Homework 3

16. Let $\mathcal{S}$ be a sample space of an experiment, and $\mathcal{B}$ the smallest Borel field of sets in $\mathcal{S}$. Let A and B be any two sets in $\mathcal{B}$ such that $P(B) > 0$. Then the conditional probability of A given that B has occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Prove the probability function $P(\cdot|B)$ satisfies Kolmogorov's Axioms.

17. General Strong is in the process of determining a strategy for the army. Intelligence has reliably narrowed down the location of an upcoming attack to three areas. The general's staff has informed him that the probabilities of attack at the three areas are 0.5, 0.3, and 0.2, respectively. Suddenly, a report arrives at army headquarters indicating that the enemy is taking certani actions (call these actions event E). A new round of consultations with his staff leaves the general with the feeling that - depending on whether the enemy plans to attack at locations 1, 2, or 3 - the probabilities of action are 0.6, 0.25, and 0.15, respectively. With this information, how can the genreal revise his initial probabilities concerning the location of the next attack from the enemy? That is, based on this new information, find the revised probability of an attack for each of the three locations.

18. Let A and B be two mutually exclusive events with $P(A) > 0$ and $P(B) > 0$. Prove that A and B are statistically dependent.

19. Let A and B be two independent events with $P(A) > 0$ and $P(B) > 0$. Prove that A and B cannot be mutually exclusive.

20. Suppose we have a finite sample space $\mathcal{S} = \{\omega_1, \omega_2, ..., \omega_n\}$ with a probability function $\mathcal{P}(\mathcal{S}) \to [0, 1]$. Define a random variable X with range being some subset $\mathcal{X}$ of the real numbers and that maps $\mathcal{S}$ to the subset; that is, $X(\omega_i) = x_j$ for some $x_j \in \mathcal{X} \in\subset \mathbb{R}$, so that $X(\mathcal{S}) \to \mathcal{X}$. We say that we observe $X = x_j$ if and only if the outcome of the random experiment is an $\omega_j \in \mathcal{S}$ such that $X(\omega_j) = x_i$. Then the induced probability function on X is

$$P_X(X = x_i) = P(\{\omega_j \in \mathcal{S} : X(\omega_j) = x_i\}).$$

Prove $P_X(\cdot)$ satisfies Kolmogorov's Axioms.

21. Show the function

$$f(x) = \mu^x \frac{e^{-\mu}}{x!} \qquad x = 0, 1, 2, ...$$

is a probability mass function, where $\mu > 0$.

22. For the following functions, prove they are cumulative distribution functions (CDFs) and find the corresponding probability density functions (PDFs). Use R to create side-by-side graphs of the CDFs and PDFs.

    (a) $F(X) = \frac{1}{2} + \frac{1}{\pi}\arctan(x), x \in (-\infty, \infty)$
    (b) $F(X) = (1 + e^{-x})^{-1}, -\infty < x < \infty$

23. Let X be a continuous random variable with probability density function (PDF) $f_X(x)$ and CDF $F_X(x)$. For a fixed number x_0 define the function

$$g(x) = \begin{cases} \frac{f_X(x)}{1-F_X(x_0)}, & x \geq x_0 \\ 0, & x < x_0 \end{cases}$$

Prove $g(x)$ is a PDF.

24. For each of the following, determine the value of c that makes $f(x)$ a PDF.

    (a) $f(x) = c \sin x, 0 < x < \pi/2$
    (b) $f(x) = ce^{-|x|}, -\infty < x < \infty$