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Homework 5

Chapter 5.1

5.1.3 True or false, with a reason if true or a counterexample if false:

(a) The determinant of $I + A$ is $1 + \det A$.

False, $A = \begin{bmatrix} 7 & 1 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad A + I = \begin{bmatrix} 8 & 1 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad 1 + |A| = 29 \neq |I + A| = 80$

(b) The determinant of ABC is $|A||B||C|$.

True, because of property #9.

(c) The determinant of $4A$ is $4|A|$.

False, $\det\left(4\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 8 * 8 \neq 4\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 * 4.$

(d) The determinant of $AB - BA$ is zero. Try an example with $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

False, $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which is invertible, meaning the determinant is not zero.

5.1.24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of $L, U, A, U^{-1}, L^{-1}, and U^{-1}L^{-1}A$.

If one reduces L to its reduced row echelon form it becomes I. So $|L| = 1$

$|U| = 3 * 2 * 1 = -1$

$|A| = |U| = -6$

$|U^{-1}L^{-1}| = \frac{1}{|U|} * \frac{1}{|L|} = -\frac{1}{6}$

$|U^{-1}L^{-1}A| = |A| = 1$

5.1.27 Compute the determinants of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

A: Two row swaps are required to get A in the form $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ so $|A| = (-1)(-1)abc = abc$.

B: Three row swaps are required to get B in the form $\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$

so $|B| = (-1)(-1)(-1)abcd = abcd$.

C: You get RREF(C) as $\begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & c-b \end{bmatrix}$ so $|C| = a(b-a)(c-b)$.

5.1.28 True or false (give a reason if true or a 2 by 2 example if false):

(a) If A is not invertible then AB is not invertible.

True, det(AB)=det(A)$*$det(B)=0$*$det(B)=0.

(b) The determinant of A is always the products of its pivots.

False, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ requires a row swap therefore the product is the products of its pivots times -1.

(c) The determinant of $A - B$ equals det(A) - det(B).

False, $\left| \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \neq \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 1 - 0 = 1$

(d) AB and BA have the same determinant.

True, since multiplication is commutative. $|AB| = |A||B| = |B||A| = |BA|$

Chapter 5.2

5.2.9 Show that 4 is the largest determinant for a 3 by 3 matrix of 1's and -1's.

The determinant for a 3x3 matrix has six terms, half of which are positive. Each value from the matrix is apart of two of those terms, once in a positive and once in a negative term. If all values where all positive 1's or negative 1's the determinant would be three. If there are an even number of negative 1's then the largest the determinant could be would be four.

5.2.23 With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A|\,|D| - |C|\,|B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

Any entry from B times and entry from the zero block is zero, so B can be excluded.

(b) Show by example that equality fails (as shown) when C enters.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A||D| - |C||B| = 0 \neq \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$$

(c) Show by example that the answer det($AD - CB$) is also wrong.

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \neq \begin{vmatrix} A & B \\ C & D \end{vmatrix} = -1$$

5.2.33 The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain).}$$

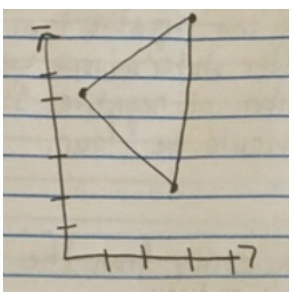
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 19 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1 + (-1) = 0$$

Chapter 5.3

5.3.16 (a) Find the area of the parallelogram with edges $v = (3, 2)$ and $w = (1, 4)$

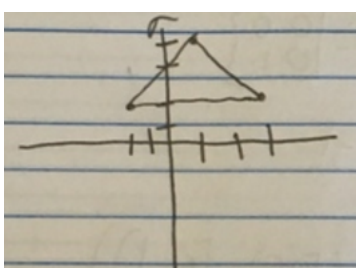
$$\text{area} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 12 - 2 = 10$$

(b) Find the area of the triangle with sides v, w, and $v + w$. Draw it.



$$\text{area} = \frac{1}{2} \begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 4 & 6 & 1 \end{vmatrix} = \frac{1}{2}(3(-2) + 2(3) - 10) = -5 = |-5| = 5$$

(c) find the area of the triangle with sides v, w, and $w - v$. Draw it



$$\text{area} = \frac{1}{2} \begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ -2 & 2 & 1 \end{vmatrix} = \frac{1}{2}(3(2) + 2(-3) - 10) = -5 = |-5| = 5$$

5.3.20 The Hadamard matrix H has orthogonal rows. The box is hypercube!

$$\text{What is } |H| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \text{volume of a hypercube in } R^4$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$= 1 \cdot -2 \cdot -2 \cdot -4 = 16$

5.3.21 If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volumne)? If all entries of the matrix are 1 or -1, what are those lengths and the maximum determinant?

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$$

$L_1 = (1,1,1,1)$, $L_2 = (1,1,-1,-1)$, $L_3 = (1,-1,-1,1)$, $L_4 = (1,-1,1,-1)$
max det $= 16$

5.3.27 Polar coordinates satisfy $x = rcos\theta$ and $y = rsin\theta$. Polar area is $J\ dr\ d\theta$:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

The two columns are orthogonal. Their lengths are ____. Thus $J =$ ____.

1 and r, $J = r$

5.3.28 Spherical coordinates ρ, ϕ, θ satisfy $x = \rho sin\phi cos\theta$ and $y = \rho sin\phi sin\theta$ and $z = \rho cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial \rho$, $\partial x/\partial \phi$, $\partial x/\partial \theta$ in row 1. Simplify its determinant to $J = \rho^2 sin\phi$. Then dV in spherical coordinates is $\rho^2 sin\phi dp d\phi d\theta$, the volume of an infinitesimal "coordinate box".

$$J = \begin{vmatrix} sin\phi cos\theta & \rho cos\phi cos\theta & -\rho sin\phi sin\theta \\ sin\phi sin\theta & \rho cos\phi sin\theta & \rho sin\phi cos\theta \\ cos\theta & -\rho sin\phi & 0 \end{vmatrix} = \rho^2 sin\phi$$

5.3.29 The matrix that connects r, θ to x, y is Problem 27. Invert that 2 by 2 matrix:

$$J^{-1} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} cos\theta & ? \\ ? & ? \end{vmatrix} = ?$$

If is surprising that $\partial r/\partial x = \partial x/\partial r$ (Calculus, Gilbert Strang, p. 501). Multiplying the matrices J and J^{-1} gives the chain rule $\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x} = 1$.

$$J^{-1} = \begin{vmatrix} cos\theta & sin\theta \\ \frac{-sin\theta}{r} & \frac{cos\theta}{r} \end{vmatrix} = \frac{1}{r}cos^2\theta + \frac{1}{r}cos^2\theta = \frac{1}{r}$$