65. Let $W_1, ..., W_n$ be a random sample from a Bernoulli(p) distribution, where $0 < p < 1$ is $p = P(W_i = 1)$ for each $i = 1, 2, ...$ The maximum likelihood estimator (MLE) of p is

$$\hat{p}_n = \frac{1}{n} \sum_{i=1}^{n} W_i.$$

(a) Find the limiting distribution of $\hat{p}_n$.

(b) Consider the odds, defined as $\tau(p) = p/(1-p)$.

i. What is the MLE of $\tau(p)$?

ii. What is the asymptotic distribution of the MLE of $\tau(p)$?

(c) Now consider testing

$$H_0 : p = p_0 \quad \text{versus} \quad H_1 : p \neq p_0.$$

i. The likelihood ratio test (LRT) statistic, written as a function of $\hat{p}_n$, is

$$\lambda(\hat{p}_n) = \left(\frac{p_0}{\hat{p}_n}\right)^{n\hat{p}_n} \left(\frac{1-p_0}{1-\hat{p}_n}\right)^{n(1-\hat{p}_n)}$$

What is the large-sample version of the likelihood ratio test?

ii. The finite sample version of the LRT is to reject H_0 if $\lambda(\hat{p}_n) \leq c$, where $0 \leq c \leq 1$ is determined so that $P_{p_0}(\lambda(\hat{p_n}) \leq c) \leq \alpha$, where α is the level of the test. Explain which version of the test is preferred, providing an argument to support your choice.

iii. Find the Wald test.

iv. Find the score test.