Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

Victor Zhang
UNSW Australia

April 9, 2023

Contents

Chapter 1

Introduction

This page was empty.

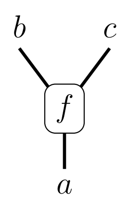
Chapter 2

Background

2.1 Drawing Monoidal Categories

A monoidal category $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold[1]. The bifunctor $\otimes$ is called the *tensor* or *monoidal product*. A monoidal category is *strict* if $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ and $A = \mathbb{1} \otimes A = A \otimes \mathbb{1}$ for objects and similarly for morphisms. In this paper, we will assume that monoidal categories are strict, since all monoidal categories are monoidally equivalent to a strict one[2].

The morphisms of $\mathcal{C}$ can be drawn as string diagrams, where the morphism maps from the bottom to the top. Functions that make up the morphism are drawn as tokens or boxes. For example



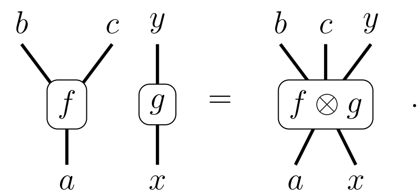
depicts a morphism $f : a \to b \otimes c$. For identity morphisms we drop the box and only draw a vertical line, so id_a is the diagram

.

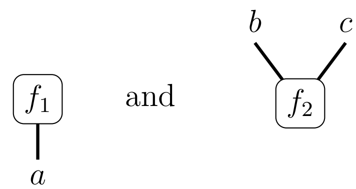
The tensor product of morphisms is the horizontal concatenation of diagrams, such that strings from separate functions don't interact. For example, given $g : x \to y$, the tensor product $f \otimes g : a \otimes x \to b \otimes c \otimes y$ is drawn as

[1] For more details see [Eti+15].

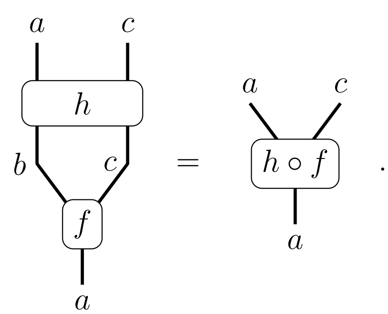
[2] See [ML98, VII.2] or [Eti+15, Thm 2.8.5]



By convention, $\mathbb{1}$ is blank and unlabelled, and strings that would join to $\mathbb{1}$ are blank. Particularly, $\mathrm{id}_{\mathbb{1}}$ is an empty diagram, and we have diagrams such as



for morphisms $f_1 : a \to \mathbb{1}$ and $f_2 : \mathbb{1} \to b \otimes c$. The compositions of morphisms is the vertical stacking of diagrams where domains and codomains match. For example, the composition $h \circ f : a \to b \otimes c \to a \otimes c$ of $f : a \to b \otimes c$ with $h : b \otimes c \to a \otimes c$ has the diagram



Before looking at our main example of a diagrammatic monoidal category, we first define some terminology.

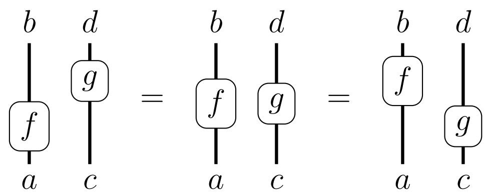
Definition 2.1.1. For a commutative ring R, an *R-linear category* is a category enriched over the category of R-modules. That is, for objects a, b, the set of morphisms $\mathrm{Hom}(a, b)$ is an R-module and the composition of morphisms is R-bilinear. A *R-linear monoidal category* is a category that is both monoidal and R-linear such that the monoidal product on morphisms is R-bilinear.

Example 2.1.2. Let $\mathbb{k}$ be a field. The category of vector spaces over $\mathbb{k}$, $\mathbf{Vect}_{\mathbb{k}}$, is a $\mathbb{k}$-linear monoidal category. This makes sense by the classical theory of linear algebra.

For a monoidal category $\mathcal{C}$, the bifunctoriality of $- \otimes -$ implies the following *interchange law*. For morphisms $f : a \to b$ and $g : c \to d$, $(\mathrm{id}_b \otimes g) \circ (f \otimes \mathrm{id}_c) = f \otimes g = (f \otimes \mathrm{id}_d) \circ (\mathrm{id}_a \otimes g)$. In other words the following diagram commutes.

$$\begin{tikzcd}
a \otimes c \arrow[r, "f\otimes \mathrm{id}_c"] \arrow[d, "\mathrm{id}_a \otimes g"'] \arrow[dr, "f\otimes g"] & b \otimes c \arrow[d, "\mathrm{id}_b \otimes g"] \\
a \otimes d \arrow[r, "f\otimes \mathrm{id}_d"] & b \otimes d
\end{tikzcd}$$

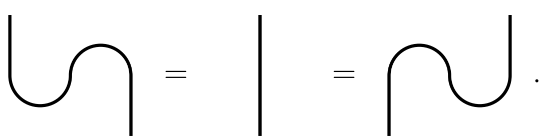
Written with string diagrams, this is



which holds up to deformation of the diagram.

Definition 2.1.3. A monoidal category $\mathcal{C}$ is *generated* by finite set S_o of objects and S_m of morphisms, when all non-unit objects are a finite tensor of objects in S_o and all non-identity morphisms are a finite combination of tensors and compositions of morphisms in S_m.

Example 2.1.4. Our first example of a diagrammatic monoidal category is the *Temperley-Lieb category*. The Temperley-Lieb category $\mathcal{TL}$ is a strict R-linear monoidal category whose objects are generated by the vertical line I and morphisms generated by the cup $\cup : \mathbb{1} \to \mathsf{I} \otimes \mathsf{I}$ and cap $\cap : \mathsf{I} \otimes \mathsf{I} \to \mathbb{1}$, with relations



Mention that composition and tensor product is as explained above
Some example
Mention bubbles and specialisation to some $\delta \in R$
Mention that these are crossingless matchings
Comment on isotopy

2.2 Module Categories

2.3 Frobenius Objects

Something something about Many relations in categorical structures can be written in diagrammatic terms - adjunctions, monoid

Let $\mathcal{C}$ be a (strict) monoidal category. We can define the following objects.

Definition 2.3.1. A *monoid object* in $\mathcal{C}$ is a triple (M, μ, η) for an object $M \in \mathcal{C}$, a *multiplication* map $\mu : M \otimes M \to M$ and a *unit* map $\eta : \mathbb{1} \to M$, such that

$$\begin{tikzcd}
 & M \otimes M \otimes M \arrow[dl, "\mu\otimes\mathrm{id}_M"'] \arrow[dr, "\mathrm{id}_M \otimes\mu"] & \\
M \otimes M \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\
 & M &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{1} \otimes M \arrow[r, "\eta\otimes\mathrm{id}_M"] \arrow[dr, "\mathrm{id}_M"'] & M \otimes M \arrow[d, "\mu"] & M \otimes \mathbb{1} \arrow[l, "\mathrm{id}_M \otimes\eta"'] \arrow[dl, "\mathrm{id}_M"] \\
 & M &
\end{tikzcd}$$

commute. The first diagram is the *associativity* relation $\mu \circ (\mu \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \mu)$ and the second diagram is the *unit* relation $\mathrm{id}_M = \mu \circ (\eta \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \eta)$.

Dually, a *comonoid* object in $\mathcal{C}$ is a triple (M, δ, ϵ) for an object $M \in \mathcal{C}$, a *comultiplication* map $\delta : M \to M \otimes M$ and a *counit* map $\epsilon : M \to \mathbb{1}$, satisfying the *coassociativity* relation

$$\begin{tikzcd}
 & M \otimes M \otimes M & \\
M \otimes M \arrow[ur, "\delta\otimes\mathrm{id}_M"] & & M \otimes M \arrow[ul, "\mathrm{id}_M \otimes\delta"'] \\
 & M \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

and *counit* relation

$$\begin{tikzcd}
\mathbb{1} \otimes M & M \otimes M \arrow[l, "\epsilon\otimes\mathrm{id}_M"'] \arrow[r, "\mathrm{id}_M \otimes\epsilon"] & M \otimes \mathbb{1} \\
 & M \arrow[ul, "\mathrm{id}_M"] \arrow[u, "\delta"] \arrow[ur, "\mathrm{id}_M"'] &
\end{tikzcd} .$$

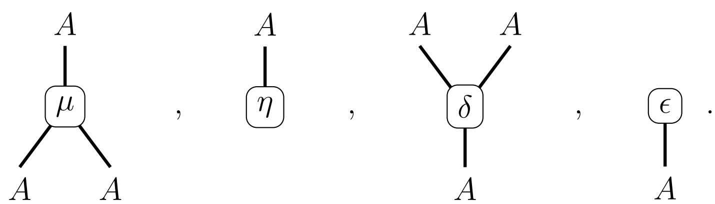
Monoid objects generalise monoids, i.e. sets with an identity equipped with an associative binary operation.

Definition 2.3.2. A Frobenius object in $\mathcal{C}$ is a quintuple $(A, \mu, \eta, \delta, \epsilon)$ such that (A, μ, η) is a monoid object, (A, δ, ϵ) is a comonoid object, and the maps satisfy the *Frobenius relations*

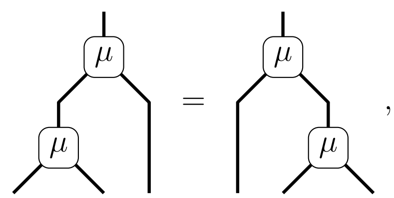
$$
\begin{tikzcd}
 & A \otimes A \arrow[dl, "\delta \otimes \mathrm{id}_A"'] \arrow[d, "\mu"] \arrow[dr, "\mathrm{id}_A \otimes \delta"] & \\
A \otimes A \otimes A \arrow[dr, "\mathrm{id}_A \otimes \mu"'] & A \arrow[d, "\delta"] & A \otimes A \otimes A \arrow[dl, "\mu \otimes \mathrm{id}_A"] \\
 & A \otimes A &
\end{tikzcd}\quad,
$$

that is $(\mathrm{id}_A \otimes \mu) \circ (\delta \otimes \mathrm{id}_A) = \delta \circ \mu = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta)$.

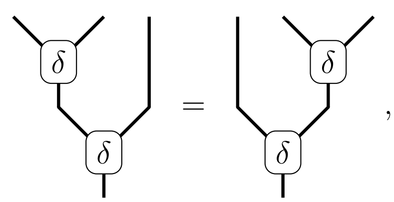
The maps and relations for a Frobenius object $(A, \mu, \eta, \delta, \epsilon)$ have a nice description with the diagrams given in Section 2.1. The structure maps are drawn as



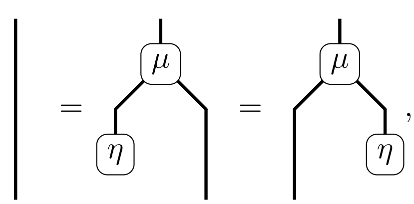
For the rest of this section, we only work with the Frobenius object A and $\mathbb{1}$, so we can identify the identity strand $| = \mathrm{id}_A$ so we can stop putting the label A. Diagrammatically, the associativity relation $\mu \circ (\mu \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \mu)$ is



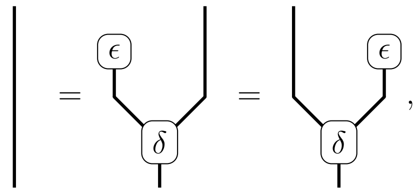
the coassociativity relation $(\delta \otimes \mathrm{id}_A) \circ \delta = (\mathrm{id}_A \otimes \delta) \circ \delta$ is



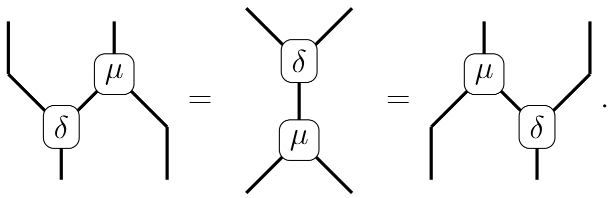
the unit relation $\mathrm{id}_A = \mu \circ (\eta \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \eta)$ is



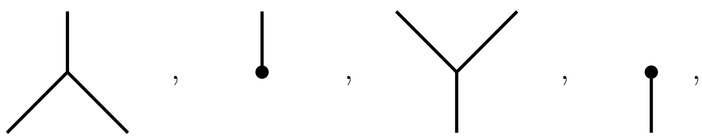
the counit relation $\mathrm{id}_A = (\epsilon \otimes \mathrm{id}_A) \circ \delta = (\mathrm{id}_A \otimes \epsilon) \circ \delta$ is



and the Frobenius relation $(\mathrm{id}_A \otimes \mu) \circ (\delta \otimes \mathrm{id}_A) = \delta \circ \mu = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta)$ is



If we stop labelling the functions and draw the structure maps as



then the relations become... Talk about isotopy

Maybe something about the (diagrammatic?) category Frob, capturing the data of a frobenius object

Chapter 3

One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = e \rangle$. At the end of this section, we see that this diagrammatic category is equivalent to the category of Soergel Bimodules under additive Karoubian closure.

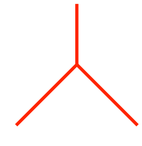
Remark 3.1.1. All diagrammatics below and in Chapter 4 can be defined in the language of planar algebras, without mentioning (monoidal) categories, e.g. in [Jon21]. Nevertheless, we define them in the context of categories as we will see them as diagrammatic versions of important categories in representation theory.

What do we do about $\mathbb{C}$? Do the theorems (at the end) apply over $\mathbb{Z}$ or $\mathbb{C}$ or both? If we define over $\mathbb{Z}$, how do we use it over $\mathbb{C}$ for the next section?

Definition 3.1.2. The *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ is a $\mathbb{Z}$-linear monoidal category with the following presentation.

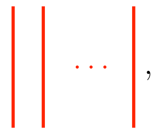
The objects are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is concatenation. The empty tensor product, i.e. the monoidal identity, will be denoted $\varnothing =: s^0$.

The morphisms are generated, up to isotopy, by univalent and trivalent vertices

 ,  (3.1.3)

that are maps $s \to \varnothing$ and $ss \to s$ respectively. Note that we put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). The free $\mathbb{Z}$-module structure on each morphism space $\operatorname{Hom}(s^n, s^m)$ produces $\mathbb{Z}$-linear combinations of such diagrams.

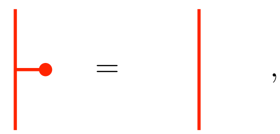
Something about composition/tensor and addition commuting Then, composition or tensors with the zero morphism 0 result in 0. To abuse notation, the empty diagram $\varnothing \to \varnothing$ will be denoted $\varnothing$. The identity morphism in $\mathrm{Hom}(s^n, s^n)$ is the diagram consisting of n (red) vertical lines



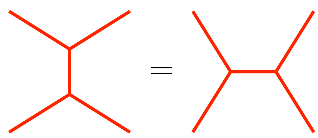
(3.1.4)

which we may identify with s^n.

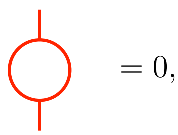
Such diagrams are subject to the following local relations



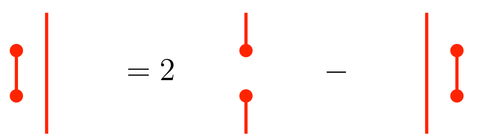
(3.1.5a)



(3.1.5b)



(3.1.5c)

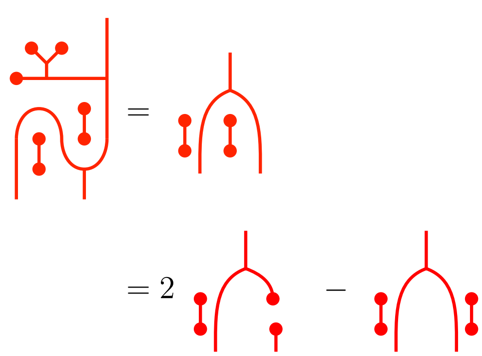


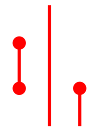
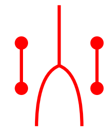
(3.1.5d)

Remark 3.1.6. The object s is a Frobenius object in $\mathcal{H}(S_2)$. The generators (3.1.3) and their horizontal reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.5a) and (3.1.5b).

Put a definition of frob object in intro

Example 3.1.7. Using the relations in (3.1.5) we can simplify the morphism in $\mathrm{Hom}(ss, s)$,



$= 2$  $-$  .

Add example of using frob associativity

The morphism space $\mathrm{Hom}(s^n, s^m)$ has a left (or right) $\mathbb{Z}[$$]$-basis called the *double leaves* basis, as described in [EW16]. To define this basis, we must first define morphisms known as *light leaves*.

To make use of the group structure of S_2, we need to translate between words in $\mathcal{H}(S_2)$ and elements in S_2. Let $\phi : (\mathrm{ob}(\mathcal{H}(S_2)), \otimes) \to (S_2, *)$ be the monoid homomorphism[1] mapping $s \mapsto s$ and $\varnothing \mapsto 1$, and $\psi : S_2 \to \mathrm{ob}(\mathcal{H}(S_2))$ be the function that maps $s \mapsto s$ and $1 \mapsto \varnothing$. Should this be a definition? The maps ϕ allows words $w = s^n$ to be seen as elements of S_2, and ψ allows $1, s \in S_2$ to be seen as the objects $\varnothing, s \in \mathcal{H}(S_2)$. Clearly, $\phi\psi$ is the identity map on S_2, and the map $\psi\phi : \mathcal{H}(S_2) \to \mathcal{H}(S_2)$ takes objects to one of $\varnothing$ or s in $\mathcal{H}(S_2)$ by considering them as elements in S_2.

Definition 3.1.8. (Subexpression for S_2) Given a word $w = s^n$, a *subexpression* e is a binary string of length n. We can *apply* a subexpression to produce an object $w(e) \in \mathcal{H}(S_2)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $0 \le i \le n$, write $w(e, i)$ for the resultant object of the first i terms in e applied to the first i terms in w. Particularly $w(e, 0) = \varnothing$ and $w(e, n) = w(e)$.

For example, $0000, 0110$ and 1011 are subexpressions of $s^4 = ssss$. Applying the third subexpression gives $ssss(1011) = s\varnothing ss = sss$ and $ssss(1011, 3) = sss(101) = s\varnothing s = \varnothing$, by strictness of the monoidal category. Here, each term of the subexpression is a decision to include or exclude the corresponding s in the word, where excluding an s amounts to tensoring with $\varnothing$.

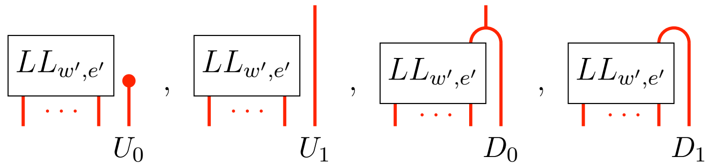
For a word w and subexpression e, we label each term by U_0, U_1, D_0 or D_1. The i-th term is labelled U_* if $\phi(w(e, i-1)) = 1 \in S_2$, and labelled D_* if $\phi(w(e, i-1)) = s \in S_2$. The label's subscript is the corresponding term in e.

Example 3.1.9. For the object $w = ssss$ and subexpression $e = 0101$, we find the labels as recorded in the following table.

Term i	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
$w(e, i)$	$\varnothing$	$\varnothing s = s$	$\varnothing s \varnothing = s$	$\varnothing s \varnothing s = ss$
Labels	U_0	$U_0 U_1$	$U_0 U_1 D_0$	$U_0 U_1 D_0 D_1$

Definition 3.1.10. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, the light leaf $LL_{w's,e'i}$ is one of

[1] A map that preserves the monoidal product and identity element.

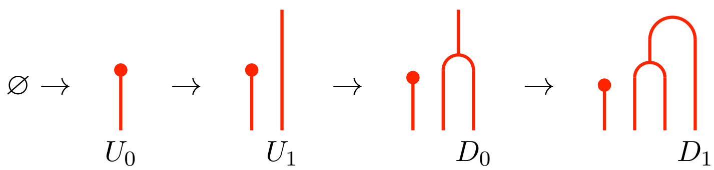


(3.1.11)

corresponding to the next label, where w' and e' are appropriate subwords[2] of w and e respectively.

Here, the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ is $\varnothing$, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. This implies that the recursive definition is consistent.

Example 3.1.12. Following from Example 3.1.9 for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.

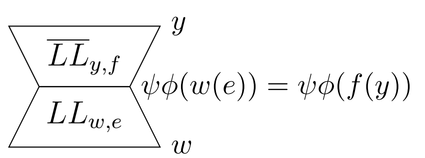


Definition 3.1.13. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y in $\mathcal{H}(S_2)$ is a composition

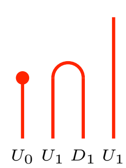
$$\mathbb{LL}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

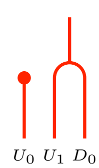
for subexpressions e of w and f of y such that $\psi\phi(w(e)) = \psi\phi(f(y))$.

Visually these are diagrams from w to y factoring through $\psi\phi(w(e)) = \psi\phi(f(y)) \in \{\varnothing, s\}$,



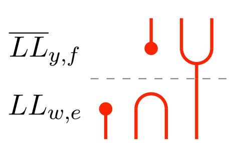
Example 3.1.14. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

.

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : ssss \to sss$, factoring through s, is



[2] A word with some letters removed.

Note that these double leaves have no floating diagrams such as . In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{Z}[\ ]$-module, where the (left) -action is left concatenation by . Since we can move barbells to the right via. the relation (3.1.5d) and double leaves cut down the middle are double leaves factoring through $\varnothing$, we can equivalently act by $\mathbb{Z}[\ ]$ on the right. This leads us to the following theorem.

Theorem 3.1.15 (Elias-Williamson [EW16, Theorem 1.2])**.** *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{LL}(w, y)$ be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $\psi\phi(w(e)) = \psi\phi(y(f))$. Then $\mathbb{LL}(w, y)$ is a basis for $\mathrm{Hom}(w, y)$ as a left (or right) $\mathbb{Z}[\ ]$-module.*

A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.16. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky's construction in [Lib08].

The morphisms in this category can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1. The degree of a diagram is the sum of the degrees of the generators that appear in it. This induces a grading for the morphism spaces of $\mathcal{H}(S_2)$. Maybe mention what a grading is.

Put example

The double leaves bases allow us to show that the Karoubi envelope of $\mathcal{H}(S_2)$ is equivalent to the category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$ over S_2 as monoidal categories.

Theorem 3.1.17 (Elias-Williamson [EW16, Theorem 6.30])**.** *The category $\mathrm{Kar}_{\oplus}(\mathcal{H}(S_2))$ and the category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$ over S_2 are equivalent as graded $\mathbb{Z}$-linear monoidal categories.*

The proof in [EW16] gives an equivalence of graded $\mathbb{Z}$-linear monoidal categories $\mathcal{H}(S_2) \cong \mathbb{BS}\mathrm{Bim}$ where $\mathbb{BS}\mathrm{Bim}$ is the category of Bott-Samelson bimodules over S_2. This was done by comparing the graded dimensions of morphism spaces using double leaves bases. Since $\mathrm{Kar}_{\oplus}(\mathbb{BS}\mathrm{Bim}) \cong \mathbb{S}\mathrm{Bim}$ and Karoubi envelope preserves equivalences, we obtain $\mathrm{Kar}_{\oplus}(\mathcal{H}(S_2)) \cong \mathbb{S}\mathrm{Bim}$.

3.2 Diagrammatic $\mathcal{O}_0(\mathfrak{sl}_2)$

A little bit about category $\mathcal{O}$, and our example of $\mathfrak{sl}_2$

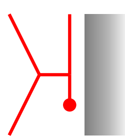
For this section, our category of interest is $\mathcal{O}$ for the semisimple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. A description of the category $\mathcal{O}$ can be found in general in [Hum08, Sections 3.8–3.10] or in [Maz09, Section 5.2] for the case of $\mathfrak{sl}_2(\mathbb{C})$, however we will only give a brief overview. The category $\mathcal{O}$ is a category of certain modules (or representations) over a semisimple Lie algebra. It is a direct sum of subcategories, where, in the case of $\mathfrak{sl}_2$ over $\mathbb{C}$, the non-trivial summands are equivalent as abelian categories to the subcategory $\mathcal{O}_0$. Within this, we look to the full subcategory $\mathrm{proj}(\mathcal{O}_0)$ of projective modules in $\mathcal{O}_0$, which is in particular additive and contains all direct summands.

In [Soe90, Section 2.4], Soergel shows that the category $\mathcal{O}$, and hence the subcategory $\mathrm{proj}(\mathcal{O}_0)$, is a Soergel module category, i.e. it has an action of the monoidal category $\mathbb{S}\mathrm{Bim}$. By the equivalence in Theorem 3.1.17 we will view $\mathrm{proj}(\mathcal{O})$ as a $\mathcal{H}(S_2)$-module category, extending via the additive Karoubi envelope. Since $\mathcal{H}(S_2)$ is diagrammatic, this action allows us to describe $\mathrm{proj}(\mathcal{O}_0)$ (thus essentially $\mathcal{O}_0$ and $\mathcal{O}$) diagrammatically.

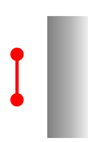
Remark 3.2.1. We can pass from $\mathrm{proj}(\mathcal{O}_0)$ to $\mathcal{O}_0$ by observing that $K^b(\mathrm{proj}(\mathcal{O}_0))$ is equivalent to $D^b(\mathcal{O}_0)$ as graded $\mathbb{Z}$-linear Should this be $\mathbb{C}$? monoidal triangulated categories. This is a standard trick in the field, for example see the introduction of [RW18][3]. However for our purposes it is not important to understand how this works.

Remark 3.2.2. For $\mathfrak{sl}_2(\mathbb{C})$, the morphism spaces in $\mathrm{proj}(\mathcal{O}_0)$ are $\mathbb{C}$-modules so the diagrammatic category $\mathcal{H}(S_2)$ from Section 3.1 must be extended from $\mathbb{Z}$ to $\mathbb{C}$. Formally this is just tensoring the morphism spaces on the left by the $\mathbb{C}$-$\mathbb{Z}$-bimodule $\mathbb{C}$, where the right action is induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$. For the remainder of this section, $\mathcal{H}(S_2)$ will refer to this $\mathbb{C}$-linear extension. Note that the above process does little to the category. In particular, double leaves in $\mathcal{H}(S_2)$ become $\mathbb{C}[\ ]$-bases[4] for the morphisms and the equivalence in Theorem 3.1.17 still holds.

Definition 3.2.3. Let $\mathcal{DO}_0(\mathfrak{sl}_2)$ be the $\mathbb{C}$-linear (Define this in background) left $\mathcal{H}(S_2)$-module category with elements generated (Define what this means.) by the monoidal identity $\varnothing$ of $\mathcal{H}(S_2)$ and morphisms generated by the empty diagram $\varnothing$. The action of $\mathcal{H}(S_2)$ on the left is left concatenation for both objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0,1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0,1] \times \mathbb{R}$. For example a morphism may be

 .

We impose the relation that diagrams are related to the wall by

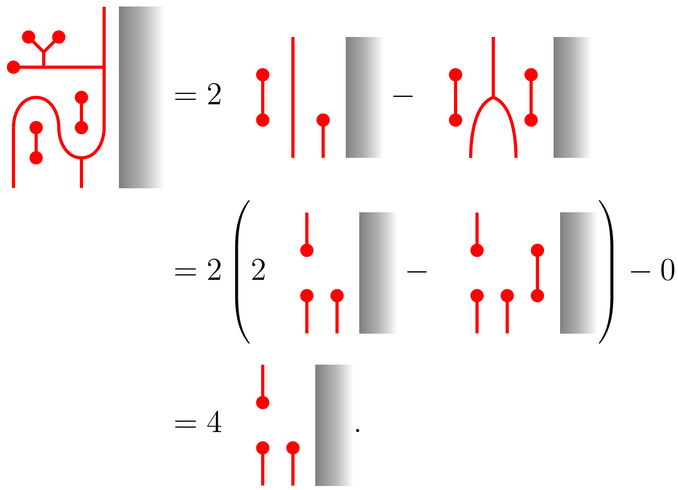
 $= 0.$ (3.2.4)

In this section we may write $\mathcal{DO}_0$ for this category. Talk about the $\mathbb{C}$-linear structure and how that works.

[3] A self-contained summary of how diagrammatic categories can be related to abelian categories.

[4] It is easy to see that double leaves in $\mathcal{H}(S_2)$...

Example 3.2.5. Using the new relation (3.2.4), we can further simplify the morphism in Example (3.1.7) by



The objects of this category are identical to objects in $\mathcal{H}(S_2)$ and the morphisms are the same modulo the wall relation (3.2.4). A natural question to ask is whether double leaves still form bases for the morphism spaces here. Notice that double leaves appear in $\mathcal{DO}_0$ by acting on $\varnothing$ by double leaves in $\mathcal{H}(S_2)$. All morphisms in $\mathcal{DO}_0$ are morphisms in $\mathcal{H}(S_2)$ so they can be written as $\mathbb{C}[\,]$-linear combinations of double leaves, though some have collapsed to 0. Thus double leaves span the morphism spaces of $\mathcal{DO}_0$ as (left) $\mathbb{C}[\,]$-modules. However they may not be linearly independent as neither left nor right modules. For example, any pair of double leaves that factor through $\varnothing$ become 0 when multiplied by on either side (by translating the barbell to the right). Although double leaves are not always a basis for its respective morphism space as $\mathbb{C}[\,]$-modules, it turns out they are a basis over $\mathbb{C}$.

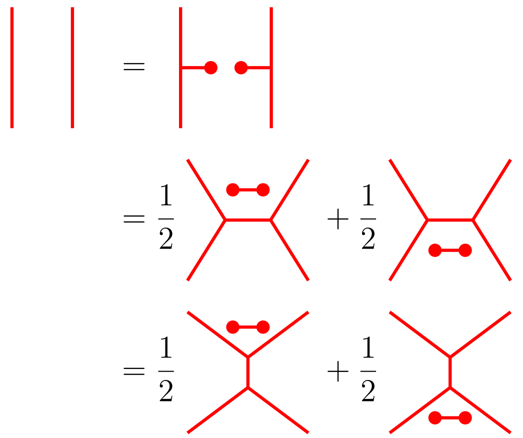
Lemma 3.2.6. *Let* $\pi : \mathrm{mor}(\mathcal{H}(S_2)) \to \mathrm{mor}(\mathcal{DO}_0)$ *be the projection map which takes a morphism to the result of its action on the empty diagram* $\varnothing$*. Then the image* $\pi(\mathbb{LL}(w,y))$ *is a basis for* $\mathrm{Hom}_{\mathcal{DO}_0}(w,y)$ *as a* $\mathbb{C}$*-module.*

Proof. We consider morphisms $\mathrm{Hom}(w,y)$ in $\mathcal{DO}_0$ for fixed objects w, y, and write $\mathbb{LL} :=$ $\pi(\mathbb{LL}(w,y))$ for the set of double leaves in $\mathcal{DO}_0$. Any diagram in $\mathcal{DO}_0$ can be written as a $\mathbb{C}$-linear combination of morphisms without floating diagrams, by simplifying them to barbells, pulling them to the right and killing them with (3.2.4). We can write each of these as a $\mathbb{C}[\,]$-linear combination of double leaves by (3.1.15) with the right action, and reduce it to a $\mathbb{C}$-linear combination by (3.2.4). This implies that $\mathbb{LL}$ spans $\mathrm{Hom}(w,y)$ as a $\mathbb{C}$-module. Since the barbell-wall relation (3.2.4) has no effect on $\mathbb{C}$-linear combinations of $\mathbb{LL}$, it follows from linear independence over $\mathbb{C}[\,]$ that they are linearly independent over $\mathbb{C}$ in $\mathcal{DO}_0$. Check the proof. ∎

Maybe put this next bit in section 3.1
Say more about what this is, and why we say it here

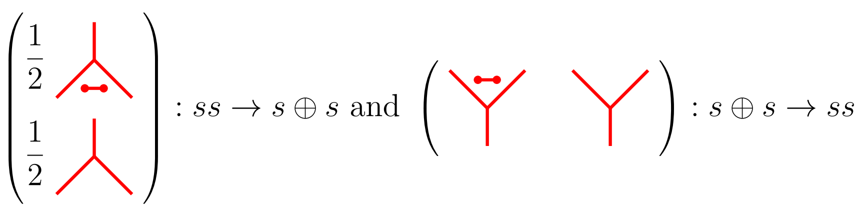
Lemma 3.2.7. *In the additive closure of $\mathcal{H}(S_2)$ we have an explicit isomorphisms $s \otimes s \cong s \oplus s$, as detailed in the proof. Particularly, these are isomorphisms in the additive closure of $\mathcal{DO}_0$.*

Proof. In $\mathcal{H}(S_2)$ we have the relation



(3.2.8)

Note that this $\mathcal{H}(S_2)$ is $\mathbb{C}$-linear, so division by 2 is allowed. This implies we have maps



It follows from (3.1.5d), (3.1.5c) and the calculation (3.2.8), that these maps are inverses. Maybe put the inverse calculation here. ☐

The following result shows that our diagrammatic category $\mathcal{DO}_0$ indeed describes $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$. (Reword this:) This is essentially due to Soergel [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4] (see also [Soe98]) but was not originally formulated as such. The key arguments are in [Soe90] so we attribute this theorem to Soergel.

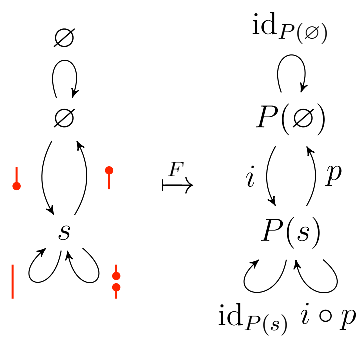
Be clear that I don't understand category $\mathcal{O}$ very well.

Theorem 3.2.9 (Soergel, [Soe90, Endomorhihsmensatz 7, Struktursatz 9 and Section 2.4])**.** *The diagrammatic category* $\mathrm{Kar}_{\oplus}(\mathcal{DO}_0(\mathfrak{sl}_2))$ *and* $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$ *are equivalent as* $\mathbb{C}$*-linear* $\mathcal{H}(S_2)$*-module categories.*

Check all of this & Put precise references
Maybe write description as a soergel module outside the proof

Proof. As a shorthand, we write $\mathrm{proj}(\mathcal{O}_0)$ for $\mathrm{proj}(\mathcal{O}_0(\mathfrak{sl}_2))$. The work of Soergel in [Soe90, Section 2.4] shows that $\mathrm{proj}(\mathcal{O}_0)$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_\varnothing, \Theta_s \in \mathrm{End}(\mathcal{O})$ (corresponding to elements in S_2). Explains what this means, how its related to the $\mathcal{H}(S_2)$ module category We will construct a functor that will map faithfully into a full subcategory of $\mathrm{proj}(\mathcal{O}_0)$, which will become the whole projective category under the additive Karoubi envelope. This mimics the strategy in the proof for Theorem 3.1.17.

Define the functor $F : \mathcal{DO}_0 \to \mathrm{proj}(\mathcal{O}_0)$ that sends the empty object $\varnothing$ to the trivial module $P(\varnothing)$, and the Soergel module action corresponding to s to the translation functor Θ_s. Then the object s maps to $\Theta_s(P(\varnothing)) =: P(s)$, and for example s^3 maps to $\Theta_s^3(P(\varnothing)) = \Theta_s\Theta_s\Theta_s(P(\varnothing))$. In order for F to be functorial, it must map identity diagrams $s^n \to s^n$ to $\mathrm{id}_{\Theta_s^n(P(\varnothing))}$. On non-identity maps, we let $F(\ ) = i$ be the inclusion $P(\varnothing) \to P(s)$ and $F(\ ) = p$ be the projection $P(s) \to P(\varnothing)$. The mapping of F is depicted by the following picture.



Now [Maz09, Proposition 5.90] shows that there is a natural isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ analogous to the isomorphism $s \otimes s \cong s \oplus s$ given in the proof of Lemma 3.2.7. We will eventually take the additive closure of $\mathcal{DO}_0$, so it does not hurt to use these isomorphisms. Given a morphism in $\mathcal{DO}_0$ from s^n to s^m, repeated precomposition and postcomposition with $s \to s \oplus s$ and $s \oplus s \to s$ from Lemma 3.2.7 results in a matrix of morphisms with domain and codomain in $\{\varnothing, s\}$. By (3.1.5d) and (3.2.4) we can draw the entries of the matrix without floating diagrams, so the only diagrams are $\circ$ and up to linear combinations. Therefore, extending by linearity, the picture above completely describes the image of F. We can similarly pull back the matrices of morphisms in $\mathrm{proj}(\mathcal{O}_0)$ to a morphism between $\Theta_s^n(P(\varnothing))$ and $\Theta_s^m(P(\varnothing))$ via the analogous maps defining $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$.

From classical results e.g. [Maz09, Proposition 5.84 and Lemma 5.87], it follows that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$. Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure. Applying these to $P(\varnothing)$ result in the same relations in $\mathrm{proj}(\mathcal{O}_0)$ for $P(\varnothing), P(s)$ and $\Theta_s^2(P(\varnothing))$. Note that the projection and inclusion maps above are exactly the unit and counit of Θ_s evaluated at $P(\varnothing)$, and the trivalent vertices provided by projecting the isomorphisms in Lemma 3.2.7 are exactly the multiplication and comultiplication maps. Furthermore, in [Soe90, Section 2.4] we see that $p \circ i = 0$ in $\operatorname{proj}(\mathcal{O}_0)$ which is analogous[5] to the barbell-wall relation (3.2.4). Hence all the relations in $\mathcal{DO}_0$ are preserved by F. By construction, F preserves $\mathbb{C}$-linear combinations and the Soregel module structure in [Soe90], so F is well defined as a functor between $\mathbb{C}$-linear $\mathcal{H}(S_2)$-module categories.

We now prove that F is fully faithful. It follows from Lemma 3.2.7 and the description of $P(\varnothing)$ and $P(s)$ in [Maz09, Section 5.2] that the image of ⫯ and ⫰ generate all morphisms of the form $\Theta_s^n(P(\varnothing)) \to \Theta_s^m(P(\varnothing))$. Hence F is full. Now the mapping of F on all morphism spaces are determined by those depicted in the above picture. So, for faithfulness, it suffices to compare the $\mathbb{C}$-dimensions of morphism spaces between objects shown in the picture. By Lemma 3.2.6, $\operatorname{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_\varnothing\}$, $\operatorname{Hom}(s, \varnothing)$ has a basis {⫯}, $\operatorname{Hom}(\varnothing, s)$ has a basis {⫰}, and $\operatorname{Hom}(s, s)$ has a basis {id_s, ⫰ $\circ$ ⫯}. The bases for the corresponding morphism spaces in $\operatorname{proj}(\mathcal{O}_0)$ are exactly those in the image Ref? - that these are actually the bases of the hom spaces, so these dimensions coincide. Therefore F is fully faithful.

All objects in $\operatorname{proj}(\mathcal{O}_0)$ appear as direct sums and direct summands of the elements $\Theta_s^n(P(\varnothing))$ for non-negative integers n. Therefore the additive Karoubi envelope induces an equivalence $\operatorname{Kar}_\oplus(\mathcal{DO}_0) \cong \operatorname{proj}(\mathcal{O}_0)$ as $\mathbb{C}$-linear left $\mathcal{H}(S_2)$-module categories. □

Maybe talk about Soergel modules and $\mathcal{H}(S_2)$-modules vs $\operatorname{Kar}_\oplus(\mathcal{H}(S_2))$-modules

Remark 3.2.10. The morphisms spaces in $\mathcal{DO}_0$ are graded by the same grading as $\mathcal{H}(S_2)$ in Section 3.1. The equivalence $\operatorname{Kar}_\oplus(\mathcal{DO}_0) \cong \operatorname{proj}(\mathcal{O}_0)$ includes a grading of morphisms in $\operatorname{proj}(\mathcal{O}_0)$ Check! and hence a grading morphisms of $\mathcal{O}$, which is otherwise ungraded.

[5]This relation extends to the analogue of the local barbell-wall relation, as all 'barbell on the right' morphisms in $\operatorname{proj}(\mathcal{O}_0)$ are linear combinations of applications of Θ_s to $p \circ i$, which is 0.

Chapter 4

Two-colour Diagrammatics

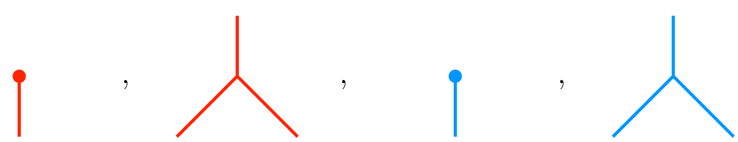
The previous chapter had its focus on the symmetric group generated by one element S_2, which brought about one-colour diagrammatics. We shift our attention to a more complex example by adding an extra generator, that is, another colour. In particular, we consider the case for the affine symmetric group on two elements $\widetilde{S}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$. Refine this

4.1 Two-colour Diagrammatic Hecke Category

Corresponding to $\widetilde{S}_2$, we define the two-colour (diagrammatic) Hecke category $\mathcal{H}(\widetilde{S}_2)$. This is a (strict) $\mathbb{C}$-linear monoidal category given by the following isotopy presentation.

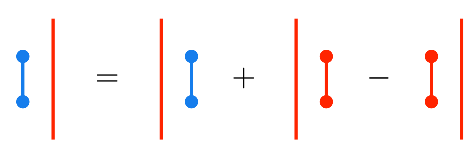
Objects in $\mathcal{H}(\widetilde{S}_2)$ are generated by formal tensor products of the non-identity elements $s, t \in \widetilde{S}_2$. As before, we write objects as words such as $sstttst =: s^2t^3st$ where the tensor product is concatenation, and associate the colour red to s and blue to t. The empty word is the monoidal identity, which we write as $\varnothing$.

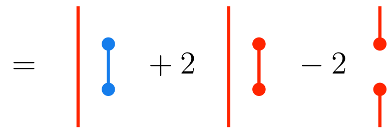
The morphisms are generated by the univalent and trivalent vertices



(4.1.1)

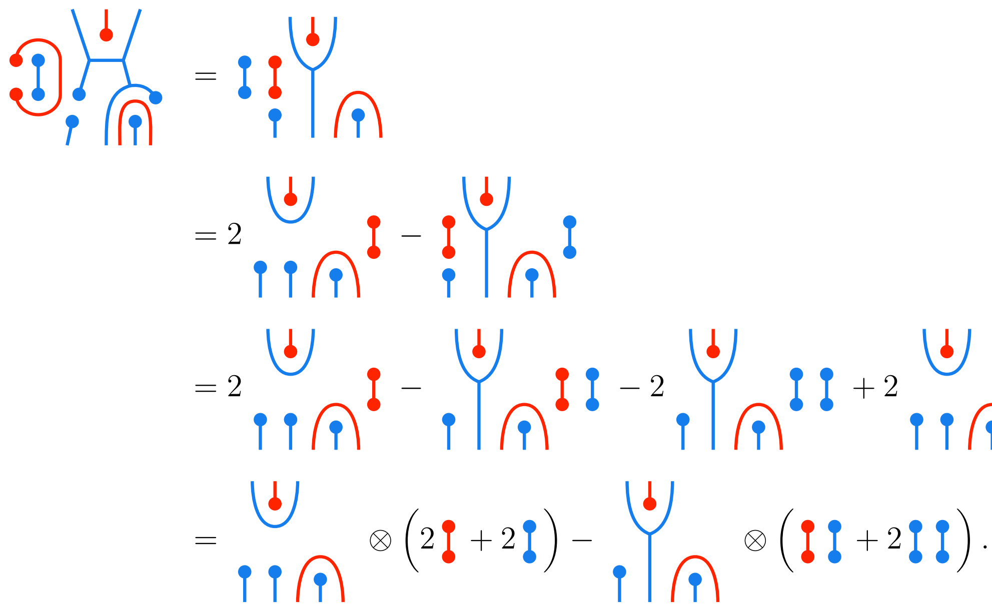
that are maps $s \to \varnothing$, $ss \to s$, $t \to \varnothing$ and $tt \to t$ respectively. As in the one-colour case, tensor product is horizontal concatenation, composition is appropriate vertical stacking, and we denote the empty diagram $\varnothing \to \varnothing$ by $\varnothing$. For each colour, these diagrams have the one-colour relations given by (3.1.5). As we have another colour, we need to describe how different colours interact. This is given by the *two-colour relation*



 (4.1.2)

and with red and blue swapped.

Example 4.1.3. Using the one-colour and two-colour relations on the following morphism in $\mathrm{Hom}(ttsts, tst)$ we have



Talk about this containing $\mathcal{H}(S_2)$

Remark 4.1.4. Notice that the red and blue lines never cross as no generators that allow crossings. This is a consequence of working over affine S_2 in which the generators s and t have no relation. Mention example of crossing and S_3.

Definition 4.1.5. For a group with a presentation in terms of generators and relations, the *length* of a product of generators is the number of generators in the product. We say that a product of generators is *reduced* if it's length cannot be shortened with relations.

In $\widetilde{S}_2$ products can be shortened by the relation $s^2 = t^2 = 1$. For instance, $sttsts$ is not reduced because it is equal to ts which is reduced. Notice that for $\widetilde{S}_2$ each element can be written uniquely as a reduced product of generators. This is true since otherwise we have two distinct reduced products for the same element in $\widetilde{S}_2$ so they must be related by $s^2 = t^2$. This means they can be reduced further by $s^2 = t^2 = 1$, which contradicts minimality of their length. Note that the reduced products in $\widetilde{S}_2$ are either the identity or alternating products of s and t.

Notice that there is a notational similarity between products in the group and words in $\mathcal{H}(\widetilde{S}_2)$. This motivates the following definitions. Let $\phi : (\mathrm{ob}(\mathcal{H}(\widetilde{S}_2)), \otimes) \to (\widetilde{S}_2, *)$ be the monoid homomorphism mapping $\varnothing \mapsto 1$, $s \mapsto s$ and $t \mapsto t$. Also define the function

$\psi : \widetilde{S}_2 \to \text{ob}(\mathcal{H}(\widetilde{S}_2))$ to map elements $x \in \widetilde{S}_2$ to the tensor product of s and t in $\mathcal{H}(\widetilde{S}_2)$ corresponding to the reduced product of x in $\widetilde{S}_2$. This is well defined because reduced products are unique and two different reduced products cannot equal the same element of $\widetilde{S}_2$. Note that the image $\psi(\widetilde{S}_2)$ is the set containing $\varnothing$ and words of alternating s and t. The composition $\psi\phi : \mathcal{H}(\widetilde{S}_2) \to \mathcal{H}(\widetilde{S}_2)$ maps words w to the tensor of s and t corresponding to the reduced product of $\phi(w)$, and $\phi\psi$ is the identity map on $\widetilde{S}_2$.

The following definition is a more general version of Definition 3.1.8.

Definition 4.1.6 (Subexpression)**.** Given a word w of length n, a *subexpression* e is a binary string of length n. A subexpression can be *applied* to produce an word $w(e)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$. For $1 \leq i \leq n$, we write $w(e, i)$ for the result of the first i terms of e applied to the first i terms in w. Particularly $w(e, 0) = \varnothing$ and $w(e, n) = w(e)$.

For example, in $\mathcal{H}(\widetilde{S}_2)$, if $w = sttts$ and $e = 11001$ then $w(e) = st\varnothing\varnothing s = sts$ and $w(e, 3) = sts(110) = st\varnothing = st$ in $\mathcal{H}(\widetilde{S}_2)$.

Let the *length* of a word be the number of generators in its tensor product. As before, given an object w and a subexpression e of w, we label each of the n terms by one of U_0, U_1, D_0, D_1. Let $i \geq 0$, and write x for the i-th term of w. We label the i-th term U_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. In other words we write U_* if the next term of w will make $\psi\phi$ applied to the partially evaluated subexpression longer, regardless of the i-term of e. We label D_* if $\psi\phi(w(e, i-1) \otimes x)$ is longer than $\psi\phi(w(e, i-1))$. The label's subscript is the i-th term of e. Note that this construction is well defined because $\psi\phi(w(e, i-1) \otimes x) = \psi(\phi(w(e, i-1)) * \phi(x)) = \psi(\phi(w(e, i-1)) * x)$ is always either longer or shorter, since the last element of the reduced product is either the same as x or different. When they are the same, the word is shorter via $s^2 = t^2 = 1$, and when they are different it is longer as no relations can make it shorter.

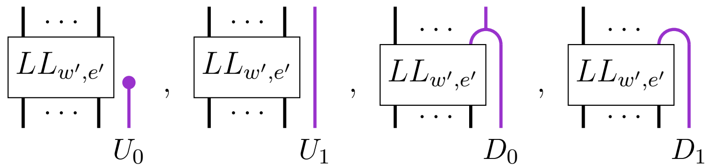
Remark 4.1.7. This description of the labels (via. reduced products) is more akin to the definition for general Coxeter groups than in Section 3.1.

Example 4.1.8. Consider the word $w = sttst$ and subexpression $e = 10011$. The labels can be constructed as in the following table.

Term i	1	2	3	4	5
Partial w	s	st	stt	$stts$	$sttst$
Partial e	1	10	100	1001	10011
$w(e,i)$	s	$s\varnothing$	$s\varnothing\varnothing = s$	$s\varnothing\varnothing s = ss$	$s\varnothing\varnothing st = sst$
Labels	U_1	U_1U_0	$U_1U_0U_0$	$U_1U_0U_0D_1$	$U_1U_0U_0D_1U_1$

Definition 4.1.9. The *light leaf* $LL_{w,e} \in \text{Hom}(w, \psi\phi(w(e)))$ for a word w and a subexpression e is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given appropriate subwords w' and e' of w and e respectively, and if the next terms are x in

w and i in e, the light leaf $LL_{w'x,e'i}$ is one of

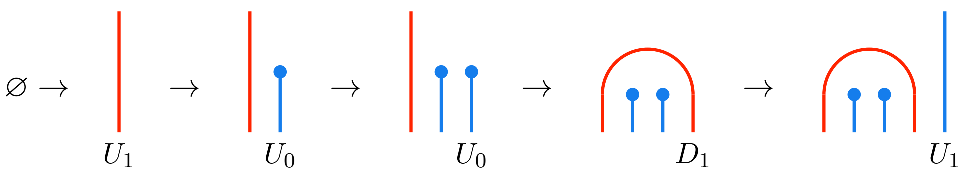


(4.1.10)

corresponding to the next label. The purple strands are red if $x = s$ and blue if $x = t$.

Notice that the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ does not end with the colour corresponding to x, and if the next label is D_* the codomain of $LL_{w',e'}$ ends with a strand with the colour corresponding to x. This implies the recursive definition in the diagram above is consistent. Note that in the case of D_*, one of the black strands in the domain of $LL_{w',e'}$ must have the colour of x in order for the colour to appear in its codomain.

Example 4.1.11. Following from Example 4.1.8, with $w = sttst$, $e = 10011$ and labels $U_1U_0U_0D_1U_1$, the light leaf $LL_{w,e}$ is build as follows.



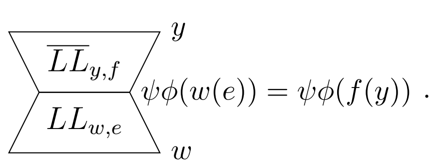
We can define double leaves exactly as we did in Definition 3.1.13.

Definition 4.1.12. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y in $\mathcal{H}(\widetilde{S}_2)$ is a composition

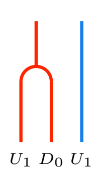
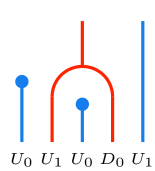
$$\mathbb{LL}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

for subexpressions e of w and f of y such that $\psi\phi(w(e)) = \psi\phi(f(y))$.

Diagrammatically these are morphisms from w to y factoring through $\psi\phi(w(e)) = \psi\phi(f(y)) \in \psi(\widetilde{S}_2)$,



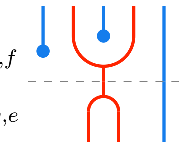
Example 4.1.13. Let $w = sst$ with the subexpression $e = 101$ and $y = tstst$ with the subexpression $f = 01001$. The corresponding light leaves are

$LL_{w,e} =$  and $LL_{y,f} =$  .

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : sst \to tstst$, factoring through st, is

$\overline{LL}_{y,f}$

$LL_{w,e}$



.

As with the one-colour case, the set of double leaves $\mathbb{LL}(w, y)$ from words w to y in $\mathcal{H}(\widetilde{S}_2)$ form a basis for $\mathrm{Hom}(w, y)$ over $\mathbb{Z}[\ ,\ ]$. The Hom spaces are graded such that the univalent vertices have degree 1 and trivalent vertices have degree -1 for either colour. Finally we have a similar theorem to Theorem 3.1.17 for $\widetilde{S}_2$.

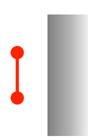
Remark 4.1.14. The construction of the diagrammatic Hecke category, light leaves, Theorem 3.1.15 and Theorem 3.1.17 all generalise to general Coxeter groups. The details are found in [EW16].

4.2 Diagrammatic $\mathrm{Tilt}(\mathfrak{sl}_2)$

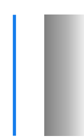
Something something about Tilt

Something something about extending $\mathcal{H}(\widetilde{S}_2)$ from $\mathbb{Z}$ to $\mathbb{C}$.

Definition 4.2.1. Let $\mathcal{DT}(\mathfrak{sl}_2)$ be the $\mathbb{C}$-linear Maybe have this $\mathbb{Z}$ left $\mathcal{H}(\widetilde{S}_2)$-module category with elements generated by the monoidal identity $\varnothing$ of $\mathcal{H}(\widetilde{S}_2)$, and morphisms generated by the empty diagram $\varnothing$. The action of $\mathcal{H}(\widetilde{S}_2)$ on the left is left concatenation for objects and morphisms. The relations on diagrams in $\mathcal{H}(\widetilde{S}_2)$ follow through to diagrams in $\mathcal{DT}(\mathfrak{sl}_2)$. Additionally, we imagine a wall on the right of diagrams and impose the local wall-annihilation relations



$=$



$= 0. \qquad (4.2.2)$

In this section we just write $\mathcal{DT}$ for this category.

Example 4.2.3. The morphism in Example 4.1.3 collapses to 0 because all the diagrams have either blue or barbell on the right.

TODO: Another example clarifying 'blue on the right'

The objects of this category are identical to objects in $\mathcal{H}(\widetilde{S}_2)$ and the morphisms are the same modulo the wall relations (4.2.2). Naturally, we wonder whether double leaves form bases for the morphism spaces in $\mathcal{DT}$. It is easy to see that double leaves appear in $\mathcal{DT}$ by acting on $\varnothing$ by double leaves in $\mathcal{H}(\widetilde{S}_2)$. All morphisms in $\mathcal{DT}$ are morphisms in $\mathcal{H}(\widetilde{S}_2)$ so they can be written as $\mathbb{C}[\ ,\ ]$-linear combinations of double leaves, though some of these leaves have collapsed to 0. This makes it clear that double leaves span the morphism spaces of $\mathcal{DT}$ as (left) $\mathbb{C}[\ ,\ ]$-modules. However they may not be linearly independent as neither left nor right modules as with the one-colour case. Although double leaves are not always a basis for its respective morphism space as $\mathbb{C}[\ ,\ ]$-modules, it turns out a subset of them are a basis over $\mathbb{C}$.

Lemma 4.2.4. *Let $\pi : \mathrm{mor}(\mathcal{H}(\widetilde{S}_2)) \to \mathrm{mor}(\mathcal{DT})$ be the projection map which takes a morphism to the result of its action on the empty diagram $\varnothing$. Then the image $\pi(\mathbb{LL}(w, y))$ without zero morphisms is a basis for $\mathrm{Hom}_{\mathcal{DT}}(w, y)$ as a $\mathbb{C}$-module.*

Proof. We consider morphisms $\mathrm{Hom}(w, y)$ in $\mathcal{DT}$ for fixed objects w, y, and write $\mathbb{LL} \coloneqq \pi(\mathbb{LL}(w, y))$ for the set of double leaves in $\mathcal{DT}$. Any diagram in $\mathcal{DT}$ can be written as a $\mathbb{C}$-linear combination of morphisms without floating diagrams by pulling floating diagrams to the right with (3.1.5d) and (4.1.2) then applying the wall relation (4.2.2). We can write each of these as a $\mathbb{C}[\ ,\ ]$-linear combination of double leaves with a right action, and reduce it to a $\mathbb{C}$-linear combination by (4.2.2). This implies that $\mathbb{LL}$ spans $\mathrm{Hom}(w, y)$ as a $\mathbb{C}$-module. Now $\mathbb{LL}$ may not be linearly independent because the two-colour wall relation (4.2.2) reduces all diagrams factoring through a word ending with t to 0. The set of light leaves after removing morphisms killed by (4.2.2), i.e. $\mathbb{LL} \setminus \{0\}$, still spans $\mathrm{Hom}(w, y)$ by the argument above. This set is linearly independent since, by construction, (4.2.2) has no effect on $\mathbb{C}$-linear combinations of $\mathbb{LL} \setminus \{0\}$. Then it follows from linear independence over $\mathbb{C}[\ ,\ ]$ that this set is linearly independent over $\mathbb{C}$ in $\mathcal{DT}$. ∎

Since $\mathcal{H}(S_2)$ appears inside $\mathcal{H}(\widetilde{S}_2)$ as both colours, Lemma 3.2.7 gives explicit isomorphisms $s \otimes s \cong s \oplus s$ and $t \otimes t \cong t \oplus t$.

Say something here?

The following result states that $\mathcal{DT}$ is indeed a diagrammatic incarnation of $\mathrm{Tilt}(\mathfrak{sl}_2)$.

Be clear that I don't understand Tilt very well.

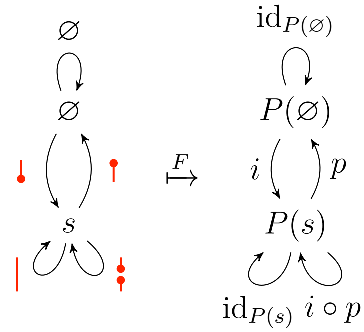
Theorem 4.2.5 (???)**.** *The diagrammatic category $\mathrm{Kar}_{\oplus}(\mathcal{DT}(\mathfrak{sl}_2))$ and $\mathrm{Tilt}(\mathfrak{sl}_2)$ are equivalent as $\mathbb{C}$-linear $\mathcal{H}(\widetilde{S}_2)$-module categories.*

Check all of this & Put precise references

Maybe write description as a soergel module outside the proof

Proof. As a shorthand, we write $\mathcal{T}$ for $\mathrm{Tilt}(\mathfrak{sl}_2)$. The work of Soergel in [Soe90, Section 2.4] shows that $\mathrm{proj}(\mathcal{O}_0)$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_\varnothing, \Theta_s \in \mathrm{End}(\mathcal{O})$ (corresponding to elements in S_2). Explains what this means, how its related to the $\mathcal{H}(S_2)$ module category We will construct a functor that will map faithfully into a full subcategory of $\mathrm{proj}(\mathcal{O}_0)$, which will become the whole projective category under the additive Karoubi envelope. This mimics the strategy in the proof for Theorem 3.1.17.

Define the functor $F : \mathcal{DO}_0 \to \mathrm{proj}(\mathcal{O}_0)$ that sends the empty object $\varnothing$ to the trivial module $P(\varnothing)$, and the Soergel module action corresponding to s to the translation functor Θ_s. Then the object s maps to $\Theta_s(P(\varnothing)) =: P(s)$, and for example s^3 maps to $\Theta_s^3(P(\varnothing)) = \Theta_s\Theta_s\Theta_s(P(\varnothing))$. In order for F to be functorial, it must map identity diagrams $s^n \to s^n$ to $\mathrm{id}_{\Theta_s^n(P(\varnothing))}$. On non-identity maps, we let $F(\ ) = i$ be the inclusion $P(\varnothing) \to P(s)$ and $F(\ ) = p$ be the projection $P(s) \to P(\varnothing)$. The mapping of F is depicted by the following picture.



Now [Maz09, Proposition 5.90] shows that there is a natural isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ analogous to the isomorphism $s \otimes s \cong s \oplus s$ given in the proof of Lemma 3.2.7. We will eventually take the additive closure of $\mathcal{DO}_0$, so it does not hurt to use these isomorphisms. Given a morphism in $\mathcal{DO}_0$ from s^n to s^m, repeated precomposition and postcomposition with $s \to s \oplus s$ and $s \oplus s \to s$ from Lemma 3.2.7 results in a matrix of morphisms with domain and codomain in $\{\varnothing, s\}$. By (3.1.5d) and (3.2.4) we can draw the entries of the matrix without floating diagrams, so the only diagrams are $\circ$ and up to linear combinations. Therefore, extending by linearity, the picture above completely describes the image of F. We can similarly pull back the matrices of morphisms in $\mathrm{proj}(\mathcal{O}_0)$ to a morphism between $\Theta_s^n(P(\varnothing))$ and $\Theta_s^m(P(\varnothing))$ via the analogous maps defining $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$.

From classical results e.g. [Maz09, Proposition 5.84 and Lemma 5.87], it follows that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$. Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure. Applying these to $P(\varnothing)$ result in the same relations in $\mathrm{proj}(\mathcal{O}_0)$ for $P(\varnothing), P(s)$ and $\Theta_s^2(P(\varnothing))$. Note that the projection and inclusion maps above are exactly the unit and counit of Θ_s evaluated at $P(\varnothing)$, and the trivalent vertices provided by projecting the isomorphisms in Lemma 3.2.7 are exactly the multiplication and comultiplication maps. Furthermore, in [Soe90, Section 2.4] we see that $p \circ i = 0$ in $\mathrm{proj}(\mathcal{O}_0)$ which is analogous[1] to the barbell-wall relation (3.2.4). Hence all the relations in $\mathcal{DO}_0$ are preserved by F. By construction, F preserves $\mathbb{C}$-linear combinations and the Soregel module structure in [Soe90], so F is well defined as a functor between $\mathbb{C}$-linear $\mathcal{H}(S_2)$-module categories.

We now prove that F is fully faithful. It follows from Lemma 3.2.7 and the description of $P(\varnothing)$ and $P(s)$ in [Maz09, Section 5.2] that the image of and generate all morphisms of the form $\Theta_s^n(P(\varnothing)) \to \Theta_s^m(P(\varnothing))$. Hence F is full. Now the mapping of F on all morphism spaces are determined by those depicted in the above picture. So, for faithfulness, it suffices to compare the $\mathbb{C}$-dimensions of morphism spaces between objects

[1]This relation extends to the analogue of the local barbell-wall relation, as all 'barbell on the right' morphisms in $\mathrm{proj}(\mathcal{O}_0)$ are linear combinations of applications of Θ_s to $p \circ i$, which is 0.

shown in the picture. By Lemma 3.2.6, $\mathrm{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_\varnothing\}$, $\mathrm{Hom}(s, \varnothing)$ has a basis $\{ \ \}$, $\mathrm{Hom}(\varnothing, s)$ has a basis $\{ \ \}$, and $\mathrm{Hom}(s, s)$ has a basis $\{\mathrm{id}_s, \ \circ \ \}$. The bases for the corresponding morphism spaces in $\mathrm{proj}(\mathcal{O}_0)$ are exactly those in the image Ref? - that these are actually the bases of the hom spaces, so these dimensions coincide. Therefore F is fully faithful.

All objects in $\mathrm{proj}(\mathcal{O}_0)$ appear as direct sums and direct summands of the elements $\Theta_s^n(P(\varnothing))$ for non-negative integers n. Therefore the additive Karoubi envelope induces an equivalence $\mathrm{Kar}_\oplus(\mathcal{DO}_0) \cong \mathrm{proj}(\mathcal{O}_0)$ as $\mathbb{C}$-linear left $\mathcal{H}(S_2)$-module categories. $\square$

Comment on grading?

Bibliography

[EW16] Ben Elias and Geordie Williamson. "Soergel Calculus". In: *Representation Theory of the American Mathematical Society* 20 (Oct. 2016). DOI: 10.1090/ert/481.

[Eti+15] Pavel Etingof et al. *Tensor Categories*. Vol. 205. Mathematical Surveys and Monographs. American Mathematical Society, 2015. DOI: http://dx.doi.org/10.1090/surv/205.

[Hum08] James E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*. Vol. 94. Graduate Studies in Mathematics. American Mathematical Society, 2008. DOI: http://dx.doi.org/10.1090/gsm/094.

[Jon21] Vaughan F. R. Jones. "Planar algebras". In: *New Zealand Journal of Mathematics* 52 (2021), pp. 1–107. DOI: 10.53733/172. URL: https://nzjmath.org/index.php/NZJMATH/article/view/172.

[Lib08] Nicolas Libedinsky. "Sur la catégorie des bimodules de Soergel". In: *Journal of Algebra* 320.7 (2008). (French), pp. 2675–2694. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2008.05.027.

[ML98] Saunders Mac Lane. *Categories for the Working Mathematician*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. Springer, 1998. DOI: https://doi.org/10.1007/978-1-4757-4721-8.

[Maz09] Volodymyr Mazorchuk. *Lectures on $\mathfrak{sl}_2(\mathbb{C})$-Modules*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. IMPERIAL COLLEGE PRESS, 2009. DOI: 10.1142/p695.

[RW18] Simon Riche and Geordie Williamson. "Tilting Modules and the p-Canonical Basis". In: *Asterisque* 397 (2018). DOI: 10.24033/ast.1043.

[Soe90] Wolfgang Soergel. "Kategorie $\mathcal{O}$, Perverse Garben Und Moduln Uber Den Koinvariantez Zur Weylgruppe". In: *Journal of the American Mathematical Society* 3.2 (1990), pp. 421–445. ISSN: 08940347, 10886834. URL: http://www.jstor.org/stable/1990960.

[Soe98] Wolfgang Soergel. "Combinatorics of Harish-Chandra modules". In: *Representation Theories and Algebraic Geometry*. Ed. by Abraham Broer, A. Daigneault, and Gert Sabidussi. Springer Netherlands, 1998, pp. 401–412. ISBN: 978-94-015-9131-7. DOI: 10.1007/978-94-015-9131-7_10. URL: https://doi.org/10.1007/978-94-015-9131-7_10.