144. If $S^2$ is the sample variance based on a sample of size n from a normal population, we know that $(n-1)S^2/\sigma^2$ has a $\chi^2_{n-1}$ distribution. The conjugate prior for $\sigma^2$ is the inverted gamma PDF, IG$(\alpha, \beta)$ given by

$$\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left\{-\frac{1}{\beta\sigma^2}\right\}, 0 < \sigma^2 < \infty$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

(a) If Y is distributed as inverted gamma with shape parameter $\alpha$ and scale parameter $\beta$, use transformation methods to show that $W = Y^{-1}$ has a gamma distribution with shape parameter $\alpha$ and scale parameter $\beta$.

(b) Show that the posterior distribution of $\sigma^2$ is the inverted gamma distribution with shape and scale parameters given by

$$\frac{n-1}{2} + \alpha, \quad \text{and} \quad \left\{\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right\}^{-1},$$

respectively.

(c) Based on observing $S^2 = s^2$, a decision about the hypothesis

$$H_0 : \sigma \leq 1 \text{ versus } H_1 : \sigma > 1$$

is to be made. Find the region of the sample space for which $P(\sigma \leq 1 \mid S^2 = s^2) > P(\sigma > 1 \mid S^2 = s^2)$, the region for which a Bayes test will decide $\sigma \leq 1$. Hint: Consider a transformation of the posterior that will result in a gamma distribution with scale parameter equal to two.

(d) For the same hypotheses, derive the likelihood ratio test. You may use that $\hat{\mu} = \bar{X}$ is the MLE of the mean and $\hat{\sigma}^2 = (n-1)S^2/n$ is the MLE of the variance without proof.

(e) Compare the region in part (c) with the acceptance region of the likelihood ratio test. Is there any choice of prior parameters for which the two regions would agree?