Dustin Ginos:
A01233669
Chandler Kinch:
A01662772
Jeff Wasden:
A01657029

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Homework 5

Chapter 5.1

5.1.3 True or false, with a reason if true or a counterexample if false:

(a) The determinant of $I + A$ is $1 + \det A$.

False, $A = \begin{bmatrix} 7 & 1 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad A + I = \begin{bmatrix} 8 & 1 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad 1 + |A| = 29 \neq |I + A| = 80$

(b) The determinant of ABC is $|A||B||C|$.

True, because of property #9.

(c) The determinant of $4A$ is $4|A|$.

False, $\det\left(4\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 8 * 8 \neq 4\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 * 4.$

(d) The determinant of $AB - BA$ is zero. Try an example with $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

False, $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which is invertible, meaning the determinant is not zero.

5.1.24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of $L, U, A, U^{-1}, L^{-1}, and U^{-1}L^{-1}A$.

If one reduces L to its reduced row echelon form it becomes I. So $|L| = 1$

$|U| = 3 * 2 * 1 = -1$

$|A| = |U| = -6$

$|U^{-1}L^{-1}| = \frac{1}{|U|} * \frac{1}{|L|} = -\frac{1}{6}$

$|U^{-1}L^{-1}A| = |A| = 1$

5.1.27 Compute the determinants of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

A: Two row swaps are required to get A in the form $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ so $|A| = (-1)(-1)abc = abc$.

B: Three row swaps are required to get B in the form $\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$

so $|B| = (-1)(-1)(-1)abcd = abcd$.

C: You get RREF(C) as $\begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & c-b \end{bmatrix}$ so $|C| = a(b-a)(c-b)$.

5.1.28 True or false (give a reason if true or a 2 by 2 example if false):

(a) If A is not invertible then AB is not invertible.

True, det(AB)=det(A)$*$det(B)=0$*$det(B)=0.

(b) The determinant of A is always the products of its pivots.

False, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ requires a row swap therefore the product is the products of its pivots times -1.

(c) The determinant of $A - B$ equals det(A) - det(B).

False, $\left| \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \neq \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 1 - 0 = 1$

(d) AB and BA have the same determinant.

True, since multiplication is commutative. $|AB| = |A||B| = |B||A| = |BA|$

Chapter 5.2

5.2.9 Show that 4 is the largest determinant for a 3 by 3 matrix of 1's and -1's.

5.2.23 With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \qquad\qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A|\,|D| - |C|\,|B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

(b) Show by example that equality fails (as shown) when C enters.

(c) Show by example that the answer det($AD - CB$) is also wrong.

5.2.33 The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

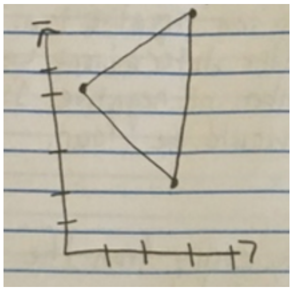
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad\qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain)}.$$

Chapter 5.3

5.3.16 (a) Find the area of the parallelogram with edges $v = (3, 2)$ and $w = (1, 4)$

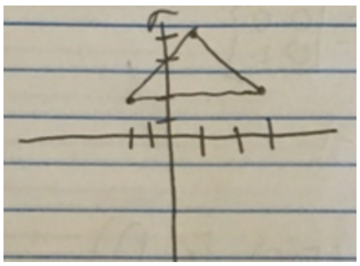
area $= \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 12 - 2 = 10$

(b) Find the area of the triangle with sides v, w, and $v+w$. Draw it.



area= $\frac{1}{2}\begin{vmatrix}3 & 2 & 1\\1 & 4 & 1\\4 & 6 & 1\end{vmatrix} = \frac{1}{2}(3(-2)+2(3)-10) = -5 = |-5| = 5$

(c) find the area of the triangle with sides v, w, and $w-v$. Draw it



area $= \frac{1}{2}\begin{vmatrix}3 & 2 & 1\\1 & 4 & 1\\-2 & 2 & 1\end{vmatrix} = \frac{1}{2}(3(2)+2(-3)-10) = -5 = |-5| = 5$

5.3.20 The Hadamard matrix H has orthogonal rows. The box is hypercube!

$$\text{What is } |H| = \begin{vmatrix}1 & 1 & 1 & 1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\\1 & -1 & 1 & -1\end{vmatrix} = \text{volume of a hypercube in } R^4$$

$$\begin{vmatrix}1 & 1 & 1 & 1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\\1 & -1 & 1 & -1\end{vmatrix} = \begin{vmatrix}1 & 1 & 1 & 1\\0 & 0 & -2 & -2\\0 & -2 & -2 & 0\\0 & -2 & 0 & -2\end{vmatrix} = -\begin{vmatrix}1 & 1 & 1 & 1\\0 & -2 & -2 & 0\\0 & 0 & -2 & -2\\0 & -2 & 0 & -2\end{vmatrix} = -\begin{vmatrix}1 & 1 & 1 & 1\\0 & -2 & -2 & 0\\0 & 0 & -2 & -2\\0 & 0 & 2 & -2\end{vmatrix} = \begin{vmatrix}1 & 1 & 1 & 1\\0 & -2 & -2 & 0\\0 & 0 & -2 & -2\\0 & 0 & 0 & -4\end{vmatrix}$$
$= 1 \cdot -2 \cdot -2 \cdot -4 = 16$

5.3.21 If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volumne)? If all entries of the matrix are 1 or -1, what are those lengths and the maximum determinant?

$$\begin{vmatrix}1 & 1 & 1 & 1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\\1 & -1 & 1 & -1\end{vmatrix}$$

$L_1 = (1,1,1,1)$, $L_2 = (1,1,-1,-1)$, $L_3 = (1,-1,-1,1)$, $L_4 = (1,-1,1,-1)$
max det $= 16$

5.3.27 Polar coordinates satisfy $x = rcos\theta$ and $y = rsin\theta$. Polar area is $J\ dr\ d\theta$:

$$J = \begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix}cos\theta & -rsin\theta\\ sin\theta & rcos\theta\end{vmatrix}$$

The two columns are orthogonal. Their lengths are ____. Thus $J =$ ____.

1 and r, $J = r$

5.3.28 Spherical coordinates ρ, ϕ, θ satisfy $x = \rho sin\phi cos\theta$ and $y = \rho sin\phi sin\theta$ and $z = \rho cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial\rho$, $\partial x/\partial\phi$, $\partial x/\partial\theta$ in row 1. Simplify its determinant to $J = \rho^2 sin\phi$. Then dV in spherical coordinates is $\rho^2 sin\phi dpd\phi d\theta$, the volume of an infinitesimal "coordinate box".

$$J = \begin{vmatrix} sin\phi cos\theta & \rho cos\phi cos\theta & -\rho sin\phi sin\theta \\ sin\phi sin\theta & \rho cos\phi sin\theta & \rho sin\phi cos\theta \\ cos\theta & -\rho sin\phi & 0 \end{vmatrix} = \rho^2 sin\phi$$

5.3.29 The matrix that connects r, θ to x, y is Problem 27. Invert that 2 by 2 matrix:

$$J^{-1} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} cos\theta & ? \\ ? & ? \end{vmatrix} =?$$

If is surprising that $\partial r/\partial x = \partial x/\partial r$ (Calculus, Gilbert Strang, p. 501). Multiplying the matrices J and J^{-1} gives the chain rule $\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x} = 1$.

$$J^{-1} = \begin{vmatrix} cos\theta & sin\theta \\ \frac{-sin\theta}{r} & \frac{cos\theta}{r} \end{vmatrix} = \frac{1}{r}cos^2\theta + \frac{1}{r}cos^2\theta = \frac{1}{r}$$