Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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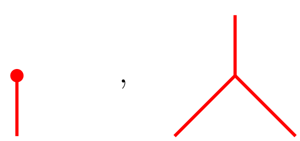
One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the one-colour diagrammatic Hecke category $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = 1 \rangle$.

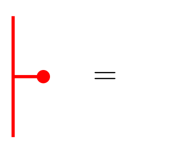
The objects of this category are generated by taking formal tensor products of the non-identity element $s \in S_2$. For example the tensor product of four s's which denote with the expression (s, s, s, s).

The morphisms in this category have a presentation in terms of generators and relations. For convenience, we will describe them up to isotopy. The generators are the following univalent and trivalent vertices, which can be rotated and flipped vertically using isotopy.

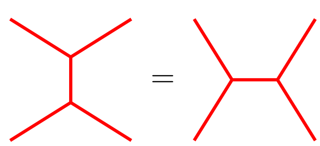


(3.1.1)

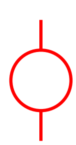
These morphisms are subject to the following local relations.



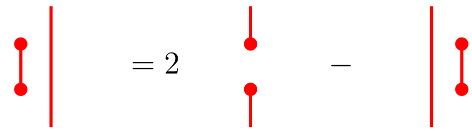
(3.1.2a)



(3.1.2b)



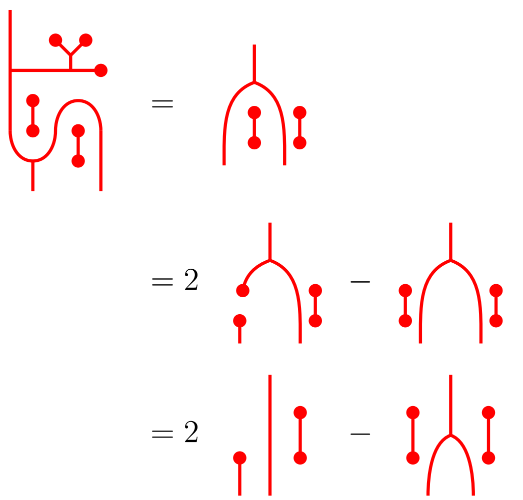
$= 0$ (3.1.2c)



(3.1.2d)

Remark 3.1.3. The object s is a Frobenius algebra object in $\mathcal{H}(S_2)$. The generators (3.1.1) and their horizontal reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.2a) and (3.1.2b).

Example 3.1.4. Let us use the relations in (3.1.2) to simplify the following morphism in $\mathrm{Hom}((s,s),(s))$.



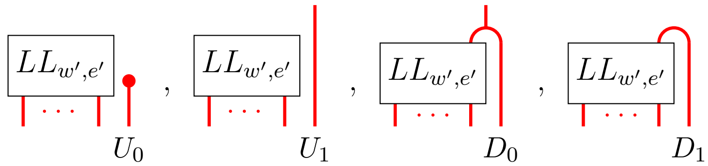
There is a $\mathbb{Z}[\ ]$-bimodule basis for $\mathrm{Hom}(s^n, s^m)$ introduced by Libedinsky [Lib13] called the Double Leaves basis. We first define morphisms known as Light leaves. Given a word $w = s^n$, a subexpression is a binary string of length n. For example, $0000, 0110$ and 1011 are subexpressions of $s^4 = ssss$. Given a subexpression e of an object w, we can apply it for an element $w^e \in S_2$, e.g. $ssss^{1011} = s * 1 * s * s = s$. Maybe use subscript here to avoid confusion with $s^n = ss...s$. Each term of the subexpression is a decision of whether to include the corresponding s in the word, where the decision to exclude an s amounts to multiplying by 1.

For a subexpression e of an expression w, we can label each term by U_0, U_1, D_0 or D_1. The label is U_* if the partial subexpression up to the current term evaluates to $1 \in S_2$ and D_* if it evaluates to $s \in S_2$, where the subscript corresponds to the term in e.

Example 3.1.5. For the object $ssss$ and subexpression 0101, we can find the labels:

Choice	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
Partial w^e	1	$1 * s = s$	$1 * s * 1 = s$	$1 * s * 1 * s = 1$
Labels	U_0	U_0U_1	$U_0U_1D_0$	$U_0U_1D_0D_1$

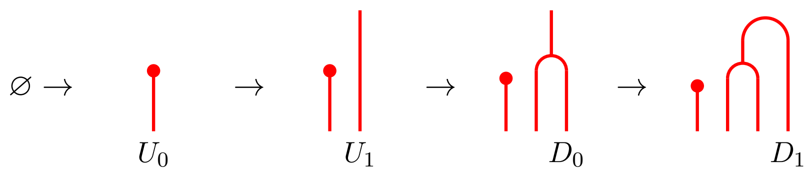
The light leaf $LL_{w,e} \in \mathrm{Hom}(w, w^e)$, corresponding to the object w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0,1\}$, $LL_{w's,e'i}$ is one of



(3.1.6)

depending on the next label, where w' and e' are appropriate subwords of w and e. Observe that the codomain of a light leaf $LL_{w,e}$ corresponds to the evaluation $w^e \in S_2$ of the subexpression. The recursive definition is consistent, since if the next label is U_* then the codomain of $LL_{w',e'}$ (the evaluation of the partial subexpression $w'^{e'}$ up to the label) is 1, and when the next label is D_* the codomain is s. Rewrite this to make sense. Do we need to talk about 'degree' of light leaves?

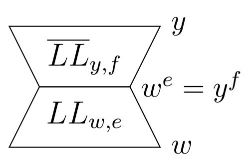
Example 3.1.7. Following from Example (3.1.5) for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.



Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. A double leaf associated to expressions w, y is a composition

$$\mathbb{LL}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

for subexpressions e of w and f of y such that $w^e = f^y$. Visually this looks like a morphism from w to y factoring through $w^e = y^f \in \{1, s\}$,



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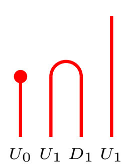
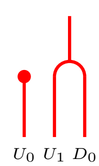
Example 3.1.8. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

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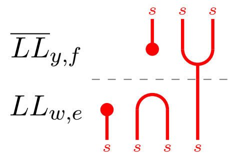
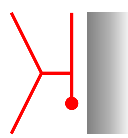
Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$ is



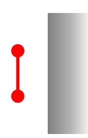
3.2 Diagrammatic $\mathcal{O}(\mathrm{SL}(2))$

With the diagrammatic category $\mathcal{H}(S_2)$, we can describe diagrammatics for the category $\mathcal{O}(\mathrm{SL}(2))$. In particular, we define a modular category *[what do we call this cat?]* over $\mathcal{H}(S_2)$.

This module category has elements copied from $\mathcal{H}(S_2)$ and morphisms are generated by the empty diagram $\varnothing$, with $\mathcal{H}(S_2)$ acting on the left by left concatenation on objects and morphisms. Additionally, the morphisms have one new relation, where diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0,1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0,1] \times \mathbb{R}$. For example a morphism may be



Then, diagrams are related to the wall by

 $= 0.$ (3.2.1)

Chapter 4

Two-colour Diagrammatics

4.1 Two-colour Diagrammatic Hecke Category

Blah

4.2 Diagrammatic $\mathrm{Tilt}(\mathrm{SL}(2))$

Blah

Bibliography

[Lib13] Nicolas Libedinsky. *Light leaves and Lusztig's conjecture*. 2013. DOI: 10.48550/ARXIV.1304.1448. arXiv: 1304.1448 [math.RT].