Dustin Ginos:
A01233669
Chandler Kinch:
A01662772
Jeff Wasden:
A01657029

April 15, 2017

Homework 6

Chapter 6.1

6.1.2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the ______ eigenvectors as A. Its eigenvalues are ______ by 1.

$$A) \quad \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = \pm 5, -1 \qquad \text{eigenvectors=} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A+I) \quad \begin{vmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 8 = \lambda^2 - 6\lambda = 0$$

$$\lambda = 0, 6 \qquad \text{eigenvectors=} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A + I$ has the <u>same</u> eigenvectors as A. Its eigenvalues are <u>increased</u> by 1.

6.1.4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same ______ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues ______. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

$$A) \quad \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2, -3 \qquad \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} x_1 = 0 \qquad \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} x_2 = 0 \qquad \text{eigenvectors=} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

$$A^2) \quad \begin{vmatrix} 7-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 6 = \lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = 4, 9 \qquad \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} x_1 = 0 \qquad \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} x_2 = 0 \qquad \text{eigenvectors=} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

A^2 has the same <u>eigenvectors</u> as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues <u>λ_1^2 and λ_2^2</u>. $\lambda_1^2 + \lambda_2^2 = 13$ because that is the trace of A^2.

6.1.9 What do you do to the equation $Ax = \lambda x$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

Multiply both sides by A.
$AAx = A\lambda x \quad \rightarrow \quad A^2x = \lambda Ax \quad \rightarrow \quad A^2x = \lambda\lambda x \quad \rightarrow \quad A^2x = \lambda^2 x.$

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

Multiply both sides by A^{-1}.
$A^{-1}Ax = A^{-1}\lambda x \quad \rightarrow \quad x = \lambda A^{-1}x \quad \rightarrow \quad \frac{1}{\lambda}x = A^{-1}x.$

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

Add $Ix = x$ to both sides.
$Ix + Ax = x + \lambda x \quad \rightarrow \quad (A + I)x = (\lambda + 1)x.$

6.1.12 Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

$$\lambda = 1 \quad \begin{bmatrix} -.8 & .4 & 0 \\ .4 & -.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{eigenvectors=} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\lambda = 0 \quad \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{eigenvector=} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Combine the eigenvectors when $\lambda = 1$ to get an eigenvector of P with no zero components: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

6.1.13 From the unit vector $u = (\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6})$ construct the rank one projection matrix $P = uu^T$. This matrix has $P^2 = P$ because $u^Tu = 1$.

(a) $Pu = u$ comes from $(uu^T)u = u(\_\_\_\_\_\_)$. Then u is an eigenvector with $\lambda = 1$.

$(uu^T)u = u(\underline{\ u^Tu\ })$

(b) If v is perpendicular to u show that $Pv = 0$. Then $\lambda = 0$.

$Pv = (uu^T)v = u(u^Tv) = u * 0 = 0$

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ all have $\lambda = 0$ and are independent.

6.1.15 Every permutation matrix leaves $x = (1, 1, ..., 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0 \qquad \lambda = \frac{-1 \pm i\sqrt{\text{âĂć}}}{2}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1 = 0 \qquad \lambda = 1, 1, -1$$

6.1.19 A 3 by 3 matrix B is known to have eigenvalues 0,1,2. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

B is a rank two because it has a $\lambda = 0$.

(b) the determinate of $B^T B$

$|B^T B|$ because $B^T B$ is singular.

(c) the eigenvalues of $B^T B$

Can't determine.

(d) the eigenvalues of $(B^2 + I)^-1$.

λ's of $(B^2 + I)^{-1}$ are $\lambda = 1, \frac{1}{2}, \frac{1}{5}$.

6.1.21 **The eigenvalues of A equal the eigenvalues of A^T.** This is because $det(A - \lambda I)$ equals $det(A^T - \lambda I)$. That is true because ______. Show by an example that the eigenvectors of A and A^T are *not* the same.

It is true because every square matrix has the property $|A| = |A^T|$.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ do not have the same eigen vectors.}$$

$$\text{Eigenvectors of } A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ while } A^T \text{ has eigenvectors } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6.1.29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

$$A) \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(6-\lambda) = 0 \quad \lambda = 1, 4, 6$$

$$B) \quad |B - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = (\lambda^2 - 3)(\lambda + 2) = 0 \quad \lambda = 2, \pm\sqrt{3}$$

C is a rank one matrix, meaning that two of its λ's are zero. The last λ is the sum of the diagonals.

$\lambda = 0, 0, 6$

Chapter 6.2

6.2.2 If A has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{bmatrix}1\\1\end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

$$S\Lambda S^{-1} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 5\\0 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 3\\0 & 5\end{bmatrix} = A$$

6.2.8 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda S^{-1}\begin{bmatrix}1\\0\end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

$$S^{-1} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}$$

$$S\Lambda^k S^{-1} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^k & 0\\0 & \lambda_2^k\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \to \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1^{k+1} & \lambda_2^{k+1}\\\lambda_1^k & \lambda_2^k\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$
$$\to \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1^{k+1} - \lambda_2^{k+1}\\\lambda_1^k - \lambda_2^k\end{bmatrix}$$

6.2.9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$G_{k+2} = \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \qquad G_{k+1} = G_{k+1} \qquad \text{and} \qquad \begin{bmatrix}G_{k+2}\\G_{k+1}\end{bmatrix} = \begin{bmatrix}A\end{bmatrix}\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\1 & 0\end{bmatrix} \qquad |A-\lambda| = \begin{vmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\1 & -\lambda\end{vmatrix}$$
$$\to \quad (\lambda-1)(\lambda+\tfrac{1}{2}) = 0 \quad \lambda = 1, \tfrac{1}{2}$$
$$\lambda = 1 \qquad \begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\1 & -1\end{bmatrix}x_1 = 0 \qquad x_1 = \begin{bmatrix}1\\1\end{bmatrix}$$
$$\lambda = \tfrac{1}{2} \qquad \begin{bmatrix}1 & \frac{1}{2}\\1 & \frac{1}{2}\end{bmatrix}x_2 = 0 \qquad x_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda S^{-1}$.

$$A^\infty = \begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\frac{1}{3}\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}$$

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

$$G^{k+1} = A^k\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$$

6.2.10 Prove that every third Fibonacci number in 0,1,1,2,3,... is even.

The fibonacci pattern is odd number + even number then odd number plus odd number, which produces an even number every third term.

6.2.11 True or false: If the eigenvalues of A are 2,2,5 then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

(a) true, no zero eigenvalue (b) false, eigenvalues are repeated
(c) false, repeated eigenvalues may have different eigenvectors

6.2.15 $A^k = S\Lambda S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ______. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

$A^k = S\Lambda S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than $\underline{\;\;1\;\;}$. A_2 has $A_2^k \to 0$ with $\lambda = .3, .9$.

6.2.16 (Recommended) Find Λ and S to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ______.

$$|A - \lambda I| = \lambda^2 - .7\lambda - .3 = 0 \qquad \lambda = 1, -.3$$

$$\lambda = 1 \qquad \begin{bmatrix} -.4 & .9 \\ .4 & -.9 \end{bmatrix} x_1 = 0 \qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -.3 \qquad \begin{bmatrix} .9 & .9 \\ .4 & .4 \end{bmatrix} x_2 = 0 \qquad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -.3 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$S\Lambda^k S^{-1} \to \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ In the columns of this limiting matrix you see the $\underline{\text{ steady state }}$.

6.2.19 Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \qquad \text{has} \qquad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

$$|B - \lambda I| = (5 - \lambda)(4 - \lambda) = 0 \qquad \lambda = 4, 5$$

$$\lambda = 4 \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x_1 = 0 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = 5 \quad \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x_2 = 0 \quad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad S\Lambda^k S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$$

6.2.36 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = S\Lambda S^{-1}$. The eigenvectors (columns of S) are (1,i) and (i,1). You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

$$|A-\lambda I| = (\cos\theta - \lambda)^2 + \sin^2\theta = 0 \quad \rightarrow \quad \lambda^2 - 2\cos\theta\lambda + 1 = 0 \qquad \lambda = e^{-i\theta}, e^{i\theta}$$
$$A^n = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{bmatrix} \tfrac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$$
$$\rightarrow \quad \tfrac{1}{2i} \begin{bmatrix} ie^{in\theta} + ie^{-in\theta} & ie^{-in\theta} - ie^{in\theta} \\ e^{in\theta} - e^{-in\theta} & 2e^{in\theta} \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Chapter 6.3

6.3.1 .

Chapter 6.4

6.4.4 Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ?

$$|A-\lambda I| = (\lambda - 10)(\lambda + 5) = 0 \qquad \lambda = 10, -5$$
$$\lambda = 10 \qquad \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} x_1 = 0 \quad x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\lambda = -5 \qquad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} x_2 = 0 \quad x_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix} \qquad Q = \tfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

6.4.6 Find *all* orthogonal matrices that diagonalize $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

$$|A-\lambda I| = \lambda^2 - 25\lambda = 0 \qquad \lambda = 0, 25$$
$$\lambda = 0 \qquad \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} x_1 = 0 \quad x_1 = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$
$$\lambda = 25 \qquad \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} x_2 = 0 \quad x_2 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

$Q = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ and all other combination of those columns with and without their signs reversed.

6.4.11 Write A and B in the form $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad \text{(keep } ||x_1|| = ||x_2|| = 1\text{)}.$$

$$|A-\lambda I| = (\lambda - 4)(\lambda - 2) = 0 \qquad \lambda = 4, 2$$
$$\lambda = 4 \qquad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_1 = 0 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \hat{x}_1 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda = 2 \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2 = 0 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \hat{x}_2 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$A = 4\tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} 2\tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} + 2\tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2\tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

B's eigenvalues and vectors were found in problem 6.4.6

$$B = 0 * \tfrac{1}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix} \tfrac{1}{5} \begin{bmatrix} 4 & -3 \end{bmatrix} + 25 * \tfrac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tfrac{1}{5} \begin{bmatrix} 3 & 4 \end{bmatrix} = 0 * 1/25 \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

6.4.21 **True** (with reason) or **false** (with example). "Orthonormal" is not assumed.

(a) A matrix with real eigenvalues and eigenvectors is symmetric.

False, $\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$ has λ=1,4 with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

True, $A = Q\Lambda Q^T \to A^T = (Q\Lambda Q^T)^T = Q^{T^T}\Lambda^T Q^T = Q\Lambda Q^T$.

(c) The inverse of a symmetric matrix is symmetric.

True, $A = Q\Lambda Q^T \to A^{-1} = (Q\Lambda Q^T)^{-1} = Q^{T^{-1}}\Lambda^{-1}Q^{-1} = Q\Lambda^{-1}Q^T$.

(d) The eigenvector matrix S of a symmetric matrix is symmetric.

False, $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \neq S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Chapter 6.5

6.5.7 .

Chapter 6.6

6.6.17 .

Chapter 6.7

6.7.4 .