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Homework 4

Chapter 4.1

4.1.3 Construct a matrix with the required property or say why that is impossible:

(a)Column space contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(b) Row space contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(c) $Ax = \begin{bmatrix}1\\1\\1\end{bmatrix}$ has a solution and $A^T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(d) Every row is orthogonal to every column (A is not the zero matrix)

(e) Columns add up to a column of zeros, rows add to a row of 1's.

(a) $\begin{bmatrix}1 & 2 & -3\\2 & -3 & 1\\-3 & 5 & -2\end{bmatrix}$.

(b) The matrix is impossible to construct because $\begin{bmatrix}2\\-3\\5\end{bmatrix}$ is not perpendicular to $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

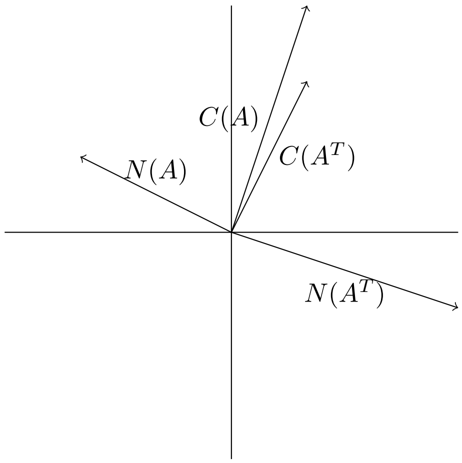
(c) Can't construct a matrix with $\begin{bmatrix}1\\1\\1\end{bmatrix}$ in the column space of A and with $\begin{bmatrix}1\\0\\0\end{bmatrix}$ in the left nullspace of A because the two vectors aren't orthogonal.

(d) $\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$.

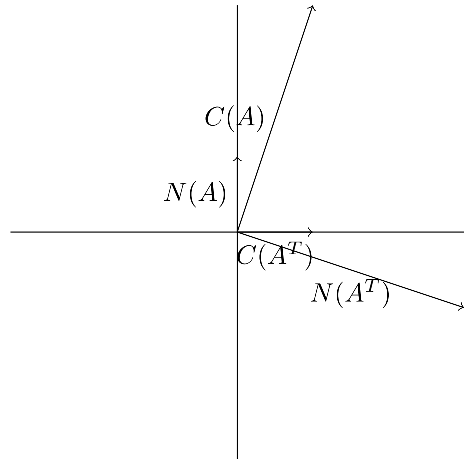
(e) The vector (1,1,1) would have to exist in both the nullspace and the row space which is impossible.

4.1.11 (Recommended) Draw Figure 4.2 to show each subspace correctly for

$$A = \begin{bmatrix}1 & 2\\3 & 6\end{bmatrix} \text{ and } B = \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}.$$



A:



B:

4.1.16 Prove that every y in $N(A^T)$ is perpendicular to every Ax in the column space, using the matrix shorthand of equation (2). Start from $A^Ty = 0$.

$$A^Ty = 0 \qquad Ax = b$$
$$A^Ty = 0 \rightarrow (Ax)^Ty = 0 \rightarrow x^TA^Ty = 0 \rightarrow y \perp Ax$$

4.1.17 If S is the subspace of R^3 containing only the zero vector, what is $S^\perp$? If S is spanned by (1, 1, 1), what is $S^\perp$? If S is spanned by (1, 1, 1) and (1, 1, -1), what is a basis for $S^\perp$?

$S^\perp$ is every vector in R^3.

$$IF\ S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ then } S^\perp = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$
$$IF\ S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \text{ then } S^\perp = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

4.1.18 Suppose S only contains two vectors (1,5,1) and (2,2,2) (not a subspace), Then $S^\perp$ is the nullspace of the matrix $A =$ _____. $S^\perp$ is a subspace even if S is not.

$$S^\perp = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

4.1.21 Suppose S is spanned by the vectors (1,2,2,3) and (1,3,3,2). Find two vectors that span $S^\perp$, This is the same as solving $Ax = 0$ for which A?

$$S^\perp = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$$

4.1.22 If P is the plane of vectors in R^4 satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $P^\perp$. Construct a matrix that has P as its nullspace.

$$P^{\perp} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \text{ has } P \text{ as its nullspace.}$$

4.1.24 Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

The first column of A^{-1} is orthogonal to every row of A except for row 1.

4.1.25 Find $A^T A$ if the columns of A are unit vectors, all mutually perpendicular.

If the columns of A are unit vectors and mutually perpendicular then $A^T A = I$.

4.1.28 Why is each of these statements false?
(a) (1, 1, 1) is perpendicular to (1,1, 2) so the planes $x+y+z = 0$ and $x+y-2z = 0$ are orthogonal subspaces.
(b) The subspace spanned by (1,1,0,0,0) and (0,0,0,1,1) is the orthogonal complement of the subspace spanned by (1,-1,0,0,0) and (2,-2,3,4,-4).
(c) Two subspaces that meet only in the zero vector are orthogonal.

(a) The planes intersect at a line, so the planes can't be orthogonal.
(b) Three vectors are needed to span the whole orthogonal complement.
(c) Lines don't have to be orthogonal to meet at the zero vector.

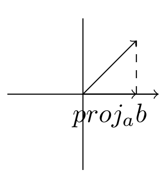
4.1.33 Suppose I give you eight vectors $r_1, r_2, n_l, n_2, c_1, c_2, l_1, l_2$ in R^4.
(a) What are the conditions for those pairs to be bases for the four fundamental subspaces of a 4 by 4 matrix?
(b) What is one possible matrix A?

(a) The r's need to be orthogonal to the n's and the c's orthogonal to the L's.
(b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

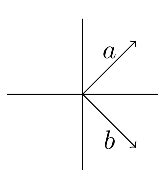
Chapter 4.2

4.2.2 Draw the projection of b onto a and also compute it from $p = \hat{x}a$:

$$\text{(a) } b = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(b) } b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) 

$$P = \hat{x}a \qquad \hat{x} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \frac{cos\theta}{1} \qquad P = \begin{bmatrix} cos\theta \\ 0 \end{bmatrix}.$$

(b) 

$$P = \hat{x}a \qquad \hat{x} = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{0}{2} \qquad P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

4.2.13 (Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $b = (1,2,3,4)$ onto the column space of A. What shape is the projection matrix P and what is P?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ so } p = P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

4.2.16 What linear combination of (1,2,-1) and (1,0,1) is closest to $b = (2,1,1)$?

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad P = x_1 a_1 + x_2 a_2$$

To find x_1 and x_2 we time both sides of $Ax = b$ by A^T.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \rightarrow x_1 = \tfrac{1}{2} \qquad x_2 = \tfrac{3}{2}$$

$$P = \tfrac{1}{2} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \tfrac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

4.2.17 (Important) If $P^2 = P$ show that $(I-P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the _____.

$$(I-P)^2 = (I-P)(I-P) = I^2 - PI - IP + P^2 = I - P$$

$I - P$ projects onto the left nullspace.

4.2.18 (a) If P is the 2 by 2 projection matrix onto the line through (1,1), then $I - P$ is the projection matrix onto _____.
(b) If P is the 3 by 3 projection matrix onto the line through (1,1,1), then $I - P$ is the projection matrix onto _____.

(a) $I - P$ projects onto $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $I - P$ projects onto $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

4.2.19 To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of A. The plane should be the column space. Then compute $P = A(A^TA)^{-l}A^T$.

$$A^T = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

4.2.26 If an m by m matrix has $A^2 = A$ and its rank is m, prove that $A = I$.

The matrix is a full rank matrix therefore A^{-1} exists.
$A^2 = A \quad \rightarrow \quad A^{-1}(AA) = A^{-1}A \quad \rightarrow \quad A = I$

4.2.27 The important fact that ends the section is this: If $A^TAx = 0$ then $Ax = 0$. New Proof: The vector Ax is in the nullspace of _____. Ax is always in the column space of _____. To be in both of those perpendicular spaces, Ax must be zero.

The vector Ax is in the nullspace of $\underline{\quad A^T \quad}$.
Ax is always in the column space of $\underline{\quad A \quad}$.
So A and A^TA have the same nullspace.

4.2.29 If B has rank m (full row rank, independent rows) show that BB^T is invertible.

$A = B^T$ has independent columns, then A^TA must be invertible. From that we can deduce that since $A = B^T$ then $A^T = B$ and therefore $A^TA = BB^T$ meaning that BB^T must be invertible.

4.2.30 (a) Find the projection matrix P_c onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

(b) Find the 3 by 3 projection matrix P_R onto the row space of A. Multiply $B = P_CAP_R$. Your answer B should be a little surprising-can you explain it?

(a) $A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix} \quad A^T = \begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 6 & 8 \end{bmatrix} \quad C(A) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$P_C = \frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \end{bmatrix}}{\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \tfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$$

(b) $C(A^T) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad P_R = \dfrac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \tfrac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$

$$B = P_CAP_R = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

This is because the columns of A projected onto themselves will just be A and by similar logic AP_R will just be A as well. Therefore $P_CAP_R = A$.

Chapter 4.3

4.3.6 Project $\boldsymbol{b} = (0, 8, 8, 20)$ onto the line through $\boldsymbol{a} = (1, 1, 1, 1)$. Find $\hat{x} = \boldsymbol{a}^T\boldsymbol{b}/\boldsymbol{a}^T\boldsymbol{a}$ and the projection $\boldsymbol{p} = \hat{x}\boldsymbol{a}$. Check that $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and find the shortest distance $\|\boldsymbol{e}\|$ from $\boldsymbol{b}$ to the line through $\boldsymbol{a}$

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\8\\8\\20\end{bmatrix}}{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} = \frac{36}{4} = 9 \quad p = \hat{x}a = \begin{bmatrix}9\\9\\9\\9\end{bmatrix}$$

$$b - p = e \begin{bmatrix}0\\8\\8\\20\end{bmatrix} - \begin{bmatrix}-9\\-1\\-1\\11\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 0 \checkmark$$

$\|e\| = \sqrt{204}$

4.3.9 For the closest parabola $b = C + Dt + Et^2$ to the same four points, write down the unsolvable equations $Ax = b$ in three unknowns $x = (C, D, E)$. Set up the three normal equations $A^T A\hat{x} = A^T b$ (solution not required). In Figure 4.9a you are now fitting a parabola to 4 points - what is happening in Figure 4.9b?

$$\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 3 & 9\\1 & 4 & 16\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix} = \begin{bmatrix}0\\8\\8\\20\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & 4\\0 & 1 & 9 & 16\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 3 & 9\\1 & 4 & 16\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & 4\\0 & 1 & 9 & 16\end{bmatrix}\begin{bmatrix}0\\8\\8\\20\end{bmatrix} \rightarrow$$

$$\begin{bmatrix}4 & 8 & 26\\8 & 26 & 92\\26 & 92 & 338\end{bmatrix}\begin{bmatrix}C & D & E\end{bmatrix} = \begin{bmatrix}36\\112\\400\end{bmatrix}$$

4.3.12 (Recommended) This problem projects $\boldsymbol{b} = (b_1, ..., b_m)$ onto the line through $a = (1, ..., 1)$. We solve m equations $ax = b$ in 1 unknown (by least squares).
(a) Solve $a^T a\hat{x} = a^T b$ to show that $\hat{x}$ is the *mean* (the average) of the b's.

$$\begin{bmatrix}11 \cdots 11\end{bmatrix}\begin{bmatrix}1\\1\\\cdot\\\cdot\\\cdot\\1\\1\end{bmatrix}\hat{x} = \begin{bmatrix}11 \cdots 11\end{bmatrix}\begin{bmatrix}b_1\\\cdot\\\cdot\\\cdot\\b_m\end{bmatrix}$$

$= (\text{number of components in a})\ \hat{x} = \sum_{i=1}^m b_i \rightarrow \frac{\hat{x} = \sum_{i=1}^m b_i}{number\ of\ components} = \text{average}$

(b) Find $e = b - a\hat{x}$ and the *variance* $\|e\|^2$ and the *standard deviation* $\|e\|$.

$$e = \begin{bmatrix}b_1\\\cdot\\\cdot\\\cdot\\b_m\end{bmatrix} - \begin{bmatrix}\hat{x}\\\cdot\\\cdot\\\cdot\\\hat{x}\end{bmatrix} = \begin{bmatrix}b_1 - \hat{x}\\\cdot\\\cdot\\\cdot\\b_m - \hat{x}\end{bmatrix}$$

$\|e\|^2 = \sum_{i=1}^m (b_i - \hat{x})^2 = \text{variance}$

(c) The horizontal line $\hat{b} = 3$ is closest to $b = (1, 2, 6)$. Check that $p = (3, 3, 3)$ is perpendicular to e and find the 3 by 3 projection matrix P.

$$e = b - p = \begin{bmatrix}1\\2\\6\end{bmatrix} - \begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}-2\\-1\\3\end{bmatrix} \cdot \begin{bmatrix}3\\3\\3\end{bmatrix} = -6 - 3 + 9 = 0 \checkmark$$

$P = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

4.3.22 Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

$$\begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 1 & 1 & 1\\-2 & -1 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\-2 & -1 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix} \rightarrow$$

$$\begin{bmatrix}5 & 0\\0 & 10\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}5\\-10\end{bmatrix} \quad C = 1 \quad D = -1$$

4.3.26 Find the *plane* that gives the best fit to the 4 values $b = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0) and (0, -1)$ of a square. The equations $C + Dx + Ey = b$ at those 4 points are $Ax = b$ with 3 unknowns $x = (C, D, E)$. What is A? At the center $(0, 0)$ of the square, show that $C + Dx + Ey =$ average of the b's.

$$\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix} = \begin{bmatrix}0\\1\\3\\4\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 1 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 0 & -1 & -\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\\3\\4\end{bmatrix}$$

$$\begin{bmatrix}4 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix} = \begin{bmatrix}8\\-3\\-3\end{bmatrix} \quad C = 2 \quad D = \frac{-3}{2} = E$$

$p = 2 - \frac{3}{2}x - \frac{3}{2}y$

at $x = y = 0$ $\;p = 2$, which is the average of the square

Chapter 4.4

4.4.1 Are these pairs of vectors orthonormal or only orthogonal or only independent?

(a) $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$ (b) $\left[\begin{smallmatrix}.6\\.8\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}.4\\-.3\end{smallmatrix}\right]$ (c) $\left[\begin{smallmatrix}cos\theta\\sin\theta\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-sin\theta\\cos\theta\end{smallmatrix}\right]$.

Change the second vector when necessary to produce orthonormal vectors.

(a) independent, the second vector would be $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ to be orthonormal

(b) orthogonal, the second vector would be $\left[\begin{smallmatrix}.8\\-.6\end{smallmatrix}\right]$ to be orthonoraml and independent

(c) orthonormal and independent

4.4.4 Give an example of each of the following:

(a) A matrix Q that has orthonormal columns but $QQ^T \neq I$.

$$Q = \begin{bmatrix}2 & 1\\2 & -1\end{bmatrix} QQ^T = \begin{bmatrix}2 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}2 & 2\\1 & -1\end{bmatrix} = \begin{bmatrix}5 & 3\\3 & 5\end{bmatrix} \neq I$$

(b) Two orthogonal vectors that are not linearly independent.

$$\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}$$

(c) An orthonormal basis for $\boldsymbol{R}^3$, including the vector $q_1 = (1, 1, 1)/\sqrt{3}$.

$$\begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\\0\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{-1}{\sqrt{2}}\end{bmatrix}$$

4.4.10 Orthonormal vectors are automatically linearly independent.
(a) Vector proof: When $c_1 q_1 + c_2 + q_2 + c_3 q_3 = 0$, what dot product leads to $c_1 = 0$? Similarly $c_2 = 0$ and $c_3 = 0$. Thus the q's are independent.

If all the q's are orthonormal then the dot product of q_1 with $c_1 q_1 + c_2 q_2 + c_3 q_3 = 0$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$.

(b) Matrix proof: Show that $Qx = 0$ leads to $x = 0$. Since Q may be rectangular, you can use Q^T but not Q^{-1}.

$Qx = 0 \to Q^T Qx = 0 \to x = 0$

4.4.11 (a) Gram-Shmidt: Find orthonormal vectors q_1 and q_2 in the plane spanned by $a = (1, 3, 4, 5, 7)$ and $b = (-6, 6, 8, 0, 8)$.

$\frac{1}{10}(1, 3, 4, 5, 7)$ $\frac{1}{10}(-7, 3, 4, -5, 1)$

(b) Which vector in this plane is closest to $(1, 0, 0, 0, 0)$?

$$\frac{1}{10}\begin{bmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{bmatrix} \frac{1}{10}\begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ -7 & 3 & 4 & -5 & 1 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 50 & -18 & -24 & 40 & 0 \\ -18 & 18 & 24 & 0 & 24 \\ -24 & 24 & 32 & 0 & 32 \\ 40 & 0 & 0 & 50 & 30 \\ 0 & 24 & 32 & 30 & 50 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 50 \\ -18 \\ -24 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ -.18 \\ -.24 \\ .4 \\ 0 \end{bmatrix}$$

4.4.15 (a) Find orthonormal vectors q_1, q_2, q_3 such that q_1, q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

$$q_1 = \tfrac{1}{3}(1, 2, -2) \quad q_2 = \tfrac{1}{3}(2, 1, 2) \quad q_3 = \tfrac{1}{3}(2, 2, -1)$$

(b) Which of the four fundamental subspaces contains q_3?

the left nullspace

(c) Solve $Ax = (1, 2, 7)$ by least squares.

$$\hat{x} = (A^T A)^{-1} A^T \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \left(\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \to \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix}^{-1}\begin{bmatrix} -9 \\ 27 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -9 \\ 27 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4.4.23 Find q_1, q_2, q_3 (orthonormal) as combinations of a, b, c (independent columns). Then write A as QR:

$$c = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

A is an invertible matrix so the vectors $q_1 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$, $q_2 = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$, $q_3 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$ are in the column space and are orthonormal.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

4.4.24 (a) Find a basis for the subspace S in R^4 spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0$$

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(b) Find a basis for the orthogonal comblement $S^{\perp}$.

Since all vectors and all their linear combinations contained in S are orthogonal to the orginal matrix $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $S^{\perp}$ is the original matrix $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

(c) Find b_1 in S and b_2 in $S^{\perp}$ so that $b_1 + b_2 = b = (1, 1, 1, 1)$.

$b_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ $\quad b_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2})$

4.4.34 $Q = I - 2uu^T$ is a reflection matrix when $u^T u = 1$. Two reflections give $Q^2 = I$.
(a) Show that $Qu = -u$. The mirror is perpendicular to u.

$Q = I - 2uu^T \rightarrow Qu = Iu - 2uu^T u$ Since $u^T u = 1 \rightarrow Qu = -u$

(b) Find Qv when $u^T v = 0$. The mirror contains v. It reflects to itself.

$Q = I - 2uu^T \rightarrow Qv = Iv - 2uu^T v$ since $u^T v = 0 \rightarrow Qv = v$