

Diagrammatic Categories

in Representation Theory

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Kazhdan–Lusztig Conjecture

(Motivation)

- Relates Kazhdan–Lusztig polynomials to Jordan–Hölder multiplicities of Verma modules
- First proof by Beilinson and Bernstein (1981) using D -modules
- Algebraic proof by Elias and Williamson (2010's) using Soergel bimodules and diagrammatics

Talk Overview

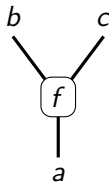
1. Diagrammatic Monoidal Categories
2. Diagrammatic Soergel bimodules
3. Further Applications

Monoidal Categories

A *monoidal category* is a category with an associative multiplication \otimes for objects and morphisms, and a unit object $\mathbb{1}$, such that the multiplication works well with composition.

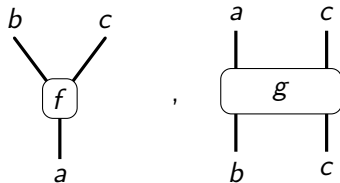
Monoidal Categories

$$f : a \rightarrow b \otimes c$$



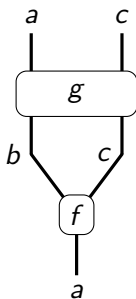
Monoidal Categories: Composition

$$f : a \rightarrow b \otimes c$$
$$g : b \otimes c \rightarrow a \otimes c$$



Monoidal Categories: Composition

$$f : a \rightarrow b \otimes c$$
$$g : b \otimes c \rightarrow a \otimes c$$



Monoidal Categories: Identity

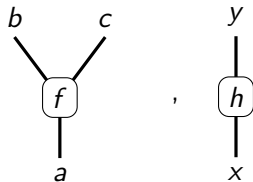
$$\mathrm{id}_a : a \rightarrow a$$

$$\begin{array}{c} a \\ | \\ a \end{array}$$

Monoidal Categories: Tensor

$$f : a \rightarrow b \otimes c$$

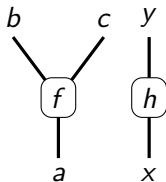
$$h : x \rightarrow y$$



Monoidal Categories: Tensor

$$f : a \rightarrow b \otimes c$$

$$h : x \rightarrow y$$



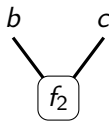
Monoidal Categories: Unit

$$f_1 : a \rightarrow \mathbb{1}$$



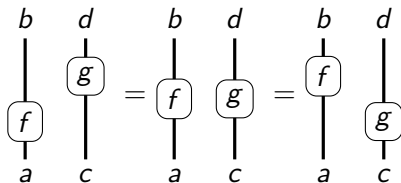
Monoidal Categories: Unit

$$f_2 : \mathbb{1} \rightarrow b \otimes c$$

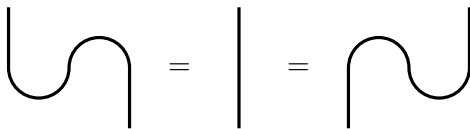


Monoidal Categories

Interchange Law



Isotopy



\mathbb{Z} -linear Monoidal Category

$$3 \cdot f(a, b, c) - 2 \cdot (f_1(a) \cdot f_2(b, c))$$

Diagrammatic Soergel Bimodules

Let \mathcal{D} be the (diagrammatic) \mathbb{Z} -linear monoidal category with:

Generating object I .

Generating morphisms



and local relations...

Relations

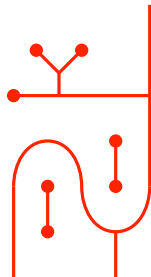
$$\begin{array}{c} | \\ \text{---} \bullet \end{array} = \begin{array}{c} | \\ | \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array},$$

$$\begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array},$$

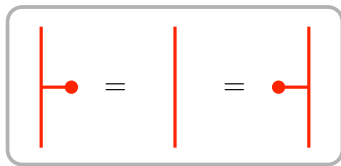
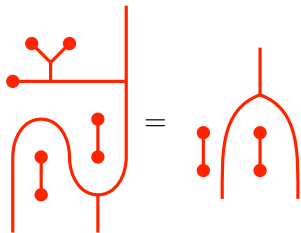
$$\begin{array}{c} | \\ \bigcirc \\ | \end{array} = 0,$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

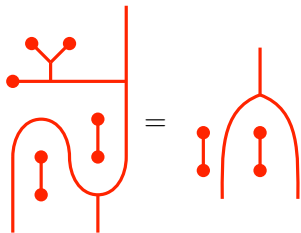
Example 1



Example 1

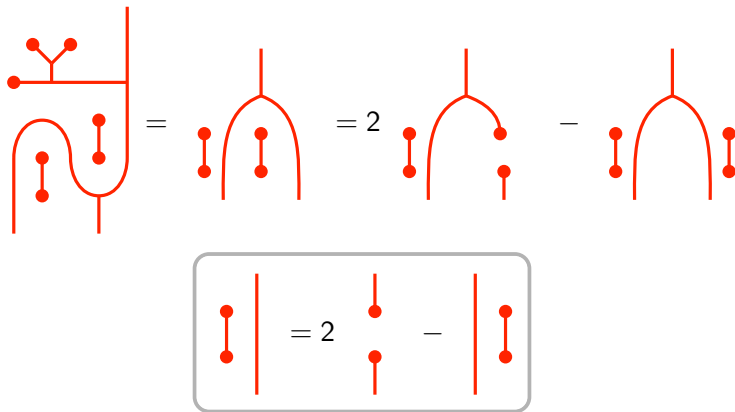


Example 1

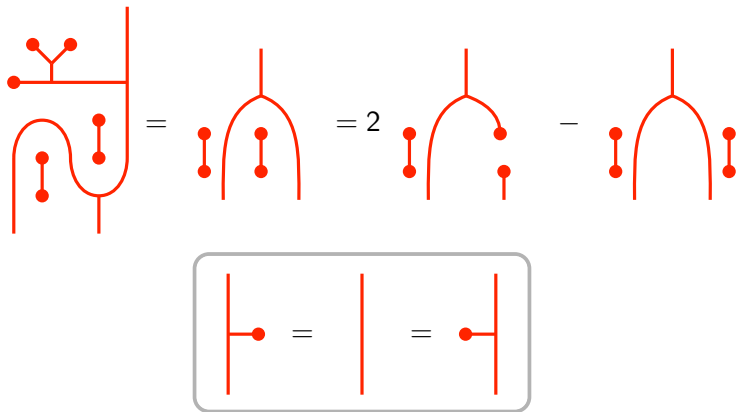


$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} | \\ \bullet \end{array}$$

Example 1



Example 1

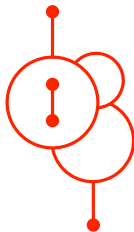


Example 1

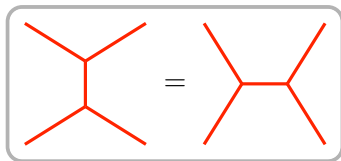
$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = 2 \times \text{Diagram 3} - \text{Diagram 4} \\
 & \qquad \qquad \qquad = 2 \times \text{Diagram 5} - \text{Diagram 6}
 \end{aligned}$$

The diagrams are composed of red lines and dots, representing a sequence of operations or a combinatorial structure. The first diagram shows a complex structure with a horizontal line, a vertical line, and several curved lines. The subsequent diagrams show simpler structures, including a single vertical line, a single curved line, and a single dot.

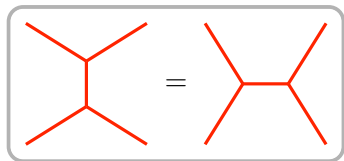
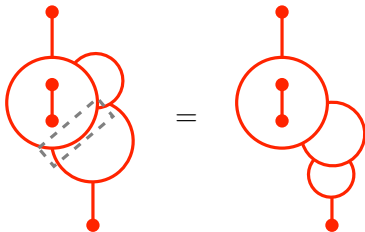
Example 2



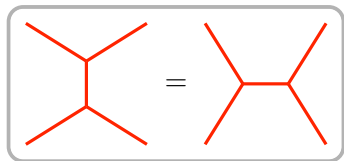
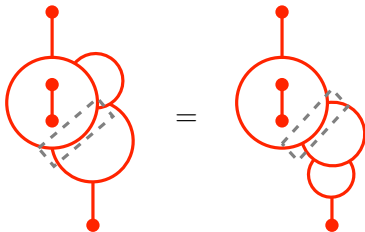
Example 2



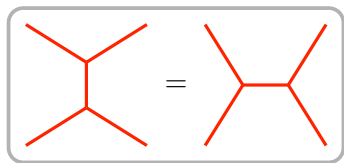
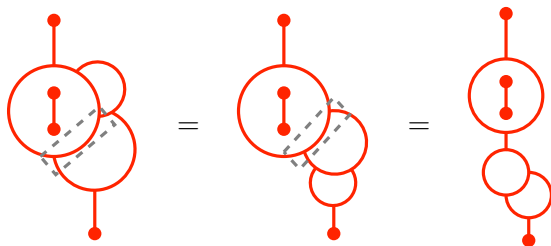
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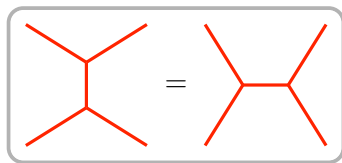
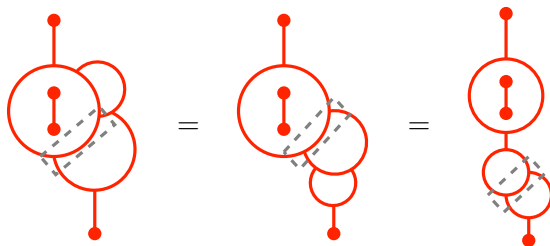
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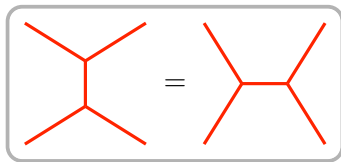
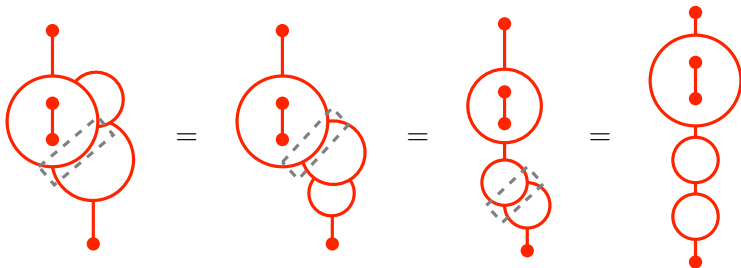
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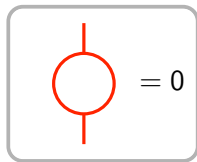
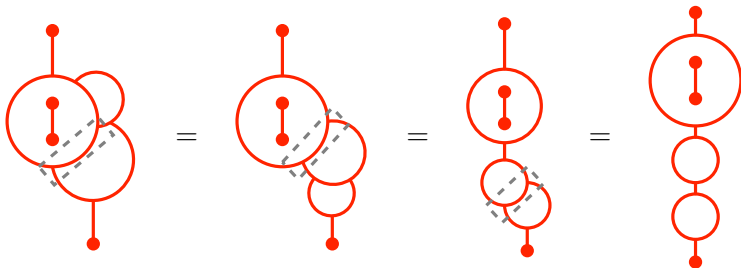
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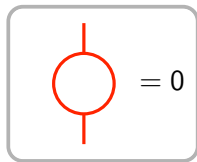
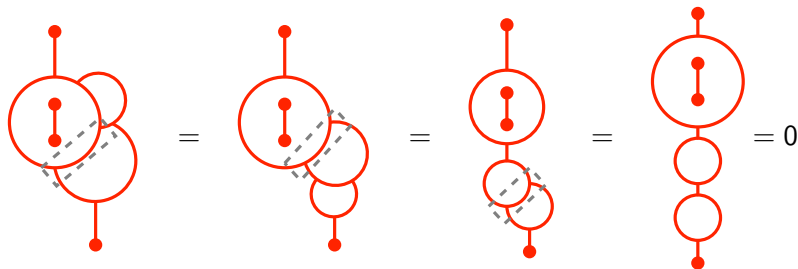
Example 2



Example 2



Example 2



Soergel Bimodules

Theorem (Elias–Williamson, 2013)

The additive Karoubi envelope of \mathcal{D} is equivalent to the category of Soergel Bimodules $\mathbb{S}\text{Bim}$ over S_2 as graded additive \mathbb{C} -linear monoidal categories.

Generalisations

- Over general Coxeter groups e.g. S_n , D_n
- Diagrammatics for other categories of representations...

Further Applications

- Diagrammatics for BGG Category \mathcal{O} , Tilting modules
- Changing scalars to fields of characteristic p