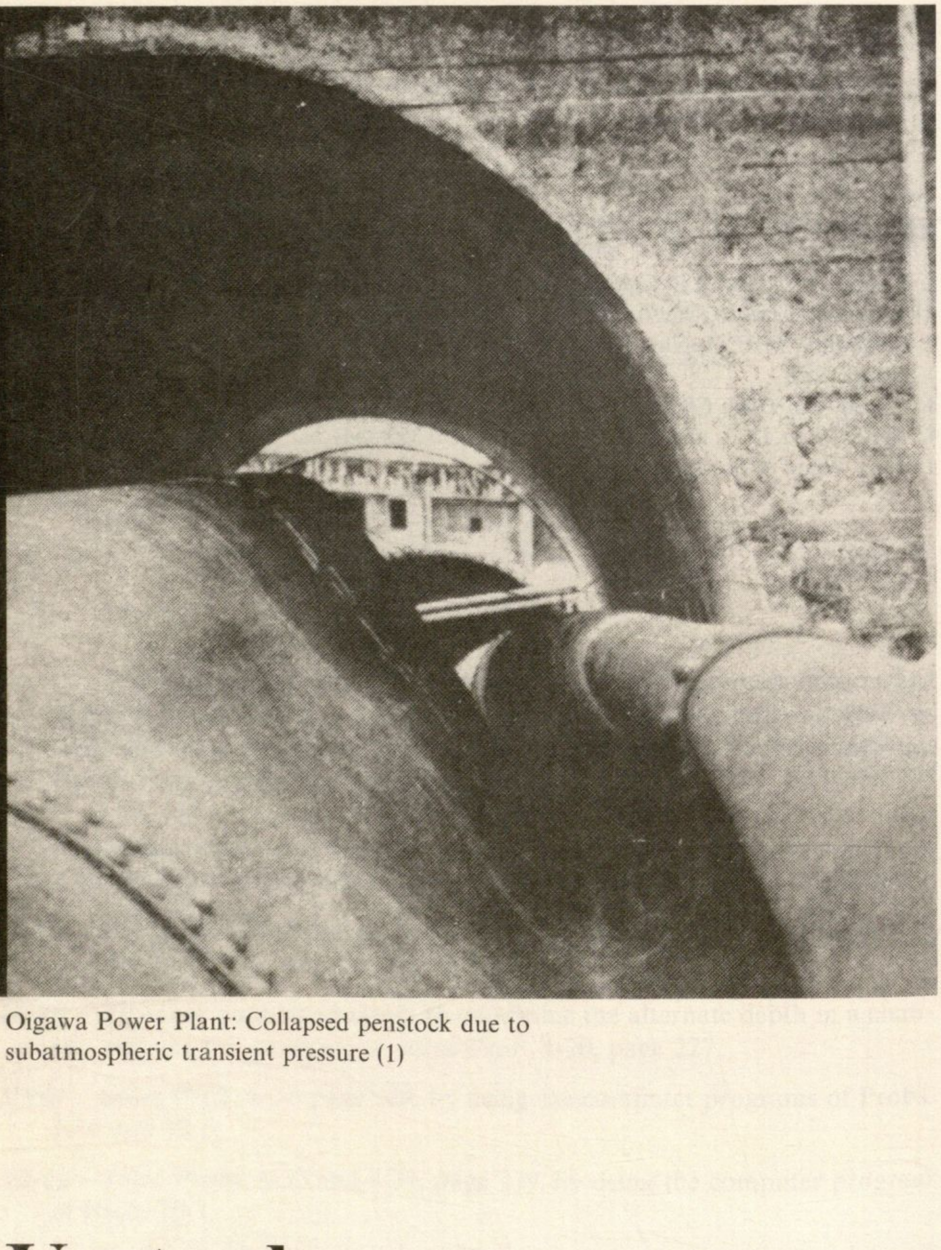

11

Oigawa Power Plant: Collapsed penstock due to subatmospheric transient pressure (1)

Unsteady Closed-Conduit Flows

Spectacular accidents have occurred because of transient-state pressures (intermediate flow while changing from one steady state to another is called transient flow) exceeding the design pressure of a conduit (1, 2, 4, 7, 9). These accidents, which are due to design or operating errors or equipment malfunction, have resulted in loss of life and money. Photos of damage due to transient-state pressures are shown in Fig. 11-1.

In this chapter, we discuss what transient flows are, how they are produced, how they are analyzed, and the different methods available for their control. Several commonly used terms are first defined. Expressions are derived for the wave speed and for the pressure rise in a closed conduit caused by an instantaneous change in flow velocity. Equations describing the unsteady flows are developed and the method of characteristics for their numerical integration is then presented. Transients caused by pumps are then discussed. The chapter concludes with a discussion of various methods for keeping the transient conditions within prescribed limits.

11-1 Definitions

As we discussed previously, a flow is *steady* if the flow velocity at a given location does not vary with time. If the flow velocity at a point does vary with time, the flow is *unsteady.*

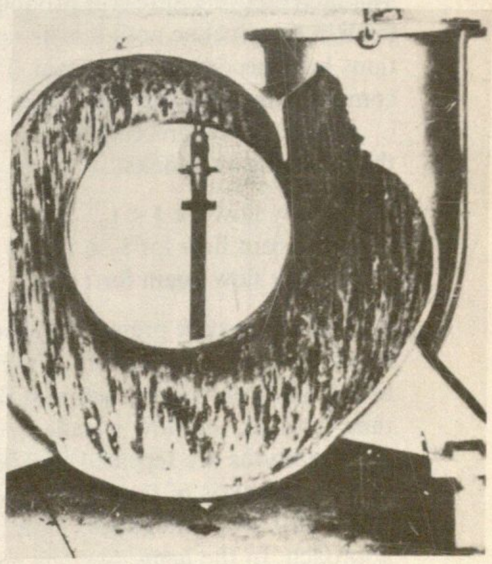

Figure 11-1 Damage caused by transient-state pressures: (left) Burst pipe due to high transient pressures (1) (right) Failed pump casing due to high transient pressures. (Courtesy of A.B. Almeida)

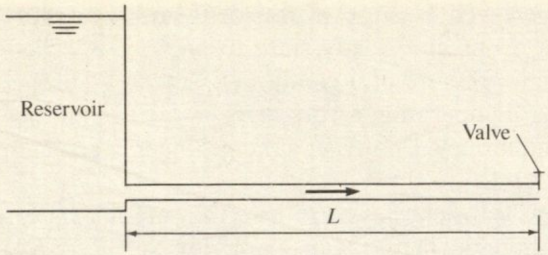

Figure 11-2 Piping system

When the flow conditions are changed from one steady state to another, the intermediate-stage flow is referred to as *transient flow* (2, 11). In the past, the terms *water hammer* and *oil hammer* were used for transient flows; at present the terms *hydraulic transients* and *fluid transients* are more commonly used.

The following discussion of different flow conditions in a piping system will help you understand the preceding definitions.

Let us consider a pipeline of length L in which water is flowing from a constant-level, upstream reservoir to a valve located at the downstream end, as shown in Fig. 11-2. Assume that the valve is instantaneously closed at time $t = t_0$ from the full-open position to the half-open position. This will reduce the flow velocity through the valve, thereby increasing the pressure at the valve. The increased pressure will produce a pressure wave that will travel back and forth in the pipeline until it is dissipated because of friction and the flow conditions have become steady again. This time, when the flow conditions have become steady again, we will call t_f.

Based on the preceding definitions, we may classify these flow regimes into the following categories:

1. Steady flow for $t < t_0$
2. Transient flow for $t_0 \leqslant t < t_f$
3. Steady flow again for $t \geqslant t_f$

Transient-state pressures are sometimes reduced to the vapor pressure of a liquid that results in separating the liquid column at that section; this is referred to as *liquid-column separation*. If the flow conditions are repeated after a fixed time interval, the flow is called *periodic flow*, and the time interval at which the conditions are repeated is called *period*.

The analysis of transient-state conditions in closed conduits may be classified into two categories: lumped-system approach and distributed-system approach. In the *lumped-system approach*, the conduit walls are assumed rigid, and the liquid in the conduit is assumed incompressible, that is, it behaves like a rigid mass so that the flow velocity at any given instant of time is the same from one end of the conduit to the other. In other words, the flow variables are functions of time only. Therefore, ordinary differential equations describe the system behavior. The flow velocity in each conduit may be considered in-

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dividually in a multiconduit system. In the *distributed-system approach,* the liquid is assumed to be slightly compressible. Therefore, the flow velocity may vary along the length of the conduit in addition to the variation in time. That is, the flow variables are now functions of not only time but also of distance. Partial differential equations therefore describe the system behavior. If the rate of change of flow velocity is slow, a lumped-system approach yields acceptable results; for rapid changes, however, a distributed-system approach must be used. The distributed-system approach is somewhat more complex than the lumped-system approach. In Sec. 11-2, we apply the lumped-system approach to derive an expression for the time required to establish flow in a conduit, and in Sec. 11-13 we apply this approach for the analysis of water level oscillations in a surge tank. In the remainder of this chapter, we deal with the analyses based on the distributed-system approach.

11-2 Time for Flow Establishment in a Pipe

Let us consider the piping system shown in Fig. 11-3, in which the valve is fully opened from the fully closed position at time $t = 0$. As a result, the pressure force acting on the liquid in the pipe is greater at the reservoir end than that at the valve end. Therefore, liquid in the pipeline begins to accelerate. The flow velocity will keep on increasing until the unbalanced pressure force is equal to the frictional resistance in the pipe. We are interested in determining the time when the flow is fully established, that is, when the flow becomes steady. To do this, we will apply the momentum equation to the control volume shown in Fig. 11-3.

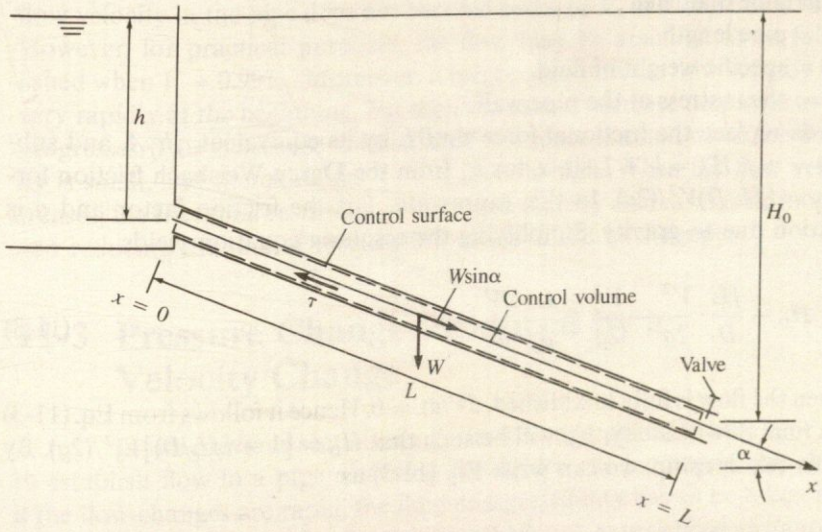


Figure 11-3 Control volume for flow establishment

Assume that the control volume is fixed and does not change shape. Let the x axis be along the pipe axis, and let the positive flow direction be from the reservoir to the valve. The one-dimensional form of momentum equation for a control volume that is fixed in space and does not change shape may be written as (6)

$$\sum F = \frac{d}{dt}\int_{x_1}^{x_2} \rho VA\,dx + (\rho AV^2)_{\text{out}} - (\rho AV^2)_{\text{in}} \tag{11-1}$$

where $\sum F$ is the sum of all forces acting on the system in the x direction, V is the flow velocity in the x direction, ρ is the mass density of the fluid, A is the flow area, and the subscripts in and out refer to the inflow and outflow quantities from the control volume.

If the liquid is assumed incompressible and the pipe is rigid, then at any instant, the velocity along the pipe length from $x = 0$ to $x = L$ will be the same. Since the flow velocity, flow area, and mass density of the liquid are assumed the same along the pipe length $(\rho AV^2)_{\text{in}} = (\rho AV^2)_{\text{out}}$. The term on the left-hand side is the sum of all the forces acting in the x direction on the system within the control volume. Substituting expressions for these forces and for the first term on the right-hand side, and noting that the end sections of the control volume are fixed, we obtain

$$pA + \gamma AL\sin\alpha - \tau_0\pi DL = \frac{d}{dt}(V\rho AL) \tag{11-2}$$

where $p = \gamma\left(h - \dfrac{V^2}{2g}\right)$ (if the entrance losses are neglected)

α = pipe slope
D = pipe diameter
L = pipe length
γ = specific weight of fluid
τ_0 = shear stress at the pipe wall

Let us replace the frictional force $\tau_0\pi DL$ by its equivalent, $\gamma h_f A$, and substitute $\rho = \gamma/g$, $H_0 = h + L\sin\alpha$, for h_f from the Darcy-Weisbach friction formula, $h_f = (fL/D)V^2/(2g)$. In this expression, f is the friction factor and g is acceleration due to gravity. Simplifying the resulting equation yields

$$H_0 - \frac{fL}{D}\cdot\frac{V^2}{2g} - \frac{V^2}{2g} = \frac{L}{g}\cdot\frac{dV}{dt} \tag{11-3}$$

When the flow is fully established, $dV/dt = 0$. Hence it follows from Eq. (11-3) that the final flow velocity, V_0, will be such that $H_0 = [1 + (fL/D)]V_0^2/(2g)$. By using this relationship, we can write Eq. (11-3) as

$$dt = \frac{2LD}{D + fL}\cdot\frac{dV}{V_0^2 - V^2}$$

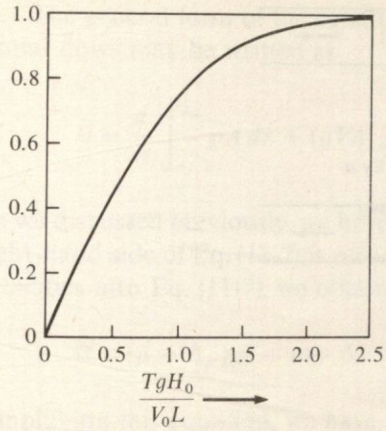


Figure 11-4 Velocity-time relation for flow establishment

Integrating both sides, noting that $V = 0$ at $t = 0$, and designating the time for flow establishment as T, we obtain

$$T = \frac{LV_0}{2gH_0} \ln \left[\frac{1 + \dfrac{V}{V_0}}{1 - \dfrac{V}{V_0}} \right] \tag{11-4}$$

A nondimensional plot of this equation is shown in Fig. 11-4. It is clear from this figure and from Eq. (11-4) that V tends to V_0 asymptotically. That is, flow velocity in the pipe does not become equal to V_0 in a finite interval of time. However, for practical purposes, the flow may be assumed to be fully established when $V = 0.99V_0$. Moreover, it can be seen that the flow velocity increases very rapidly at the beginning, but then the rate of increase decreases as the time progresses. This is because the frictional resistance is small when the flow velocity is small, and therefore, the motive force is high. When the flow velocity increases, the frictional resistance is increased thereby decreasing the motive force and reducing the rate of acceleration of the liquid in the pipe.

11-3 Pressure Change Produced by a Velocity Change

In the previous section, we derived an expression for the time required to establish flow in a pipe assuming the fluid to be incompressible. However, if the flow changes are rapid, the fluid compressibility has to be taken into consideration. As a result, the flow changes are not experienced instantaneously throughout the system; rather pressure waves move back and forth in the piping

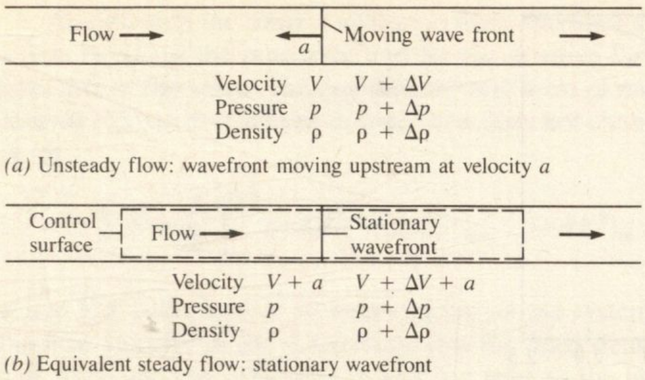


Figure 11-5 Definition sketch

system. In this section, we will derive an expression for the change in pressure produced by an instantaneous change in the flow velocity by assuming the pipe walls to be rigid and the liquid to be slightly compressible. The walls are rigid if the pipe diameter does not increase or decrease with a change in the pressure and the liquid is slightly compressible if the mass density of the liquid changes due to a change in pressure, although this change is very small.

Let us consider a pipeline (Fig. 11-5) in which the flow velocity at the downstream end is changed from V to $V + \Delta V$, thereby changing the pressure from p to $p + \Delta p$. This change in pressure will produce a pressure wave that will propagate in the upstream direction (upstream and downstream are with respect to undisturbed initial flow). The pressure on the upstream side of this wave will be p, whereas the pressure on the downstream side of this wave will be $p + \Delta p$. Let us denote the speed of this wave by a, and let us consider the downstream flow direction positive.

We can transform the unsteady-flow situation of Fig. 11-5*a* to a steady-flow situation by letting the velocity reference system move with the pressure wave. Then, the flow velocities will be as shown in Fig. 11-5*b*. We will use the momentum equation (Eq. 11-1) with control volume approach to solve for Δp. We let the control surface move with the wave front so that we have steady flow with respect to the moving coordinate system.

Because we have steady flow, the first term on the right-hand side of the momentum equation (Eq. 11-1) is zero. Referring to Fig. 11-5*b* and introducing the forces and velocities into Eq. (11-1) yield

$$pA - (p + \Delta p)A = (V + a + \Delta V)(\rho + \Delta\rho)(V + a + \Delta V)A - (V + a)\rho(V + a)A \quad (11\text{-}5)$$

By simplifying and discarding terms of higher order, this equation becomes

$$-\Delta p = 2\rho V \Delta V + 2\rho \Delta V a + \Delta\rho(V^2 + 2Va + a^2) \quad (11\text{-}6)$$

The general form of the equation for conservation of mass for one-dimensional flows may be written as

$$0 = \frac{d}{dt}\int_{x_1}^{x_2} \rho A\,dx + (\rho VA)_{\text{out}} - (\rho VA)_{\text{in}} \tag{11-7}$$

As we discussed previously, we have steady flow. Therefore, the first term on the right-hand side of Eq. (11-7) is zero. Referring to Fig. 11-5*b* and introducing the velocities into Eq. (11-7), we obtain

$$0 = (\rho + \Delta\rho)(V + a + \Delta V)A - \rho(V + a)A \tag{11-8}$$

Simplifying this equation, we have

$$\Delta\rho = -\frac{\rho\,\Delta V}{V + a} \tag{11-9}$$

In most of the real-life situations, $V \ll a$ for example, V is usually less than 20 m/s, and a is usually in the range of 1000 to 1400 m/s (2, 4, 11). Hence we may approximate $(V + a)$ as a, and Eq. (11-9) may be written as

$$\Delta\rho = -\frac{\rho\,\Delta V}{a} \tag{11-10}$$

Now, by substituting Eq. (11-10) into Eq. (11-6), discarding terms of higher order, and simplifying, we obtain

$$\Delta p = -\rho\,\Delta V a \tag{11-11}$$

or, since $\Delta p = \gamma\,\Delta H = \rho g\,\Delta H$, we can write Eq. (11-11) as

$$\Delta H = -\frac{a}{g}\,\Delta V \tag{11-12}$$

In other words, the change in pressure head due to an instantaneous change in flow velocity is approximately 100 times the change in the flow velocity. No wonder very high transient pressures occurred that caused the failures shown in Fig. 11-1.

Equation (11-12) gives the pressure head change caused by an instantaneous velocity change at the downstream end of a pipe. By doing a similar derivation, it can be shown that the pressure change caused by an instantaneous change in the flow velocity at the upstream end of a pipeline and the wave moving in the downstream direction is given by the equation

$$\Delta H = \frac{a}{g}\,\Delta V \tag{11-13}$$

In this case, there is no negative sign; in other words, the pressure rises for an increase in the flow velocity, and it reduces by a decrease in the flow velocity.

11-4 Wave Speed

From Eq. (11-10), it follows that

$$\Delta V = -\frac{\Delta\rho}{\rho}\, a \tag{11-14}$$

Now, the bulk modulus of elasticity of a liquid may be defined (6) as

$$K = \frac{\Delta p}{\dfrac{\Delta\rho}{\rho}} = \frac{\rho g\,\Delta H}{\dfrac{\Delta\rho}{\rho}} \tag{11-15}$$

Hence it follows from Eqs. (11-14) and (11-15) that

$$a = -\frac{K\,\Delta V}{\rho g\,\Delta H} \tag{11-16}$$

By substituting the expression for $\Delta V/\Delta H$ from Eq. (11-12) into Eq. (11-16), and simplifying the resulting equation, we obtain

$$a = \sqrt{\frac{K}{\rho}} \tag{11-17}$$

If we had assumed the conduit walls to be slightly deformable instead of rigid, then as shown in Sec. 11-6, Eq. (11-17) would be modified to

$$a = \sqrt{\frac{\dfrac{K}{\rho}}{1 + \left(\dfrac{KD}{eE}\right)}} \tag{11-18}$$

where D is the inside diameter of the conduit, e is the wall thickness, and E is the modulus of elasticity of the conduit-wall material (2, 11).

EXAMPLE 11-1 Determine the rise in pressure head in a 1-m diameter pipeline if a valve is instantaneously closed at the downstream end. The initial steady-state flow velocity in the pipeline is 1.5 m/s. The pipeline is carrying water. The bulk modulus of elasticity and the mass density for water are 2.19 GPa and 999 kg/m^3. Assume the pipe walls are rigid.

SOLUTION To compute the increase in pressure head, we have to first determine the value for the wave speed, a. This value may be determined from Eq. (11-17):

$$a = \sqrt{\frac{K}{\rho}}$$

$$= \sqrt{\frac{2.19 \times 10^9}{999}}$$

$$= 1480.6 \text{ m/s}$$

Now, we determine the instantaneous change in the flow velocity ΔV:

$$\Delta V = 0 - 1.5$$

$$= -1.5 \text{ m/s}$$

By substituting the values for ΔV, a, and $g = 9.81$ m/s^2 into Eq. (11-12), we obtain

$$\Delta H = -\frac{1480.6}{9.81}(-1.5)$$

$$= 226.4 \text{ m}$$

The positive sign indicates that the pressure head increases. ■

11-5 Pressure Wave Propagation and Reflections

Once a pressure wave is produced in a pipeline, it propagates back and forth in the pipeline until it is dissipated because of friction. This wave is reflected and transmitted at different boundaries. To illustrate the propagation and reflection of pressure waves in a pipeline, let us consider the piping system shown in Fig. 11-6a on the next page. Let the pipe length be L, wave speed be a, and initial piezometric head and flow velocity be H_0 and V_0. Let us assume the system is frictionless, the pipe walls are elastic, and the liquid inside the pipeline is slightly compressible. The transient-state conditions are produced by instantaneously closing a downstream valve at time $t = 0$.

Figure 11-6 shows the sequence of events following the valve closure. These events may be divided into the following four parts:

1. (Fig. 11-6a) Pressure wave propagation toward the reservoir ($0 < t < L/a$): The pressure wave will reach the upstream reservoir in L/a seconds. For any time less than L/a, the wave will be between the valve and the reservoir. On the reservoir side of the wave, flow will be undisturbed, flow velocity will be V_0, the piezometric head will be H_0 (assuming $V_0^2/2g$ is negligible

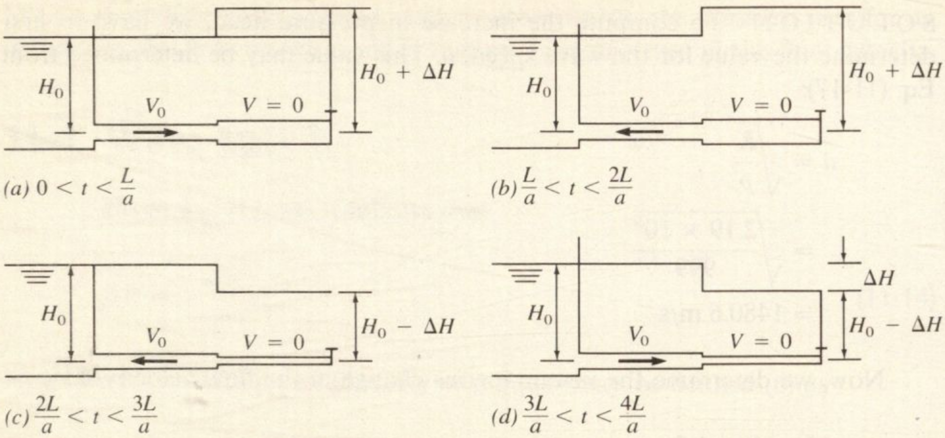


Figure 11-6 Sequence of events following valve closure

compared to H_0), and the pipe diameter will be the same as during the initial steady-state conditions. However, on the valve side of the wave front, that is, behind the wave, the flow velocity will be zero, piezometric head will be $H_0 + \Delta H$, and the pipe diameter will be increased because of inside pressure being higher than the initial steady-state value.

2. (Fig. 11-6*b*) Pressure wave reflection at the reservoir and propagation toward the valve ($L/a < t < 2L/a$):

   At time $t = L/a$, the pressure wave will reach the upstream reservoir, and the piezometric head throughout the pipe length will be $H_0 + \Delta H$. However, the head on the upstream side of the pipe entrance will be H_0, since the reservoir level is assumed to remain constant. Physically, it is not possible to have the pressure head on one side of a fluid section H_0 and, on the other side, $H_0 + \Delta H$ and be in stable equilibrium. Thus, the fluid will begin to flow toward the reservoir with velocity V_0, and the head will drop to H_0. A pressure wave will therefore now propagate toward the valve. In front of this wave (on the valve side), the flow velocity will be zero, pressure head will be $H_0 + \Delta H$, and the pipe will be expanded. Behind the wave (on the reservoir side), however, the flow velocity will be $-V_0$, that is, toward the reservoir, the pressure head will be H_0, and the pipe diameter will be the same as that during the steady state.

3. (Fig. 11-6*c*) Pressure wave reflection at the valve and propagation toward the reservoir ($2L/a < t < 3L/a$):

   The wave reflected from the reservoir will reach the valve at $t = 2L/a$. Since the valve is completely closed, it is not possible to maintain a flow velocity of $-V_0$ at the valve. Therefore, the flow velocity instantaneously becomes zero (that is, $\Delta V = V_0$), and the pressure head drops to $H_0 - \Delta H$. This can be seen by substituting $\Delta V = V_0$ into Eq. (11-12). Now, this negative pressure wave propagates toward the reservoir. On the front side of the wave, the head is H_0, the flow velocity is $-V_0$, and the pipe diameter is the same as that during the initial steady-state conditions; behind the wave, however, the pressure head is $H_0 - \Delta H$, flow velocity is zero, and the pipe diameter is reduced.

4. (Fig. 11-6*d*) Pressure wave reflection at the reservoir and propagation toward the valve ($3L/a < t < 4L/a$):

   As the negative wave reaches the upstream reservoir, we have an unstable situation again; that is, the pressure head on the reservoir side of the entrance is H_0, and the pressure head on the valve side is $H_0 - \Delta H$. Therefore, the negative pressure wave is now reflected as a positive pressure wave. On the valve side of this wave, the pressure head is $H_0 - \Delta H$, the flow velocity is zero, and the pipe diameter is reduced. On the reservoir side of the wave, however, the pressure head is H_0, the flow velocity is V_0, and the pipe diameter is the same as that during the initial steady-state conditions. As this wave reaches the valve at $t = 4L/a$, we have the same conditions as at $t = 0$ except that the valve is now closed. Therefore, the above sequence of events starts all over again.

Since, we are assuming a frictionless system, the pressure wave travels back and forth in the pipeline indefinitely with the same flow conditions being repeated every $4L/a$ seconds. The time interval, $4L/a$, after which conditions are repeated, is referred to as the theoretical period of the pipeline (2, 11). Figure 11-7 shows the variation of pressure head at the valve with respect to time. The variation of pressure head at other locations may similarly be plotted (see Prob. 11-2, page 606).

11-6 Governing Equations

To analyze the transient-state conditions in a pipeline, we need the equations describing these flows. In this section, we derive the equations by making the following assumptions: The fluid is slightly compressible, the walls of the conduit are linearly elastic and are slightly deformable, and the head losses during the transient state may be computed by using the steady-state formula.

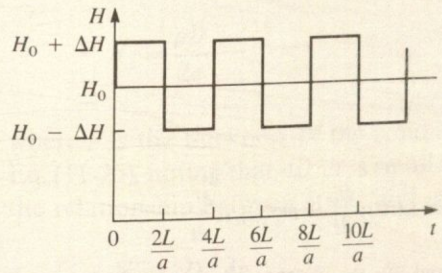


Figure 11-7 Variation of pressure at valve

Continuity Equation

To derive the continuity equation, we will apply the law of conservation of mass to the control volume shown in Fig. 11-8. Let the velocity (with respect to the coordinate axes) of sections 1 and 2, because of the contraction or expansion of the control volume, be W_1 and W_2, respectively. The distance x, flow velocity V, and discharge Q will be considered positive in the downstream direction.

Hence applying Eq. (11-7), page 579, to the control volume shown in Fig. 11-8, and using relative flow velocity at sections 1 and 2 to allow for velocity of these two sections, we obtain

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho A\,dx + \rho_2 A_2(V_2 - W_2) - \rho_1 A_1(V_1 - W_1) = 0 \tag{11-19}$$

Applying Leibnitz's rule* to the first term on the left-hand side of this equation, and noting that $dx_2/dt = W_2$ and $dx_1/dt = W_1$, this equation simplifies to

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho A)\,dx + (\rho A V)_2 - (\rho A V)_1 = 0 \tag{11-20}$$

By using the mean value theorem,† dividing throughout by $\Delta x = x_2 - x_1$ and letting Δx approach zero, Eq. (11-20) may be written as

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A V) = 0 \tag{11-21}$$

Expanding the terms in the parentheses, rearranging various terms, using expressions for total derivatives, and dividing throughout by ρA, we obtain

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{A}\frac{dA}{dt} + \frac{\partial V}{\partial x} = 0 \tag{11-22}$$

Let us express the derivatives of ρ and A in terms of commonly used variables p and V as follows.

---

* According to this rule (10),

$$\frac{d}{dt}\int_{f_1(t)}^{f_2(t)} F(x, t)\,dx = \int_{f_1(t)}^{f_2(t)} \frac{\partial}{\partial t} F(x, t)\,dx + F(f_2(t), t)\frac{df_2}{dt} - F(f_1(t), t)\frac{df_1}{dt}$$

provided f_1 and f_2 are differentiable functions of t, and $F(x, t)$ and $\partial F/\partial t$ are continuous in x and t.

† According to this theorem, $\int F(x)\,dx = (x_2 - x_1)F(\xi)$, where $x_1 < \xi < x_2$.

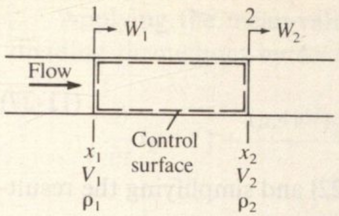


Figure 11-8 Notation for continuity equation

We define the bulk modulus of elasticity, K, of a fluid Eq. (11-15) as

$$K = \frac{dp}{\frac{d\rho}{\rho}}$$

We can write this equation as

$$\frac{d\rho}{dt} = \frac{\rho}{K}\frac{dp}{dt} \tag{11-23}$$

Now, area of the conduit, $A = \pi R^2$, where R is the radius of the conduit. Hence $dA/dt = 2\pi R\, dR/dt$. In terms of strain, ϵ, this may be written as (8)

$$\frac{dA}{dt} = 2A\frac{d\epsilon}{dt}$$

or

$$\frac{1}{A}\frac{dA}{dt} = 2\frac{d\epsilon}{dt} \tag{11-24}$$

To simplify the derivation, let us assume the conduit has expansion joints throughout its length so that the axial stress will be zero. Now, the hoop stress (8) in a thin-walled conduit having inside pressure p is given by the expression

$$\sigma_2 = \frac{pD}{2e} \tag{11-25}$$

where e is the thickness of the conduit walls. By taking the time derivative of Eq. (11-25), noting that dD/dt is small and therefore may be neglected, and using the relationship between stress and strain ($\epsilon = \sigma_2/E$), we obtain

$$\frac{d\epsilon}{dt} = \frac{D}{2eE}\frac{dp}{dt} \tag{11-26}$$

It follows from Eqs. (11-24) and (11-26) that

$$\frac{1}{A}\frac{dA}{dt} = \frac{D}{eE}\frac{dp}{dt} \tag{11-27}$$

Substituting Eqs. (11-23) and (11-27) into Eq. (11-22) and simplifying the resulting equation yields

$$\frac{\partial V}{\partial x} + \frac{1}{K}\left[1 + \frac{1}{eE/DK}\right]\frac{dp}{dt} = 0 \tag{11-28}$$

Let us define $a^2 = \dfrac{\dfrac{K}{\rho}}{1 + \dfrac{DK}{eE}}$ (11-29)

As we will see in the next section, a is the wave speed with which pressure waves travel back and forth.

Substituting Eq. (11-29) and the expression for the total derivative into Eq. (11-28) yields

$$\frac{\partial p}{\partial t} + V\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial V}{\partial x} = 0 \tag{11-30}$$

This equation is called the *continuity equation.*

Momentum Equation

For an expanding or contracting control volume (see Fig. 11-9, page 588), Eq. (11-1) is modified to

$$\frac{d}{dt}\int_{x_1}^{x_2} AV\rho\, dx + [\rho A(V - W)V]_2 - [\rho A(V - W)V]_1 = \sum F \tag{11-31}$$

Applying Leibnitz's rule to the first term on the left-hand side of this equation and noting that $dx_1/dt = W_1$ and $dx_2/dt = W_2$, we obtain

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho AV)\, dx + (\rho AV)_2 W_2 - (\rho AV)_1 W_1 + [\rho A(V - W)V]_2 - [\rho A(V - W)]_1 = \sum F \tag{11-32}$$

Applying the mean-value theorem to the first term of this equation and dividing throughout by Δx yield

$$\frac{\partial}{\partial t}(\rho A V) + \frac{(\rho A V^2)_2 - (\rho A V^2)_1}{\Delta x} = \sum \frac{F}{\Delta x} \tag{11-33}$$

The following forces are acting on the system in the control volume (pipe is assumed horizontal):

$$\text{Pressure force at section 1, } F_{p_1} = p_1 A \tag{11-34a}$$

where p is pressure intensity, and the subscript, 1, refers to the section.

$$\text{Pressure force at section 2, } F_{p_2} = p_2 A \tag{11-34b}$$

If the Darcy-Weisbach friction formula is used for computing the losses due to friction, the shear stress between the fluid and the conduit walls, $\tau_0 = \rho f V|V|/8$, where f is the Darcy-Weisbach friction factor, and V^2 is written as $V|V|$ to automatically account for reverse flows. Therefore,

$$\text{Shear force, } F_s = \tau_0 \pi D \, \Delta x$$

$$= \rho \frac{fV|V|}{8} \pi D \, \Delta x \tag{11-34c}$$

Substituting these expressions for the various forces yields

$$\sum F = p_1 A - p_2 A - \rho \frac{fV|V|}{8} \pi D \, \Delta x \tag{11-35}$$

Substituting Eq. (11-35) into Eq. (11-33) and letting Δx approach zero in the limit yield

$$\frac{\partial}{\partial t}(\rho A V) + \frac{\partial}{\partial x}(\rho A V^2) + A\frac{\partial p}{\partial x} + \rho \frac{fV|V|}{8} \pi D = 0 \tag{11-36}$$

Expanding the terms in parentheses and rearranging the terms of the resulting equation gives

$$V\left[\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A V)\right] + \rho A \frac{\partial V}{\partial t} + \rho A V \frac{\partial V}{\partial x} + A \frac{\partial p}{\partial x} + \rho A f \frac{V|V|}{2D} = 0 \tag{11-37}$$

According to the continuity equation (Eq. 11-21, page 584), the sum of the two terms in the brackets is zero. Hence dropping these terms and dividing the resulting equation by ρA, we obtain

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{fV|V|}{2D} = 0 \tag{11-38}$$

This equation is called the *momentum equation.*

Simplified Equations

In most engineering applications, the terms $V\dfrac{\partial p}{\partial x}$ and $V\dfrac{\partial V}{\partial x}$ of the governing equations (Eqs. 11-30 and 11-38) are very small as compared to the other terms. Therefore, these terms may be neglected. In hydraulic engineering applications, the pressure in the pipeline is expressed in terms of the piezometric head, H, above a specified datum, and the discharge, Q, is used as the second variable instead of flow velocity V. Now, the pressure intensity $p = \rho g H$. If we assume the fluid is slightly compressible and the conduit walls are slightly deformable, we may neglect the variation of ρ and flow area A caused by variation of the inside pressure. Hence we can write, $\dfrac{\partial p}{\partial t} = \rho g\dfrac{\partial H}{\partial t}$, and $\dfrac{\partial p}{\partial x} = \rho g\dfrac{\partial H}{\partial x}$. Substituting these relationships and $Q = VA$ into Eqs. (11-30) and (11-38), and neglecting $V\dfrac{\partial V}{\partial x}$ and $\rho g V\dfrac{\partial H}{\partial x}$, we obtain

$$\frac{\partial H}{\partial t} + \frac{a^2}{gA}\frac{\partial Q}{\partial x} = 0 \tag{11-39}$$

$$\frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + RQ|Q| = 0 \tag{11-40}$$

where $R = f/(2DA)$.

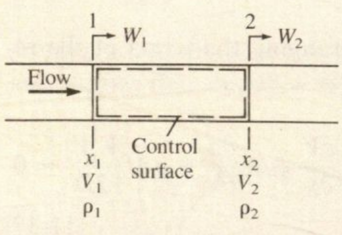


Figure 11-9 Notation for momentum equation

Steady-state equations corresponding to these equations may be obtained by substituting $\partial H/\partial t = 0$ and $\partial Q/\partial t = 0$. Hence it follows from Eq. (11-39) that $dQ/dx = 0$, that is, Q is constant along the pipe length. Similarly, by substituting $\partial Q/\partial t = 0$ into Eq. (11-40), and writing it in finite-difference form, we obtain $\Delta H = f\, \Delta x Q^2/(2gDA^2)$. This is the same as the Darcy-Weisbach friction formula.

11-7 Solution of Momentum and Continuity Equations

The continuity and momentum equations (Eqs. 11-39 and 11-40) are a set of partial differential equations; in other words, the piezometric head and discharge are functions of time t and distance x. The time and distance are referred to as the independent variables, whereas the head and discharge are referred to as the dependent variables. In transient-flow computations, we are interested in determining the variation of H and Q with respect to both x and t.

Assume the piping system is frictionless, that is, $f = 0$. Let us also eliminate Q from the continuity and momentum equations as follows: Multiply Eq. (11-40) by $a^2/(gA)$, substitute $R = 0$, differentiate it with respect to x, and then subtract it from an equation obtained by differentiating Eq. (11-39) with respect to t. Then,

$$\frac{\partial^2 H}{\partial t^2} = a^2 \frac{\partial^2 H}{\partial x^2} \tag{11-41}$$

This is the well-known wave equation describing propagation of pressure waves at wave velocity a. Most of the computational procedures for water hammer analysis—graphical, arithmetical integration, or closed-form solutions in one form or another—before the availability of digital computer, used solutions of this equation. Various innovations were devised to include the head losses in the analysis. With the availability of high-speed digital computers, however, it is now possible to obtain a numerical solution of the governing equations with the nonlinear head-loss term included. We present such a method in the next section. This method, called the method of characteristics, was introduced in 1789 by Monge (5) for a graphical solution of partial differential equations. Gray (3) was the first to use it in 1956 for the analysis of water hammer; however, several of Streeter's publications (11) made it popular for such analyses.

11-8 Method of Characteristics

Multiplying Eq. (11-39) by an unknown multiplier, λ, and adding the resulting equation to Eq. (11-40), we obtain the following equation.

$$\left[\frac{\partial Q}{\partial t} + \frac{\lambda a^2}{gA}\frac{\partial Q}{\partial x}\right] + \lambda\left[\frac{\partial H}{\partial t} + \frac{gA}{\lambda}\frac{\partial H}{\partial x}\right] + RQ|Q| = 0 \tag{11-42}$$

As we discussed above, both Q and H are functions of x and t, that is, $Q = Q(x, t)$, and $H = H(x, t)$. Therefore, the total derivatives may be written as

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dx}{dt} \tag{11-43a}$$

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{dx}{dt} \tag{11-43b}$$

Now, let us compare Eqs. (11-43a) and (11-43b) with the expressions in the brackets of Eq. (11-42). If we select the unknown multiplier, λ, so that

$$\frac{\lambda a^2}{gA} = \frac{dx}{dt} = \frac{gA}{\lambda} \tag{11-44}$$

then the right-hand side of Eq. (11-43b) is the same as the first expression in the brackets on the left-hand side of Eq. (11-42), and the right-hand side of Eq. (11-43a) is the same as the second expression in the brackets on the left-hand side of Eq. (11-42). It follows from Eq. (11-44) that

$$\lambda = \pm\frac{gA}{a} \tag{11-45}$$

and
$$\frac{dx}{dt} = \pm a \tag{11-46}$$

By specifying λ as given by Eq. (11-45), we may write Eq. (11-42) as follows.

If
$$\frac{dx}{dt} = a \tag{11-47}$$

then
$$\frac{dQ}{dt} + \frac{gA}{a}\frac{dH}{dt} + RQ|Q| = 0 \tag{11-48}$$

And, if
$$\frac{dx}{dt} = -a \tag{11-49}$$

then
$$\frac{dQ}{dt} - \frac{gA}{a}\frac{dH}{dt} + RQ|Q| = 0 \tag{11-50}$$

We now have two ordinary differential equations, Eqs. (11-48) and (11-50) in H and Q instead of the partial differential equations, since we have eliminated the independent variable, x. However, in obtaining this simplification, we have paid a price. Equations (13-39) and (11-40) were valid everywhere in the x-t plane; this is not the case with Eqs. (11-48) and (11-50). Equation (11-48) is valid

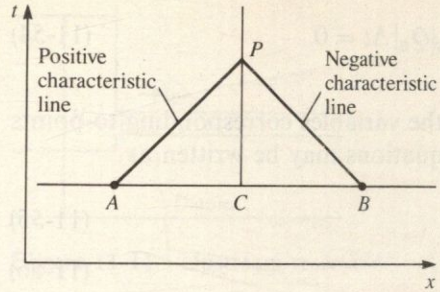


Figure 11-10 Characteristic lines

only if Eq. (11-47) is satisfied, and Eq. (11-50) is valid only if Eq. (11-49) is satisfied. In the x-t plane, Eqs. (11-47) and (11-49) describe two straight lines AP and BP, respectively, as shown in Fig. 11-10. These two lines are referred to as the characteristic lines; line AP is called the positive characteristic, and line BP is called the negative characteristic. Moreover, during the transformation from the partial differential equations to the ordinary differential equations, we have not made any approximation at all; that is, Eqs. (11-48) and (11-50) are as valid as the original governing equations (Eqs. 11-39 and 11-40) except that the former are valid only along the characteristic lines.

We will now discuss how to solve Eqs. (11-48) and (11-50). Let us multiply these equations by dt and integrate along the characteristic lines AP and BP. This procedure yields:

Along the positive characteristic line AP:

$$\int_A^P dQ + \frac{gA}{a}\int_A^P dH + R\int_A^P Q|Q|\,dt = 0 \tag{11-51}$$

Along the negative characteristic line BP:

$$\int_B^P dQ - \frac{gA}{a}\int_B^P dH + R\int_B^P Q|Q|\,dt = 0 \tag{11-52}$$

The first two integrals on the left-hand sides of these equations can be exactly evaluated. However, this is not the case with the friction-loss term, since we do not a priori know the variation of either Q or H along the characteristic lines. Therefore, we have to make some approximations to evaluate the integral of the friction-loss term. Several procedures have been proposed in the literature for this purpose. The simplest of these is to use the value of Q at point A for the positive characteristic line, and to use the value of Q at point B for the negative characteristic line. Then, Eqs. (11-51) and (11-52) are simplified to

$$Q_P - Q_A + \frac{gA}{a}(H_P - H_A) + RQ_A|Q_A|\,\Delta t = 0 \tag{11-53}$$

and $$Q_P - Q_B - \frac{gA}{a}(H_P - H_B) + RQ_B|Q_B|\,\Delta t = 0 \tag{11-54}$$

where the subscripts A, B, and P refer to the variables corresponding to points in the x-t plane (Fig. 11-10). These two equations may be written as

$$Q_P = C_p - C_a H_P \tag{11-55}$$

and $$Q_P = C_n + C_a H_P \tag{11-56}$$

where $$C_p = Q_A + C_a H_A - RQ_A|Q_A|\,\Delta t \tag{11-57}$$

$$C_n = Q_B - C_a H_B - RQ_B|Q_B|\,\Delta t \tag{11-58}$$

$$C_a = \frac{gA}{a} \tag{11-59}$$

Assume that we know the values of H and Q at points A and B and that we want to determine their values at point P (Fig. 11-10). They may be determined from Eqs. (11-55) and (11-56). The following discussion for the analysis of transient-state conditions in a single pipeline (Fig. 11-2) should help you to understand the computational procedure.

The pipeline is divided into a number of reaches. The ends of a reach are called *sections, nodes,* or *grid points.* The nodes at the upstream end and at the downstream end of a pipe are called *boundary nodes,* and the remaining nodes are called the *interior nodes.* To start the calculations, the piezometric head and discharge at $t = t_0$ are determined at the computational nodes. These are called the *initial conditions.* Then, by using Eqs. (11-55) and (11-56), the conditions at the interior nodes at time $t_0 + \Delta t$ are computed. At the boundaries, however, we have only one equation: Eq. (11-55) at the downstream end and Eq. (11-56) at the upstream end. To determine the second unknown from these equations at the boundary nodes, we need another equation. This additional equation is provided by the condition imposed by the boundary. By solving this equation simultaneously with the positive or negative characteristic equations, we develop the boundary conditions, which are then used to determine the transient conditions at the boundaries. To illustrate this procedure, we will develop in the following section boundary conditions for a constant-level upstream reservoir, for a downstream reservoir, for a dead end, for an opening or closing valve, and for a series junction.

11-9 Boundary Conditions

As we mentioned in Sec. 11-8 we may develop the boundary conditions by solving the positive or negative characteristic equations simultaneously with the condition imposed by the boundary. This condition may be in the form of

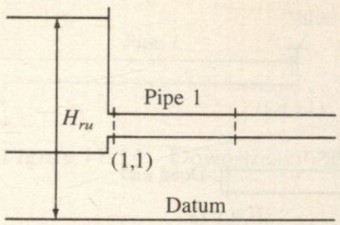


Figure 11-11 Upstream reservoir

specifying head, discharge, or a relationship between the head and discharge. For example, head is constant in the case of a constant-level reservoir, flow is always zero at a dead end, and the flow through an orifice is related to the head loss through the orifice. The following simple examples should clarify the development of the boundary conditions. In these derivations, we will use two subscripts to denote variables at different nodes: The first subscript will denote the number of the pipe and the second subscript will refer to the number of the node on that pipe. If a pipe is divided into n reaches, and the first node is numbered as 1, the last node will be $n + 1$.

Constant-Level Upstream Reservoir

In this case (Fig. 11-11), we are assuming that the water surface in the reservoir or tank remains at the same level independent of the flow conditions in the pipeline. This will be true if the reservoir volume is large. Hence if we refer to the pipe at the upstream end of the pipeline as 1, we may write that

$$H_{P1,1} = H_{ru} \tag{11-60}$$

where H_{ru} is the elevation of the water level in the reservoir above the datum.

Now, at the upstream end, we have the negative characteristic equation. Hence substituting Eq. (11-60) into Eq. (11-56), we obtain

$$Q_{P1,1} = C_n + C_a H_{ru} \tag{11-61}$$

Thus, we determine the head at an upstream reservoir from Eq. (11-60) and the discharge from Eq. (11-61).

Constant-Level Downstream Reservoir

In this case (Fig. 11-12), the head at the last node of pipe i will always be equal to the height of the water level in the tank above the datum, H_{rd}:

$$H_{Pi,n+1} = H_{rd} \tag{11-62}$$

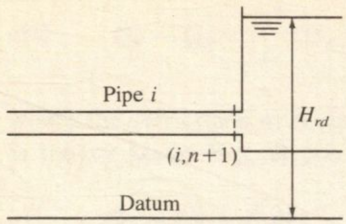


Figure 11-12 Downstream reservoir

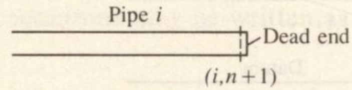


Figure 11-13 Dead end

At the downstream end, we have the positive characteristic equation linking the boundary node to the rest of the pipeline. Substituting Eq. (11-62) into the positive characteristic equation, Eq. (11-55), we obtain

$$Q_{Pi,n+1} = C_p - C_a H_{rd} \tag{11-63}$$

Dead End

At a dead end located at the end of pipe *i*, see Fig. 11-13 at right above, the discharge is always zero:

$$Q_{Pi,n+1} = 0 \tag{11-64}$$

At the last node of pipe *i*, we have the positive characteristic equation. Hence it follows from Eqs. (11-55) and (11-64) that

$$H_{Pi,n+1} = \frac{C_p}{C_a} \tag{11-65}$$

Downstream Valve

In the previous three boundaries, either the head or discharge was specified. However, for a valve, we specify a relationship between the head (see Fig. 11-14) losses through the valve and the discharge. Denoting the steady-state values by subscript 0, the discharge through a valve is given by the following equation:

$$Q_0 = C_d A_{vo} \sqrt{2gH_0} \tag{11-66}$$

where C_d is the coefficient of discharge, A_{vo} is the area of the valve opening, and H_0 is the drop in head for a discharge of Q_0. By assuming that a similar relationship is valid for the transient-state conditions, we may write

$$Q_{Pi,n+1} = (C_d A_v)_P \sqrt{2gH_{Pi,n+1}} \tag{11-67}$$

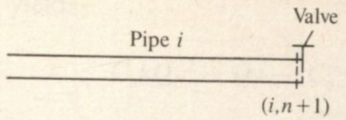


Figure 11-14 Downstream valve

where subscript P denotes values of Q and H at the end of a computational time interval.

Dividing Eq. (11-67) by Eq. (11-66), and squaring both sides, we obtain

$$Q^2_{Pi,n+1} = (Q_0\tau)^2 \frac{H_{Pi,n+1}}{H_0} \tag{11-68}$$

where the effective valve opening is $\tau = (C_dA_v)_P/(C_dA_v)_0$. For the last section on pipe i, we have the positive characteristic equation. Eliminating $H_{Pi,n+1}$ from Eq. (11-68) and the positive characteristic equation (Eq. 11-55), and simplifying the resulting equation, we obtain

$$Q^2_{Pi,n+1} + C_vQ_{Pi,n+1} - C_pC_v = 0 \tag{11-69}$$

where $C_v = (\tau Q_0)^2/(C_aH_0)$. Solving for $Q_{Pi,n+1}$ and neglecting the negative sign with the radical term, we obtain

$$Q_{Pi,n+1} = 0.5(-C_v + \sqrt{C_v{}^2 + 4C_pC_v}) \tag{11-70}$$

Now, $H_{Pi,n+1}$ may be determined from Eq. (11-55).

To compute the transient-state conditions caused by an opening or closing valve, the variation of τ with respect to time is needed. This relationship may be specified either by describing the variation by an expression or by giving the values of τ at discrete times. At intermediate times, the value of τ may be interpolated from the tabulated values. For example, if the values of τ are stored at an interval of 1 s, the value of τ at $t = 4.3$ s may be interpolated from the specified τ values at 4 and 5 s.

Series Junction

A boundary where two pipes having different diameters, wall materials, or friction factors are connected is referred to as a series junction. There are two nodes at a series junction, as shown in Fig. 11-15. Hence, there are four unknowns—head and discharge for each node—and we need four equations for a unique solution. These equations are as follows.

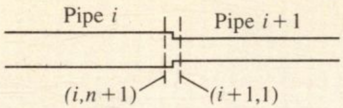


Figure 11-15 Series junction

Positive characteristic equation:

$$Q_{Pi,n+1} = C_p - C_{ai} H_{Pi,n+1} \tag{11-71}$$

Negative characteristic equation:

$$Q_{Pi+1,1} = C_n + C_{ai+1} H_{Pi+1,1} \tag{11-72}$$

Continuity equation:

$$Q_{Pi,n+1} = Q_{Pi+1,1} \tag{11-73}$$

Energy equation:

$$H_{Pi,n+1} = H_{Pi+1,1} \tag{11-74}$$

In Eq. (11-74) we have neglected the head losses at the junction and the difference in the velocity heads in pipes i and $i + 1$.

Eliminating $Q_{Pi,n+1}$, $Q_{Pi+1,1}$, and $H_{Pi+1,1}$ from Eqs. (11-71–11-74), we obtain

$$H_{Pi,n+1} = \frac{C_p - C_n}{C_{ai} + C_{ai+1}} \tag{11-75}$$

The remaining variables, $H_{Pi+1,1}$, $Q_{Pi,n+1}$, and $Q_{Pi+1,1}$ may now be determined from Eqs. (11-71–11-74).

EXAMPLE 11-2 Develop the boundary conditions for a centrifugal pump operating at constant speed. Assume the pipe between the upstream reservoir and the suction flange of the pump is short.

SOLUTION The head-discharge relationship for a centrifugal pump operating at constant speed may be approximated as

$$H_P = H_{sh} - kQ_P^{\,2} \tag{11-76}$$

where H_{sh} is the shutoff head, which is the head developed by the pump when there is no discharge, and k is a constant.

For the node on the discharge flange of the pump, we have the negative characteristic equation (Eq. 11-56). Eliminating H_P from Eqs. (11-56) and (11-76) yields

$$C_a k Q_P^2 + Q_P - (C_n + C_a H_{sh}) = 0 \tag{11-77}$$

Solving this equation and neglecting the negative sign with the radical term, we obtain

$$Q_P = \frac{-1 + \sqrt{1 + 4\,C_a k(C_n + C_a H_{sh})}}{2C_a k} \tag{11-78}$$

Now, H_P may be determined from Eq. (11-76). ■

11-10 Computational Procedure

To compute the transient-state conditions in a pipeline, each pipe is divided into a number of reaches. For short pipes, only one reach may be used. It is necessary that the same computational time interval be used for all pipes so that conditions at the junction may be computed without interpolation or extrapolation. For the computational procedure presented in the previous sections to be stable, the computational time interval and reach length in each pipe must satisfy the following stability condition, commonly referred to as Courant's condition:

$$\Delta x \geqslant a\,\Delta t \tag{11-79}$$

While selecting the computational time interval and the number of reaches in which a pipe is subdivided, it is necessary to satisfy Eq. (11-79). However, better results are obtained if $\Delta x = a\,\Delta t$. For this purpose, the wave speed may be slightly adjusted, if necessary, so that $\Delta x = a\,\Delta t$ for each pipe in the system.

Now, the initial conditions (the piezometric head and discharge) at time t_0, at all the grid points are computed. From these conditions, the head and discharge at time $= t_0 + \Delta t$ at the interior points are computed by using Eqs. (11-55) and (11-56). The boundary conditions are used to compute the flow conditions at the boundaries. Thus, we now know the head and discharge at time $t_0 + \Delta t$. By repeating the above procedure, the head and discharge may be computed for any length of time.

If the computational time interval is short, the computed conditions may be printed after a number of time steps.

Figure 11-16, pages 598–601, lists a FORTRAN computer program for the analysis of transient-state conditions in a pipe generated by opening or closing a downstream valve. The valve is discharging into atmosphere, and there is a constant-level reservoir at the upstream end. An equation is used to specify the variation of τ with time, and the friction losses are computed using the Darcy-Weisbach formula. The input data and the output of the program are given at the end of the program.

Figure 11-17, page 602, shows a plot of the variation of pressure head at the valve with respect to time. To show the difference between the transient pressures produced by instantaneous and gradual valve closure, transient pressures generated by instantaneous valve closure are shown in this figure by a broken line.

Figure 11-16 Computer program for analysis of transients in a pipeline caused by opening or closing a valve

Program Listing

```
C
C    ANALYSIS OF TRANSIENTS IN A PIPELINE CAUSED BY OPENING OR
C      CLOSING OF A DOWNSTREAM VALVE
C
C    *************** NOTATION ********************
C      A = WAVE SPEED;
C      AR = PIPE CROSS-SECTIONAL AREA;
C      D = PIPE DIAMETER;
C      DT = COMPUTATIONAL TIME INTERVAL;
C      F = DARCY-WEISBACH FRICTION FACTOR;
C      H = PIEZOMETRIC HEAD AT BEGINNING OF TIME INTERVAL;
C      HP = PIEZOMETRIC HEAD AT END OF TIME INTERVAL;
C      HRES = RESERVOIR LEVEL ABOVE DATUM;
C      HS = VALVE HEAD LOSS FOR FLOW OF QS;
C      IPRINT = TIME INTERVALS AFTER WHICH CONDITIONS ARE TO BE
C     PRINTED;
C      L = PIPE LENGTH;
C      N = NUMBER OF REACHES INTO WHICH PIPE IS SUB-DIVIDED;
C      Q = DISCHARGE AT THE BEGINNING OF TIME INTERVAL;
C      QO = STEADY-STATE DISCHARGE;
C      QP = DISCHARGE AT END OF TIME INTERVAL;
C      QS = VALVE DISCHARGE;
C      TAU = RELATIVE VALVE OPENING;
C      TAUF = FINAL VALVE OPENING;
C      TAUO = INITIAL VALVE OPENING;
C      TLAST = TIME UPTO WHICH CONDITIONS ARE TO BE COMPUTED;
C      TV = VALVE OPENING OR CLOSING TIME.
C
       REAL L
       DIMENSION H(100),Q(100),HP(100),QP(100)
C
C      READING AND WRITING OF INPUT DATA
C
       READ(5,*) N, IPRINT,QO,HRES,TLAST
       WRITE(6,20) N,IPRINT,QO,HRES,TLAST
20     FORMAT(3X,'N = ',I2,2X,'IPRINT =',I2,2X,'QO =',F6.3,' M3/S',
      1     2X,'HRES =',F7.2, ' M',2X,'TLAST =',F6.1,' S'/)
       READ (5,*) L,D,A,F
       WRITE (6,40) L,D,A,F
40     FORMAT(3X,'L = ',F7.1, ' M',2X,'D =',F5.2,' M',2X,'A =',
      1    F7.1,' M/S'/2X,'F =',F6.3/)
       READ(5,*) TV,TAUO,TAUF,HS,QS
       WRITE(6,60) TV,TAUO,TAUF,HS,QS
60     FORMAT(3X,'TV =',F5.2,2X,'TAUO =',F6.3,2X,'TAUF =',F5.3,2X,
      1    'HS =',F7.2,' M',2X,'QS =',F6.3,' M3/S'/)
C
C     COMPUTATION OF PIPE CONSTANTS
C
       AR = .7854*D*D
       CA=9.81*AR/A
       DT=L/(N*A)
       CF=F*DT/(2.*D*AR)
       F=F*L/(19.62*D*N*AR*AR)
C
```

```
C     STEADY-STATE CONDITIONS
C
      H(1)=HRES
      NN = N+1
      DH=F*QO*QO
      DO 80 I=1,NN
      H(I)=HRES-(I-1)*DH
      Q(I)=QO
80    CONTINUE
      K=0
      TAU=TAUO
      T=0.
100   WRITE(6,110) T,TAU
110   FORMAT(/3X,'T =',F6.2,' S',2X,'TAU =',F5.3)
      WRITE(6,120) (H(I),I=1,NN)
120   FORMAT(5X,'H =',15F8.2)
      WRITE(6,130) (Q(I),I=1,NN)
      K=0
130   FORMAT(5X,'Q =',15F8.3)
150   T=T+DT
      K=K+1
      IF (T.GT.TLAST) STOP
C
C     UPSTREAM RESERVOIR
C
      HP(1)=HRES
      CN=Q(2)-H(2)*CA-CF*Q(2)*ABS(Q(2))
      QP(1)=CN+CA*HRES
C
C     INTERIOR POINTS
C
      DO 160 J=2,N
      CN=Q(J+1)-CA*H(J+1)-CF*Q(J+1)*ABS(Q(J+1))
      CP=Q(J-1)+CA*H(J-1)-CF*Q(J-1)*ABS(Q(J-1))
      QP(J)=0.5*(CP+CN)
      HP(J)=(CP-QP(J))/CA
160   CONTINUE
C
C     DOWNSTREAM VALVE
C
      CP=Q(N)+CA*H(N)-CF*Q(N)*ABS(Q(N))
      IF (T.GE.TV) GO TO 180
      TAU=TAUO+T*(TAUF-TAUO)/TV
      GO TO 190
180   TAU=TAUF
190   IF (TAU.LE.0.0) GO TO 200
      CV=(QS*TAU)**2/(HS*CA)
      QP(NN)=0.5*(-CV+SQRT(CV*CV+4.*CP*CV))
      HP(NN)=(CP-QP(NN))/CA
      GO TO 210
200   QP(NN)=0.
      HP(NN)=CP/CA
C     STORING VARIABLES FOR NEXT TIME STEP

210   DO 230 J=1,NN
      Q(J)=QP(J)
      H(J)=HP(J)
230   CONTINUE
      IF (K.EQ.IPRINT) GO TO 100
      GO TO 150
300   STOP
      END
```

Input Data

```
8,4,1.,200.99,16.
1000.,1.,1000.,0.012
4.0,1.,0.,200.,1.
```

Program Output

```
N =  8  IPRINT = 4  QO = 1.000 M3/S  HRES = 200.99 M  TLAST =  16.0 S

L = 1000.0 M  D = 1.00 M  A = 1000.0 M/S
F =  .012
```

(continued)

Figure 11-16
Program Output (continued)

```
TV = 4.00  TAUO = 1.000  TAUF = .000  HS = 200.00 M  QS = 1.000 M3/S

T =    .00 S  TAU =1.000
 H =  200.99  200.87  200.74  200.62  200.49  200.37  200.25  200.12  200.00
 Q =   1.000   1.000   1.000   1.000   1.000   1.000   1.000   1.000   1.000

T =    .50 S  TAU = .875
 H =  200.99  200.87  200.74  200.62  200.49  203.45  206.47  209.55  212.69
 Q =   1.000   1.000   1.000   1.000   1.000    .976    .952    .927    .902

T =   1.00 S  TAU = .750
 H =  200.99  203.93  206.94  210.01  213.14  216.33  219.59  222.92  226.31
 Q =   1.000    .976    .952    .928    .903    .877    .851    .825    .798

T =   1.50 S  TAU = .625
 H =  200.99  207.42  213.84  220.28  226.72  230.17  233.68  237.27  240.93
 Q =    .806    .806    .804    .802    .798    .771    .743    .715    .686

T =   2.00 S  TAU = .500
 H =  200.99  207.92  214.85  221.79  228.73  235.68  242.64  249.61  256.59
 Q =    .598    .598    .596    .594    .590    .586    .580    .574    .566

T =   2.50 S  TAU = .375
 H =  200.99  208.45  215.92  223.38  230.86  236.01  241.05  245.95  250.72
 Q =    .376    .375    .373    .371    .367    .380    .393    .407    .420

T =   3.00 S  TAU = .250
 H =  200.99  206.69  212.26  217.69  222.98  228.14  233.14  238.00  242.71
 Q =    .137    .154    .171    .188    .206    .223    .240    .258    .275

T =   3.50 S  TAU = .125
 H =  200.99  203.98  206.96  209.92  212.85  217.94  222.87  227.62  232.19
 Q =    .036    .037    .038    .041    .045    .067    .089    .112    .135

T =   4.00 S  TAU = .000
 H =  200.99  203.32  205.64  207.93  210.20  212.42  214.59  216.69  218.72
 Q =   -.046   -.045   -.043   -.040   -.035   -.028   -.020   -.011    .000

T =   4.50 S  TAU = .000
 H =  200.99  202.49  203.98  205.44  206.86  206.35  205.98  205.76  205.69
 Q =   -.106   -.105   -.102   -.098   -.091   -.069   -.046   -.023    .000

T =   5.00 S  TAU = .000
 H =  200.99  199.57  198.35  197.32  196.48  195.83  195.37  195.09  195.00
 Q =   -.137   -.121   -.105   -.088   -.071   -.053   -.036   -.018    .000

T =   5.50 S  TAU = .000
 H =  200.99  198.00  195.03  192.07  189.14  188.32  187.74  187.39  187.28
 Q =   -.036   -.037   -.038   -.041   -.045   -.034   -.023   -.012    .000

T =   6.00 S  TAU = .000
 H =  200.99  198.66  196.35  194.05  191.79  189.57  187.40  185.30  183.27
 Q =    .046    .045    .043    .040    .035    .028    .020    .011    .000

T =   6.50 S  TAU = .000
 H =  200.99  199.49  198.00  196.54  195.12  195.64  196.00  196.22  196.30
 Q =    .106    .105    .102    .098    .091    .069    .046    .023    .000

T =   7.00 S  TAU = .000
 H =  200.99  202.41  203.63  204.66  205.50  206.14  206.61  206.88  206.98
 Q =    .136    .121    .105    .088    .071    .053    .036    .018    .000

T =   7.50 S  TAU = .000
 H =  200.99  203.97  206.95  209.91  212.84  213.65  214.23  214.58  214.69
 Q =    .036    .037    .038    .041    .045    .034    .023    .012    .000

T =   8.00 S  TAU = .000
 H =  200.99  203.32  205.63  207.92  210.19  212.41  214.57  216.67  218.70
 Q =   -.046   -.045   -.043   -.040   -.035   -.028   -.020   -.011    .000

T =   8.50 S  TAU = .000
 H =  200.99  202.49  203.97  205.43  206.85  206.34  205.97  205.75  205.68
 Q =   -.106   -.105   -.102   -.098   -.091   -.069   -.046   -.023    .000
```

```
T =  9.00 S  TAU = .000
  H =  200.99  199.57  198.35  197.32  196.49  195.84  195.38  195.10  195.01
  Q =   -.136   -.121   -.105   -.088   -.071   -.053   -.036   -.018    .000

T =  9.50 S  TAU = .000
  H =  200.99  198.01  195.04  192.08  189.15  188.34  187.76  187.41  187.30
  Q =   -.036   -.037   -.038   -.041   -.045   -.034   -.023   -.012    .000

T = 10.00 S  TAU = .000
  H =  200.99  198.67  196.35  194.06  191.80  189.58  187.42  185.32  183.30
  Q =    .046    .045    .043    .040    .035    .028    .020    .011    .000

T = 10.50 S  TAU = .000
  H =  200.99  199.49  198.01  196.55  195.13  195.65  196.01  196.23  196.30
  Q =    .105    .105    .102    .097    .091    .069    .046    .023    .000

T = 11.00 S  TAU = .000
  H =  200.99  202.41  203.63  204.65  205.49  206.14  206.60  206.88  206.97
  Q =    .136    .121    .105    .088    .071    .053    .036    .018    .000

T = 11.50 S  TAU = .000
  H =  200.99  203.97  206.94  209.89  212.82  213.63  214.21  214.56  214.67
  Q =    .036    .037    .038    .041    .045    .034    .023    .012    .000

T = 12.00 S  TAU = .000
  H =  200.99  203.31  205.62  207.92  210.17  212.39  214.55  216.65  218.67
  Q =   -.046   -.045   -.043   -.040   -.035   -.028   -.020   -.011    .000

T = 12.50 S  TAU = .000
  H =  200.99  202.49  203.97  205.42  206.84  206.33  205.97  205.75  205.67
  Q =   -.105   -.105   -.102   -.097   -.091   -.069   -.046   -.023    .000

T = 13.00 S  TAU = .000
  H =  200.99  199.58  198.36  197.33  196.49  195.85  195.38  195.11  195.02
  Q =   -.136   -.121   -.104   -.088   -.071   -.053   -.036   -.018    .000

T = 13.50 S  TAU = .000
  H =  200.99  198.01  195.04  192.09  189.17  188.36  187.78  187.43  187.32
  Q =   -.036   -.037   -.038   -.041   -.045   -.034   -.023   -.012    .000

T = 14.00 S  TAU = .000
  H =  200.99  198.67  196.36  194.07  191.81  189.60  187.44  185.34  183.32
  Q =    .046    .045    .043    .040    .035    .028    .020    .011    .000

T = 14.50 S  TAU = .000
  H =  200.99  199.50  198.01  196.56  195.14  195.65  196.02  196.24  196.31
  Q =    .105    .104    .102    .097    .091    .069    .046    .023    .000

T = 15.00 S  TAU = .000
  H =  200.99  202.40  203.62  204.65  205.48  206.13  206.59  206.87  206.96
  Q =    .136    .121    .104    .088    .071    .053    .036    .018    .000

T = 15.50 S  TAU = .000
  H =  200.99  203.97  206.93  209.88  212.80  213.62  214.19  214.54  214.66
  Q =    .036    .037    .038    .041    .045    .034    .023    .012    .000

T = 16.00 S  TAU = .000
  H =  200.99  203.31  205.62  207.91  210.16  212.38  214.53  216.63  218.64
  Q =   -.046   -.045   -.043   -.040   -.035   -.028   -.020   -.011    .000
```

11-11 Transients Caused by Pumps

Power failure to the pump motors usually produces the most critical conditions in a pipeline. After a power failure, pump discharge and speed reduce rapidly. Usually, flow reduces to zero and then reverses although the pump may still be rotating in the positive direction. This reverse flow causes rapid deceleration of the pump, and the pump speed reverses as well if no protective devices, such as a ratchet, are installed to prevent the reverse rotation. The

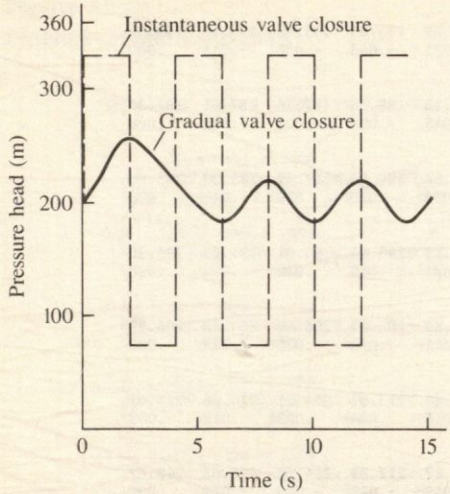


Figure 11-17 Pressure head at valve following valve closure

pump speed keeps on increasing in the reverse direction until it reaches the runaway speed. Because of increasing reverse speed, the reverse flow through the pump is reduced, thereby increasing the pressure at the pump end.

Figure 11-18 shows the hydraulic grade line at different times during the transient-state conditions produced by power failure. When the hydraulic grade line falls below the pipeline, the liquid column may separate if the pressures fall to the vapor pressure of the liquid. The subatmospheric pressure may collapse the pipe, or the high pressures generated by rejoining of the separated liquid columns may burst the pipe. Therefore, if the analysis shows that the transient-state hydraulic grade line falls below the centerline of the pipeline,

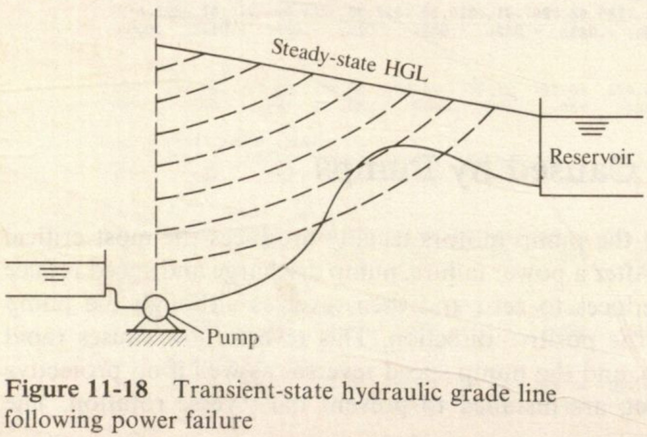


Figure 11-18 Transient-state hydraulic grade line following power failure

either the pipe should be designed to withstand both the maximum and minimum pressures or control devices should be provided to prevent liquid-column separation.

11-12 Control Devices

Control devices are installed in a pipeline to keep the transient-state conditions within prescribed limits. These conditions may be maximum and minimum pressures, pump and turbine overspeed, water-level oscillations in a surge tank, and so forth. The main function of these devices is to reduce the rates of acceleration and deceleration of the liquid column in the pipeline.

Some common control devices are

Surge tanks
Air chambers
Valves
Flywheels

A surge tank is a standpipe connected to the pipeline. This tank stores excess liquid and provides it when the hydraulic grade line in the pipeline falls below the liquid level in the tank. Several types of surge tanks—simple, orifice, differential, closed, and one-way—are shown in Fig. 11-19.

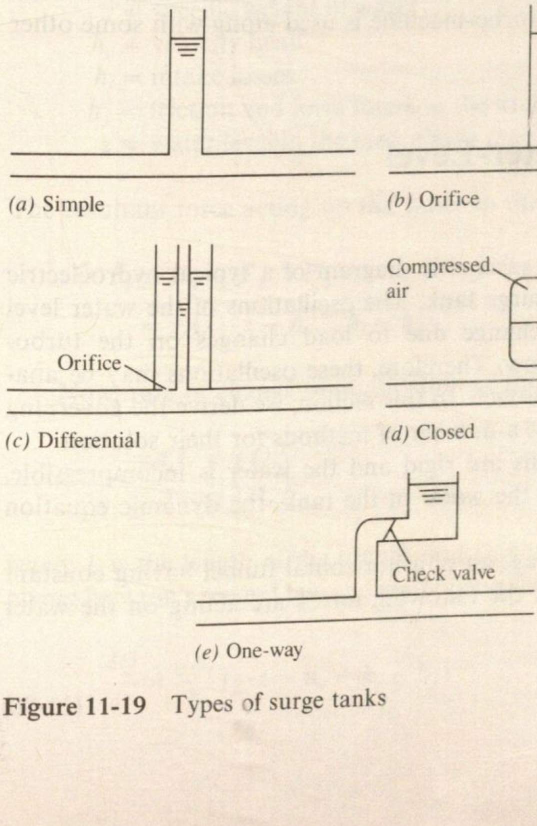


Figure 11-19 Types of surge tanks

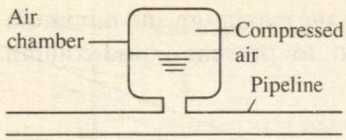


Figure 11-20 Air chamber

An air chamber has compressed air at the top (Fig. 11-20). The air acts as a cushion. When the pressure inside the pipeline falls, liquid flows out of the air chamber and the air expands. When the pressure rises, the liquid flows back into the chamber and the air is compressed. Thus, the air reduces the rates of acceleration and deceleration of the liquid column in the pipeline.

Valves are used to provide by-pass facilities so that the flow can be diverted to prevent sudden flow changes. The valve opening and closing rates are set by the pressure in the pipeline, or they are prespecified. Several different types of valves are available for transient control, such as pressure relief valves, pressure regulating valves, and safety valves. To prevent the pressure from falling too much below the atmospheric pressure, air valves are installed. These valves admit air into the pipeline when the pressure falls below the atmospheric pressure.

By increasing the inertia of the pump motor or turbogenerator set or by installing a flywheel, the transients may be kept within prescribed limits. Usually, an increase in the inertia of turbo-machine is used along with some other control methods.

11-13 Surge Tank Water-Level Oscillations

Figure 11-21 shows the schematic diagram of a typical hydroelectric power plant with an upstream surge tank. The oscillations of the water level in the tank following a flow change due to load changes on the turbogenerator set are usually very slow. Therefore, these oscillations may be analyzed using a lumped-system approach. In this section, we derive the governing equations for this system and list a number of methods for their solution.

By assuming the tunnel walls are rigid and the water is incompressible, and by neglecting the inertia of the water in the tank, the dynamic equation may be derived as follows.

Referring to the freebody diagram of a horizontal tunnel having constant cross-sectional area (Fig. 11-21), the following forces are acting on the water in the conduit:

$$F_1 = \gamma A_t(H_r - h_i - h_v) \tag{11-80}$$

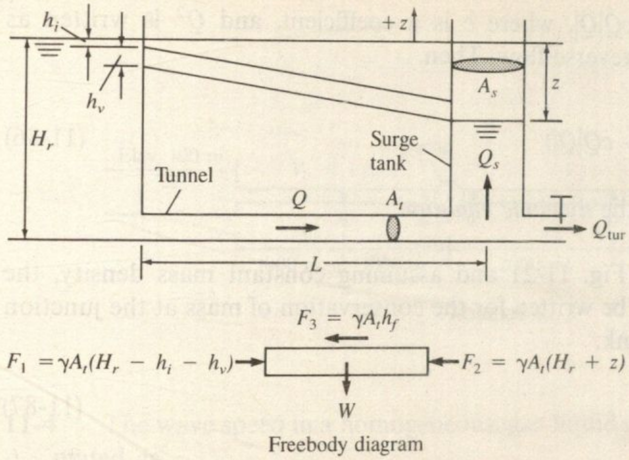


Figure 11-21 Surge-tank system

$$F_2 = \gamma A_t(H_r + z) \tag{11-81}$$

$$F_3 = \gamma A_t h_f \tag{11-82}$$

where A_t = cross-sectional area of the tunnel
H_r = static head
γ = specific weight of water
h_v = velocity head
h_i = intake losses
h_f = friction and form losses in the tunnel
z = water level in the tank above the reservoir level

The resultant force acting on the water in the tunnel is

$$\begin{aligned} F_r &= F_1 - F_2 - F_3 \\ &= \gamma A_t(-z - h_v - h_i - h_f) \end{aligned} \tag{11-83}$$

Now, rate of change of momentum of the water in the tunnel

$$= \frac{\gamma A_t L}{g}\frac{d}{dt}\left(\frac{Q}{A_t}\right) \tag{11-84}$$

where L is the length of the tunnel, and Q is the flow in the tunnel. Hence applying Newton's second law and simplifying the resulting equation, we can write

$$\frac{dQ}{dt} = \frac{gA_t}{L}(-z - h_v - h_i - h_f) \tag{11-85}$$

Let $h = h_v + h_i + h_f = cQ|Q|$, where c is a coefficient, and Q^2 is written as $Q|Q|$ to account for the reverse flow. Then,

$$\frac{dQ}{dt} = \frac{gA_t}{L}(-z - cQ|Q|) \tag{11-86}$$

This equation is called the *dynamic equation.*

Referring again to Fig. 11-21 and assuming constant mass density, the following equation may be written for the conservation of mass at the junction of the tunnel and the tank:

$$Q = Q_s + Q_{\text{tur}} \tag{11-87}$$

where Q_s is the flow into the tank, considered positive into the tank, and Q_{tur} is the turbine flow. Substituting $Q_s = A_s dz/dt$ into this equation and rearranging, we obtain

$$\frac{dz}{dt} = \frac{Q - Q_{\text{tur}}}{A_s} \tag{11-88}$$

where A_s = area of the surge tank.
This equation is called the *continuity equation.*

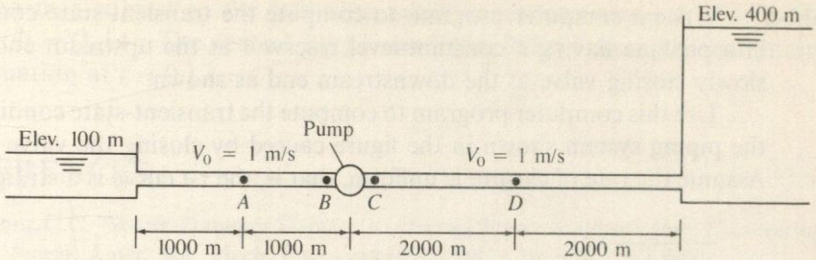
Equations (11-86) and (11-88) describe the water-level oscillations in the surge tank system shown in Fig. 11-21. Because of the presence of nonlinear terms (the expression for Q_{tur} may also be nonlinear), a closed form solution is not usually available. Therefore, the numerical methods we discussed in Chapter 10 may be used for the numerical integration of these equations.

PROBLEMS

11-1 Compute the pressure rise caused by instantaneous closure of a valve located at the downstream end of a 2-m diameter, 1962-m long pipeline conveying 10 m^3/s of water. The walls of the pipe are 20 mm thick and made of steel.

11-2 Plot the variation of pressure at midlength of the pipeline of Prob. 11-1, assuming the system is frictionless and a constant-level upstream reservoir has water surface 100 m above the centerline of the pipeline.

11-3 A pipeline connects two constant-level reservoirs with a pump located at midlength of the pipeline. If the pump instantaneously stops pumping, plot the pressure variation at points A, B, C, and D. Assume there are no losses in the pipeline and the wave speed is 1000 m/s.



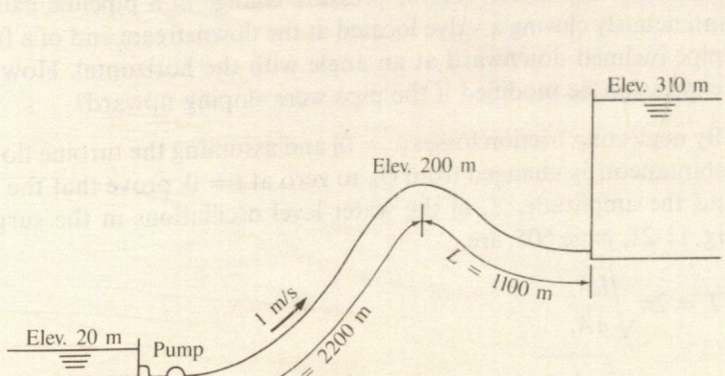
PROBLEM 11-3

11-4 The wave speed in a homogeneous, gas-liquid mixture may be approximated as

$$a = \sqrt{\frac{p^*}{\rho_l \alpha (1 - \alpha)}}$$

where p^* is absolute pressure, ρ_l is mass density of the liquid, void fraction $\alpha = \forall_g / \forall_m$, $\forall_g$ is volume of gas, and $\forall_m$ is volume of gas-liquid mixture. For standard temperature and pressure, compute the wave speed in an air-water mixture for different values of α, and plot a graph between a and α.

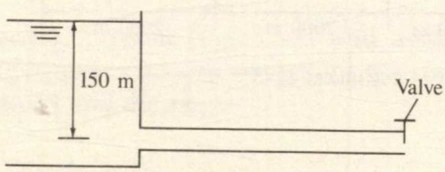
11-5 The inertia of the pump motor in this pipeline is very small so that the pump stops instantly upon power failure at time $t = 0$. A mechanical ratchet does not allow reverse pump rotation. The initial flow velocity in the pipeline is 1 m/s, and the wave speed is 1100 m/s. Determine the amount by which the pipeline would have to be lowered so that the hydraulic grade line at the summit does not fall below the centerline of the pipeline during the transient-state conditions. Assume the system is frictionless.



PROBLEM 11-5

11-6 Write a computer program to compute the transient-state conditions in a pipeline having a constant-level reservoir at the upstream end and a slowly closing valve at the downstream end as shown.

Use this computer program to compute the transient-state conditions in the piping system shown in the figure caused by closing the valve in 10 s. Assume the rate of closure is uniform, that is, the τ-t curve is a straight line.



$L = 4000$ m
$a = 1000$ m/s
$D = 0.5$ m
$V_0 = 2$ m/s
$f = 0.012$

PROBLEM 11-6

11-7 Compute the time of flow establishment by instantaneously opening the downstream valve of the piping system of the figure in Prob. 11-6 by using the computer program of Prob. 11-6. Compare the results with those obtained from Eq. (11-4, page 577).

11-8 Develop the boundary conditions for an orifice located at the downstream end of a pipe.

11-9 Develop the boundary conditions for an opening or closing valve located at the junction of two pipes having the same diameter, wall material, and wall thickness.

11-10 Derive an expression for the pressure change in a pipeline caused by instantaneously closing a valve located at the downstream end of a frictionless pipe inclined downward at an angle with the horizontal. How would this expression be modified if the pipe were sloping upward?

11-11 By neglecting friction losses ($c = 0$) and assuming the turbine flow, Q_{tur}, is instantaneously changed from Q_0 to zero at $t = 0$, prove that the period, T, and the amplitude, Z, of the water-level oscillations in the surge tank of Fig. 11-21, page 605, are

$$T = 2\pi \sqrt{\frac{LA_s}{gA_t}}$$

$$Z = Q_0 \sqrt{\frac{L}{gA_sA_t}}$$

[*Hint*: Substitute $c = 0$ in Eq. (11-86) and eliminate Q from this equation and Eq. (11-88). Then, solve the resulting second-order differential equation in z subject to the following initial conditions at $t = 0$: $z = 0$, and $dz/dt = Q_0/A_s$. The second initial condition follows from the continuity equation at $t = 0$.]

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