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Homework 6

Chapter 6.1

6.1.2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the ______ eigenvectors as A. Its eigenvalues are ______ by 1.

$$A) \quad \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = 5, -1 \qquad \text{eigenvectors=} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A + I) \quad \begin{vmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 8 = \lambda^2 - 6\lambda = 0$$

$$\lambda = 0, 6 \qquad \text{eigenvectors=} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A + I$ has the <u>same</u> eigenvectors as A. Its eigenvalues are <u>increased</u> by 1.

6.1.4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same ______ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues ______. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

$$A) \quad \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2, -3 \qquad \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} x_1 = 0 \qquad \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} x_2 = 0 \qquad \text{eigenvectors=} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

$$A^2) \quad \begin{vmatrix} 7-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 6 = \lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = 4, 9 \qquad \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} x_1 = 0 \qquad \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} x_2 = 0 \qquad \text{eigenvectors=} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

A^2 has the same <u>eigenvectors</u> as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues <u>λ_1^2 and λ_2^2</u>. $\lambda_1^2 + \lambda_2^2 = 13$ because that is the trace of A^2.

6.1.9 What do you do to the equation $Ax = \lambda x$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

Multiply both sides by A.
$AAx = A\lambda x \quad \rightarrow \quad A^2x = \lambda Ax \quad \rightarrow \quad A^2x = \lambda\lambda x \quad \rightarrow \quad A^2x = \lambda^2 x.$

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

Multiply both sides by A^{-1}.
$A^{-1}Ax = A^{-1}\lambda x \quad \rightarrow \quad x = \lambda A^{-1}x \quad \rightarrow \quad \frac{1}{\lambda}x = A^{-1}x.$

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

Add $Ix = x$ to both sides.
$Ix + Ax = x + \lambda x \quad \rightarrow \quad (A + I)x = (\lambda + 1)x.$

6.1.12 Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

$$\lambda = 1 \qquad \begin{bmatrix} -.8 & .4 & 0 \\ .4 & -.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \text{eigenvectors=} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\lambda = 0 \qquad \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \text{eigenvector=} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Combine the eigenvectors when $\lambda = 1$ to get an eigenvector of P with no zero components: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

6.1.13 From the unit vector $u = (\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6})$ construct the rank one projection matrix $P = uu^T$. This matrix has $P^2 = P$ because $u^Tu = 1$.

$$P = \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 1 & 1 & 3 & 5 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ 3 & 3 & 9 & 15 \\ 5 & 5 & 15 & 25 \end{bmatrix}$$

(a) $Pu = u$ comes from $(uu^T)u = u(\_\_\_\_\_\_)$. Then u is an eigenvector with $\lambda = 1$.

$(uu^T)u = u(\underline{\quad u^Tu \quad})$

(b) If v is perpendicular to u show that $Pv = 0$. Then $\lambda = 0$.

$Pv = (uu^T)v = u(u^Tv) = u * 0 = 0$

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ all have $\lambda = 0$ and are independent.

6.1.15 Every permutation matrix leaves $x = (1, 1, ..., 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0 \quad \lambda = \tfrac{1 \pm i\sqrt{3}}{2}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1 = 0 \quad \lambda = 1, 1, -1$$

6.1.19 A 3 by 3 matrix B is known to have eigenvalues 0,1,2. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

B is a rank two because it has a $\lambda = 0$.

(b) the determinate of $B^T B$

$|B^T B| = 0$ because $B^T B$ is singular.

(c) the eigenvalues of $B^T B$

Can't determine.

(d) the eigenvalues of $(B^2 + I)^-1$.

λ's of $(B^2 + I)^{-1}$ are $\lambda = 1, \frac{1}{2}, \frac{1}{5}$.

6.1.21 **The eigenvalues of A equal the eigenvalues of A^T.** This is because $det(A-\lambda I)$ equals $det(A^T - \lambda I)$. That is true because ______. Show by an example that the eigenvectors of A and A^T are *not* the same.

It is true because every square matrix has the property $|A| = |A^T|$.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ do not have the same eigen vectors.}$$

$$\text{Eigenvectors of } A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ while } A^T \text{ has eigenvectors } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6.1.29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

$$A) \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(6-\lambda) = 0 \quad \lambda = 1, 4, 6$$

$$B) \quad |B - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = (\lambda^2 - 3)(\lambda + 2) = 0 \quad \lambda = 2, \pm\sqrt{3}$$

C is a rank one matrix, meaning that two of its λ's are zero. The last λ is the sum of the diagonals.
$\lambda = 0, 0, 6$

Chapter 6.2

6.2.2 If A has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{bmatrix}1\\1\end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

$$S\Lambda S^{-1} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 5\\0 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 3\\0 & 5\end{bmatrix} = A$$

6.2.8 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda S^{-1}\begin{bmatrix}1\\0\end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

$$S^{-1} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}$$

$$S\Lambda^k S^{-1} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^k & 0\\0 & \lambda_2^k\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1^{k+1} & \lambda_2^{k+1}\\\lambda_1^k & \lambda_2^k\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$
$$\rightarrow \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1^{k+1} - \lambda_2^{k+1}\\\lambda_1^k - \lambda_2^k\end{bmatrix}$$

6.2.9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$G_{k+2} = \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \qquad G_{k+1} = G_{k+1} \qquad \text{and} \qquad \begin{bmatrix}G_{k+2}\\G_{k+1}\end{bmatrix} = \begin{bmatrix}A\end{bmatrix}\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\1 & 0\end{bmatrix} \qquad |A-\lambda| = \begin{vmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\1 & -\lambda\end{vmatrix}$$
$$\rightarrow \quad (\lambda-1)(\lambda+\tfrac{1}{2}) = 0 \quad \lambda = 1, -\tfrac{1}{2}$$
$$\lambda = 1 \qquad \begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\1 & -1\end{bmatrix}x_1 = 0 \quad x_1 = \begin{bmatrix}1\\1\end{bmatrix}$$
$$\lambda = -\tfrac{1}{2} \qquad \begin{bmatrix}1 & \frac{1}{2}\\1 & \frac{1}{2}\end{bmatrix}x_2 = 0 \quad x_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda S^{-1}$.

$$A^\infty = \begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\frac{1}{3}\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}$$

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

$$G^{k+1} = A^k\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$$

6.2.10 Prove that every third Fibonacci number in 0,1,1,2,3,... is even.

The fibonacci pattern is odd number + even number then odd number plus odd number, which produces an even number every third term.

6.2.11 True or false: If the eigenvalues of A are 2,2,5 then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

(a) true, no zero eigenvalue (b) false, eigenvalues are repeated
(c) false, repeated eigenvalues may have different eigenvectors

6.2.15 $A^k = S\Lambda S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ______. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

$A^k = S\Lambda S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than $\underline{\ 1\ }$. A_2 has $A_2^k \to 0$ with $\lambda = .3, .9$.

6.2.16 (Recommended) Find Λ and S to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ______.

$$|A - \lambda I| = \lambda^2 - .7\lambda - .3 = 0 \qquad \lambda = 1, -.3$$

$$\lambda = 1 \qquad \begin{bmatrix} -.4 & .9 \\ .4 & -.9 \end{bmatrix} x_1 = 0 \qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -.3 \qquad \begin{bmatrix} .9 & .9 \\ .4 & .4 \end{bmatrix} x_2 = 0 \qquad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -.3 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Lambda^k \quad \to \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$S\Lambda^k S^{-1} \quad \to \quad \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ In the columns of this limiting matrix you see the $\underline{\text{ steady state }}$.

6.2.19 Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \qquad \text{has} \qquad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

$$|B - \lambda I| = (5 - \lambda)(4 - \lambda) = 0 \qquad \lambda = 4, 5$$

$$\lambda = 4 \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x_1 = 0 \qquad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad\qquad \lambda = 5 \qquad \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x_2 = 0 \qquad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad S\Lambda^k S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$$

6.2.36 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = S\Lambda S^{-1}$. The eigenvectors (columns of S) are $(1,i)$ and $(i,1)$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

$$|A - \lambda I| = (\cos\theta - \lambda)^2 + \sin^2\theta = 0 \quad \rightarrow \quad \lambda^2 - 2\cos\theta\lambda + 1 = 0 \qquad \lambda = e^{-i\theta}, e^{i\theta}$$
$$A^n = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{bmatrix} \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$$
$$\rightarrow \quad \frac{1}{2i} \begin{bmatrix} ie^{in\theta} + ie^{-in\theta} & ie^{-in\theta} - ie^{in\theta} \\ e^{in\theta} - e^{-in\theta} & 2e^{in\theta} \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Chapter 6.3

6.3.1 Find two λ's and x's so that $u = e^{\lambda t}x$ solves

$$\frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u$$

What combination $u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ starts from $u(0) = (5, -2)$?

$A = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \quad \lambda = 1, 4 \quad A - 4I = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \quad N(A - 4I) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_1 = e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A - I = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \quad N(A - I) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u_2 = e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $u(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ the $u(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6.3.4 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $\frac{du}{dt} = Au$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

$\frac{dv}{dt} + \frac{dw}{dt} = w - w + v - v = 0$, so $v + w$ is constant at 40.

$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad |A - \lambda I| = \begin{bmatrix} -1 - \lambda & 1 \\ 1 & -1 - \lambda \end{bmatrix} = \lambda^2 + 2\lambda \;\; \lambda = -2, 0 = 0$

$A + 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad N(A + 2I) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$N(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = -2 \quad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 0 \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$V(1) = 20 + 10e^{-2} \qquad V(\infty) = 20$

$W(1) = 20 - 10e^{-2} \qquad W(\infty) = 20$

6.3.5 Reverse the diffusion of people in Problem 4 to $\frac{du}{dt} = -Au$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

$-A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $|A - \lambda I| = \lambda^2 - 2\lambda \to \lambda = 0, 2$

$\lambda_1 = 0 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$v(t) = 20 + 10e^{2t} = \infty \omega t \to \infty$$

$\lambda_2 = 2 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6.3.8 The rabbit population shows fast growth (from 6r) but loss to wolves (from -2w). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \quad \text{and} \quad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectos. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

$A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ $\quad |A - \lambda I| = (6 - \lambda)(1 - \lambda) + 4 = 0 \to \lambda^2 - 7\lambda + 10 \to (\lambda - 5)(\lambda - 2) \quad \lambda = 2, 5$

$A - 2I = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \quad N(A - 2I) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A - 5I = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \quad N(A - 5I) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_1 = 2 \quad x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad u(t) = 10e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 10e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 5 \quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad r(t) = 10e^{2t} + 20e^{5t}$

$w(t) = 20e^{2t} + 10e^{5t}$

Ratio of wolves to rabbits will be $\frac{1}{2}$ as $t \to \infty$, e^{5t} dominates.

6.3.10 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $u = (y, y')$:

$$\frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

$\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$

$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 5 & 4 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda - 4 = 0$

$y = e^{\lambda t} \qquad y'' = 5y' + 4y' \to \frac{1}{25} + -\sqrt{25 + 16} = \lambda$

$\lambda^2 e^{\lambda t} = 5\lambda e^{\lambda t} + 4e^{\lambda t}$

$\lambda^2 - 5\lambda - 4 = 0$

$\lambda = \frac{1}{2} 5 + -\sqrt{41}$

6.3.21 Write $A = \left[\begin{smallmatrix}1 & 4\\ 0 & 0\end{smallmatrix}\right]$ in the form $S\Lambda S^{-1}$. Find e^{At} from $Se^{\Lambda t}S^{-1}$.

$A = \begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix} \quad A - \lambda I = \begin{vmatrix}1-\lambda & 4\\ 0 & -\lambda\end{vmatrix} = \lambda^2 - \lambda \quad \lambda = 0, 1$

$N(A) = \begin{bmatrix}4\\ -1\end{bmatrix}$

$\lambda_1 = 1 \quad x_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$

$\lambda_2 = 0 \quad x_2 = \begin{bmatrix}4\\ -1\end{bmatrix}$

$A = \begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}$

$e^{At} = \begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}e^t & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix} = \begin{bmatrix}e^t & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix} = \begin{bmatrix}e^t & 4e^t - 4\\ 0 & 1\end{bmatrix}$

Chapter 6.4

6.4.4 Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix}-2 & 6\\ 6 & 7\end{bmatrix}$. What is λ?

$$|A - \lambda I| = (\lambda - 10)(\lambda + 5) = 0 \qquad \lambda = 10, -5$$

$$\lambda = 10 \qquad \begin{bmatrix}-12 & 6\\ 6 & -3\end{bmatrix} x_1 = 0 \quad x_1 = \begin{bmatrix}1\\ 2\end{bmatrix}$$

$$\lambda = -5 \qquad \begin{bmatrix}3 & 6\\ 6 & 12\end{bmatrix} x_2 = 0 \quad x_2 = \begin{bmatrix}2\\ -1\end{bmatrix}$$

$$\Lambda = \begin{bmatrix}10 & 0\\ 0 & -5\end{bmatrix} \qquad Q = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

6.4.6 Find *all* orthogonal matrices that diagonalize $A = \begin{bmatrix}9 & 12\\ 12 & 16\end{bmatrix}$.

$$|A - \lambda I| = \lambda^2 - 25\lambda = 0 \qquad \lambda = 0, 25$$

$$\lambda = 0 \qquad \begin{bmatrix}9 & 12\\ 12 & 16\end{bmatrix} x_1 = 0 \quad x_1 = \begin{bmatrix}\frac{4}{5}\\ -\frac{3}{5}\end{bmatrix}$$

$$\lambda = 25 \qquad \begin{bmatrix}-16 & 12\\ 12 & -9\end{bmatrix} x_2 = 0 \quad x_2 = \begin{bmatrix}\frac{3}{5}\\ \frac{4}{5}\end{bmatrix}$$

$Q = \frac{1}{5}\begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}$ and all other combination of those columns with and without their signs reversed.

6.4.11 Write A and B in the form $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} \qquad B = \begin{bmatrix}9 & 12\\ 12 & 16\end{bmatrix} \qquad (\text{keep } ||x_1|| = ||x_2|| = 1).$$

$$|A - \lambda I| = (\lambda - 4)(\lambda - 2) = 0 \qquad \lambda = 4, 2$$

$$\lambda = 4 \qquad \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} x_1 = 0 \quad x_1 = \begin{bmatrix}1\\ 1\end{bmatrix} \quad \hat{x}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\lambda = 2 \qquad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} x_2 = 0 \quad x_2 = \begin{bmatrix}1\\ -1\end{bmatrix} \quad \hat{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$A = 4\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} 2\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} + 2\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} 2\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix} = 2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

B's eigenvalues and vectors were found in problem 6.4.6

$$B = 0 * \frac{1}{5}\begin{bmatrix}4\\ -3\end{bmatrix}\frac{1}{5}\begin{bmatrix}4 & -3\end{bmatrix} + 25 * \frac{1}{5}\begin{bmatrix}3\\ 4\end{bmatrix}\frac{1}{5}\begin{bmatrix}3 & 4\end{bmatrix} = 0 * 1/25\begin{bmatrix}16 & -12\\ -12 & 9\end{bmatrix} + \begin{bmatrix}9 & 12\\ 12 & 16\end{bmatrix}$$

6.4.21 **True** (with reason) or **false** (with example). "Orthonormal" is not assumed.

(a) A matrix with real eigenvalues and eigenvectors is symmetric.

False, $\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$ has λ=1,4 with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

True, $A = Q\Lambda Q^T \rightarrow A^T = (Q\Lambda Q^T)^T = {Q^T}^T \Lambda^T Q^T = Q\Lambda Q^T$.

(c) The inverse of a symmetric matrix is symmetric.

True, $A = Q\Lambda Q^T \rightarrow A^{-1} = (Q\Lambda Q^T)^{-1} = {Q^T}^{-1} \Lambda^{-1} Q^{-1} = Q\Lambda^{-1} Q^T$.

(d) The eigenvector matrix S of a symmetric matrix is symmetric.

False, $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \neq S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Chapter 6.5

6.5.7 Test to see if R^R is positive definite in each case:

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad R^T R = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

$x^T \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} x = x^T \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 13x_2 \end{bmatrix} = x_1^2 + 4x_2x_1 + 13x_2^2 > 0$ positive definite

$R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad R^T R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$\begin{vmatrix} 6-\lambda & 5 \\ 5 & 6-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 11 = 0 \rightarrow (\lambda - 1)(\lambda - 11) \rightarrow \lambda = 1, 11$ All eigenvalues are positive so $R^T R$ is positive definite

$R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad R^T R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 11\lambda = 0 \rightarrow \lambda = 0, 1, 11$ Positive Semi-definite

6.5.10 Which 3 by 3 symmetric matrices A and B produce these quadratics?
$x^T Ax = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$. Why is A positive definite?
$x^T Bx = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$. Why is B semidefinite?

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ After elimination $\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$

A has all positive pivots so A is positive definite

$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = \lambda^3 + 6\lambda^2 + 9\lambda = 0 \quad \lambda = 0, 3, 3$

B is semi definite because all of its λ's are either positive or zero.

6.5.17 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have ____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a ____ on the main diagonal.

positive - first blank
zero - second blank

6.5.18 If $Ax = \lambda x$ then $x^T Ax =$ ____. If $x^T Ax > 0$, prove that $\lambda > 0$.
$x^T Ax = x^T \lambda x \rightarrow \lambda = \frac{x^T Ax}{x^T x}$ if $x^T Ax > 0$

6.5.19 Reverse Problem 18 to show that if all $\lambda > 0$ then $x^T Ax > 0$. We must do this for every nonzero x, not just the eigenvectors. So write x as a combination of the eigenvectors and explain why all "cross terms" are $x_i^T x_j = 0$. Then $x^T Ax$ is

$(c_1 x_1 + \cdots + c_n x_n)^T (c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n) = c_1^2 \lambda_1 x_1^T x_1 + \cdots + c_n^2 \lambda_n x_n^T x_n > 0$. $x^T Ax = x^T \lambda x \rightarrow$ $x^T Ax = \lambda x^T x \rightarrow x^T Ax = \lambda \|x\|^2 > 0$ if $\lambda > 0$
$x_i^T x_j = 0$ because evectors of symmetric matrices are orthogonal.

6.5.20 Give a quick reason why each of these statements is true:
(a) Every positive definite matrix is invertible.
Positive definite matrices have non-zero evalues

(b) The only positive definite porjection matrix is $P = I$.
All projection matrices are singular except for I

(c) A diagonal matrix with positive diagonal entries is positive definite.
It has positive evalues and pivots

(d) A symmetric matrix with a positive determinant might not be positive definite!
It could have two negative evalues because the det $= \pi_i \lambda_i$

6.5.28 Without multiplying $A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$
(a) The determinant of A (b) the eigenvalues of A
(c) the eigenvectors of A (d) a reason why A is symmetric positive definite.
(a) $|A| = 2 \cdot 5 = 10$
(b) Values along the diagonal of middle matrix, so $\lambda = 2, 5$
(c) Columns of the first matrix $\begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$
(d) A has all positive evalues

Chapter 6.6

6.6.17 True of False, with a good reason:
(a) A symmetri matrix can't be similar to a nonsymmetric matrix.
False $A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ $\lambda_1 = 2$ $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda_2 = 5$ $x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} - y_3$ A is similar to $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

(b) An invertible matrix can't be similar to a singular matrix.
True, Ranks of similar matrices are the same.

(c) A can't be similar to $-A$ unless $A = 0$.
False $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar with $\lambda = -1, 1$

(d) A can't be similar to $A + I$.
True, Adding I increases the evalues by 1

6.6.18 If B is invertible, prove that AB is similar to BA. *They have the ame eigenvalues.*
$AB = \lambda x \to MAB = \lambda Mx \to$
$$MAB = MBA \to AB = MBAM^{-1}$$
$BA = \lambda x \to MBA = \lambda Mx \to$

6.6.20 Why are these statements all true?
(a) If A is similar to B then A^2 is similar to B^2.
$Ax = \lambda x \to A^2x = \lambda^2 x$ both evalues are squared
$Bx = \lambda x \to B^2x = \lambda^2 x$

(b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).
$A^2 = (-A)^2$ but $B \neq -A$

(c) $\left[\begin{smallmatrix}3 & 0\\ 0 & 4\end{smallmatrix}\right]$ is not similar to $\left[\begin{smallmatrix}3 & 1\\ 0 & 4\end{smallmatrix}\right]$.
Both have $\lambda = 3, 4$

(d) $\left[\begin{smallmatrix}3 & 0\\ 0 & 3\end{smallmatrix}\right]$ is not similar to $\left[\begin{smallmatrix}3 & 1\\ 0 & 3\end{smallmatrix}\right]$.
Both have $\lambda = 3, 3$ so there are not two evectors to construct a invertible matrix with

(e) If we echange rows 1 and 2 of A, and then exchange columns 1 and 2, **the eigenvalues stay the same**. In this case $M =$ ____.
$$A = PAP^T \qquad M = P = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Chapter 6.7

6.7.4 Find the eigenvalues and unit egienvectors of A^TA and AA^T. Keep each $Av = \sigma \boldsymbol{u}$:
$$\textbf{Fibonacci matrix} \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$$
Construct the singular value decomposition and verify that A equals $U\Sigma V^T$.
$A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} AA^T = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$
$|AA^T - \lambda I| = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 \;\; \lambda = \frac{3+=\sqrt{5}}{2}$
$AA^T - (\frac{3}{2} + \frac{\sqrt{5}}{2})I = \begin{bmatrix}\frac{1}{2} - \frac{\sqrt{5}}{2} & 1\\ 1 & \frac{-1}{2} - \frac{\sqrt{5}}{2}\end{bmatrix} \quad N(AA^T - (\frac{3}{2} + \frac{\sqrt{5}}{2}I) = \begin{bmatrix}1\\ \frac{5}{2} - \frac{1}{2}\end{bmatrix}$
$AA^T - (\frac{3}{2} - \frac{\sqrt{5}}{2})I = \begin{bmatrix}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1\\ 1 & \frac{-1}{2} + \frac{\sqrt{5}}{2}\end{bmatrix} \quad N(AA^T - (\frac{3}{2} - \frac{\sqrt{5}}{2}I) = \begin{bmatrix}1\\ \frac{1}{2} - \frac{\sqrt{5}}{2}\end{bmatrix}$
$\sigma_1 = \frac{1}{2} + \frac{\sqrt{5}}{2} = \lambda_1(A)\sigma_2 = \frac{\sqrt{5}}{2} - \frac{1}{2} = \lambda_2(A)$

6.7.6 Compute A^TA and AA^T and their eigenvalues and unit eigenvectors for V and U.
$$\textbf{Rectangular matrix } A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$
Check $AV = U\Sigma$ (this will decide $+ =$ signs in U). Σ has the same shape as A.
$$A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} \qquad A^TA = \begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}$$
$$AA^T = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$
$|AA^T - \lambda I| = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

$$AA^T - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad N(AA^T - I) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$$
$$AA^T - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad N(AA^T - 3I) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$
$$A^TA - I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad N(A^TA - I) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$$
$$A^TA - 3I = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad N(A^TA - 3I) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{6}}$$
$$N(A^TA) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}}$$
$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

6.7.7 .

The closest rank one matrix will be the combination of $u_i \sigma_i v_i^T$, where i is determined by the i value of the largest evalue.

$$\sqrt{3} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$$

6.7.10 Construct the matrix with rank one that has $Av = 12u$ for $v = \frac{1}{2}(1,1,1,1)$ and $u = \frac{1}{3}(2,2,1)$. It only sigular value is $\sigma_1 =$ ____.

$$A = 12uv^T = 12 \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = 2 \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

6.7.11 Suppose A has orthogonal columns $w_1, w_2, \cdots, w_n$ of lengths $\sigma_1, \sigma_2, \cdots, \sigma_n$. What are U, Σ, and V in the SVD?

$A^TA = I = V$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdots & \\ & & & \sigma_n \end{bmatrix} \quad U = \begin{bmatrix} \frac{AV_i}{\sigma_i} & & \\ & \cdots & \\ & & \frac{AV_n}{\sigma_n} \end{bmatrix}$$