1. Model Setup

We consider the multivariate linear model

$$\mathbf{Y} = \mathbf{X}\mathbf{B} + \mathbf{E},$$

where: - $\mathbf{Y}$ is an $n \times q$ matrix of responses, - $\mathbf{X}$ is an $n \times p$ design matrix, - $\mathbf{B}$ is a $p \times q$ coefficient matrix, - $\mathbf{E}$ is an $n \times q$ error matrix.

Assume rows of $\mathbf{E}$ are i.i.d.

$$\mathbf{E}_i \sim N_q(\mathbf{0}, \mathbf{\Sigma}).$$

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2. General Linear Hypothesis

Hypotheses are expressed in matrix form:

$$H_0 : \ \mathbf{C}\mathbf{B}\mathbf{M} = \mathbf{0},$$

where: - $\mathbf{C}$ specifies linear combinations of predictors, - $\mathbf{M}$ specifies linear combinations of responses.

This framework allows: - testing a single predictor across all responses, - testing multiple predictors jointly, - testing contrasts among responses.

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3. Hypothesis and Error SSCP Matrices

Define: - **Hypothesis SSCP matrix**

$$\mathbf{H} = (\mathbf{C}\hat{\mathbf{B}}\mathbf{M})^\top \left[\mathbf{C}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{C}^\top\right]^{-1}(\mathbf{C}\hat{\mathbf{B}}\mathbf{M})$$

- **Error SSCP matrix**

$$\mathbf{E} = \hat{\mathbf{E}}^\top\hat{\mathbf{E}}$$

Both are $q \times q$ symmetric positive semi-definite matrices.

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4. Multivariate Test Statistics

Inference is based on the eigenvalues $\lambda_1, \ldots, \lambda_q$ of

$$\mathbf{E}^{-1}\mathbf{H}.$$

Common test statistics:

Wilks’ Lambda

$$\Lambda = \frac{|\mathbf{E}|}{|\mathbf{E}+\mathbf{H}|} = \prod_{j=1}^{q} \frac{1}{1+\lambda_j}$$

- Most commonly reported. - Sensitive to departures from assumptions.

Pillai’s Trace

$$V = \sum_{j=1}^{q} \frac{\lambda_j}{1+\lambda_j}$$

- Most robust to assumption violations. - Often preferred in applied work.

Hotelling–Lawley Trace

$$U = \sum_{j=1}^{q} \lambda_j$$

- Emphasizes large eigenvalues.

Roy’s Largest Root

$$\theta = \max_j \lambda_j$$

- Focuses on strongest single dimension of separation.

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5. Distributional Results

- Exact null distributions are complicated.
- Test statistics are transformed to approximate **F distributions**.
- Degrees of freedom depend on:
  - number of responses q,
  - rank of $\mathbf{C}$,
  - sample size n,
  - number of predictors p.

In large samples, results are typically similar across statistics.

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6. Interpretation of Hypothesis Tests

A significant multivariate test implies: - at least one **linear combination of responses** differs across predictor levels or changes with predictors.

It does **not** identify: - which response(s) drive the effect, - or which combination is responsible.

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7. Follow-up Analyses

Common follow-ups include: - univariate regressions with multiplicity control, - confidence intervals for $\mathbf{CB}$, - examination of canonical variates, - profile plots of fitted response means.

Follow-up analyses should be **secondary** to the global test.

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8. Comparison to Multiple Univariate Tests

Multivariate testing: - controls Type I error automatically, - uses response covariance for efficiency, - can detect joint effects missed by univariate tests.

Separate univariate tests: - ignore response correlation, - inflate familywise error, - may lose power when responses are correlated.

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9. Special Cases

- **MANOVA**: predictors are group indicators.
- **ANCOVA**: mix of continuous and categorical predictors.
- **Repeated measures**: responses correspond to time points.

All fit within the same hypothesis-testing framework.

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10. Key Takeaways

- Hypothesis testing is about **subspaces**, not individual coefficients.
- Test statistics compare explained vs unexplained covariance.
- A significant result implies **joint response differences**, not marginal ones.
- Interpretation requires care and often structured follow-up analysis.