

Diagrammatic Categories in Representation Theory  
Honours Thesis  
(Draft)

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# Chapter 1

## Introduction

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# Chapter 2

## Background

To do

# Chapter 3

## One-colour Diagrammatics

### 3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the one-colour diagrammatic Hecke category  $\mathcal{H}$  for the symmetric group  $S_2 = \langle s \mid s^2 = 1 \rangle$ .

The objects of this category are generated by taking formal tensor products of the non-identity element  $s \in S_2$ . For example the tensor product of four  $s$ 's which denote with the expression  $(s, s, s, s)$ .

The morphisms in this category have a presentation in terms of generators and relations. For convenience, we will describe them up to isotopy. The generators are the following univalent and trivalent vertices, which can be rotated and flipped vertically using isotopy.

$$\begin{array}{|c|} \hline \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \diagup \\ \hline \diagdown \end{array} \quad (3.1.1)$$

These morphisms are subject to the following local relations.

$$\begin{array}{|c|} \hline \diagup \\ \hline \diagdown \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \quad (3.1.2a)$$

$$\begin{array}{|c|} \hline \diagup \\ \hline \diagdown \end{array} = \begin{array}{|c|} \hline \diagdown \\ \hline \diagup \end{array} \quad (3.1.2b)$$

$$\begin{array}{|c|} \hline \bullet \\ \hline \bullet \end{array} = \boxed{\alpha} \quad (3.1.2c)$$

$$\begin{array}{|c|} \hline \bigcirc \\ \hline \end{array} = 0 \quad (3.1.2d)$$

$$\boxed{\text{line with segment}} = 2 \boxed{\text{line with dots}} - \boxed{\text{line with segment}}$$

To simplify the boxes, we can write

$$\boxed{f} \boxed{g} = \boxed{fg} \quad (3.1.3)$$

where  $f, g$  and  $fg$  are polynomials in  $\alpha$ . Polynomials with what coefficients?

The first four generators with the relations (3.1.2a) and (3.1.2b) describes a Frobenius algebra object structure on the object  $s$ . Here the generators correspond to the unit, counit, multiplication and comultiplication maps respectively.

*Example 3.1.4.* Let us use the relations in (3.1.2) to simplify the following morphism in  $\text{Hom}((s, s), (s))$ .

Diagrammatic proof of the identity (1.10) using the graphical calculus of the free fermion Fock space. The proof shows the equality of two diagrams representing the same operator. The first diagram is a complex network of lines and dots. The second diagram is a simpler one. The third diagram is the difference of two diagrams. The fourth diagram is the difference of two diagrams.

### 3.2 Diagrammatic $\mathcal{O}(\mathrm{SL}(2))$

We will describe one-colour diagrammatics for  $\mathcal{O}(\mathrm{SL}(2))$  via generators and relations up to isotopy.

The elements of this category are generated by taking tensor products of an element  $s$ , coloured red.

Similar to one colour diagrammatics for  $\mathbb{B}\text{SBim}$ , the morphisms in this category are generated by horizontal concatenation, vertical concatenation, and sums of the following univalent and trivalent vertices, along with boxes where  $f$  is a homogeneous polynomial in  $\mathbb{F}$ .

(3.2.1)

The morphisms are subject to the following local relations, up to isotopy.

$$\begin{array}{|c|} \hline \text{Diagram: A vertical red line with a horizontal red line segment extending to the right, ending in a red dot.} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram: A single vertical red line.} \\ \hline \end{array} \left( = \begin{array}{|c|} \hline \text{Diagram: A vertical red line with a horizontal red line segment extending to the left, starting from a red dot.} \\ \hline \end{array} \right) \quad (3.2.2a)$$

$$\begin{array}{|c|} \hline \text{Diagram: A red line entering from the bottom and splitting into two lines exiting to the top-left and top-right.} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram: Two red lines entering from the bottom-left and bottom-right and meeting at a central point, from which a single red line exits to the top.} \\ \hline \end{array} \quad (3.2.2b)$$

$$\begin{array}{|c|} \hline \text{Diagram: Two adjacent boxes labeled } f \text{ and } g. \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram: A single box labeled } fg. \\ \hline \end{array} \quad (3.2.2c)$$

$$\begin{array}{|c|} \hline \text{Diagram: A vertical red line segment with red dots at both ends.} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram: A box labeled } \alpha_s. \\ \hline \end{array} \quad (3.2.2d)$$

$$\begin{array}{|c|} \hline \text{Diagram: A red circle with a vertical red line passing through its center.} \\ \hline \end{array} = 0 \quad (3.2.2e)$$

$$\begin{array}{|c|} \hline \text{Diagram: A box labeled } f \text{ followed by a vertical red line.} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram: A vertical red line followed by a box labeled } sf. \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Diagram: A box labeled } \partial_s f \text{ with red dots at the top and bottom.} \\ \hline \end{array} \quad (3.2.2f)$$

Additionally, we must impose a right  $R$ -module relation following from the lack of a left action. Instead of embedding in the double-sided strip  $[0, 1] \times \mathbb{R}$ , we embed the diagrams in the one-sided strip  $[0, 1] \times \mathbb{R}_{\geq 0}$  (Check the inequality), where the left side is an imaginary wall. For example a morphism may be



Then, diagrams are related to the wall by

$$\begin{array}{|c|} \hline \text{Diagram: A solid gray vertical bar followed by a box labeled } f. \\ \hline \end{array} = 0 \quad (3.2.3)$$

where  $f$  is a homogeneous polynomial in  $R$  with non-zero degree. That is, if a diagram has a non-constant homogeneous polynomial on its far left, then the entire diagram dies.

# Chapter 4

## Two-colour Diagrammatics

### 4.1 Two-colour Diagrammatic Hecke Category

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### 4.2 Diagrammatic $\text{Tilt}(\text{SL}(2))$

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