33. Let $X_1, ..., X_n$ be a random sample from a normal$(\theta, \sigma^2)$ population, and suppose the hypotheses to be tested are

$$H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta > 0.$$

The prior distribution on $\theta$ is normal(0, $\tau^2$) with $\tau^2$ known, which is symmetric about the hypothesis in the sense that $P(\theta \leq 0) = P(\theta > 0) = 0.5$.

(a) It can be shown that the normal distribution is a conjugate prior for estimating the mean of a normal distribution. The mean and variance of the posterior are

$$\begin{aligned}
\mathrm{E}[\theta|\boldsymbol{x}] &= \frac{\tau^2}{\tau^2 + (\sigma^2/n)}\bar{x} + \frac{\sigma^2/n}{(\sigma^2/n) + \tau^2}\theta \\
\mathrm{Var}(\theta|\boldsymbol{x}) &= \frac{\sigma^2\tau^2/n}{(\sigma^2/n) + \tau^2}
\end{aligned}$$

Find an expression for the posterior probability that $H_0$ is true, $P(\theta \leq 0|\boldsymbol{x})$.

(b) Find an expression for the $p$-value of the test corresponding to a value of $\bar{x}$, using tests that reject for large values of $\bar{X}$.

(c) For the special case of $\sigma^2 = \tau^2 = 1$, compute $P(\theta \leq 0|\boldsymbol{x})$ and the p-value for values of $\bar{x} > 0$. Show that in this case the Bayes probability is always greater than the $p$-value. This shows that the Bayes probability is always greater than the $p$-value.

(d) Using the expression derived in parts (a) and (b), show that

$$\lim_{\tau^2 \to \infty} P(\theta \leq 0|\boldsymbol{x}) = p\text{-value}.$$