Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

Victor Zhang
UNSW Australia

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Contents

Chapter 1

Introduction

Visual interpretations of functions simplify calculations and often provides insights into the mathematical objects they encode. More detail about what this is This general philosophy takes form in various settings. A simple example are string diagrams for finite permutations. A permutation can be drawn as strings between two copies of a set determining how the objects are permuted. Compositions of these permutations is the operation of joining corresponding strings start to end in order to create a larger string diagram representing their product.

A significant example are *planar algebras* in the work of Jones. These are certain algebras of planar diagrams that describe operators. His study of the Temperley-Lieb planar algebra lead to the discovery of an important invariant in knot theory in the 1983, which we know now as the Jones polynomial. For this and surrounding works he received a Fields medal. This technology of planar algebras have been since used to study subfactors in functional analysis [Jon21][1] and have consequences in for example statistical mechanics and mathematical physics.

In representation theory, our main motivational example is given by the proof of the Kazhdan–Lusztig conjecture through the diagrammatics of Soergel bimodules. The conjecture relates Kazhdan–Lusztig polynomials, arising from the Weyl group associated with a Lie algebra, to Jordan–Hölder multiplicities of particular representations of Lie algebras called Verma modules. Proofs were discovered independently by Beilinson–Bernstein and Brylinski–Kashiwara in 1981 but by geometric methods, which was unsatisfying to many. Around this time, Soergel was working toward an algebraic proof by Soergel bimodules, however he hit a technical road block. In 2010's, Elias and Williamson Ref? developed planar diagrams for morphisms on Soergel bimodules and were able to overcome the technical point where Soergel got stuck, to prove the conjecture diagrammatically. The diagrams can greatly simplify algebraic calculations and the diagrammatic category can be considered independently from Soergel bimodules. We explore this diagrammatics for S_2 in Section 3.1.

One of the advantages of the diagrammatic Soergel bimodules is that it can be defined over $\mathbb{Z}$ and extended to fields of characteristic p where classical Soergel bimodules are

[1] Originally from 1999, and was recently published.

ill-behaved. Characters in the category of tilting modules (certain representations of a Lie algebra) can be calculated via Kazhdan–Lusztig polynomial in characteristic zero, however these polynomials were unknown in characteristic p. Riche and Williamson in [RW18] were able to construct these characteristic p Kazhdan–Lusztig polynomials by considering diagrammatic Soergel bimodules in characteristic p.

In this paper we give an introduction to drawing morphisms in monoidal categories, put more here and define some mechanisms to form an additive and idempotent complete category. In Chapter 3 we define diagrammatic Soergel bimodules associated with the symmetric group S_2, construct a basis for its morphism spaces and state the theorem for its equivalence to the category of algebraic Soergel bimodules. We use this diagrammatic category to construct a diagrammatic module category with an extra relation, then prove its equivalence to the category of projective objects in the principle block of the category $\mathcal{O}$. In Chapter 4 we consider the affine symmetric group $\widetilde{S}_2$ to define the diagrammatic Soergel bimodules associated it, construct a basis for its morphism spaces and state the theorem for its equivalence to the category of algebraic Soergel bimodules. The extra generator in $\widetilde{S}_2$ compared with S_2 provides some additional complexity to the structure of the category. We then form a module category with two extra relations and provide a proof of its equivalence to the category of tilting modules for $\mathfrak{sl}_2$. In the last chapter we discuss the consequences of diagrammatics in relation to Chapter 3 and Chapter 4, mention generalisations of the results and further areas of interest.

The contents of this thesis are for honours students and future readers that are interested in this topic. The reader is assumed to have some familiarity with undergraduate algebra (such as groups, rings, algebras and fields), basic ideas in representation theory, basic category theory and monoidal categories.

Talk about we dont need to know about category $\mathcal{O}$ and Tilt. One of the advantages of diagrammatics is that we don't need to understand these complex categories in representation theory to study them. For this reason, we will defer the details in the proofs involving them to other sources.

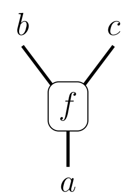
Chapter 2

Background

2.1 Drawing Monoidal Categories

A *monoidal category* $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold[1]. The bifunctor $\otimes$ is called the *tensor* or *monoidal product*. A monoidal category is *strict* if $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ and $A = \mathbb{1} \otimes A = A \otimes \mathbb{1}$ for objects and similarly for morphisms. In this paper, we will assume that monoidal categories are strict, since all monoidal categories are monoidally equivalent to a strict one[2].

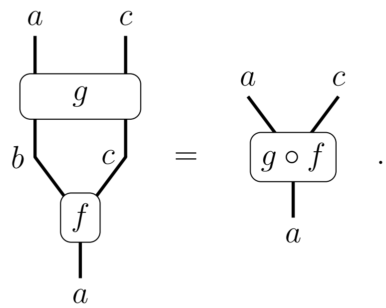
The morphisms of $\mathcal{C}$ can be drawn as string diagrams, where the morphism maps from the bottom to the top. Functions that make up the morphism are drawn as tokens or boxes. For example



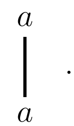
depicts a morphism $f : a \to b \otimes c$. Notice here that tensor products are displayed with its factors laid out horizontally. The compositions of morphisms is the vertical stacking of diagrams whenever labels on domains and codomains match. For example, the composition $g \circ f : a \to b \otimes c \to a \otimes c$ of $f : a \to b \otimes c$ with $g : b \otimes c \to a \otimes c$ has the diagram

[1]For more details see [Eti+15].

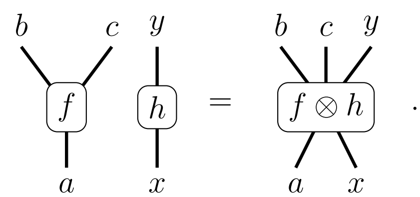
[2]See [ML98, VII.2] or [Eti+15, Thm 2.8.5]



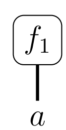
For identity morphisms we just draw a vertical line, so id_a is the diagram



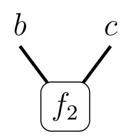
This is sensible as composing a function with identities should not change the function, and this is clearly evident with diagrams. The tensor product of morphisms is the horizontal concatenation of diagrams, such that strings from separate functions don't interact. For example, given $h : x \to y$, the tensor product $f \otimes h : a \otimes x \to b \otimes c \otimes y$ is drawn as



We let the monoidal unit $\mathbb{1}$ be blank and unlabelled, and strings that would join to $\mathbb{1}$ are blank. Particularly, $\mathrm{id}_{\mathbb{1}}$ is an empty diagram. It makes sense to display $\mathbb{1}$ in this way since tensoring with $\mathbb{1}$ does nothing to objects and tensoring with $\mathrm{id}_{\mathbb{1}}$ does nothing to morphisms in a strict monoidal category. Then we have diagrams such as



and

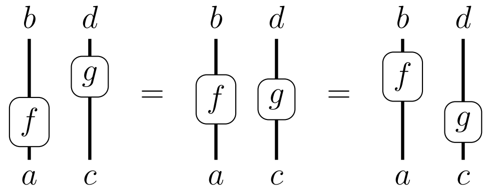


for morphisms $f_1 : a \to \mathbb{1}$ and $f_2 : \mathbb{1} \to b \otimes c$.

For a monoidal category $\mathcal{C}$, the bifunctoriality of $- \otimes -$ implies the following *interchange law*. For morphisms $f : a \to b$ and $g : c \to d$, $(\mathrm{id}_b \otimes g) \circ (f \otimes \mathrm{id}_c) = f \otimes g = (f \otimes \mathrm{id}_d) \circ (\mathrm{id}_a \otimes g)$. In other words the following diagram commutes.

$$\begin{tikzcd}
a \otimes c \arrow[r, "f\otimes \mathrm{id}_c"] \arrow[d, "\mathrm{id}_a \otimes g"'] \arrow[dr, "f\otimes g"] & b \otimes c \arrow[d, "\mathrm{id}_b \otimes g"] \\
a \otimes d \arrow[r, "f\otimes \mathrm{id}_d"] & b \otimes d
\end{tikzcd}$$

Written with string diagrams, this is



which holds up to deformation of the diagram.

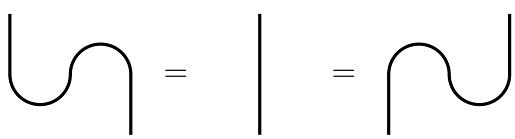
Before looking at our main example of a diagrammatic monoidal category, we first define some terminology.

Definition 2.1.1. For a commutative ring R, an *R-linear category* is a category enriched over the category of R-modules. That is, for objects a, b, the set of morphisms $\mathrm{Hom}(a, b)$ is an R-module and the composition of morphisms is R-bilinear. A *R-linear monoidal category* is a category that is both monoidal and R-linear such that the monoidal product on morphisms is R-bilinear.

Example 2.1.2. Let $\Bbbk$ be a field. The category of vector spaces over $\Bbbk$, $\mathbf{Vect}_{\Bbbk}$, is a $\Bbbk$-linear monoidal category. This follows by the classical theory of linear algebra.

Definition 2.1.3. A monoidal category $\mathcal{C}$ is *generated* by finite set S_o of objects and S_m of morphisms, when all non-unit objects are a finite tensor of objects in S_o and all non-identity morphisms are a finite combination of tensors and compositions of morphisms in S_m.

Example 2.1.4. Our first example of a diagrammatic monoidal category is the *Temperley-Lieb category*. The Temperley-Lieb category $\mathcal{TL}$ is a strict R-linear monoidal category whose objects are generated by the vertical line $|$ and morphisms generated by the cup $\cup : \mathbb{1} \to | \otimes |$ and cap $\cap : | \otimes | \to \mathbb{1}$, with relations



.

Mention that composition and tensor product is as explained above
Some example
Mention bubbles and specialisation to some $\delta \in R$
Mention that these are crossingless matchings
Comment on isotopy

2.2 Frobenius Objects

Something something about Many relations in categorical structures can be written in diagrammatic terms - adjunctions, monoid

Something about isotopy

Let $\mathcal{C}$ be a (strict) monoidal category. We can define the following objects.

Definition 2.2.1. A *monoid object* in $\mathcal{C}$ is a triple (M, μ, η) for an object $M \in \mathcal{C}$, a *multiplication* map $\mu : M \otimes M \to M$ and a *unit* map $\eta : \mathbb{1} \to M$, such that

$$\begin{tikzcd} & M \otimes M \otimes M \arrow[dl, "\mu\otimes\mathrm{id}_M"'] \arrow[dr, "\mathrm{id}_M \otimes\mu"] & \\ M \otimes M \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "\eta\otimes\mathrm{id}_M"] \arrow[dr, "\mathrm{id}_M"'] & M \otimes M \arrow[d, "\mu"] & M \otimes \mathbb{1} \arrow[l, "\mathrm{id}_M \otimes\eta"'] \arrow[dl, "\mathrm{id}_M"] \\ & M & \end{tikzcd}$$

commute. The first diagram is the *associativity* relation $\mu \circ (\mu \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \mu)$ and the second diagram is the *unit* relation $\mathrm{id}_M = \mu \circ (\eta \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \eta)$.

Dually, a *comonoid object* in $\mathcal{C}$ is a triple (M, δ, ϵ) for an object $M \in \mathcal{C}$, a *comultiplication* map $\delta : M \to M \otimes M$ and a *counit* map $\epsilon : M \to \mathbb{1}$, satisfying the *coassociativity* relation

$$\begin{tikzcd} & M \otimes M \otimes M & \\ M \otimes M \arrow[ur, "\delta\otimes\mathrm{id}_M"] & & M \otimes M \arrow[ul, "\mathrm{id}_M \otimes\delta"'] \\ & M \arrow[ul, "\delta"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

and *counit* relation

$$\begin{tikzcd} \mathbb{1} \otimes M & M \otimes M \arrow[l, "\epsilon\otimes\mathrm{id}_M"'] \arrow[r, "\mathrm{id}_M \otimes\epsilon"] & M \otimes \mathbb{1} \\ & M \arrow[ul, "\mathrm{id}_M"] \arrow[u, "\delta"'] \arrow[ur, "\mathrm{id}_M"'] & \end{tikzcd} .$$

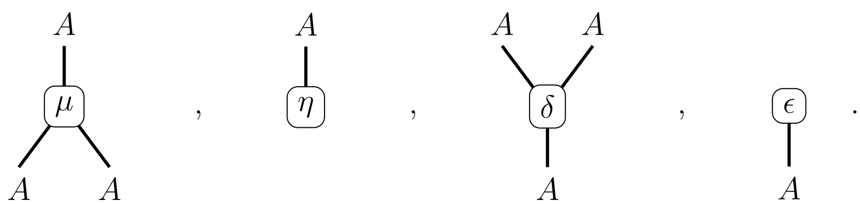
Monoid objects generalise monoids, i.e. sets with an identity equipped with an associative binary operation.

Definition 2.2.2. A *Frobenius object* in $\mathcal{C}$ is a quintuple $(A, \mu, \eta, \delta, \epsilon)$ such that (A, μ, η) is a monoid object, (A, δ, ϵ) is a comonoid object, and the maps satisfy the *Frobenius relations*

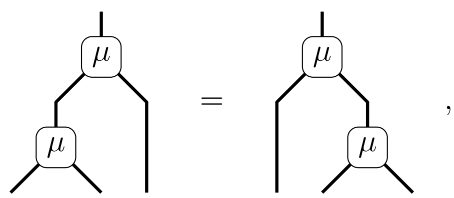
$$\begin{tikzcd}
& A\otimes A \arrow[dl, "\delta\otimes\mathrm{id}_A"'] \arrow[d, "\mu"] \arrow[dr, "\mathrm{id}_A\otimes\delta"] & \\
A\otimes A\otimes A \arrow[dr, "\mathrm{id}_A\otimes\mu"'] & A \arrow[d, "\delta"] & A\otimes A\otimes A \arrow[dl, "\mu\otimes\mathrm{id}_A"] \\
& A\otimes A &
\end{tikzcd},$$

that is $(\mathrm{id}_A\otimes\mu)\circ(\delta\otimes\mathrm{id}_A)=\delta\circ\mu=(\mu\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\delta)$.

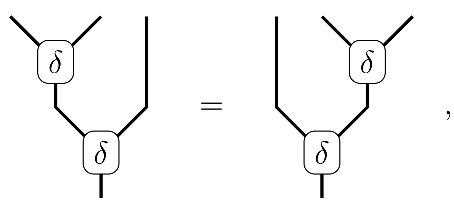
The maps and relations for a Frobenius object $(A,\mu,\eta,\delta,\epsilon)$ have a nice description with the diagrams given in Section 2.1. The structure maps are drawn as



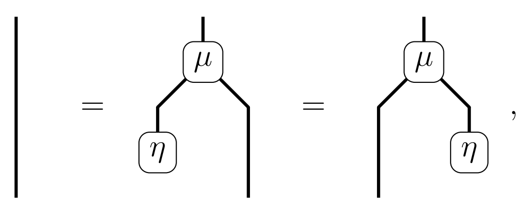
For the rest of this section, we only work with the Frobenius object A and $\mathbb{1}$. We can stop putting the label A by identifying A with the identity strand $\mathsf{I}=\mathrm{id}_A$. Diagrammatically, the associativity relation $\mu\circ(\mu\otimes\mathrm{id}_M)=\mu\circ(\mathrm{id}_M\otimes\mu)$ is



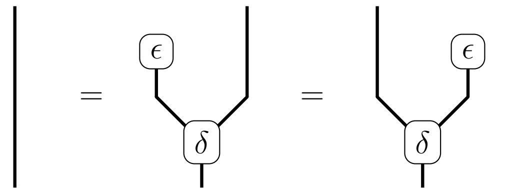
the coassociativity relation $(\delta\otimes\mathrm{id}_A)\circ\delta=(\mathrm{id}_A\otimes\delta)\circ\delta$ is



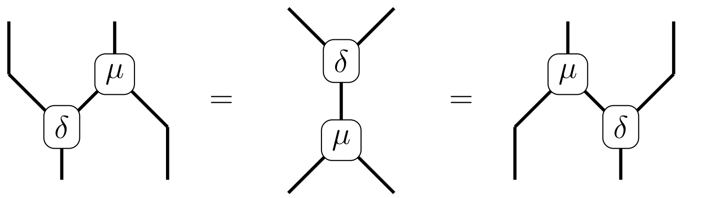
the unit relation $\mathrm{id}_A=\mu\circ(\eta\otimes\mathrm{id}_A)=\mu\circ(\mathrm{id}_A\otimes\eta)$ is



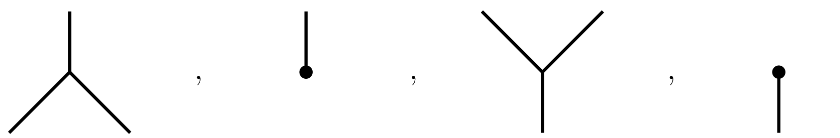
the counit relation $\mathrm{id}_A=(\epsilon\otimes\mathrm{id}_A)\circ\delta=(\mathrm{id}_A\otimes\epsilon)\circ\delta$ is



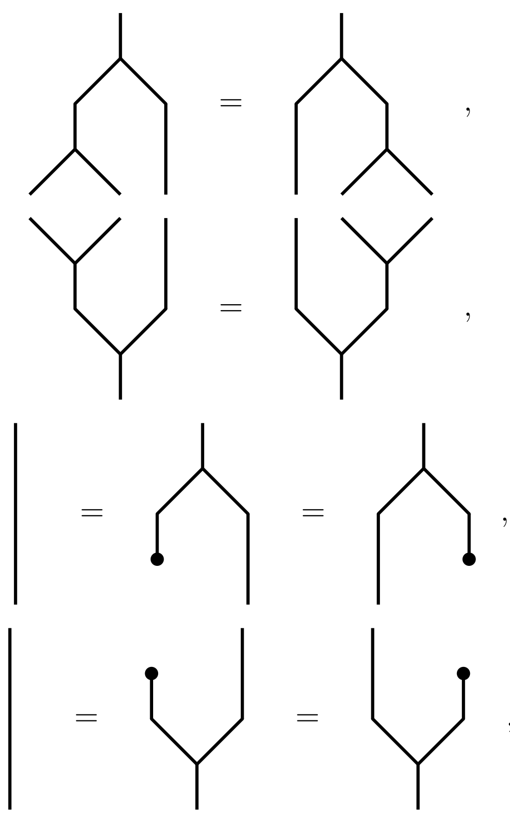
and the Frobenius relation $(\mathrm{id}_A \otimes \mu) \circ (\delta \otimes \mathrm{id}_A) = \delta \circ \mu = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta)$ is



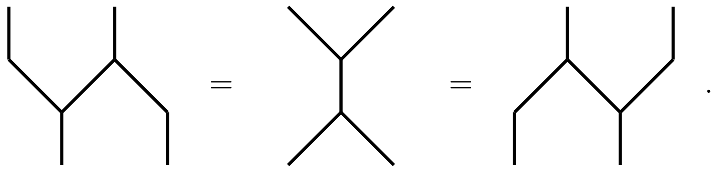
To simplify the diagrams, we stop labelling the functions and draw the structure maps as



So the relations become



and



Talk about isotopy

Maybe something about the (diagrammatic?) category Frob, capturing the data of a frobenius object

2.3 Module Categories

Chapter 3

One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = e \rangle$. At the end of this section, we see that this diagrammatic category is equivalent to the category of Soergel Bimodules under additive Karoubian closure.

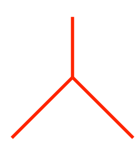
Remark 3.1.1. All diagrammatics below and in Chapter 4 can be defined in the language of planar algebras, without mentioning (monoidal) categories, e.g. in [Jon21]. Nevertheless, we define them in the context of categories as we will see them as diagrammatic versions of important categories in representation theory.

What do we do about $\mathbb{C}$? Do the theorems (at the end) apply over $\mathbb{Z}$ or $\mathbb{C}$ or both? If we define over $\mathbb{Z}$, how do we use it over $\mathbb{C}$ for the next section?

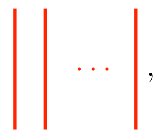
Definition 3.1.2. The *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ is a $\mathbb{Z}$-linear monoidal category with the following presentation.

The objects are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is concatenation. The empty tensor product, i.e. the monoidal identity, will be denoted $\varnothing =: s^0$.

The morphisms are generated, up to isotopy, by univalent and trivalent vertices

 ,  , (3.1.3)

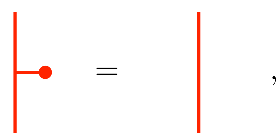
that are maps $s \to \varnothing$ and $ss \to s$ respectively, and their vertical reflections. We put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). The free $\mathbb{Z}$-module structure on each morphism space $\mathrm{Hom}(s^n, s^m)$ produces $\mathbb{Z}$-linear combinations of such diagrams. Something about composition/tensor and addition commuting Then, composition or tensors with the zero morphism 0 result in 0. To abuse notation, the empty diagram $\varnothing \to \varnothing$ will be denoted $\varnothing$. The identity morphism in $\mathrm{Hom}(s^n, s^n)$ is the diagram consisting of n (red) vertical lines



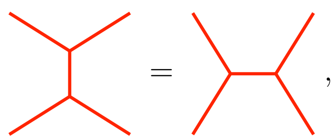
(3.1.4)

which we may identify with s^n.

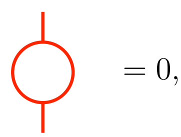
Such diagrams are subject to the following local relations



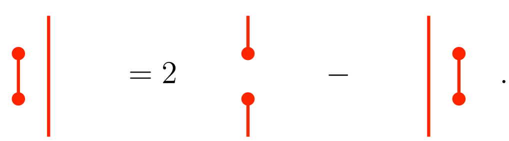
(3.1.5a)



(3.1.5b)



(3.1.5c)

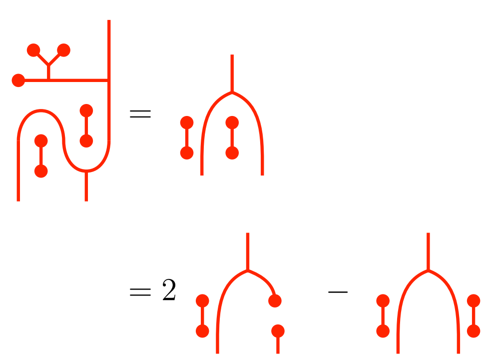


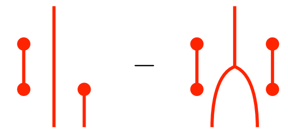
(3.1.5d)

Remark 3.1.6. The object s is a Frobenius object in $\mathcal{H}(S_2)$. The generators (3.1.3) and their vertical reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.5a) and (3.1.5b).

Put a definition of frob object in intro

Example 3.1.7. Using the relations in (3.1.5) we can simplify the morphism in $\mathrm{Hom}(ss, s)$,

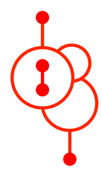


$= 2$  .

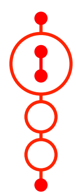
Add example of using frob associativity

Proposition 3.1.8. *All diagrams with floating diagrams, i.e. diagrams not connected to the domain or codomain by a red strand, are linear combination of diagrams where all floating diagrams are barbells.*

Proof. By isotopy and (3.1.5a), floating diagrams can be drawn as barbells with 'bubbles' and possibly floating subdiagrams inside each bubble. For example,

 .

The Frobeius relation (3.1.5b) allows us to 'straighten out' the bubbles to a chain of individual bubbles. The diagram above becomes

 .

For a floating diagram without floating subdiagrams, it is either 0 by (3.1.5c), or which can be removed from any diagrams containing it via (3.1.5d). Repeating this process produces a linear combination of diagrams where all floating diagrams are barbells. $\square$

The morphism space $\mathrm{Hom}(s^n, s^m)$ has a left (or right) $\mathbb{Z}[\ ]$-basis called the *double leaves* basis, as described in [EW16]. To define this basis, we must first define morphisms known as *light leaves*.

To make use of the group structure of S_2, we need to translate between words in $\mathcal{H}(S_2)$ and elements in S_2. Let $\phi : (\mathrm{ob}(\mathcal{H}(S_2)), \otimes) \to (S_2, *)$ be the monoid homomorphism[1] mapping $s \mapsto s$ and $\varnothing \mapsto 1$, and $\psi : S_2 \to \mathrm{ob}(\mathcal{H}(S_2))$ be the function that maps $s \mapsto s$ and $1 \mapsto \varnothing$. Should this be a definition? The maps ϕ allows words $w = s^n$ to be seen as elements of S_2, and ψ allows $1, s \in S_2$ to be seen as the objects $\varnothing, s \in \mathcal{H}(S_2)$. Clearly, $\phi\psi$ is the identity map on S_2, and the map $\psi\phi : \mathcal{H}(S_2) \to \mathcal{H}(S_2)$ takes objects to one of $\varnothing$ or s in $\mathcal{H}(S_2)$ by considering them as elements in S_2.

Definition 3.1.9. (Subexpression for S_2) Given a word $w = s^n$, a *subexpression* e is a binary string of length n. We can *apply* a subexpression to produce an object $w(e) \in \mathcal{H}(S_2)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $0 \leq i \leq n$, write $w(e, i)$ for the resultant object of the first i terms in e applied to the first i terms in w. Particularly $w(e, 0) = \varnothing$ and $w(e, n) = w(e)$.

[1]A map that preserves the monoidal product and identity element.

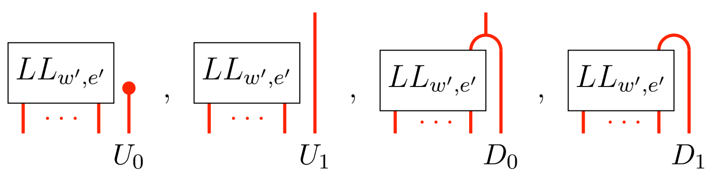
For example, 0000, 0110 and 1011 are subexpressions of $s^4 = ssss$. Applying the third subexpression gives $ssss(1011) = s\varnothing ss = sss$ and $ssss(1011, 3) = sss(101) = s\varnothing s = \varnothing$, by strictness of the monoidal category. Here, each term of the subexpression is a decision to include or exclude the corresponding s in the word, where excluding an s amounts to tensoring with $\varnothing$.

For a word w and subexpression e, we label each term by U_0, U_1, D_0 or D_1. The i-th term is labelled U_* if $\phi(w(e, i-1)) = 1 \in S_2$, and labelled D_* if $\phi(w(e, i-1)) = s \in S_2$. The label's subscript is the corresponding term in e.

Example 3.1.10. For the object $w = ssss$ and subexpression $e = 0101$, we find the labels as recorded in the following table.

Term i	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
$w(e,i)$	$\varnothing$	$\varnothing s = s$	$\varnothing s \varnothing = s$	$\varnothing s \varnothing s = ss$
Labels	U_0	U_0U_1	$U_0U_1D_0$	$U_0U_1D_0D_1$

Definition 3.1.11. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, the light leaf $LL_{w's,e'i}$ is one of

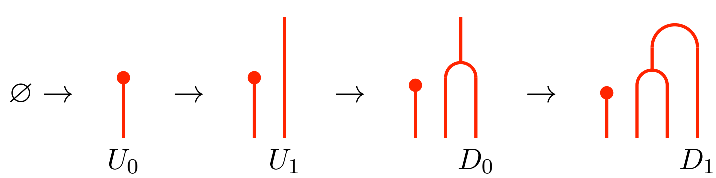


(3.1.12)

corresponding to the next label, where w' and e' are appropriate subwords[2] of w and e respectively.

Here, the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ is $\varnothing$, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. This implies that the recursive definition is consistent.

Example 3.1.13. Following from Example 3.1.10 for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.



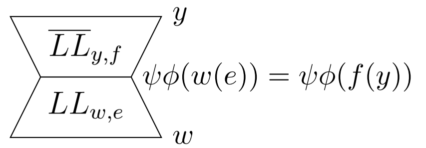
Definition 3.1.14. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y in $\mathcal{H}(S_2)$ is a composition

$$\mathbb{LL}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

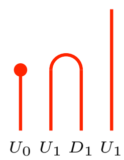
[2]A word with some letters removed.

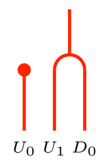
for subexpressions e of w and f of y such that $\psi\phi(w(e)) = \psi\phi(f(y))$.

Visually these are diagrams from w to y factoring through $\psi\phi(w(e)) = \psi\phi(f(y)) \in \{\varnothing, s\}$,



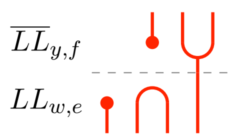
Example 3.1.15. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

.

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : ssss \to sss$, factoring through s, is



.

Note that these double leaves have no floating diagrams, which are by Proposition 3.1.8. In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{Z}[\ ]$-module, where the (left) -action is left concatenation by . Since we can move barbells to the right via. the relation (3.1.5d) and double leaves cut down the middle are double leaves factoring through $\varnothing$, we can equivalently act by $\mathbb{Z}[\ ]$ on the right. This leads us to the following theorem.

Theorem 3.1.16 (Elias-Williamson [EW16, Theorem 1.2])**.** *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{LL}(w,y)$ be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $\psi\phi(w(e)) = \psi\phi(y(f))$. Then $\mathbb{LL}(w,y)$ is a basis for* Hom(w,y) *as a left (or right) $\mathbb{Z}[\ ]$-module.*

A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.17. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky's construction in [Lib08].

The morphisms in this category can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1. The degree of a diagram is the sum of the degrees of the generators that appear in it. This makes $\mathcal{H}(S_2)$ a $\mathbb{Z}$-graded category. Maybe mention what a grading is.

Put graded category definition in background

Put example

The category $\mathcal{H}(S_2)$ (under the additive Karoubi Envelope) is a diagrammatic version of the category of Soergel bimodules $\mathbb{S}\mathrm{Bim}$ for S_2. However $\mathbb{S}\mathrm{Bim}$ is not well behaved with morphisms over $\mathbb{Z}$, so we must first alter the morphisms spaces in $\mathcal{H}(S_2)$ to be over $\mathbb{C}$ instead[3]. Formally we merely tensor the morphism spaces on the left by the $\mathbb{C}$-$\mathbb{Z}$-bimodule $\mathbb{C}$, where the right action is induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$. We write $\mathcal{H}_{\mathbb{C}}(S_2)$ for this $\mathbb{C}$-linear monoidal category. This process is quite simple and does little to the category itself. In particular, double leaves in $\mathcal{H}_{\mathbb{C}}(S_2)$ remain as $\mathbb{C}[\;]$-bases[4] for the morphisms.

Theorem 3.1.18 (Elias-Williamson [EW16, Theorem 6.30])**.** *The diagrammatic category* $\mathrm{Kar}^{\oplus}(\mathcal{H}_{\mathbb{C}}(S_2))$ *and the category of Soergel Bimodules* $\mathbb{S}\mathrm{Bim}$ *over* S_2 *are equivalent as graded* $\mathbb{C}$*-linear monoidal categories.*

The proof in [EW16] gives an equivalence of graded $\mathbb{C}$-linear monoidal categories $\mathcal{H}_{\mathbb{C}}(S_2) \cong \mathbb{BS}\mathrm{Bim}$ where $\mathbb{BS}\mathrm{Bim}$ is the category of Bott-Samelson bimodules over S_2. Since the additive Karoubi envelope preserves equivalences, $\mathrm{Kar}^{\oplus}(\mathbb{BS}\mathrm{Bim}) \cong \mathbb{S}\mathrm{Bim}$ implies $\mathrm{Kar}^{\oplus}(\mathcal{H}_{\mathbb{C}}(S_2)) \cong \mathbb{S}\mathrm{Bim}$.

3.2 Diagrammatic $\mathcal{O}_0(\mathfrak{sl}_2)$

A little bit about category $\mathcal{O}$, and our example of $\mathfrak{sl}_2$

For this section, our category of interest is $\mathcal{O}$ for the semisimple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. A description of the category $\mathcal{O}$ can be found in general in [Hum08, Sections 3.8–3.10] or in [Maz09, Section 5.2] for the case of $\mathfrak{sl}_2(\mathbb{C})$, however we will only give a brief overview. The category $\mathcal{O}$ is a category of certain modules (or representations) over a semisimple Lie algebra. It is a direct sum of subcategories, where, in the case of $\mathfrak{sl}_2$ over $\mathbb{C}$, the non-trivial summands are equivalent as abelian categories to the subcategory $\mathcal{O}_0$. Within this, we look to the full subcategory $\mathrm{proj}(\mathcal{O}_0)$ of projective modules in $\mathcal{O}_0$, which is in particular additive and contains all direct summands.

In [Soe90, Section 2.4], Soergel shows that the category $\mathcal{O}$, and hence the subcategory $\mathrm{proj}(\mathcal{O}_0)$, is a Soergel module category, i.e. it has an action of the monoidal category $\mathbb{S}\mathrm{Bim}$. By the equivalence in Theorem 3.1.18 we will view $\mathrm{proj}(\mathcal{O})$ as a $\mathcal{H}_{\mathbb{C}}(S_2)$-module category, extending via the additive Karoubi envelope. Since $\mathcal{H}_{\mathbb{C}}(S_2)$ is diagrammatic, this action allows us to describe $\mathrm{proj}(\mathcal{O}_0)$ (thus essentially $\mathcal{O}_0$ and $\mathcal{O}$) diagrammatically.

Remark 3.2.1. We can pass from $\mathrm{proj}(\mathcal{O}_0)$ to $\mathcal{O}_0$ by observing that $K^b(\mathrm{proj}(\mathcal{O}_0))$ is equivalent to $D^b(\mathcal{O}_0)$ as graded $\mathbb{Z}$-linear Should this be $\mathbb{C}$? monoidal triangulated categories. This is a standard trick in the field, for example see the introduction of [RW18][5]. However for our purposes it is not important to understand how this works.

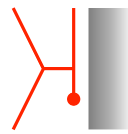
[3]The equivalence actually holds in more generality, but we choose $\mathbb{C}$ because it is easy to work with.

[4]It is not hard to check that double leaves tensored with $1 \in \mathbb{C}$ on the left form a basis.

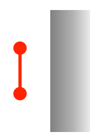
[5]A self-contained summary of how diagrammatic categories can be related to abelian categories.

Although we need to work over $\mathbb{C}$ for $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$, the diagrammatic category can be defined more simply, that is over $\mathbb{Z}$.

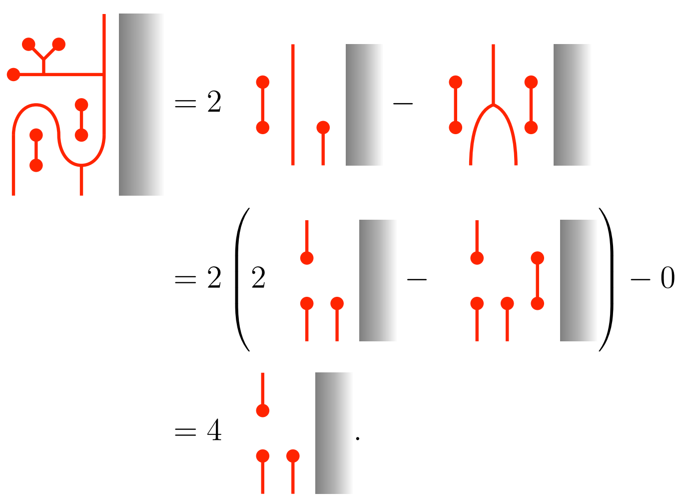
Definition 3.2.2. Let $\mathcal{DO}_0 := \mathcal{DO}_0(\mathfrak{sl}_2)$ be the $\mathbb{Z}$-linear (Define this in background) left $\mathcal{H}(S_2)$-module category with elements generated (Define what this means.) by the monoidal identity $\varnothing$ of $\mathcal{H}(S_2)$ and morphisms generated by the empty diagram $\varnothing$. The action of $\mathcal{H}(S_2)$ on the left is left concatenation for both objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0, 1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0, 1] \times \mathbb{R}$. For example a morphism may be

 .

We impose the relation that diagrams are related to the wall by

 $= 0.$ (3.2.3)

Example 3.2.4. Using the new relation (3.2.3), we can further simplify the morphism in Example (3.1.7) by



The objects of this category are identical to objects in $\mathcal{H}(S_2)$ and the morphisms are the same modulo the wall relation (3.2.3). A natural question to ask is whether double leaves still form bases for the morphism spaces here. Notice that double leaves appear in $\mathcal{DO}_0$ by acting on $\varnothing$ by double leaves in $\mathcal{H}(S_2)$. All morphisms in $\mathcal{DO}_0$ are morphisms in $\mathcal{H}(S_2)$ so they can be written as $\mathbb{Z}[$$]$-linear combinations of double leaves, though some have collapsed to 0. Thus double leaves span the morphism spaces of $\mathcal{DO}_0$ as (left)

$\mathbb{Z}[\ ]$-modules. However they may not be linearly independent as neither left nor right modules. For example, any pair of double leaves that factor through $\varnothing$ become 0 when multiplied by on either side (by translating the barbell to the right). Although double leaves are not always a basis for its respective morphism space as $\mathbb{Z}[\ ]$-modules, it turns out they are a basis over $\mathbb{Z}$.

Lemma 3.2.5. *Let $\pi : \mathrm{mor}(\mathcal{H}(S_2)) \to \mathrm{mor}(\mathcal{DO}_0)$ be the projection map which takes a morphism to the result of its action on the empty diagram $\varnothing$. Then the image $\pi(\mathbb{LL}(w, y))$ is a basis for $\mathrm{Hom}_{\mathcal{DO}_0}(w, y)$ as a $\mathbb{Z}$-module.*

Proof. We consider morphisms $\mathrm{Hom}(w, y)$ in $\mathcal{DO}_0$ for fixed objects w, y, and write $\mathbb{LL} := \pi(\mathbb{LL}(w, y))$ for the set of double leaves in $\mathcal{DO}_0$. Any diagram in $\mathcal{DO}_0$ can be written as a $\mathbb{Z}$-linear combination of morphisms without floating diagrams, by simplifying them to barbells, pulling them to the right and killing them with (3.2.3). We can write each of these as a $\mathbb{Z}[\ ]$-linear combination of double leaves by (3.1.16) with the right action, and reduce it to a $\mathbb{Z}$-linear combination by (3.2.3). This implies that $\mathbb{LL}$ spans $\mathrm{Hom}(w, y)$ as a $\mathbb{Z}$-module. Since the barbell-wall relation (3.2.3) has no effect on $\mathbb{Z}$-linear combinations of $\mathbb{LL}$, it follows from linear independence over $\mathbb{Z}[\ ]$ that they are linearly independent over $\mathbb{Z}$ in $\mathcal{DO}_0$. Check the proof. $\square$

Our goal is to prove that this diagrammatic category is equivalent to $\mathrm{proj}(\mathcal{O}_0)$. To that end, we will shift our focus from $\mathbb{Z}$ to $\mathbb{C}$ for the remainder of this section. From now on we write $\mathcal{DO}_0$ for the $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(S_2)$-module category obtained by replacing $\mathbb{Z}$ with $\mathbb{C}$ and $\mathcal{H}(S_2)$ with $\mathcal{H}_{\mathbb{C}}(S_2)$ in Definition 3.2.2. The above discussion and Lemma 3.2.5 still apply to $\mathcal{DO}_0$ over $\mathbb{C}$.

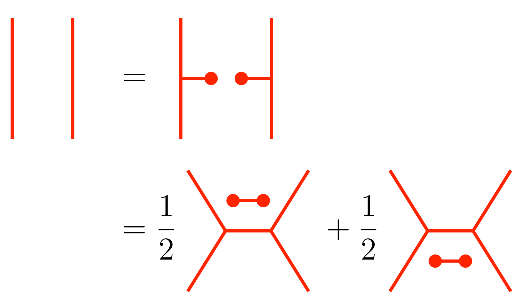
Maybe put this next bit in section 3.1

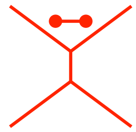
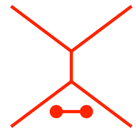
Say more about what this is, and why we say it here

The next result allows us to reduce all morphisms to a matrix of diagrams only involving $\varnothing$ and s.

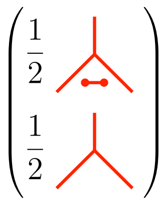
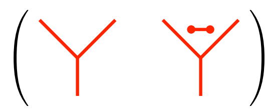
Lemma 3.2.6. *In the additive closure of $\mathcal{H}_{\mathbb{C}}(S_2)$ we have an explicit isomorphisms $ss \cong s \oplus s$, as detailed in the proof. Particularly, these are isomorphisms in the additive closure of $\mathcal{DO}_0$.*

Proof. In $\mathcal{H}(S_2)$ we have the relation



$= \frac{1}{2}$  $+ \frac{1}{2}$  . (3.2.7)

Note that this $\mathcal{H}(S_2)$ is $\mathbb{C}$-linear, so division by 2 is allowed. This implies we have maps

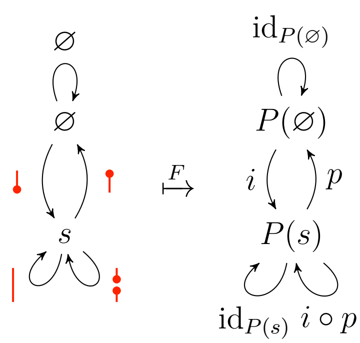
 $: ss \to s \oplus s$ and  $: s \oplus s \to ss.$

It follows from (3.1.5d), (3.1.5c) and the calculation (3.2.7), that these maps are inverses. Maybe put the inverse calculation here. □

Be clear that I don't understand category $\mathcal{O}$ very well.

As a shorthand, we write $\mathrm{proj}(\mathcal{O}_0)$ for $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$. The work of Soergel in [Soe90, Section 2.4] shows that $\mathrm{proj}(\mathcal{O}_0)$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_\varnothing, \Theta_s \in \mathrm{End}(\mathcal{O})$ (corresponding to elements in S_2). Explains what this means, how its related to the $\mathcal{H}(S_2)$ module category We construct a functor that maps faithfully into a full subcategory of $\mathrm{proj}(\mathcal{O}_0)$, which will become the whole of $\mathrm{proj}(\mathcal{O}_0)$ under the additive Karoubi envelope. This is the same strategy as in the proof for Theorem 3.1.18.

Definition 3.2.8. Let $F : \mathcal{DO}_0 \to \mathrm{proj}(\mathcal{O}_0)$ be the $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(S_2)$-module functor that sends the empty object $\varnothing$ to the trivial module $P(\varnothing)$, and the Soergel module action corresponding to s to the translation functor Θ_s. Then the object s maps to $\Theta_s(P(\varnothing)) =: P(s)$, and for example s^3 maps to $\Theta_s^3(P(\varnothing)) = \Theta_s\Theta_s\Theta_s(P(\varnothing))$. In order for F to be functorial, it must map identity diagrams $s^n \to s^n$ to $\mathrm{id}_{\Theta_s^n(P(\varnothing))}$. On non-identity maps, we let $F(\ )$ be the inclusion $i : P(\varnothing) \to P(s)$ and $F(\ )$ be the projection $p : P(s) \to P(\varnothing)$. The mapping of F is depicted by the following picture. Need to talk about compositions, why is F well defined if its generated by compositions of these? Are there any clashes? – Actually preserves compositions by construction



(3.2.9)

Actually refer to the picture

Lemma 3.2.10. *The functor F is well defined.*

Proof. From [Maz09, Proposition 5.90], there is a natural isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ analogous to the isomorphism $ss \cong s \oplus s$ given in the proof of Lemma 3.2.6. We consider the additive closure $\mathcal{DO}_0^{\oplus}$, which does not cause problems since we will eventually take this anyway Maybe just do additive karoubi closure in the first place. Given a morphism in $\mathcal{DO}_0$ from s^n to s^m, repeated precomposition and postcomposition with $ss \to s \oplus s$ and $ss \oplus s \to s$ from Lemma 3.2.6 results in an isomorphic matrix of diagrams with domain and codomain in $\{\varnothing, s\}$. By Lemma 3.2.5 over $\mathbb{C}$, $\mathrm{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_\varnothing\}$, $\mathrm{Hom}(s, \varnothing)$ has a basis $\{\ \}$, $\mathrm{Hom}(\varnothing, s)$ has a basis $\{\ \}$, and $\mathrm{Hom}(s, s)$ has a basis $\{\mathrm{id}_s, \ \circ\ \}$. Therefore, extending by linearity, the picture above completely describes the image of F. We can similarly pull back the image, i.e. matrices of morphisms in $\mathrm{proj}(\mathcal{O}_0)$, to a morphism between $\Theta_s^n(P(\varnothing))$ and $\Theta_s^m(P(\varnothing))$ via the analogous maps defining $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$. Therefore (3.2.9) is enough to define F for $\mathcal{DO}_0$.

Next we check that all the relations are preserved. From classical results e.g. [Maz09, Proposition 5.84 and Lemma 5.87], it follows that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$. Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure. Applying these to $P(\varnothing)$ result in the same relations in $\mathrm{proj}(\mathcal{O}_0)$ for $P(\varnothing), P(s)$ and $\Theta_s^2(P(\varnothing))$. Note that the projection and inclusion maps above are exactly the unit and counit of Θ_s evaluated at $P(\varnothing)$, and the trivalent vertices provided by projecting the isomorphisms in Lemma 3.2.6 map exactly to the multiplication and comultiplication maps. Furthermore, in [Soe90, Section 2.4] we see that $p \circ i = 0$ in $\mathrm{proj}(\mathcal{O}_0)$ which is analogous[6] to the barbell-wall relation (3.2.3). Hence all the relations in $\mathcal{DO}_0$ are preserved by F. By construction, F preserves $\mathbb{C}$-linear combinations and the Soregel module structure in [Soe90], so F is well defined as a functor between $\mathbb{C}$-linear $\mathcal{H}(S_2)$-module categories. □

Theorem 3.2.11 (Soergel, [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4])**.** *The diagrammatic category* $\mathrm{Kar}^{\oplus}(\mathcal{DO}_0(\mathfrak{sl}_2))$ *and* $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$ *are equivalent as* $\mathbb{C}$*-linear* $\mathcal{H}(S_2)$*-module categories.*

Check all of this & Put precise references
Maybe write description as a soergel module outside the proof

Proof. First we show that F is full and faithful. It follows from Lemma 3.2.6 and the description of $P(\varnothing)$ and $P(s)$ in [Maz09, Section 5.2] that the image of and generate all morphisms of the form $\Theta_s^n(P(\varnothing)) \to \Theta_s^m(P(\varnothing))$. Hence F is full. Now the mapping of F on all morphism spaces are determined by those depicted in the above picture. So, for faithfulness, it suffices to compare the $\mathbb{C}$-dimensions of morphism spaces between objects shown in the picture. As mentioned above, the double leaves basis are precisely the diagrams depicted in the image. The bases for the corresponding morphism spaces

[6]This relation extends to the analogue of the local barbell-wall relation, as all 'barbell on the right' morphisms in $\mathrm{proj}(\mathcal{O}_0)$ are linear combinations of applications of Θ_s to $p \circ i$, which is 0.

in $\mathrm{proj}(\mathcal{O}_0)$ are also those in the image Ref? - that these are actually the bases of the hom spaces, so the dimensions of Hom spaces coincide. Therefore F is fully faithful.

All objects in $\mathrm{proj}(\mathcal{O}_0)$ appear as direct sums and direct summands of the elements $\Theta_s^n(P(\varnothing))$ for non-negative integers n. Therefore the additive Karoubi envelope induces an equivalence $\mathrm{Kar}^{\oplus}(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ as $\mathbb{C}$-linear left $\mathcal{H}(S_2)$-module categories. $\square$

This result is essentially due to Soergel [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4] (see also [Soe98]) but this was not its original formulation. Nevertheless we attribute this theorem to Soergel.

Maybe talk about Soergel modules and $\mathcal{H}(S_2)$-modules vs $\mathrm{Kar}^{\oplus}(\mathcal{H}(S_2))$-modules

Remark 3.2.12. The morphisms spaces in $\mathcal{DO}_0$ are graded by the same grading as $\mathcal{H}(S_2)$ in Section 3.1. The equivalence $\mathrm{Kar}^{\oplus}(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ includes a grading of morphisms in $\mathrm{proj}(\mathcal{O}_0)$ Check! and hence a grading morphisms of $\mathcal{O}$, which is otherwise ungraded.

Some more consequences

Chapter 4

Two-colour Diagrammatics

The previous chapter had its focus on the symmetric group generated by one element S_2, which brought about one-colour diagrammatics. We now explore a more complex example by adding an extra generator, that is, another colour. In particular, we consider the affine symmetric group on two elements $\widetilde{S}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$. Refine this

4.1 Two-colour Diagrammatic Hecke Category

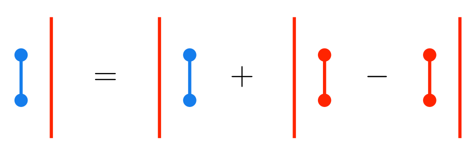
Definition 4.1.1. The *two-colour (diagrammatic) Hecke category* $\mathcal{H}(\widetilde{S}_2)$ is a (strict) $\mathbb{Z}$-linear monoidal category given by the following isotopy presentation.

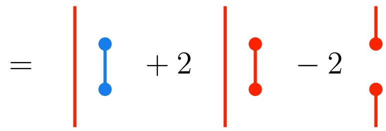
Objects in $\mathcal{H}(\widetilde{S}_2)$ are generated by formal tensor products of the non-identity elements $s, t \in \widetilde{S}_2$. As before, we write objects as words such as $sstttst =: s^2t^3st$ where the tensor product is concatenation, and associate the colour red to s and blue to t. The empty word is the monoidal identity, which we write as $\varnothing$.

The morphisms are generated by the univalent and trivalent vertices

(4.1.2)

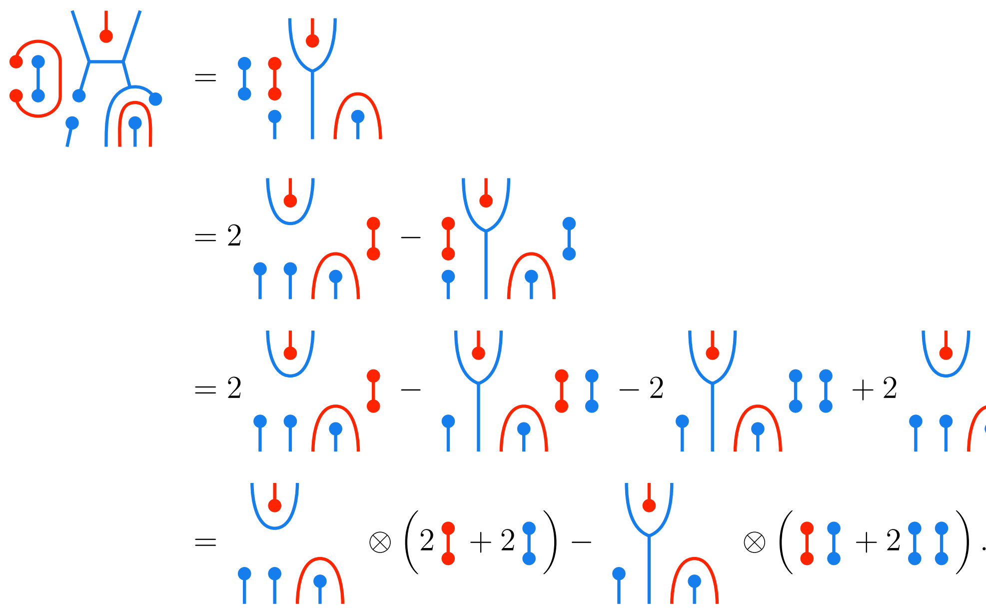
that are maps $s \to \varnothing$, $ss \to s$, $t \to \varnothing$ and $tt \to t$ respectively, and their vertical reflections. As in the one-colour case, tensor product is horizontal concatenation, composition is appropriate vertical stacking, and we denote the empty diagram $\varnothing \to \varnothing$ by $\varnothing$. For each colour, these diagrams have the one-colour relations given by (3.1.5). As we have another colour, we need to describe how different colours interact. This is given by the *two-colour relation*



 (4.1.3)

and with red and blue swapped.

Example 4.1.4. Using the one-colour and two-colour relations on the following morphism in $\mathrm{Hom}(ttsts, tst)$ we have



Talk about this containing $\mathcal{H}(S_2)$

Remark 4.1.5. Notice that the red and blue lines never cross as no generators that allow crossings. This is a consequence of working over affine S_2 in which the generators s and t have no relation. Mention example of crossing and S_3.

In this two-colour case, Proposition 3.1.8 holds by replacing (3.1.5d) with (4.1.3) in the proof. This handles the new possibility of floating subdiagrams with alternating colours.

Definition 4.1.6. For a group with a presentation in terms of generators and relations, the *length* of a product of generators is the number of generators in the product. We say that a product of generators is *reduced* if it's length cannot be shortened with relations.

In $\widetilde{S}_2$ products can be shortened by the relation $s^2 = t^2 = 1$. For instance, $sttsts$ is not reduced because it is equal to ts which is reduced. Notice that for $\widetilde{S}_2$ each element can be written uniquely as a reduced product of generators. This is true since otherwise we have two distinct reduced products for the same element in $\widetilde{S}_2$ so they must be related by $s^2 = t^2$. This means they can be reduced further by $s^2 = t^2 = 1$, which contradicts minimality of their length. Note that the reduced products in $\widetilde{S}_2$ are either the identity or alternating products of s and t.

Notice that there is a notational similarity between products in the group and words in $\mathcal{H}(\widetilde{S}_2)$. This motivates the following definitions. Let $\phi : (\mathrm{ob}(\mathcal{H}(\widetilde{S}_2)), \otimes) \to (\widetilde{S}_2, *)$ be the monoid homomorphism mapping $\varnothing \mapsto 1$, $s \mapsto s$ and $t \mapsto t$. Also define the function $\psi : \widetilde{S}_2 \to \mathrm{ob}(\mathcal{H}(\widetilde{S}_2))$ to map elements $x \in \widetilde{S}_2$ to the tensor product of s and t in $\mathcal{H}(\widetilde{S}_2)$ corresponding to the reduced product of x in $\widetilde{S}_2$. This is well defined because reduced products are unique and two different reduced products cannot equal the same element of $\widetilde{S}_2$. Note that the image $\psi(\widetilde{S}_2)$ is the set containing $\varnothing$ and words of alternating s and t. The composition $\psi\phi : \mathcal{H}(\widetilde{S}_2) \to \mathcal{H}(\widetilde{S}_2)$ maps words w to the tensor of s and t corresponding to the reduced product of $\phi(w)$, and $\phi\psi$ is the identity map on $\widetilde{S}_2$.

The following definition is a more general version of Definition 3.1.9.

Definition 4.1.7 (Subexpression)**.** Given a word w of length n, a *subexpression* e is a binary string of length n. A subexpression can be *applied* to produce an word $w(e)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $1 \leq i \leq n$, we write $w(e, i)$ for the result of the first i terms of e applied to the first i terms in w. Particularly $w(e, 0) = \varnothing$ and $w(e, n) = w(e)$.

For example, in $\mathcal{H}(\widetilde{S}_2)$, if $w = sttts$ and $e = 11001$ then $w(e) = st\varnothing\varnothing s = sts$ and $w(e, 3) = sts(110) = st\varnothing = st$ in $\mathcal{H}(\widetilde{S}_2)$.

Let the *length* of a word be the number of generators in its tensor product. As before, given an object w and a subexpression e of w, we label each of the n terms by one of U_0, U_1, D_0, D_1. Let $i \geq 0$, and write x for the i-th term of w. We label the i-th term U_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. In other words we write U_* if the next term of w will make $\psi\phi$ applied to the partially evaluated subexpression longer, regardless of the i-term of e. We label D_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. The label's subscript is the i-th term of e. Note that this construction is well defined because $\psi\phi(w(e, i-1) \otimes x) = \psi(\phi(w(e, i-1)) * \phi(x)) = \psi(\phi(w(e, i-1)) * x)$ is always either longer or shorter, since the last element of the reduced product is either the same as x or different. When they are the same, the word is shorter via $s^2 = t^2 = 1$, and when they are different it is longer as no relations can make it shorter.

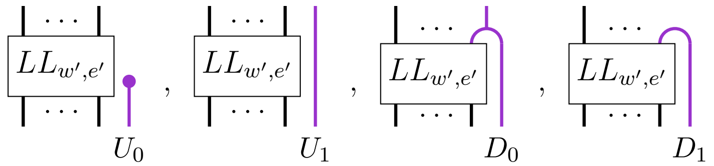
Remark 4.1.8. This description of the labels (via. reduced products) is more akin to the definition for general Coxeter groups than in Section 3.1.

Example 4.1.9. Consider the word $w = sttst$ and subexpression $e = 10011$. The labels can be constructed as in the following table.

Term i	1	2	3	4	5
Partial w	s	st	stt	$stts$	$sttst$
Partial e	1	10	100	1001	10011
$w(e,i)$	s	$s\varnothing$	$s\varnothing\varnothing = s$	$s\varnothing\varnothing s = ss$	$s\varnothing\varnothing st = sst$
Labels	U_1	U_1U_0	$U_1U_0U_0$	$U_1U_0U_0D_1$	$U_1U_0U_0D_1U_1$

Definition 4.1.10. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and a subexpression e is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram.

Given appropriate subwords w' and e' of w and e respectively, and if the next terms are x in w and i in e, the light leaf $LL_{w'x,e'i}$ is one of

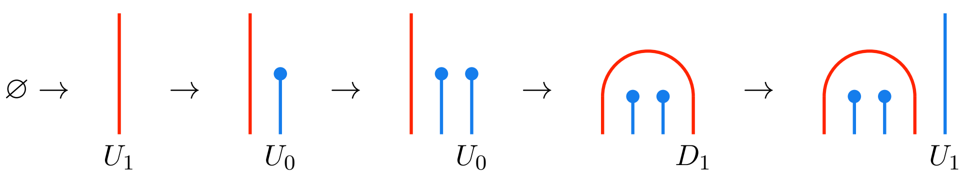


(4.1.11)

corresponding to the next label. The purple strands are red if $x = s$ and blue if $x = t$.

Notice that the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ does not end with the colour corresponding to x, and if the next label is D_* the codomain of $LL_{w',e'}$ ends with a strand with the colour corresponding to x. This implies the recursive definition in the diagram above is consistent. Note that in the case of D_*, one of the black strands in the domain of $LL_{w',e'}$ must have the colour of x in order for the colour to appear in its codomain.

Example 4.1.12. Following from Example 4.1.9, with $w = sttst$, $e = 10011$ and labels $U_1U_0U_0D_1U_1$, the light leaf $LL_{w,e}$ is build as follows.



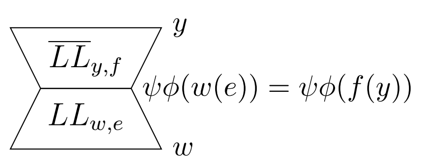
We can define double leaves exactly as we did in Definition 3.1.14.

Definition 4.1.13. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y in $\mathcal{H}(\widetilde{S}_2)$ is a composition

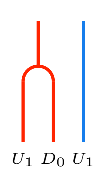
$$\mathbb{LL}_{f,e} \coloneqq \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

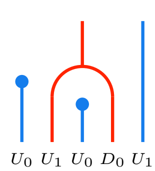
for subexpressions e of w and f of y such that $\psi\phi(w(e)) = \psi\phi(f(y))$.

Diagrammatically these are morphisms from w to y factoring through $\psi\phi(w(e)) = \psi\phi(f(y)) \in \psi(\widetilde{S}_2)$,



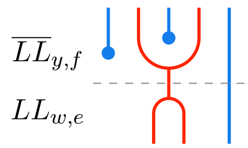
Example 4.1.14. Let $w = sst$ with the subexpression $e = 101$ and $y = tstst$ with the subexpression $f = 01001$. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

.

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : sst \to tstst$, factoring through st, is



.

As with the one-colour case, the set of double leaves $\mathbb{LL}(w, y)$ from words w to y in $\mathcal{H}(\widetilde{S}_2)$ form a basis for $\mathrm{Hom}(w, y)$ over $\mathbb{Z}[\ ,\ ]$. The Hom spaces are graded such that the univalent vertices have degree 1 and trivalent vertices have degree -1 for either colour.

Let $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$ be the $\mathbb{C}$-linear monoidal category obtained by extending the scalars of morphisms spaces in $\mathcal{H}(\widetilde{S}_2)$ from $\mathbb{Z}$ with $\mathbb{C}$. All the results above also hold for $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$. Additionally, a result similar to Theorem 3.1.18 states that $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$ is equivalent to $\mathbb{S}\mathrm{Bim}$ over $\widetilde{S}_2$ as graded is this graded? $\mathbb{C}$-linear monoidal categories.

Remark 4.1.15. The construction of the diagrammatic Hecke category, light leaves, Theorem 3.1.16 and Theorem 3.1.18 all generalise to general Coxeter groups. The details can be found in [EW16].

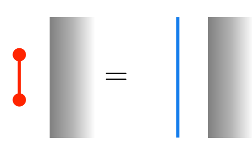
4.2 Diagrammatic $\mathrm{Tilt}(\mathfrak{sl}_2)$

Something something about Tilt

Something something about extending $\mathcal{H}(\widetilde{S}_2)$ from $\mathbb{Z}$ to $\mathbb{C}$.

Although need to work over $\mathbb{C}$ for $\mathrm{Tilt}(\mathfrak{sl}_2)$, the diagrammatic category below c an be defined more simply over $\mathbb{Z}$.

Definition 4.2.1. Let $\mathcal{DT} \coloneqq \mathcal{DT}(\mathfrak{sl}_2)$ be the $\mathbb{Z}$-linear left $\mathcal{H}(\widetilde{S}_2)$-module category with elements generated by the monoidal identity $\varnothing$ of $\mathcal{H}(\widetilde{S}_2)$, and morphisms generated by the empty diagram $\varnothing$. The action of $\mathcal{H}(\widetilde{S}_2)$ on the left is left concatenation for objects and morphisms. The relations on diagrams in $\mathcal{H}(\widetilde{S}_2)$ follow through to diagrams in $\mathcal{DT}(\mathfrak{sl}_2)$. Additionally, we imagine a wall on the right of diagrams and impose the local wall-annihilation relations



$$= 0. \tag{4.2.2}$$

Example 4.2.3. The morphism in Example 4.1.4 collapses to 0 because all the diagrams have either blue or barbell on the right.

TODO: Another example clarifying 'blue on the right'

The objects of this category are identical to objects in $\mathcal{H}(\widetilde{S}_2)$ and the morphisms are the same modulo the wall relations (4.2.2). Naturally, we wonder whether double leaves form bases for the morphism spaces in $\mathcal{DT}$. It is easy to see that double leaves appear in $\mathcal{DT}$ by acting on $\varnothing$ by double leaves in $\mathcal{H}(\widetilde{S}_2)$. All morphisms in $\mathcal{DT}$ are morphisms in $\mathcal{H}(\widetilde{S}_2)$ so they can be written as $\mathbb{Z}[\ ,\ ]$-linear combinations of double leaves, though some of these leaves have collapsed to 0. This makes it clear that double leaves span the morphism spaces of $\mathcal{DT}$ as (left) $\mathbb{Z}[\ ,\ ]$-modules. However they may not be linearly independent as neither left nor right modules as with the one-colour case. Although double leaves are not always a basis for its respective morphism space as $\mathbb{Z}[\ ,\ ]$-modules, it turns out a subset of them are a basis over $\mathbb{Z}$.

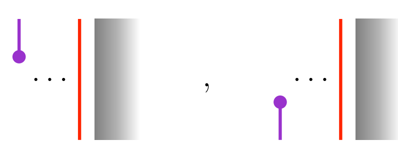
Lemma 4.2.4. *Let $\pi : \mathrm{mor}(\mathcal{H}(\widetilde{S}_2)) \to \mathrm{mor}(\mathcal{DT})$ be the projection map which takes a morphism to the result of its action on the empty diagram $\varnothing$. Then the image $\pi(\mathbb{LL}(w,y))$ without zero morphisms is a basis for $\mathrm{Hom}_{\mathcal{DT}}(w,y)$ as a $\mathbb{C}$-module.*

Proof. We consider morphisms $\mathrm{Hom}(w,y)$ in $\mathcal{DT}$ for fixed objects w,y, and write $\mathbb{LL} \coloneqq \pi(\mathbb{LL}(w,y))$ for the set of double leaves in $\mathcal{DT}$. Any diagram in $\mathcal{DT}$ can be written as a $\mathbb{C}$-linear combination of morphisms without floating diagrams by pulling floating diagrams to the right with (3.1.5d) and (4.1.3) then applying the wall relation (4.2.2). We can write each of these as a $\mathbb{Z}[\ ,\ ]$-linear combination of double leaves with a right action, and reduce it to a $\mathbb{Z}$-linear combination by (4.2.2). This implies that $\mathbb{LL}$ spans $\mathrm{Hom}(w,y)$ as a $\mathbb{Z}$-module. Now $\mathbb{LL}$ may not be linearly independent because the two-colour wall relation (4.2.2) reduces all diagrams factoring through a word ending with t to 0. The set of light leaves after removing morphisms killed by (4.2.2), i.e. $\mathbb{LL} \setminus \{0\}$, still spans $\mathrm{Hom}(w,y)$ by the argument above. This set is linearly independent since, by construction, (4.2.2) has no effect on $\mathbb{Z}$-linear combinations of $\mathbb{LL} \setminus \{0\}$. Then it follows from linear independence over $\mathbb{Z}[\ ,\ ]$ that this set is linearly independent over $\mathbb{Z}$ in $\mathcal{DT}$. $\square$

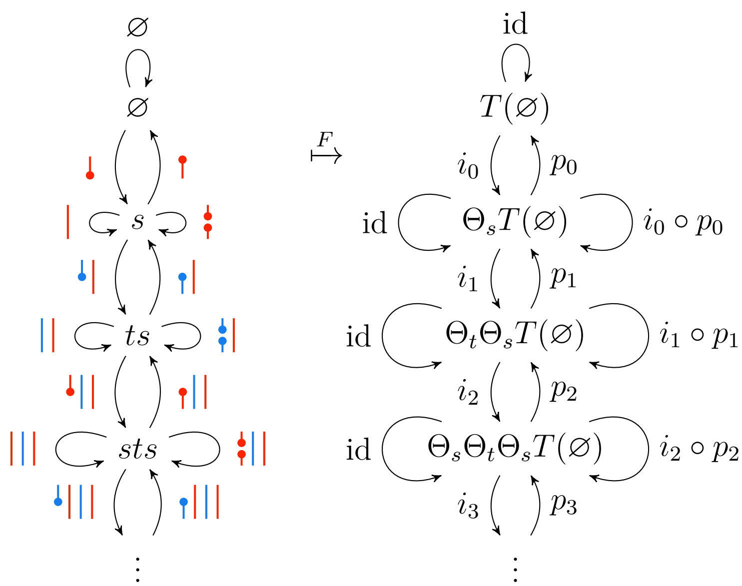
We aim to show that this diagrammatic category is equivalent to $\mathrm{Tilt}(\mathfrak{sl}_2)$. So, from now on, we write $\mathcal{DT}$ for the $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$-module category obtained by replacing $\mathbb{Z}$ with $\mathbb{C}$ and $\mathcal{H}(\widetilde{S}_2)$ with $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$ in Definition 4.2.1. The above discussion and Lemma 4.2.4 still apply to $\mathcal{DT}$ over $\mathbb{C}$.

Since $\mathcal{H}_{\mathbb{C}}(S_2)$ appears inside $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$ for each colour, Lemma 3.2.6 provides explicit isomorphisms $ss \cong s \oplus s$ and $tt \cong t \oplus t$ in the additive closure of $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$.

Definition 4.2.5. Let $F : \mathrm{Kar}^{\oplus}(\mathcal{DT}) \to \mathrm{Tilt}(\mathfrak{sl}_2)$ to be the additive $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$-module functor defined as follows What about gradings?. Map the empty word $\varnothing$ to the check: trivial module $T(\varnothing)$. Given a general word $s_1 \ldots s_n$ in $\mathcal{DT}$, for $s_i \in \{s,t\}$, map $F(s_1 \ldots s_n) = \Theta_{s_n} \ldots \Theta_{s_1} T(\varnothing)$ where Θ_s, Θ_t are translation functors associated to generators of $\widetilde{S}_2$. On morphisms, map identity to the corresponding identity, to the inclusion map $i_0 : T(\varnothing) \to \Theta_s T(\varnothing)$ and to the projection map $p_0 : \Theta_s T(\varnothing) \to T(\varnothing)$. Consider diagrams of the form



where the colours of the diagram alternate, "…" stands for zero or more identity strands and the purple is either red or blue such that the colours alternate. The functor F maps them to the inclusion $i_k : \Theta_x \ldots \Theta_s T(\varnothing) \to \Theta_y \ldots \Theta_s T(\varnothing)$ and projection $p_k : \Theta_y \ldots \Theta_s T(\varnothing) \to \Theta_x \ldots \Theta_s T(\varnothing)$, respectively, where $x, y \in \{s, t\}$ are different and k is the number of identity strands. The mappings are summarised in the picture below.

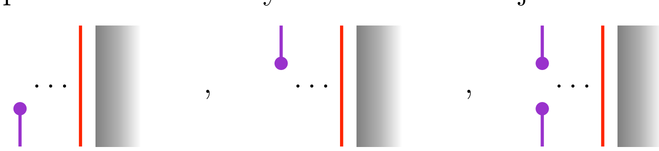


(4.2.6)

The right wall on each diagram is not shown to reduce clutter. We extend the functor by additivity, composition and linearity. Do we need to define where JW-projectors go?

Lemma 4.2.7. *The functor F is well defined.*

Proof. CHECK Consider the additive closure $\mathcal{DT}^{\oplus}$ of $\mathcal{DT}$. For red and blue, the isomorphisms given in Lemma 3.2.6 imply that any diagram in $\mathcal{DT}$ is isomorphic to a matrix of $\mathbb{C}$-linear combinations of diagrams where the domain and codomain have alternating colours. Due to (4.2.2) non-trivial diagrams do not have blue strands next to the wall, so the alternating colours must end in red, i.e. the domain and codomain end in s. We constructed F to be additive, so it suffices to consider morphisms between indecomposable summands of words alternating s, t and ending in s. Recall that these alternating words have idempotents given by their Jones-Wenzl projectors. The discussion in [Eli16, Section 5.4.2] states that the images of Jones-Wenzl projectors give all the indecomposables Is this right? Is the ref right–maybe see book thm9.22 and [AT17, Corollary 4.21] shows that all morphisms not killed by Jones-Wenzl Projectors are of the form



with alternating colours, purple being either red or blue, and "..." are zero or more identity strands. Therefore the morphisms in the picture (4.2.6) are enough to define a mapping on $\mathcal{DT}$ by going through $\mathrm{Kar}^{\oplus}(\mathcal{DT})$ Word this better.

Next, we check that all relations are preserved. From [Maz09, Proposition 5.84 and Lemma 5.87] Is this the right ref?, we know that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$ and there are unit, counit, multiplication and comultiplication natural transformations from the Frobenius object structure. Applying these to $T(\varnothing)$ result in the same relations in $\mathrm{Tilt}(\mathfrak{sl}_2)$ for $T(\varnothing), \Theta_s T(\varnothing)$ and $\Theta_s^2(T(\varnothing))$. Note that the projection and inclusion maps in the picture (4.2.6) are exactly the unit and counit of Θ_s evaluated at $T(\varnothing)$, and the trivalent vertices provided by projecting the isomorphisms in Lemma 3.2.6 map exactly to the multiplication and comultiplication maps. Furthermore, in [Soe90, Section 2.4] Ref? we see that $p_0 \circ i_0 = 0$ and Is this right: $\mathrm{id}_{\Theta_t(T(\varnothing))} = 0$ in $\mathrm{Tilt}(\mathfrak{sl}_2)$ which is analogous to the two-colour wall relations (4.2.2). Furthermore there is a relation $i_k \circ p_k = p_{k+1} \circ i_{k+1}$ up to a scalar multiple ref? which is analogous to [PUT THE RELATION HERE]. What about $p_{k+1} \circ p_k = 0$? Hence all the relations in $\mathcal{DT}$ are preserved by F. By construction, F preserves direct sums, $\mathbb{C}$-linear combinations and the Soregel module structure in [Soe90], so F is well defined as a functor between $\mathbb{C}$-linear $\mathcal{H}(S_2)$-module categories.

Talk about how extending by composition makes sense. $\square$

Say something here?

The following result states that $\mathcal{DT}$ is indeed a diagrammatic incarnation of $\mathrm{Tilt}(\mathfrak{sl}_2)$.

Be clear that I don't understand Tilt very well.

Theorem 4.2.8 (???)**.** *The diagrammatic category* $\mathrm{Kar}^{\oplus}(\mathcal{DT}(\mathfrak{sl}_2))$ *and* $\mathrm{Tilt}(\mathfrak{sl}_2)$ *are equivalent as* $\mathbb{C}$*-linear* $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$*-module categories.*

Proof. As a shorthand, we write $\mathcal{T}$ for $\mathrm{Tilt}(\mathfrak{sl}_2)$. We now prove that F is fully faithful. It follows from Lemma 3.2.6 and the description of $P(\varnothing)$ and $P(s)$ in [Maz09, Section 5.2] that the image of and generate all morphisms of the form $\Theta_s^n(P(\varnothing)) \to \Theta_s^m(P(\varnothing))$. Hence F is full. Now the mapping of F on all morphism spaces are determined by those depicted in the above picture. So, for faithfulness, it suffices to compare the $\mathbb{C}$-dimensions of morphism spaces between objects shown in the picture. By Lemma 3.2.5, $\mathrm{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_\varnothing\}$, $\mathrm{Hom}(s, \varnothing)$ has a basis $\{\ \}$, $\mathrm{Hom}(\varnothing, s)$ has a basis $\{\ \}$, and $\mathrm{Hom}(s, s)$ has a basis $\{\mathrm{id}_s, \ \circ\ \}$. The bases for the corresponding morphism spaces in $\mathrm{proj}(\mathcal{O}_0)$ are exactly those in the image Ref? - that these are actually the bases of the hom spaces, so these dimensions coincide. Therefore F is fully faithful.

All objects in $\mathrm{proj}(\mathcal{O}_0)$ appear as direct sums and direct summands of the elements $\Theta_s^n(P(\varnothing))$ for non-negative integers n. Therefore the additive Karoubi envelope induces an equivalence $\mathrm{Kar}^{\oplus}(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ as $\mathbb{C}$-linear left $\mathcal{H}(S_2)$-module categories. $\square$

Comment on grading?

Comment on consequences

Bibliography

[AT17] Henning Haahr Andersen and Daniel Tubbenhauer. “Diagram Categories for $\mathbf{U}_q$-Tilting Modules at Roots of Unity”. In: *Transformation Groups* 22 (2017), pp. 29–89. DOI: 10.1007/s00031-016-9363-z.

[Eli16] Ben Elias. “The two-color Soergel calculus”. In: *Compositio Mathematica* 152.2 (2016), pp. 327–398. DOI: 10.1112/S0010437X15007587.

[EW16] Ben Elias and Geordie Williamson. “Soergel Calculus”. In: *Representation Theory of the American Mathematical Society* 20 (Oct. 2016). DOI: 10.1090/ert/481.

[Eti+15] Pavel Etingof et al. *Tensor Categories*. Vol. 205. Mathematical Surveys and Monographs. American Mathematical Society, 2015. DOI: http://dx.doi.org/10.1090/surv/205.

[Hum08] James E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*. Vol. 94. Graduate Studies in Mathematics. American Mathematical Society, 2008. DOI: http://dx.doi.org/10.1090/gsm/094.

[Jon21] Vaughan F. R. Jones. “Planar algebras”. In: *New Zealand Journal of Mathematics* 52 (2021), pp. 1–107. DOI: 10.53733/172. URL: https://nzjmath.org/index.php/NZJMATH/article/view/172.

[Lib08] Nicolas Libedinsky. “Sur la catégorie des bimodules de Soergel”. In: *Journal of Algebra* 320.7 (2008). (French), pp. 2675–2694. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2008.05.027.

[ML98] Saunders Mac Lane. *Categories for the Working Mathematician*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. Springer, 1998. DOI: https://doi.org/10.1007/978-1-4757-4721-8.

[Maz09] Volodymyr Mazorchuk. *Lectures on $\mathfrak{sl}_2(\mathbb{C})$-Modules*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. IMPERIAL COLLEGE PRESS, 2009. DOI: 10.1142/p695.

[RW18] Simon Riche and Geordie Williamson. “Tilting Modules and the p-Canonical Basis”. In: *Asterisque* 397 (2018). DOI: 10.24033/ast.1041.

[Soe90] Wolfgang Soergel. “Kategorie $\mathcal{O}$, Perverse Garben Und Moduln Uber Den Koinvariantez Zur Weylgruppe”. In: *Journal of the American Mathematical Society* 3.2 (1990), pp. 421–445. ISSN: 08940347, 10886834. URL: http://www.jstor.org/stable/1990960.

[Soe98] Wolfgang Soergel. “Combinatorics of Harish-Chandra modules”. In: *Representation Theories and Algebraic Geometry*. Ed. by Abraham Broer, A. Daigneault, and Gert Sabidussi. Springer Netherlands, 1998, pp. 401–412. ISBN: 978-94-015-9131-7. DOI: 10.1007/978-94-015-9131-7_10. URL: https://doi.org/10.1007/978-94-015-9131-7_10.