41. Consider the function

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, ..., n$$

where $n \in \mathbb{Z}^+$ and $0 < p < 1$.

(a) Prove $f_X(x)$ is a probability mass function (PMF).

(b) Prove the mean of X is $\mu = np$ using the PMF.

(c) Prove the moment generating function (MGF) of the distribution of X is $M_X(t) = \{pe^t + (1-p)\}^n$.

(d) Use the moment generating function to prove the second moment of X is $\mathrm{E}[X^2] = np - np^2 + (np)^2$.

(e) Use any relevant previous results in this problem to find an expression for the variance of X.

Hint: The Binomial Theorem. Let a and b be any two real numbers and r a positive integer. Then

$$(a+b)^r = \sum_{j=0}^{r} \binom{r}{j} a^j b^{r-j}$$