123. Let $X_1, X_2, ..., X_n$ be a random sample from a normal population with unknown mean μ and unknown variance σ^2. Define the sample mean $\bar{X}$ and sample variance S^2 as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

(a) Prove the sample variance S^2 is ancillary for μ.

(b) Prove $T(\mathbf{X}) = \left(\sum_{j=1}^{n} X_j, \sum_{j=1}^{n} X_j^2\right)$ is a complete minimal sufficient statistic for $\boldsymbol{\theta} = (\mu, \sigma^2)$.

(c) Are $\bar{X}$ and S^2 independent? Why? (Use a result other than the sample is from a normal distribution).