Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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March 30, 2023

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Introduction

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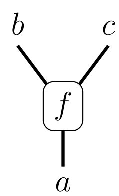
Chapter 2

Background

2.1 Drawing Monoidal Categories

A monoidal category $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold[1]. We assume that monoidal categories are strict, since all monoidal categories are monoidally equivalent to a strict one[2].

The morphisms of $\mathcal{C}$ can be drawn as string diagrams, where the morphism maps from the bottom to the top. Functions that make up the morphism are drawn as tokens or boxes. For example

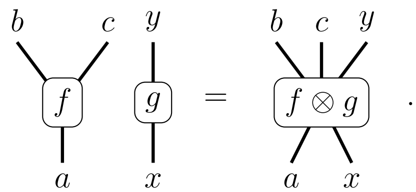


depicts a morphism $f : a \to b \otimes c$. For identity morphisms we drop the box and only draw a vertical line, so id_a is the diagram

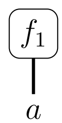
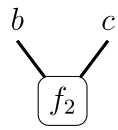
.

The tensor product of morphisms is the horizontal concatenation of diagrams, such that strings from separate functions don't interact. For example, given $g : x \to y$, the tensor product $f \otimes g : a \otimes x \to b \otimes c \otimes y$ is drawn as

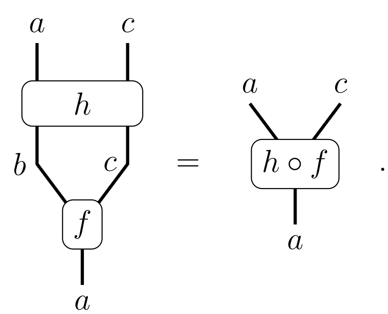
[1] For more details see [Eti+15].
[2] See [ML98, VII.2] or [Eti+15, Thm 2.8.5]



By convention, $\mathbb{1}$ is blank and unlabelled, and strings that would join to $\mathbb{1}$ are blank. Particularly, $\mathrm{id}_{\mathbb{1}}$ is an empty diagram, and we have diagrams such as

 $: a \to \mathbb{1}$ and  $: \mathbb{1} \to b \otimes c.$

The compositions of morphisms is the vertical stacking of diagrams where domains and codomains match. For example, the composition $h \circ f : a \to b \otimes c \to a \otimes c$ of $f : a \to b \otimes c$ with $h : b \otimes c \to a \otimes c$ has the diagram



Before looking at our main example of a diagrammatic monoidal category, we first define some terminology.

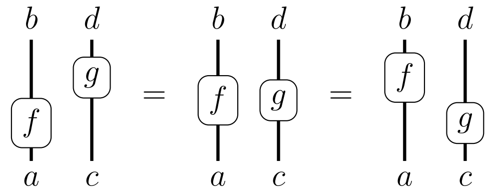
Definition 2.1.1. For a commutative ring R, an *R-linear category* is a category enriched over the category of R-modules. That is, for objects a, b, the set of morphisms $\mathrm{Hom}(a, b)$ is an R-module and the composition of morphisms is R-bilinear.

Example 2.1.2. Let $\Bbbk$ be a field. The category of vector spaces over $\Bbbk$, $\mathbf{Vect}_{\Bbbk}$, is a $\Bbbk$-linear category. This makes sense by the classical theory of linear algebra.

For a strict R-linear monoidal category $\mathcal{C}$, the bifunctoriality of $- \otimes -$ implies the following *interchange law*. For morphisms $f : a \to b$ and $g : c \to d$, $(\mathrm{id}_b \otimes g) \circ (f \otimes \mathrm{id}_c) = f \otimes g = (f \otimes \mathrm{id}_d) \circ (\mathrm{id}_a \otimes g)$. In other words the following diagram commutes.

$$\begin{tikzcd}
a \otimes c \arrow[r, "f \otimes \mathrm{id}_c"] \arrow[d, "\mathrm{id}_a \otimes g"'] \arrow[dr, "f \otimes g"] & b \otimes c \arrow[d, "\mathrm{id}_b \otimes g"] \\
a \otimes d \arrow[r, "f \otimes \mathrm{id}_d"] & b \otimes d
\end{tikzcd}$$

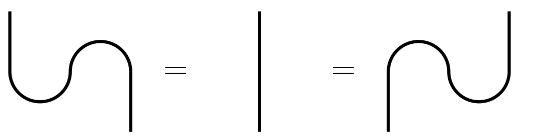
Written with string diagrams, this is



which holds up to deformation of the diagram.

Definition 2.1.3. A monoidal category $\mathcal{C}$ is *generated* by finite set S_o of objects and S_m of morphisms, when all non-unit objects are a finite tensor of objects in S_o and all non-identity morphisms are a finite combination of tensors and compositions of morphisms in S_m.

Example 2.1.4. Our first example of a diagrammatic monoidal category is the *Temperley-Lieb category*. The Temperley-Lieb category $\mathcal{TL}$ is a strict R-linear monoidal category whose objects are generated by the vertical line $\mathbf{|}$ and morphisms generated by the cup $\cup : \mathbb{1} \to \mathbf{|} \otimes \mathbf{|}$ and cap $\cap : \mathbf{|} \otimes \mathbf{|} \to \mathbb{1}$, with relations



Mention that composition and tensor product is as explained above
Some example
Mention bubbles and specialisation to some $\delta \in R$
Mention that these are crossingless matchings
Comment on isotopy

Chapter 3

One-colour Diagrammatics

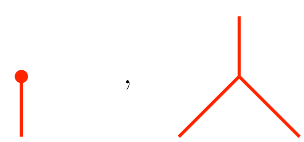
3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the one-colour (diagrammatic) Hecke category $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = 1 \rangle$. This is a monoidal category with a presentation in terms of generators and relations.

Remark 3.1.1. All the following diagrams could be defined without the language of categories, as planar pictures with appropriate vertical stacking and horizontal concatenation. Nevertheless, we define them in a category because they will eventually be seen as diagrammatic versions of important categories in representation theory.

The objects of $\mathcal{H}(S_2)$ are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is just concatenation. The empty tensor product (or empty word) will be denoted by $\varnothing =: s^0$.

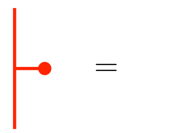
The morphisms are generated, up to isotopy, by univalent and trivalent vertices

 (3.1.2)

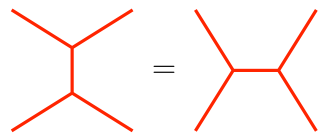
which are maps $s \to \varnothing$ and $ss \to s$ respectively. Note that we put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). Additionally[1], we allow formal sums of diagrams by putting a $\mathbb{Z}$-module structure on each morphism space $\mathrm{Hom}(s^n, s^m)$, for non-negative integers n, m. Composition or tensor with the zero morphism 0 in this $\mathbb{Z}$-module result in 0. To abuse notation, the empty diagram $\varnothing \to \varnothing$ will be denoted $\varnothing$. The identity morphism in $\mathrm{Hom}(s^n, s^n)$ is the diagram consisting of n vertical lines.

[1] Pun intended.

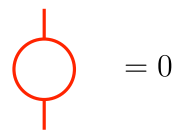
Such diagrams are subject to the following local relations.



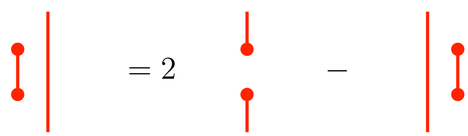
(3.1.3a)



(3.1.3b)



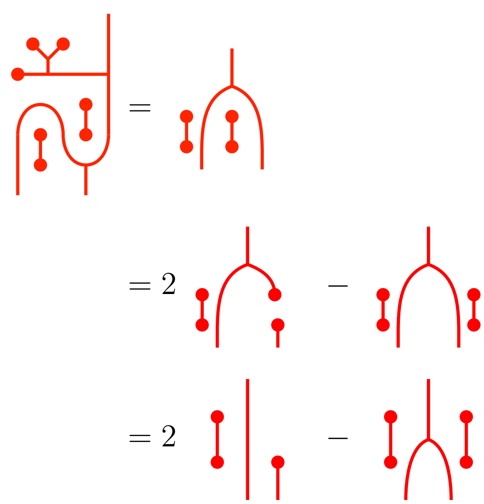
(3.1.3c)



(3.1.3d)

Remark 3.1.4. The object s is a Frobenius algebra object in $\mathcal{H}(S_2)$. The generators (3.1.2) and their horizontal reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.3a) and (3.1.3b).

Example 3.1.5. Let us use the relations in (3.1.3) to simplify the following morphism in $\mathrm{Hom}((s,s),(s))$.



Add example of frob associativity

There is a right (or left) $\mathbb{Z}[$$]$-basis for $\mathrm{Hom}(s^n, s^m)$ described in [EW16] called the Double Leaves basis. To define this basis we must first look at morphisms known as Light Leaves. Given a word $w = s^n$, a subexpression is a binary string of length n. For example, $0000, 0110$ and 1011 are subexpressions of $s^4 = ssss$. Given a subexpression

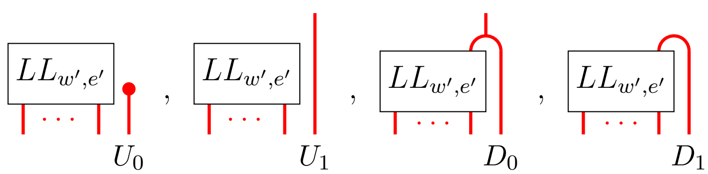
e of an object w, we can apply it to produce an element $w(e) \in S_2$, e.g. $ssss(1011) = s * 1 * s * s = s$. Each term of the subexpression is a decision of whether to include the corresponding s in the word, where excluding an s amounts to multiplying by 1. Talk about how s could mean an object of $\mathcal{H}$ or an object of S_2.

For a subexpression e of an expression w, we can label each term by U_0, U_1, D_0 or D_1. The label is U_* if the partial subexpression up to the current term evaluates to $1 \in S_2$ and D_* if it evaluates to $s \in S_2$, where the subscript is the corresponding term in e.

Example 3.1.6. For the object $ssss$ and subexpression 0101, we can find the labels:

Choice	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
Partial $w(e)$	1	$1 * s = s$	$1 * s * 1 = s$	$1*s*1*s = 1$
Labels	U_0	U_0U_1	$U_0U_1D_0$	$U_0U_1D_0D_1$

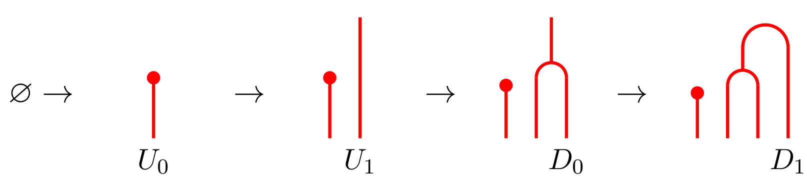
The light leaf $LL_{w,e} \in \mathrm{Hom}(w, w(e))$ $w(e)$ here is an element of S_2 that we identify with one of the objects $\varnothing$ or s, corresponding to the object w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, $LL_{w's,e'i}$ is one of



(3.1.7)

depending on the next label, where w' and e' are appropriate subwords of w and e. Observe that the codomain of a light leaf $LL_{w,e}$ is the object corresponding to the evaluation $w(e) \in S_2$ of the subexpression. The recursive definition is consistent, since if the next label is U_* then the codomain of $LL_{w',e'}$ (the evaluation of the partial subexpression $w'(e')$ up to the label) is 1, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. Rewrite this to make sense. Do we need to talk about 'degree' of light leaves?

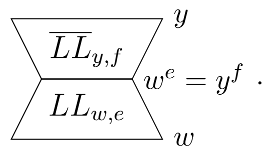
Example 3.1.8. Following from Example (3.1.6) for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.



Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. A *double leaf* associated to expressions w, y is a composition

$$\mathbb{LL}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

for subexpressions e of w and f of y such that $w(e) = f(y)$. Visually this looks like a morphism from w to y factoring through $w(e) = y(f) \in \{1, s\}$,



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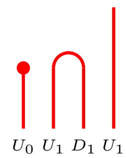
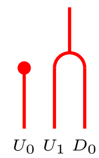
Example 3.1.9. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

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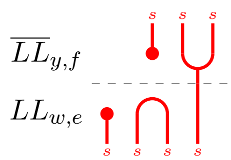
Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$, factoring through s, is



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Notice that these double leaves have no floating diagrams such as . In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{Z}[\ ]$-module. Here, the left -action on a diagram is just concatenation by on the left. Since we can move barbells to the right via. the relation (3.1.3d), we can equivalently act by $\mathbb{Z}[\ ]$ on the right. This leads us to the following theorem.

Theorem 3.1.10 (Elias-Williamson [EW16], Theorem 1.2)**.** *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{LL}_{w,y}$ [2] be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $w(e) = y(f)$. Then $\mathbb{LL}_{w,y}$ is a left (or right) $\mathbb{Z}[\ ]$-module basis for* $\mathrm{Hom}(w, y)$.

A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.11. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky's work in [Lib08].

The morphisms can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1. The degree of a general diagram is the sum of the degrees of the generators that appear in it. How do you do degree of a $\mathbb{Z}$-linear combination?

Put example

The double leaves bases allow us to show that the Karoubi envelope of $\mathcal{H}(S_2)$ is equivalent to the category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$ over S_2 as monoidal categories.

[2] this can be confused with the double leaves themselves, maybe write $\mathbb{LL}(w, y)$

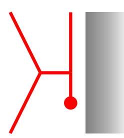
Theorem 3.1.12 (Elias-Williamson [EW16], Theorem 6.30)**.** *There is an equivalence of categories between* $\mathrm{Kar}(\mathcal{H}(S_2))$ *and the category of Soergel Bimodules* $\mathbb{S}\mathrm{Bim}$ *over* S_2*.*

The proof gives an equivalence of categories $\mathcal{H}(S_2) \cong \mathbb{BS}\mathrm{Bim}$ of Bott-Samelson bimodules over S_2, by comparing the graded dimensions of morphism spaces using double leaves. Since $\mathrm{Kar}(\mathbb{BS}\mathrm{Bim}) \cong \mathbb{S}\mathrm{Bim}$ and Karoubi envelope preserves equivalences, we obtain $\mathrm{Kar}(\mathcal{H}(S_2)) \cong \mathbb{S}\mathrm{Bim}$.

3.2 Diagrammatic $\mathcal{O}(\mathrm{SL}(2))$

With the diagrammatic category $\mathcal{H}(S_2)$, we can describe diagrammatics for the category $\mathcal{O}(\mathrm{SL}(2))$. In particular, we define a modular category $\mathcal{DO}(\mathrm{SL}(2))$ with a left-action of $\mathcal{H}(S_2)$.

The category $\mathcal{DO}(\mathrm{SL}(2))$ has elements that are generated (Define what this means.) by the identity element $\varnothing$ of $\mathcal{H}(S_2)$ and morphisms are generated by the empty diagram $\varnothing$, where $\mathcal{H}(S_2)$ acts on the left by left concatenation for objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0, 1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0, 1] \times \mathbb{R}$. For example a morphism may be



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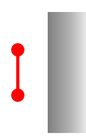
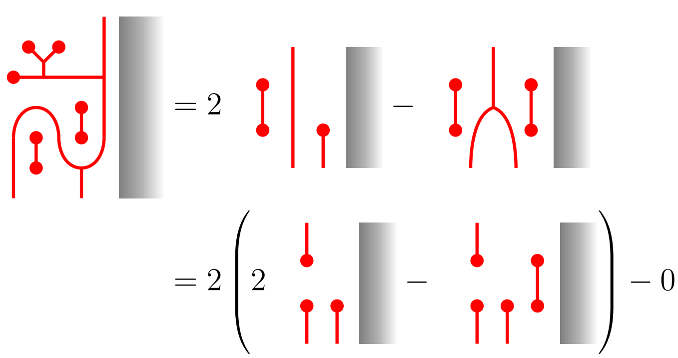
We impose the relation that diagrams are related to the wall by

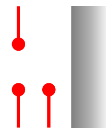


$$= 0. \tag{3.2.1}$$

Notice that all the morphisms in $\mathcal{H}(S_2)$ appear in this modular category, although they may have been annihilated by (3.2.1).

Example 3.2.2. We use the new relation (3.2.1) to further simplify the morphism in Example (3.1.5).

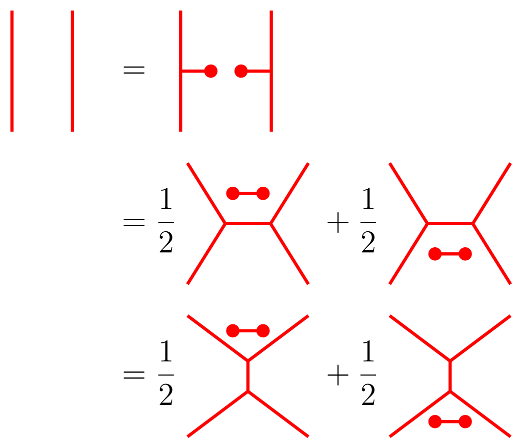


$= 4$ 

Since the objects and morphisms in this category are that of $\mathcal{H}(S_2)$ modulo the wall relation (3.2.1), double leaves still span the morphism spaces of $\mathcal{DO}(\mathrm{SL}(2))$. We see these as left $\mathbb{Z}[\;]$-module bases since right multiplication by annihilates the morphism and adds no new diagrams. Are these basis? They might not be linearly independent e.g. double leaves with a gap means that multiplying with kills it.

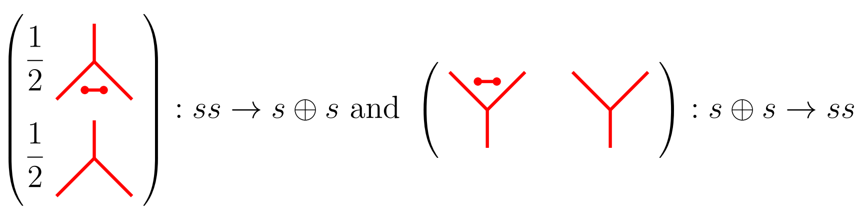
Only allowing $\mathbb{Z}$-module should make them linearly independent, and they should still span.

In the diagrammatic category $\mathcal{H}(S_2)$ from Section 3.1, we have the relation



(3.2.3)

In the additive closure of this category, this shows there is an isomorphism $s \otimes s \cong s \oplus s$ by



Its is not hard to check that these are inverses.

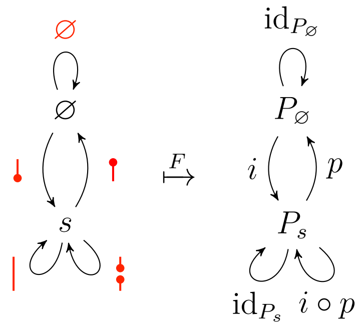
Theorem 3.2.4 (???)**.** *The diagrammatic category* $\mathrm{Kar}(\mathcal{DO}(\mathrm{SL}(2)))$ *and* $\mathcal{O}(\mathrm{SL}(2))$ *are equivalent as categories.*

Check all of this & Put precise references

Proof. As a shorthand, we write $\mathcal{DO}$ for $\mathcal{DO}(\mathrm{SL}(2))$ and $\mathcal{O}$ for $\mathcal{O}(\mathrm{SL}(2))$. The work of Soergel in [Soe90] shows that $\mathcal{O}$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_\varnothing, \Theta_s \in \mathrm{End}(\mathcal{O})$, corresponding to elements in S_2 Check this. Classical results, e.g. [Hum08], show that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$.

Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure. Additionally, Soergel's work (References?) shows that there is a relation in $\mathcal{O}$ analogous to the barbell-wall relation (3.2.1), and that there is an isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ (Is the direct sum here correct?) which is analogous to the isomorphism given by (3.2.3).

Define the functor $F : \mathcal{DO} \to \mathcal{O}$ that sends the empty object $\varnothing$ to the trivial module $P_\varnothing$, and the Soergel module action corresponding to s to the translation functor Θ_s. Then the object s maps to $\Theta_s(P_\varnothing) =: P_s$, and s^3 maps to $\Theta_s^3(P_\varnothing) = \Theta_s\Theta_s\Theta_s(P_\varnothing)$. Functoriality forces F to map identity diagrams $s^n \to s^n$ to $\mathrm{id}_{\Theta_s^n(P_\varnothing)}$. For non-identity maps, we let $F(\ ) = i$ be the inclusion map $P_\varnothing \to P_s$ and $F(\ ) = p$ be the projection map $P_s \to P_\varnothing$. The mapping is depicted by the following diagram.



Note that the projection and inclusion maps are exactly the unit and counit of Θ_s evaluated at $P_\varnothing$. This is enough to completely determine the image of F, since $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$. Now Θ_s is a Frobenius object and the barbell-wall relation is satisfied in $\mathcal{O}$, so the functor F is well defined.

Now we show that F is fully faithful. We know (From Soergel, EW, Libedinsky? Explain this more) that the image of and generate all morphisms of the form $\Theta_s^n(P_\varnothing) \to \Theta_s^m(P_\varnothing)$, so F is full. For the faithfulness of F, it suffices to match the dimensions of $\mathbb{Z}$-bases for hom-spaces involving $P_\varnothing$ and P_s. By double leaves in $\mathcal{DO}$, as $\mathbb{Z}$-modules, $\mathrm{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_\varnothing\}$, $\mathrm{Hom}(s, \varnothing)$ has a basis $\{\ \}$, $\mathrm{Hom}(\varnothing, s)$ has a basis $\{\ \}$, and $\mathrm{Hom}(s, s)$ has a basis $\{\mathrm{id}_s, \ \circ\ \}$. The dimensions match exactly with the corresponding images of F. Therefore F is fully faithful.

Since objects in $\mathcal{O}$ are direct sums and direct summands of the elements $\Theta_s^n(P_\varnothing)$ for non-negative integer n, taking the Karoubi envelope $\mathrm{Kar}(\mathcal{DO})$ induces an equivalence of categories $\mathrm{Kar}(\mathcal{DO}) \cong \mathcal{O}$.

$\square$

Old Proof. As a shorthand, we write $\mathcal{DO}$ for $\mathcal{DO}(\mathrm{SL}(2))$ and $\mathcal{O}$ for $\mathcal{O}(\mathrm{SL}(2))$. Let $F : \mathcal{DO} \to \mathcal{O}$ be a functor that sends the empty object $\varnothing$ to the trivial module $P_\varnothing$ and $s \mapsto P_s$, the indecomposable objects in $\mathcal{O}$ corresponding to elements in S_2. On morphisms, F sends the identity morphism on s (the red strand) to the translation functor Θ_s in $\mathcal{O}$ corresponding to $s \in S_2$. This completely determines the action of F (Why?). Due to classical results in [Hum08], the translation functors are Frobenius objects, so there have unit, counit, multiplication and comultiplication maps with appropriate relations in $\mathcal{O}$.

These the image of which are the image of the generators (3.1.2) under F, that satisfy the analogous relations (3.1.3). Furthermore, the work of Soergel in [Soe90] shows that there is a relation in $\mathcal{O}$ analogous to the barbell-wall relation (3.2.1). This F is well defined as all the generators and relations in $\mathcal{DO}$ are accounted for (Word this better).

Next we show that F is a fully faithful functor. By results from [EW16] and [Lib08], the inclusion $\mathcal{H}(S_2) \to \mathbb{S}\mathrm{Bim}$ is fully faithful, so we have a copy of double leaves bases in $\mathbb{S}\mathrm{Bim}$. By the work of Soergel in [Soe90], the category $\mathcal{O}$ is a Soergel module (Explain what this is) with certain bases for the morphism. Thus (Why?) it suffices to compare the dimension of morphism spaces between $\mathcal{DO}$ and $\mathcal{O}$, as Soergel modules. [Comparison?]

The functor F mapped objects of $\mathcal{DO}$ to objects ??? in $\mathcal{O}$, which generate all other objects by direct sums and direct summands Is this right?. Now F is fully faithful, Kar preserves equivalences of categories and taking the Karoubi envelope of the image of $\mathcal{DO}$ gives exactly $\mathcal{O}$ (Is this right?), we obtain an equivalence of categories between $\mathrm{Kar}(\mathcal{DO})$ and $\mathcal{O}$. □

Note on induced grading

Chapter 4

Two-colour Diagrammatics

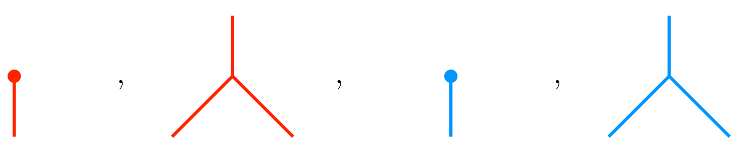
The previous chapter had its focus on the symmetric group generated by one element S_2, which brought about one-colour diagrammatics. We shift our attention to a more complex example by adding an extra generator, that is, another colour. In particular, we consider the case for the affine symmetric group on two elements $\widetilde{S}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$.

4.1 Two-colour Diagrammatic Hecke Category

Corresponding to $\widetilde{S}_2$, we define the two-colour (diagrammatic) Hecke category $\mathcal{H}(\widetilde{S}_2)$. This is a (strict) $\mathbb{Z}$-linear monoidal category given by the following isotopy presentation.

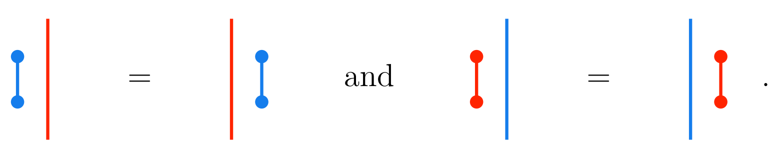
Objects in $\mathcal{H}(\widetilde{S}_2)$ are generated by formal tensor products of the non-identity elements $s, t \in \widetilde{S}_2$. As before, we write objects as words such as $ssttst =: s^2t^3st$ where the tensor product is concatenation, and associate the colour red to s and blue to t. The empty word is the monoidal identity, which we write as $\varnothing$.

The morphisms are generated by the univalent and trivalent vertices



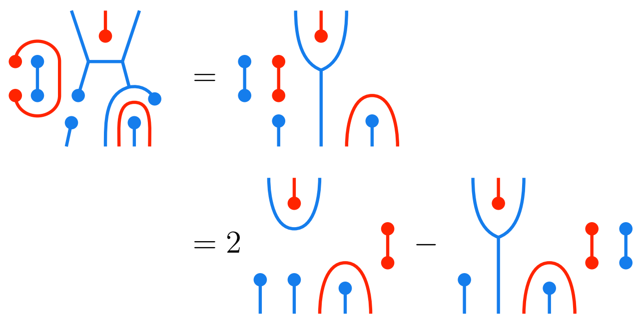
(4.1.1)

that are maps $s \to \varnothing$, $ss \to s$, $t \to \varnothing$ and $tt \to t$ respectively. As in the one-colour case, tensor product is horizontal concatenation, composition is appropriate vertical stacking, and we denote the empty diagram $\varnothing \to \varnothing$ by $\varnothing$. For each colour, these diagrams have the one-colour relations given by (3.1.3). Since we have two colours now, we also need to describe how the colours interact. This is given by the *two-colour* relations



(4.1.2)

Example 4.1.3. The following morphism in $\mathrm{Hom}(ttsts, tst)$ can be simplified using the one-colour and two-colour relations.



Remark 4.1.4. Notice that, in this category, red and blue lines never cross as no generators that allow crossings. This is a consequence of working over affine S_2 in which the generators s and t have no relation. Mention example of crossing and S_3.

4.2 Diagrammatic Tilt(SL(2))

Blah

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