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Homework 3

Chapter 3.1

3.1.10 Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

yes

(b) The plane of vectors with $b_1 = 1$.

no

(c) The vectors with $b_1 b_2 b_3 = 0$.

no

(d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

yes

(e) All vectors that satisfy $b_1 + b_2 + b_3 = O$.

yes

(f) All vectors with $b_1 < b_2 < b_3$.

no

3.1.17 (a) Show that the set of *invertible* matrices in M is not a subspace.

No because there is no zero matrix

(b) Show that the set of *singular* matrices in M is not a subspace.

No

ex: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 12 & 24 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 14 & 28 \end{bmatrix}$

3.1.18 True or false (check addition in each case by an example):

(a) The symmetric matrices in M (with $A^T = A$) form a subspace.

True: $\begin{bmatrix} 1 & 2 & 7 \\ 2 & 1 & 2 \\ 7 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 11 \\ 3 & 4 & 4 \\ 11 & 4 & 4 \end{bmatrix}$

(b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

True: $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 1 \\ -5 & 0 & 5 \\ -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3 \\ -6 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

(c) The unsymmetric matrices in M (with $A^T \neq -A$) form a subspace.

False: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3.1.27 True or false (with a counterexample if false):

(a) The vectors b that are not in the column space $C(A)$ form a subspace.

False: Vectors that aren't in a column space can't form a subspace.

(b) If $C(A)$ contains only the zero vector, then A is the zero matrix.

True

(c) The column space of $2A$ equals the column space of A.

True

(d) The column space of $A - I$ equals the column space of A (test this).

False

3.1.28 Construct a 3 by 3 matrix whose column space contains (1, 1,0) and (1,0,1) but not (1,1, 1). Construct a 3 by 3 matrix whose column space is only a line.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & 0 \\ 4 & 8 & 0 \end{bmatrix}$$

Chapter 3.2

3.2.9 True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

False: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(b) An invertible matrix has no free variables.

True because an invertible matrix has three pivots

(c) An m by n matrix has no more than n pivot variables.

True because a matrix can't have more pivot variables than it has pivot columns

(d) An m by n matrix has no more than m pivot variables.

True because a matrix can't have more pivot variables than it has variables

3.2.19 Prove that U and $A = LU$ have the same nullspace when L is invertible:

$$\text{If } Ux = 0 \text{ then } LUx = 0. \text{ If } LUx = 0, \text{ how do you know } Ux = 0?$$

3.2.21 Construct a matrix whose nullspace consists of all combinations of (2,2,1,0) and (3,1,0,1).

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 5 & 3 & 1 & 1 \end{bmatrix}$$

3.2.22 Construct a matrix whose nullspace consists of all multiples of (4, 3, 2,1).

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

3.2.23 Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose nullspace contains (1, 1,2).

$$\begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

3.2.24 Construct a matrix whose column space contains (1, 1,0) and (0,1,1) and whose nullspace contains (1,0,1) and (0,0,1).

The matrix is impossible to construct because there is only one free column, and there are two independent vectors in the nullspace. This means there would need to be two free columns.

3.2.25 Construct a matrix whose column space contains (1, 1, 1) and whose nullspace is the line of multiples of (1, 1, 1, 1).

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

3.2.26 Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible.

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad C(A) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$
$$N(A) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

3.2.27 Why does no 3 by 3 matrix have a nullspace that equals its column space?

If a $3x3$ matrix has one pivot, it has one vector in the column space and two in the nullspace vice versa, therefore the column space can never equal the nullspace.

3.2.28 If $AB = 0$ then the column space of B is contained in the _____ of A. Give an example of A and B.

Nullspace

$$\underset{\text{A}}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \underset{\text{B}}{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}}$$

3.2.29 The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be _____. What R is virtually certain if the random A is 4 by 3?

I

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Chapter 3.3

3.3.10 Choose vectors u and v so that $A = uv^T$ = column times row:

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} .$$

$A = uv^T$ is the natural form for every matrix that has rank $r = 1$.

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -3 & -2 \end{bmatrix}$$

3.3.23 Answer the same questions as in Worked Example **3.3 C** for

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix} .$$

$$\text{If c} = 1,\ R_A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ then } N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank 1

$$\text{If c} \neq 1,\ R_A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ then } N = \begin{bmatrix} -1 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rank 2

$$\text{If c} = 1,\ R_B = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \text{ then } N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rank 1

$$\text{If c} = 2,\ R_B = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \text{ then } N = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

If c $\neq 1, 2$ R = I

Chapter 3.4

3.4.4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 6 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

y and t free variables $y = t = 0$ then $x = \frac{1}{2} = z$

$$X_p = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \qquad X_n = c_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$X_{complete} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

3.4.19 Find the rank of A and also of $A^T A$ and also of AA^T:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$R_A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix} \text{ rank} = 2$$

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 5 \\ 6 & 5 & 26 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & 6 \\ 6 & 5 & 26 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = R_{A^T A}$$

rank $= 2$

$$AA^T = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ 6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{2}{9} \\ 6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{2}{9} \\ 0 & \frac{2}{3} \end{bmatrix} = R_{AA^T} \text{ rank} = 2$$

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R_A \text{ rank} = 2$$

$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & \frac{7}{2} \end{bmatrix} = R_{A^T A} \text{ rank} = 2$$

$$AA^T = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R_{AA^T} \text{ rank: } 2$$

3.4.21 Find the complete solution in the form $x_p + x_n$ to these full rank systems:

(a) $x+y+z=4$ (b) $x+y+z=4$
$x-y+z=4$

$x+y+z=4$

$Ax=b \rightarrow \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=4 \quad X_p=\begin{bmatrix}4\\0\\0\end{bmatrix} X_n = c_1\begin{bmatrix}1 & -1 & 0\end{bmatrix}+c_2\begin{bmatrix}1 & 0 & -1\end{bmatrix}$

y and z free variables

$X_c=\begin{bmatrix}4\\0\\0\end{bmatrix}+c_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$x+y+z=4$
$x-y+z=4$

$Ax=b \rightarrow \begin{bmatrix}1 & 1 & 1\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 1\\0 & -2 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$

$X_p=\begin{bmatrix}4\\0\\0\end{bmatrix} \quad X_n = c\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad X_c = c\begin{bmatrix}4\\0\\0\end{bmatrix}+c\begin{bmatrix}1\\0\\-1\end{bmatrix}$

3.4.24 Give examples of matrices A for which the number of solutions to $Ax=b$ is

(a) 0 or 1, depending on $\boldsymbol{b}$

$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$

(b) ∞, regardless of $\boldsymbol{b}$

$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

(c) 0 or ∞, depending on $\boldsymbol{b}$

$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

(d) 1, regardless of $\boldsymbol{b}$.

$\begin{bmatrix}2 & 3\\3 & 4\end{bmatrix}$

3.4.33 The complete solution to $Ax=\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ is $x=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]+c\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Find A.

$Ax=\begin{bmatrix}1\\3\end{bmatrix} x=\begin{bmatrix}1\\0\end{bmatrix}+c\begin{bmatrix}0\\1\end{bmatrix}$

$c_1=\begin{bmatrix}1\\3\end{bmatrix} \quad c_2 = A_{12}-A_{22}$

$A=\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}$

Chapter 3.5

3.5.16 Find a basis for each of these subspaces of $\mathbf{R}^4$
(a) All vectors whose components are equal.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ Only one vector so therefore it is linearly independent.}$$

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} \text{ all components are equal.}$$

(b) All vectors whose components add to zero.

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$a \neq 0, b \neq 0, c \neq 0 \qquad a \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a+b+c \\ -a \\ -b \\ -c \end{bmatrix} \neq 0$$

There is no linear combination of the vectors that will equal the zero vector. The above equation also shows that if all components of the right hand side vector are added together the result is zero.

(c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$a \neq 0, b \neq 0 \qquad a \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a+b \\ -a-b \\ -b \\ -c \end{bmatrix} \neq 0 \text{ It is linearly independent.}$$

$$\begin{bmatrix} a+b \\ -a-b \\ -b \\ -c \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \qquad \begin{bmatrix} a+b \\ -a-b \\ -b \\ -c \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

This shows that any vector from the set will be perpendicular to (1,1,0,0) and (1,0,1,1)

(d) The column space and the nullspace of $\boldsymbol{I}$ (4 by 4).

$$C(I) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$N(I) = 0$$

Each vector in the set is orthogonal to each other vector in the set. This can be seen by taking the dot product between each vector.

The basis spans the entire set by definition of the column space of a matrix.

3.5.24 True of false (give a good reason):
(a) If the columns of a matrix are dependent, so are the rows.

False. In the matrix $\begin{bmatrix}1 & 1 & 2\\ 3 & 7 & 10\end{bmatrix}$ the last column is dependent on the first two but the rows are independent.

(b) The column space of a 2 by 2 matrix is the same as its row space.

$$\text{False. } A = \begin{bmatrix}1 & 1\\ 3 & 7\end{bmatrix} \qquad C(A) = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\7\end{bmatrix}\right\}$$

$$C(A^T) \neq C(A) \qquad C(A^T) = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\7\end{bmatrix}\right\}$$

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

$$\text{True. } A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \qquad C(A) = \left\{\begin{bmatrix}a\\c\end{bmatrix}, \begin{bmatrix}b\\d\end{bmatrix}\right\} \qquad C(A^T) = \left\{\begin{bmatrix}a\\b\end{bmatrix}, \begin{bmatrix}c\\d\end{bmatrix}\right\}$$

Both $C(A)$ and $C(A^T)$ have the same dimension.

(d) The columns of a matrix are a basis for the column space.

$$\text{False. } A = \begin{bmatrix}1 & 3 & 4\\ 1 & 4 & 5\\ 1 & 5 & 6\end{bmatrix} \qquad C(A) = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix}\right\}$$

$C(A)$ only has two vectors, meaning a column was excluded from A.

3.5.26 Find a basis (and the dimension) for each of these subspaces of 3 by 3 matrices:
(a) All diagonal matrices.

$$S = \left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$$

Dimension = 3

(b) All symmetric matrices ($A^T = A$).

$$S = \left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\right\}$$

Dimension = 6

(c) All skew-symmetric matrices ($A^T = -A$).

$$S = \left\{\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}\right\}$$

Dimension = 3

Chapter 3.6

3.6.2 Find bases and dimensions for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} = R$$

$C(A)$ =SPAN $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ 1 dimensional $\qquad$ $C(A^T)$ =SPAN $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$ 1 dimensional

$N(A)$ =SPAN $\left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ 2 dimensional $\qquad$ $N(A^T)$ =SPAN $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ 1 dimensional

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} = R$$

$C(B)$ =SPAN $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ 2 dimensional $\qquad$ $C(B^T)$ =SPAN $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$ 2 dimensional

$N(B)$ =SPAN $\left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ 1 dimensional $\qquad$ $N(B^T)$ =SPAN $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ 1 dimensional

3.6.3 Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C(A)$ =SPAN $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$ $\qquad$ $C(A^T)$ =SPAN $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \right\}$

$N(A)$ =SPAN $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ $\qquad$ $N(A^T)$ =SPAN $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

3.6.13 True of false (with a reason or a counterexample):
(a) If $m = n$ then the row space of A equals the column space.

False. Proof: $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad$ $C(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$ $\qquad$ $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} \right\}$ $\qquad$ $C(A) \neq C(A^T)$

(b) The matrices A and $-A$ share the same four subspaces.

True. Since -A is a multiple of A they have the same four subspaces because each each subspace has closure under multiplication.

(c) If A and B share the same four subspaces then A is a multiple of B.

False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 \\ 0 & 6 \end{bmatrix}$

$$C(A) = C(B) \qquad C(A^T) = C(B^T) \qquad N(A) = N(B) \qquad N(A^T) = N(B^T) \qquad A \neq cB$$

3.6.21 Suppose A is the sum of two matrices of rank one: $A = \boldsymbol{u}\boldsymbol{v}^T + \boldsymbol{w}\boldsymbol{z}^T$.
(a) Which vectors span the column space of A?
u and v

(b) Which vectors span the row space of A?
v and z

(c) The rank is less than 2 if ____ or if ____.
if u and w are dependent or if z and w are dependent

(d) Compute A and its rank if $\boldsymbol{u} = \boldsymbol{z} = (1, 0, 0)$ and $\boldsymbol{v} = \boldsymbol{w} = (0, 0, 1)$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{rank=2}$$

3.6.22 Construct $A = uv^T + wz^T$ whose column space has basis $(1, 2, 4), (2, 2, 1)$ and whose row space has basis $(1, 0), (1, 1)$. Write A as (3 by 2) times (2 by 2).

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$$

3.6.23 Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

How do you know from these shapes that A cannot be invertible?

$$C(A) = \left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \right\} \qquad C(A^T) = \left\{ \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

The rank of A cannot be larger then the rank of either of its factors and to be invertible it would need a rank of 3 which is greater than that of either of its factors.

3.6.25 True or false (with a reason or a counterexample):
(a) A and A^T have the same number of pivots.

True. Both A and A^T have the same rank.

(b) A and A^T have the same left nullspace.

False.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}$$

The dimensions of $N(A^T)$ is 1x3 and $N((A^T)^T)$ is 1x2

(c) If the row space equals the column space then $A^T = A$.

False. If A is invertible and unsymmetric then $C(A) = C(A^T)$ while $A \neq A^T$

(d) If $A^T = -A$ then the row space of A equals the column space.

True. A and $-A$ always have the same row and column space. So if $A^T = -A$, A's row and column space are equal to each other.

3.6.26 (***Rank of AB***) if $AB = C$, the rows of C are combinations of the rows of $\underline{\quad B \quad}$. So the rank of C is not greater than the rank of $\underline{\quad B \quad}$. Since $B^T A^T = C^T$, the rank of C is also not greater than the rank of $\underline{\quad A \quad}$.

3.6.28 Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & four & zero & rows & & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and left nullspace of B and C. Challenge problem: Find a basis for the nullspace of C.

B - rank 2 $\qquad$ C - rank 2 or rank 1 if $p = 0$

The bases for the row space of B and C are the top two rows if $p \neq 0$ (if $p = 0$ then its just the top row of C).

$$N(B^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$N(C^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$