117. Let $X_1, ..., X_n$ be independent and identically distributed (IID) random variables with mean μ and variance σ^2. Define the sample mean $\bar{X}$ and sample variance S^2 as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

(a) For this part only, suppose the X_i are normally distributed. Use the Derived Distribution approach to find an expression that has a Student's T distribution with $n-1$ degrees of freedom.

(b) It is now not necessarily the case that X_i are normal. Consider $\bar{X}$ and S^2 as a sequence in the sample size n; that is, consider the sample mean and sample variance as

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Prove that

$$Z_n = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{S_n}\right) \xrightarrow{\mathcal{D}} Z$$

where Z is a standard normal random variable.

Hint: The variance of S_n^2 is $\text{Var}(S_n^2) = 2\sigma^4/(n-1)$.