

# Diagrammatic Categories

## in Representation Theory

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# Kazhdan–Lusztig Conjecture

(Motivation)

- Relates Kazhdan–Lusztig polynomials to Jordan–Hölder multiplicities of Verma modules
- First proof by Beilinson and Bernstein (1981) using  $D$ -modules
- Algebraic proof by Elias and Williamson (2010's) using Soergel bimodules and diagrammatics

# Talk Overview

(Motivation)

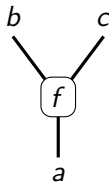
1. Diagrammatic Monoidal Categories
2. Diagrammatic Soergel bimodules
3. Further Applications

# Monoidal Categories

A *monoidal category* is a category with an associative multiplication  $\otimes$  for objects and morphisms, and a unit object  $\mathbb{1}$ , such that the multiplication works well with composition.

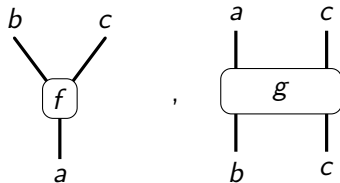
# Monoidal Categories

$$f : a \rightarrow b \otimes c$$



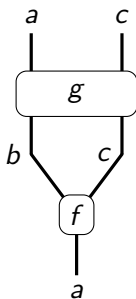
# Monoidal Categories: Composition

$$f : a \rightarrow b \otimes c$$
$$g : b \otimes c \rightarrow a \otimes c$$



# Monoidal Categories: Composition

$$f : a \rightarrow b \otimes c$$
$$g : b \otimes c \rightarrow a \otimes c$$



# Monoidal Categories: Identity

$$\mathrm{id}_a : a \rightarrow a$$

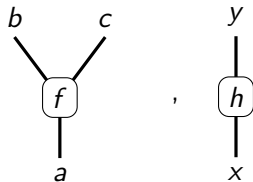
$$\begin{array}{c} a \\ | \\ a \end{array}$$



# Monoidal Categories: Tensor

$$f : a \rightarrow b \otimes c$$

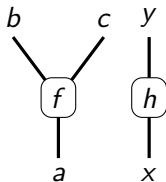
$$h : x \rightarrow y$$



# Monoidal Categories: Tensor

$$f : a \rightarrow b \otimes c$$

$$h : x \rightarrow y$$



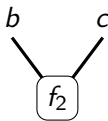
# Monoidal Categories: Unit

$$f_1 : a \rightarrow \mathbb{1}$$



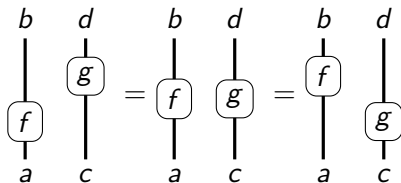
# Monoidal Categories: Unit

$$f_2 : \mathbb{1} \rightarrow b \otimes c$$

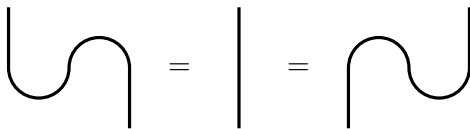


# Monoidal Categories

*Interchange Law*



# Isotopy



# $\mathbb{Z}$ -linear Monoidal Category

$$3 \cdot f(a, b, c) = -2 \cdot (f_1(a), f_2(b, c))$$

# Diagrammatic Soergel Bimodules

Let  $\mathcal{D}$  be the (diagrammatic)  $\mathbb{Z}$ -linear monoidal category with:

Generating object  $I$ .

Generating morphisms



and local relations...



# Relations

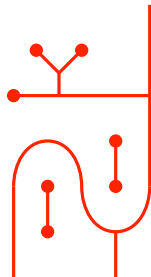
$$\begin{array}{c} | \\ \text{---} \bullet \end{array} = \begin{array}{c} | \\ \\ \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array},$$

$$\begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array},$$

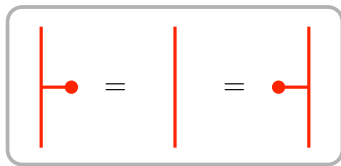
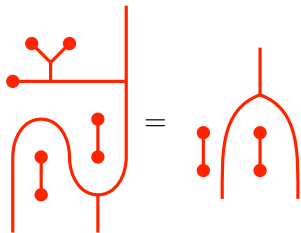
$$\begin{array}{c} | \\ \bigcirc \\ | \end{array} = 0,$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

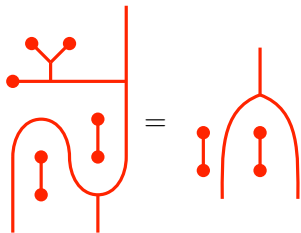
# Example 1



# Example 1

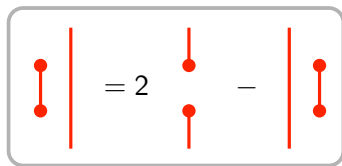
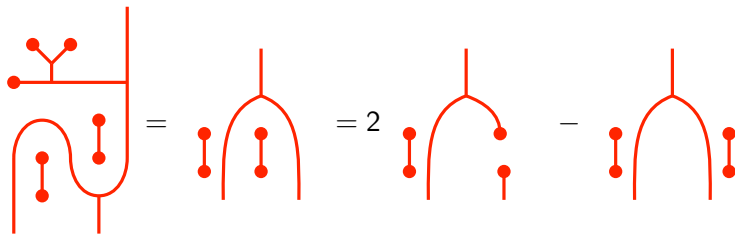


# Example 1

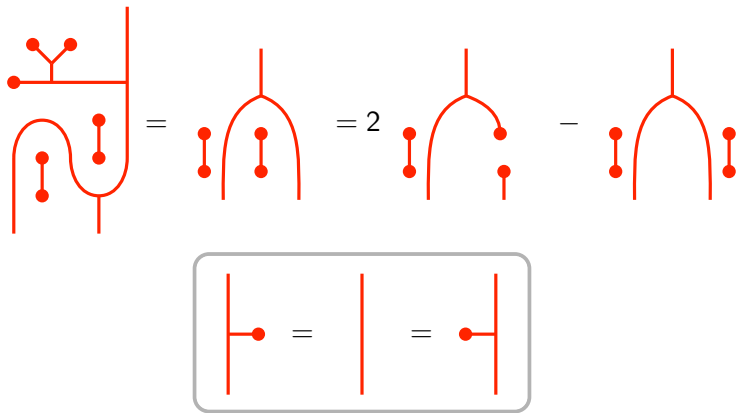


$$\begin{array}{|c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{|c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{|c} \bullet \\ | \\ \bullet \end{array}$$

# Example 1



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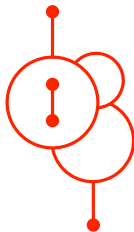


# Example 1

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = 2 \times \text{Diagram 3} - \text{Diagram 4} \\
 & \qquad \qquad \qquad = 2 \times \text{Diagram 5} - \text{Diagram 6}
 \end{aligned}$$

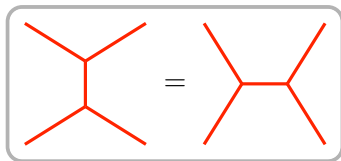
The diagrams are composed of red lines and dots, representing a sequence of operations or a combinatorial structure. The first diagram shows a complex structure with a horizontal line, a vertical line, and several loops and dots. The subsequent diagrams show simpler structures, including a single vertical line, a loop, and a combination of these elements.

## Example 2

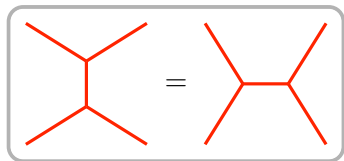
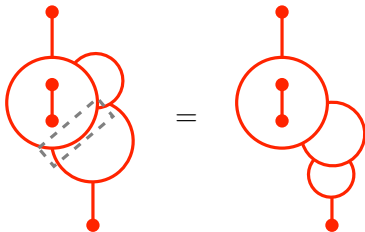




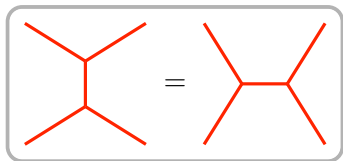
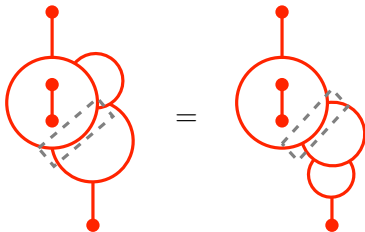
## Example 2



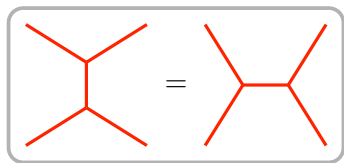
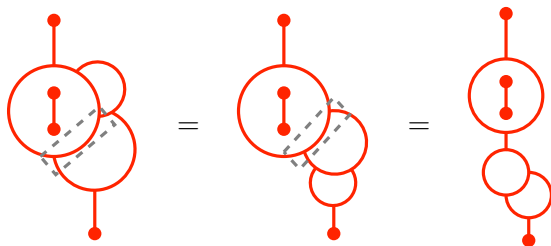
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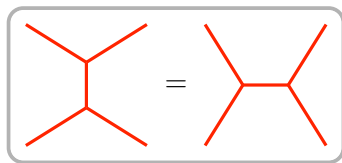
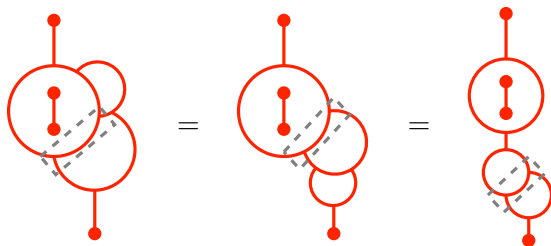
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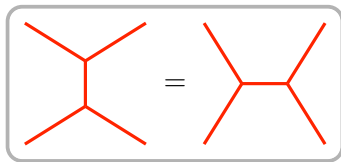
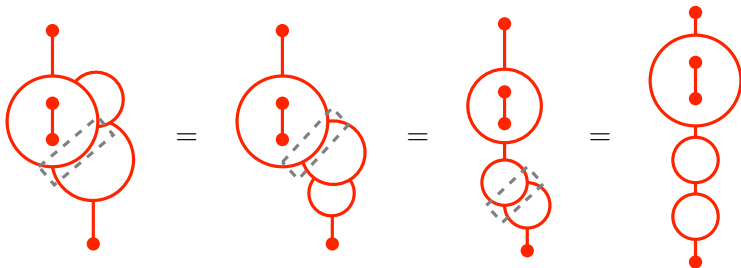
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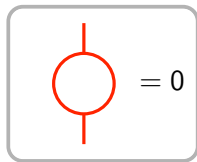
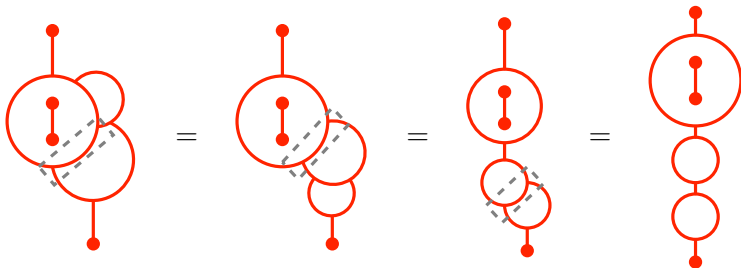
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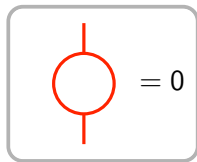
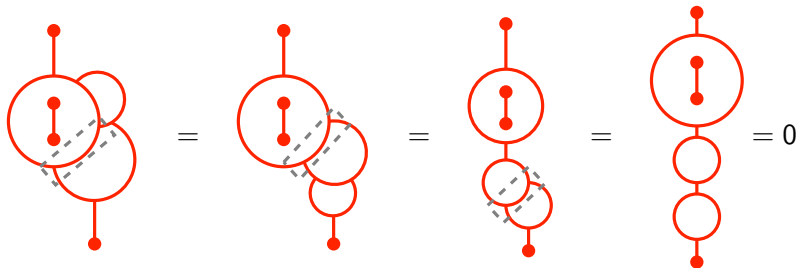
## Example 2



## Example 2



## Example 2





# Soergel Bimodules

## Theorem (Elias–Williamson, 2013)

*The additive Karoubi envelope of  $\mathcal{D}$  is equivalent to the category of Soergel Bimodules  $\mathbb{S}\text{Bim}$  over  $S_2$  as graded additive  $\mathbb{C}$ -linear monoidal categories.*

# Generalisations

- Over general Coxeter groups e.g.  $S_n$ ,  $D_n$
- Diagrammatics for other categories of representations...

# Further Applications

- Diagrammatics for BGG Category  $\mathcal{O}$ , Tilting modules
- Changing scalars to fields of characteristic  $p$