Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

Victor Zhang
UNSW Australia

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Introduction

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Background

2.1 Drawing Monoidal Categories

A monoidal category $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$, such that certain associativity and unit relations hold[1]. We will assume that monoidal categories are strict, since all monoidal categories are monoidally equivalent to a strict one [Reference?]. The morphisms of C can be drawn as string diagrams, where we read from bottom to top and functions are labelled dots. For example
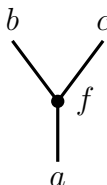
describes a morphism $f : a \to b \otimes c$. By convention, $\mathbb{1}$ is blank and morphisms to $\mathbb{1}$ are blank. For example

The vertical stacking of diagrams depicts composition.

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[1]For more details see [Eti+15].

Chapter 3

One-colour Diagrammatics

3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the one-colour (diagrammatic) Hecke category $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = 1 \rangle$. This is a monoidal category which we will describe via generators and relations.

Remark 3.1.1. All the following diagrams could be defined without the language of categories, as planar pictures with appropriate vertical stacking and horizontal concatenation. Nevertheless, we define them in a category because they will eventually be seen as diagrammatic versions of important categories in representation theory.

The objects of $\mathcal{H}(S_2)$ are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is just concatenation. The empty tensor product (or empty word) will be denoted by $\varnothing =: s^0$.

The morphisms are generated, up to isotopy, by univalent and trivalent vertices

$$\tag{3.1.2}$$

which are maps $s \to \varnothing$ and $ss \to s$ respectively. Note that we put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). Additionally[1], we allow formal sums of diagrams by putting a $\mathbb{Z}$-module structure on each morphism space $\mathrm{Hom}(s^n, s^m)$, for non-negative integers n, m. Composition or tensor with the zero morphism 0 in this $\mathbb{Z}$-module result in 0. To abuse notation, the empty diagram $\varnothing \to \varnothing$ will be denoted $\varnothing$. The identity morphism in $\mathrm{Hom}(s^n, s^n)$ is the diagram consisting of n vertical lines.

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[1]Pun intended.

Such diagrams are subject to the following local relations.

$$\text{(3.1.3a)}$$

$$\text{(3.1.3b)}$$

$$= 0 \qquad \text{(3.1.3c)}$$

$$= 2 \qquad - \qquad \text{(3.1.3d)}$$

Remark 3.1.4. The object s is a Frobenius algebra object in $\mathcal{H}(S_2)$. The generators (3.1.2) and their horizontal reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.3a) and (3.1.3b).

Example 3.1.5. Let us use the relations in (3.1.3) to simplify the following morphism in $\mathrm{Hom}((s,s),(s))$.

$$= $$

$$= 2 \qquad - $$

$$= 2 \qquad - $$

There is a right (or left) $\mathbb{Z}[\,]$-basis for $\mathrm{Hom}(s^n, s^m)$ described in [EW16] called the Double Leaves basis. To define this basis we must first look at morphisms known as Light Leaves. Given a word $w = s^n$, a subexpression is a binary string of length n. For example, $0000, 0110$ and 1011 are subexpressions of $s^4 = ssss$. Given a subexpression e of an object w, we can apply it to produce an element $w(e) \in S_2$, e.g. $ssss(1011) =$

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$s * 1 * s * s = s$. Each term of the subexpression is a decision of whether to include the corresponding s in the word, where excluding an s amounts to multiplying by 1. Talk about how s could mean an object of $\mathcal{H}$ or an object of S_2.

For a subexpression e of an expression w, we can label each term by U_0, U_1, D_0 or D_1. The label is U_* if the partial subexpression up to the current term evaluates to $1 \in S_2$ and D_* if it evaluates to $s \in S_2$, where the subscript is the corresponding term in e.

Example 3.1.6. For the object $ssss$ and subexpression 0101, we can find the labels:

Choice	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
Partial $w(e)$	1	$1 * s = s$	$1 * s * 1 = s$	$1 * s * 1 * s = 1$
Labels	U_0	$U_0 U_1$	$U_0 U_1 D_0$	$U_0 U_1 D_0 D_1$

The light leaf $LL_{w,e} \in \mathrm{Hom}(w, w(e))$ $w(e)$ here is an element of S_2 that we identify with one of the objects $\varnothing$ or s, corresponding to the object w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0,1\}$, $LL_{w's,e'i}$ is one of

$$\tag{3.1.7}$$

depending on the next label, where w' and e' are appropriate subwords of w and e. Observe that the codomain of a light leaf $LL_{w,e}$ is the object corresponding to the evaluation $w(e) \in S_2$ of the subexpression. The recursive definition is consistent, since if the next label is U_* then the codomain of $LL_{w',e'}$ (the evaluation of the partial subexpression $w'(e')$ up to the label) is 1, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. Rewrite this to make sense. Do we need to talk about 'degree' of light leaves?

Example 3.1.8. Following from Example (3.1.6) for $w = ssss$ and $e = 0101$, we have labels $U_0 U_1 D_0 D_1$ so the light leaf $LL_{w,e}$ is built as follows.

Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. A *double leaf* associated to expressions w, y is a composition

$$\mathbb{LL}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

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for subexpressions e of w and f of y such that $w(e) = f(y)$. Visually this looks like a morphism from w to y factoring through $w(e) = y(f) \in \{1, s\}$,

$$\overline{LL}_{y,f} \Big/ LL_{w,e} \Big\backslash \quad w^e = y^f \ .$$

Example 3.1.9. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$$LL_{w,e} = \quad \underset{U_0 \ U_1 \ D_1 \ U_1}{} \qquad \text{and } LL_{y,f} = \quad \underset{U_0 \ U_1 \ D_0}{} \ .$$

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$, factoring through s, is

$$\overline{LL}_{y,f} \qquad \underset{s \ s \ s}{}$$
$$LL_{w,e} \qquad \underset{s \ s \ s \ s}{} .$$

   Notice that these double leaves have no floating diagrams such as ❗. In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a left $\mathbb{Z}[\,❗\,]$-module. Here, the left ❗-action on a diagram is just concatenation by ❗ on the left. Since we can move barbells to the right via. the relation (3.1.3d), we can equivalently act by $\mathbb{Z}[\,❗\,]$ on the right. This leads us to the following theorem.

Theorem 3.1.10 (Elias-Williamson [EW16], Theorem 1.2). *Given objects $w, y \in \mathcal{H}(S_2)$, let $\mathbb{LL}_{w,y}$ [2] be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $w(e) = y(f)$. Then $\mathbb{LL}_{w,y}$ is a left (or right) $\mathbb{Z}[\,❗\,]$-module basis for $\mathrm{Hom}(w, y)$.*

   A purely diagrammatic proof (of a more general theorem) can be found in [EW16].

Remark 3.1.11. The above light leaves and double leaves, introduced in [EW16], are diagrammatic analogues of Libedinsky's work in [Lib08].

   The morphisms can be graded such that the univalent vertices has degree 1 and trivalent vertices have degree -1. The degree of a general diagram is the sum of the degrees of the generators that appear in it. How do you do degree of a $\mathbb{Z}$-linear combination? Put example

   The double leaves bases allow us to show that the Karoubi envelope of $\mathcal{H}(S_2)$ is equivalent to the category of Soergel Bimodules $\mathbb{S}$Bim over S_2 as monoidal categories.

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[2] this can be confused with the double leaves themselves, maybe write $\mathbb{LL}(w, y)$

Theorem 3.1.12 (Elias-Williamson [EW16], Theorem 6.30)**.** *There is an equivalence of categories between* $\mathrm{Kar}(\mathcal{H}(S_2))$ *and the category of Soergel Bimodules* $\mathbb{S}\mathrm{Bim}$ *over* S_2.

The proof gives an equivalence of categories $\mathcal{H}(S_2) \cong \mathbb{BS}\mathrm{Bim}$ of Bott-Samelson bimodules over S_2, by comparing the dimensions of morphism spaces using double leaves.

3.2 Diagrammatic $\mathcal{O}(\mathrm{SL}(2))$

With the diagrammatic category $\mathcal{H}(S_2)$, we can describe diagrammatics for the category $\mathcal{O}(\mathrm{SL}(2))$. In particular, we define a modular category $\mathcal{DO}(\mathrm{SL}(2))$ with a left-action of $\mathcal{H}(S_2)$.

The category $\mathcal{DO}(\mathrm{SL}(2))$ has elements that are generated (Define what this means.) by the identity element $\varnothing$ of $\mathcal{H}(S_2)$ and morphisms are generated by the empty diagram $\varnothing$, where $\mathcal{H}(S_2)$ acts on the left by left concatenation for objects and morphisms. In addition to the relations from $\mathcal{H}(S_2)$, the morphisms have one new relation in which diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0,1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0,1] \times \mathbb{R}$. For example a morphism may be
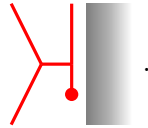
We impose the relation that diagrams are related to the wall by

$$\text{} = 0. \tag{3.2.1}$$

Notice that all the morphisms in $\mathcal{H}(S_2)$ appear in this modular category, although they may have been annihilated by (3.2.1).

Example 3.2.2. We use the new relation (3.2.1) to further simplify the morphism in Example (3.1.5).

$$= 4 \quad \text{(diagram)}$$

Since the objects and morphisms in this category are that of $\mathcal{H}(S_2)$ modulo the wall relation (3.2.1), double leaves still span the morphism spaces of $\mathcal{DO}(\mathrm{SL}(2))$. We see these as left $\mathbb{Z}[\,\text{(diagram)}\,]$-module bases since right multiplication by (diagram) annihilates the morphism and adds no new diagrams. Are these basis? They might not be linearly independent e.g. double leaves with a gap means that multiplying with (diagram) kills it.

Only allowing $\mathbb{Z}$-module should make them linearly independent, and they should still span.

Theorem 3.2.3 (???). *The diagrammatic category* $\mathrm{Kar}(\mathcal{DO}(\mathrm{SL}(2)))$ *and* $\mathcal{O}(\mathrm{SL}(2))$ *are equivalent as categories.*

Check all of this Put precise references

Proof. As a shorthand, we write $\mathcal{DO}$ for $\mathcal{DO}(\mathrm{SL}(2))$ and $\mathcal{O}$ for $\mathcal{O}(\mathrm{SL}(2))$. Let $F : \mathcal{DO} \to \mathcal{O}$ be a functor that sends objects ???. On morphisms, F sends | (What is non-trivial strand?) to a relevant translation functor in $\mathcal{O}$. This completely determines the action of F (Why?). Due to classical results in Humphrey's book [Hum08], the translation functors are maps between Frobenius objects, so there are unit, counit, multiplication and comultiplication maps with appropriate relations in $\mathcal{O}$. These are the image of the generators (3.1.2) under F, that satisfy the analogous relations (3.1.3). Also, the work of Soergel in [Soe90] imply that there is also a relation analogous to the barbell-wall relation (3.2.1) in $\mathcal{O}$. Thus F is well defined as all the generators and relations in $\mathcal{DO}$ are accounted for (Word this better).

Next we show that F is a fully faithful functor. By results from [EW16] and [Lib08], the inclusion $\mathcal{H}(S_2) \to \mathbb{S}\mathrm{Bim}$ is fully faithful, so we have a copy of double leaves bases in $\mathbb{S}\mathrm{Bim}$. By the work of Soergel in [Soe90], the category $\mathcal{O}$ is a Soergel module (What does this mean?) with certain bases for the morphism. Thus (Why?) it suffices to compare the (graded??) dimensions of the morphism spaces between $\mathcal{DO}$ and $\mathcal{O}$. [Comparison?]

The functor F mapped objects of $\mathcal{DO}$ to objects ??? in $\mathcal{O}$, which generate all other objects by direct sums, direct summands and grading shifts (Is this right?). Now F is fully faithful, Kar preserves equivalences of categories and taking the Karoubi envelope of the image of $\mathcal{DO}$ gives exactly $\mathcal{O}$ (Is this right?), we obtain an equivalence of categories between $\mathrm{Kar}(\mathcal{DO})$ and $\mathcal{O}$. $\square$

Chapter 4

Two-colour Diagrammatics

4.1 Two-colour Diagrammatic Hecke Category

Blah

4.2 Diagrammatic $\mathrm{Tilt}(SL(2))$

Blah

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