE PRESS, 2009. DOI: [10.1142/p695](https://doi.org/10.1142/p695).
- [RW18] Simon Riche and Geordie Williamson. “Tilting Modules and the p -Canonical Basis”. In: *Asterisque* 397 (2018). DOI: [10.24033/ast.1041](https://doi.org/10.24033/ast.1041).

- [Soe90] Wolfgang Soergel. “Kategorie \mathcal{O} , Perverse Garben Und Moduln Uber Den Koinvarianten Zur Weylgruppe”. In: *Journal of the American Mathematical Society* 3.2 (1990), pp. 421–445. ISSN: 08940347, 10886834. URL: <http://www.jstor.org/stable/1990960>.
- [Soe98] Wolfgang Soergel. “Combinatorics of Harish-Chandra modules”. In: *Representation Theories and Algebraic Geometry*. Ed. by Abraham Broer, A. Daigneault, and Gert Sabidussi. Springer Netherlands, 1998, pp. 401–412. ISBN: 978-94-015-9131-7. DOI: [10.1007/978-94-015-9131-7_10](https://doi.org/10.1007/978-94-015-9131-7_10). URL: https://doi.org/10.1007/978-94-015-9131-7_10.