

# Diagrammatic Categories

## in Representation Theory

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# Kazhdan–Lusztig Conjecture

(Motivation)

- Relates Kazhdan–Lusztig polynomials to Jordan–Hölder multiplicities of Verma modules
- First proof by Beilinson and Bernstein (1981) using  $D$ -modules
- Algebraic proof by Elias and Williamson (2010's) using Soergel bimodules and diagrammatics

# Talk Overview

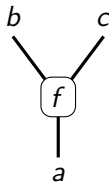
1. Diagrammatic Monoidal Categories
2. Diagrammatic Soergel bimodules
3. Further Applications

# Monoidal Categories

A *monoidal category* is a category with an associative multiplication  $\otimes$  for objects and morphisms, and a unit object  $\mathbb{1}$ , such that the multiplication works well with composition.

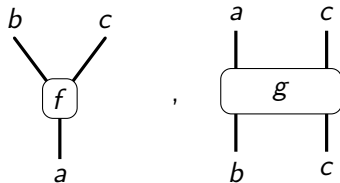
# Monoidal Categories

$$f : a \rightarrow b \otimes c$$



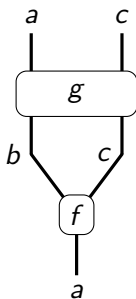
# Monoidal Categories: Composition

$$f : a \rightarrow b \otimes c$$
$$g : b \otimes c \rightarrow a \otimes c$$



# Monoidal Categories: Composition

$$f : a \rightarrow b \otimes c$$
$$g : b \otimes c \rightarrow a \otimes c$$



# Monoidal Categories: Identity

$$\mathrm{id}_a : a \rightarrow a$$

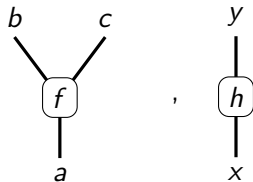
$$\begin{array}{c} a \\ | \\ a \end{array}$$



# Monoidal Categories: Tensor

$$f : a \rightarrow b \otimes c$$

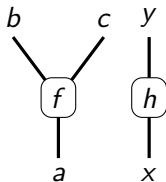
$$h : x \rightarrow y$$



# Monoidal Categories: Tensor

$$f : a \rightarrow b \otimes c$$

$$h : x \rightarrow y$$



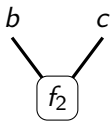
# Monoidal Categories: Unit

$$f_1 : a \rightarrow \mathbb{1}$$



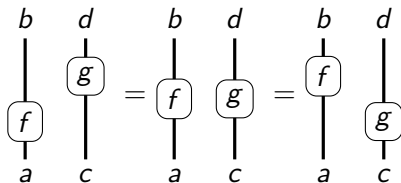
# Monoidal Categories: Unit

$$f_2 : \mathbb{1} \rightarrow b \otimes c$$

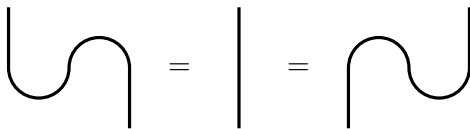


# Monoidal Categories

*Interchange Law*



# Isotopy



# $\mathbb{Z}$ -linear Monoidal Category

A diagrammatic equation in a  $\mathbb{Z}$ -linear monoidal category. The left side consists of the integer 3 followed by a node labeled  $f$ . The node  $f$  has a single input labeled  $a$  from below and two outputs labeled  $b$  and  $c$  going upwards and to the left and right respectively. The right side consists of the integer  $-2$  followed by two nodes,  $f_1$  and  $f_2$ . Node  $f_1$  has a single input labeled  $a$  from below and one output labeled  $b$  going upwards and to the left. Node  $f_2$  has a single input from node  $f_1$  (the output of  $f_1$  is connected to the input of  $f_2$ ) and one output labeled  $c$  going upwards and to the right.

# Diagrammatic Soergel Bimodules

Let  $\mathcal{D}$  be the (diagrammatic)  $\mathbb{Z}$ -linear monoidal category with:

Generating object  $I$ .

Generating morphisms



and local relations...



# Relations

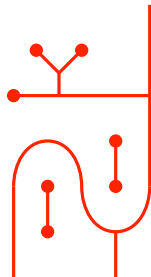
$$\begin{array}{c} | \\ \text{---} \bullet \end{array} = \begin{array}{c} | \\ \\ \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array},$$

$$\begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array},$$

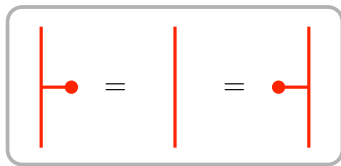
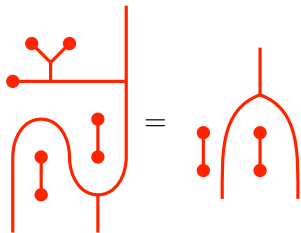
$$\begin{array}{c} | \\ \bigcirc \\ | \end{array} = 0,$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

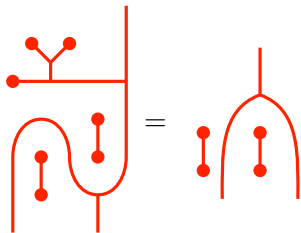
# Example 1



# Example 1

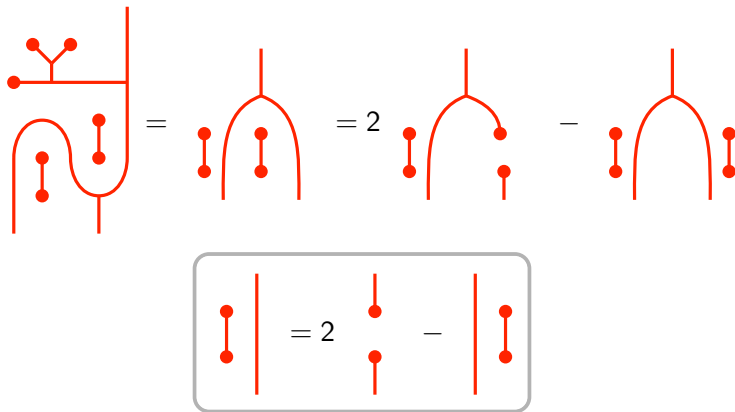


# Example 1

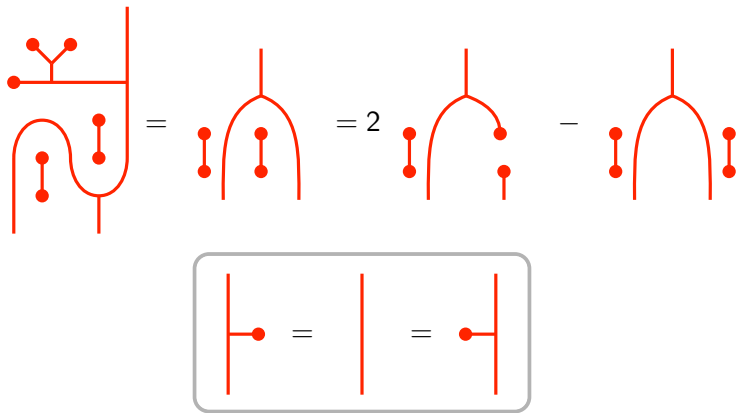


$$\begin{array}{|c} \bullet \\ \bullet \\ \hline \end{array} = 2 \begin{array}{|c} \bullet \\ \bullet \\ \hline \end{array} - \begin{array}{|c} \bullet \\ \bullet \\ \hline \end{array}$$

# Example 1



# Example 1



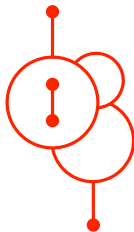
# Example 1

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = 2 \times \text{Diagram 3} - \text{Diagram 4} \\
 & \qquad \qquad \qquad = 2 \times \text{Diagram 5} - \text{Diagram 6}
 \end{aligned}$$

The diagrams are composed of red lines and dots, representing a sequence of transformations in a topological or combinatorial context.

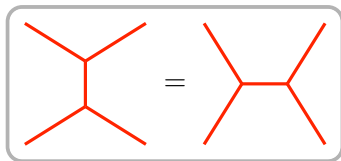
- Diagram 1:** A complex structure with a horizontal line at the top, a vertical line on the right, and a loop at the bottom. It contains several dots and internal connections.
- Diagram 2:** A simpler structure with a vertical line on the right and a loop at the bottom, with dots at the top and bottom.
- Diagram 3:** A structure with a vertical line on the right and a loop at the bottom, with dots at the top and bottom.
- Diagram 4:** A structure with a vertical line on the right and a loop at the bottom, with dots at the top and bottom.
- Diagram 5:** A structure with a vertical line on the right and a loop at the bottom, with dots at the top and bottom.
- Diagram 6:** A structure with a vertical line on the right and a loop at the bottom, with dots at the top and bottom.

## Example 2

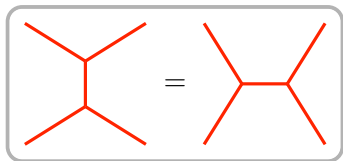
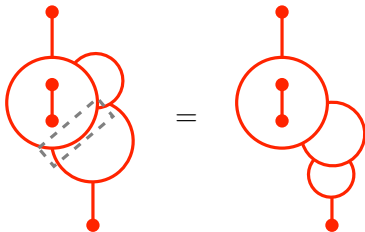




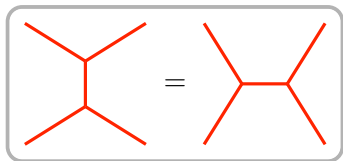
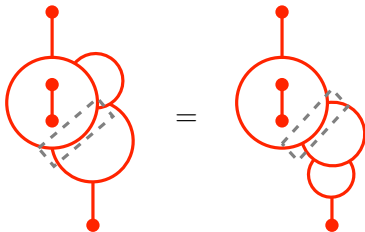
## Example 2



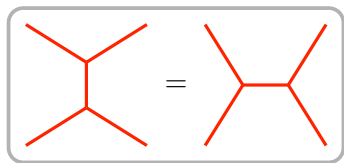
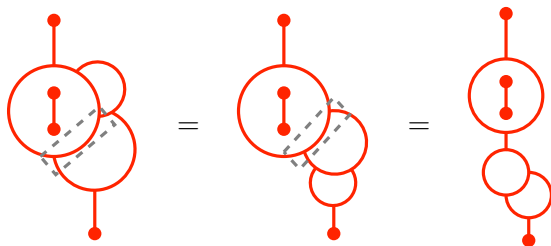
## Example 2



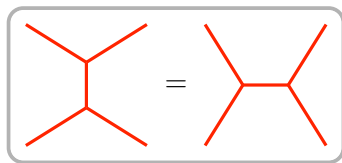
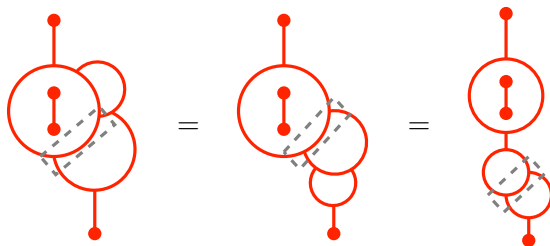
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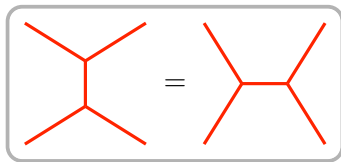
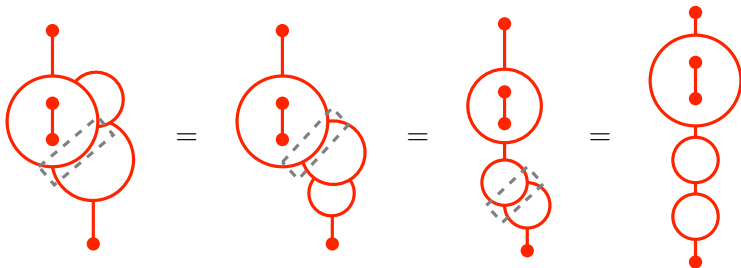
## Example 2



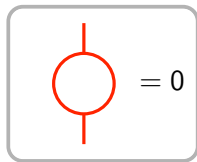
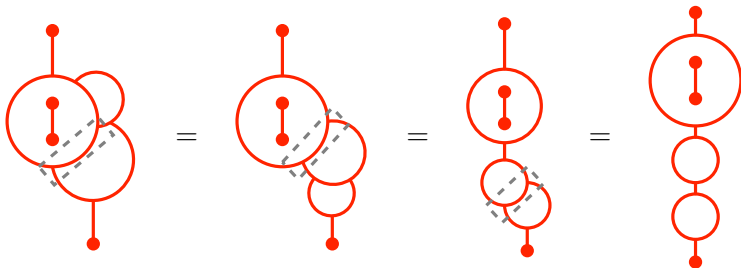
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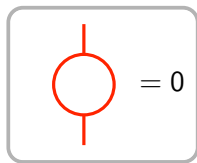
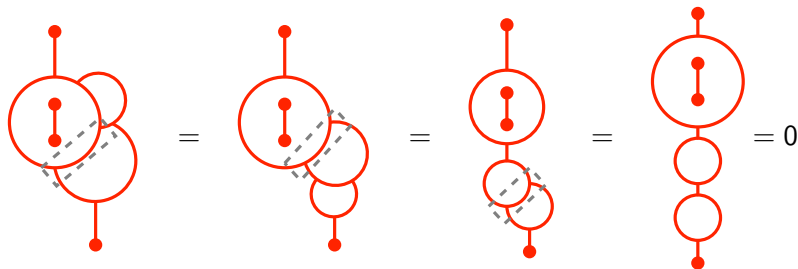
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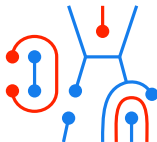
# Soergel Bimodules

## Theorem (Elias–Williamson, 2013)

*The additive Karoubi envelope of  $\mathcal{D}$  is equivalent to the category of Soergel Bimodules  $\mathbb{S}\text{Bim}$  over  $S_2$  as (idempotent complete) graded additive  $\mathbb{C}$ -linear monoidal categories.*

# Generalisations

- Over general Coxeter groups e.g.  $S_n$ ,  $D_n$



- Diagrammatics for other categories of representations...

# Further Applications

- Diagrammatics for BGG Category  $\mathcal{O}$ , Tilting modules
- Changing scalars to fields of characteristic  $p$