19. A random sample $X_1, ..., X_n$ is drawn from a Pareto population with PDF

$$f(x \mid v, \theta) = \frac{\theta v^\theta}{x^{\theta+1}}, v < x < \infty, v > 0, \theta > 0$$

(a) Use the Factorization Theorem to find a sufficient statistic for the parameter vector (v, θ)

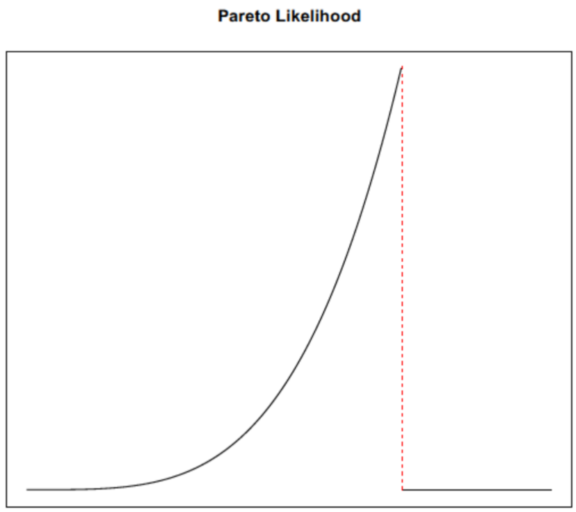
(b) Find the maximum likelihood estimators (MLE) of v and θ

(c) Figure 1 on page 4 contains a graph of the likelihood $L(v, \theta \mid \mathbf{x})$ as a function of v (with θ held at a fixed value)

i. Label the horizontal and vertical axes.

ii. The curve reaches a maximum at the red dashed vertical line. Label this line with the statistic it represents, and use this to explain why the MLE of v is $\hat{v} = X_{(1)}$, along with the indicator function factor in the likelihoods.

Figure 1: Likelihood function $L(\nu, \theta \mid \boldsymbol{x})$ of the Pareto(ν, θ) family as a function of ν for fixed θ.

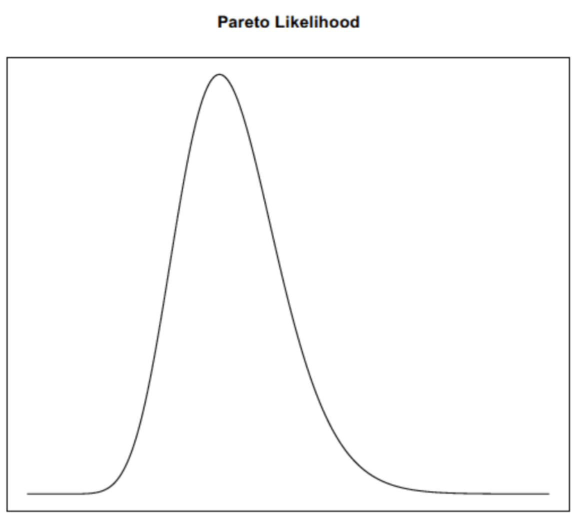


(d) Figure 2 on page 5 contains a graph of the likelihood $L(v, \theta \mid \mathbf{x})$ as a function of θ (with v held at a fixed value)

i. Label the horizontal and vertical axes

ii. Draw and label a vertical line that would represent the location of the MLE $\hat{\theta}$ of θ.

Figure 2: Likelihood function $L(\nu, \theta \mid \boldsymbol{x})$ of the Pareto(ν, θ) family as a function of θ for fixed ν.



(e) Consider testing

$$H_0 : \theta = 1, v \text{ unknown, versus } H_1 : \theta \neq 1, v \text{ unknown}$$

Show the LRT has a critical region of the form $\{\mathbf{x} : T(\mathbf{x}) \leq c_1 \text{ or } T(\mathbf{x}) \geq c_2\}$ where $0 \leq c_1 < c_2 \leq 1$ and

$$T(\mathbf{x}) = \log\left\{\frac{\prod_{i=1}^n X_i}{X_{(1)}^n}\right\}$$

where $X_{(1)}$ is the minimum order statistic

(f) Figure 3 on page 6 is a graph of the likelihood ratio test statistic $\lambda(t)$ graphed as a function of t. Use this graph to illustrate the relationship just derived in part (e). Label where the value of n would occur on the horizontal axis.

Figure 3: Likelihood ratio test statistic $\lambda(t)$ as a function of the sufficient statistic $T(\boldsymbol{x})$ given in equation (1).