12

Surge waves in Amenagement d'Oraison (Courtesy Eletricitie de France)

Unsteady Free-Surface Flows

We discussed steady open-channel flows in Chapter 4 and unsteady closed-conduit flows in Chapter 11. In this chapter, we discuss unsteady free-surface flows. These flows occur in natural and manmade channels and are produced by changes in the water levels or changes in the inflow or outflow rates. Typical examples of these flows are floods in streams and rivers, surges in power canals, tidal flows in estuaries, unsteady flows in irrigation canals and channels, and storm runoff in sewers.

In this chapter, commonly used terms are first defined. Causes which produce these flows are discussed. The governing equations are derived, and numerical methods for their solution are presented.

12-1 Definitions

If the flow velocity at a point varies with time, the flow is called *unsteady flow*. Depending on the rate of variation of depth, unsteady flows may be classified as *gradually varied* (flood waves) or *rapidly varied* (surges). In the case of rapidly varied flows, steep fronts or discontinuities may occur in the flow depth. These discontinuities are referred to as shocks or bores.

A *wave* is a temporal or spatial variation of flow depth or rate of discharge. The wave velocity with respect to the flow it is traveling in is referred to as the *celerity*, *c*. In one-dimensional flows, the absolute wave velocity, V_w, is given by the equation

$$V_w = V \pm c \tag{12-1}$$

where V = flow velocity.

Considering the direction of downstream flow positive, the plus sign is used if the wave is traveling in the downstream direction, and the negative sign is used if the wave is traveling in the upstream direction.

The *wave length* is the distance between two consecutive crests. If the wave length is more than 20 times the undisturbed flow depth, the wave is called *shallow-water wave*; if the wave length is less than twice the flow depth, the wave is called *deep-water wave*. The ratio of the wave length and the flow depth defines whether the wave is shallow water or deep water. That is, a ripple in shallow water can be a deep-water wave, and a tide in the ocean can be a shallow-water wave. This distinction between shallow- and deep-water waves is important because each type has certain characteristics that dictate selection of the governing equations. In this chapter, we discuss only shallow-water waves.

Waves may be classified as positive or negative. A *positive wave* occurs when the flow depth is greater than the undisturbed flow depth, and a *negative wave* occurs when the flow depth is smaller than the undisturbed flow depth.

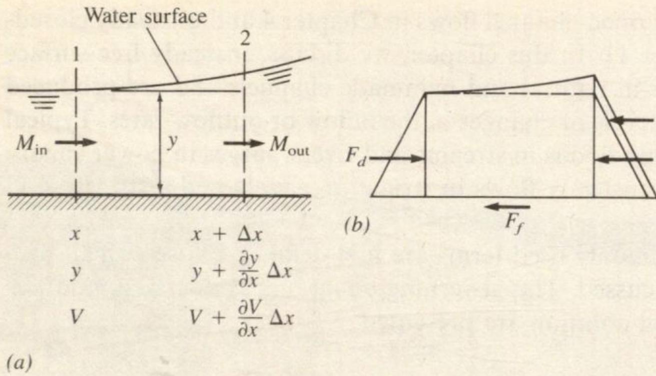

Figure 12-1 Definition sketches

12-2 Governing Equations

The continuity and dynamic equations describe the unsteady free-surface flows. These equations may be derived by making the following assumptions (3, 4, 7, 8):

1. The pressure distribution is hydrostatic. This is usually true if the flow surface does not have a sharp curvature.
2. The channel bottom slope is small so that the flow depth measured vertically is almost the same as the flow depth normal to the channel bottom and $\sin \theta \approx \tan \theta \approx \theta$, where θ is the angle between the channel bottom and horizontal datum.
3. The velocity distribution at a channel cross section is uniform.
4. The channel is prismatic, that is, the bottom slope and cross section remain unchanged with distance.
5. The friction losses in unsteady flow may be computed using the empirical formulas for steady-state flows.

Let us consider a segment of water between two cross sections located at distance x and $x + \Delta x$ (Fig. 12-1). Consider the downstream flow direction positive, and measure the flow depth vertically. Let the flow depth and the flow velocity at distance x be y and V, respectively. Then their values at distance $x + \Delta x$ will be $y + (\partial y/\partial x)\Delta x$ and $V + (\partial V/\partial x)\Delta x$.

Continuity Equation

To derive the continuity equation, we apply the law of conservation of mass to the segment of water between sections 1 and 2, as shown in Fig. 12-1*a*. Referring to this figure, the mass inflow into the segment of water during time Δt, is expressed by:

$$M_{\text{in}} = \frac{\gamma}{g} AV\,\Delta t \tag{12-2a}$$

where γ = specific weight of water. The mass outflow during time Δt, is:

$$M_{\text{out}} = \frac{\gamma}{g}\left(A + \frac{\partial A}{\partial x}\Delta x\right)\left(V + \frac{\partial V}{\partial x}\Delta x\right)\Delta t \tag{12-2b}$$

Hence, the net mass inflow into the segment of water

$$\begin{aligned} &= M_{\text{in}} - M_{\text{out}} \\ &= -\frac{\gamma}{g}\left(V\frac{\partial A}{\partial x} + A\frac{\partial V}{\partial x}\right)\Delta x\,\Delta t \end{aligned} \tag{12-3}$$

where higher-order terms have been neglected.

Now, we can also write an expression for the increase in the mass of segment during time interval Δt as

$$= \frac{\gamma}{g}\frac{\partial A}{\partial t}\Delta x\,\Delta t \tag{12-4}$$

Equating the net mass inflow into the segment of water (given by Eq. 12-3) to the increase in its mass during time interval Δt (given by Eq. 12-4) and dividing throughout by $(\gamma/g)\Delta x\,\Delta t$, we obtain

$$\frac{\partial A}{\partial t} + V\frac{\partial A}{\partial x} + A\frac{\partial V}{\partial x} = 0 \tag{12-5}$$

By combining the second and the third terms of Eq. (12-5) we obtain

$$\frac{\partial A}{\partial t} + \frac{\partial (AV)}{\partial x} = 0 \tag{12-6}$$

This equation is referred to as the *continuity equation in the conservation form.*

We can express the variation of A with respect to x and t as

$$\frac{\partial A}{\partial x} = \frac{dA}{dy}\frac{\partial y}{\partial x} = B\frac{\partial y}{\partial x} \tag{12-7}$$

and
$$\frac{\partial A}{\partial t} = \frac{dA}{dy}\frac{\partial y}{\partial t} = B\frac{\partial y}{\partial t} \tag{12-8}$$

where B = top water surface width.

Substituting Eqs. (12-7) and (12-8) into Eq. (12-5) and simplifying, we obtain

$$\frac{\partial y}{\partial t} + D\frac{\partial V}{\partial x} + V\frac{\partial y}{\partial x} = 0 \tag{12-9}$$

where D = the hydraulic depth defined as $D = A/B$.

Dynamic Equation

To derive the dynamic equation, we apply the law of conservation of momentum to the segment of water between sections 1 and 2. To do this, we equate the rate of change of momentum of the segment of water to the resultant of the external forces acting on the segment.

Four forces act on the segment of water, as shown in Fig. 12-1*b*. The pressure force acting on the downstream face has been divided into two parts. Expressions for these forces are as follows:

Pressure force acting on the upstream face is

$$F_d = \gamma A\bar{z} \tag{12-10}$$

where $\bar{z}$ = the depth of the centroid below the water surface.

Similarly, $F_{u1} = \gamma A\bar{z}$. Neglecting the higher-order terms (these will correspond to the small shaded triangle shown in Fig. 12-1*b*), the second part of the pressure force acting on the downstream face is

$$F_{u2} = \gamma A \frac{\partial y}{\partial x}\Delta x \tag{12-11}$$

If S_f is the slope of the energy grade line, then the force due to friction, F_f, may be written as

$$F_f = \gamma A S_f \Delta x \tag{12-12}$$

This force will be acting in the upstream direction. S_f may be computed by using the Manning or Chezy formula. Now,

$$\text{Weight of segment of water} = \gamma A\,\Delta x \tag{12-13}$$

Since the slope of the channel bottom is assumed to be small, $S_0 = \sin\theta$. Therefore, the component of the weight of water, F_w, acting in the downstream direction is

$$F_w = \gamma A\,\Delta x S_0 \tag{12-14}$$

Hence, the resultant force, F_r, acting on the water segment is

$$F_r = F_d - F_{u1} - F_{u2} - F_f + F_w \tag{12-15}$$

Substituting the expressions for different forces into Eq. (12-15), we obtain

$$F_r = \gamma A\left(-\frac{\partial y}{\partial x} - S_f + S_0\right)\Delta x \tag{12-16}$$

Now, the rate of momentum inflow into the segment of water is

$$M_i = \frac{\gamma}{g} AV^2 \tag{12-17}$$

Rate of momentum outflow is

$$M_o = \frac{\gamma}{g}\left[AV^2 + \frac{\partial}{\partial x}(AV^2)\Delta x\right] \tag{12-18}$$

Hence, the rate of net momentum influx

$$= M_i - M_o$$

$$= -\frac{\gamma}{g}\frac{\partial}{\partial x}(AV^2)\Delta x \tag{12-19}$$

And time rate of increase of momentum of the segment of water may be written as

$$= \frac{\partial}{\partial t}\left[\frac{\gamma}{g} AV\,\Delta x\right] \tag{12-20}$$

Now, the time rate of increase of momentum

= Rate of net influx of momentum + Resultant force acting on the control volume

Substituting into this equation expressions for various terms from Eqs. (12-16), (12-19), and (12-20), dividing throughout by $(\gamma/g)\Delta x$, and simplifying, we obtain

$$\frac{\partial}{\partial t}(AV) + \frac{\partial}{\partial x}(AV^2) + gA\frac{\partial y}{\partial x} = gA(S_0 - S_f) \tag{12-21}$$

By expanding the terms on the left-hand side of Eq. (12-21), dividing by A, and rearranging the resulting equation, we obtain the following.

$$g\frac{\partial y}{\partial x} + V\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} + \frac{V}{A}\left[\frac{\partial A}{\partial t} + V\frac{\partial A}{\partial x} + A\frac{\partial V}{\partial x}\right] = g(S_0 - S_f) \quad (12\text{-}22)$$

According to Eq. (12-5), the sum of the terms within the brackets is zero. Hence Eq. (12-22) may be written as

$$g\frac{\partial y}{\partial x} + \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = g(S_0 - S_f) \quad (12\text{-}23)$$

This is referred to as the *dynamic equation*. By rearranging the terms of this equation, we can write the equation to indicate the significance of each term for a particular type of flow (7):

$$S_f = S_0 - \frac{\partial y}{\partial x} - \frac{V}{g}\frac{\partial V}{\partial x} - \frac{1}{g}\frac{\partial V}{\partial t} \quad (12\text{-}24)$$

Steady, uniform ⟶

Steady, nonuniform ⟶

Unsteady, nonuniform ⟶

12-3 Methods of Solution

Equations (12-9) and (12-23) are a set of hyperbolic, partial differential equations. Because of the presence of nonlinear terms, a closed-form solution of these equations is not available except for very simplified cases. Therefore, numerical methods are used for their solution (1, 2, 6, 10). There are several different types of these methods, notably, *method of characteristics*, *finite-difference methods*, *finite-element method*, and *spectral method*. The method of characteristics and various finite-difference methods have been used more extensively than the other methods for the analysis of unsteady free-surface flows; therefore, we discuss only these methods.

12-4 Method of Characteristics

This method, popular in the 1960s for the analysis of unsteady flows in open channels, has been replaced by various finite-difference schemes. It is, however, still being used in the explicit finite-difference schemes to simulate the boundary nodes (3). Details of this method follow.

Multiplying Eq. (12-9) by an unknown multiplier, λ, and adding to Eq. (12-23), we obtain

$$\left[\frac{\partial V}{\partial t} + (V + \lambda D)\frac{\partial V}{\partial x}\right] + \lambda\left[\frac{\partial y}{\partial t} + \left(V + \frac{g}{\lambda}\right)\frac{\partial y}{\partial x}\right] = g(S_0 - S_f) \quad (12\text{-}25)$$

Now, if we define the unknown multiplier so that

$$V + \lambda D = \frac{dx}{dt} = V + \frac{g}{\lambda} \tag{12-26a}$$

Since $D = A/B$, it follows from this equation that

$$\lambda = \pm\sqrt{\frac{gB}{A}} \tag{12-26b}$$

The celerity of a shallow-water wave in free-surface flows is given by the expression

$$c = \sqrt{\frac{gA}{B}} \tag{12-27}$$

Therefore, by defining $\lambda = g/c$, we can write Eq. (12-25) as

$$\frac{dx}{dt} = V + c \tag{12-28}$$

Using Eq. 12-28 and expressions for the total derivatives of V and y, we can write Eq. (12-25) as

$$\frac{dV}{dt} + \frac{g}{c}\frac{dy}{dt} = g(S_0 - S_f) \tag{12-29}$$

Similarly, by defining $\lambda = -g/c$, Eq. (12-25) may be written as

$$\frac{dx}{dt} = V - c \tag{12-30}$$

Then, using Eqs. (12-25) and (12-30), Eq. (12-24) becomes

$$\frac{dV}{dt} - \frac{g}{c}\frac{dy}{dt} = g(S_0 - S_f) \tag{12-31}$$

Equation (12-29) is valid if Eq. (12-28) is satisfied, and Eq. (12-31) is valid if Eq. (12-30) is satisfied. Equations (12-28) and (12-30) plot as characteristic curves (Fig. 12-2) in the x–t plane. Referring to this figure and noting the above conditions for the validity of Eqs. (12-29) and (12-31), we may say that Eq. (12-29) is valid along the positive characteristic curve, C^+, and Eq. (12-31) is valid along the negative characteristic curve, C^-.

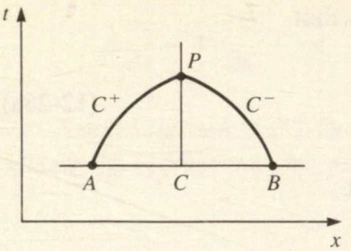

Figure 12-2 Characteristic curves

Multiplying Eqs. (12-29) and (12-31) by dt and integrating along the characteristic curves AP and BP, we obtain

$$\int_A^P dV + \int_A^P \frac{g}{c}\, dy = \int_A^P g(S_0 - S_f)\, dt \tag{12-32}$$

and

$$\int_B^P dV - \int_B^P \frac{g}{c}\, dy = \int_B^P g(S_0 - S_f)\, dt \tag{12-33}$$

In the preceding derivation of Eqs. (12-32) and (12-33), we have not made any approximation whatsoever. However, approximations become necessary to integrate the various terms, as we discuss in the following paragraph.

To determine the integrals of the second term on the left-hand side and the terms on the right-hand side of Eqs. (12-32) and (12-33), the variation of V and y along the characteristic curves should be known. However, V and y are the unknowns we want to compute. Therefore, we cannot directly evaluate the integrals of these terms, and some approximation has to be made for their evaluation. To do this, we may use the values of c and S_f computed by using the values of V and y at the known time level and assume these computed values of c and S_f remain unchanged from A to P and from B to P (Fig. 12-2). Then, we may write Eqs. (12-32) and (12-33) as

$$V_P - V_A + \left(\frac{g}{c}\right)_A (y_P - y_A) = g(S_0 - S_f)_A\, \Delta t \tag{12-34}$$

and

$$V_P - V_B - \left(\frac{g}{c}\right)_B (y_P - y_B) = g(S_0 - S_f)_B\, \Delta t \tag{12-35}$$

where subscripts P, A, and B refer to the grid points in the x–t plane. By combining the known quantities, Eqs. (12-34) and (12-35) may be written as

$$V_P = C_p - K_A y_P \tag{12-36}$$

and

$$V_P = C_n + K_B y_P \tag{12-37}$$

where $C_p = V_A + K_A y_A + g(S_0 - S_f)_A \Delta t$ (12-38)

$$C_n = V_B - K_B y_B + g(S_0 - S_f)_B \Delta t \tag{12-39}$$

$$K = \frac{g}{c} \tag{12-40}$$

12-5 Finite-Difference Methods

In the finite-difference methods, the channel is divided into a number of reaches, usually having equal length, Δx. The ends of each reach are called *computational nodes* or *grid points*. If the channel is divided into N reaches and the first node (upstream end) is numbered as 1, then the last node (downstream end) will be $N + 1$. The nodes at the upstream and downstream ends are called *boundary nodes*, and the remaining nodes are called *interior nodes*. Computations are performed at discrete times. The difference between two consecutive times is called *computational time interval* or *computational time step*. Thus, the x–t plane is divided into a grid (Fig. 12-3), usually referred to as the *computational grid* or *lattice*.

We replace the partial derivatives of the governing equations, Eqs. (12-9) and (12-23), with finite-difference approximations and then solve the resulting algebraic equations at each grid point or computational node. Referring to Fig. 12-3, let us say we know the flow velocity and flow depth at all grid points at time t_0, and we want to determine their values at time $t_0 + \Delta t$. The known values may be the initial conditions from which the unsteady flow conditions begin, or they may be computed during the previous time interval.

We will use a subscript to denote the grid point in the x direction and a superscript to denote the grid point in the t direction. For example, V_i^k refers to the flow velocity at the ith section and at the kth time level. We use superscript k for the time level at which flow conditions are known (referred to as the known time level) and $k + 1$ for the time level at which flow conditions are unknown (referred to as the unknown time level).

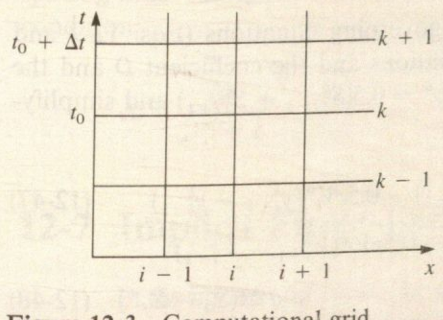

Figure 12-3 Computational grid

Depending on the type of finite-difference approximation, two different schemes are produced. If the finite-difference approximations for the spatial derivatives—that is, partial derivative with respect to x—are in terms of the quantities at the known time level, the resulting equations can be directly solved for each computational node one at a time. These methods are referred to as the *explicit methods.* In the *implicit methods,* the finite-difference approximation for the spatial derivatives are in terms of the unknown variables, and the algebraic equations for the entire system have to be solved simultaneously.

Details of each class of these methods are presented in the following sections.

12-6 Explicit Methods

Several explicit finite-difference methods have been used for unsteady free-surface flows. We will present details of one of these methods, the *Lax diffusive scheme* (3, 6, 8). It is simple to program and yields satisfactory results even when flows have bores.

In this scheme, the partial derivatives of the governing equations are replaced as follows:

$$\frac{\partial y}{\partial t} = \frac{y_i^{k+1} - y_i^*}{\Delta t} \tag{12-41}$$

$$\frac{\partial V}{\partial t} = \frac{V_i^{k+1} - V_i^*}{\Delta t} \tag{12-42}$$

$$\frac{\partial y}{\partial x} = \frac{y_{i+1}^k - y_{i-1}^k}{2\,\Delta x} \tag{12-43}$$

$$\frac{\partial V}{\partial x} = \frac{V_{i+1}^k - V_{i-1}^k}{2\,\Delta x} \tag{12-44}$$

where $y_i^* = 0.5(y_{i-1}^k + y_{i+1}^k)$ (12-45)

$$V_i^* = 0.5(V_{i-1}^k + V_{i+1}^k) \tag{12-46}$$

Replacing the partial derivatives of the governing equations (Eqs. 12-9 and 12-23) by these finite-difference approximations and the coefficient D and the slope S_f by $D^* = 0.5(D_{i-1}^k + D_{i+1}^k)$ and $S_f{}^* = 0.5(S_{fi-1}^k + S_{fi+1}^k)$ and simplifying, we obtain

$$y_i^{k+1} = y_i^* - 0.5rD_i^*(V_{i+1}^k - V_{i-1}^k) - 0.5rV_i^*(y_{i+1}^k - y_{i-1}^k) \tag{12-47}$$

$$V_i^{k+1} = V_i^* - 0.5rg(y_{i+1}^k - y_{i-1}^k) - 0.5rV_i^*(V_{i+1}^k - V_{i-1}^k) + g\,\Delta t(S_0 - S_f{}^*) \tag{12-48}$$

where $r = \Delta t/\Delta x$.

Equations (12-47) and (12-48) yield the values at the interior nodes only, that is, at $i = 2, 3, \ldots, N$. Boundary nodes need special treatment, as discussed in the following paragraphs.

Boundary Conditions

The characteristic equations along with the conditions imposed by the boundaries determine the *conditions at the boundary nodes.* The conditions imposed by the boundary may be in the form of specifying the variation of discharge or depth with respect to time, or a relationship between the discharge and depth. For example, for a constant-level reservoir at the upstream end (node 1), we specify

$$y_1^{k+1} = y_{\text{res}} \tag{12-49}$$

where entrance and the velocity head at node 1 are assumed to be small and neglected, and y_{res} is the flow depth in the reservoir above the channel bottom at node 1.

At the upstream end, the negative characteristic equation, Eq. (12-37), and at the downstream end, the positive characteristic equation, Eq. (12-36), are used.

Stability

It is usually necessary in the explicit finite-difference schemes that the ratio of Δx and Δt satisfies a condition for stability. A scheme is said to be *stable* if an error introduced in the solution does not grow as the computations progress in time, and the scheme is said to be *unstable* if the error is amplified with time. In the case of an unstable scheme, the error is amplified very rapidly and masks the true solution in a few time intervals.

For the Lax scheme to be stable, the computational time step and the grid spacing must satisfy the following condition, referred to as Courant's stability condition:

$$\Delta t \leqslant \frac{\Delta x}{|V| + c} \tag{12-50}$$

12-7 Implicit Finite-Difference Methods

In the implicit methods, the spatial derivatives are replaced by the finite-difference approximations in terms of the variables at the unknown time level. Depending on the finite-difference approximations and the coefficients used, several different formulations are possible. Of these schemes, the *Preissmann scheme* has been used extensively for the analysis of unsteady free-surface flows.

In the Preissmann scheme, the partial derivatives and other variables are approximated as follows:

$$\frac{\partial f}{\partial t} = \frac{[(f_i^{k+1} + f_{i+1}^{k+1}) - (f_i^k + f_{i+1}^k)]}{2\,\Delta t} \tag{12-51}$$

$$\frac{\partial f}{\partial x} = \frac{\alpha[f_{i+1}^{k+1} - f_i^{k+1}]}{\Delta x} + \frac{(1-\alpha)[f_{i+1}^k - f_i^k]}{\Delta x} \tag{12-52}$$

$$f(x,t) = \frac{\alpha(f_{i+1}^{k+1} + f_i^{k+1})}{2} + \frac{(1-\alpha)(f_{i+1}^k + f_i^k)}{2} \tag{12-53}$$

where α = the weighting coefficient, $0.5 < \alpha \leqslant 1$.

Substituting these equations into Eqs. (12-9) and (12-23) and simplifying the resulting equations, we obtain

$$\begin{aligned} y_i^{k+1} + y_{i+1}^{k+1} &+ \frac{\Delta t}{\Delta x}[\alpha(D_{i+1}^{k+1} + D_i^{k+1}) + (1-\alpha)(D_{i+1}^k + D_i^k)] \\ &\cdot [\alpha(V_{i+1}^{k+1} - V_i^{k+1}) + (1-\alpha)(V_{i+1}^k + V_i^k)] + \frac{\Delta t}{\Delta x} \\ &\cdot [\alpha(V_{i+1}^{k+1} + V_i^{k+1}) + (1-\alpha)(V_{i+1}^k + V_i^k)] \\ &\cdot [\alpha(y_{i+1}^{k+1} - y_i^{k+1}) + (1-\alpha)(y_{i+1}^k - y_i^k)] \\ &= y_i^k + y_{i+1}^k \end{aligned} \tag{12-54}$$

$$\begin{aligned} V_i^{k+1} + V_{i+1}^{k+1} &+ 2g\frac{\Delta t}{\Delta x}[\alpha(y_{i+1}^{k+1} - y_i^{k+1}) + (1-\alpha)(y_{i+1}^k - y_i^k)] \\ &+ \frac{\Delta t}{\Delta x}[\alpha(V_{i+1}^{k+1} + V_i^{k+1}) + (1-\alpha)(V_{i+1}^k + V_i^k)] \\ &\cdot [\alpha(V_{i+1}^{k+1} - V_i^{k+1}) + (1-\alpha)(V_{i+1}^k - V_i^k)] \\ &= (V_i^k + V_{i+1}^k) + 2g\,\Delta t S_0 - g\,\Delta t \\ &\cdot [\alpha(S_{fi+1}^{k+1} + S_{fi}^{k+1}) + (1-\alpha)(S_{fi+1}^k + S_{fi}^k)] \end{aligned} \tag{12-55}$$

There are four unknowns in Eqs. (12-54) and (12-55). By writing similar equations for grid points $i = 1, 2, \ldots, N$, we will have a total of $2N$ equations. We cannot write Eqs. (12-54) and (12-55) for node $N + 1$, since we do not have node $N + 2$. However, there are two unknowns, V and y, per node. Therefore, we have $2(N + 1)$ unknowns, and to obtain a unique solution, we need two more equations. These equations are provided by the boundary conditions.

Boundary Conditions

Unlike the explicit methods, equations representing the conditions imposed by the boundary are directly included in the system of equations and not in combination with the characteristic equations. For example, for a constant-level reservoir at the upstream end, y will be constant at all times if the velocity head and entrance losses are neglected. Similarly, other conditions may be specified at the upstream and downstream ends.

The resulting system of equations are nonlinear algebraic equations. These may be solved by using the Newton-Raphson method.

Stability

The implicit methods are usually unconditionally stable. This means there is no restriction on the size of the grid spacing Δx and Δt for the stability of the numerical scheme. However, accuracy dictates that the computational time step nearly equal to that given by the Courant condition be used in the computations.

EXAMPLE 12-1 Analyze unsteady flow in a 1000-m long trapezoidal channel having bottom width of 20 m, side slope of 2 H to 1 V, the bottom slope, S_0 of 0.0001, and Manning's n of 0.013. Initial flow and flow depth in the channel are 110 m^3/s and 3.069 m, respectively. Unsteady flow is produced by instantaneously closing a downstream gate at $t = 0$.

SOLUTION Figure 12-4 lists the computer program based on the Lax scheme. To ensure that Courant's condition is always satisfied at each node, the time interval required to satisfy this condition was determined at the end of each time step. If necessary, computations are repeated with a reduced value of the time interval; the value of the interval is increased for the next cycle of computations if it is found to be too small as compared to that required for stability.

Figure 12-4, pages 623–627, shows the program input and output, and Fig. 12-5, page 627, shows the computed water levels. ■

Figure 12-4 Program listing: Computation of unsteady, free-surface flows

Program Listing

```
C
C     COMPUTATION OF UNSTEADY, FREE-SURFACE FLOWS BY LAX'S
C      DIFFUSIVE SCHEME
C
C     ******************** NOTATION **********************
C
C     A = FLOW AREA;
C     B = TOP WATER-SURFACE WIDTH;
C     BO = CHANNEL-BOTTOM WIDTH;
C     P = WETTED PERIMETER;
C     Q = DISCHARGE;
```

(continued)

```
C     S = CHANNEL-SIDE SLOPE, S HORIZONTAL : 1 VERTICAL;
C     SO = CHANNEL-BOTTOM SLOPE;
C     X = DISTANCE ALONG CHANNEL BOTTOM, POSITIVE IN THE
C         DOWNSTREAM DIRECTION;
C     Y = FLOW DEPTH;
C     YU = FLOW DEPTH AT UPSTREAM END.
C
      REAL L, MN,MN2
      DIMENSION Y(100),YP(100),V(100),VP(100)
      BT(YY)=BO+2.*S*YY
      AR(YY)=YY*(BO+S*YY)
      WP(YY)=BO+2.*YY*SQRT(1.+S*S)
      G=9.81
      READ (5,*) N,IPRINT,QO,YD,MN,BO,SO,S,L,TMAX
      WRITE(6,10) N,QO,YD,MN,BO,SO,S,L
 10   FORMAT(5X,'N =',I3,' QO =',F8.3,' M3/S',' YD =',F6.2,' M',
     1 2X, ' MN =',F6.3,' BO =',F6.2,' M'/5x,' SO =',F6.4,'S =',F8.4,
     2  ' L =',F8.2,' M')
C
C    STEADY-STATE CONDITIONS
C
     MN2=MN*MN
     NN=N+1
     A=AR(YD)
     VO=QO/A
     DO 30 I = 1,NN
     Y(I) = YD
     V(I)=VO
 30  CONTINUE
     YU= Y(1)
     B=BT(YD)
     C=SQRT(G*A/B)
     DX=L/N
     DT=DX/(VO+C)
     T=0.0
 35  K = 0
     WRITE(6,40) T
 40  FORMAT(/5X,'T =',F8.3,' S')
     WRITE(6,50) (Y(I),I=1,NN)
 50  FORMAT(6X,'Y =',(12F10.2))
     WRITE (6,60) (V(I),I=1,NN)
 60  FORMAT(6X,' V=',(12F10.3))
 70  T=T+DT
     K=K+1
     R=0.5*DT/DX
     IF (T.GT.TMAX) GO TO 160
C
C    UPSTREAM END
C
     YP(1) = YU
     AB=AR(Y(2))
     BB=BT(Y(2))
     CB=SQRT(G*BB/AB)
     RB=AB/WP(Y(2))
     SFB=(MN2*V(2)*V(2))/(RB**1.333)
     CN=V(2)-CB*Y(2)+G*(SO-SFB)*DT
     VP(1)=CN+CB*YP(1)
C
C    DOWNSTREAM END
C
     VP(NN)=0.
     AA=AR(Y(N))
     BA=BT(Y(N))
     CA=SQRT(G*BA/AA)
     RA=AA/WP(Y(N))
     SFA=(MN2*V(N)*V(N))/(RA**1.333)
     CP=V(N)+CA*Y(N)+G*(SO-SFA)*DT
     YP(NN)=CP/CA
C
C    INTERIOR NODES
C
     DO 80 I=2,N
     I1=I-1
     IP1=I+1
     AA=AR(Y(I1))
     PA=WP(Y(I1))
     RA=AA/PA
     SFA=(MN2*V(I1)*V(I1))/(RA**1.333)
     BA=BT(Y(I1))
     AB=AR(Y(IP1))
```

```
      BB=BT(Y(IP1))
      P=WP(Y(IP1))
      RB=AB/P
      SFB=(MN2*V(IP1)*V(IP1))/(RB**1.333)
      DM=0.5*(AA/BA + AB/BB)
      SFM=0.5*(SFA+SFB)
      VM=0.5*(V(I1)+V(IP1))
      YM=0.5*(Y(I1)+Y(IP1))
      VP(I)=VM-R*G*(Y(IP1)-Y(I1)) - R*VM*(V(IP1)-V(I1))
     1      + G*DT*(SO-SFM)
      YP(I)=YM-R*DM*(V(IP1)-V(I1))-R*VM*(Y(IP1)-Y(I1))
80    CONTINUE
C
C     CHECK FOR STABILITY
C
      DO 100 I=1,NN
      A=AR(YP(I))
      B=BT(YP(I))
      C=SQRT(G*A/B)
      DTN=DX/(ABS(VP(I))+C)
      IF (DTN.LE.DT) GO TO 110
      IF (DT.LT.0.75*DTN) DTNEW=1.15*DT
      IF (DTN.GE.0.75*DT) DTNEW=DT
100   CONTINUE
      GO TO 120
C
C     REDUCE DT FOR STABILITY AND RE-CALCULATE
C
110   T=T-DT
      DT=.9*DTN
      K=K-1
      GO TO 70
120   DT=DTNEW
      DO 130 I=1,NN
      V(I)=VP(I)
      Y(I)=YP(I)
130   CONTINUE
      IF (K.EQ.IPRINT) GO TO 35
      GO TO 70
160   STOP
      END
```

Input Data

```
10,1,110.,3.069,0.013,20.,0.0001,2.,1000.,400.
```

Program Output

```
N = 10 QO = 110.000 M3/S YD =  3.07 M   MN =  .013 BO = 20.00 M
 SO = .0001S =  2.0000 L = 1000.00 M

T =    .000 S
 Y =      3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07
  V=     1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371

T =  15.850 S
 Y =      3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.76
  V=     1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371     .000

T =  31.701 S
 Y =      3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.67     3.76
  V=     1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371     .231     .000

T =  47.551 S
 Y =      3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.58     3.67     3.81
  V=     1.371    1.371    1.371    1.371    1.371    1.371    1.371    1.371     .412     .231     .000

T =  63.401 S
 Y =      3.07     3.07     3.07     3.07     3.07     3.07     3.07     3.49     3.58     3.78     3.81
  V=     1.371    1.371    1.371    1.371    1.371    1.371    1.371     .573     .412     .049     .000

T =  79.252 S
 Y =      3.07     3.07     3.07     3.07     3.07     3.07     3.41     3.49     3.75     3.78     3.82
  V=     1.371    1.371    1.371    1.371    1.371    1.371     .714     .573     .110     .049     .000
```

(continued)

Figure 12-4
Program Output (continued)

```
T =  95.102 S
 Y =      3.07     3.07     3.07     3.07     3.07     3.35     3.41     3.70     3.75     3.81     3.82
  V=     1.371    1.371    1.371    1.371    1.371     .836     .714     .185     .110     .017     .000

T = 110.952 S
 Y =      3.07     3.07     3.07     3.07     3.29     3.35     3.66     3.70     3.79     3.81     3.83
  V=     1.371    1.371    1.371    1.371     .939     .836     .272     .185     .038     .017     .000

T = 126.802 S
 Y =      3.07     3.07     3.07     3.25     3.29     3.60     3.66     3.78     3.79     3.82     3.83
  V=     1.371    1.371    1.371    1.024     .939     .368     .272     .065     .038     .011     .000

T = 142.653 S
 Y =      3.07     3.07     3.21     3.25     3.55     3.60     3.76     3.78     3.81     3.82     3.83
  V=     1.371    1.371    1.095    1.024     .467     .368     .101     .065     .022     .011     .000

T = 158.503 S
 Y =      3.07     3.18     3.21     3.49     3.55     3.73     3.76     3.80     3.81     3.83     3.83
  V=     1.371    1.152    1.095     .568     .467     .145     .101     .035     .022     .009     .000

T = 174.353 S
 Y =      3.07     3.18     3.44     3.49     3.70     3.73     3.79     3.80     3.82     3.83     3.84
  V=      .936    1.152     .667     .568     .197     .145     .051     .035     .019     .009     .000

T = 190.204 S
 Y =      3.07     3.29     3.44     3.67     3.70     3.78     3.79     3.81     3.82     3.84     3.84
  V=      .936     .540     .667     .258     .197     .071     .051     .028     .019     .009     .000

T = 206.054 S
 Y =      3.07     3.29     3.53     3.67     3.77     3.78     3.81     3.81     3.83     3.84     3.85
  V=      .134     .540     .127     .258     .094     .071     .038     .028     .018     .009     .000

T = 221.904 S
 Y =      3.07     3.29     3.53     3.65     3.77     3.80     3.81     3.82     3.83     3.84     3.85
  V=      .134    -.210     .127    -.060     .094     .050     .038     .026     .018     .009     .000

T = 237.755 S
 Y =      3.07     3.29     3.44     3.65     3.70     3.80     3.82     3.82     3.84     3.84     3.86
  V=     -.629    -.210    -.395    -.060    -.104     .050     .035     .026     .017     .009     .000

T = 253.605 S
 Y =      3.07     3.22     3.44     3.51     3.70     3.73     3.82     3.83     3.84     3.85     3.86
  V=     -.629    -.781    -.395    -.428    -.104    -.109     .035     .026     .017     .009     .000

T = 269.455 S
 Y =      3.07     3.22     3.31     3.51     3.55     3.73     3.75     3.83     3.85     3.85     3.87
  V=    -1.070    -.781    -.800    -.428    -.414    -.109    -.108     .026     .017     .009     .000

T = 285.306 S
 Y =      3.07     3.15     3.31     3.36     3.55     3.58     3.75     3.77     3.85     3.86     3.87
  V=    -1.070   -1.091    -.800    -.766    -.414    -.392    -.108    -.106     .017     .009     .000

T = 301.156 S
 Y =      3.07     3.15     3.20     3.36     3.40     3.58     3.61     3.77     3.79     3.86     3.87
  V=    -1.244   -1.091   -1.057    -.766    -.725    -.392    -.371    -.106    -.104     .009     .000

T = 317.006 S
 Y =      3.07     3.11     3.20     3.24     3.40     3.44     3.61     3.64     3.79     3.81     3.87
  V=    -1.244   -1.232   -1.057   -1.012    -.725    -.686    -.371    -.352    -.104    -.103     .000

T = 332.857 S
 Y =      3.07     3.11     3.14     3.24     3.28     3.44     3.47     3.64     3.67     3.81     3.76
  V=    -1.309   -1.232   -1.202   -1.012    -.967    -.686    -.651    -.352    -.335    -.103     .000

T = 348.707 S
 Y =      3.07     3.09     3.14     3.18     3.28     3.32     3.47     3.51     3.67     3.64     3.76
  V=    -1.309   -1.301   -1.202   -1.167    -.967    -.925    -.651    -.619    -.335    -.217     .000

T = 364.557 S
 Y =      3.07     3.09     3.12     3.18     3.21     3.32     3.36     3.51     3.49     3.64     3.53
  V=    -1.346   -1.301   -1.285   -1.167   -1.132    -.925    -.886    -.619    -.490    -.217     .000
```

```
T = 380.408 S
 Y =      3.07     3.09     3.12     3.14     3.21     3.25     3.36     3.35     3.49     3.40     3.53
 V=     -1.346   -1.345   -1.285   -1.264   -1.132   -1.097    -.886    -.752    -.490    -.253     .000

T = 394.657 S
 Y =      3.07     3.09     3.11     3.15     3.17     3.25     3.24     3.35     3.28     3.41     3.27
 V=     -1.378   -1.342   -1.336   -1.259   -1.235   -1.088    -.962    -.745    -.511    -.252     .000
```

12-8 Comparison of Explicit and Implicit Methods

The explicit and implicit finite-difference methods have their advantages and disadvantages and none of them is suitable for universal applications. We will discuss the advantages and disadvantages of these methods in this section so that they may be taken into consideration during their selection for a particular application.

The *implicit methods* are usually unconditionally stable whereas the Courant condition has to be satisfied for the explicit methods. This condition restricts the size of the computational time interval, thereby making the method uneconomical for the analysis of unsteady flows in a large system for long durations.

The *explicit methods* are easier to program and debug than the implicit methods. Therefore if the available time for the development of a computer program is short, then explicit methods should be selected.

The computer storage requirements for the implicit methods are usually more than those for the explicit methods. If a large system has to be analyzed and the computer memorary is limited, then there may be no choice except to use the explicit method.

It might be necessary to use small time steps if the flow variables to be analyzed have sharp peaks. In such a case explicit methods are superior because the computational effort per time step is more for the implicit methods than that for the explicit methods.

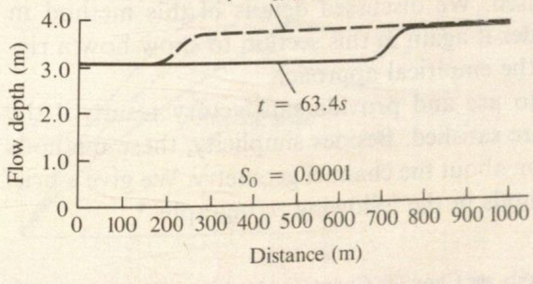

Figure 12-5 Flow depth along channel

12-9 Approximate Methods for Flood Routing

The term *flood routing* refers to the computation of the speed, height, and discharge of a flood wave as it propagates in waterways, such as rivers, lakes, reservoirs, and streams. For the routing of a flood wave, two numerical methods (presented in the preceding sections) that solve the complete continuity and momentum equations may be used. This is referred to as *hydraulic routing*. The mathematical models based on the numerical solution of complete equations are called *dynamic models*, and the waves computed by these models are called *dynamic waves*. In several situations, the relative effects of some terms of the governing equations are very small as compared with the other terms and may be neglected. For example, Henderson (7) gave the following values for different terms of Eq. (12-24), page 616, for a very fast-rising flood in a river in a steep, alluvial country:

Term	S_0	$\dfrac{\partial y}{\partial x}$	$\dfrac{V}{g}\dfrac{\partial V}{\partial x}$	$\dfrac{1}{g}\dfrac{\partial V}{\partial t}$
Value (ft/mi)	26	0.5	0.125–0.25	0.05

Therefore, the inertial terms in this case may be neglected without introducing large errors. Approximate procedures in which the continuity equation is solved simultaneously with a simplified form of the dynamic equation are called *hydrologic routing*. There are several of these procedures depending on the terms retained in the simplified form of the dynamic equation and the computational procedure used for solving the governing equations. For example, in a kinematic wave model, the steady uniform flow equation is used for the momentum equation; whereas, an additional term representing the slope of the water surface is included in the diffusion models. By approximating the complex relationships between storage capacity of a reach, inflow, outflow, stage, and so on, several procedures known as coefficient methods have been developed. Of these procedures, one developed by the U.S. Army Corps of Engineers on the Muskingum River in 1934 has been widely used. We discussed details of this method in Sect. 4-5, page 223, and we consider it again in this section to show how a rigorous derivation compares with the empirical approach.

Simplified models are easy to use and provide satisfactory results if the assumptions they are based on are satisfied. Besides simplicity, these methods do not require detailed information about the channel geometry. We give a brief description of some of these methods in the following paragraphs.*

* For a detailed description of these methods, see Chow (4), Cunge (5), Henderson (7), Mahmood (8), Ponce (9), and Weinmann (11).

A general form of the resistance formulas, such as Chezy or Manning, may be written as

$$Q = C'AR^m\sqrt{S_f} \tag{12-56}$$

where C' is an empirical constant, R is the hydraulic radius, m is an empirical exponent, and S_f is the slope of the energy grade line. In Manning's formula (in SI units), $C' = 1/n$ and $m = 2/3$; whereas, in Chezy's formula, $C' = C$, and $m = 1/2$. For steady-uniform flows, Q is the normal discharge, Q_n, and $S_f = S_0$. Hence, for steady-uniform flows, Eq. (12-56) becomes

$$Q_n = C'AR^m\sqrt{S_0} \tag{12-57}$$

Eliminating $C'AR^m$ from Eqs. (12-56) and (12-57), we obtain

$$Q = \frac{Q_n}{\sqrt{S_0}}\sqrt{S_f} \tag{12-58}$$

Now, substitution of expression for S_f from Eq. (12-24) into this equation yields

$$Q = Q_n\sqrt{1 - \frac{1}{S_0}\frac{\partial y}{\partial x} - \frac{V}{gS_0}\frac{\partial V}{\partial x} - \frac{1}{gS_0}\frac{\partial V}{\partial t}} \tag{12-59}$$

Kinematic →

Diffusion →

Dynamic →

In this equation, terms that are included in different types of models are marked.

Figure 12-6 shows a plot of typical relationship between Q and y, as given by Eq. (12-59). For steady-uniform flows, there is a unique, single-valued relationship between Q and y, shown by the dashed line in this figure. Simply, this

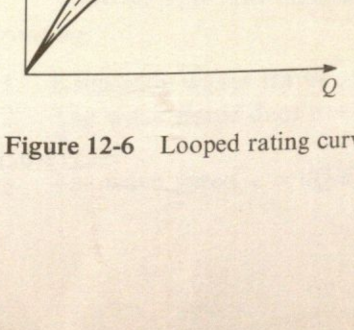

Figure 12-6 Looped rating curve

means only one value of normal discharge is possible for a given y. Closely examining Eq. (12-59) shows that the discharge for a specified y at a channel section during a rising stage will be more than the normal discharge for that y because $\partial y/\partial x$ and $\partial V/\partial x$ are negative during a rising stage, and the last term is usually small. Following a similar reasoning, discharge for a specified flow depth during a falling stage is less than the corresponding normal discharge, Q_n. This difference between the flows corresponding to a given flow depth during rising and falling stages is due to the effects of the above three terms (Kinematic, Diffusion, and Dynamic). In other words, the larger the hysteresis effect between the rising and the falling stages at a channel section, the more important it will be to include these terms in the analysis.

We will now briefly discuss three commonly used approximate methods for flood routing.

Kinematic Models

As we mentioned, kinematic models are based on the solution of the continuity equation and the steady-uniform equation for the dynamic equation. The waves propagated using these models are called *kinematic waves*, and routing is called *kinematic routing*. To study the properties of kinematic waves, we consider the solution of the governing equations.

The continuity equation, Eq. (12-6), page 613, may be written as

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{12-60}$$

where Q = discharge

For the momentum equation, we use the equation describing steady-uniform flows, $Q = Q_n$. Since Q_n is a single-valued function of flow depth y, and the flow area is a function of flow depth, we can write

$$Q = Q(A)$$

or

$$A = A(Q) \tag{12-61}$$

Because both A and Q are functions of x and t, we can write the following equation by applying the chain rule:

$$\begin{aligned}\frac{\partial A}{\partial t} &= \frac{\partial A}{\partial Q}\frac{\partial Q}{\partial t}\\ &= \frac{\partial Q}{\partial t}\frac{dA}{dQ}\end{aligned} \tag{12-62}$$

Substituting Eq. (12-62) into Eq. (12-60) and multiplying throughout by $\dfrac{dQ}{dA}$ yield

$$\frac{\partial Q}{\partial t} + \frac{dQ}{dA}\frac{\partial Q}{\partial x} = 0 \tag{12-63}$$

or

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = 0 \tag{12-64}$$

where $c = dQ/dA$

From the following discussion, we will see that c describes the speed of a kinematic wave and is called kinematic wave speed.

Eq. (12-64) is a first-order partial differential equation with Q as the dependent variable and x and t as the independent variables. For a general solution of Eq. (12-64), let us try the function

$$Q = f(x - ct) \tag{12-65}$$

and assume that the partial derivatives of f with respect to x and t exist and that c is a constant. Then,

$$\begin{aligned} \frac{\partial f}{\partial x} &= f'(x - ct) \\ \frac{\partial f}{\partial t} &= -cf'(x - ct) \end{aligned} \tag{12-66}$$

Substituting Eqs. (12-66) into Eq. (12-64) shows that the function $Q = f(x - ct)$ is a general solution of Eq. (12-64). This is called D'Alembert's solution (12); it is useful in understanding the characteristics of the kinematic waves.

At time $t = 0$, the general solution of Eq. (12-64), $Q = f(x - ct)$, defines the curve $Q = f(x)$. This curve describes the initial condition (Fig. 12-7, page 632). At any later time t_1, the solution defines the curve $Q = f(x - ct_1)$. These two curves are identical in shape except that the curve representing the solution at time t_1 is translated to the right a distance equal to ct_1. That is, the entire curve moves in the positive x direction without distortion a distance equal to ct_1 during time t_1. Therefore, the velocity of the wave is $ct_1/t_1 = c$.

Thus we can now summarize the following three *properties of kinematic routing*:

1. Kinematic waves travel only in the positive x direction.
2. The wave shape does not change, and there is no attenuation of the wave height.
3. The wave speed, $c = dQ/dA$

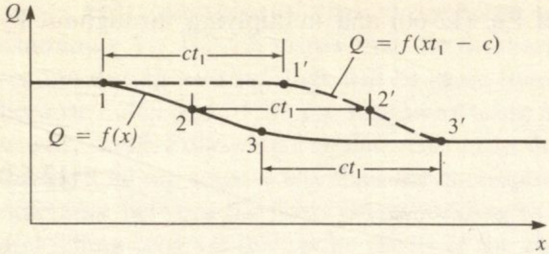

Figure 12-7 Propogation of kinematic wave

Kinematic wave models are based on an analytical solution or on a numerical solution of Eqs. (12-64) and (12-65). The value of c may be determined from the observed flood hydrographs, or it may be computed from the slope of the rating curve for a given section, that is, $c = dQ/dA = (1/B)\, dQ/dy$, where B = the width of the water surface at depth y.

Some attenuation of the flood waves has reportedly been computed by certain kinematic models. This attenuation either is caused by dispersion introduced by the numerical scheme or is due to the value of Courant number, $C_N = c\,\Delta t/\Delta x$, being different from 1 in the computations.

Diffusion Routing

In the diffusion routing, the following simplified equation is used for the momentum equation:

$$S_f = S_0 - \frac{\partial y}{\partial x} \tag{12-67}$$

Replacing S_f in terms of conveyance factor, K, this equation may be written as

$$\frac{Q^2}{K^2} = S_0 - \frac{\partial y}{\partial x} \tag{12-68}$$

Differentiating Eq. (12-68) with respect to t yields

$$\frac{2Q}{K^2}\frac{\partial Q}{\partial t} - \frac{2Q^2}{K^3}\frac{\partial K}{\partial t} = -\frac{\partial^2 y}{\partial t\,\partial x} \tag{12-69}$$

By differentiating Eq. (12-60) with respect to x and noting that

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial y}\frac{\partial y}{\partial x} = B\frac{\partial y}{\partial x}$$

we obtain
$$B\frac{\partial^2 y}{\partial x\,\partial t}+\frac{\partial^2 Q}{\partial x^2}=0 \tag{12-70}$$

Dividing this equation by B and subtracting from Eq. (12-69) yields

$$\frac{2Q}{K^2}\frac{\partial Q}{\partial t}-\frac{2Q^2}{K^3}\frac{\partial K}{\partial t}=\frac{1}{B}\frac{\partial^2 Q}{\partial x^2} \tag{12-71}$$

Now,
$$\frac{\partial K}{\partial t}=\frac{\partial K}{\partial A}\frac{\partial A}{\partial t} \tag{12-72}$$

Substitution for $\partial A/\partial t$ from Eq. (12-60) into this equation yields

$$\frac{\partial K}{\partial t}=\frac{dK}{dA}\left(-\frac{\partial Q}{\partial x}\right) \tag{12-73}$$

To determine the expression for dK/dA, assume $Q = K\sqrt{S_0}$ instead of $Q = K\sqrt{S_f}$. Then, $dQ/dA = dK/dA\sqrt{S_0}$. Substituting this relationship into Eq. (12-73), we obtain

$$\frac{\partial K}{\partial t}=\frac{1}{\sqrt{S_0}}\frac{dQ}{dA}\left(-\frac{\partial Q}{\partial x}\right) \tag{12-74}$$

Eliminating $\partial K/\partial t$ from Eqs. (12.71) and (12.74) gives

$$\frac{2BQ}{K^2}\frac{\partial Q}{\partial t}+\frac{2BQ^2}{K^3\sqrt{S_0}}\frac{dQ}{dA}\frac{\partial Q}{\partial x}=\frac{\partial^2 Q}{\partial x^2}$$

Multiplying throughout by $K^2/(2BQ)$, noting that $Q = K\sqrt{S_0}$, and simplifying the resulting equation, we obtain

$$\frac{\partial Q}{\partial t}+\frac{dQ}{dA}\frac{\partial Q}{\partial x}=\frac{Q}{2BS_0}\frac{\partial^2 Q}{\partial x^2} \tag{12-75}$$

Comparing this equation with the equation describing a kinematic wave (Eq. 12-64) shows that, except for the term on the right-hand side of Eq. (12-75), both equations are identical. The diffusion wave travels at the same speed, $c = dQ/dA$, as the kinematic wave. However, the additional term results in the diffusion of a wave. We can rewrite Eq. (12-75) compactly as

$$\frac{\partial Q}{\partial t}+c\frac{\partial Q}{\partial x}=D\frac{\partial^2 Q}{\partial x^2} \tag{12-76}$$

where $D = Q/(2BS_0)$.

D is called the diffusion coefficient that simulates the attenuation of the wave as it propagates in the channel. If D and C are determined from the observed flood hydrographs in a channel, they account for the effects of channel storage and other factors on the movement of flood waves.

Muskingum-Cunge Method

In Sec. 4-5, page 223, we presented details of the Muskingum method in which the coefficients for the method were determined from the observed flood records. In this section, we show that this method is actually a particular finite-difference approximation of the kinematic wave equations and present expressions for the Muskingum coefficients in terms of the physical properties of the channel.

Referring to Fig. 12-8 the spatial grid points i and $i + 1$ refer to the ends of a channel reach, and the temporal grid points k and $k + 1$ refer to the beginning and the end of the routing interval. Let us use the following finite-difference approximations for the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{(Q_{i+1}^{k} + Q_{i+1}^{k+1}) - (Q_i^{k} + Q_i^{k+1})}{2\,\Delta x} \tag{12-77}$$

$$\frac{\partial Q}{\partial t} = \frac{\alpha(Q_i^{k+1} - Q_i^{k}) + (1-\alpha)(Q_{i+1}^{k+1} - Q_{i+1}^{k})}{\Delta t} \tag{12-78}$$

where α is the weighting coefficient used for the temporal partial derivative. By substituting Eqs. (12-77) and (12-78) into Eq. (12-64) and simplifying the resulting equation, we obtain

$$\begin{aligned}&[(1-\alpha) + 0.5cr]Q_{i+1}^{k+1} + (\alpha - 0.5cr)Q_i^{k+1}\\ &\quad + (-\alpha - 0.5cr)Q_i^{k} + [-(1-\alpha) + 0.5cr]Q_{i+1}^{k} = 0\end{aligned} \tag{12-79}$$

where $r = \Delta t/\Delta x$

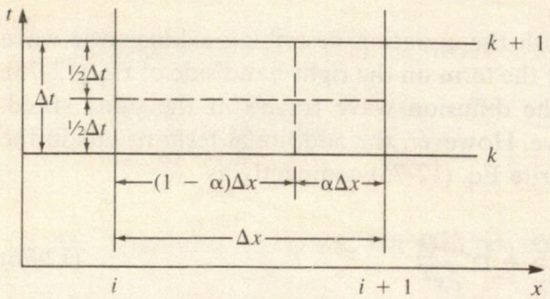

Figure 12-8 Definition sketch

Therefore, in terms of the notation of Fig. 12-8 and the terminology of Sec. 4-4, $Q_{i+1}^{k+1} = O_{i+1}$, $Q_{i+1}^{k} = O_i$, $Q_i^k = I_i$, and $Q_i^{k+1} = I_{i+1}$. Then we can write Eq. (12-79) as

$$Q_{i+1}^{k+1} = C_0 Q_i^{k+1} + C_1 Q_i^k + C_2 Q_{i+1}^k \tag{12-80}$$

where
$$C_0 = \frac{0.5\,\Delta t - \alpha\,\Delta x/c}{0.5\,\Delta t + (1-\alpha)\,\Delta x/c}$$

$$C_1 = \frac{0.5\,\Delta t + (\alpha\,\Delta x/c)}{0.5\,\Delta t + (1-\alpha)\,\Delta x/c}$$

$$C_2 = \frac{-0.5\,\Delta t + (1-\alpha)\,\Delta x/c}{0.5\,\Delta t + (1-\alpha)\,\Delta x/c} \tag{12-81}$$

Comparing Eqs. (12-81) with Eqs. (4-58) through (4-60) shows that the expressions for the Muskingum coefficients are identical if we assume $\alpha = X$ and $\Delta x/c = K$. Thus, K is the travel time for the flood wave in the reach. By expanding Eq. (12-80) in a Taylor series and comparing it with the diffusion equation, Eq. (12-76), the following expression may be obtained for X in terms of the physical properties of the channel:

$$X = \frac{1 - Q_0}{2S_0 Bc\,\Delta x} \tag{12-82}$$

We can show that the Muskingum routing becomes equivalent to kinematic routing for certain values of α and $c\,\Delta t/\Delta x$. For example, let us substitute $\alpha = 0.5$ and $c\,\Delta t/\Delta x = 1$ into Eqs. (12-81). This process gives $C_0 = 0$, $C_1 = 1$, and $C_2 = 0$. Substituting these values of Muskingum coefficients into Eq. (12-80) yields

$$Q_{i+1}^{k+1} = Q_i^k \tag{12-83}$$

This equation shows us that the flood wave is not attenuated as it propagates through the channel reach under consideration; that is, we get the same result that we get for kinematic routing.

PROBLEMS

12-1 Develop the following boundary conditions for the Lax scheme:

a. Constant-level upstream reservoir
b. Constant-level downstream reservoir
c. Rating curve for the downstream end

Assume the entrance losses at the reservoir are small, and neglect the velocity head.

12-2 Write a computer program to analyze unsteady flows in a channel 5000 m long having a channel bottom slope of 0.0001 and Manning's $n = 0.025$. The channel cross section is trapezoidal in shape with bottom width of 10 m and side slopes of 1 vertical to 1.5 horizontal. The initial flow conditions are uniform at a flow depth of 3 m. There is a constant-level reservoir at the downstream end, and the flow velocity at the upstream end is instantaneously reduced to zero at $t = 0$. Use the Lax scheme.

12-3 Compute the celerity of a wave in a circular tunnel having a flow depth of 6 m and the tunnel diameter of 8 m.

12-4 A 10-m wide rectangular channel has a flow depth of 5 m and flow velocity of 4 m/s. Determine the absolute velocity of a surge wave produced by instantaneously closing a control gate located at the
a. Downstream end
b. Upstream end
c. Midlength of the channel

12-5 Develop a computer program for the analysis of unsteady flow in a channel having a constant-level reservoir at the upstream end and a slowly closing gate at the downstream end. Use the Preissmann scheme. Run this program for the channel of Prob. 12-2 assuming there is a constant-level reservoir at the upstream end and a control gate at the downstream end. Assume the downstream gate is closed in 45 s.

12-6 Run the programs of Probs. 12-2 and 12-5 using different values of time interval, and compare the computed results.

12-7 Using the computer program of Prob. 12-2, show that the computations become unstable if the Courant's stability condition is not satisfied. (*Hint*: Use a computational time interval larger than that required by the Courant condition.)

12-8 Lax scheme becomes unstable if the time derivative is replaced by $\partial f/\partial t = [f_i^{k+1} - f_i^k]/\Delta t$ instead of that given by Eqs. (12-41) and (12-42), page 620, (f stands for both y and V) even if the Courant condition is satisfied. By modifying the program of Prob. 12-2, prove that this is the case.

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