13. Suppose that the random variables $Y_1, ..., Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, 2, ..., n$$

where $x_1, ..., x_n$ are fixed constants and $\epsilon_i, i = 1, ..., n$ are IID N(0, σ^2), where $\sigma^2 > 0$ is unknown.

(a) In the previous assignment, the MLE for β was found to be

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}.$$

Show $\hat{\beta}$ is an unbiased estimator for β, and find the MSE of $\hat{\beta}$.

(b) Consider the estimator

$$\tilde{\beta} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}.$$

Show this is an unbiased estimator for β and calculate the MSE of $\tilde{\beta}$.

(c) Consider the estimator

$$\beta^* = \frac{1}{n} \sum_{i=1}^n \left(\frac{Y_i}{x_i} \right).$$

Show β^* is unbiased, calculate its MSE.

(d) Which of these three estimators would be the best estimator based on these comparisons? Explain.