47. Let $X_1, ..., X_n$ be a random sample from a normal population with known mean μ and unknown variance θ. If needed, you may use the following without proof.

- The statistic $T(\boldsymbol{X}) = \sum_{i=1}^{n}(X_i - \mu)^2$ is sufficient for θ.
- The probability density function for $T(\boldsymbol{X})$ is

$$g(T(\boldsymbol{X})|\theta) = \frac{1}{\Gamma(n/2)(2\theta)^{n/2}} t^{(n/2)-1} \exp\{-t/(2\theta)\},$$

  for all $0 < t < \infty$.

(a) Prove that the maximum likelihood estimator (MLE) of θ is

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

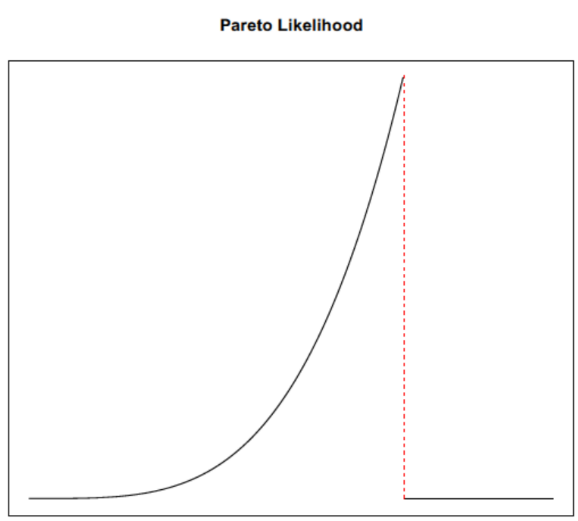
(b) For some real number $\theta_0 > 0$, consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0,$$

i. Find the likelihood ratio test (LRT).

ii. Using the axes in the figure below, sketch the LRT statistic and illustrate the rejection region. Label the axes. Show which regions are the null and alternative parameter spaces. The more relevant information, the better!

Figure 1: Likelihood function $L(\nu, \theta \mid \boldsymbol{x})$ of the Pareto(ν, θ) family as a function of ν for fixed θ.

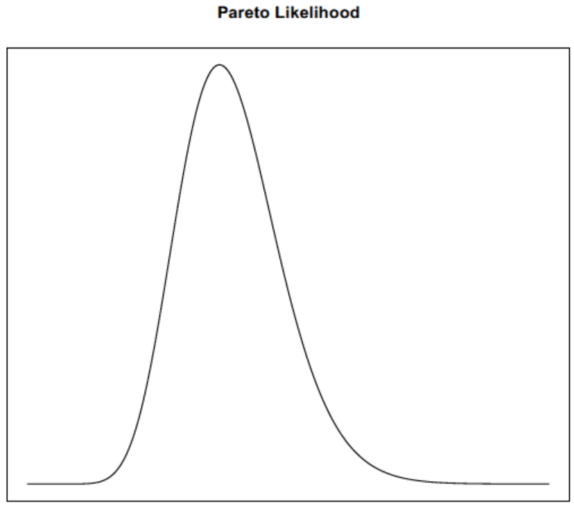


iii. Find the level α uniformly most powerful (UMP) test. The more specific your result for the rejection region, the better.

iv. Derive the power function for the level α UMP test.

v. Use the axes in the figure below to sketch the power function of the UMP test. Label the axes. Show which regions correspond to the null and alternative parameter spaces. Also illustrate type I error probability.

Figure 2: Likelihood function $L(\nu, \theta \mid \boldsymbol{x})$ of the Pareto(ν, θ) family as a function of θ for fixed ν.



vi. Find a valid p-value for the level α test. The more detail, the better.

(c) Suppose the level α UMP test for the hypotheses in (b) is the test that rejects the null hypothesis if $T(\boldsymbol{x}) > \theta_0 \chi^2_{n,\alpha}$, where $\chi^2_{\nu,\alpha}$ is the upper-tailed quantile of the chi-square distribution with ν degrees of freedom. Find the $1-\alpha$ uniformly most accurate (UMA) lower confidence bound for θ.