Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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Acknowledgements

I like to acknowledge ... blah blah blah

Contents

Chapter 1

Introduction

Visual interpretations of functions simplify calculations and often provides insights into the mathematical objects they encode. More detail about what this is This general philosophy takes form in various settings. A simple example are string diagrams for permutations. A permutation can be drawn as strings between two copies of a set determining how the objects are permuted. Compositions of these permutations is the operation of joining corresponding strings start to end in order to create a larger string diagram representing their product. Another example are *(Artin) braid groups*, whose elements can be depicted similarly to the symmetric group, but where each crossing of strings has a choice of going over or under. As suggested by the name, these string diagrams resemble braids.

A significant example are *planar algebras* in the work of Jones. These are certain algebras of planar diagrams that describe operators. His study of the Temperley-Lieb-Jones (planar[1]) algebra lead to the discovery of an important invariant in knot theory [Jon85] in 1983, which we know now as the Jones polynomial. For this and surrounding works he received a Fields medal. This technology of planar algebras have been since used to study subfactors in functional analysis [Jon21][2] and have consequences in for example statistical mechanics and mathematical physics.

In representation theory, our main motivational example is given by the proof of the Kazhdan–Lusztig conjecture through the diagrammatics of Soergel bimodules. The conjecture relates Kazhdan–Lusztig polynomials, arising from the Weyl group associated with a Lie algebra, to Jordan–Hölder multiplicities of particular representations of Lie algebras called Verma modules. Proofs were discovered independently by Beilinson–Bernstein and Brylinski–Kashiwara in 1981 but by geometric methods, which was unsatisfying to many. Around this time, Soergel was working toward an algebraic proof by Soergel bimodules, however he hit a technical road block. In 2010's, Elias and Williamson Ref? developed planar diagrams for morphisms on Soergel bimodules and were able to overcome the technical point where Soergel got stuck, to prove the conjecture diagrammatically. The diagrams can greatly simplify algebraic calculations and the

[1]The algebra was later presented as a diagram algebra by Kauffman in [Kau90]

[2]Originally from 1999, and was recently published.

diagrammatic category can be considered independently from Soergel bimodules. We explore this diagrammatics for S_2 in Section 3.1.

One of the advantages of the diagrammatic Soergel bimodules is that it can be defined over $\mathbb{Z}$ and extended to fields of characteristic p where classical Soergel bimodules are ill-behaved. Characters in the category of tilting modules (certain representations of a Lie algebra) can be calculated via Kazhdan–Lusztig polynomial in characteristic zero, however these polynomials were unknown in characteristic p. Riche and Williamson in [RW18] were able to construct these characteristic p Kazhdan–Lusztig polynomials by considering diagrammatic Soergel bimodules in characteristic p.

In this paper we give an introduction to drawing morphisms in monoidal categories, put more here and define some mechanisms to form an additive and idempotent complete category. In Chapter 3 we define diagrammatic Soergel bimodules associated with the symmetric group S_2, construct a basis for its morphism spaces and state the theorem for its equivalence to the category of algebraic Soergel bimodules. We use this diagrammatic category to construct a diagrammatic module category with an extra relation, then prove its equivalence to the category of projective objects in the principle block of the category $\mathcal{O}$. In Chapter 4 we consider the affine symmetric group $\widetilde{S}_2$ to define the diagrammatic Soergel bimodules associated it, construct a basis for its morphism spaces and state the theorem for its equivalence to the category of algebraic Soergel bimodules. The extra generator in $\widetilde{S}_2$ compared with S_2 provides some additional complexity to the structure of the category. We then form a module category with two extra relations and provide a proof of its equivalence to the category of tilting modules for $\mathfrak{sl}_2$. In the last chapter we discuss the consequences of diagrammatics in relation to Chapter 3 and Chapter 4, mention generalisations of the results and further areas of interest.

The contents of this thesis are for honours students and future readers that are interested in this topic. The reader is assumed to have some familiarity with undergraduate algebra (such as groups, rings, algebras and fields), basic ideas in representation theory, basic category theory and monoidal categories.

Talk about we dont need to know about category $\mathcal{O}$ and Tilt. One of the advantages of diagrammatics is that we don't need to understand these complex categories in representation theory to study them. For this reason, we will defer the details in the proofs involving them to other sources.

Talk about introduction to soergel bimodules

Chapter 2

Background

For a category $\mathcal{C}$ we write $\mathrm{ob}(\mathcal{C})$ for the collection of objects, $\mathrm{mor}(\mathcal{C})$ for the collection of all morphisms, and for any pair of objects A, B we write $\mathrm{Hom}(A, B)$ for the collection of morphisms from A to B. The collection of endomorphisms of an object A is written $\mathrm{End}(A) \coloneqq \mathrm{Hom}(A, A)$. Our focus of study are particular types of categories, not categories in the abstract, so we may assume that all categories we encounter are locally small.

Put some text here with where to find more details

2.1 Drawing Monoidal Categories

Definition 2.1.1. A *monoidal category* $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold (see [Eti+15, Definition 2.1.1, 2.2.8]). The bifunctor $\otimes$ is called the *tensor* or *monoidal product*. A monoidal category is *strict* if $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ and $A = \mathbb{1} \otimes A = A \otimes \mathbb{1}$ for objects and similarly for morphisms.

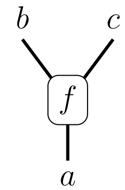
In this paper, we will assume that monoidal categories are strict. This does not pose any problems since all monoidal categories are monoidally equivalent to a strict one[1]. In this context, the details in the coherence relations are trivial.

Definition 2.1.2. A functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories is called *monoidal* if it preserves the monoidal product. That is $F(A \otimes B) = F(A) \otimes F(B)$.

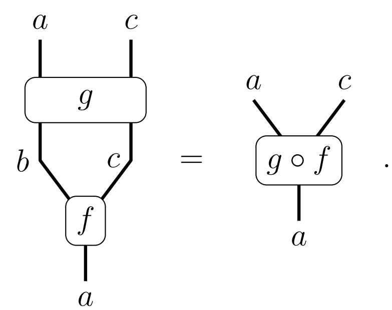
Structure preserving functors for other types of categories can be defined in a similar way.

The morphisms of a monoidal category $\mathcal{C}$ can be drawn as string diagrams embedded in a planar strip. A diagram is a morphism when read from bottom to top, that is the domain is on the bottom of the strip and the codomain on the top. Functions that make up a morphism are drawn as tokens or boxes. For example

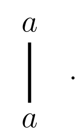
[1] See [ML98, VII.2] or [Eti+15, Thm 2.8.5]



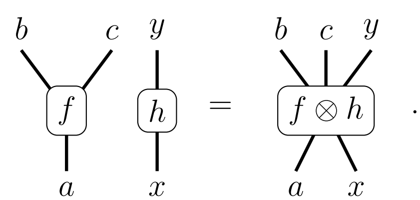
depicts a morphism $f : a \to b \otimes c$. Notice here that tensor products of objects have its factors displayed horizontally. The compositions of morphisms is the vertical stacking of diagrams whenever labels on domains and codomains match. For example, the composition $g \circ f : a \to b \otimes c \to a \otimes c$ of $f : a \to b \otimes c$ with $g : b \otimes c \to a \otimes c$ has the diagram



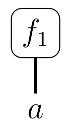
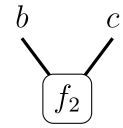
For identity morphisms we just draw a vertical line, so id_a is the diagram



This is a sensible choice since composition with the identity should not change the function, which is clear diagrammatically. The tensor product of morphisms is the horizontal concatenation of diagrams, such that strings from separate functions don't interact. For example, given $h : x \to y$, the tensor product $f \otimes h : a \otimes x \to b \otimes c \otimes y$ is drawn as



We let the monoidal unit $\mathbb{1}$ be blank and unlabelled, and strings that would join to $\mathbb{1}$ are blank. Particularly, $\mathrm{id}_{\mathbb{1}}$ is an empty diagram. It makes sense to display $\mathbb{1}$ in this way since tensoring with $\mathbb{1}$ (in a strict monoidal category) does nothing to objects and tensoring with $\mathrm{id}_{\mathbb{1}}$ does nothing to morphisms. By this convention, we also have diagrams such as

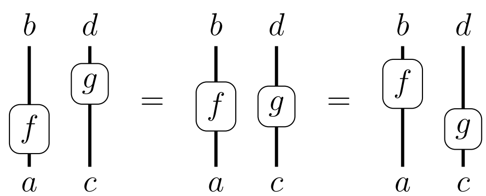
 and 

for morphisms $f_1 : a \to \mathbb{1}$ and $f_2 : \mathbb{1} \to b \otimes c$.

The bifunctoriality of $\otimes$ implies the following *interchange law*. For morphisms $f : a \to b$ and $g : c \to d$, we have $(\mathrm{id}_b \otimes g) \circ (f \otimes \mathrm{id}_c) = f \otimes g = (f \otimes \mathrm{id}_d) \circ (\mathrm{id}_a \otimes g)$. In other words the following diagram commutes.

$$\begin{tikzcd}
a \otimes c \arrow[r, "f\otimes \mathrm{id}_c"] \arrow[d, "\mathrm{id}_a \otimes g"'] \arrow[dr, "f\otimes g"] & b \otimes c \arrow[d, "\mathrm{id}_b \otimes g"] \\
a \otimes d \arrow[r, "f\otimes \mathrm{id}_d"] & b \otimes d
\end{tikzcd}$$

Written with string diagrams, this is



which holds up to vertical deformation of the diagram. This is our first glimpse of isotopy, but only in the vertical direction.

Reword this Before looking at an example of a diagrammatic monoidal category, we mention some definitions.

Definition 2.1.3. For a commutative ring R, an *R-linear category* is a category enriched over the category of R-modules. That is, for objects a, b, the set of morphisms $\mathrm{Hom}(a, b)$ is an R-module and the composition of morphisms is R-bilinear. An *R-linear monoidal category* is a category that is both monoidal and R-linear such that the monoidal product on morphisms is R-bilinear.

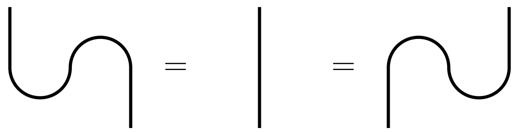
Put reference to somewhere with a better definition?

Note that since composition and tensor products are bilinear, $0 \otimes f = (0 + 0) \otimes f = 0 \otimes f + 0 \otimes f$ and similarly with composition, so composition and tensors with 0 are also zero.

Example 2.1.4. The category of vector spaces over a field $\mathbb{k}$, $\mathbf{Vect}_{\mathbb{k}}$, is a $\mathbb{k}$-linear monoidal category given by the usual tensor product of vector spaces and linear maps.

Definition 2.1.5. A monoidal category $\mathcal{C}$ is *generated* by finite set S_o of objects and S_m of morphisms, when all non-unit objects are a finite tensor of objects in S_o and all non-identity morphisms are a finite combination of tensors and compositions of morphisms in S_m.

Example 2.1.6. Our first example of a diagrammatic monoidal category is the *Temperley-Lieb-Jones category*. The Temperley-Lieb-Jones[2] category $\mathcal{TLJ}$ is a strict R-linear monoidal category whose objects are generated by the vertical line I and morphisms generated by the cup $\cup : \mathbb{1} \to \mathsf{I} \otimes \mathsf{I}$ and cap $\cap : \mathsf{I} \otimes \mathsf{I} \to \mathbb{1}$, with the relation



Mention that composition and tensor product is as explained above
Some example
Mention bubbles and specialisation to some $\delta \in R$
Mention that these are crossingless matchings
Comment on isotopy

2.2 Frobenius Objects

The structure of Frobenius objects give rise to useful diagrammatics that can be defined up to isotopy. This section gives some background to the objects we will encounter in Section 3.1 and beyond.

Let $\mathcal{C}$ be a (strict) monoidal category.

Definition 2.2.1. A *monoid object* in $\mathcal{C}$ is a triple (M, μ, η) for an object $M \in \mathcal{C}$, a *multiplication* map $\mu : M \otimes M \to M$ and a *unit* map $\eta : \mathbb{1} \to M$, such that

$$\begin{tikzcd} & M \otimes M \otimes M \arrow[dl, "\mu \otimes \mathrm{id}_M"'] \arrow[dr, "\mathrm{id}_M \otimes \mu"] & \\ M \otimes M \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "\eta \otimes \mathrm{id}_M"] \arrow[dr, "\mathrm{id}_M"'] & M \otimes M \arrow[d, "\mu"] & M \otimes \mathbb{1} \arrow[l, "\mathrm{id}_M \otimes \eta"'] \arrow[dl, "\mathrm{id}_M"] \\ & M & \end{tikzcd}$$

commute. The first diagram is the *associativity* relation $\mu \circ (\mu \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \mu)$ and the second diagram is the *unit* relation $\mathrm{id}_M = \mu \circ (\eta \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \eta)$.

[2]Originally the Temperley–Lieb–Jones algebra was used by Temperley and Lieb in [TL71] for statistical physics. Jones discovered the same structure and relations independently in his work.

Dually, a *comonoid object* in $\mathcal{C}$ is a triple (M, δ, ϵ) for an object $M \in \mathcal{C}$, a *comultiplication* map $\delta : M \to M \otimes M$ and a *counit* map $\epsilon : M \to \mathbb{1}$, satisfying the *coassociativity* relation

$$\begin{tikzcd}
& M \otimes M \otimes M & \\
M \otimes M \arrow[ur, "\delta \otimes \mathrm{id}_M"] & & M \otimes M \arrow[ul, "\mathrm{id}_M \otimes \delta"'] \\
& M \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

and *counit* relation

$$\begin{tikzcd}
\mathbb{1} \otimes M & M \otimes M \arrow[l, "\epsilon \otimes \mathrm{id}_M"'] \arrow[r, "\mathrm{id}_M \otimes \epsilon"] & M \otimes \mathbb{1} \\
& M \arrow[ul, "\mathrm{id}_M"] \arrow[u, "\delta"'] \arrow[ur, "\mathrm{id}_M"'] &
\end{tikzcd} .$$

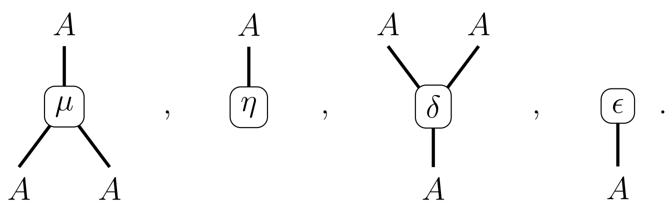
Monoid objects generalise monoidsin algebra, i.e. sets with an identity equipped with an associative binary operation.

Definition 2.2.2. A *Frobenius object* in $\mathcal{C}$ is a quintuple $(A, \mu, \eta, \delta, \epsilon)$ such that (A, μ, η) is a monoid object, (A, δ, ϵ) is a comonoid object, and the maps satisfy the *Frobenius relations*

$$\begin{tikzcd}
& A \otimes A \arrow[dl, "\delta \otimes \mathrm{id}_A"'] \arrow[d, "\mu"] \arrow[dr, "\mathrm{id}_A \otimes \delta"] & \\
A \otimes A \otimes A \arrow[dr, "\mathrm{id}_A \otimes \mu"'] & A \arrow[d, "\delta"] & A \otimes A \otimes A \arrow[dl, "\mu \otimes \mathrm{id}_A"] \\
& A \otimes A &
\end{tikzcd} ,$$

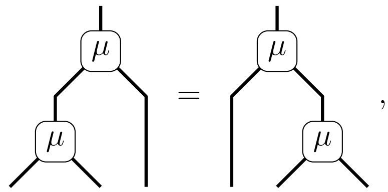
that is $(\mathrm{id}_A \otimes \mu) \circ (\delta \otimes \mathrm{id}_A) = \delta \circ \mu = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta)$.

The maps and relations for a Frobenius object $(A, \mu, \eta, \delta, \epsilon)$ have a pleasant description via the diagrams given in Section 2.1. The structure maps are drawn as

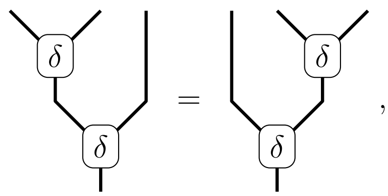


For the rest of this section, we will only work with the Frobenius object A and $\mathbb{1}$. We can stop putting the label A by identifying A with the identity strand $| = \mathrm{id}_A$.

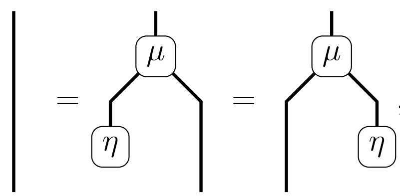
Diagrammatically, the associativity relation $\mu \circ (\mu \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \mu)$ is



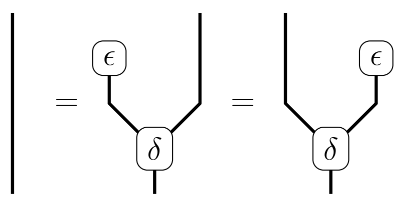
the coassociativity relation $(\delta \otimes \mathrm{id}_A) \circ \delta = (\mathrm{id}_A \otimes \delta) \circ \delta$ is



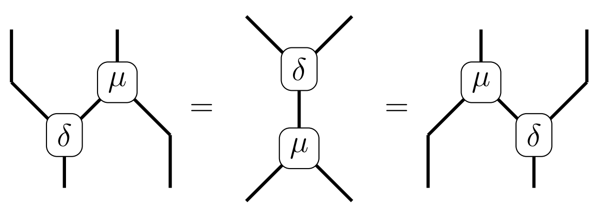
the unit relation $\mathrm{id}_A = \mu \circ (\eta \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \eta)$ is



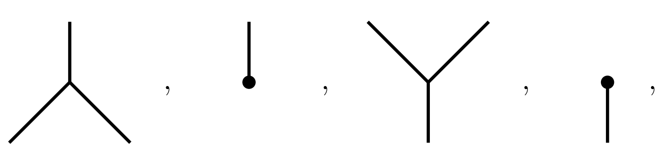
the counit relation $\mathrm{id}_A = (\epsilon \otimes \mathrm{id}_A) \circ \delta = (\mathrm{id}_A \otimes \epsilon) \circ \delta$ is



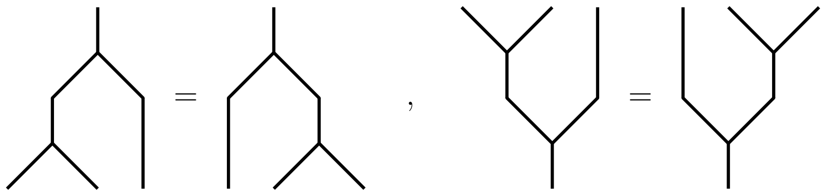
and the Frobenius relation $(\mathrm{id}_A \otimes \mu) \circ (\delta \otimes \mathrm{id}_A) = \delta \circ \mu = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta)$ is



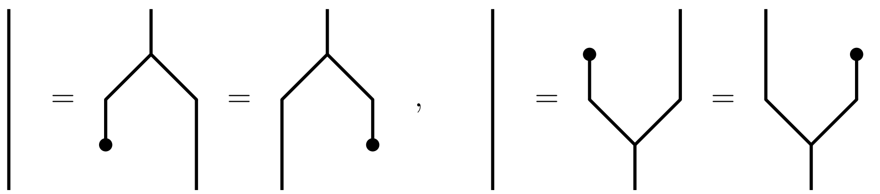
To further simply the diagrams, we stop labelling the functions and draw the structure maps as univalent and trivalent vertices



where the large dot on the unit and counit indicates that the string stops before reaching the other end. Then the relations become

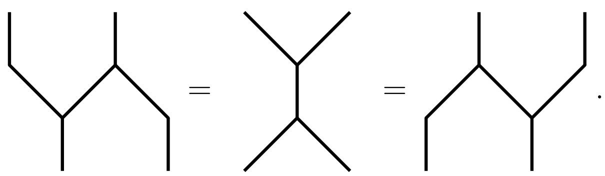


(Frob1)



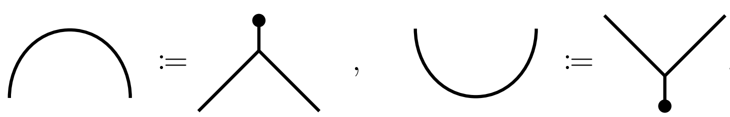
(Frob2)

and

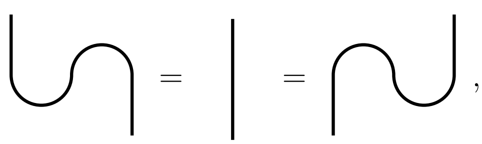


(Frob3)

Let us write cups and caps for the diagrams

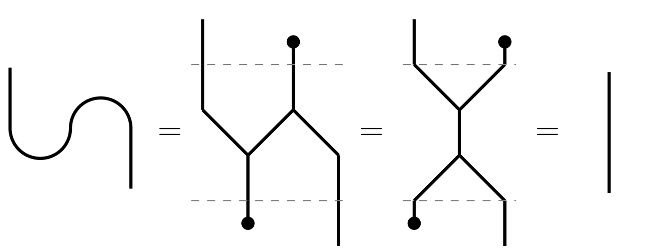


Then the Frobenius object relations admit a more familiar form of (planar) isotopy by the relations



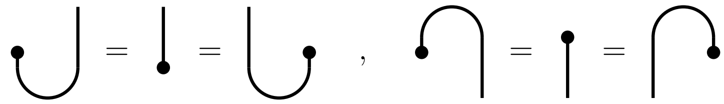
(Iso1)

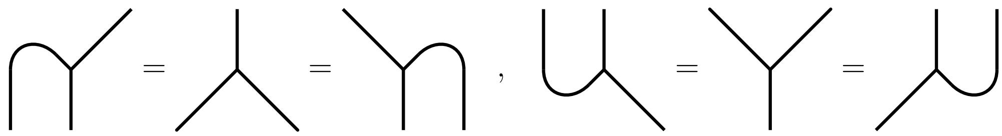
which we saw in for the Temperley-Lieb-Jones category. For instance the first equality follows from (Frob3) and (Frob2),



Remark on self adjointness and the above relation

Similarly we can deduce more isotopy relations

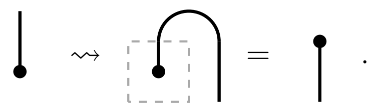

(Iso2)


(Iso3)

which can be thought of as "rotating of vertices".

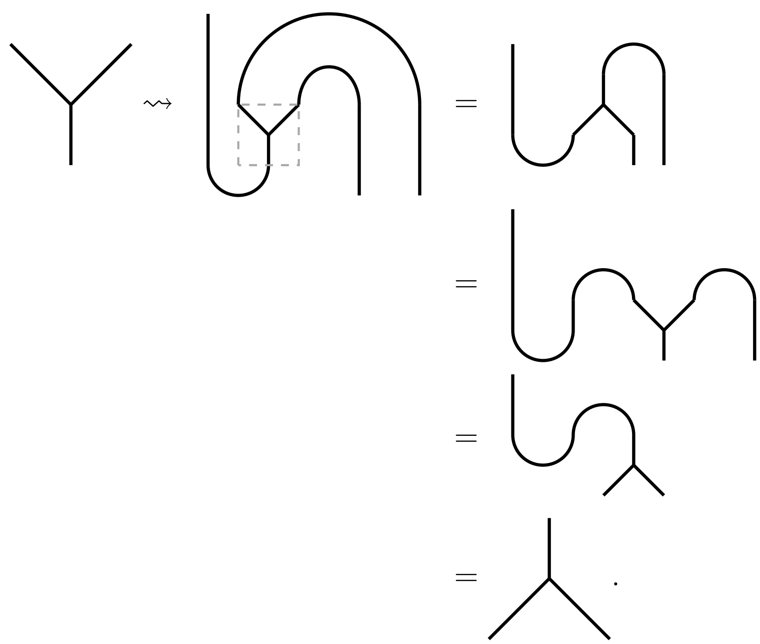
Using these maps, we can rotate diagrams by putting caps and cups on a diagram.

Example 2.2.3. The unit relation can be rotated to the counit map



where the equality follows from (Iso2).

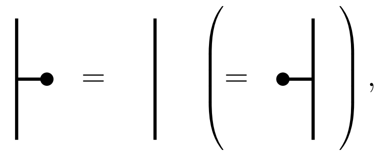
Example 2.2.4. The comultiplication map can be rotated to the multiplication map



where the equality follows from applying (Iso3) three times then (Iso1).

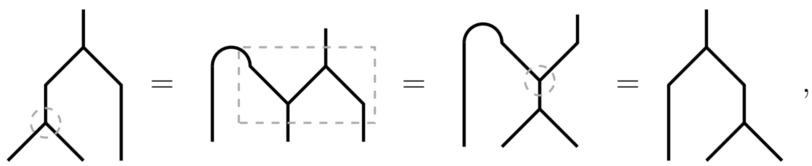
We can therefore consider the diagrams generated by concatenations of Frobenius structure maps up to planar isotopy. That is, we equate two diagrams if one diagram can be continuous deformed to the other in the plane without crossing itself. In this way, we can just use our visual intuition in place of applying specific isotopy relations (Iso1)-(Iso3).

The Frobenius object relations (Frob1), (Frob2), (Frob3) can be simplified as following. The unit and counit relations are



where the second equality follows from rotating the first one with cups and caps. Here the horizontal line has no innate meaning in the category but isotopically asserts equality between the "bent up" and "bent down" diagrams in (Frob2).

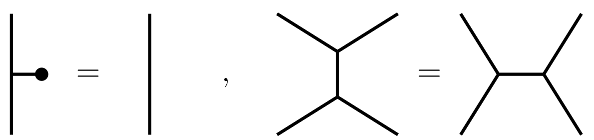
Note that the Frobenius relation (Frob3) implies the associativity and coassociativity relations (Frob1) by isotopy. For instance, we have



(2.2.5)

where the second equality is the Frobenius relation. For preciseness, this calculation shows the trivalent rotations (Iso3), but the reader is encouraged to think of the first and third equalities as isotopic deformations.

Therefore, up to isotopy, the Frobenius object relations are summed by the unit and Frobenius relation



(2.2.6)

These objects will appear again in the context of diagrammatic Soergel bimodules in Section 3.1

2.3 Module Categories

Something about what this is used for

Something about where this is seen

Definition 2.3.1. Let $(\mathcal{M}, \otimes, \mathbb{1})$ be a (strict) monoidal category. A *(left) module category over* $\mathcal{M}$ or $\mathcal{M}$*-module category* is a category $\mathcal{C}$ and a bifunctor $\odot : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ such that there are natural isomorphisms $(X \otimes Y) \odot A \cong X \odot (Y \odot A)$ and $\mathbb{1} \odot A \cong A$ for $X, Y \in \mathcal{M}$ and $A \in \mathcal{C}$, satisfying coherence relations analogous to those for monoidal categories (see [Eti+15, Definition 7.1.2]). A (left) $\mathcal{M}$-module category is *strict* if the natural isomorphisms above are identity natural isomorphisms, i.e. $(X \otimes Y) \odot A = X \odot (Y \odot A)$ and $\mathbb{1} \odot A = A$. We call $\odot$ the *action of* $\mathcal{M}$ or *the module product.*

In the following examples, the module action is essentially the monoidal product, which we may denote by the same symbol $\otimes$. Note that module actions may not always be an underlying monoidal product.

Example 2.3.2. A monoidal category is a module category over itself, where the action is its tensor product.

Example 2.3.3. Let G be a finite group and $H \subseteq G$ a subgroup. Consider the categories of group representations $\mathbf{Rep}(G)$ and $\mathbf{Rep}(H)$ over a field $\Bbbk$. Recall that $\mathbf{Rep}(G)$ is a category where objects are pairs (V, ρ) for V a $\Bbbk$-vector space and $\rho : G \to \mathrm{GL}(V)$ is a representation of G, and morphisms are equivariant maps i.e. linear maps that preserve the group action. There is a monoidal structure on $\mathbf{Rep}(G)$ (and similarly $\mathbf{Rep}(H)$) given by

$$(V, \rho_V) \otimes (W, \rho_W) = (V \otimes W, \rho_{V\otimes W})$$

where $V \otimes V$ is the usual tensor of vector spaces, and $\rho_{V\otimes W}$ is defined such that for $v \in V_1, w \in V_2$ and $g \in G$,

$$(\rho_1 \otimes \rho_2)(g)(v \otimes w) = (\rho_1(g)v) \otimes (\rho_2(g)w)$$

extended linearly. This is well defined by the universal property of tensor products. The monoidal unit is $\Bbbk$ with the trivial representation. The tensor product on morphisms f and g is defined by component-wise application, which is equivariant by equivariance of f and g.

We have that $\mathbf{Rep}(H)$ is a left module category over $\mathbf{Rep}(G)$ with the following action. For an object (V, ρ) in $\mathbf{Rep}(G)$, we can consider it as a representation over H by the restriction

$$\rho|_H : H \hookrightarrow G \xrightarrow{\rho} \mathrm{GL}(V).$$

The left action of (V, ρ) is the left tensor of $(V, \rho|_H)$ in $\mathbf{Rep}(H)$. On morphisms we apply a similar restriction of equivariant maps.

Definition 2.3.4. A (strict) module category $\mathcal{C}$ over a monoidal category $\mathcal{M}$ is *generated* by finite set S_o of objects and S_m of morphisms in $\mathcal{C}$, when all non-unit objects are of the form $X \odot A$ for $X \in \mathcal{M}$ and $A \in S_o$, and non-identity morphisms in $\mathcal{C}$ are defined similarly.

Definition 2.3.5. Let $\mathcal{M}$ be a (strict) R-linear monoidal category, and $\mathcal{C}$ be a (strict) module category over $\mathcal{M}$. We say that $\mathcal{C}$ is a *(strict) R-linear module category* if $\odot$ is R-bilinear on morphisms.

2.4 Additive Karoubi Envelope

Something about why this is needed

Mention that this technical and mostly a formal process

Additive and Karoubian Categories

Definition 2.4.1. A *preadditive category* is a category enriched over the category of abelian groups. That is, for objects A and B, $\mathrm{Hom}(A, B)$ has the structure of an abelian group and the composition of morphisms is bilinear (over the abelian group operation).

Remark 2.4.2. In particular R-linear categories are preadditive because R-modules are defined over abelian groups.

Definition 2.4.3. A *biproduct* of objects of a category is both a product and a coproduct. An *additive category* is a preadditive category that admits all finite biproducts.

Biproducts are a generalisation of direct sums of modules, so we often write $\oplus$ and say "direct sum". In other words, additive categories are preadditive categories containing all direct sums.

Definition 2.4.4. An *idempotent* is a endomorphism e such that $e \circ e = e$. We say that a preadditive category is *Karoubian* or *idempotent complete* if for every idempotent $e : X \to X$ there is a direct sum decomposition $X \cong Y \oplus Z$ such that e is a projection onto Y.

This is a formal way to say that a category contains all direct sums, as every direct summand is an image of an idempotent given by projection.

Additive Closure and Karoubi Envelope

We can formally add direct sums and direct summands into a preadditive category, by the additive closure and the Karoubi envelope.

Definition 2.4.5. Let $\mathcal{C}$ be a preadditive category. The *additive closure* $\mathcal{C}^{\oplus}$ of $\mathcal{C}$ is the category where objects are finite (possibly empty) formal direct sums $\bigoplus_{i=1}^{n} A_i$ for $A_i \in \mathrm{ob}(\mathcal{C})$. We call the empty direct sum the *zero object* 0. A morphism f of $\mathrm{Hom}_{\mathcal{C}^{\oplus}}(\bigoplus_{i=1}^{n} A_i, \bigoplus_{i=1}^{m} B_i)$ is an $m \times n$ matrix $f = (f_{j,i})$ of morphisms $f_{j,i} \in \mathrm{Hom}_{\mathcal{C}}(A_i, B_j)$.

It is clear that $\mathcal{C}$ is a category that embeds in $\mathcal{C}^{\oplus}$ and $\mathcal{C}^{\oplus}$ is additive. In the case where $\mathcal{C}$ is monoidal, $\mathcal{C}^{\oplus}$ is monoidal by extending the monoidal product to be an additive functor in each input. If $\mathcal{C}$ is R-linear, then $\mathcal{C}$ is an R-linear category by assuming that the R-action on morphisms applies componentwise. If $\mathcal{C}$ is a $\mathcal{M}$-module category, then $\mathcal{C}$ is a $\mathcal{M}$-module category by additionally assuming that the module action applies componentwise.

Lemma 2.4.6. *The additive closure satisfies the following universal property. For every preadditive functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is an additive category, there is a unique additive functor $F' : \mathcal{C}^{\oplus} \to \mathcal{D}$ such that the composition $\mathcal{C} \hookrightarrow \mathcal{C}^{\oplus} \xrightarrow{F'} \mathcal{D}$ is F.*

This is a classical result so we will not provide a proof. It can be observed by extending F to a functor $F^{\oplus} : \mathcal{C}^{\oplus} \to \mathcal{D}^{\oplus}$ defined componentwise with F.

Definition 2.4.7. Let $\mathcal{C}$ be a category. The *Karoubi envelope* $\mathrm{Kar}(\mathcal{C})$ of $\mathcal{C}$ is the category where objects are ordered pairs (A, e) for an object A in $\mathcal{C}$ and an idempotent $e \in \mathrm{End}_{\mathcal{C}}(A)$. Morphisms $f : (A, e) \to (A', e')$ are morphisms $f : A \to A'$ in $\mathcal{C}$ such that $f = f \circ e = e' \circ f$, where composition is composition in $\mathcal{C}$. Equivalently, morphisms $f : (A, e) \to (A', e')$ are of the form $e' \circ f \circ e$ for some (not necessarily unique) morphism $f : A \to A'$. The identity morphism on (A, e) is e.

The objects (A, e) should be seen as "the image of e". This is sometimes called the *Karoubian closure* or *idempotent completion*. The *additive Karoubi envelope* of a category $\mathcal{C}$ is $\mathrm{Kar}(\mathcal{C}^{\oplus})$ which we may denote $\mathrm{Kar}^{\oplus}(\mathcal{C})$.

Proposition 2.4.8. *The Karoubi envelope* $\mathrm{Kar}(\mathcal{C})$ *is Karoubian.*

A proof can be found at [Eli+20, Lemma 11.17].

Lemma 2.4.9. *Every functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is Karoubian, extends uniquely (up to isomorphism) to a functor $F' : \mathrm{Kar}(\mathcal{C}) \to \mathcal{D}$.*

This is another classical result. See [Bor94, Proposition 6.5.9 (1)] for a proof.

What happens when we do this on monoidal, R-linear or additive categories?

Talk about diagrammatics, $\mathcal{C}^{\oplus}$ is easy to describe diagrammatically (just matrices of diagrams) but Kar is not easy to describe in general (we need to find idempotents and put them before and after a diagram)

Chapter 3

One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = e \rangle$. At the end of this section, we see that this diagrammatic category is equivalent to the category of Soergel Bimodules under additive Karoubian closure.

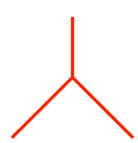
Remark 3.1.1. All diagrammatics below and in Chapter 4 can be defined in the language of planar algebras, without mentioning (monoidal) categories, e.g. in [Jon21]. Nevertheless, we define them in the context of categories as we will see them as diagrammatic versions of important categories in representation theory.

What do we do about $\mathbb{C}$? Do the theorems (at the end) apply over $\mathbb{Z}$ or $\mathbb{C}$ or both? If we define over $\mathbb{Z}$, how do we use it over $\mathbb{C}$ for the next section?

Definition 3.1.2. The *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ is a $\mathbb{Z}$-linear monoidal category with the following presentation.

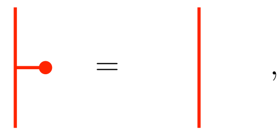
The objects are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is concatenation. The empty tensor product, i.e. the monoidal identity, will be denoted $\varnothing =: s^0$.

The morphisms are generated, up to isotopy, by univalent and trivalent vertices

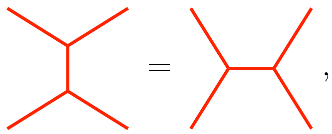
 ,  , (G1)

that are maps $s \to \varnothing$ and $ss \to s$ respectively, and their vertical reflections. We put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). The free $\mathbb{Z}$-module structure on each morphism space $\mathrm{Hom}(s^n, s^m)$ produces $\mathbb{Z}$-linear combinations of such diagrams. Something about composition/tensor and addition commuting Then, composition or tensors with the zero morphism 0 result in 0. To abuse notation, the empty diagram $\varnothing \to \varnothing$ will be denoted $\varnothing$. We may sometimes identify the identity morphism id_{s^n} (consisting of n red strings) with s^n.

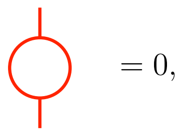
Such diagrams are subject to the following local relations



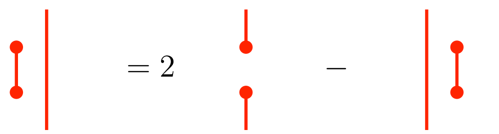
(R1a)



(R1b)



(R1c)



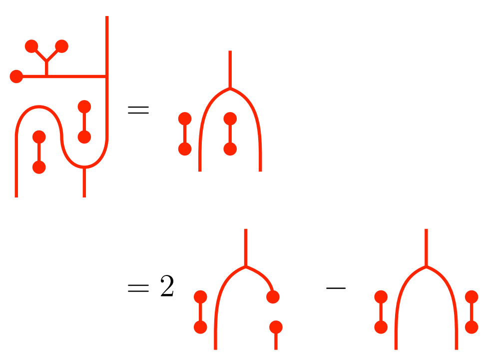
(R1d)

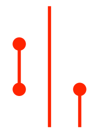
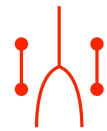
Note that the local relations apply to any subdiagram.

Remark 3.1.3. The object s is a Frobenius object in $\mathcal{H}(S_2)$. The generators (G1) and their vertical reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (R1a) and (R1b).

Put a definition of frob object in intro Talk about how the underlying thing that inspired these diagrams is frobenius object s. Isotopy is natural for frob objects. Talk about isotopy

Example 3.1.4. Using isotopy and the relations in (R1) we can simplify the morphism in $\mathrm{Hom}(ss, s)$,

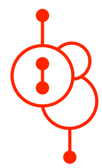


$= 2$  $-$  .

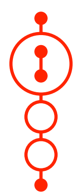
Add example of using frob associativity

Proposition 3.1.5. *All diagrams with floating diagrams, i.e. diagrams not connected to the domain or codomain by a red strand, are linear combination of diagrams where all floating diagrams are barbells.*

Proof. By isotopy and (R1a), floating diagrams can be drawn as barbells with "bubbles" and possibly floating subdiagrams inside each bubble. For example,

 .

The Frobenius relation (R1b) allows us to "straighten out" the bubbles to a chain of individual bubbles. The diagram above becomes

 .

For a floating diagram without floating subdiagrams, it is either 0 by (R1c), or which can be removed from any diagrams containing it via (R1d). Repeating this process produces a linear combination of diagrams where all floating diagrams are barbells. $\square$

The morphism space $\mathrm{Hom}(s^n, s^m)$ has a left (or right) $\mathbb{Z}[\ ]$-basis called the *double leaves* basis, as described in [EW16]. To define this basis, we must first define morphisms known as *light leaves*.

To make use of the group structure of S_2, we need to translate between words in $\mathcal{H}(S_2)$ and elements in S_2. Let $\phi : (\mathrm{ob}(\mathcal{H}(S_2)), \otimes) \to (S_2, *)$ be the monoid homomorphism[1] mapping $s \mapsto s$ and $\varnothing \mapsto 1$, and $\psi : S_2 \to \mathrm{ob}(\mathcal{H}(S_2))$ be the function that maps $s \mapsto s$ and $1 \mapsto \varnothing$. Should this be a definition? The maps ϕ allows words $w = s^n$ to be seen as elements of S_2, and ψ allows $1, s \in S_2$ to be seen as the objects $\varnothing, s \in \mathcal{H}(S_2)$. Clearly, $\phi\psi$ is the identity map on S_2, and the map $\psi\phi : \mathcal{H}(S_2) \to \mathcal{H}(S_2)$ takes objects to one of $\varnothing$ or s in $\mathcal{H}(S_2)$ by considering them as elements in S_2.

Definition 3.1.6. (Subexpression for S_2) Given a word $w = s^n$, a *subexpression* e is a binary string of length n. We can *apply* a subexpression to produce an object $w(e) \in \mathcal{H}(S_2)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $0 \leq i \leq n$, write $w(e, i)$ for the resultant object of the first i terms in e applied to the first i terms in w. Particularly $w(e, 0) = \varnothing$ and $w(e, n) = w(e)$.

[1] A map that preserves the monoidal product and identity element.

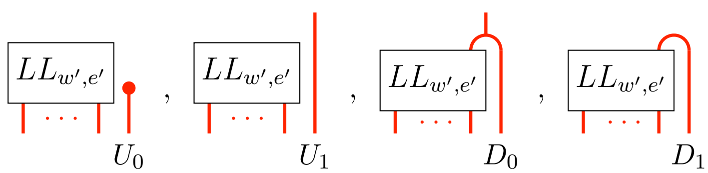
For example, 0000, 0110 and 1011 are subexpressions of $s^4 = ssss$. Applying the third subexpression gives $ssss(1011) = s\varnothing ss = sss$ and $ssss(1011, 3) = sss(101) = s\varnothing s = \varnothing$, by strictness of the monoidal category. Here, each term of the subexpression is a decision to include or exclude the corresponding s in the word, where excluding an s amounts to tensoring with $\varnothing$.

For a word w and subexpression e, we label each term by U_0, U_1, D_0 or D_1. The i-th term is labelled U_* if $\phi(w(e, i-1)) = 1 \in S_2$, and labelled D_* if $\phi(w(e, i-1)) = s \in S_2$. The label's subscript is the corresponding term in e.

Example 3.1.7. For the object $w = ssss$ and subexpression $e = 0101$, we find the labels as recorded in the following table.

Term i	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
$w(e, i)$	$\varnothing$	$\varnothing s = s$	$\varnothing s \varnothing = s$	$\varnothing s \varnothing s = ss$
Labels	U_0	U_0U_1	$U_0U_1D_0$	$U_0U_1D_0D_1$

Definition 3.1.8. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, the light leaf $LL_{w's,e'i}$ is one of

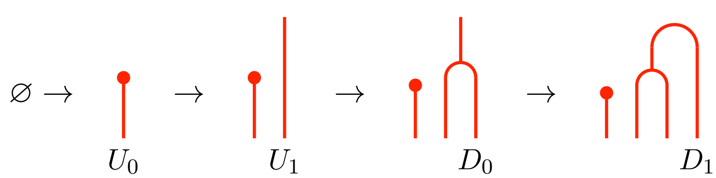


(LL1)

corresponding to the next label, where w' and e' are appropriate subwords[2] of w and e respectively.

Here, the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ is $\varnothing$, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. This implies that the recursive definition is consistent.

Example 3.1.9. Following from Example 3.1.7 for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.



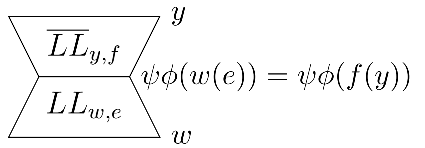
Definition 3.1.10. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y in $\mathcal{H}(S_2)$ is a composition

$$\mathbb{LL}_{f,e} \coloneqq \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

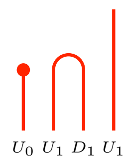
[2]A word with some letters removed.

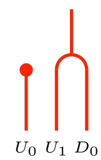
for subexpressions e of w and f of y such that $\psi\phi(w(e)) = \psi\phi(f(y))$.

Visually these are diagrams from w to y factoring through $\psi\phi(w(e)) = \psi\phi(f(y)) \in \{\varnothing, s\}$,



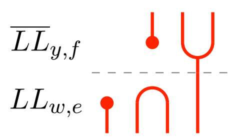
Example 3.1.11. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

.

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : ssss \to sss$, factoring through s, is



Note that these double leaves have no floating diagrams, which are by Proposition 3.1.5. In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{Z}[\ ]$-module, where the (left) -action is left concatenation by . Since we can move barbells to the right via. the relation (R1d) and double leaves cut down the middle are double leaves factoring through $\varnothing$, we can equivalently act by $\mathbb{Z}[\ ]$ on the right. This leads us to the following theorem.

Theorem 3.1.12 (Elias-Williamson [EW16, Theorem 1.2])**.** *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{LL}(w, y)$ be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $\psi\phi(w(e)) = \psi\phi(y(f))$. Then $\mathbb{LL}(w, y)$ is a basis for* $\mathrm{Hom}(w, y)$ *as a left (or right) $\mathbb{Z}[\ ]$-module.*

A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.13. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky's construction in [Lib08].

The morphisms in this category can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1. The degree of a diagram is the sum of the degrees of the generators that appear in it. This makes $\mathcal{H}(S_2)$ a $\mathbb{Z}$-graded category. Maybe mention what a grading is.

Put graded category definition in background

Put example

The category $\mathcal{H}(S_2)$ (under the additive Karoubi Envelope) is a diagrammatic version of the category of Soergel bimodules $\mathbb{S}\mathrm{Bim}$ for S_2. However $\mathbb{S}\mathrm{Bim}$ is not well behaved with morphisms over $\mathbb{Z}$, so we must first alter the morphisms spaces in $\mathcal{H}(S_2)$ to be over $\mathbb{C}$ instead[3]. Formally we merely tensor the morphism spaces on the left by the $\mathbb{C}$-$\mathbb{Z}$-bimodule $\mathbb{C}$, where the right action is induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$. We write $\mathcal{H}_{\mathbb{C}}(S_2)$ for this $\mathbb{C}$-linear monoidal category. This process is quite simple and does little to the category itself. In particular, double leaves in $\mathcal{H}_{\mathbb{C}}(S_2)$ remain as $\mathbb{C}[\ ]$-bases[4] for the morphisms.

Theorem 3.1.14 (Elias-Williamson [EW16, Theorem 6.30])**.** *The diagrammatic category* $\mathrm{Kar}^{\oplus}(\mathcal{H}_{\mathbb{C}}(S_2))$ *and the category of Soergel Bimodules* $\mathbb{S}\mathrm{Bim}$ *over* S_2 *are equivalent as graded* $\mathbb{C}$*-linear monoidal categories.*

The proof in [EW16] gives an equivalence of graded $\mathbb{C}$-linear monoidal categories $\mathcal{H}_{\mathbb{C}}(S_2) \cong \mathbb{B}\mathbb{S}\mathrm{Bim}$ where $\mathbb{B}\mathbb{S}\mathrm{Bim}$ is the category of Bott-Samelson bimodules over S_2. Since the additive Karoubi envelope preserves equivalences, $\mathrm{Kar}^{\oplus}(\mathbb{B}\mathbb{S}\mathrm{Bim}) \cong \mathbb{S}\mathrm{Bim}$ implies $\mathrm{Kar}^{\oplus}(\mathcal{H}_{\mathbb{C}}(S_2)) \cong \mathbb{S}\mathrm{Bim}$.

3.2 Diagrammatic $\mathcal{O}(\mathfrak{sl}_2)$

A little bit about category $\mathcal{O}$, and our example of $\mathfrak{sl}_2$

For this section, our category of interest is $\mathcal{O}$ for the semisimple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. A description of the category $\mathcal{O}$ can be found in general in [Hum08, Sections 3.8–3.10] or in [Maz09, Section 5.2] for the case of $\mathfrak{sl}_2(\mathbb{C})$, however we will only give a brief overview. The category $\mathcal{O}$ is a category of certain modules (or representations) over a semisimple Lie algebra. It is a direct sum of subcategories, where, in the case of $\mathfrak{sl}_2$ over $\mathbb{C}$, the non-trivial summands are equivalent as abelian categories to the subcategory $\mathcal{O}_0$. Within this, we look to the full subcategory $\mathrm{proj}(\mathcal{O}_0)$ of projective modules in $\mathcal{O}_0$, which, in particular, is additive and contains all direct summands.

In [Soe90, Section 2.4], Soergel shows that the category $\mathcal{O}$, and hence the subcategory $\mathrm{proj}(\mathcal{O}_0)$, is a Soergel module category, i.e. it has an action of the monoidal category $\mathbb{S}\mathrm{Bim}$. By the equivalence in Theorem 3.1.14 we will view $\mathrm{proj}(\mathcal{O})$ as a $\mathcal{H}_{\mathbb{C}}(S_2)$-module category, extending via the additive Karoubi envelope. Since $\mathcal{H}_{\mathbb{C}}(S_2)$ is diagrammatic, this action allows us to describe $\mathrm{proj}(\mathcal{O}_0)$ (thus essentially $\mathcal{O}_0$ and $\mathcal{O}$) diagrammatically.

Remark 3.2.1. We can pass from $\mathrm{proj}(\mathcal{O}_0)$ to $\mathcal{O}_0$ by observing that $K^b(\mathrm{proj}(\mathcal{O}_0))$ is equivalent to $D^b(\mathcal{O}_0)$ as graded $\mathbb{C}$-linear monoidal triangulated categories. This is a standard trick in the field, for example see the introduction of [RW18][5]. However for our purposes it is not important to understand how this works.

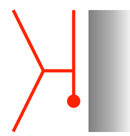
[3] The equivalence actually holds in more generality, but we choose $\mathbb{C}$ because it is easy to work with.

[4] It is not hard to check that double leaves tensored with $1 \in \mathbb{C}$ on the left form a basis.

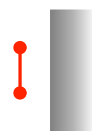
[5] A self-contained summary of how diagrammatic categories can be related to abelian categories.

Although we need to work over $\mathbb{C}$ for $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$, the diagrammatic category can be defined more simply, that is over $\mathbb{Z}$.

Definition 3.2.2. Let $\mathcal{DO}_0 \coloneqq \mathcal{DO}_0(\mathfrak{sl}_2)$ be the $\mathbb{Z}$-linear (Define this in background) left $\mathcal{H}(S_2)$-module category with elements generated (Define what this means.) by the monoidal identity $\varnothing$ of $\mathcal{H}(S_2)$ and morphisms generated by the empty diagram $\varnothing$. The action of $\mathcal{H}(S_2)$ on the left is left concatenation for both objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0,1] \times \mathbb{R}_{<0}$ instead of in the double-sided strip $[0,1] \times \mathbb{R}$. For example a morphism may be

 .

We impose the local relation that diagrams are related to the wall by

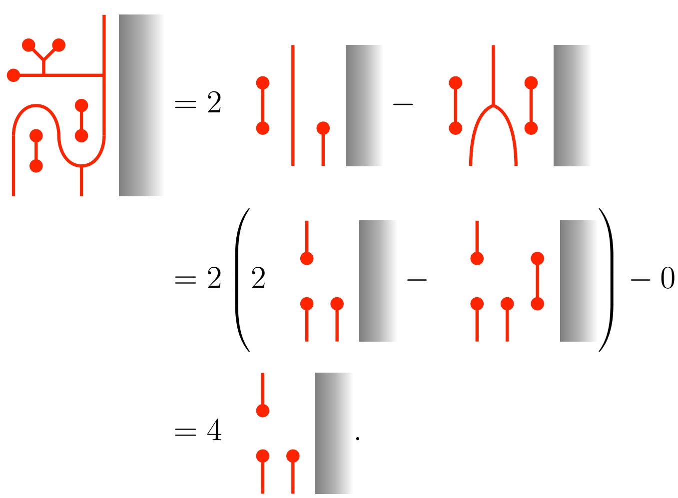


$$= 0. \tag{W1}$$

Note that this local relations applies to any subdiagram involving the wall.

The objects of this category are identical to objects in $\mathcal{H}(S_2)$ and the morphisms are the same modulo the wall relation (W1). The left module category means that we can only concatenate diagrams on the left by means of the module action. This may seem no different from $\mathcal{H}(S_2)$, however the wall relation (W1) makes right tensors inconsistent. For instance, a barbell diagram is 0 however tensoring by id_s on the right gives a non-zero diagram. In particular $\mathcal{DO}_0$ is not a monoidal category.

Example 3.2.3. Using the new relation (W1), we can further simplify the morphism in Example 3.1.4 by



A natural question to ask is whether double leaves still form bases for the morphism spaces here. Notice that double leaves appear in $\mathcal{DO}_0$ by acting on $\varnothing$ by double leaves in $\mathcal{H}(S_2)$. All morphisms in $\mathcal{DO}_0$ are morphisms in $\mathcal{H}(S_2)$ so they can be written as $\mathbb{Z}[\ ]$-linear combinations of double leaves, though some have collapsed to 0. Thus double leaves span the morphism spaces of $\mathcal{DO}_0$ as (left) $\mathbb{Z}[\ ]$-modules. However they may not be linearly independent as neither left nor right modules. For example, any pair of double leaves that factor through $\varnothing$ become 0 when multiplied by on either side (by translating the barbell to the right). Although double leaves are not always a basis for its respective morphism space as $\mathbb{Z}[\ ]$-modules, it turns out they are a basis over $\mathbb{Z}$.

Lemma 3.2.4. *Let $\pi : \mathrm{mor}(\mathcal{H}(S_2)) \to \mathrm{mor}(\mathcal{DO}_0)$ be the projection map which takes a morphism to the result of its action on the empty diagram $\varnothing$. Then the image $\pi(\mathbb{LL}(w, y))$ is a basis for $\mathrm{Hom}_{\mathcal{DO}_0}(w, y)$ as a $\mathbb{Z}$-module.*

Proof. We consider morphisms $\mathrm{Hom}(w, y)$ in $\mathcal{DO}_0$ for fixed objects w, y, and write $\mathbb{LL} := \pi(\mathbb{LL}(w, y))$ for the set of double leaves in $\mathcal{DO}_0$. Any diagram in $\mathcal{DO}_0$ can be written as a $\mathbb{Z}$-linear combination of morphisms without floating diagrams, by simplifying them to barbells, pulling them to the right and killing them with (W1). We can write each of these as a $\mathbb{Z}[\ ]$-linear combination of double leaves by (3.1.12) with the right action, and reduce it to a $\mathbb{Z}$-linear combination by (W1). This implies that $\mathbb{LL}$ spans $\mathrm{Hom}(w, y)$ as a $\mathbb{Z}$-module. Since the barbell-wall relation (W1) has no effect on $\mathbb{Z}$-linear combinations of $\mathbb{LL}$, it follows from linear independence over $\mathbb{Z}[\ ]$ that they are linearly independent over $\mathbb{Z}$ in $\mathcal{DO}_0$. Check the proof. $\square$

Our goal is to prove that this diagrammatic category is equivalent to $\mathrm{proj}(\mathcal{O}_0)$. To that end, we will shift our focus from $\mathbb{Z}$ to $\mathbb{C}$ for the remainder of this section. From now on we write $\mathcal{DO}_0$ for the $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(S_2)$-module category obtained by replacing $\mathbb{Z}$ with $\mathbb{C}$ and $\mathcal{H}(S_2)$ with $\mathcal{H}_{\mathbb{C}}(S_2)$ in Definition 3.2.2. The above discussion and Lemma 3.2.4 still apply to $\mathcal{DO}_0$ over $\mathbb{C}$.

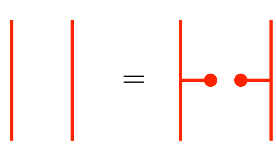
Maybe put this next bit in section 3.1

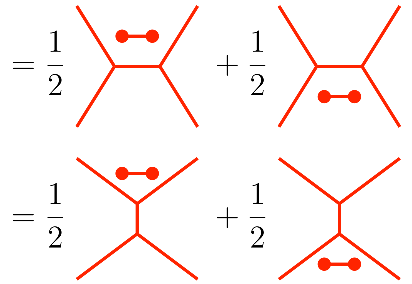
Say more about what this is, and why we say it here

The next result allows us to reduce all morphisms to a matrix of diagrams only involving $\varnothing$ and s.

Lemma 3.2.5. *In the additive closure of $\mathcal{H}_{\mathbb{C}}(S_2)$ we have an explicit isomorphisms $ss \cong s \oplus s$, as detailed in the proof. Particularly, these are isomorphisms in the additive closure of $\mathcal{DO}_0$.*

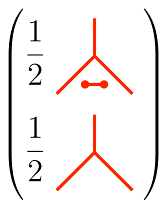
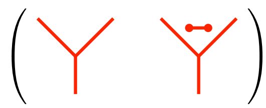
Proof. In $\mathcal{H}_{\mathbb{C}}(S_2)$ we have the relation





(3.2.6)

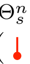
This implies that in $\mathcal{H}_{\mathbb{C}}(S_2)^{\oplus}$, we have maps

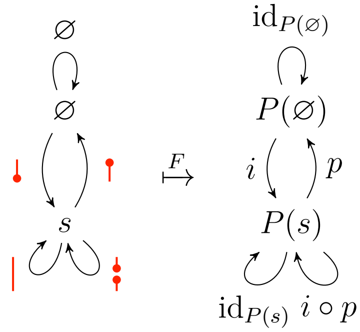
 $: ss \to s \oplus s$ and  $: s \oplus s \to ss.$

It follows from (R1d), (R1c) and the calculation (3.2.6), that these maps are inverses. Maybe put the inverse calculation here. □

Be clear that I don't understand category $\mathcal{O}$ very well.

As a shorthand, we write $\mathrm{proj}(\mathcal{O}_0)$ for $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$. The work of Soergel in [Soe90, Section 2.4] shows that $\mathrm{proj}(\mathcal{O}_0)$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_\varnothing, \Theta_s \in \mathrm{End}(\mathcal{O})$ (corresponding to elements in S_2). Explains what this means, how its related to the $\mathcal{H}(S_2)$ module category We construct a functor that maps faithfully into a full subcategory of $\mathrm{proj}(\mathcal{O}_0)$, which will become the whole of $\mathrm{proj}(\mathcal{O}_0)$ under the additive Karoubi envelope. This is the same strategy as in the proof for Theorem 3.1.14.

Definition 3.2.7. Let $F : \mathcal{DO}_0^{\oplus} \to \mathrm{proj}(\mathcal{O}_0)$ be the $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(S_2)$-module functor that sends the empty object $\varnothing$ to the trivial module $P(\varnothing)$, and the Soergel module action corresponding to s to the translation functor Θ_s. Then the object s maps to $\Theta_s(P(\varnothing)) =: P(s)$, and for example s^3 maps to the composition of three Θ_s on $P(\varnothing)$, $\Theta_s^3(P(\varnothing)) = \Theta_s\Theta_s\Theta_s(P(\varnothing))$. In order for F to be functorial, it must map identity diagrams $s^n \to s^n$ to $\mathrm{id}_{\Theta_s^n(P(\varnothing))}$. On non-identity maps, we let $F($$)$ be the inclusion $i : P(\varnothing) \to P(s)$ and $F($$)$ be the projection $p : P(s) \to P(\varnothing)$. We then extend F by composition, additivity and linearity. The mapping of F is depicted by the following picture. Need to talk about compositions, why is F well defined if its generated by compositions of these? Are there any clashes? – Actually preserves compositions by construction



(3.2.8)

Actually refer to the picture

Lemma 3.2.9. *The functor F is well defined.*

Proof. From [Maz09, Proposition 5.90], there is a natural isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ analogous to the isomorphism $ss \cong s \oplus s$ given in the proof of Lemma 3.2.5. We consider the additive closure $\mathcal{DO}_0^{\oplus}$, which does not cause problems since we will eventually take this anyway Maybe just do additive karoubi closure in the first place. Given a morphism in $\mathcal{DO}_0$ from s^n to s^m, repeated precomposition and postcomposition with $ss \to s \oplus s$ and $ss \oplus s \to s$ from Lemma 3.2.5 results in an isomorphic matrix of diagrams with domain and codomain in $\{\varnothing, s\}$. By Lemma 3.2.4 over $\mathbb{C}$, $\mathrm{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_{\varnothing}\}$, $\mathrm{Hom}(s, \varnothing)$ has a basis $\{\ \}$, $\mathrm{Hom}(\varnothing, s)$ has a basis $\{\ \}$, and $\mathrm{Hom}(s, s)$ has a basis $\{\mathrm{id}_s, \ \circ\ \}$. Therefore, extending by linearity, the picture above completely describes the image of F.

Next we check that all the relations are preserved. From classical results e.g. [Maz09, Proposition 5.84 and Lemma 5.87], it follows that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$. Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure. Applying these to $P(\varnothing)$ result in the same relations in $\mathrm{proj}(\mathcal{O}_0)$ for $P(\varnothing), P(s)$ and $\Theta_s^2(P(\varnothing))$. Note that the projection and inclusion maps above are exactly the unit and counit of Θ_s evaluated at $P(\varnothing)$, and the trivalent vertices provided by projecting the isomorphisms in Lemma 3.2.5 map exactly to the multiplication and comultiplication maps. Hence the Frobenius relations (R1a) and (R1b) are satisfied. Ref for needle and barbell? Furthermore, in [Soe90, Section 2.4] we see that $p \circ i = 0$ in $\mathrm{proj}(\mathcal{O}_0)$ which is analogous[6] to the barbell-wall relation (W1). Hence all the relations in $\mathcal{DO}_0$ are preserved by F. By construction, F preserves $\mathbb{C}$-linear combinations and the Soregel module structure in [Soe90], so F is well defined as a functor between $\mathbb{C}$-linear $\mathcal{H}(S_2)$-module categories. □

Theorem 3.2.10 (Soergel, [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4])**.** *The diagrammatic category* $\mathrm{Kar}^{\oplus}(\mathcal{DO}_0(\mathfrak{sl}_2))$ *and* $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$ *are equivalent as* $\mathbb{C}$*-linear* $\mathcal{H}(S_2)$*-module categories.*

[6] This relation extends to the analogue of the local barbell-wall relation, as all "barbell on the right" morphisms in $\mathrm{proj}(\mathcal{O}_0)$ are linear combinations of applications of Θ_s to $p \circ i$, which is 0.

Check all of this & Put precise references

Maybe write description as a soergel module outside the proof

Proof. First we show that F is full and faithful. It follows from Lemma 3.2.5 and the description of $P(\varnothing)$ and $P(s)$ in [Maz09, Section 5.2] that the image of  and generate all morphisms of the form $\Theta_s^n(P(\varnothing)) \to \Theta_s^m(P(\varnothing))$. Hence F is full. Now the mapping of F on all morphism spaces are determined by those depicted in the above picture. So, for faithfulness, it suffices to compare the $\mathbb{C}$-dimensions of morphism spaces between objects shown in the picture. As mentioned above, the double leaves basis are precisely the diagrams depicted in the image. The bases for the corresponding morphism spaces in $\mathrm{proj}(\mathcal{O}_0)$ are also those in the image Ref? - that these are actually the bases of the hom spaces, so the dimensions of Hom spaces coincide. Therefore F is fully faithful.

All objects in $\mathrm{proj}(\mathcal{O}_0)$ appear as direct sums and direct summands of the elements $\Theta_s^n(P(\varnothing))$ for non-negative integers n. Therefore the additive Karoubi envelope induces an equivalence $\mathrm{Kar}^{\oplus}(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ as $\mathbb{C}$-linear left $\mathcal{H}(S_2)$-module categories. □

This result is essentially due to Soergel [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4] (see also [Soe98]) but this was not its original formulation. Nevertheless we attribute this theorem to Soergel.

Maybe talk about Soergel modules and $\mathcal{H}(S_2)$-modules vs $\mathrm{Kar}^{\oplus}(\mathcal{H}(S_2))$-modules

Remark 3.2.11. The morphisms spaces in $\mathcal{DO}_0$ are graded by the same grading as $\mathcal{H}(S_2)$ in Section 3.1. The equivalence $\mathrm{Kar}^{\oplus}(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ includes a grading of morphisms in $\mathrm{proj}(\mathcal{O}_0)$ Check! and hence a grading morphisms of $\mathcal{O}$, which is otherwise ungraded.

Some more consequences

Chapter 4

Two-colour Diagrammatics

The previous chapter had its focus on the symmetric group generated by one element S_2, which brought about one-colour diagrammatics. We now explore a more complex example by adding an extra generator, that is, another colour. In particular, we consider the affine symmetric group on two elements[1] $\widetilde{S}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$. Refine this

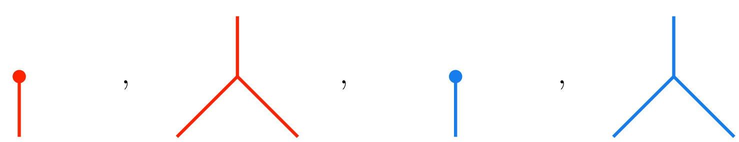
4.1 Two-colour Diagrammatic Hecke Category

The definition is similar to the one-colour case, so we will be brief.

Definition 4.1.1. The *two-colour (diagrammatic) Hecke category* $\mathcal{H}(\widetilde{S}_2)$ is a (strict) $\mathbb{Z}$-linear monoidal category given by the following presentation.

Objects in $\mathcal{H}(\widetilde{S}_2)$ are generated by formal tensor products of the non-identity elements $s, t \in \widetilde{S}_2$. As before, we write objects as words such as $sstttst =: s^2t^3st$ where the tensor product is concatenation, and associate the colour red to s and blue to t. The empty word is the monoidal identity, which we write as $\varnothing$.

The morphisms are generated by the univalent and trivalent vertices

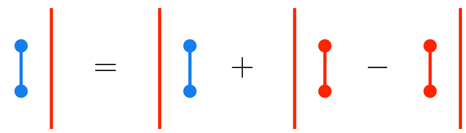


(G2)

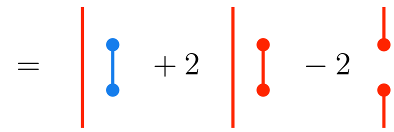
that are maps $s \to \varnothing$, $ss \to s$, $t \to \varnothing$ and $tt \to t$ respectively, and their vertical reflections. As in the one-colour case, tensor product is horizontal concatenation, composition is appropriate vertical stacking, and we denote the empty diagram $\varnothing \to \varnothing$ by $\varnothing$. For each colour, these diagrams have the one-colour relations given by (R1). As we have another colour, we need to describe how different colours interact. This is given by the

[1] Also known as the infinite dihedral group.

two-colour relations



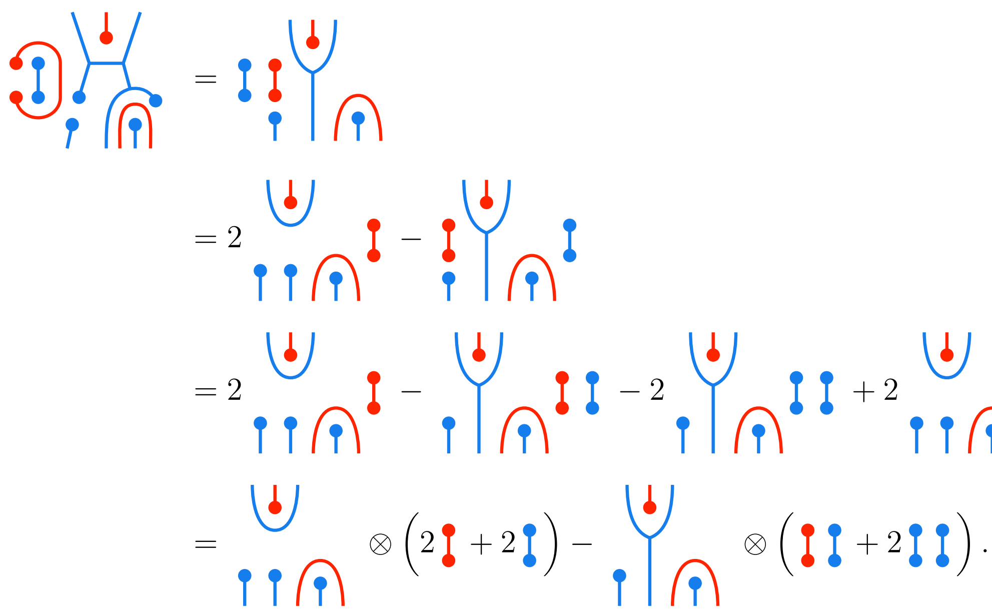
(R2)



(4.1.2)

and with red and blue swapped.

Example 4.1.3. Using the one-colour and two-colour relations on the following morphism in $\mathrm{Hom}(ttsts, tst)$ we have



Talk about this containing $\mathcal{H}(S_2)$

Remark 4.1.4. Notice that the red and blue lines never cross as no generators that allow crossings. This is a consequence of working over affine S_2 in which the generators s and t have no relation. Mention example of crossing and S_3.

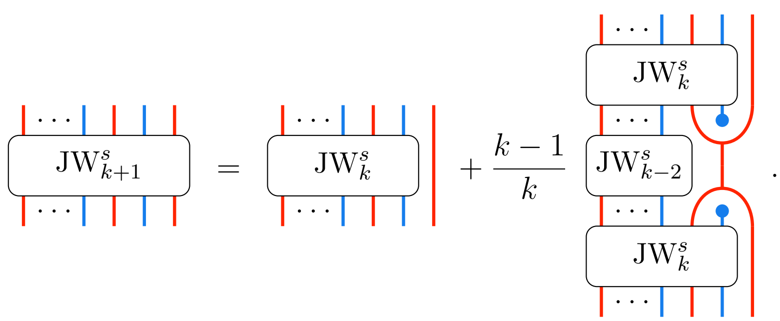
In this two-colour case, Proposition 3.1.5 holds by replacing (R1d) with (R2) in the proof. This handles the new possibility of floating subdiagrams with alternating colours.

Definition 4.1.5. For a group with a presentation in terms of generators and relations, the *length* of a product of generators is the number of generators in the product. We say that a product of generators is *reduced* if it's length cannot be shortened with relations.

In $\widetilde{S}_2$ products can be shortened by the relation $s^2 = t^2 = 1$. For instance, $sttsts$ is not reduced because it is equal to ts which is reduced. Notice that for $\widetilde{S}_2$ each element can be written uniquely as a reduced product of generators. This is true since otherwise we have two distinct reduced products for the same element in $\widetilde{S}_2$ so they must be related by $s^2 = t^2$. This means they can be reduced further by $s^2 = t^2 = 1$, which contradicts minimality of their length. Note that the reduced products in $\widetilde{S}_2$ are either the identity or alternating products of s and t.

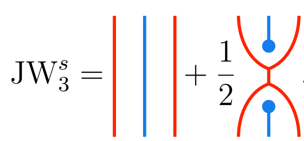
Notice that there is a notational similarity between products in the group and words in $\mathcal{H}(\widetilde{S}_2)$. This motivates the following definitions. Let $\phi : (\mathrm{ob}(\mathcal{H}(\widetilde{S}_2)), \otimes) \to (\widetilde{S}_2, *)$ be the monoid homomorphism mapping $\varnothing \mapsto 1$, $s \mapsto s$ and $t \mapsto t$. Also define the function $\psi : \widetilde{S}_2 \to \mathrm{ob}(\mathcal{H}(\widetilde{S}_2))$ to map elements $x \in \widetilde{S}_2$ to the tensor product of s and t in $\mathcal{H}(\widetilde{S}_2)$ corresponding to the reduced product of x in $\widetilde{S}_2$. This is well defined because reduced products are unique and two different reduced products cannot equal the same element of $\widetilde{S}_2$. Note that the image $\psi(\widetilde{S}_2)$ is the set containing $\varnothing$ and words of alternating s and t. The composition $\psi\phi : \mathcal{H}(\widetilde{S}_2) \to \mathcal{H}(\widetilde{S}_2)$ maps words w to the tensor of s and t corresponding to the reduced product of $\phi(w)$, and $\phi\psi$ is the identity map on $\widetilde{S}_2$.

Definition 4.1.6. (Jones–Wenzl Projectors) Consider words w in $\mathcal{H}(\widetilde{S}_2)$ corresponding to reduced products in $\widetilde{S}_2$ i.e. alternating s and t. Suppose that the leftmost generator in w is s. Then *Jones–Wenzl projector* $\mathrm{JW}_k^s \in \mathrm{Hom}(w, w)$ is defined recursively such that $\mathrm{JW}_0^s = \mathrm{id}_\varnothing$, $\mathrm{JW}_1^s = \mathrm{id}_s$, $\mathrm{JW}_2^s = \mathrm{id}_{st}$ and for $k \geq 2$ even,

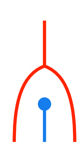


For k odd, we just swap red and blue to the right of the ellipsis. If w starts with t, we can define JW_i^t by swapping all reds and blues in the recursive formula.

Example 4.1.7. The first non-trivial JW-projector is



Definition 4.1.8. A *pitchfork* is the diagram of the form



possibly with the colours swapped or vertically reflected.

Proposition 4.1.9. *The Jones–Wenzl projector is an idempotent, i.e.* $\mathrm{JW}_k \circ \mathrm{JW}_k = \mathrm{JW}_k$*, and is killed by pitchforks on the top or the bottom.*

The JW-projectors are important idempotents in our category, as their images are all the indecomposables in the additive Karoubi envelope (see [Eli16, Section 5.4.2]).

Remark 4.1.10. Jones–Wenzl projectors are originally defined to be elements in the Temperley–Lieb algebra satisfying certain properties. The above recursive formula was first shown in [Wen87], which we just take for its definition. The functor given in [Eli16, Section 5.3.2] sends them into our diagrammatic category. The proof of the Temperley–Lieb version of Proposition 4.1.9 is a classical result and can be found in for example [Wen87] or [Mor15].

As in the one-colour case, there are bases for morphism spaces in $\mathcal{H}(\widetilde{S}_2)$ given by double leaves, which we will build up to. The following definition is a more general version of Definition 3.1.6.

Definition 4.1.11 (Subexpression)**.** Given a word w of length n, a *subexpression* e is a binary string of length n. A subexpression can be *applied* to produce an word $w(e)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $1 \leq i \leq n$, we write $w(e,i)$ for the result of the first i terms of e applied to the first i terms in w. Particularly $w(e,0) = \varnothing$ and $w(e,n) = w(e)$.

For example, in $\mathcal{H}(\widetilde{S}_2)$, if $w = sttts$ and $e = 11001$ then $w(e) = st\varnothing\varnothing s = sts$ and $w(e,3) = sts(110) = st\varnothing = st$ in $\mathcal{H}(\widetilde{S}_2)$.

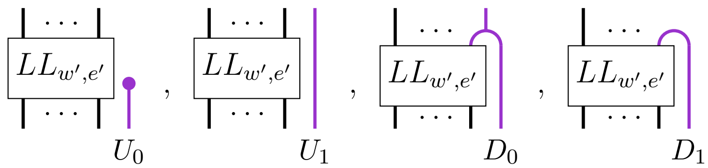
Let the *length* of a word be the number of generators in its tensor product. As before, given an object w and a subexpression e of w, we label each of the n terms by one of U_0, U_1, D_0, D_1. Let $i \geq 0$, and write x for the i-th term of w. We label the i-th term U_* if $\psi\phi(w(e,i-1) \otimes x)$ is longer than $\psi\phi(w(e,i-1))$. In other words we write U_* if the next term of w will make $\psi\phi$ applied to the partially evaluated subexpression longer, regardless of the i-term of e. We label D_* if $\psi\phi(w(e,i-1) \otimes x)$ is longer than $\psi\phi(w(e,i-1))$. The label's subscript is the i-th term of e. Note that this construction is well defined because $\psi\phi(w(e,i-1)\otimes x) = \psi(\phi(w(e,i-1)) * \phi(x)) = \psi(\phi(w(e,i-1)) * x)$ is always either longer or shorter, since the last element of the reduced product is either the same as x or different. When they are the same, the word is shorter via $s^2 = t^2 = 1$, and when they are different it is longer as no relations can make it shorter.

Remark 4.1.12. This description of the labels (via. reduced products) is more akin to the definition in general for Coxeter groups, than in Section 3.1. Mention the words coxeter group in intro

Example 4.1.13. Consider the word $w = sttst$ and subexpression $e = 10011$. The labels can be constructed as in the following table.

Term i	1	2	3	4	5
Partial w	s	st	stt	$stts$	$sttst$
Partial e	1	10	100	1001	10011
$w(e,i)$	s	$s\varnothing$	$s\varnothing\varnothing = s$	$s\varnothing\varnothing s = ss$	$s\varnothing\varnothing st = sst$
Labels	U_1	U_1U_0	$U_1U_0U_0$	$U_1U_0U_0D_1$	$U_1U_0U_0D_1U_1$

Definition 4.1.14. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and a subexpression e is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given appropriate subwords w' and e' of w and e respectively, and if the next terms are x in w and i in e, the light leaf $LL_{w'x,e'i}$ is one of

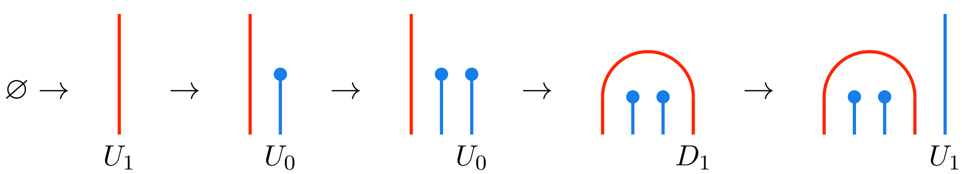


(LL2)

corresponding to the next label. The purple strands are red if $x = s$ and blue if $x = t$.

Notice that the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ does not end with the colour corresponding to x, and if the next label is D_* the codomain of $LL_{w',e'}$ ends with a strand with the colour corresponding to x. This implies the recursive definition in the diagram above is consistent. Note that in the case of D_*, one of the black strands in the domain of $LL_{w',e'}$ must have the colour of x in order for the colour to appear in its codomain.

Example 4.1.15. Following from Example 4.1.13, with $w = sttst$, $e = 10011$ and labels $U_1U_0U_0D_1U_1$, the light leaf $LL_{w,e}$ is build as follows.



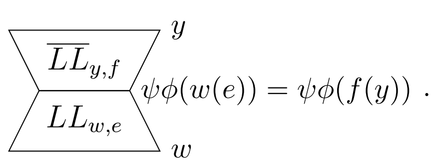
We can define double leaves exactly as we did in Definition 3.1.10.

Definition 4.1.16. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y in $\mathcal{H}(\widetilde{S}_2)$ is a composition

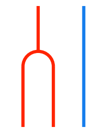
$$\mathbb{LL}_{f,e} \coloneqq \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

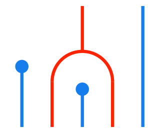
for subexpressions e of w and f of y such that $\psi\phi(w(e)) = \psi\phi(f(y))$.

Diagrammatically these are morphisms from w to y factoring through $\psi\phi(w(e)) = \psi\phi(f(y)) \in \psi(\widetilde{S}_2)$,



Example 4.1.17. Let $w = sst$ with the subexpression $e = 101$ and $y = tstst$ with the subexpression $f = 01001$. The corresponding light leaves are

$LL_{w,e} =$ 

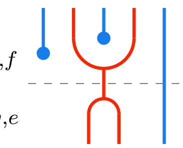
and $LL_{y,f} =$ 

.

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : sst \to tstst$, factoring through st, is

$\overline{LL}_{y,f}$

$LL_{w,e}$


.

As with the one-colour case, the set of double leaves $\mathbb{LL}(w, y)$ from words w to y in $\mathcal{H}(\widetilde{S}_2)$ form a basis for $\mathrm{Hom}(w, y)$ over $\mathbb{Z}[\ ,\ ]$. The Hom spaces are graded such that the univalent vertices have degree 1 and trivalent vertices have degree -1 for either colour.

Let $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$ be the $\mathbb{C}$-linear monoidal category obtained by extending the scalars of morphisms spaces in $\mathcal{H}(\widetilde{S}_2)$ from $\mathbb{Z}$ with $\mathbb{C}$. All the results above also hold for $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$. Additionally, a result similar to Theorem 3.1.14 states that $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$ is equivalent to $\mathbb{S}\mathrm{Bim}$ over $\widetilde{S}_2$ as graded is this graded? $\mathbb{C}$-linear monoidal categories.

Remark 4.1.18. The construction of the diagrammatic Hecke category, light leaves, Theorem 3.1.12 and Theorem 3.1.14 all generalise to general Coxeter groups. The details can be found in [EW16].

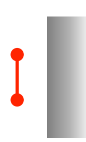
4.2 Diagrammatic $\mathrm{Tilt}(\mathfrak{sl}_2)$

Something something about Tilt

Something something about extending $\mathcal{H}(\widetilde{S}_2)$ from $\mathbb{Z}$ to $\mathbb{C}$.

Although need to work over $\mathbb{C}$ for $\mathrm{Tilt}(\mathfrak{sl}_2)$, the diagrammatic category below can be defined more simply over $\mathbb{Z}$.

Definition 4.2.1. Let $\mathcal{DT} := \mathcal{DT}(\mathfrak{sl}_2)$ be the $\mathbb{Z}$-linear left $\mathcal{H}(\widetilde{S}_2)$-module category with elements generated by the monoidal identity $\varnothing$ of $\mathcal{H}(\widetilde{S}_2)$, and morphisms generated by the empty diagram $\varnothing$. The action of $\mathcal{H}(\widetilde{S}_2)$ on the left is left concatenation for objects and morphisms. The relations on diagrams in $\mathcal{DT}(\mathfrak{sl}_2)$ are inherited from those in $\mathcal{H}(\widetilde{S}_2)$. Additionally, we imagine a wall on the right of diagrams and impose the local wall-annihilation relations

 $=$  $= 0.$ (W2)

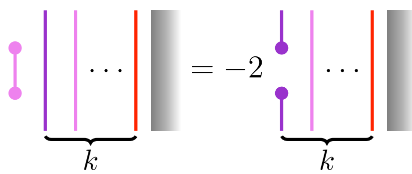
In other words, if a red barbell or blue string can come close to the wall without anything in between, then the diagram is 0. Note that local relations in (W2) involve the wall.

Example 4.2.2. The morphism in Example 4.1.3 collapses to 0 because all the diagrams have either blue or barbell on the right.

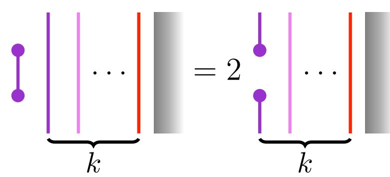
TODO: Another example clarifying 'blue on the right'

Using (W2) we can extend (R2).

Proposition 4.2.3. *In the following diagrams, the domain and codomain alternate colours and to the left of the ellipsis, pink and purple represent different colours in* $\{$*red*, *blue*$\}$*. For integers* $k \geq 1$

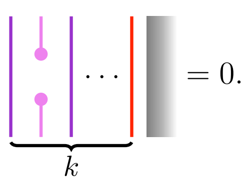


(4.2.3a)



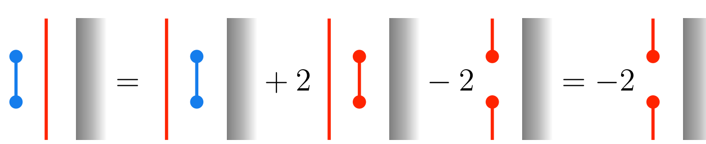
(4.2.3b)

and for $k \geq 3$

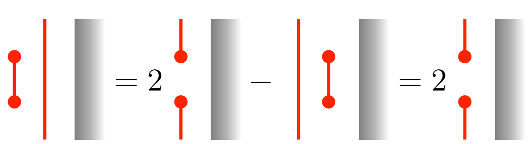


(4.2.3c)

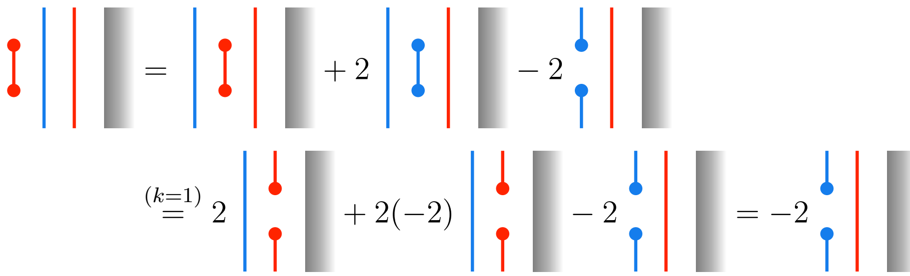
Proof. For $k \in \{1, 2\}$, we check the second two relations by hand. For $k = 1$, pulling the barbell through the line using (R1d) and (R2), then applying (W2) gives us



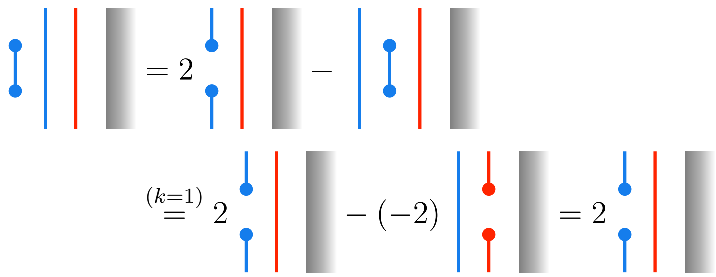
and



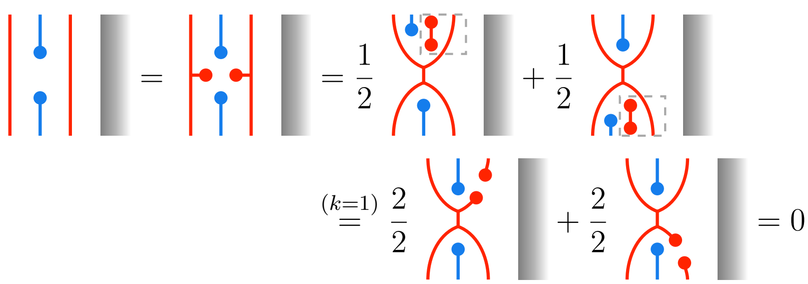
By a similar proof, using the $k = 1$ relations locally, we have for $k = 2$,



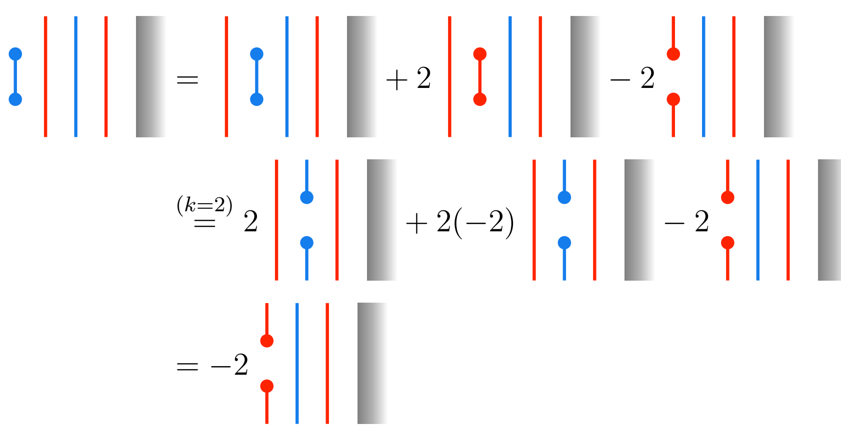
and



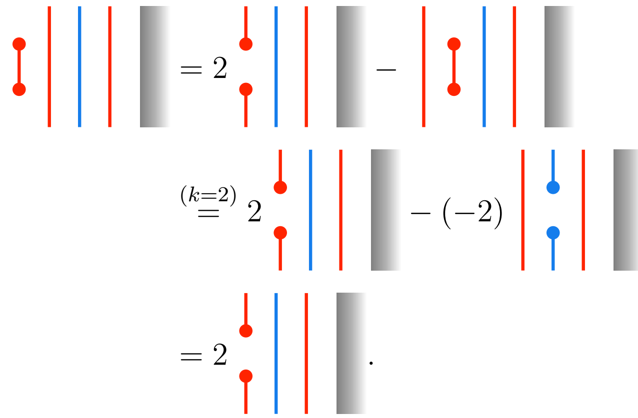
Now we proceed by induction on k. For $k = 3$ we first show (4.2.3c). By a similar argument to (3.2.6) we have



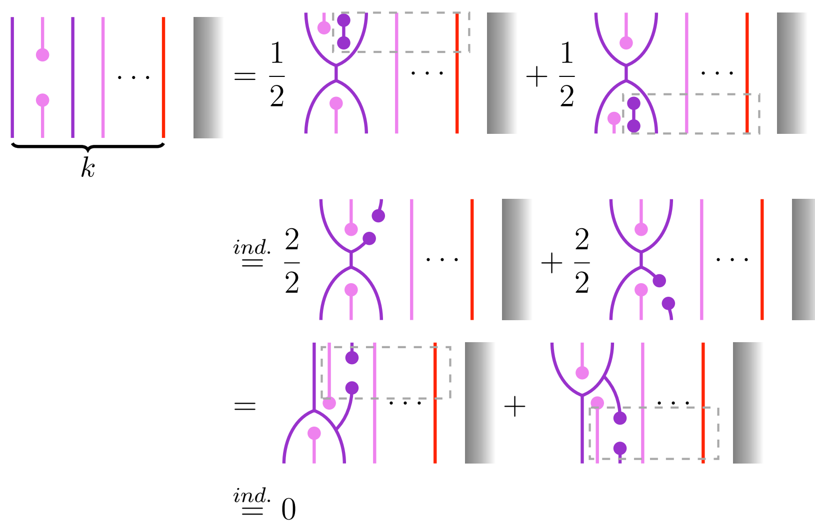
since the wall is accessible by the blue dot. Then



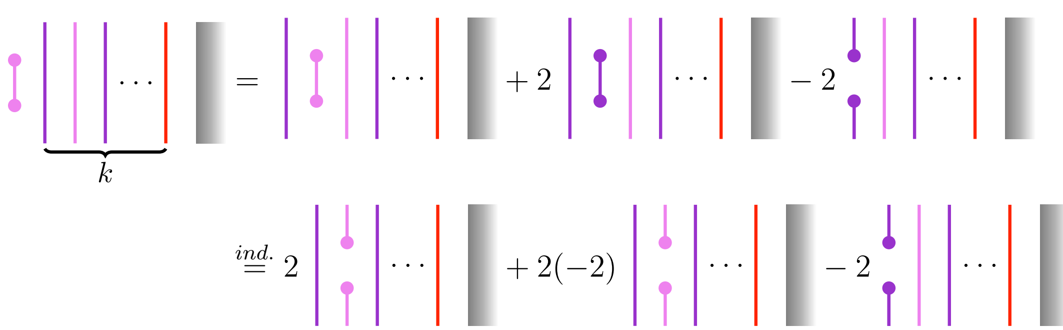
and

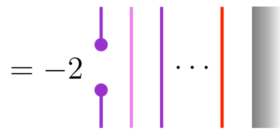


Let $k \geq 4$ and assume the relations hold for diagrams with $k-1, k-2, ..., 1$. Again, the argument to (3.2.6) implies

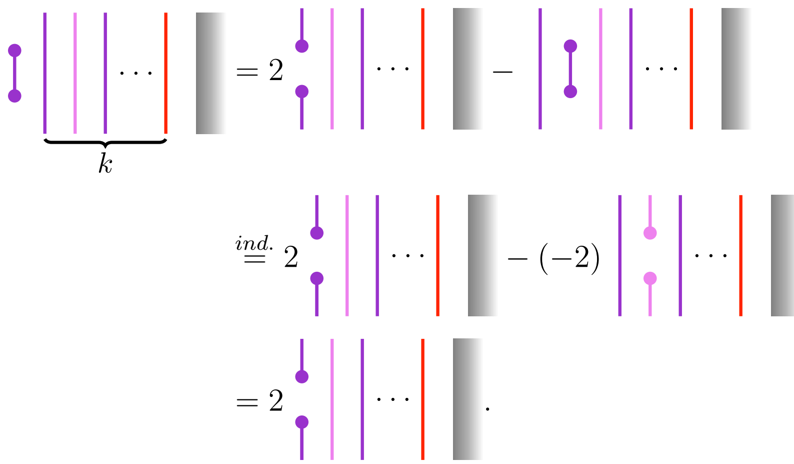


where the rightmost pink string is the right red string when $k = 4$. Furthermore, we have





and



∎

The objects of this category are identical to objects in $\mathcal{H}(\widetilde{S}_2)$ and the morphisms are the same modulo the wall relations (W2). Naturally, we wonder whether double leaves form bases for the morphism spaces in $\mathcal{DT}$. It is easy to see that double leaves appear in $\mathcal{DT}$ by acting on $\varnothing$ by double leaves in $\mathcal{H}(\widetilde{S}_2)$. All morphisms in $\mathcal{DT}$ are morphisms in $\mathcal{H}(\widetilde{S}_2)$ so they can be written as $\mathbb{Z}[\;,\;]$-linear combinations of double leaves, though some of these leaves have collapsed to 0. This makes it clear that double leaves span the morphism spaces of $\mathcal{DT}$ as (left) $\mathbb{Z}[\;,\;]$-modules. However they may not be linearly independent as neither left nor right modules as with the one-colour case. Although double leaves are not always a basis for its respective morphism space as $\mathbb{Z}[\;,\;]$-modules, it turns out a subset of them are a basis over $\mathbb{Z}$.

Lemma 4.2.6. *Let $\pi : \mathrm{mor}(\mathcal{H}(\widetilde{S}_2)) \to \mathrm{mor}(\mathcal{DT})$ be the projection map which takes a morphism to the result of its action on the empty diagram $\varnothing$. Then the image $\pi(\mathbb{LL}(w,y))$ without zero morphisms is a basis for $\mathrm{Hom}_{\mathcal{DT}}(w,y)$ as a $\mathbb{C}$-module.*

Proof. Consider morphisms $\mathrm{Hom}(w,y)$ in $\mathcal{DT}$ for fixed objects w, y, and write $\mathbb{LL} := \pi(\mathbb{LL}(w,y))$ for the set of double leaves in $\mathcal{DT}$. Any diagram in $\mathcal{DT}$ can be written as a $\mathbb{C}$-linear combination of morphisms without floating diagrams by pulling floating diagrams to the right with (R1d) and (R2) then applying the wall relation (W2). We can write each of these as a $\mathbb{Z}[\;,\;]$-linear combination of double leaves with a right action, and reduce it to a $\mathbb{Z}$-linear combination by (W2). This implies that $\mathbb{LL}$ spans $\mathrm{Hom}(w,y)$ as a $\mathbb{Z}$-module. Now $\mathbb{LL}$ may not be linearly independent because the two-colour wall relation (W2) reduces all diagrams factoring through a word ending with t

to 0. The set of light leaves after removing morphisms killed by (W2), i.e. $\mathbb{LL} \setminus \{0\}$, still spans $\mathrm{Hom}(w, y)$ by the argument above. This set is linearly independent since, by construction, (W2) has no effect on $\mathbb{Z}$-linear combinations of $\mathbb{LL} \setminus \{0\}$. Then it follows from linear independence over $\mathbb{Z}[\ ,\ ]$ that this set is linearly independent over $\mathbb{Z}$ in $\mathcal{DT}$. $\square$

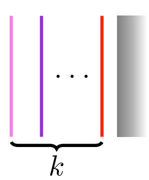
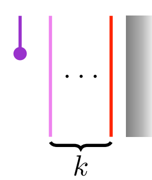
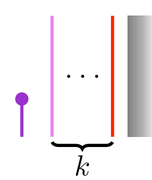
Since there exists light leaves with unbroken red strands on the right, this lemma implies that our category does not collapse by adding the module category structure and the wall relation (W2).

We aim to show that the additive Karoubi envelope of this diagrammatic category is equivalent to $\mathrm{Tilt}(\mathfrak{sl}_2)$. From now on, we write $\mathcal{DT}$ for the $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$-module category obtained by extending scalars from $\mathbb{Z}$ with $\mathbb{C}$. All the above discussion and results still apply to $\mathcal{DT}$ over $\mathbb{C}$. For brevity we also write $\mathcal{T}$ for $\mathrm{Tilt}(\mathfrak{sl}_2)$.

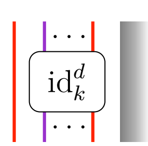
Since $\mathcal{H}_{\mathbb{C}}(S_2)$ appears inside $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$ for each colour, Lemma 3.2.5 provides explicit isomorphisms $ss \cong s \oplus s$ and $tt \cong t \oplus t$ in the additive closure of $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$.

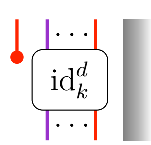
Definition 4.2.7. Let $F : \mathcal{DT}^{\oplus} \to \mathrm{Tilt}(\mathfrak{sl}_2)$ to be the additive $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$-module functor defined as follows. Map the empty word $\varnothing$ to the trivial module $T(\varnothing)$ Is that right to say?. Given a general word $s_n \dots s_1$ in $\mathcal{DT}$, for $s_i \in \{s, t\}$, map $F(s_n \dots s_1) = \Theta_{s_n} \dots \Theta_{s_1} T(\varnothing)$ where Θ_s, Θ_t are translation functors associated to generators of $\widetilde{S}_2$.

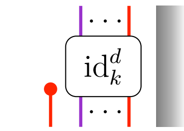
On morphisms, we define F recursively. Note that we only have red strands on the right since otherwise (W2) reduces it to 0. For $k \geq 0$, define

$\mathrm{id}_k^d :=$  , $i_k^d :=$  , $p_k^d :=$ 

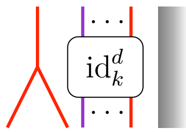
where colours alternate and a red strand on the right when $k \neq 0$. Similarly for $k \geq 0$, we define $\mathrm{id}_k : \Theta_x \dots \Theta_s(T(\varnothing)) \to \Theta_x \dots \Theta_s(T(\varnothing))$, $i_k : \Theta_x \dots \Theta_s(T(\varnothing)) \to \Theta_y\Theta_x \dots \Theta_s(T(\varnothing))$ and $p_k : \Theta_y\Theta_x \dots \Theta_s(T(\varnothing)) \to \Theta_x \dots \Theta_s(T(\varnothing))$ to be the identity, inclusion and projection maps in $\mathcal{T}$, where the subscripts alternate s, t and $\Theta_x \dots \Theta_s$ is a composition of k translation functors. Further we write $\tilde{p}_k := (-1)^{k+1}\frac{1}{2^{k+1}} p_k$. Let $F(\mathrm{id}_k^d) = \mathrm{id}_k$. On the generators (G2) of $\mathcal{DT}$, map

 $\overset{F}{\longmapsto} \begin{cases} \mathrm{id}_{k+1}, & \text{if } k \text{ even}, \\ \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & \mathrm{id}_k \end{pmatrix}, & \text{if } k \text{ odd}, \end{cases}$

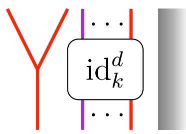
 $\overset{F}{\longmapsto} \begin{cases} \tilde{p}_k, & \text{if } k \text{ even}, \\ \begin{pmatrix} i_{k-1} \circ \tilde{p}_{k-1} \\ \mathrm{id}_k \end{pmatrix}, & \text{if } k \text{ odd}, \end{cases}$



$$\xmapsto{F} \begin{cases} i_k, & \text{if } k \text{ even},\\ \begin{pmatrix}\mathrm{id}_k & i_{k-1}\circ\tilde{p}_{k-1}\end{pmatrix}, & \text{if } k \text{ odd},\end{cases}$$

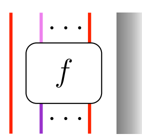


$$\xmapsto{F} \begin{cases} \begin{pmatrix}0 & \mathrm{id}_{k+1}\end{pmatrix}, & \text{if } k \text{ even},\\ \begin{pmatrix}0 & 0 & \mathrm{id}_k & 0\\ 0 & 0 & 0 & \mathrm{id}_k\end{pmatrix}, & \text{if } k \text{ odd},\end{cases}$$



$$\xmapsto{F} \begin{cases} \begin{pmatrix}\mathrm{id}_{k+1}\\ 0\end{pmatrix}, & \text{if } k \text{ even},\\ \begin{pmatrix}\mathrm{id}_k & 0\\ 0 & \mathrm{id}_k\\ 0 & 0\\ 0 & 0\end{pmatrix}, & \text{if } k \text{ odd},\end{cases}$$

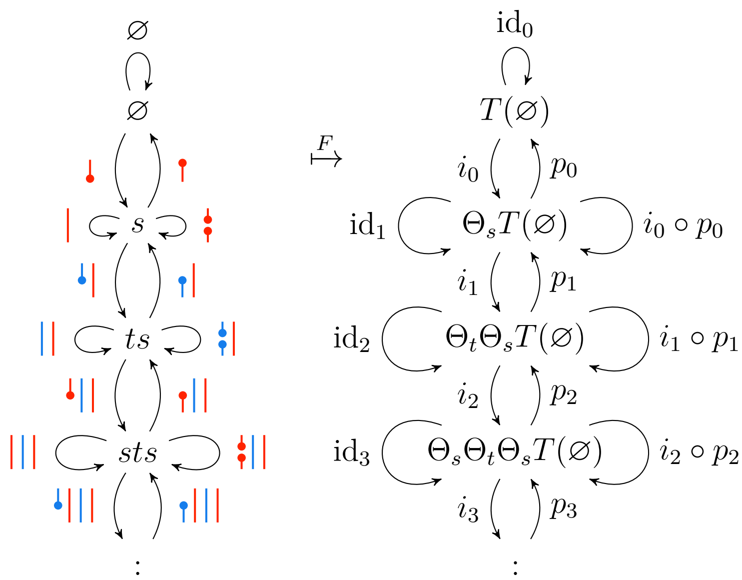
where each entry in the matrix are matrices themselves. CHECK THE SCALING!!!!!!! For blue generators, the definition is the same with the words even and odd swapped. Putting a red (resp. blue) identity strands on the left of a diagram is applying Θ_s (resp. Θ_t) to the output morphism. Pictorially, for a morphism f in $\mathcal{DT}$,



$$\xmapsto{F} \Theta_s F(f).$$

We extend the functor by composition, additivity and linearity.

The mappings that don't involve matrices are summarised in the picture below.



(4.2.8)

The right wall on each diagram is not shown to reduce clutter.

The definition on generators is a consequence of the isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ analogous to $ss \cong s \oplus s$ (and respectively for t) from Lemma 3.2.5.

Remark 4.2.9. The action of an arbitrary morphism of $\mathcal{H}(\widetilde{S}_2)$ on the left of a morphism in $\mathcal{DT}$ is sent to the Godement product[2] of the natural transformations underlying the image of morphisms under F. Taking the Godement product of natural transformations $\Theta_x \ldots \Theta_s \to \Theta_y \ldots \Theta_s$, when viewed as diagrams in $\mathcal{DT}$, is just a left tensor of the corresponding diagrams. Visually, the construction of looks like putting identity morphisms on the left of one morphism on the right of the other, so that the codomains align, and then composing them. In $\mathcal{T}$, this is the Kronecker product of matrices.

Lemma 4.2.10. *The functor F is well defined as a functor between additive $\mathbb{C}$-linear $\mathcal{H}(S_2)$-module categories.*

Proof. By Remark 4.2.9, the definition of F defines an action of every morphism in $\mathcal{DT}^{\oplus}$. It remains to check that all relations are preserved. It follows from [AT17, Proposition 2.35] Check ref translation functors Θ_s, Θ_t are Frobenius objects in the category of endofunctors of $\mathcal{T}$ and there are unit, counit, multiplication and comultiplication natural transformations and corresponding relations from the Frobenius object structure. Applying these to $T(\varnothing)$ result in Frobenius object relations in $\mathcal{T}$ for $T(\varnothing), \Theta_s T(\varnothing)$ and $\Theta_s^2(T(\varnothing))$. Note that and map to i_0 and $\tilde{p}_0$ Is the scaling here right? which are exactly the unit and counit of Θ_s evaluated at $T(\varnothing)$ (up to scaling), and the trivalent vertices defined with id_0^d are mapped exactly to the multiplication and comultiplication maps. The isomorphism Lemma 3.2.5 we use to reduce domain and codomain has an analogue $\Theta_s \circ \Theta_s \cong \Theta_s \oplus \Theta_s$ as in [AT17, Corollary 2.35(a)], and similarly for t. Furthermore, in [AT17, Proposition 2.30] we see that $p_0 \circ i_0 = 0$, $p_{k+1} \circ i_{k+1} = i_k \circ p_k$ that are analogous to the relations in Proposition 4.2.3, up to an adjusting scalar given in the definition. From [AT17, Corollary 2.35] the translation functors satisfy properties analogous to the two-colour wall relations (W2). Checking that the remaining relations (R1c), (R1d), (R2) and (W2) hold in $\mathcal{T}$ is straightforward, see [AT17, Lemma 4.26]. Hence all the relations in $\mathcal{DT}$ are preserved by F. By construction, F preserves direct sums, $\mathbb{C}$-linear combinations and the Soergel module structure, so F is well defined as a functor between additive $\mathbb{C}$-linear $\mathcal{H}(S_2)$-module categories.

Talk about how extending by composition makes sense. □

Say something here?

The following result states that $\mathcal{DT}$ is indeed a diagrammatic incarnation of $\mathrm{Tilt}(\mathfrak{sl}_2)$.
Be clear that I don't understand Tilt very well.

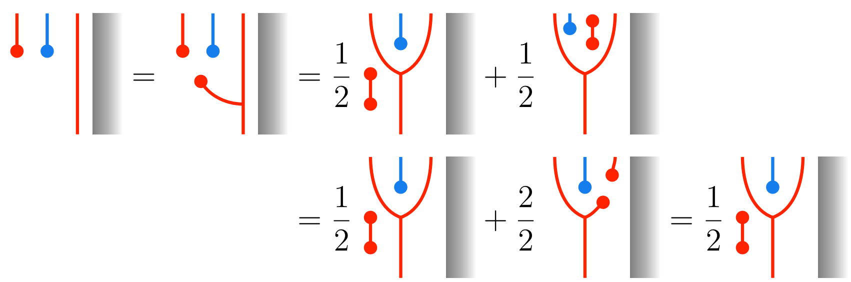
Theorem 4.2.11 (Andersen–Tubbenhauer, [AT17, Theorem 4.27])**.** *The diagrammatic category* $\mathrm{Kar}^{\oplus}(\mathcal{DT}(\mathfrak{sl}_2))$ *and* $\mathrm{Tilt}(\mathfrak{sl}_2)$ *are equivalent as additive Karoubian $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$-module categories.*

Proof. Since $\mathcal{T}$ is additive and Karoubian, our functor F extends uniquely to an additive functor $F' : \mathrm{Kar}^{\oplus}(\mathcal{DT}) \to \mathcal{T}$. By the argument in [AT17, Theorem 4.27], every element

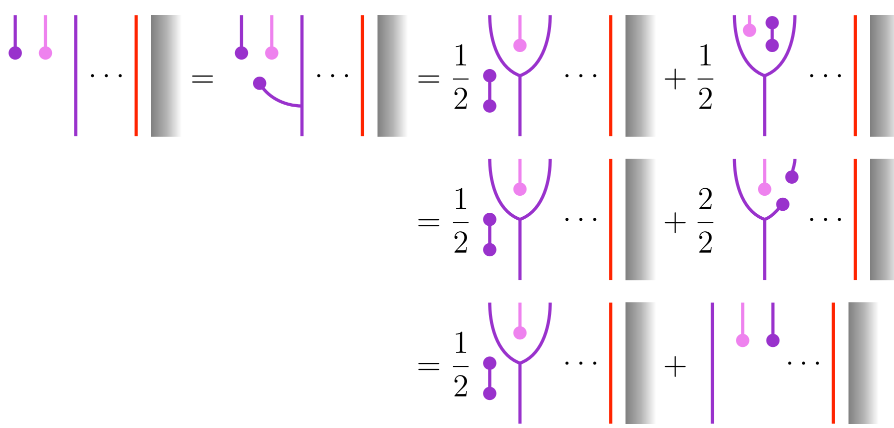
[2] The horizontal composition of natural transformations.

in $\mathcal{T}$ is isomorphic to F' applied to a direct sum of images of Jones–Wenzl projectors, so F' is essentially surjective. Particularly, this shows that the images of JW-projectors map exactly to the indecomposable "leading" tilting modules.

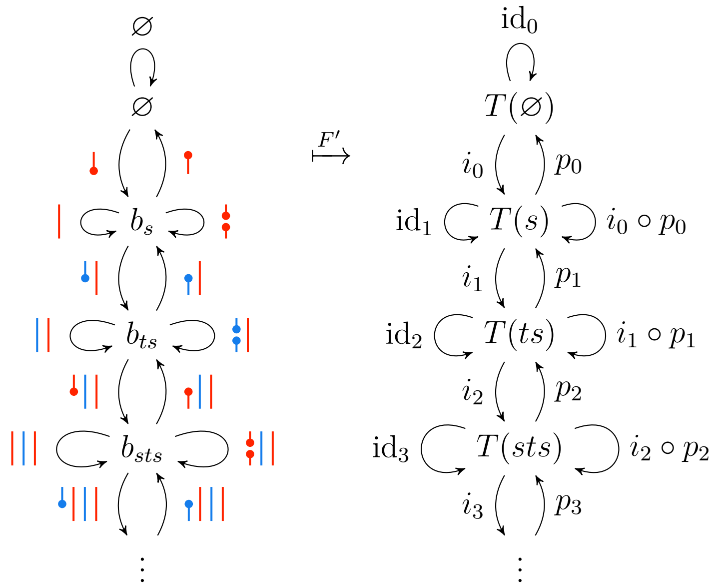
By Lemma 3.2.5 and (W2), we just consider words with alternating generators and ending with s. Write $T(\ldots ts)$ for the leading indecomposable summand of $\ldots\Theta_t\Theta_s(T(\varnothing))$ in $\mathcal{T}$, and write $b_{\ldots ts}$ for the image of $\mathrm{JW}_{\ldots ts}$. By [Eli16, Section 5.4.2], Jones–Wenzl projectors are primitive idempotents and their images are all the indecomposables in $\mathcal{DT}$, and as mentioned above they map to the leading indecomposables in $\mathcal{T}$. For full and faithfulness, it is sufficient to check that the dimensions of the morphism spaces between indecomposables $\mathrm{Hom}_{\mathcal{DT}}(b_{x\ldots ts}, b_{y\ldots ts})$ and $\mathrm{Hom}_{\mathcal{T}}(T(x\ldots ts), T(y\ldots ts))$ coincide. On the diagrammatic side, a morphism $b_{x\ldots ts} \to b_{y\ldots ts}$ is given by $\mathrm{JW}_{y\ldots ts}\, f\, \mathrm{JW}_{x\ldots ts}$ where $f : x\ldots ts \to y\ldots ts$. Since morphisms can be written as a linear combination of double leaves, we consider f to be a double leaf. By Proposition 4.1.9, all double leaves in which pitchfork appear on the top or bottom of the diagram are killed. Since the domain and codomain alternate colours, the remaining diagrams are a tensor and composition of , and identity strands. Notice that we have the relation



Proceeding inductively on the number of identity strands on the right, we have



where the third equality follows from Proposition 4.2.3. By induction the second term is a linear combination of diagrams with pitchforks, hence this diagram is a linear combinations of diagrams with pitchforks. Particularly, these are killed by Jones–Wenzl projectors. The same holds for the vertically reflected diagram. Along with Proposition 4.2.3, we conclude that the only double leaves we should consider are id_k, i_k^d, p_k^d and their composition $i^d_k \circ p^d_k$. This is informally summarised by the diagram below (similar to (4.2.8)).



(4.2.12)

Although not drawn, all the diagrams are flanked by Jones–Wenzl projectors, and the matching morphisms in $\mathcal{T}$ are pre and post-composed with the idempotents corresponding to the appropriate JW-projectors. Putting JW-projectors above and below any of these diagrams clearly do not result in zero. Moreover, in the endomorphism space of each non-trivial indecomposable, the morphisms id^d_k and $i^d_{k-1} \circ p^d_{k-1}$, with JW-projectors before and after, can easily be checked to be linearly independent. Hence the bases for the spaces can be read off the picture (4.2.12). In $\mathcal{T}$, the analogous bases for the morphism spaces in [AT17, Corollary 2.3.1] have matching dimensions, hence F' is fully faithful. Therefore the categories $\mathrm{Kar}^{\oplus}(\mathcal{DT})$ and $\mathrm{Tilt}(\mathfrak{sl}_2)$ are equivalent as additive Karoubian $\mathbb{C}$-linear $\mathcal{H}_{\mathbb{C}}(\widetilde{S}_2)$-module categories. □

Comment on grading?
Comment on consequences
Say similaritied with proj(O)

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