Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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Chapter 1

Introduction

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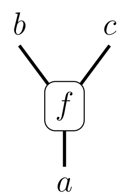
Chapter 2

Background

2.1 Drawing Monoidal Categories

A monoidal category $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold[1]. We assume that monoidal categories are strict, since all monoidal categories are monoidally equivalent to a strict one[2].

The morphisms of $\mathcal{C}$ can be drawn as string diagrams, where the morphism maps from the bottom to the top. Functions that make up the morphism are drawn as tokens or boxes. For example

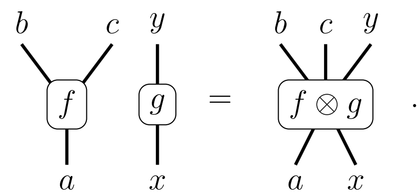


depicts a morphism $f : a \to b \otimes c$. For identity morphisms we drop the box and only draw a vertical line, so id_a is the diagram

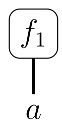
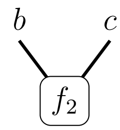
.

The tensor product of morphisms is the horizontal concatenation of diagrams, such that strings from separate functions don't interact. For example, given $g : x \to y$, the tensor product $f \otimes g : a \otimes x \to b \otimes c \otimes y$ is drawn as

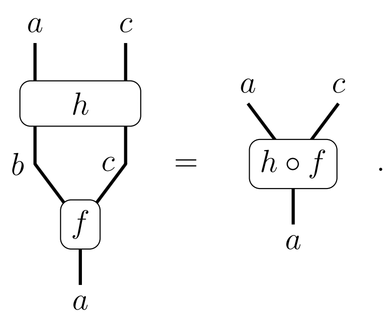
[1] For more details see [Eti+15].
[2] See [ML98, VII.2] or [Eti+15, Thm 2.8.5]



By convention, $\mathbb{1}$ is blank and unlabelled, and strings that would join to $\mathbb{1}$ are blank. Particularly, $\mathrm{id}_{\mathbb{1}}$ is an empty diagram, and we have diagrams such as

 $: a \to \mathbb{1}$ and  $: \mathbb{1} \to b \otimes c.$

The compositions of morphisms is the vertical stacking of diagrams where domains and codomains match. For example, the composition $h \circ f : a \to b \otimes c \to a \otimes c$ of $f : a \to b \otimes c$ with $h : b \otimes c \to a \otimes c$ has the diagram



Before looking at our main example of a diagrammatic monoidal category, we first define some terminology.

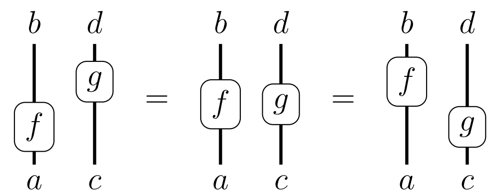
Definition 2.1.1. For a commutative ring R, an *R-linear category* is a category enriched over the category of R-modules. That is, for objects a, b, the set of morphisms $\mathrm{Hom}(a, b)$ is an R-module and the composition of morphisms is R-bilinear.

Example 2.1.2. Let $\Bbbk$ be a field. The category of vector spaces over $\Bbbk$, $\mathbf{Vect}_{\Bbbk}$, is a $\Bbbk$-linear category. This makes sense by the classical theory of linear algebra.

For a strict R-linear monoidal category $\mathcal{C}$, the bifunctoriality of $- \otimes -$ implies the following *interchange law*. For morphisms $f : a \to b$ and $g : c \to d$, $(\mathrm{id}_b \otimes g) \circ (f \otimes \mathrm{id}_c) = f \otimes g = (f \otimes \mathrm{id}_d) \circ (\mathrm{id}_a \otimes g)$. In other words the following diagram commutes.

$$\begin{tikzcd}
a \otimes c \arrow[r, "f \otimes \mathrm{id}_c"] \arrow[d, "\mathrm{id}_a \otimes g"'] \arrow[dr, "f \otimes g"] & b \otimes c \arrow[d, "\mathrm{id}_b \otimes g"] \\
a \otimes d \arrow[r, "f \otimes \mathrm{id}_d"] & b \otimes d
\end{tikzcd}$$

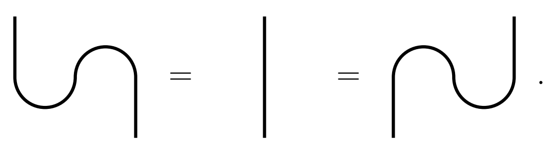
Written with string diagrams, this is



which holds up to deformation of the diagram.

Definition 2.1.3. A monoidal category $\mathcal{C}$ is *generated* by finite set S_o of objects and S_m of morphisms, when all non-unit objects are a finite tensor of objects in S_o and all non-identity morphisms are a finite combination of tensors and compositions of morphisms in S_m.

Example 2.1.4. Our first example of a diagrammatic monoidal category is the *Temperley-Lieb category*. The Temperley-Lieb category $\mathcal{TL}$ is a strict R-linear monoidal category whose objects are generated by the vertical line I and morphisms generated by the cup $\cup : \mathbb{1} \to \mathsf{I} \otimes \mathsf{I}$ and cap $\cap : \mathsf{I} \otimes \mathsf{I} \to \mathbb{1}$, with relations



Mention that composition and tensor product is as explained above
Some example
Mention bubbles and specialisation to some $\delta \in R$
Mention that these are crossingless matchings
Comment on isotopy

Chapter 3

One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

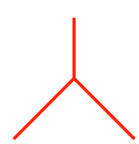
The first one-colour diagrammatic we explore is the *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = e \rangle$. At the end of this section, we see that this diagrammatic category is equivalent to the category of Soergel Bimodules under additive Karoubian closure.

Remark 3.1.1. All diagrammatics below and in Chapter 4 can be defined in the language of planar algebras, without the additional structure of categories, e.g. in [Jon21]. Nevertheless, we define them in the context of categories as we will see them as diagrammatic versions of important categories in representation theory.

Definition 3.1.2. The *one-colour (diagrammatic) Hecke category* $\mathcal{H}(S_2)$ is a $\mathbb{C}$-linear monoidal category with the following presentation.

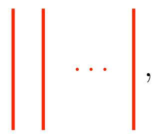
The objects are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is concatenation. The empty tensor product, i.e. the monoidal identity, will be denoted $\varnothing =: s^0$.

The morphisms are generated, up to isotopy, by univalent and trivalent vertices

 ,  (3.1.3)

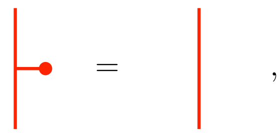
that are maps $s \to \varnothing$ and $ss \to s$ respectively. Note that we put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). The free $\mathbb{C}$-module structure on each morphism space $\mathrm{Hom}(s^n, s^m)$ produces $\mathbb{C}$-linear combinations of such diagrams. Something about composition/tensor and addition commuting Then, composition or tensors with the zero morphism 0 result in 0. To abuse notation, the empty diagram

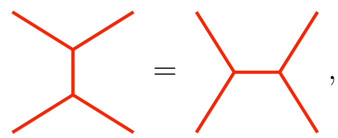
$\varnothing \to \varnothing$ will be denoted $\varnothing$. The identity morphism in $\mathrm{Hom}(s^n, s^n)$ is the diagram consisting of n (red) vertical lines

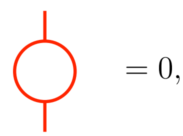
 (3.1.4)

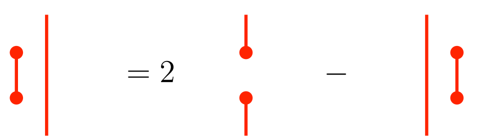
which we may identify with s^n.

Such diagrams are subject to the following local relations

 (3.1.5a)

 (3.1.5b)

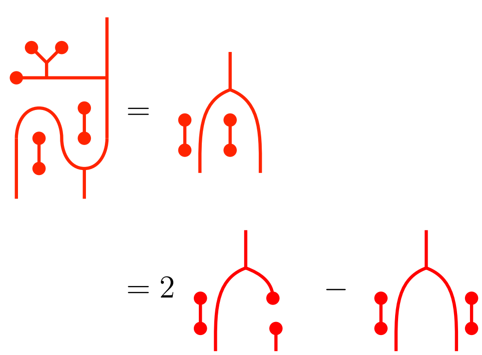
 (3.1.5c)

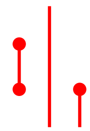
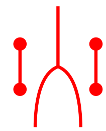
 (3.1.5d)

Remark 3.1.6. The object s is a Frobenius object in $\mathcal{H}(S_2)$. The generators (3.1.3) and their horizontal reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.5a) and (3.1.5b).

Put a definition of frob object in intro

Example 3.1.7. Using the relations in (3.1.5) we can simplify the morphism in $\mathrm{Hom}(ss, s)$,



$= 2$  $-$  .

Add example of using frob associativity

The morphism space $\mathrm{Hom}(s^n, s^m)$ has a left (or right) $\mathbb{C}[$$]$-basis called the *double leaves* basis, as described in [EW16]. To define this basis, we must first define morphisms known as *light leaves*.

Define $\phi : (\mathrm{ob}(\mathcal{H}(S_2)), \otimes) \to (S_2, *)$ to be the monoid homomorphism[1] mapping $s \mapsto s$ and $\varnothing \mapsto 1$, and $\psi : S_2 \to \mathrm{ob}(\mathcal{H}(S_2))$ to be the function that maps $s \to s$ and $1 \to \varnothing$. Should this be a definition? The maps ϕ and ψ allow words $w = s^n$ to be seen as elements of S_2, and $1, s \in S_2$ to be seen as the objects $\varnothing, s \in \mathcal{H}(S_2)$. Clearly, $\phi\psi$ is the identity map on S_2, and the map $\psi\phi : \mathcal{H}(S_2) \to \mathcal{H}(S_2)$ takes objects to one of $\varnothing$ or s in $\mathcal{H}(S_2)$ by considering them as elements in S_2.

Definition 3.1.8. (Subexpression for S_2) Given a word $w = s^n$, a *subexpression* e is a binary string of length n. We can *apply* a subexpression to produce an object $w(e) \in \mathcal{H}(S_2)$, which is w where terms corresponding to 0 in e are replaced with $\varnothing$.

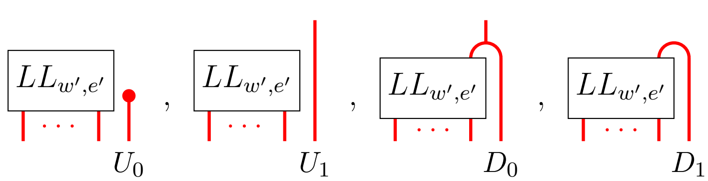
For example, $0000, 0110$ and 1011 are subexpressions of $s^4 = ssss$. Applying the third subexpression gives $ssss(1011) = s\varnothing ss = sss$, by strictness of the monoidal category. Here, each term of the subexpression is a decision to include or exclude the corresponding s in the word, where excluding an s amounts to tensoring with $\varnothing$.

For a word w and subexpression e, we label each term by U_0, U_1, D_0 or D_1. A term is labelled U_* if ϕ applied to the partial subexpression up to the current term is $1 \in S_2$, and labelled D_* if it evaluates to $s \in S_2$. The label's subscript is the corresponding term in e.

Example 3.1.9. For the object $ssss$ and subexpression 0101, we can find the labels:

Choice	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
Partial $w(e)$	$\varnothing$	$\varnothing s = s$	$\varnothing s \varnothing = s$	$\varnothing s \varnothing s = ss$
Labels	U_0	U_0U_1	$U_0U_1D_0$	$U_0U_1D_0D_1$

Definition 3.1.10. The *light leaf* $LL_{w,e} \in \mathrm{Hom}(w, \psi\phi(w(e)))$ for a word w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, the light leaf $LL_{w's,e'i}$ is one of

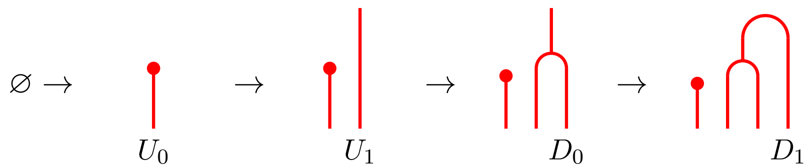


(3.1.11)

[1] A map that preserves the monoidal product and identity element.

corresponding to the next label, where w' and e' are appropriate subwords[2] of w and e respectively. Here, the codomain of a light leaf $LL_{w,e}$ is the object $\psi\phi(w(e))$. So if the next label is U_* then the codomain of $LL_{w',e'}$ is $\varnothing$, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. This implies that the recursive definition is consistent.

Example 3.1.12. Following from Example (3.1.9) for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.

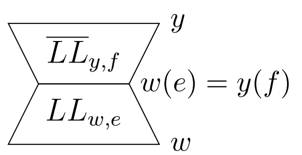


Definition 3.1.13. Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. The *double leaf* for words w, y is a composition

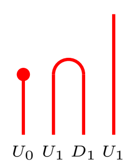
$$\mathbb{LL}_{f,e} \coloneqq \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

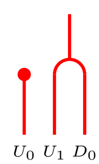
for subexpressions e of w and f of y such that $w(e) = f(y)$.

Visually this looks like a morphism from w to y factoring through $w(e) = y(f) \in \{\varnothing, s\}$,



Example 3.1.14. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$ 

and $LL_{y,f} =$ 

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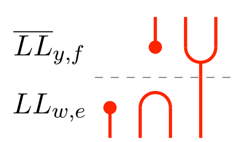
Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$, factoring through s, is



Note that these double leaves have no floating diagrams such as . In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{C}[\ ]$-module. We define the (left) -action as left concatenation by . Since we can move barbells to the right via. the relation (3.1.5d), we can equivalently act by $\mathbb{C}[\ ]$ on the right. This leads us to the following theorem.

[2]A word with some letters removed.

Theorem 3.1.15 (Elias-Williamson [EW16, Theorem 1.2])**.** *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{LL}(w, y)$ be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $w(e) = y(f)$. Then $\mathbb{LL}(w, y)$ is a basis for* $\mathrm{Hom}(w, y)$ *as a left (or right)* $\mathbb{C}[$  $]$*-module.*

A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.16. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky's construction in [Lib08].

The morphisms in this category can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1. The degree of a general diagram is the sum of the degrees of the generators that appear in it.

Put example

The double leaves bases allow us to show that the Karoubi envelope of $\mathcal{H}(S_2)$ is equivalent to the category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$ over S_2 as monoidal categories.

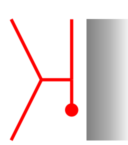
Theorem 3.1.17 (Elias-Williamson [EW16, Theorem 6.30])**.** *The category* $\mathrm{Kar}(\mathcal{H}(S_2))$ *and the category of Soergel Bimodules* $\mathbb{S}\mathrm{Bim}$ *over S_2 are equivalent as graded $\mathbb{C}$-linear monoidal categories.*

The proof in [EW16] gives an equivalence of graded $\mathbb{C}$-linear monoidal categories $\mathcal{H}(S_2) \cong \mathbb{BS}\mathrm{Bim}$ where $\mathbb{BS}\mathrm{Bim}$ is the category of Bott-Samelson bimodules over S_2. This was done by comparing the graded dimensions of morphism spaces using double leaves bases. Since $\mathrm{Kar}(\mathbb{BS}\mathrm{Bim}) \cong \mathbb{S}\mathrm{Bim}$ and Karoubi envelope preserves equivalences, we obtain $\mathrm{Kar}(\mathcal{H}(S_2)) \cong \mathbb{S}\mathrm{Bim}$.

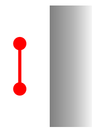
3.2 Diagrammatic $\mathcal{O}(\mathrm{SL}(2))$

With the diagrammatic category $\mathcal{H}(S_2)$, we can describe diagrammatics for the category $\mathcal{O}(\mathrm{SL}(2))$. In particular, we define a modular category $\mathcal{DO}(\mathrm{SL}(2))$ with a left-action of $\mathcal{H}(S_2)$. At the end, we give a useful description of category $\mathcal{O}$ for $\mathrm{SL}(2)$ and a proof for the equivalence of these categories (up to idempotent completion).

Let $\mathcal{DO}(\mathrm{SL}(2))$ be a category with elements generated (Define what this means.) by the identity element $\varnothing$ of $\mathcal{H}(S_2)$ and morphisms generated by the empty diagram $\varnothing$, where $\mathcal{H}(S_2)$ acts on the left by left concatenation for objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0, 1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0, 1] \times \mathbb{R}$. For example a morphism may be

 .

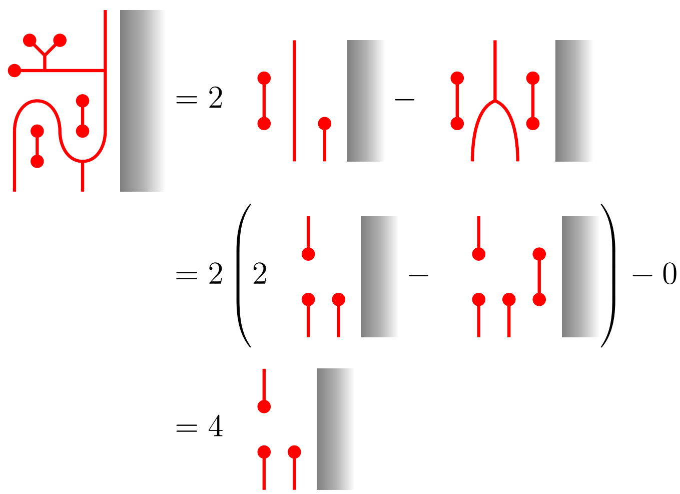
We impose the relation that diagrams are related to the wall by



$$= 0. \tag{3.2.1}$$

Notice that all the morphisms in $\mathcal{H}(S_2)$ appear in this modular category, although they may have been annihilated by (3.2.1).

Example 3.2.2. We use the new relation (3.2.1) to further simplify the morphism in Example (3.1.7).



The objects and morphisms in this category are essentially that of $\mathcal{H}(S_2)$ modulo the wall relation (3.2.1), so the double leaves bases of $\mathcal{H}(S_2)$ remain a spanning set of the morphism spaces in $\mathcal{DO}(\mathrm{SL}(2))$ over $\mathbb{Z}[\ ]$. Note that is not a $\mathbb{Z}[\ ]$-basis in $\mathcal{DO}(\mathrm{SL}(2))$ as linear independence is not preserved by the new relation. For example, any double leaf factoring through $\varnothing$ is 0 when multiplied by . It turns out that double leaves are a $\mathbb{Z}$-module basis for $\mathcal{DO}(\mathrm{SL}(2))$. Is this right?

Before moving on, we give a useful description of $\mathcal{O}(\mathrm{SL}(2))$ from classical theory (see [Hum08] Find precise ref). Category $\mathcal{O}$ is an abelian category that can be decomposed into a direct sum[3] of subcategories called *blocks*. There is a particularly important block called the *principle block*, which we write as $\mathcal{O}_0$. Over SL(2), the principle block of $\mathcal{O}$ has exactly two simple modules $L(\varnothing)$ and $L(s)$, corresponding to the elements of S_2, and there are projective covers[4] $P(\varnothing) \twoheadrightarrow L(\varnothing)$ and $P(s) \twoheadrightarrow L(s)$. Here, all elements of $\mathcal{O}_0$ are generated from direct sums of $P(s)$ and their filtrations. Note that that simple modules $L(\varnothing)$ and $L(s)$ appear as factors in the filtration of their respective projective modules, and that $P(\varnothing) = L(\varnothing)$ and $P(s) \supset L(s)$. In this case, all blocks of $\mathcal{O}$ are isomorphic to either the principle block $\mathcal{O}_0$ or the block generated by the trivial module $L(\varnothing)$ (isomorphic to finite dimensional vector spaces over(?) This contains no

[3]Put reference; put quick explanation

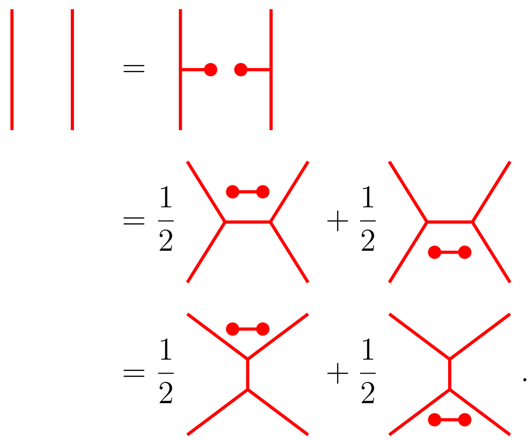
[4]A projective cover is a projective module and a surjection onto our module, which is the "smallest".

information about SL(2)). This means that a description of the modules $P(\varnothing)$ and $P(s)$ in $\mathcal{O}_0$ induces a description of the entire category $\mathcal{O}(\mathrm{SL}(2))$.

Maybe put this next bit in section 3.1

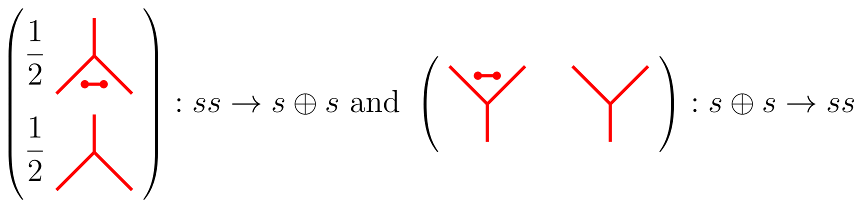
Say more about what this is, and why we say it here

In the diagrammatic category $\mathcal{H}(S_2)$ from Section 3.1, we have the relation



(3.2.3)

In the additive closure of this category, this shows there is an isomorphism $s \otimes s \cong s \oplus s$ by



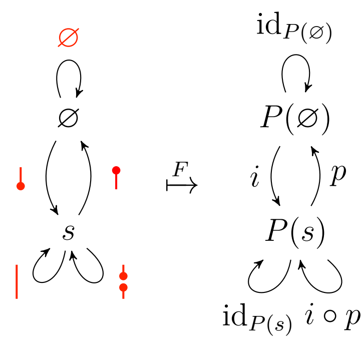
It follows from (3.1.5d), (3.1.5c) and the above calculation (3.2.3), that these maps are inverses.

Theorem 3.2.4 (???)**.** *The diagrammatic category* $\mathrm{Kar}(\mathcal{DO}(\mathrm{SL}(2)))$ *and* $\mathcal{O}(\mathrm{SL}(2))$ *are equivalent as categories.*

Check all of this & Put precise references

Proof. As a shorthand, we write $\mathcal{DO}$ for $\mathcal{DO}(\mathrm{SL}(2))$ and $\mathcal{O}$ for $\mathcal{O}(\mathrm{SL}(2))$. The work of Soergel in [Soe90] shows that $\mathcal{O}$ is a Soergel module, i.e. it has a left action of the category of Soergel bimodules defined by applications of the translation functors $\Theta_\varnothing, \Theta_s \in \mathrm{End}(\mathcal{O})$, corresponding to elements in S_2 Check this. Classical results, e.g. [Hum08], show that Θ_s is a Frobenius object in the category of endofunctors of $\mathcal{O}$. Then there are unit, counit, multiplication and comultiplication natural transformations satisfying coherence relations in the Frobenius object structure. Additionally, Soergel's work (References?) shows that there is a relation in $\mathcal{O}$ analogous to the barbell-wall relation (3.2.1), and that there is an isomorphism $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$ (Is the direct sum here correct?) which is analogous to the isomorphism given by (3.2.3).

Define the functor $F : \mathcal{DO} \to \mathcal{O}$ that sends the empty object $\varnothing$ to the trivial module $P(\varnothing)$, and the Soergel module action corresponding to s to the translation functor Θ_s. Then the object s maps to $\Theta_s(P(\varnothing)) =: P(s)$, and s^3 maps to $\Theta_s^3(P(\varnothing)) = \Theta_s\Theta_s\Theta_s(P(\varnothing))$. Functoriality forces F to map identity diagrams $s^n \to s^n$ to $\mathrm{id}_{\Theta_s^n(P(\varnothing))}$. For non-identity maps, we let $F(\ ) = i$ be the inclusion $P(\varnothing) \to P(s)$ and $F(\ ) = p$ be the projection $P(s) \to P(\varnothing)$. The mapping of F is depicted by the following diagram.



Note that the projection and inclusion maps are exactly the unit and counit of Θ_s evaluated at $P(\varnothing)$. This is enough to completely determine the image of F, since $\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s$. Now Θ_s is a Frobenius object and the barbell-wall relation is satisfied in $\mathcal{O}$, so the functor F is well defined.

Now we show that F is fully faithful. We know (From Soergel, EW, Libedinsky? Explain this more) that the image of and generate all morphisms of the form $\Theta_s^n(P(\varnothing)) \to \Theta_s^m(P(\varnothing))$, so F is full. For the faithfulness of F, it suffices to match the dimensions of $\mathbb{Z}$-bases for hom-spaces involving $P(\varnothing)$ and $P(s)$. By double leaves in $\mathcal{DO}$, as $\mathbb{Z}$-modules, $\mathrm{Hom}(\varnothing, \varnothing)$ has a basis $\{\varnothing = \mathrm{id}_\varnothing\}$, $\mathrm{Hom}(s, \varnothing)$ has a basis $\{ \ \}$, $\mathrm{Hom}(\varnothing, s)$ has a basis $\{ \ \}$, and $\mathrm{Hom}(s, s)$ has a basis $\{\mathrm{id}_s, \ \circ \ \}$. The dimensions match exactly with the corresponding images of F. Therefore F is fully faithful.

Since objects in $\mathcal{O}$ are direct sums and direct summands (How does this fit into the description of $\mathcal{O}$ we mentioned before the proof?) of the elements $\Theta_s^n(P(\varnothing))$ for non-negative integer n, taking the Karoubi envelope $\mathrm{Kar}(\mathcal{DO})$ induces an equivalence of categories $\mathrm{Kar}(\mathcal{DO}) \cong \mathcal{O}$.

□

Old Proof. As a shorthand, we write $\mathcal{DO}$ for $\mathcal{DO}(\mathrm{SL}(2))$ and $\mathcal{O}$ for $\mathcal{O}(\mathrm{SL}(2))$. Let $F : \mathcal{DO} \to \mathcal{O}$ be a functor that sends the empty object $\varnothing$ to the trivial module $P(\varnothing)$ and $s \mapsto P(s)$, the indecomposable objects in $\mathcal{O}$ corresponding to elements in S_2. On morphisms, F sends the identity morphism on s (the red strand) to the translation functor Θ_s in $\mathcal{O}$ corresponding to $s \in S_2$. This completely determines the action of F (Why?). Due to classical results in [Hum08], the translation functors are Frobenius objects, so there have unit, counit, multiplication and comultiplication maps with appropriate relations in $\mathcal{O}$. These the image of which are the image of the generators (3.1.3) under F, that satisfy the analogous relations (3.1.5). Furthermore, the work of Soergel in [Soe90] shows that there is a relation in $\mathcal{O}$ analogous to the barbell-wall relation (3.2.1). This F is well defined as all the generators and relations in $\mathcal{DO}$ are accounted for (Word this better).

Next we show that F is a fully faithful functor. By results from [EW16] and [Lib08], the inclusion $\mathcal{H}(S_2) \to \mathbb{S}\mathrm{Bim}$ is fully faithful, so we have a copy of double leaves bases in $\mathbb{S}\mathrm{Bim}$. By the work of Soergel in [Soe90], the category $\mathcal{O}$ is a Soergel module (Explain what this is) with certain bases for the morphism. Thus (Why?) it suffices to compare the dimension of morphism spaces between $\mathcal{DO}$ and $\mathcal{O}$, as Soergel modules. [Comparison?]

The functor F mapped objects of $\mathcal{DO}$ to objects ??? in $\mathcal{O}$, which generate all other objects by direct sums and direct summands Is this right?. Now F is fully faithful, Kar preserves equivalences of categories and taking the Karoubi envelope of the image of $\mathcal{DO}$ gives exactly $\mathcal{O}$ (Is this right?), we obtain an equivalence of categories between $\mathrm{Kar}(\mathcal{DO})$ and $\mathcal{O}$. $\square$

Note on induced grading

Chapter 4

Two-colour Diagrammatics

The previous chapter had its focus on the symmetric group generated by one element S_2, which brought about one-colour diagrammatics. We shift our attention to a more complex example by adding an extra generator, that is, another colour. In particular, we consider the case for the affine symmetric group on two elements $\widetilde{S}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$.

4.1 Two-colour Diagrammatic Hecke Category

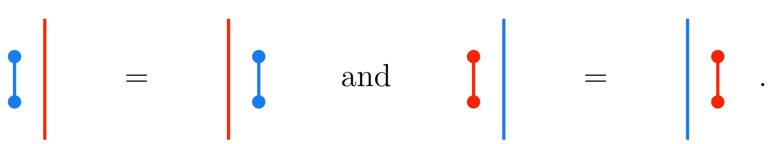
Corresponding to $\widetilde{S}_2$, we define the two-colour (diagrammatic) Hecke category $\mathcal{H}(\widetilde{S}_2)$. This is a (strict) $\mathbb{Z}$-linear monoidal category given by the following isotopy presentation.

Objects in $\mathcal{H}(\widetilde{S}_2)$ are generated by formal tensor products of the non-identity elements $s, t \in \widetilde{S}_2$. As before, we write objects as words such as $ssttst =: s^2t^3st$ where the tensor product is concatenation, and associate the colour red to s and blue to t. The empty word is the monoidal identity, which we write as $\varnothing$.

The morphisms are generated by the univalent and trivalent vertices

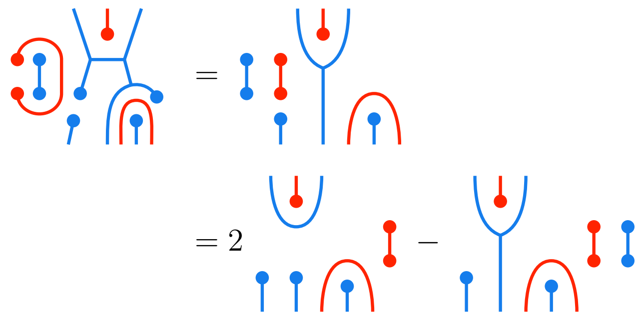
(4.1.1)

that are maps $s \to \varnothing$, $ss \to s$, $t \to \varnothing$ and $tt \to t$ respectively. As in the one-colour case, tensor product is horizontal concatenation, composition is appropriate vertical stacking, and we denote the empty diagram $\varnothing \to \varnothing$ by $\varnothing$. For each colour, these diagrams have the one-colour relations given by (3.1.5). Since we have two colours now, we also need to describe how the colours interact. This is given by the *two-colour* relations



(4.1.2)

Example 4.1.3. The following morphism in $\mathrm{Hom}(ttsts, tst)$ can be simplified using the one-colour and two-colour relations.



Remark 4.1.4. Notice that, in this category, red and blue lines never cross as no generators that allow crossings. This is a consequence of working over affine S_2 in which the generators s and t have no relation. Mention example of crossing and S_3.

4.2 Diagrammatic Tilt(SL(2))

Blah

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