Homework 4

25. In each of the following, find the PDF of Y and show that it integrates to one.

    (a) $Y = X^3$ and $f_X(x) = 42x^5(1-x), \quad 0 < x < 1.$

    (b) $Y = 4X + 3$ and $f_X(x) = 7e^{-7x}, \quad 0 < x < \infty.$

    (c) $Y = X^2$ and $f_X(x) = 30x^2(1-x)^2, \quad 0 < x < 1.$

26. Suppose X has the PMF
$$f_X(x) = p(1-p)^x, \quad x = 0, 1, 2, \ldots.$$
Determine the probability distribution of $Y = X/(X+1)$.

27. For each of the following, find the PDF of Y and show that it integrates to one.

    (a) $f_X(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty; Y = |X|^3.$

    (b) $f_X(x) = \frac{3}{8}(x+1)^2, \quad -1 < x < 1; Y = 1 - X^2.$

28. If the random variable X has PDF
$$f(x) = \begin{cases} \frac{x-1}{2}, & 1 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$
Find a monotone function $g(x)$ such that the random variable $Y = g(X)$ has uniform distribution.

29. The median of a distribution is the value m such that $P(X \leq m) = \frac{1}{2}$ and $P(X \geq m) = \frac{1}{2}$. Find the median of the following distributions. Use R to draw the PDFs with a red vertical line representing the median.

    (a) $f(x) = 3x^2, \quad 0 < x < 1.$

    (b) $f(x) = \{\pi(1+x^2)\}^{-1}, \quad -\infty < x < \infty.$

30. Consider a sequence of independent coin flips, each of which has probability p of being heads. Define a random variable X as the length of the run (of either heads or tails) started by the first trial. For example, $X = 3$ if either TTTH or HHHT is observed.

    (a) Find the PMF of X.

    (b) Find the expected value of X.

31. Compute E$[X]$ and Var(X) for the following distribution functions.

    (a) $f_X(x) = ax^{a-1}, \quad 0 < x < 1, a > 0.$

    (b) $f_X(x) = n^{-1}, \quad x = 1, 2, \ldots, n, n > 0$ where n is an integer.

    (c) $f_X(x) = \frac{3}{2}(x-1)^2, \quad 0 < x < 2.$

32. Let μ_n denote the n-th central moment of a random variable X. Two quantities of interest, in addition to the mean and variance are
$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{and} \quad \alpha_4 = \frac{\mu_4}{\mu_2^2}.$$
The value α_3 is called the skewness and α_4 is called the kurtosis. The skewness measures the lack of symmetry in the PDF. The kurtosis, although harder to interpret, measures the peakedness or flatness of the PDF. If a PDF is symmetric, then $\alpha_3 = 0$.

    (a) Calculate α_3 for $f(x) = e^{-x}, x \geq 0$, a PDF that is skewed to the right. Hint: Use the definition of the gamma function found in equations (3.3.2) through (3.3.4) on page 99 in Chapter 3 of your text.

    (b) Calculate α_4 for each of the following PDFs and comment on the peakedness of each. Use R to draw a plot for each of the PDFs. If you recognize a distribution as having a specific form (family) and know the mean and variance for that distribution, you may use that result without proof as long as you state the name of the distribution's family.
$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad -\infty < x < \infty$$