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Homework 6

Chapter 6.1

6.1.2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \text{and} \qquad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the ______ eigenvectors as A. Its eigenvalues are ______ by 1.

$$A) \quad \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = \pm 5, -1 \qquad \text{eigenvectors} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A + I) \quad \begin{vmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 8 = \lambda^2 - 6\lambda = 0$$

$$\lambda = 0, 6 \qquad \text{eigenvectors} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A + I$ has the <u>same</u> eigenvectors as A. Its eigenvalues are <u>increased</u> by 1.

6.1.4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad \text{and} \qquad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same ______ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues ______. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

$$A) \quad \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2, -3 \qquad \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} x_1 = 0 \qquad \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} x_2 = 0 \qquad \text{eigenvectors} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

$$A^2) \quad \begin{vmatrix} 7-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 6 = \lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = 4, 9 \qquad \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} x_1 = 0 \qquad \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} x_2 = 0 \qquad \text{eigenvectors} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

A^2 has the same <u>eigenvectors</u> as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues <u>λ_1^2 and λ_2^2</u>. $\lambda_1^2 + \lambda_2^2 = 13$ because that is the trace of A^2.

6.1.9 What do you do to the equation $Ax = \lambda x$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

Multiply both sides by A.
$AAx = A\lambda x \quad \rightarrow \quad A^2x = \lambda Ax \quad \rightarrow \quad A^2x = \lambda\lambda x \quad \rightarrow \quad A^2x = \lambda^2 x.$

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

Multiply both sides by A^{-1}.
$A^{-1}Ax = A^{-1}\lambda x \quad \rightarrow \quad x = \lambda A^{-1}x \quad \rightarrow \quad \frac{1}{\lambda}x = A^{-1}x.$

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

Add $Ix = x$ to both sides.
$Ix + Ax = x + \lambda x \quad \rightarrow \quad (A + I)x = (\lambda + 1)x.$

6.1.12 Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

$$\lambda = 1 \qquad \begin{bmatrix} -.8 & .4 & 0 \\ .4 & -.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \text{eigenvectors} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\lambda = 0 \qquad \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \text{eigenvector} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Combine the eigenvectors when $\lambda = 1$ to get an eigenvector of P with no zero components: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

6.1.13 From the unit vector $u = (\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6})$ construct the rank one projection matrix $P = uu^T$. This matrix has $P^2 = P$ because $u^Tu = 1$.

(a) $Pu = u$ comes from $(uu^T)u = u(\underline{\qquad})$. Then u is an eigenvector with $\lambda = 1$.

$(uu^T)u = u(\underline{\;u^Tu\;})$

(b) If v is perpendicular to u show that $Pv = 0$. Then $\lambda = 0$.

$Pv = (uu^T)v = u(u^Tv) = u * 0 = 0$

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ all have $\lambda = 0$ and are independent.

6.1.15 Every permutation matrix leaves $x = (1, 1, ..., 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0 \qquad \lambda = 1$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1 = 0 \qquad \lambda = 1, 1, -1$$

6.1.19 A 3 by 3 matrix B is known to have eigenvalues 0,1,2. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

B is a rank two because it has a $\lambda = 0$.

(b) the determinate of $B^T B$

$|B^T B|$ because $B^T B$ is singular.

(c) the eigenvalues of $B^T B$

Can't determine.

(d) the eigenvalues of $(B^2 + I)^{-}1$.

λ's of $(B^2 + I)^{-1}$ are $\lambda = 1, \frac{1}{2}, \frac{1}{5}$.

6.1.21 **The eigenvalues of A equal the eigenvalues of A^T.** This is because $det(A - \lambda I)$ equals $det(A^T - \lambda I)$. That is true because ______. Show by an example that the eigenvectors of A and A^T are *not* the same.

It is true because every square matrix has the property $|A| = |A^T|$.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ do not have the same eigen vectors.}$$

$$\text{Eigenvectors of } A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ while } A^T \text{ has eigenvectors } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6.1.29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

$$A) \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(6-\lambda) = 0 \quad \lambda = 1, 4, 6$$

$$B) \quad |B - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = (\lambda^2 - 3)(\lambda + 2) = 0 \quad \lambda = 2, \pm\sqrt{3}$$

C is a rank one matrix, meaning that two of its λ's are zero. The last λ is the sum of the diagonals.

$\lambda = 0, 0, 6$

Chapter 6.2

6.2.2 If A has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

6.2.8 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

6.2.9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$G_{k+2} = \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \qquad G_{k+1} = G_{k+1} \qquad \text{and} \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

.

(b) Find the limist as $n \to \infty$ of the matrices $A^n = S\Lambda S^{-1}$.

.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

.

6.2.10 Prove that every third Fibonacci number in 0,1,1,2,3,... is even.

6.2.11 True or false: If the eigenvalues of A are 2,2,5 then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

6.2.15 $A^k = S\Lambda S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ______. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

6.2.16 (Recommended) Find Λ and S to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ______.

6.2.19 Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \qquad \text{has} \qquad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

6.2.36 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = S\Lambda S^{-1}$. The eigenvectors (columns of S) are (1,i) and (i,1). You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Chapter 6.3

6.3.1 Find two λ's and x's so that $u = e^{\lambda t}x$ solves

$$\frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u$$

What combination $u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ starts from $u(0) = (5, -2)$?

$A = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda = 1, 4 \quad A - 4I = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \quad N(A-4I) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_1 = e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A - I = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \quad N(A-I) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u_2 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $u(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ the $u(t) = 3e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6.3.4 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $\frac{du}{dt} = Au$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

$\frac{dv}{dt} + \frac{dw}{dt} = w - w + v - v = 0$, so $v + w$ is constant at 40.

$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad |A - \lambda I| = \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda \;\; \lambda = -2, 0 = 0$

$A + 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad N(A+2I) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$N(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = -2 \quad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 0 \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$V(1) = 20 + 10e^{-2} \qquad V(\infty) = 20$

$W(1) = 20 - 10e^{-2} \qquad W(\infty) = 20$

6.3.5 Reverse the diffusion of people in Problem 4 to $\frac{du}{dt} = -Au$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

$-A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $|A - \lambda I| = \lambda^2 - 2\lambda \to \lambda = 0, 2$

$\lambda_1 = 0 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$v(t) = 20 + 10e^{2t} = \infty\omega t \to \infty$$

$\lambda_2 = 2 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6.3.8 The rabbit population shows fast growth (from 6r) but loss to wolves (from -2w). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \quad \text{and} \quad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectos. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

$A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ $\quad |A - \lambda I| = (6 - \lambda)(1 - \lambda) + 4 = 0 \to \lambda^2 - 7\lambda + 10 \to (\lambda - 5)(\lambda - 2) \quad \lambda = 2, 5$

$A - 2I = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \quad N(A - 2I) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A - 5I = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \quad N(A - 5I) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_1 = 2 \quad x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad u(t) = 10e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 10e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 5 \quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad r(t) = 10e^{2t} + 20e^{5t}$

$w(t) = 20e^{2t} + 10e^{5t}$

Ratio of wolves to rabbits will be $\frac{1}{2}$ as $t \to \infty, e^{5t}$ dominates.

6.3.10 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $u = (y, y')$:

$$\frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

$\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$

$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 5 & 4 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda - 4 = 0$

$y = e^{\lambda t} \qquad y'' = 5y' + 4y' \to \frac{1}{25} + = \sqrt{25 + 16} = \lambda$

$\lambda^2 e^{\lambda t} = 5\lambda e^{\lambda t} + 4e^{\lambda t}$

$\lambda^2 - 5\lambda - 4 = 0$

$\lambda = \frac{1}{2} 5 + = \sqrt{41}$

6.3.21 Write $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in the form $S\Lambda S^{-1}$. Find e^{At} from $Se^{\Lambda t}S^{-1}$.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \quad A - \lambda I = \begin{vmatrix} 1-\lambda & 4 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 - \lambda \quad \lambda = 0, 1$$

$$N(A) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\lambda_1 = 1 \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0 \quad x_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$$

Chapter 6.4

6.4.4 Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is λ?

6.4.6 Find *all* orthogonal matrices that diagonalize $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

6.4.11 Write A and B in the form $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } ||x_1|| = ||x_2|| = 1).$$

6.4.21 **True** (with reason) or **false** (with example). "Orthonormal" is not assumed.

(a) A matrix with real eigenvalues and eigenvectors is symmetric.

.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

.

(c) The inverse of a symmetric matrix is symmetric.

.

(d) The eigenvector matrix S of a symmetric matrix is symmetric.

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Chapter 6.5

6.5.7 Test to see if R^R is positive definite in each case:

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad R^T R = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$$

$x^T \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} x = x^T \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 13x_2 \end{bmatrix} = x_1^2 + 4x_2x_1 + 13x_2^2 \to 0$ positive definite

$R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad R^T R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$\begin{vmatrix} 6-\lambda & 5 \\ 5 & 6-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 11 = 0 \to (\lambda - 1)(\lambda - 11) \to \lambda = 1, 11$ All eigenvalues are positive so $R^T R$ is positive definite

$R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad R^T R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 11\lambda = 0 \to \lambda = 0, 1, 11$ Positive Semi-definite

6.5.10 Which 3 by 3 symmetric matrices A and B produce these quadratics?
$x^T Ax = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$. Why is A positive definite?
$x^T Bx = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$. Why is B semidefinite?

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ After elimination $\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$

A has all positive pivots so A is positive definite

$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \to \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = \lambda^3 + 6\lambda^2 + 9\lambda = 0 \quad \lambda = 0, 3, 3$

B is semi definite because all of its λ's are either positive or zero.

6.5.17 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have ____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a ____ on the main diagonal.
positive - first blank
zero - second blank

6.5.18 If $Ax = \lambda x$ then $x^T Ax =$ ____. If $x^T Ax > 0$, prove that $\lambda > 0$.
$x^T Ax = x^T \lambda x \to \lambda = \frac{x^T Ax}{x^T x}$ if $x^T Ax > 0$

6.5.19 Reverse Problem 18 to show that if all $\lambda > 0$ then $x^T Ax > 0$. We must do this for every nonzero x, not just the eigenvectors. So write x as a combination of the eigenvectors and explain why all "cross terms" are $x_i^T x_j = 0$. Then $x^T Ax$ is

$(c_1x_1 + \cdots + c_nx_n)^T(c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n) = c_1^2\lambda_1x_1^Tx_1 + \cdots + c_n^2\lambda_nx_n^Tx_n > 0$. $x^T Ax = x^T\lambda x \to x^T Ax = \lambda x^T x \to x^T Ax = \lambda\|x\|^2 > 0$ if $\lambda > 0$
$x_i^T x_j = 0$ because evectors of symmetric matrices are orthogonal.

6.5.20 Give a quick reason why each of these statements is true:
(a) Every positive definite matrix is invertible.
Positive definite matrices have non-zero evalues

(b) The only positive definite porjection matrix is $P = I$.
All projection matrices are singular except for I

(c) A diagonal matrix with positive diagonal entries is positive definite.
It has positive evalues and pivots

(d) A symmetric matrix with a positive determinant might not be positive definite!
It could have two negative evalues because the det $= \pi_i\lambda_i$

6.5.28 Without multiplying $A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$

(a) The determinant of A (b) the eigenvalues of A
(c) the eigenvectors of A (d) a reason why A is symmetric positive definite.
(a) $|A| = 2 \cdot 5 = 10$
(b) Values along the diagonal of middle matrix, so $\lambda = 2, 5$
(c) Columns of the first matrix $\begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$
(d) A has all positive evalues

Chapter 6.6

6.6.17 True of False, with a good reason:
(a) A symmetri matrix can't be similar to a nonsymmetric matrix.
False $A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ $\lambda_1 = 2$ $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda_2 = 5$ $x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} - y_3$ A is similar to $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

(b) An invertible matrix can't be similar to a singular matrix.
True, Ranks of similar matrices are the same.

(c) A can't be similar to $-A$ unless $A = 0$.
False $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar with $\lambda = -1, 1$

(d) A can't be similar to $A + I$.
True, Adding I increases the evalues by 1

6.6.18 If B is invertible, prove that AB is similar to BA. *They have the ame eigenvalues.*
$AB = \lambda x \to MAB = \lambda Mx \to$
$$MAB = MBA \to AB = MBAM^{-1}$$
$BA = \lambda x \to MBA = \lambda Mx \to$

6.6.20 Why are these statements all true?
(a) If A is similar to B then A^2 is similar to B^2.
$Ax = \lambda x \to A^2x = \lambda^2 x$ both evalues are squared
$Bx = \lambda x \to B^2x = \lambda^2 x$

(b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).
$A^2 = (-A)^2$ but $B \neq -A$

(c) $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 4 \end{smallmatrix}\right]$ is not similar to $\left[\begin{smallmatrix} 3 & 1 \\ 0 & 4 \end{smallmatrix}\right]$.
Both have $\lambda = 3, 4$

(d) $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ is not similar to $\left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$.
Both have $\lambda = 3, 3$ so there are not two evectors to construct a invertible matrix with

(e) If we echange rows 1 and 2 of A, and then exchange columns 1 and 2, **the eigenvalues stay the same**. In this case $M =$ ____.
$A = PAP^T$ $\quad M = P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Chapter 6.7

6.7.4 Find the eigenvalues and unit egienvectors of A^TA and AA^T. Keep each $Av = \sigma \boldsymbol{u}$:

$$\textbf{Fibonacci matrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^T$.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$|AA^T - \lambda I| = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 \quad \lambda = \frac{3+=\sqrt{5}}{2}$

$AA^T - (\frac{3}{2} + \frac{\sqrt{5}}{2})I = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & \frac{-1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}$ $N(AA^T - (\frac{3}{2} + \frac{\sqrt{5}}{2}I) = \begin{bmatrix} 1 \\ \frac{5}{2} - \frac{1}{2} \end{bmatrix}$

$AA^T - (\frac{3}{2} - \frac{\sqrt{5}}{2})I = \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & 1 \\ 1 & \frac{-1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}$ $N(AA^T - (\frac{3}{2} - \frac{\sqrt{5}}{2}I) = \begin{bmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}$

$\sigma_1 = \frac{1}{2} + \frac{\sqrt{5}}{2} = \lambda_1(A)\sigma_2 = \frac{\sqrt{5}}{2} - \frac{1}{2} = \lambda_2(A)$

6.7.6 Compute A^TA and AA^T and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Check $AV = U\Sigma$ (this will decide $+=$ signs in U). Σ has the same shape as A.

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$|AA^T - \lambda I| = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

$AA^T - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $N(AA^T - I) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$

$AA^T - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $N(AA^T - 3I) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$

$A^TA - I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $N(A^TA - I) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$

$A^TA - 3I = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ $N(A^TA - 3I) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{6}}$

$N(A^TA) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}}$

$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

6.7.7 .

The closest rank one matrix will be the combination of $u_i\sigma_i v_i^T$, where i is determined by the i value of the largest evalue.

$\sqrt{3} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$

6.7.10 Construct the matrix with rank one that has $Av = 12u$ for $v = \frac{1}{2}(1,1,1,1)$ and $u = \frac{1}{3}(2,2,1)$. It only sigular value is $\sigma_1 =$ ____.

$A = 12uv^T = 12 \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = 2 \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

6.7.11 Suppose A has orthogonal columns $w_1, w_2, \cdots, w_n$ of lengths $\sigma_1, \sigma_2, \cdots, \sigma_n$. What are U, Σ, and V in the SVD?

$A^T A = I = V$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdots & \\ & & & \sigma_n \end{bmatrix} \quad U = \begin{bmatrix} \frac{AV_i}{\sigma_i} & & \\ & \cdots & \\ & & \frac{AV_n}{\sigma_n} \end{bmatrix}$$