Diagrammatic Categories

in Representation Theory

Victor Zhang

Supervisors:
Dr Anna Romanov
Dr Arnaud Brothier
Dr Daniel Tubbenhauer

UNSW Sydney

Kazhdan–Lusztig Conjecture

(Motivation)

- Relates Kazhdan–Lusztig polynomials to Jordan–Hölder multiplicities of Verma modules
- First proof by Beilinson and Bernstein (1981) using D-modules
- Algebraic proof by Elias and Williamson (2010's) using Soergel bimodules and diagrammatics

Talk Overview

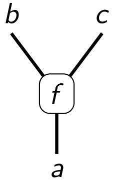
1. Diagrammatic Monoidal Categories
2. Diagrammatic Soergel bimodules
3. Further Applications

Monoidal Categories

A *monoidal category* is a category with an associative multiplication $\otimes$ for objects and morphisms, and a unit object $\mathbb{1}$, such that the multiplication works well with composition.

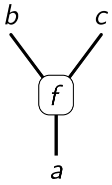
Monoidal Categories

$$f : a \to b \otimes c$$

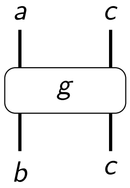


Monoidal Categories: Composition

$$f : a \to b \otimes c$$
$$g : b \otimes c \to a \otimes c$$

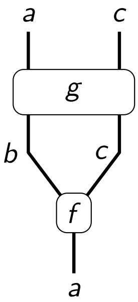


,



Monoidal Categories: Composition

$$f : a \to b \otimes c$$
$$g : b \otimes c \to a \otimes c$$

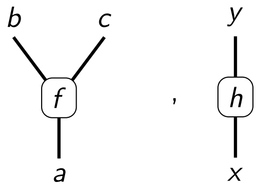


Monoidal Categories: Identity

$$\mathrm{id}_a : a \to a$$

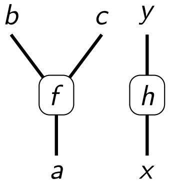
Monoidal Categories: Tensor

$$f : a \to b \otimes c$$
$$h : x \to y$$



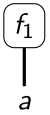
Monoidal Categories: Tensor

$$f : a \to b \otimes c$$
$$h : x \to y$$



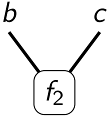
Monoidal Categories: Unit

$$f_1 : a \to \mathbb{1}$$



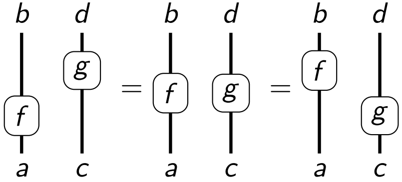
Monoidal Categories: Unit

$$f_2 : \mathbb{1} \to b \otimes c$$

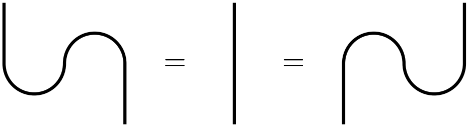


Monoidal Categories

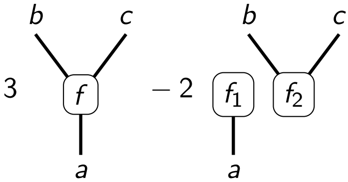
Interchange Law



Isotopy

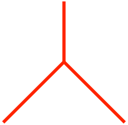
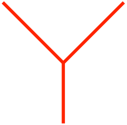


$\mathbb{Z}$-linear Monoidal Category



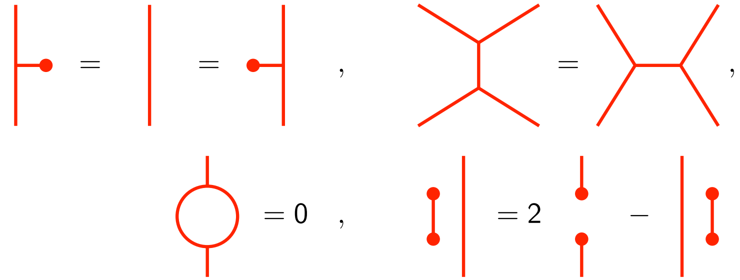
Diagrammatic Soergel Bimodules

Let $\mathcal{D}$ be the (diagrammatic) $\mathbb{Z}$-linear monoidal category with:
Generating object $\color{red}{|}$.
Generating morphisms

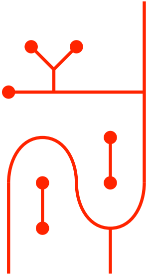
 ,  ,  , 

and local relations...

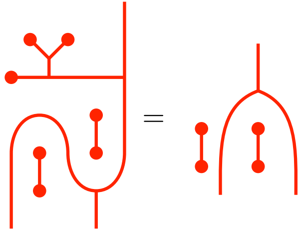
Relations

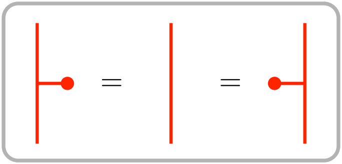


Example 1

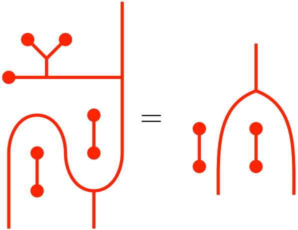


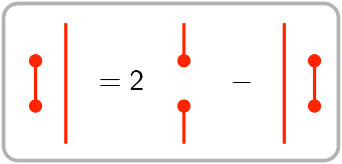
Example 1



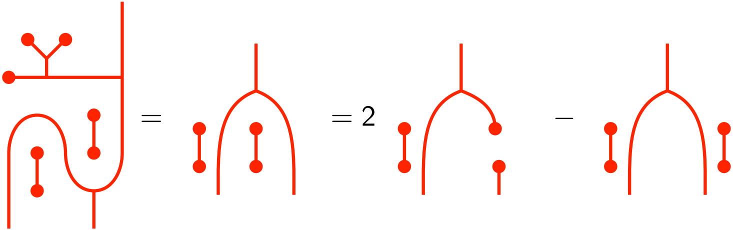


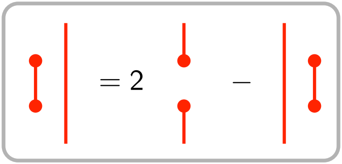
Example 1



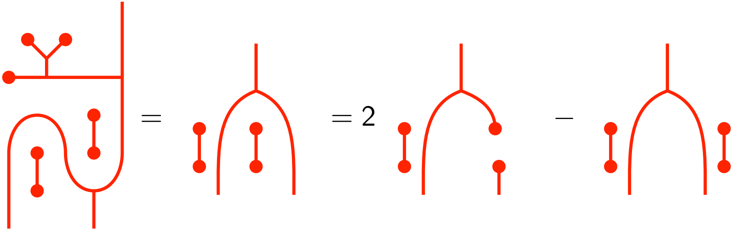


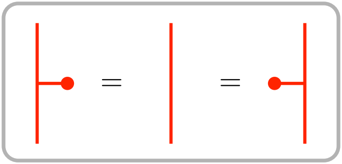
Example 1



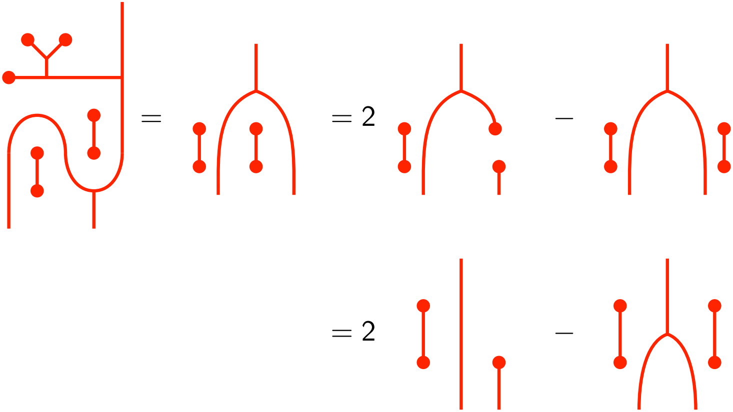


Example 1

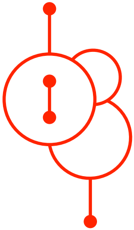




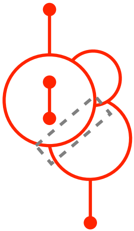
Example 1

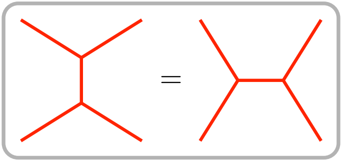


Example 2

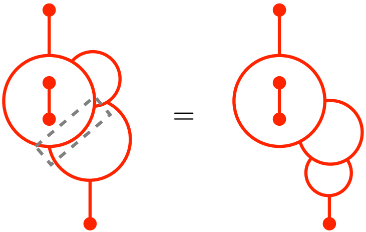


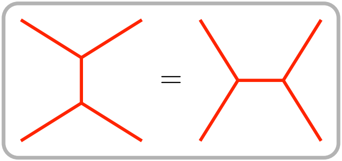
Example 2



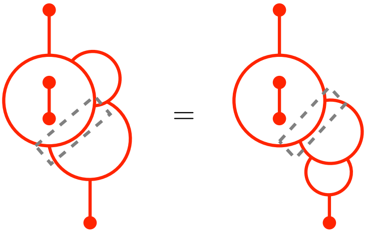


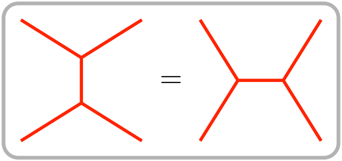
Example 2



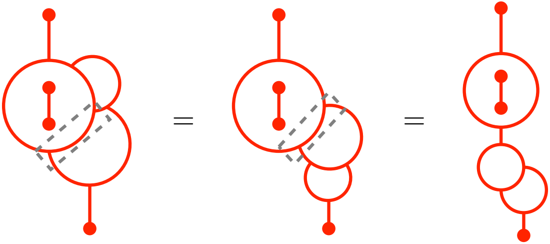


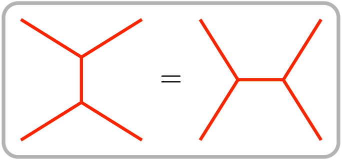
Example 2



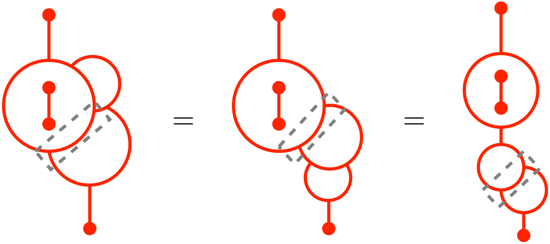


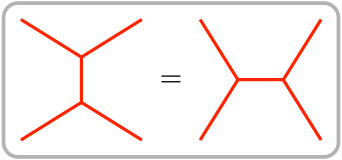
Example 2



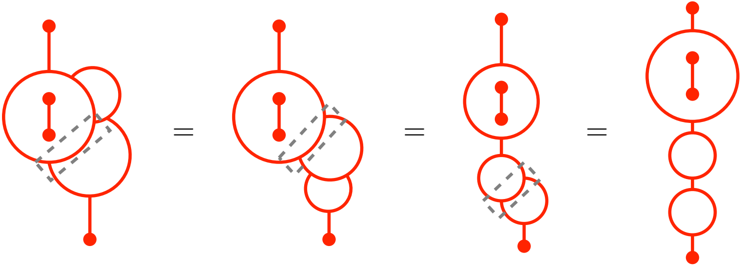


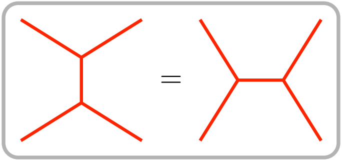
Example 2



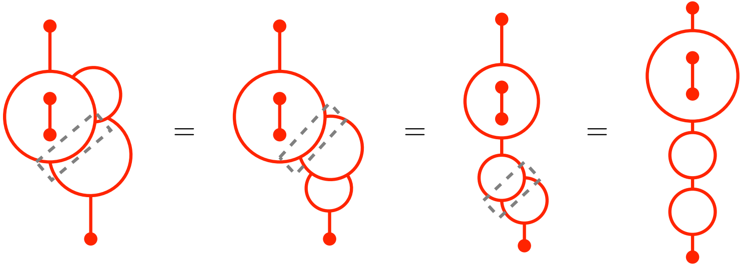


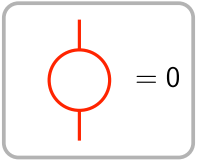
Example 2



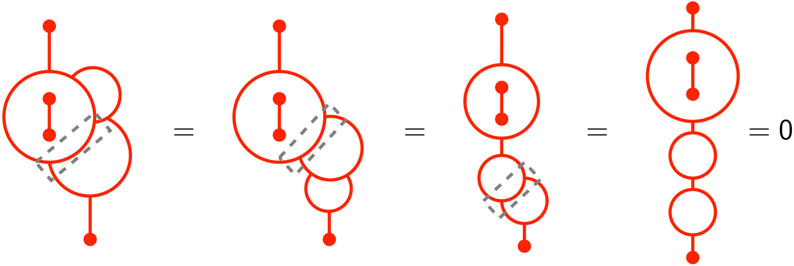


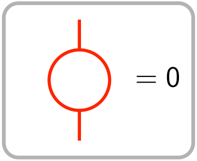
Example 2





Example 2





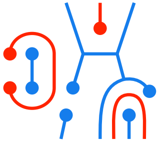
Soergel Bimodules

Theorem (Elias–Williamson, 2013)

The additive Karoubi envelope of $\mathcal{D}$ is equivalent to the category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$ *over* S_2 *as (idempotent complete) graded additive* $\mathbb{C}$*-linear monoidal categories.*

Generalisations

- Over general Coxeter groups e.g. S_n, D_n



- Diagrammatics for other categories of representations...

Further Applications

- Diagrammatics for BGG Category $\mathcal{O}$, Tilting modules
- Changing scalars to fields of characteristic p