171. Let $X_1, ..., X_n$ be independent and identically distributed according to a normal distriution with unknown mean θ and known variance σ^2. Let the prior distribution of θ be normal with mean μ and variance τ^2, where both μ and τ^2 are known. You may use without proof that the posterior distribution of θ is normal with mean and variance given by

$$\frac{n\bar{x}\tau^2 + \mu\sigma^2}{n\tau^2 + \sigma^2} = \frac{n\tau^2}{\sigma^2 + n\tau^2}\bar{x} + \frac{\sigma^2}{\sigma^2 + n\tau^2}\mu \quad \text{and} \quad \frac{\sigma^2\tau^2}{n\tau^2 + \sigma^2},$$

respectively. For the three questions below, provide as much detail as possible.

(a) Find a Bayesian hypothesis test for

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

(b) Find the $1 - \alpha$ highest posterior density (HPD) credible set for θ.

(c) What is the difference between the $1 - \alpha$ equal-tail and $1 - \alpha$ HPD credible sets? Explain.