Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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February 25, 2023

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Chapter 1

Introduction

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Chapter 2

Background

Notation: we write 1 for the neutral element of a group.

2.1 Coxeter Groups

Definition 2.1.1. A *Coxeter system* (W, S) is a group W and a finite subset $S = \{s_1, ..., s_n\} \subset W$ under the following conditions. For any $s, t \in S$, $(st)^{m_{st}} = 1$ where $m_{st} \in \mathbb{Z}_{>0} \cup \{\infty\}$ such that $m_{st} = 1$ if $s = t$, and $m_{st} = m_{ts} \in \{2, 3, ...\} \cup \{\infty\}$ if $s \neq t$. In other words, $W = \langle s \in S \mid (st)^{m_{st}} = 1 \rangle$ with generator S. We call W a *Coxeter group*.

The value $m_{st} = \infty$ indicates there are no relations of the form $(st)^m = 1$ for any $m \in \mathbb{Z}_{>0}$. We often call relations of the form $s^2 = 1$ *quadratic relations*. The quadratic relations on the generators of the Coxeter group imply that $s^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1} = ts$. Moreover, if $s \neq t$ and $m_{st} < \infty$, then we can use the quadratic relations to write $(st)^{m_{st}} = 1$ equivalently as

$$\underbrace{sts...}_{m_{st}} = \underbrace{tst...}_{m_{st}},$$

which we call *braid relations*. Coxeter systems are closely related to reflections, so we often call elements of S *simple reflections*, and elements in W that are conjugates to elements in S *reflections*.

Example 2.1.2. The permutation group of n elements S_n is a Coxeter group generated by the set of transpositions $S = \{(i, i+1) \in S_n : 1 \leq i \leq n-1\}$. Let $s_i := (i, i+1)$. We know from algebra that S generates S_n, so let us check the relations.

- For any i, $s_i^2 = (i, i+1)(i, i+1) = 1$.
- For $i > j+1$, the transpositions $(i, i+1)$ and $(j, j+1)$ are disjoint so $(s_i s_j)^2 = (i, i+1)(j, j+1)(i, i+1)(j, j+1) = 1$.
- For $i = j+1$, $(s_i s_j)^3 = ((i, i+1)(j, j+1))^3 = (i, i+1, i+2)^3 = 1$.

These are sometimes called the Coxeter system of type A_{n-1}, for $n \geq 2$.

An easy case is the Coxeter group $W \simeq S_3$ with generators $S = \{s, t\}$ where s, t correspond to transpositions (12) and (23) respectively. By the quadratic and braid relations, we find that the elements of W are exactly $1, s, t, st, ts, sts = tst$. We will frequently revisit this example.

Definition 2.1.3. Let $w \in W$. As S generates W, we can write $w = s_1 s_2 ... s_k$ for some $s_1, ..., s_k \in S$. We say the sequence $(s_1, ..., s_k)$ is an *expression* for w of *length* k. Given the relations in the definition, $w \in W$ is not uniquely expressed as such a sequence, so we write $\underline{w}$ to denote a choice of expression $(s_1, ..., s_k)$ for w.

Definition 2.1.4. Let $w \in W$. For any expression $\underline{w} = (s_1, ..., s_k)$, we say the *length of* $\underline{w}$ is k, and write $\ell(\underline{w}) = k$. The *length of* w, written $\ell(w)$ is the smallest integer k such that w admits an expression of length k. We say an expression $\underline{w}$ is *reduced* if $\ell(\underline{w}) = \ell(w)$.

Note that $\ell(w) = 0$ if and only if $w = 1$.

The following are useful results regarding reduced expressions.

Theorem 2.1.5 (Exchange condition)**.** *Let $\underline{w} = (s_1, ..., s_k)$ be a reduced expression for $w \in W$, and let $t \in S$. If $\ell(wt) < \ell(w)$, then there exists an integer $i \in \{1, 2, ..., k\}$ such that $wt = s_1 ... \hat{s}_i ... s_k$, i.e. with s_i omitted.*

Corollary 2.1.6 (Deletion Condition)**.** *Let $\underline{w} = (s_1, ..., s_k)$ be an expression for $w \in W$ where $\ell(w) < k$, i.e. not a reduced expression. Then there exists some $i < j$ such that $w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_k$.*

In other words, if an expression is not reduced, two elements in the expression may be cancelled to result in a shorter expression.

Theorem 2.1.7 (Matsumoto, 1964)**.** *Any two reduced expressions for $w \in W$ are related by braid relations.*

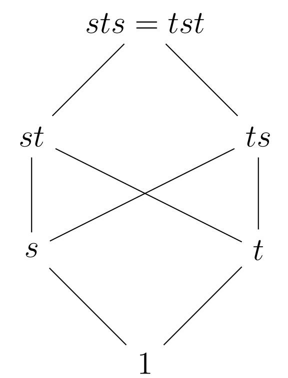
We can also define a partial order on W.

Definition 2.1.8 (Bruhat Order)**.** Let T be the set of elements of W that are conjugate to elements in S. Define a partial order $\leq$ on W such that for $x, y \in W$, $x \leq y$ if and only if there exists a chain $x = x_0, x_1, ..., x_m = y$ of elements in W such that $\ell(x_i) < \ell(x_{i+1})$ and $x_i^{-1} x_{i+1} \in T$ for each $i = 0, 1, ..., m-1$.

That is x_{i+1} is x_i multiplied on the right with a conjugate of an element in S such that its length is longer than x_i. Note that we can equivalently multiply on the left because for any $t \in T$ we can write $xt = (xtx^{-1})x$ where $xtx^{-1} \in T$.

Equivalently let $y = s_1 ... s_k$ be a reduced expression, and we can define $x \leq y$ to be if and only if there exists a reduced expression $x = s_{i_1} ... s_{i_\ell}$ such that $1 \leq i_1 < ... < i_\ell \leq k$. In other words, x is y after removing some terms from a reduced expression (we say x is a *subexpression* of y).

Example 2.1.9. The Hasse diagram for the Bruhat order on S_3 is as follows (using the labelling of elements from Example 2.1.2).



Definition 2.1.10. Given a Coxeter system (W, S), define a representation V of W as follows. Let V be a vector space over $\mathbb{R}$ generated by the basis $\{\alpha_s : s \in S\}$. Equip V with a symmetric bilinear form $(-,-)$ defined by

$$(\alpha_s, \alpha_t) = -\cos \frac{\pi}{m_{st}}.$$

If $m_{st} = \infty$ we define $\pi/m_{st} = 0$. Define the W-action on V such that for $s \in S$ and $v \in V$,

$$s \cdot v = v - 2(v, \alpha_s)\alpha_s.$$

We call this the *geometric representation* of the Coxeter system.

This is defined for both finite and infinite Coxeter groups.

Proposition 2.1.11. *For any Coxeter system, the geometric representation is faithful.*

In this paper, we will work with this representation of the Coxeter group.

2.2 Hecke Algebra

Let $\mathbb{Z}[v, v^{-1}]$ be the set of integer Laurent polynomials, for an indeterminate v.

Definition 2.2.1. The *Hecke algebra* $\mathcal{H}$ for a Coxeter system (W, S) is the unital associative algebra over $\mathbb{Z}[v, v^{-1}]$ generated by $\{\delta_s : s \in S\}$ with the following relations.

- $\delta_s^2 = (v^{-1} - v)\delta_s + 1$, for any $s \in S$.
- $\underbrace{\delta_s\delta_t\delta_s...}_{m_{st}} = \underbrace{\delta_t\delta_s\delta_t...}_{m_{st}}$, for any $s, t \in S$ where $m_{st} < \infty$.

Recall that an algebra over a commutative ring R is an R-module with an R-bilinear multiplication operation. A unital associative algebra over R is then an algebra over R for which multiplication is associative and has a multiplicative identity.

Similarly to Coxeter groups, we call the first relations *quadratic relations* and the second *braid relations*.

Note that the quadratic relation is equivalent to $(\delta_s - v^{-1})(\delta_s + v) = 0$.

For $w \in W$ with reduced expression $w = s_1 s_2 ... s_k$, define the element $\delta_w = \delta_{s_1}\delta_{s_2}...\delta_{s_k}$ of $\mathcal{H}$. Since $\mathcal{H}$ has braid relations identical to W, Matsumoto's theorem (Theorem 2.1.7) implies that this is independent of the choice of reduced expression. Note that we set $\delta_1 = 1$

Theorem 2.2.2. *The Hecke algebra is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{\delta_w : w \in W\}$.*

Definition 2.2.3. We call $\{\delta_w : w \in W\}$ the *standard basis* of $\mathcal{H}$.

Proposition 2.2.4. *The following multiplication formulae hold in $\mathcal{H}$. For $w \in W$ and $s \in S$,*

$$\delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } ws > w, \\ (v^{-1} - v)\delta_w + \delta_{ws} & \text{if } ws < w, \end{cases}$$

and

$$\delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } ws > w, \\ (v^{-1} - v)\delta_w + \delta_{sw} & \text{if } ws < w. \end{cases}$$

Proposition 2.2.5. *For any simple reflection $s \in S$,*

$$\delta_s^{-1} = \delta_s + (v - v^{-1}).$$

This follows from the quadratic relation in $\mathcal{H}$.

Proposition 2.2.6. *Since the generators $\{\delta_s : s \in S\}$ of $\mathcal{H}$ are invertible, δ_w is invertible for every $w \in W$. Moreover,*

$$\delta_{w^{-1}}^{-1} = \delta_w + \sum_{x<w} a_x \delta_x$$

for some $a_x \in \mathbb{Z}[v, v^{-1}]$.

There is another basis known as the Kazhdan-Lusztig basis.

Definition 2.2.7. The *Kazhdan-Lusztig involution* or *bar involution* is a $\mathbb{Z}$-linear involution $\mathcal{H} \to \mathcal{H}, h \mapsto \overline{h}$ defined on generators $\overline{v} = v^{-1}$ and $\overline{\delta_s} = \delta_s^{-1}$ for $s \in S$, such that it distributes across products as a ring automorphism.

Definition 2.2.8. The *Kazhdan-Lusztig basis* for $\mathcal{H}$ is the set $\{b_w : w \in W\} \subseteq \mathcal{H}$ such that for any $w \in W$,

- b_x is self-dual, i.e. $\overline{b_x} = b_x$, and

- b_x has the form
$$b_x = \delta_x + \sum_{y<x} h_{y,x}\delta_y$$
for some $h_{y,x} \in v\mathbb{Z}[v]$, where $<$ is the Bruhat order.

The coefficients $h_{y,x} \in v\mathbb{Z}[v]$ are called *Kazhdan-Lusztig polynomials.*

Additionally, we set $h_{x,x} = 1$ and $h_{y,x} = 0$ if $y \not\leq x$ in the Bruhat order. The second condition is sometimes called the *degree bound* condition.

Lemma 2.2.9. *The Kazhdan-Lusztig basis is unique.*

Furthermore, the corresponding Kazhdan-Lusztig basis element for $s \in S$ is $b_s = \delta_s + v$.

Definition 2.2.10. The *Kazhdan-Lusztig anti-involution* $\omega : \mathcal{H} \to \mathcal{H}$ is an involution defined similarly to the Kazhdan-Lusztig involution, but distributes across products as a ring *anti*-automorphism. That is for $a, b \in \mathcal{H}$, $\omega(ab) = \omega(b)\omega(a)$.

Definition 2.2.11. The *standard trace* $\epsilon : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ is a $\mathbb{Z}[v, v^{-1}]$-linear map which extracts the coefficient of δ_{id} for elements written in the standard basis.

Definition 2.2.12. The *standard form* $(-,-) : \mathcal{H}\times\mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ is a sesquilinear form (with respect to either involution restricted to $\mathbb{Z}[v, v^{-1}]$) such that $(a, b) := \epsilon(\omega(a)b)$ for $a, b \in \mathcal{H}$.

Here, sesquilinear means that the form is linear in the second variable and in the first variable, $(fa, b) = \overline{f}(a, b)$ for $a, b \in \mathcal{H}$ and $f \in \mathbb{Z}[v, v^{-1}]$. Note the restricted involution inverts each v extending linearly to $\mathbb{Z}[v, v^{-1}]$. The bar involution and anti-involution restricted to $\mathbb{Z}[v, v^{-1}]$ are the same, as this ring is commutative.

Theorem 2.2.13. *The Kazhdan-Lusztig basis is asymptotically orthonormal. That is for $x, y \in W$,*
$$(b_x, b_y) = \begin{cases} 1 + v\mathbb{Z}[v] & \text{if } x = y, \\ v\mathbb{Z}[v] & \text{otherwise.} \end{cases}$$

2.3 Soergel Bimodules

Definition 2.3.1. A $\mathbb{Z}$*-graded ring* R is a ring with a decomposition
$$R = \bigoplus_{i\in\mathbb{Z}} R^i$$
into a direct sum of additive subgroups $R_i \subseteq R$ such that $R^iR^j \subseteq R^{i+j}$.

Gradings are defined in this way to generalise a notion of 'degree', where the degree of a product is the sum of their degrees. This definition can naturally be extended to $\mathbb{Z}$-graded modules over some a $\mathbb{Z}$-graded ring.

Definition 2.3.2. Let R be a $\mathbb{Z}$-graded ring. A *$\mathbb{Z}$-graded R-module* M is a module over R with a decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M^i$$

into a direct sum of additive subgroups $M^i \subseteq M$ such that $R^i M^j \subseteq M^{i+j}$. We call the M^i *graded pieces* of M, and the elements of M^i *homogeneous of degree i*.

For the remainder of this paper we will only be working with $\mathbb{Z}$-graded objects, so we just say *graded*.

Example 2.3.3. For any ring R, the *trivial grading* of R is the decomposition where $R^0 = R$ and $R^i = 0$ for all $i \neq 0$.

Example 2.3.4. Let F be a field with the trivial grading. The vector space of real polynomials (in one or several variables) over F has a natural grading where the n-graded piece is the subspace generated by degree n monomials. For example, the $\mathbb{R}$-vector space $\mathbb{R}[x]$ has a decomposition

$$\mathbb{R}[x] = V^0 \oplus V^1 \oplus V^2 \oplus \dots$$

where V^i is the subspace spanned by $\{x^i\}$. This example is a $\mathbb{Z}$-grading where the n-graded piece is 0 for $n < 0$.

Gradings for other algebraic objects, such as algebras and bimodules, can be similarly defined. The following definitions are for general graded objects.

Definition 2.3.5. Let M and N be graded objects. For $i \in \mathbb{Z}$, define $M(i)$ to be the graded object with graded pieces $M(i)^j := M^{i+j}$. We say this is obtained by a *shift in grading* of M.

If we visualise the graded pieces horizontally in ascending order of degree, the grading of $M(i)$ is the grading of M shifted to the left by i places. Particularly, if $x \in M^n$ is homogeneous of degree n in M, then it is homogeneous of degree $n - i$ in $M(i)$.

Degree	-2	-1	0	1	2
M	M^{-2}	M^{-1}	M^0	M^1	M^2
$M(1)$	M^{-1}	M^0	M^1	M^2	M^3
$M(2)$	M^0	M^1	M^2	M^3	M^4
$M(-1)$	M^{-3}	M^{-2}	M^{-1}	M^0	M^1
$M(i)$	M^{-2+i}	M^{-1+i}	M^i	M^{1+i}	M^{2+i}

Definition 2.3.6. Let M and N be graded objects. A morphism $f : M \to N$ is *homogeneous of degree k* if $f(M^i) \subseteq N^{i+k}$ for all $i \in \mathbb{Z}$. Typically we assume morphisms between graded objects are homogeneous of degree 0, and call them *graded morphisms*. A *graded isomorphism* is a graded morphism with a graded (two-sided) inverse. We say that M and N are *isomorphic up to shift* if there is a graded isomorphism $M \simeq N(i)$ for some $i \in \mathbb{Z}$. The *graded morphism space* (or *graded Hom space*) between M and N is

$$\mathrm{Hom}^\bullet(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(M, N(i)).$$

Notice for any morphism $M \to N$ of degree k, there is a morphism $M \to N(k)$ of degree 0 that contains the same information.

Definition 2.3.7. Let M be a graded object in an additive category (i.e. direct sums are defined on M), and let $p = \sum_i p_i v^i \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ be a Laurent polynomial with positive integer coefficients. Define

$$M^{\oplus p} := \bigoplus_{i \in \mathbb{Z}} M(i)^{\oplus p_i}$$

where $M^{\oplus k} := \bigoplus_{j=1}^{k} M$ for $k \in \mathbb{Z}_{\geq 0}$.

Definition 2.3.8. Let R be a graded ring and M a graded R-module. A *graded submodule* of M is a submodule $N \subseteq M$ with the induced grading $N^i = N \cap M^i$ for all $i \in \mathbb{Z}$. A *graded direct summand* of M is a graded module N such that $M \simeq N \oplus N'$ as graded modules for some graded submodule $N' \subseteq M$. We say that M is *graded free* if it has an R-basis of homogeneous elements of M. If this basis is finite, then there is a unique $p \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ such that $M \simeq R^{\oplus p}$, and we call p the *graded rank* of M.

For our purposes, fix a Coxeter system (W, S) and consider it's geometric representation V. Let $R = \mathrm{Sym}(V) \simeq \mathbb{R}[\alpha_s : s \in S]$ be the symmetric algebra of V, which we will think of as the real polynomial ring generated, as a ring, by the basis of V. We can think of R as a graded algebra[1], such that $V \subseteq R$ is homogeneous of degree 2, i.e. $\deg \alpha_s = 2$ and the 'monomials' that are products of i basis elements are degree $2i$.

There is a natural action of W on R, induced by its action on V, that for any $w \in W$,

$$w \cdot \prod_{s \in S} \alpha_s^{k_s} = \prod_{s \in S} (w \cdot \alpha_s)^{k_s}$$

where $k_s \in \mathbb{Z}_{\geq 0}$, extending linearly to R.

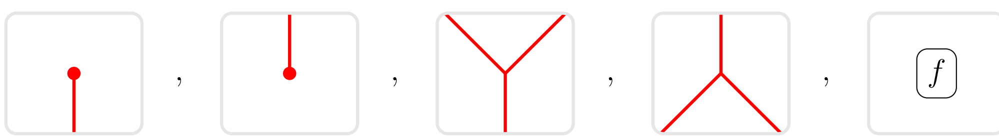
[1] A graded module that is also a graded ring.

Chapter 3

One-Colour Diagrammatics

In this section, we describe one-colour diagrammatics for morphisms in $\mathbb{BS}$Bim. The morphisms in this category have a presentation in terms of generators and relations.

The generators are the following univalent and trivalent vertices, along with boxes.



(3.0.1)

These are the unit, multiplication, counit and comultiplication maps from the Frobenius algebra structure of $B_s \in \mathbb{BS}$Bim.