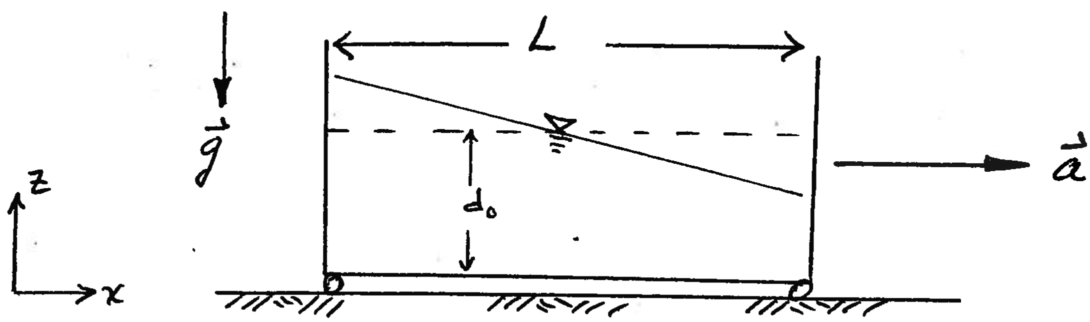
Application #1 Uniform Linear Acceleration

(EULER'S EQUATION EXAMPLE 1)

A rectangular container of water is subject to constant acceleration. Determine the shape of the free surface



Solution

Fundamental Equation: $\rho \vec{a} = \rho \vec{g} - \nabla p$

In component form:

$$\rho a_x = \rho g_x - \frac{\partial p}{\partial x}$$

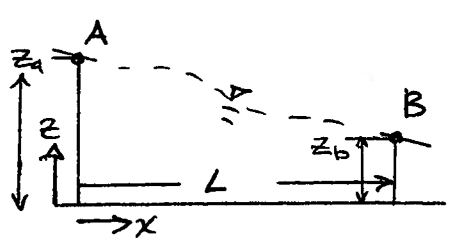
$$\rho a_z = \rho g_z - \frac{\partial p}{\partial y}$$

Observe: $a_z = 0$, $g_x = 0$

Substitute and simplify to obtain

$$\frac{\partial p}{\partial x} = -\rho a_x \quad ; \quad \frac{\partial p}{\partial z} = -\rho g_z$$

To find shape consider two points on the free surface: VITAL OBSERVATION!!



$p_A = p_B$ (both on free surface)

$p_A = p(0, z_a) = p(L, z_b)$

The pressure variation is

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz = -\rho a_x dx - \rho g_z dz$$

but on the free surface $dp = 0$

$$\therefore \rho a_x dx = -\rho g_z dz \quad \text{or} \quad \frac{dz}{dx} = -\frac{a_x}{g_z}$$

Integrate to find equation of the free surface:

$$\int dz = -\frac{a_x}{g_z}\int dx \Rightarrow z = -\frac{a_x}{g_z}x + C$$

To evaluate the constant of integration consider the depth at $x = L/2$. Suppose this depth is d_0. Then

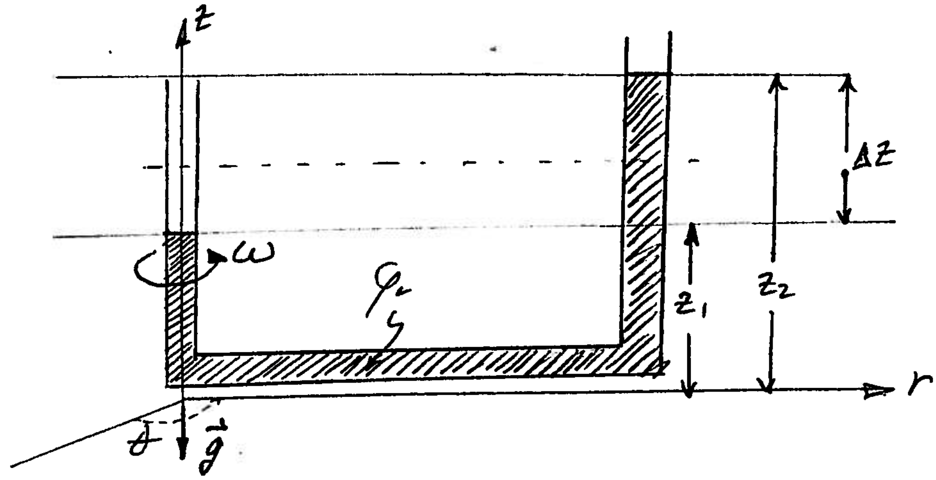
$$z_{L/2} = -\frac{a_x}{g_z}\left(\frac{L}{2}\right) + C = d_0 \Rightarrow C = d_0 + \frac{a_x}{g_z}\frac{L}{2}$$

Finally the equation of the free surface is given by

$$\boxed{z = d_0 + \frac{a_x}{g_z}\frac{L}{2} - \frac{a_x}{g_z}x}$$

Application #2 Constant Angular Velocity

A U-tube manometer is rotated about an axis coincident with one of the arms. At equilibrium, what is the difference in height of the fluid in the two arms of the manometer?



Solution

Fundamental equation of motion $\rho\vec{g} - \nabla p = \rho\vec{a}$

In component form:

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r \qquad g_r = g_\theta = 0$$

$$-\frac{\partial p}{\partial \theta} + \rho g_\theta = \rho a_\theta \qquad a_\theta = a_z = 0$$

$$-\frac{\partial p}{\partial z} + \rho g_z = \rho a_z \qquad a_r = -\omega^2 r$$

Substitute and simplify

$$-\frac{\partial p}{\partial r} = -\rho\omega^2 r \;;\; -\frac{\partial p}{\partial \theta} = 0 \;;\; -\frac{\partial p}{\partial z} = \rho g_z$$

Pressure variation

$$dp = \frac{\partial p}{\partial r}dr + \cancelto{0}{\frac{\partial p}{\partial \theta}d\theta} + \frac{\partial p}{\partial z}dz = \rho\omega^2 r\,dr - \rho g_z\,dz$$

Integrate to find $p(r,\theta,z)$

$$\int dp = \int \rho\omega^2 r\,dr - \int \rho g_z\,dz = \frac{\rho\omega^2 r^2}{2} - \rho g_z z + C_1 + C_2$$

Find constants from known pressure at liquid surface.

$@\ r=0,\ z=z_1,\ p=0 \Rightarrow \rho g_z z_1 = C_1 + C_2$ (A)

$@\ r=R,\ z=z_2,\ p=0 \Rightarrow \rho g_z z_2 = \frac{\rho\omega^2 R^2}{2} + C_1 + C_2$ (B)

(B) − (A) Gives

$$\rho g_z \underbrace{(z_2 - z_1)}_{\Delta z} = \frac{\rho\omega^2 R^2}{2} + \underbrace{C_1 + C_2 - (C_1 + C_2)}_{\equiv 0}$$

So

$$\frac{\cancel{\rho}\omega^2 R^2}{2} = \cancel{\rho} g_z \Delta z \Rightarrow \Delta z = \boxed{\frac{\omega^2 R^2}{2 g_z}}$$