Diagrammatic Categories in Representation Theory
Honours Thesis
(Draft)

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One-colour Diagrammatics

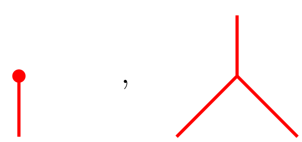
3.1 One-colour Diagrammatic Hecke Category

The first one-colour diagrammatic we explore is the one-colour (diagrammatic) Hecke category $\mathcal{H}(S_2)$ for the symmetric group $S_2 = \langle s \mid s^2 = 1 \rangle$. This is a monoidal category which we will describe via generators and relations.

Remark 3.1.1. All the following diagrams could be defined without the language of categories, as planar pictures with appropriate vertical stacking and horizontal concatenation. Nevertheless, we define them in a category because they will eventually be seen as diagrammatic versions of important categories in representation theory.

The objects of $\mathcal{H}(S_2)$ are generated by taking formal tensor products of the non-identity element $s \in S_2$. We will write these objects as words, e.g. s, $ssss =: s^4$, $sssssss =: s^7$, where the tensor product is just concatenation. The empty tensor product (or empty word) will be denoted by $\varnothing =: s^0$.

The morphisms are generated, up to isotopy, by univalent and trivalent vertices

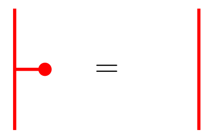


(3.1.2)

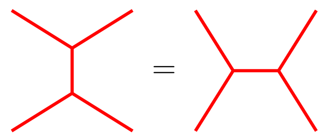
which are maps $s \to \varnothing$ and $ss \to s$ respectively. Note that we put a large dot on univalent vertices to signify that the line stops abruptly and does not connect to the top. The composition of such diagrams is appropriate vertical stacking, and the tensor product is horizontal concatenation (without intersection). Additionally[1], we allow formal sums of diagrams by putting a $\mathbb{Z}$-module structure on each morphism space $\mathrm{Hom}(s^n, s^m)$, for non-negative integers n, m. Composition or tensor with the zero morphism 0 in this $\mathbb{Z}$-module result in 0. To abuse notation, the empty diagram $\varnothing \to \varnothing$ will be denoted $\varnothing$. The identity morphism in $\mathrm{Hom}(s^n, s^n)$ is the diagram consisting of n vertical lines.

[1] Pun intended.

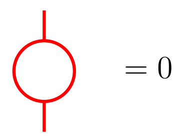
Such diagrams are subject to the following local relations.



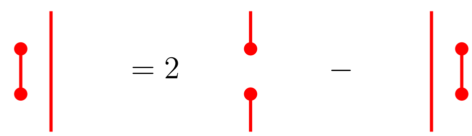
(3.1.3a)



(3.1.3b)



(3.1.3c)

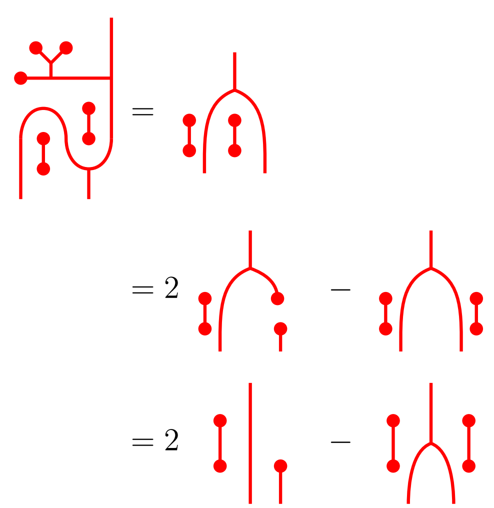


(3.1.3d)

Mention that the morphisms are enriched over the category of $\mathbb{Z}$-modules. What is 0? the "zero module"? Also is | the identity morphism? in what context?

Remark 3.1.4. The object s is a Frobenius algebra object in $\mathcal{H}(S_2)$. The generators (3.1.2) and their horizontal reflections are the unit, multiplication, counit and comultiplication maps. The unit, associativity and Frobenius associativity axioms are satisfied by the relations (3.1.3a) and (3.1.3b).

Example 3.1.5. Let us use the relations in (3.1.3) to simplify the following morphism in $\mathrm{Hom}((s,s),(s))$.



There is a right (or left) $\mathbb{Z}[\ ]$-basis for $\mathrm{Hom}(s^n, s^m)$ described in [EW13] called the Double Leaves basis. To define this basis we must first look at morphisms known as

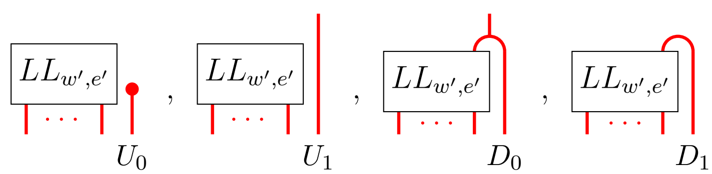
Light leaves. Given a word $w = s^n$, a subexpression is a binary string of length n. For example, $0000, 0110$ and 1011 are subexpressions of $s^4 = ssss$. Given a subexpression e of an object w, we can apply it to produce an element $w^e \in S_2$, e.g. $ssss^{1011} = s * 1 * s * s = s$. Maybe use subscript here to avoid confusion with $s^n = ss...s$. Each term of the subexpression is a decision of whether to include the corresponding s in the word, where the decision to exclude an s amounts to multiplying by 1.

For a subexpression e of an expression w, we can label each term by U_0, U_1, D_0 or D_1. The label is U_* if the partial subexpression up to the current term evaluates to $1 \in S_2$ and D_* if it evaluates to $s \in S_2$, where the subscript corresponds to the term in e.

Example 3.1.6. For the object $ssss$ and subexpression 0101, we can find the labels:

Choice	1	2	3	4
Partial w	s	ss	sss	$ssss$
Partial e	0	01	010	0101
Partial w^e	1	$1 * s = s$	$1 * s * 1 = s$	$1 * s * 1 * s = 1$
Labels	U_0	U_0U_1	$U_0U_1D_0$	$U_0U_1D_0D_1$

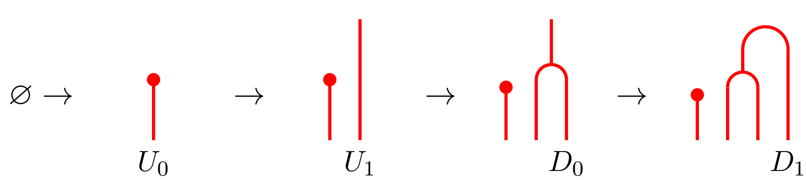
The light leaf $LL_{w,e} \in \mathrm{Hom}(w, w^e)$, corresponding to the object w and subexpression e, is defined iteratively as follows. Let $LL_{\varnothing,\varnothing} = \varnothing$ be the empty diagram. Given $LL_{w',e'}$ and $i \in \{0, 1\}$, $LL_{w's,e'i}$ is one of



(3.1.7)

depending on the next label, where w' and e' are appropriate subwords of w and e. Observe that the codomain of a light leaf $LL_{w,e}$ corresponds to the evaluation $w^e \in S_2$ of the subexpression. The recursive definition is consistent, since if the next label is U_* then the codomain of $LL_{w',e'}$ (the evaluation of the partial subexpression $w'^{e'}$ up to the label) is 1, and when the next label is D_* the codomain of $LL_{w',e'}$ is s. Rewrite this to make sense. Do we need to talk about 'degree' of light leaves?

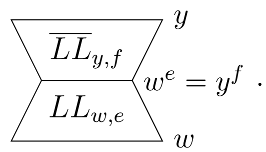
Example 3.1.8. Following from Example (3.1.6) for $w = ssss$ and $e = 0101$, we have labels $U_0U_1D_0D_1$ so the light leaf $LL_{w,e}$ is built as follows.



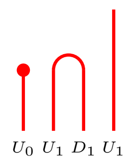
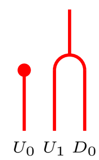
Let $\overline{LL}_{w,e}$ denote the vertical reflection of $LL_{w,e}$. A double leaf associated to expressions w, y is a composition

$$\mathbb{LL}_{f,e} := \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$$

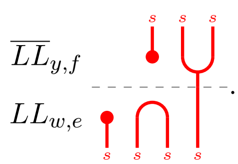
for subexpressions e of w and f of y such that $w^e = f^y$. Visually this looks like a morphism from w to y factoring through $w^e = y^f \in \{1, s\}$,



Example 3.1.9. Let $w = ssss$ and $y = sss$. Let $e = 0111$ be a subexpression of w, and $f = 010$ be a subexpression of y. The corresponding light leaves are

$LL_{w,e} =$  and $LL_{y,f} =$  .

Then the double leaf $\mathbb{LL}_{f,e} = \overline{LL}_{y,f} \circ LL_{w,e} : w \to y$, factoring through s, is



Notice that these double leaves have no floating diagrams such as . In order for these double leaves to be a basis for a morphism space, we insert these floating diagrams by taking linear combinations as a right $\mathbb{Z}[\ ]$-module. Here, the right -action on a diagram is just concatenation by on the right. Since we can move barbells to the left, via. the relation (3.1.3d), we can equivalently act by $\mathbb{Z}[\ ]$ on the left. Why do we default to right module? This leads us to the following theorem.

Theorem 3.1.10 (Elias-Williamson [EW13], Theorem 1.2)**.** *Given objects w, y in $\mathcal{H}(S_2)$, let $\mathbb{LL}_{w,y}$* [2] *be the collection of double leaves $\mathbb{LL}_{f,e}$ for subexpressions e of w and f of y, such that $w^e = y^f$. Then $\mathbb{LL}_{w,y}$ is a right (or left) $\mathbb{Z}[\ ]$-module basis for* $\mathrm{Hom}(w, y)$.

The purely diagrammatic proof (of a more general theorem) can be found in [EW13]. What is it used for?

Remark 3.1.11. The above light leaves and double leaves, introduced in [EW13], are diagrammatic analogues of Libedinsky's work in [Lib08].

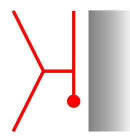
Put equivalence of karoubian additive closure with category of soergel bimodules. Is this equivalent to category of bott-samelson bimodules?

[2] this can be confused with the double leaves themselves, maybe write $\mathbb{LL}(w, y)$

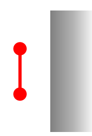
3.2 Diagrammatic $\mathcal{O}(\mathrm{SL}(2))$

With the diagrammatic category $\mathcal{H}(S_2)$, we can describe diagrammatics for the category $\mathcal{O}(\mathrm{SL}(2))$. In particular, we define a modular category *[what do we call this cat?]* over $\mathcal{H}(S_2)$.

This module category has elements copied from $\mathcal{H}(S_2)$ and morphisms are generated by the empty diagram $\varnothing$, with $\mathcal{H}(S_2)$ acting on the left by left concatenation on objects and morphisms. Additionally, the morphisms have one new relation, where diagrams collapse to 0 when there are barbells on the right. To depict this we add a wall on the right of the diagram, i.e. embedding the diagrams in the one-sided strip $[0,1] \times \mathbb{R}_{\geq 0}$ instead of in the double-sided strip $[0,1] \times \mathbb{R}$. For example a morphism may be



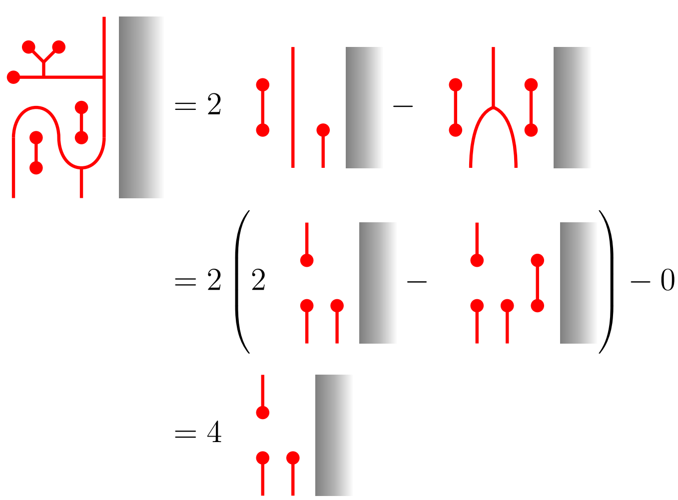
Then, diagrams are related to the wall by

 $= 0.$ (3.2.1)

What happens when i have 0 concatenated to a diagram? Is it also 0 (its a tensor product)?

Notice that all the morphisms in $\mathcal{H}(S_2)$ appear in this modular category, although they may have been annihilated by (3.2.1).

Example 3.2.2. We use the new relation (3.2.1) to further simplify the morphism in Example (3.1.5).



Chapter 4

Two-colour Diagrammatics

4.1 Two-colour Diagrammatic Hecke Category

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4.2 Diagrammatic $\mathrm{Tilt}(\mathrm{SL}(2))$

Blah

Bibliography

[EW13] Ben Elias and Geordie Williamson. "Soergel Calculus". In: *Representation Theory of the American Mathematical Society* 20 (Sept. 2013). DOI: 10.1090/ert/481.

[Lib08] Nicolas Libedinsky. "Sur la catégorie des bimodules de Soergel". In: *Journal of Algebra* 320.7 (2008). (French), pp. 2675–2694. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2008.05.027.