145. Suppose a random sample is taken from a normal distribution with known variance σ^2. For each observation, a random Y_i is known to have mean βx_i, where x_i is an easily measured observable variable; that is,

$$Y_i \sim N(\beta x_i, \sigma^2), \quad i = 1, 2, ..., n$$

where each x_i is considered a constant (not a random quantity).

(a) Find the methods of moments estimator for β.

(b) Prove the maximum likelihood estimator (MLE) for β is

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}.$$

(c) Prove the MLE is unbiased.

(d) Prove the distribution is an exponential family, and use that to find a complete sufficient statistic.

(e) Is the complete sufficient statistic minimal sufficient?

(f) What is the Cramér-Rao Lower Bound (CRLB) for the variance of any estimator of β? The MLE was proven to be unbiased. What is the CRLB for any unbiased estimator of β?

(g) What is a uniformly minimum variance unbiased estimator (UMVUE) of β?

(h) Is the UMVUE unique? Explain.