The Bayesian Lasso

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MODIFIED BAYES' THEOREM:

$$P(H|X) = P(H) \times \left(1 + P(C) \times \left(\frac{P(x|H)}{P(x)} - 1\right)\right)$$

H: HYPOTHESIS

x: OBSERVATION

P(H): PRIOR PROBABILITY THAT H IS TRUE

P(x): PRIOR PROBABILITY OF OBSERVING x

P(C): PROBABILITY THAT YOU'RE USING BAYESIAN STATISTICS CORRECTLY

The Bayesian Lasso[1]

1. Formulation

- Classical Regression
- Classical Lasso
- Bayesian Lasso

2. Selecting λ

- Classical Regression
- Classical Lasso

3. Comparison
4. Extensions
5. Synthetic Example

[1] Park and Casella (2008)

Clasical Regression

$$y = \mu 1_n + X\beta + \epsilon$$

- y is an $n \times 1$ vector of responses
- μ is the overall mean
- X is the $n \times p$ matrix of **standardized** regressors
- $\beta = (\beta_1, \dots, \beta_p)^T$
- ϵ is an $n \times 1$ vector of $\overset{iid}{\sim} N(0, \sigma^2)$

Satisfies $\min_{\beta}(\tilde{y} - X\beta)^T(\tilde{y} - X\beta)$

Clasical Lasso

Formulation

$$\min_{\beta}(\tilde{y} - X\beta)^T(\tilde{y} - X\beta) + \lambda \Sigma_{j=1}^{p} |\beta_j|$$

$$\lambda \geq 0$$

Clasical Lasso

$$\min_{\beta}(\tilde{y} - X\beta)^T(\tilde{y} - X\beta) + \lambda\Sigma_{j=1}^{p}|\beta_j|$$

Notes

1. Often called "penalized regression"
2. L1 penalty
3. "Shrinkage" β values are shrunk towards 0
4. Tune λ through cross validation

Motivation

1. Model selection - often as a precursor to other models
2. Reduce overfitting
3. Easily extendable to generalized linear models

Clasical Lasso

$$\min_{\beta}(\tilde{y} - X\beta)^T(\tilde{y} - X\beta) + \lambda\Sigma_{j=1}^{p}|\beta_j|$$

Drawbacks

1. Biases β
2. Unreliable standard errors (issues with statistical tests)
3. Correlated features
4. Tuning issues / time

Bayesian Lasso[2]

Hierarchical Specification 1 (1 of 2)

$$y|\mu, X, \beta, \sigma^2 \sim N_n(\mu 1_n + X\beta, \sigma^2 I_n)$$

$$\beta|\sigma^2, \tau_1^2, \dots, \tau_p^2 \sim N_p(0_p, \sigma^2 D_t)$$

$$D_t = diag(\tau_1^2, \dots, \tau_p^2)$$

$$\sigma^2, \tau_1^2, \dots, \tau_p^2 \sim \pi(\sigma^2) d\sigma^2 \prod_{j=1}^{P} \frac{\lambda^2}{2} e^{-\lambda^2 \frac{\tau^2}{2}} d\tau_j^2$$

$$\sigma^2, \tau_1^2, \dots, \tau_p^2 > 0$$

[2] Andrews and Mallows (1974)

Bayesian Lasso

Hierarchical Specification 1 (2 of 2)

The authors integrate out $\tau_1^2, \dots, \tau_p^2$ which yields the conditional prior for β as a Laplace (double-exponential) distribution:

$$\pi(\beta|\sigma^2) = \prod_{j=1}^{P} \frac{\lambda}{2\sqrt{\sigma^2}} e^{\frac{-\lambda|\beta_j|}{\sqrt{\sigma^2}}}$$

$$\pi(\sigma^2) = IG(\alpha, \beta)$$

$$p(\mu) = U(a, b)$$

Bayesian Lasso[3]

Hierarchical Specification 2 (1 of 2)

$$y|\mu, X, \beta, \sigma^2 \sim N_n(\mu 1_n + X\beta, \sigma^2 I_n)$$

Authors integrate out μ

$$p(\beta) = N(A^{-1} X^T \tilde{y}, \sigma^2 A^{-1})$$

where

$$A = X^T X + D_\tau^{-1}$$

[3] Bae and Mallick (2004)

Bayesian Lasso

Hierarchical Specification 2 (2 of 2)

$$p(\sigma^2) = IG(\frac{n-1}{2} + \frac{p}{2}, (\tilde{y} - X\beta)^T \frac{(\tilde{y} - X\beta)}{2} + \beta^T D_\tau^{-1} \frac{\beta}{2})$$

$$p(\tau_1^2, \ldots, \tau_p^2) = \sqrt{\frac{\lambda'}{2\pi}} x^{-\frac{3}{2}} exp\{-\frac{\lambda'(x-\mu')^2}{2(\mu')^2 x}\}$$

where

$$\mu' = \sqrt{\frac{\lambda^2 \sigma^2}{\beta_j^2}}$$

$$\lambda' = \lambda^2$$

Choosing the Lasso Parameter: Classical Lasso

Cross Validation

1. Cross validate over a grid of λ where $\lambda \geq 0$.
2. For each λ value find the error metric of interest.
3. Select the λ value that minimizes the metric of interest.

Choosing the Lasso Parameter: Bayesian Lasso

Technique 1: Empirical Bayes

Choosing the Lasso Parameter: Bayesian Lasso

Technique 1: Empirical Bayes

Choosing the Lasso Parameter: Bayesian Lasso

Technique 1: Empirical Bayes[4]

1. Solve for a marginal maximum likelihood for λ using estimates of the hyperparameters.
2. λ is updated for each iteration using the estimates from the sample of the previous iteration

$$\lambda^{(k)} = \sqrt{\frac{2p}{\Sigma_{j=1}^{p} E_{\lambda(k-1)}[\tau_j^2|\tilde{y}]}}$$

3. Recommended initial value of:

$$\lambda^{(0)} = \frac{p\sqrt{\hat{\sigma_{LS}^2}}}{\Sigma_{j=1}^{p}|\hat{\beta_j^{LS}}|}$$

4. β_j^{LS} and σ_{LS}^2 are estimated from least squares.

[4] Casella (2001)

Choosing the Lasso Parameter: Bayesian Lasso

Technique 2: Hyperpriors

Authors recommend the diffuse Gamma hyperprior of λ^2 in the following form

$$\pi(\lambda^2) = \frac{\delta^r}{\Gamma(r)}(\lambda^2)^{r-1}e^{-\delta\lambda^2}$$

$$\lambda^2 > 0, r > 0, \delta > 0$$

- Select r and δ such that there is high probability near the maximum likelihood estimate to avoid mixing problems.
- $r = 0$ and $\delta = 0$ are tempting but lead to an improper posterior.
- This formulation allows easy integration into a Gibbs sampler.

Comparison

Consider the following data:[5] [6]

obs	age	sex	bmi	bp	s1	s2	s3	s4	s5
-0.01	0.80	1.06	1.30	0.46	-0.93	-0.73	-0.91	-0.05	0.42
-1.00	-0.04	-0.94	-1.08	-0.55	-0.18	-0.40	1.56	-0.83	-1.43
-0.14	1.79	1.06	0.93	-0.12	-0.96	-0.72	-0.68	-0.05	0.06
0.70	-1.87	-0.94	-0.24	-0.77	0.26	0.52	-0.76	0.72	0.48
-0.22	0.11	-0.94	-0.76	0.46	0.08	0.33	0.17	-0.05	-0.67
-0.72	-1.95	-0.94	-0.85	-0.41	-1.45	-1.67	0.87	-1.60	-0.86

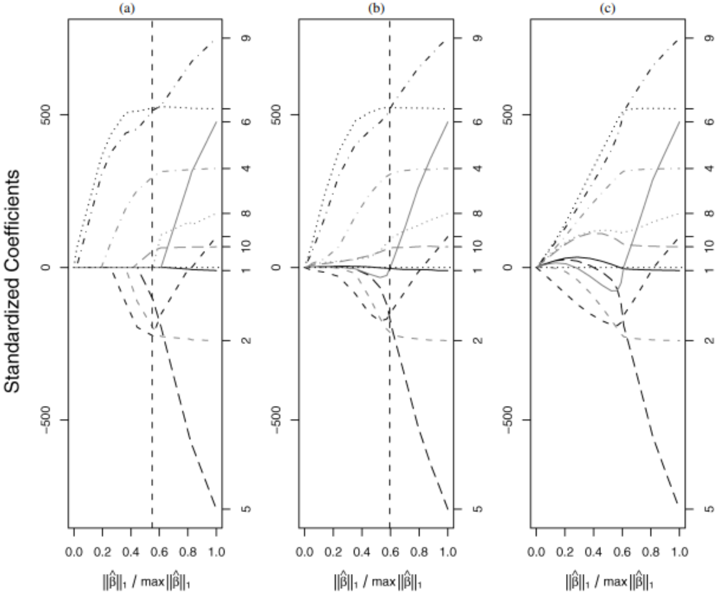
- Measurements of 440 diabetic patients
- 10 baseline variables (centered and scaled)
- Response variable is a measure of disease progression one year after baseline

[5] {care} Efron et al. (2004)
[6] Zuber and Strimmer. (2021)

Trace Plot of Coefficients by Lasso

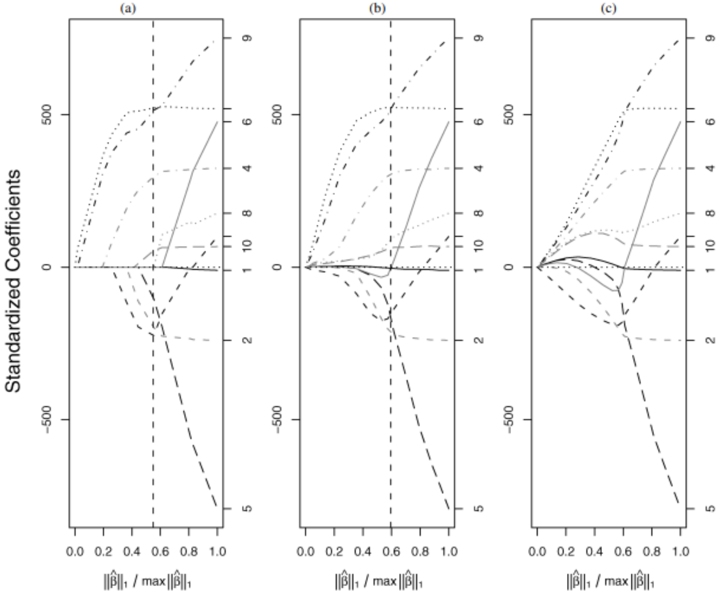
a) Lasso
b) Bayesian Lasso
c) Ridge Regression

Vertical lines for the Lasso and Bayesian Lasso indicating the estimates chosen by n-fold cross-validation and marginal maximum likelihood.



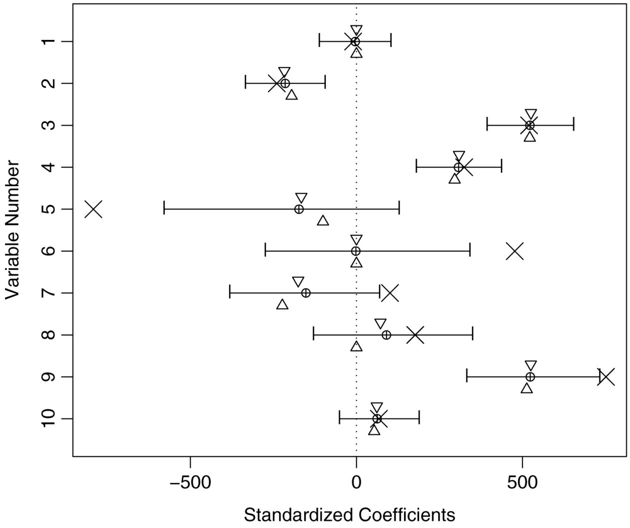
Comparison

- The Bayesian Lasso estimates appear to be a compromise between the Lasso and ridge regression estimates
- The Bayesian Lasso appears to pull the more weakly related parameters to 0 faster than ridge regression



Comparison

- Least squares estimates ($\times$),
- Lasso estimates based on n-fold cross-validation ($\triangle$),
- Lasso estimates chosen to match the L1 norm of the Bayes estimates ( )
- Posterior median Bayesian Lasso estimates ($\otimes$) and corresponding 95% credible intervals (equal-tailed) with λ selected according to marginal maximum likelihood



Extentions

"Bridge" Regression[7]

$$\min_{\beta}(\tilde{y} - X\beta)^T(\tilde{y} - X\beta) + \lambda\Sigma_{j=1}^{p}|\beta_j|^2$$

$$\pi(\beta|\sigma^2) \propto \prod_{j=1}^{P} e^{-\lambda(\frac{|\beta_j|}{\sqrt{\sigma^2}})^2}$$

Huberized Lasso[8]

$$\min_{\beta}\sum_{i=1}^{n} L(\tilde{y}_i - x_i^T\beta) + \lambda\Sigma_{j=1}^{p}|\beta_j|$$

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[7] Knight and Fu (2000)

[8] Rosset and Zhu (2007)

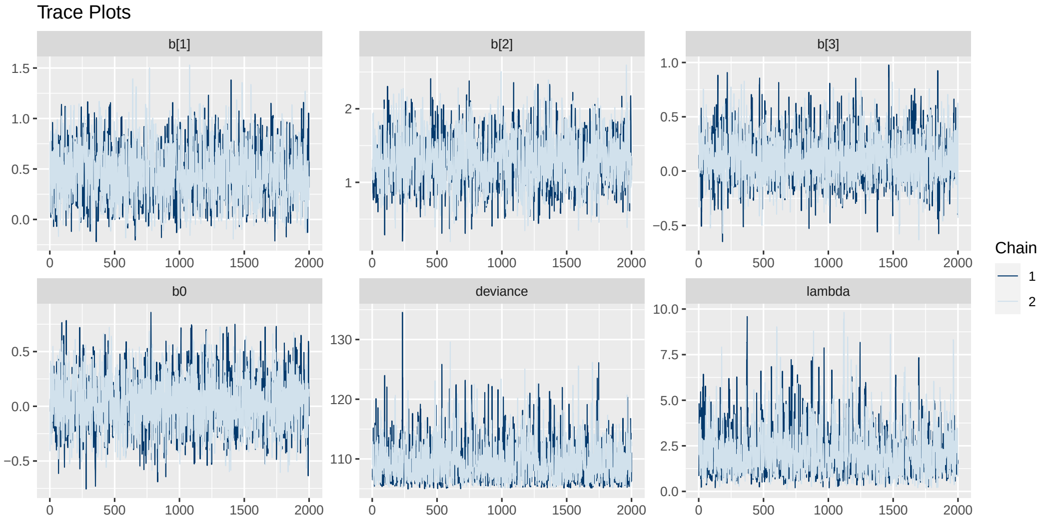
Example

To compare the results, we will generate synthetic data in the form:

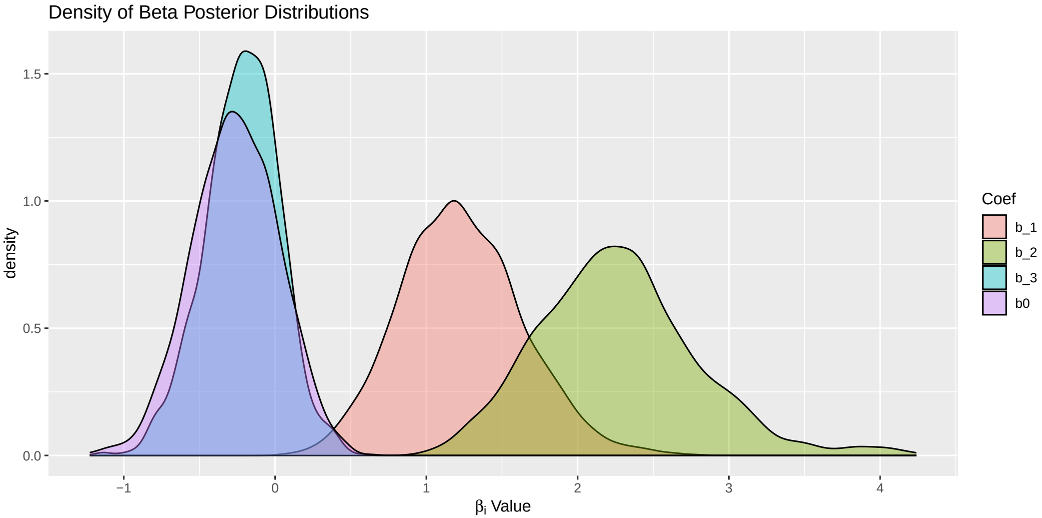
$$\text{logit}(p) = \log \frac{p}{1-p} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

```
num <- 100

x1 <- rnorm(num)
x2 <- rnorm(num)
x3 <- rnorm(num)

prob <- exp(x1+2*x2) / (1+exp(x1+2*x2))
y <- rbinom(num, 1, prob)
```

Example



Example


Density of Beta Posterior Distributions
density
1.5
1.0
0.5
0.0
−1
0
1
2
3
4
β_i Value
Coef
b_1
b_2
b_3
b0

Example

model	b0	b_1	b_2	b_3
Logistic Regression	-0.29	1.41	2.45	-0.21
Lasso	-0.21	0.87	1.73	-0.11
Bayes Lasso	-0.26	1.23	2.27	-0.22
Truth	0.00	1.00	1.00	0.00

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