169. Let $X \sim \text{normal}(\mu, \sigma^2)$ with σ^2 known ($n = 1$). For each $c \geq 0$ define an interval estimator for μ by $C(x) = [x - c\sigma, x + c\sigma]$ and consider the loss function

$$L(\theta, C) = b\text{Length}(C) - I_C(\theta),$$

where b is a positive constant that reflects the relative weight given to the two criteria, and

$$I_C(\theta) = \begin{cases} 1 & \theta \in C \\ 0 & \theta \notin C \end{cases}$$

(a) Show that the risk function $R(\mu, C)$ is given by

$$R(\mu, C) = b(2c\sigma) = 2bc\sigma - P(-c \leq Z \leq c).$$

(b) Using the Fundamental Theorem of Calculus, show that

$$\frac{d}{dc} R(\mu, C) = 2b\sigma - \frac{2}{\sqrt{2\pi}} e^{-c^2/2},$$

which is an increasing function of c for $c \geq 0$.

(c) Show that if $b\sigma > (2\pi)^{-1/2}$, the derivative is positive for all $c \geq 0$ and therefore $R(\mu, C)$ is minimized at $c = 0$. What does this imply about the "best interval" estimator?

(d) Show that if $b\sigma \leq (2\pi)^{-1/2}$, the c that minimizes the risk is

$$c = \sqrt{-2\log(b\sigma\sqrt{2\pi})}.$$

Therefore, if b is chosen so that $c = z_{\alpha/2}$ for some α then the interval estimator that minimizes the risk is just the usual $1 - \alpha$ confidence interval.