Test 1 Prep

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Week 1

Set Theory

Cardinality

$\mathcal{C}(A)$ number of elements in the set A

1) The set A is said to be finite if $\mathcal{C} < \infty$.
2) The set A is said to be countable if the elements of A can be represented in a list.
3) If A is countable, but the number of elements in the list is infinite, the set is said to be countably infinite.
4) If the set of elements cannot be represented by some type of list, then the set is said to be infinite.

Event

An event is any collection of possible outcomes of an experiment; that is, any subset of $\mathcal{S}$ including $\mathcal{S}$ itself.

1) A simple event is an event with a single outcome. A simple event can occur in only one way.
2) A compound event is the combination of two or more simple events.

Relations

In the sample space $\mathcal{B}$, let ω represent an outcome, and A and B be any two events. Then

1) A occurs if the outcome of the experiment is in A.
2) B is a subset of A, or B is contained in A if for every $\mathcal{A}$ if for every $\omega \in B \Rightarrow \omega \in A$
3) A and B are equal if and only if $A \subset B$ and $B \subset A$

Properties of Set Operations

1) Communitativity: $A \cup B = B \cup A, and B \cup A = A \cup B$
2) Associativity:
    a) $A (B C) = (A B) C $
    b) $A \cap (B \cap C) = (A \cap B) \cap C$
3) Distributive Laws:
    a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
    b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4) DeMrgan's Laws:
    a) $(A \cup B)^c = A^c \cap B^c$
    b) $(A \cap B)^c = A^c \cup B^c$

Disjoint or Mutually Exclusive

1) Two events A and B defined on S are disjoint if $A \cap B = \emptyset$
2) The events $A_1, A_2, ... $ are said to be pairwise disjoint if $A_i \cap A_j = \emptyset, \forall i \neq j$

Exhaustive

$A_1, A_2, ...$ are such that $\bigcup_{i=1}^{\infty} A_I = \mathcal{S}$

Partition of $\mathcal{S}$

$A_1, A_2, ...$ form a partition of $\mathcal{S}$ if they are mutually exclusive and exhaustive

Basics of Probability

σ-algebra or Borel Field satisfies the the following properties

1) $\emptyset \in \mathcal{B}$
2) $A \in \mathcal{B}$ then $A^C \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation), and
3) If $A_1, A_2, ... \in \mathcal{B}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions)

Kolmogrov's Axioms

Given a sample space S and an associated $\sigma-$algebra $\mathcal{B}$, a probability function is a function P with domain $\mathcal{B}$ that satisfies

1) $P(A) \geq 0, \forall A \in B$

2) $P(S) = 1$, and

3) If $A_1, A_2, ... \in B$ are pairwise disjoint, then $P(\bigcup_{i=1}^{\infty}) = \sum_{i=1}^{\infty}$

Axiom of Finite Additivity

If $A \in \mathcal{B}$ and $B \in \mathcal{B}$ are disjoint, then $P(A \cup B) = P(A) + P(B)$

The Calculus of Probabilities

For the probability space (S, B, P)

1) $P(\emptyset) = 0$

2) $P(A) \geq 0$,

3) $P(A^C) = 1 - P(A)$

Probabilities of Operations on Sets

1) $P(A \cup B^c) = P(A) - P(A \cap B)$

2) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

3) If $A \subset B$ then $P(A) \leq P(B)$

Bonferroni's Inequality

For the probability space $(\mathcal{S}, \mathcal{B}, \mathcal{P})$, with A and B any two sets in $\mathcal{B}$, $P(A \cap B) \geq P(A) + P(B) - 1$

Law of Total Probability

$P(A) = \sum_{i=t}^{\infty} P(A \cap C_i)$ for $C_1, C_2, ... \in \mathcal{B}$, where $C_1, C_2, ...$ are a partition of S_i

Boole's Inequality

$P(\bigcup_{i=1}^{\infty} A_i \leq \sum_{i=1}^{\infty} P(A_i)$ for any collection of sets $A_1, A_2, ... \in \mathcal{B}$

Week 2

Introduction to Counting Techniques

Fundamental Theorem of Counting

If a job consists of k seperate tasks, the $i - th$ of which can be done in n_i ways, $i = 1, 2, ..., k$ then the entire job can be done in $n_1 \times n_2 \times ... \times n_k$ ways.

Ordered Outcomes, Sampling without Replacement

$\frac{n!}{(n-r)!}$

n: Number in sample r: Total times selected without replacement

Ordered Outcomes, Sampling with Replacement

n^r

n: Number in sample r: Total times selecting with replacement

Unordered Outcomes, Sampling without Replacement

$\binom{n}{r} = \frac{n!}{(n-r)!(r!)}$

Unordered Outcomes, Sampling with Replacement

$\binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!r!}$

	Permutations (Ordered)	Combinations (Unordered)
Without Replacement	$\frac{n!}{(n-r)!}$	$\binom{n}{r=\frac{r!}{r!(n-r)!}}$
With Replacement	n^r	$\binom{(n+r-1)}{r}$

Week 3

Conditional Probability

$P(A|B) = \frac{P(A\cap B)}{P(B)}$

$P(A|B)P(B) = P(A|B)$

Bayes Rule

$P(A_i|B) = \frac{P(B|A_i)P(A)}{\sum_{j=1}^{\infty} P(B|A_j)P(A_j)}$

Statistical Independence

1) $P(A \cap B) = P(A)P(B)$
2) $P(A|B) = P(A)$

Independence of Complements

1) A and B^c

2) A^c and B

3) A^c and B^c

Independence of Multiple Events

If $A_1, A_2, ..., A_k$ are independent if $P(A_1 \cap A_2 \cap ... \cap A_k) = P(A_1)P(A_2)...P(A_k)$

Mutually Independent Events

$P(\bigcap_{j=1}^{k} A_{i_j}) = \prod_{j=1}^{k} P(A_{i_j})$

Random Variables

A Random variable is a function from the sample space s into the real numbers $\mathbb{R}$

Induced Probability Function

Consider a finite sample space $\mathcal{S} = \omega_1, \omega_2, ..., \omega_n$ with probability function P. Define the random variable $X(S) \to X$ where $X = x_1, x_2, ..., x_m$. Then $X = x_i$ if and only if the outcome of the experiment is an $\omega_j \in \mathcal{S}$ such that $X(\omega_j) = x_i$. The induced probability function P_X on X is

$$P_X(X = x_i) = P(\omega_j \in \mathcal{S} : X(\omega_j) = x_i)$$

Distribution Functions

Cumulative Distribution Function

The CDF of a random variable X, denoted by $F_X(x)$ is defined by $F_X(x) = P_X(X \leq x), \forall x \in \mathcal{X}$

Right Continuous Function

Let $f(x)$ represent a function. Then f is right continuous at a point x_0 if the function is defined on an interval $[x_0, a]$, lying to the right of x_0, and if $\lim_{x \to x_0^+} f(x) = f(x_0)$

To prove a function i s right-continuous, you must start with Let $\epsilon > 0$. Then for any x_0 you must show

$$\lim_{\epsilon \to 0+} F_X(x+\epsilon) = F_X(x)$$

Properties of a CDF

1) $lim_{x \to -\infty} F_X(X) = 0$ and $lim_{x \to \infty} F_X(X) = 1$
2) $F_X(X)$ is a nondecreasing function of x. (can show that the first derivative is always positive)
3) $F_X(X)$ is right-continuous.

Probability Mass Function

Probability Mass Function

The PMF of a discrete random variable X is given by $f_X(x) = P(X = x), \forall x \in \mathcal{X}$

Probability Density Function

The probability density function (PDF) of a continuous random variable X is the function $f_X(x) \geq 0$ that satisfies

$$F_X(x) = \int_{-\infty} x f_X(t), \forall x \in \mathcal{X}$$

Properties of a PDF and PMF

A function $f_X(x)$ is a PDF or PMF of a random variable X if and only if

1) $f_X(x) \geq 0, \forall x \in X$
2) If X is discrete, then $\sum_{all x \in \mathcal{X}} F_X(X) = 1$. If X is continuous, then $\int_{-\infty}^{\infty} f_X(x) dx = 1$

Comparing PMFs and PDFs

More to follow

Week 4

Distribution of a Function of a Random Variable

Defining Support for Transformations

When the transformation is from X to $Y = g(X)$, the following notation often used to assign sample spaces is

$$X = x : f_X(x) > 0$$

$$Y = y : y = g(x), x \in X$$

Monotone Functions

If g is monotone, then it is one-to-one and onto;

a) One-to-one: Each value x goes to only one y, and each value of y comes from at most one value x.

b) Onto: For each $y \in Y$, there is an $x \in X$ such that $g(x) = y$.

CDF of a Monotone Transformations

Let X have CDF $F_X(x)$, let $Y = g(X)$, and let X and Y be defined as in (1).

a) If g is an increasing function on X , $F_Y(y) = F_X(g^{-1}(y))$ for $y \in Y$.
b) If g is a decreasing function on X and X is a continuous random variable

$$F_Y(y) = 1 - FX(g^{-1}(y)), y \in Y$$

PDF of a Monotone Transformation

Let X have PDF $f_X(x)$ and let $Y = g(X)$, where g is a monotone function. Let X and Y be defined by (1). Suppose that $f_X(x)$ is continuous on X and that $g^{-1}(y)$ has a continuous derivative on Y. Then the PDF of Y is given by

$$f_Y(y) = \left\{ f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, y \in Y, 0, otherwise. \right.$$

1) show the transformed sample space
2) show that the function g(x) has first derivative that is positive and this will show that its monotone increasing on X

3) show that g^-1(x) has a first derivative which is continuous on Y

PDF of a non-monotone Transformation

Let X have PDF $f_X(x)$, let $Y = g(X)$, and define the sample space X as in (1). Suppose there exist a partition $A_0, A_1, ..., A_k$ of X such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i. Further suppose there exists functions $g_1(x), ..., g_k(x)$, defined on $A_1, ..., A_k$, respectively, satisfying

a) $g(x) = g_i(x)$ for $x \in A_i$
b) $g_i(x)$ is monotone on A_i
c) The set $\mathcal{Y} = y : y = g_i(x)$ for some $xinA_i$ for each $i = 1, 2, … , k, and
d) $g_i^{-1}(y)$ has a continuous derivative on Y for each $i = 1, ..., k$.

Then

$$f_Y(y) = \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right|, y \in Y$$

and is zero elsewhere.

Probability Integral Transformation

Let X have continuous CDF $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$; that is, $P(Y \leq y) = y$ for $0 < y < 1$.

Expected Value

1) If X is discrete

$E[X] = \sum_{x:f_X(x)>0} x * F_X(x)$

2) If X is continuous

$E[X] = \int_{x:f_X(x)>0} x * F_X(x)dx$

Law of the unconscious statistician (LOTUS)

1) If X is discrete with PMF $F_X(x)$ then

$E[g(X)] = \sum_{x:f_X(x)>0} g(x) * F_X(x)$

2) If X is continuous with PDF $F_X(x)$ then

$E[g(X)] = \int_{x:f_X(x)>0} g(x) * F_X(x)$

Properties of Expectation

Let X be a random variable and let a, b, and c be real constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectation exist

a) $E[ag_1(X) + bg_2(X) + c] = aE[g_1(X)] + bE[g_2(X)] + c$.
b) If $g_1(x) \geq 0 \forall x$, then $E[g_1(X)] \geq 0$.
c) If $g_1(x) \geq g_2(x) \forall x$, then $E[g_1(X)] \geq E[g_2(X)]$.
d) If $a \leq g_1(x) \leq b \forall x$, then $a \leq E[g_1(X)] \leq b$

Moments and Moment Generating Functions

For each integer n, the $n-th$ moment of X (or of $F_X(x)$) denoted μ_n', is defined as $\mu_n' = E[X^n]$

The $n-th$ central moment of X (or of $F_X(x)$), denoted μ_n, is defined $\mu_n = E[(X-\mu)^n]$, where $\mu = \mu_1' = E[X]$

Variance

The variance of a random variable $X(orofF_X(x))$ is its second central moment, $Var(X) = E[(X - E[X])^2]$. The positive square root of $Var(X)$ is the standard deviation of $X(orofF_X(x))$.

Alternative for of Variance

$Var(X) = E[X^2] - (E[X])^2$

Variance of a Linear Transformation

$Var(aX + b) - a^2 Var(X)$

Moment Generating Function

Let X be a random variable with CDF $F_X(x)$. The moment generating function (MGF) of X (or $F_X(x)$), denoted $M_X(t)$, is $M_X(t) = E[e^{tX}]$.

Utility of the MGF

If X has MGF $M_X(t)$, then $E[X^n] = M_X^{(n)}(0)$, where $M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t) \Big|_{t=0}$

Identically Distributed Using the MGF

Let $F_X(x)$ and $F_Y(y)$ be two CDFs all of whose moments exist.

a) If X and Y have bounded support, then $F_X(u) = F_Y(u)$ for all u if and only if $E[X^r] = E[Y^r]$ for all integers $r = 0, 1, 2, ...$

b) If the MGFs exist and $M_X(t) = M_Y(t)$ for all t in a neighborhood of zero, then $F_X(u) = F_Y(u)$ for all u.

Convergence of MGFs

not included yet

MGF of a Linear Transformation

For any constants a and b, the MGF of the random variable $aX + b$ is given by $M_{aX+b}(t) = E[e^{t(aX+b)}] = E[e^{atX}e^{bt}] = e^{bt}M_X(at)$.