139. Let $X_1, ..., X_n \sim$ IID normal(θ, σ^2), and let the prior distribution on θ be normal(μ, τ^2).

(a) Find the joint PDF of $\bar{X}$ and θ.

(b) Show that $m(\bar{x} \mid \sigma^2, \mu, \tau^2)$, the marginal distribution of $\bar{X}$, is normal with mean μ and variance $(\sigma^2/n) + \tau^2$.

(c) Show that the $\pi(\theta \mid \bar{x}, \mu, \tau^2)$, the posterior distribution of θ is normal with mean and variance

$$\mathrm{E}[\theta \mid \bar{x}, \mu, \tau^2] = \left(\frac{\tau^2}{\tau^2 + \sigma^2}\right) \bar{x} + \left(\frac{\sigma^2}{\tau^2 + \sigma^2}\right) \mu$$

$$\mathrm{Var}(\theta \mid \bar{x}, \mu, \tau^2) = \frac{\sigma^2 \tau^2}{\tau^2 + \sigma^2}$$

(d) Let δ^π denote the Bayes estimator of θ for squared error loss. Find the risk function $R(\theta, \delta^\pi)$ for the Bayes estimator of θ.

(e) Let $X \sim$ normal$(\theta$, 1) for this $n = 1$. Consider two prior distributions $\pi_1(\theta) \sim$ normal(0, 1) and $\pi_2(\theta) \sim$ normal(0, 10). Write the risk functions for each of these two priors. Use R to compute and graph the risk functions on the same axes. Comment on how the prior affects the risk function of the Bayes estimator.