22. Suppose a random sample is taken from a normal distribution with known variance $\sigma^2$. For each observation, a random $Y_i$ is known to have mean $\beta x_i$, where $x_i$ is an easily measured observable variable; that is,

$$Y_i \sim N(\beta x_i, \sigma^2), \quad i = 1, 2, ...., n$$

where each $x_i$ is considered a constant (not a random quantity).

(a) Find the methods of moments estimator for $\beta$.

(b) Prove the maximum likelihood estimator (MLE) for $\beta$ is

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}.$$

(c) Prove the MLE is unbiased.

(d) Prove the distribution is an exponential family, and use that to find a complete sufficient statistic.

(e) Is the complete sufficient statistic minimal sufficient?

(f) What is the Cramér-Rao Lower Bound (CRLB) for the variance of any estimator of $\beta$? The MLE was proven to be unbiased. What is the CRLB for any unbiased estimator of $\beta$?

(g) What is a uniformly minimum variance unbiased estimator (UMVUE) of $\beta$?

(h) Is the UMVUE unique? Explain.