195. Let $\boldsymbol{W} = (W_1, ..., W_n)$ be a collection of IID random variables from a gamma(η, θ) population with known shape parameter η and unknown rate θ (that is, the scale is θ^{-1}). The probability density function for this parameterization of the gamma distribution is

$$f(\omega|\eta, \theta) = \frac{\theta^{\eta}}{\Gamma(\eta)}\omega^{\eta-1}\exp\{-\omega\theta\}, \quad \omega > 0,$$

with parameter space $\Theta = \{(\eta, \theta) : \eta > 0, \theta > 0\}$. For modeling the uncertainty in θ, use a gamma(α, β) prior, where β is the rate.

(a) Show that the posterior distribution for Bayesian estimation of θ is a gamma with shape parameter $\alpha + n\nu$ and rate parameter $\beta + \sum_{i=1}^{n} w_i$. You do not need to find an expression for $m(\boldsymbol{w})$, the marginal distribution of $\boldsymbol{W}$.

(b) Under Bayes rule for absolute error loss, what is the Bayes estimator of θ? (Provide a specific mathematical explanation, not just one word).

(c) Under Bayes rule for squared error loss, what is the Bayes estimator of θ? (Provide a specific mathematical explanation, not just one word).

(d) What is the Bayes test of

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0?$$

Be as specific as possible.

(e) Determine the shortest $1 - \alpha$ Bayesian credible set for θ. What is the name given to this credible set?