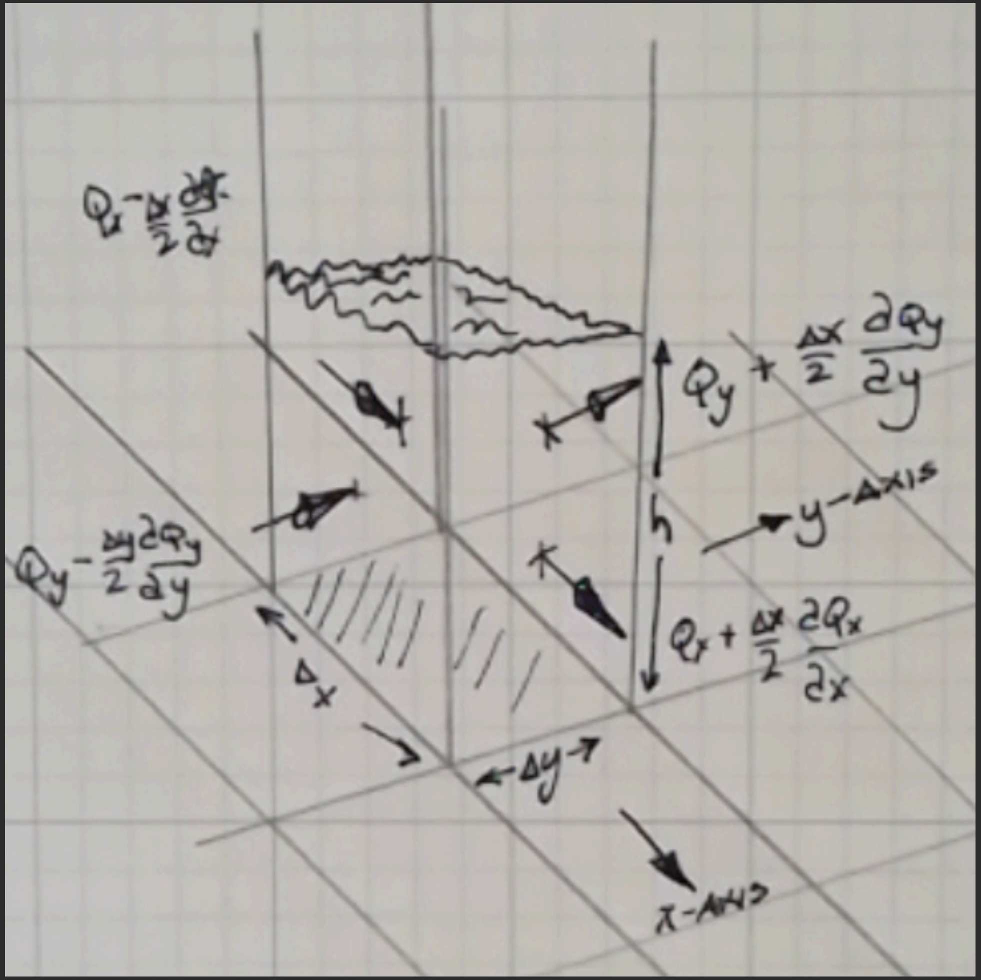
CE 5362 Surface Water Modeling

1D/2D Hydrodynamic Models

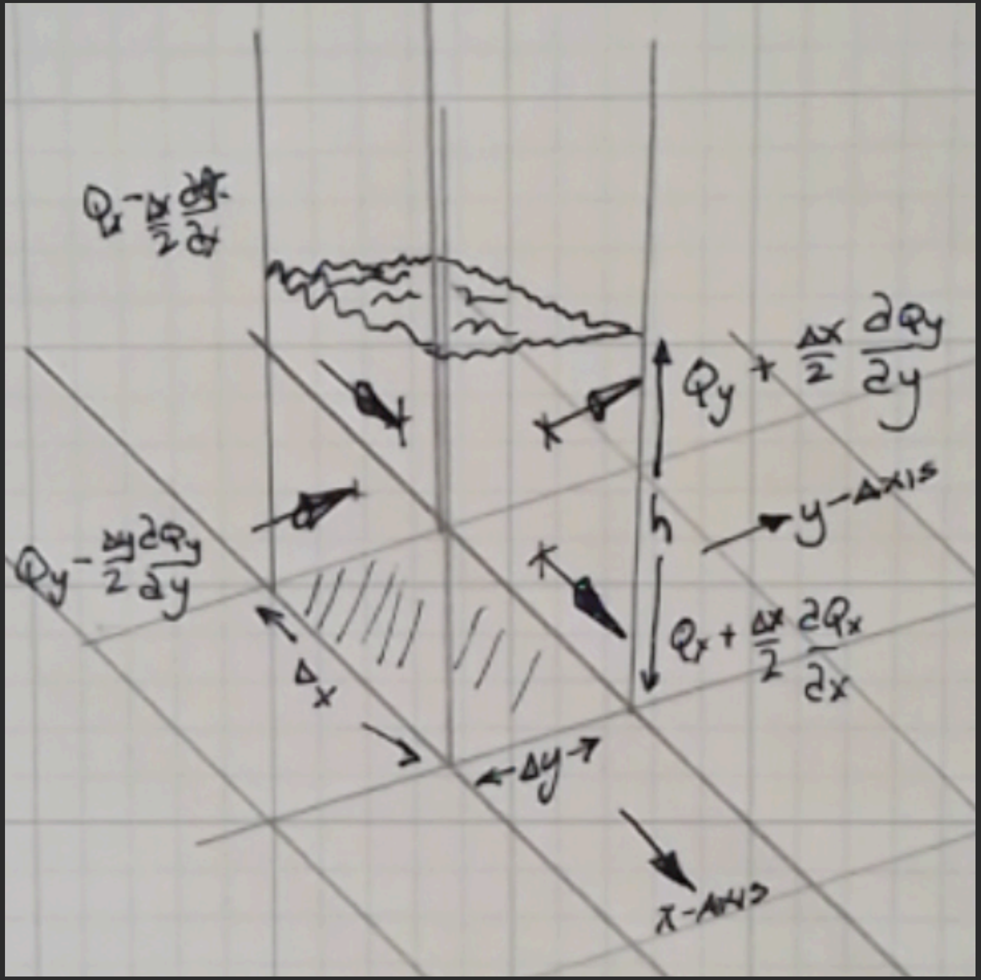
2D Diffusion Hydrodynamic Model

- 2D horizontal flow – an extension of St.Venant-type formulation
- Assume: Hydrostatic pressure distribution; constant density; planar dimension (x,y) much larger then vertical dimension (z)
- \cite{DHM-report}



2D Diffusion Hydrodynamic Model

- Sketch of a computational cell
- x, y directions shown, z is up.
- Within the cell, average flow depth is h
- On the 4 cell faces, are the flux terms Q, shown here as series expansions about the cell center.
- Apply continunity principle



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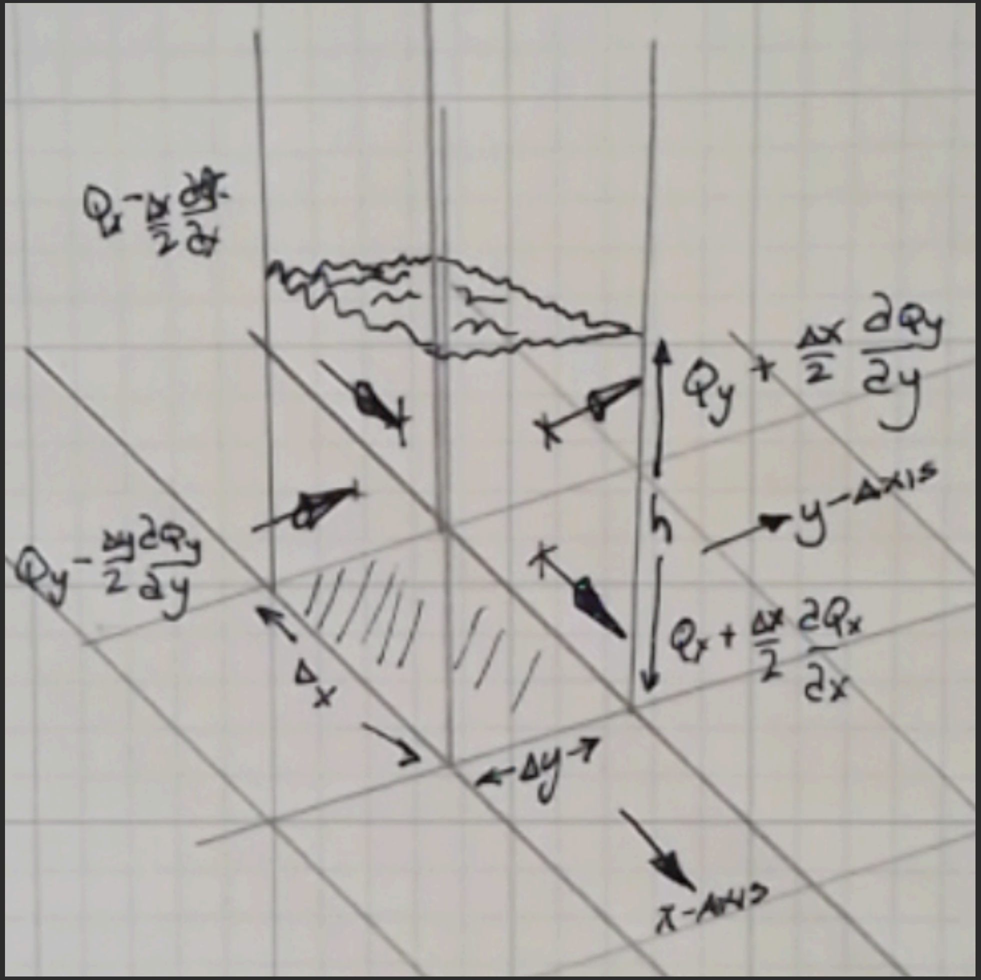


2D Diffusion Hydrodynamic Model

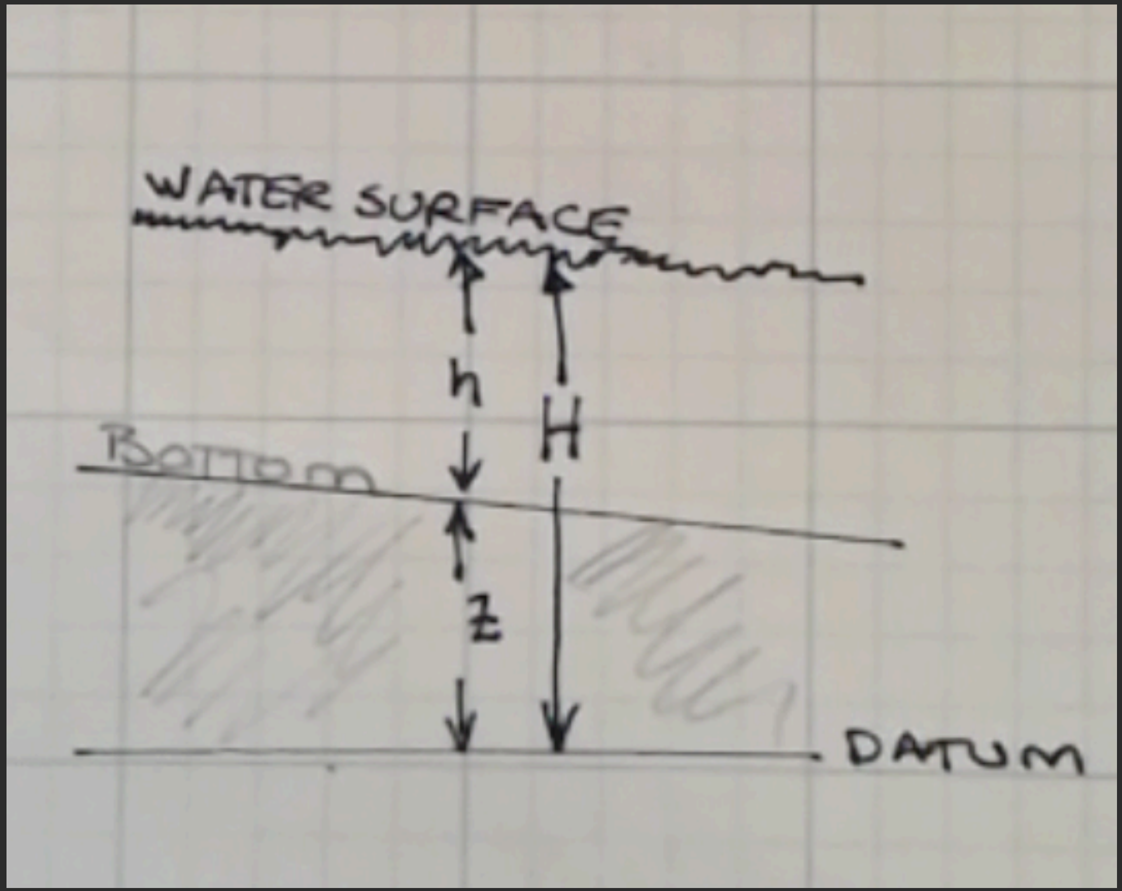
- Apply continunity principle

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial H}{\partial t} = \pm \text{ sink}$$

- Where,

$$q_x = \frac{Q_x}{\Delta y} = U \cdot h$$

$$q_y = \frac{Q_y}{\Delta x} = V \cdot h$$



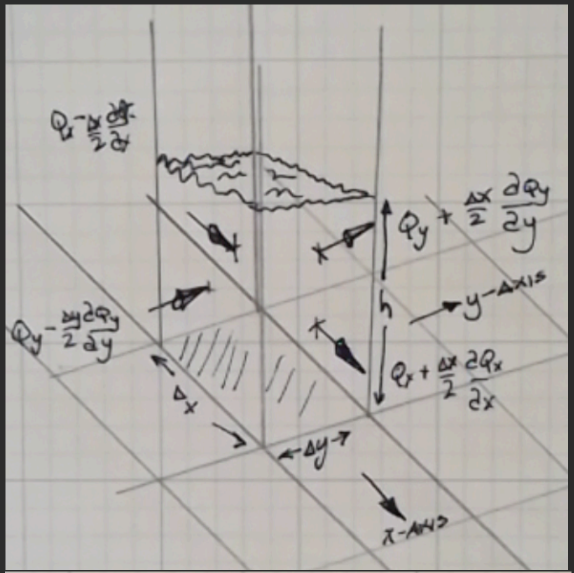
2D Diffusion Hydrodynamic Model

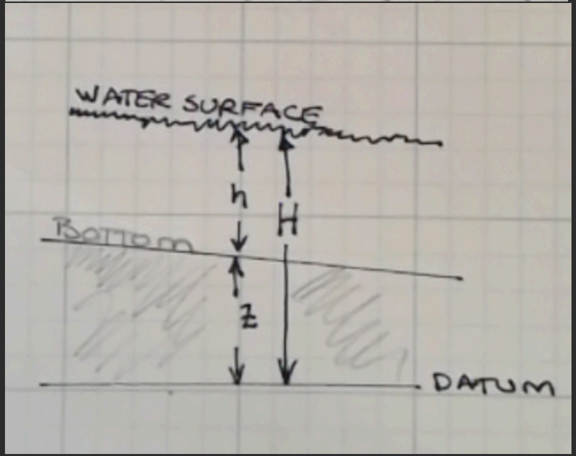
- Momentum-x

$$\frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_x q_x}{h}\right) + \frac{\partial}{\partial y}\left(\frac{q_x q_y}{h}\right) + gh\left(S_{fx} + \frac{\partial H}{\partial x}\right) = 0$$

- Momentum-y

$$\frac{\partial q_y}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_x q_y}{h}\right) + \frac{\partial}{\partial y}\left(\frac{q_y q_y}{h}\right) + gh\left(S_{fy} + \frac{\partial H}{\partial x}\right) = 0$$





2D Diffusion Hydrodynamic Model

- DHM moniker comes from next few steps, where the momentum terms are converted into linear-flux law terms so the equations of motion can be placed directly into continunity and a diffusion equation results.
- The PDE is pretty non-linear, but the construction allows use of very stable methods

$$\frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_x q_x}{h}\right) + \frac{\partial}{\partial y}\left(\frac{q_x q_y}{h}\right) + gh\left(S_{fx} + \frac{\partial H}{\partial x}\right) =$$

$$\frac{\partial q_y}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_x q_y}{h}\right) + \frac{\partial}{\partial y}\left(\right.$$

2D Diffusion Hydrodynamic Model

- First divide momentum by gh

$$\frac{\frac{\partial q_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q_x q_x}{h}\right)+\frac{\partial}{\partial y}\left(\frac{q_x q_y}{h}\right)}{gh}+\left(S_{fx}+\frac{\partial H}{\partial x}\right)=0$$

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2D Diffusion Hydrodynamic Model

- First divide momentum by gh

$$\frac{\dfrac{\partial q_y}{\partial t}+\dfrac{\partial}{\partial x}\left(\dfrac{q_x q_y}{h}\right)+\dfrac{\partial}{\partial y}\left(\dfrac{q_y q_y}{h}\right)}{gh}+\left(S_{fy}+\frac{\partial H}{\partial x}\right)=0$$

2D Diffusion Hydrodynamic Model

- Rename the leftmost three terms "m" (a function of current state)

$$m_x + \left(S_{fx} + \frac{\partial H}{\partial x} \right) = 0$$

$$m_y + \left(S_{fy} + \frac{\partial H}{\partial x} \right) = 0$$

2D Diffusion Hydrodynamic Model

- Stipulate that the friction slope is well approximated by Chezy/Manning formulation

$$q_x = \frac{1.49}{n} h^{5/3} S_{fx}^{1/2}$$

$$q_y = \frac{1.49}{n} h^{5/3} S_{fy}^{1/2}$$

2D Diffusion Hydrodynamic Model

- Rewrite in terms of momentum equations

$$q_x = -K_x \frac{\partial H}{\partial x} - K_x m_x ; K_x = \frac{1.49}{n} h^{5/3} \bigg/ \left| \frac{\partial H}{\partial S} + m_s \right|^{1/2}$$

$$q_y = -K_y \frac{\partial H}{\partial y} - K_y m_y ; K_y = \frac{1.49}{n} h^{5/3} \bigg/ \left| \frac{\partial H}{\partial S} + m_s \right|^{1/2}$$

2D Diffusion Hydrodynamic Model

- Substitute into continunity, and the result is a diffusion equation in H, quite non-linear.

$$\frac{\partial}{\partial x}\left(K_x\left(\frac{\partial H}{\partial x}+m_x\right)\right)+\frac{\partial}{\partial y}\left(K_y\left(\frac{\partial H}{\partial y}+m_y\right)\right)=\frac{\partial H}{\partial t}$$

2D Diffusion Hydrodynamic Model

- Solving takes a trial-and-error approach (hence a lot of iteration)
- General algorithm is:
  1) Between nodes points (velocity grid) compute average Manning's n, average geometric factors (h)
  2) Assume m=0,estimate nodal flow depths for next time (t+Dt) using the momentum equations (as difference equations)
  3) Use these new depths, and old depths to estimate the midtimestep value of m (including the dynamic components, this could be an elaborate computation)
  4) Recompute values of K using these midstep m values
  5) Find flow depths at next time (t+Dt) using continunity (same as step 2, but with midstep m values)
  6) Return to step 3 and repeat until K matches the midstep estimate – declare that convergence and update, proceede to next time step.

$$\frac{\partial}{\partial x}\left(K_x\left(\frac{\partial H}{\partial x}+m_x\right)\right)+\frac{\partial}{\partial y}\left(K_y\left(\frac{\partial H}{\partial y}+m_y\right)\right)=\frac{\partial H}{\partial t}$$

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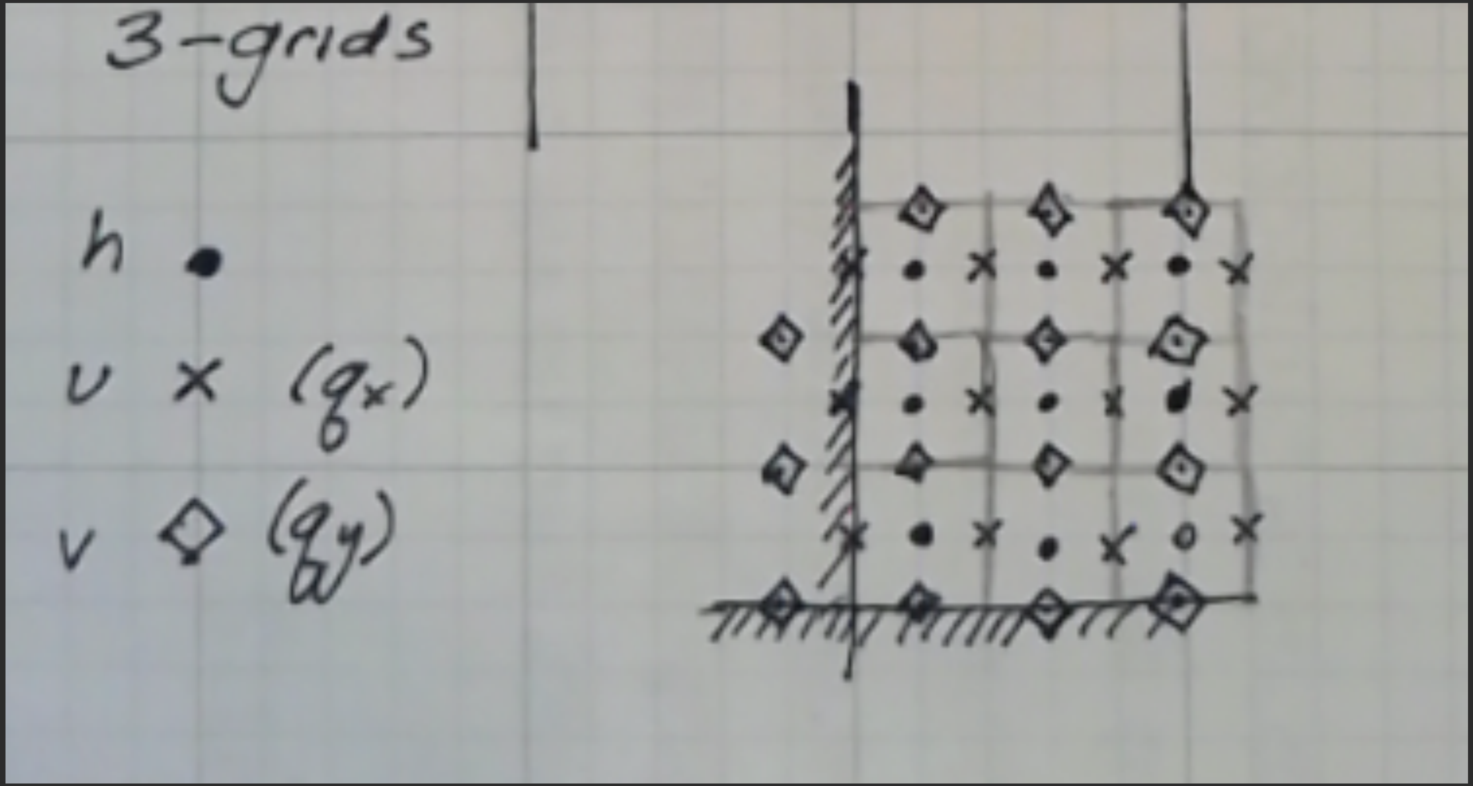
2D Diffusion Hydrodynamic Model

- Grid system is complicated, there are 3 grids
  - A grid of depth nodes,
  - A grid of Qx nodes,
  - A grid of Qy nodes



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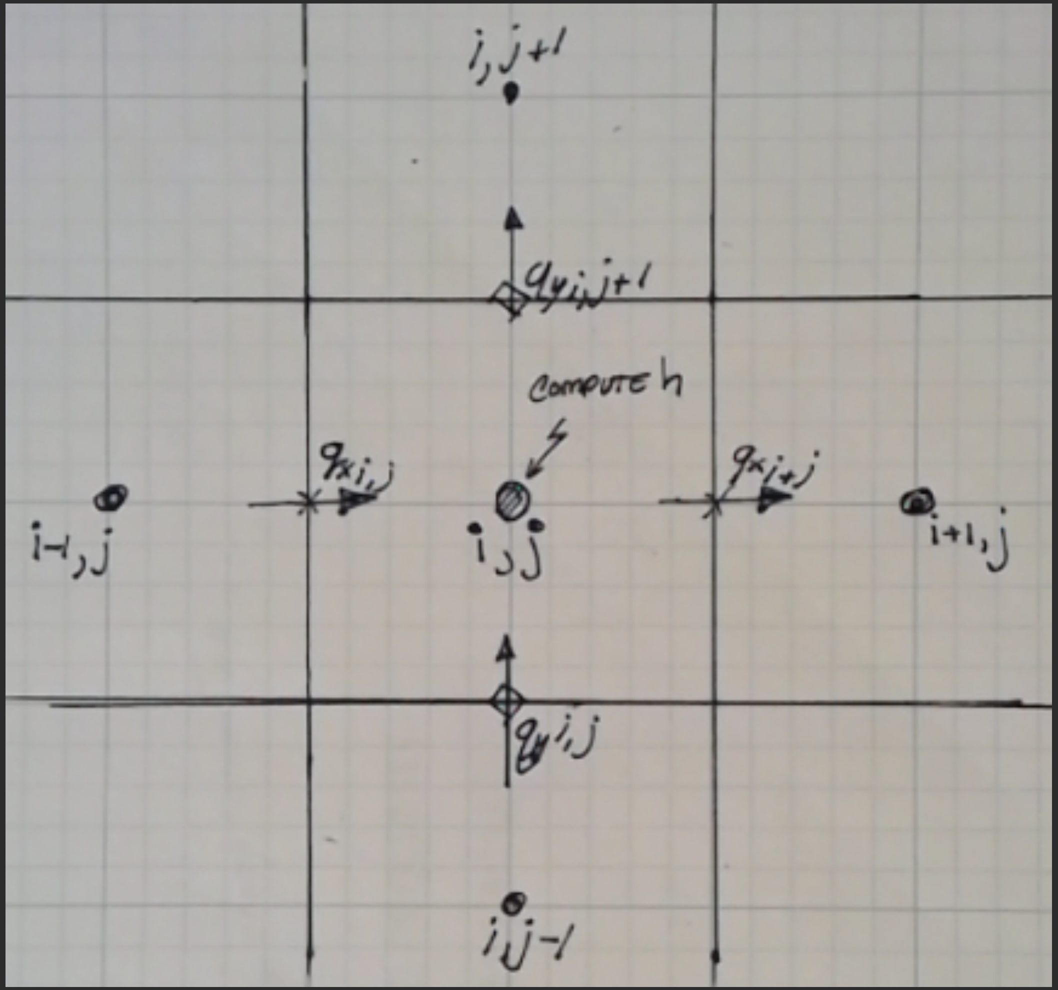

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- An effective naming scheme is important