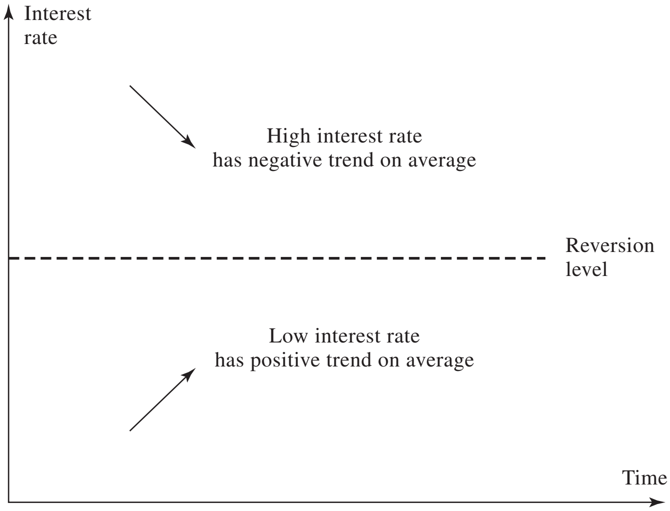
Figure 31.1 Mean reversion.



There are compelling economic arguments in favor of mean reversion. When rates are high, the economy tends to slow down and there is low demand for funds from borrowers. As a result, rates decline. When rates are low, there tends to be a high demand for funds on the part of borrowers and rates tend to rise.

The Vasicek Model

In Vasicek's model, the risk-neutral process for r is

$$dr = a(b - r)\,dt + \sigma\,dz$$

where a, b, and σ are nonnegative constants.[2] This model incorporates mean reversion. The short rate is pulled to a "reversion level" b at "reversion rate" a. Superimposed upon this pull is a normally distributed stochastic term $\sigma\,dz$.

Zero-coupon bond prices in Vasicek's model are given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{31.6}$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{31.7}$$

and

$$A(t,T) = \exp\left[\frac{(B(t,T) - T + t)(a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right] \tag{31.8}$$

(When $a = 0$, these equations become $B(t,T) = T - t$ and $A(t,T) = \exp[\sigma^2(T-t)^3/6]$.)

[2] See O. A. Vasicek, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977): 177–88. For a discrete-time version of Vasicek's model, see S. Heston, "Discrete-Time Versions of Continuous-Time Interest Rate Models." *Journal of Fixed Income*, 5, 2 (1995): 86–88.

To see this, note that $m = a(b - r)$ and $s = \sigma$ in differential equation (31.5), so that

$$\frac{\partial P(t,T)}{\partial t} + a(b - r)\frac{\partial P(t,T)}{\partial r} + \tfrac{1}{2}\sigma^2\frac{\partial^2 P(t,T)}{\partial r^2} = rP(t,T)$$

By substitution, we see that $P(t,T) = A(t,T)e^{-B(t,T)r}$ satisfies this differential equation when

$$B_t - aB + 1 = 0$$

and

$$A_t - abAB + \tfrac{1}{2}\sigma^2 AB^2 = 0$$

where subscripts denote derivatives. The expressions for $A(t,T)$ and $B(t,T)$ in equations (31.7) and (31.8) are solutions to these equations. What is more, because $A(T,T) = 1$ and $B(T,T) = 0$, the boundary condition $P(T,T) = 1$ is satisfied.

The Cox, Ingersoll, and Ross Model

Cox, Ingersoll, and Ross (CIR) have proposed the following alternative model:[3]

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

where a, b, and σ are nonnegative constants. This has the same mean-reverting drift as Vasicek, but the standard deviation of the change in the short rate in a short period of time is proportional to $\sqrt{r}$. This means that, as the short-term interest rate increases, the standard deviation increases.

Bond prices in the CIR model have the same general form as those in Vasicek's model,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

but the functions $B(t,T)$ and $A(t,T)$ are different:

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

and

$$A(t,T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2ab/\sigma^2}$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$.

To see this result, substitute $m = a(b - r)$ and $s = \sigma\sqrt{r}$ into differential equation (31.5) to get

$$\frac{\partial P(t,T)}{\partial t} + a(b - r)\frac{\partial P(t,T)}{\partial r} + \tfrac{1}{2}\sigma^2 r\frac{\partial^2 P(t,T)}{\partial r^2} = rP(t,T)$$

As in the case of Vasicek's model, we can prove the bond-pricing result by substituting $P(t,T) = A(t,T)e^{-B(t,T)r}$ into the differential equation. In this case, $A(t,T)$ and $B(t,T)$ are solutions of

$$B_t - aB - \tfrac{1}{2}\sigma^2 B^2 + 1 = 0, \quad A_t - abAB = 0$$

Furthermore, the boundary condition $P(T,T) = 1$ is satisfied.

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[3] See J. C. Cox, J. E. Ingersoll, and S. A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985): 385–407.

Properties of Vasicek and CIR

The $A(t, T)$ and $B(t, T)$ functions are different for Vasicek and CIR, but for both models

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)}$$

so that

$$\frac{\partial P(t, T)}{\partial r(t)} = -B(t, T)\, P(t, T) \tag{31.9}$$

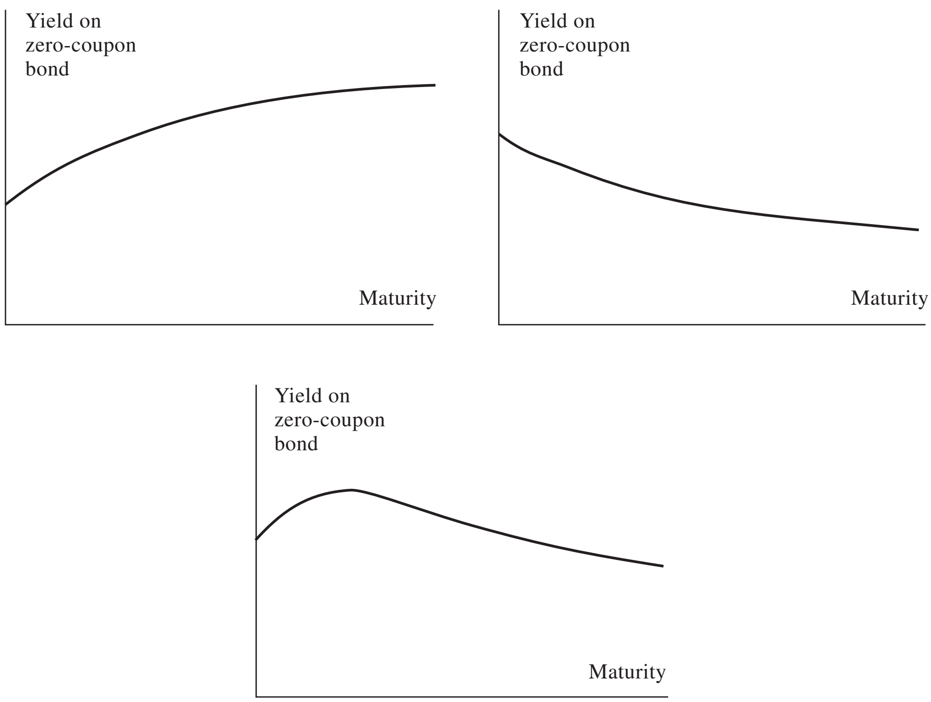
From equation (31.3), the zero rate at time t for a period of $T - t$ is

$$R(t, T) = -\frac{1}{T - t} \ln A(t, T) + \frac{1}{T - t} B(t, T)r(t) \tag{31.10}$$

This shows that the entire term structure at time t can be determined as a function of $r(t)$ once a, b, and σ have been chosen. The rate $R(t, T)$ is linearly dependent on $r(t)$. This means that the value of $r(t)$ determines the level of the term structure at time t. As shown in Figure 31.2, the shape at a particular time can be upward sloping, downward sloping, or slightly "humped."

One difference between Vasicek and CIR is that in Vasicek the short rate, $r(t)$, can become negative whereas in CIR this is not possible. If $2ab \geq \sigma^2$ in CIR, $r(t)$ is never zero; otherwise it occasionally touches zero.

Figure 31.2 Possible shapes of term structure in the Vasicek and CIR models.



From Chapter 4, the duration D of a bond that has a price of Q is given by

$$D = -\frac{1}{Q}\frac{\partial Q}{\partial y} \quad \text{so that} \quad \frac{\Delta Q}{Q} = -D\,\Delta y$$

where y is the continuously compounded bond yield. An alternative duration measure $\hat{D}$, which can be used in conjunction with Vasicek or CIR, is defined as follows:

$$\hat{D} = -\frac{1}{Q}\frac{\partial Q}{\partial r} \quad \text{so that} \quad \frac{\Delta Q}{Q} = -\hat{D}\,\Delta r$$

Here we are measuring the sensitivity of the bond price to the short rate rather than to its yield. Equation (31.9) shows that when we consider a T-maturity zero-coupon bond so that $Q = P(t, T)$, the alternative duration measure, $\hat{D}$, is equal to $B(t, T)$.

Example 31.1

Consider a zero-coupon bond lasting 4 years. In this case, $D = 4$, so that a 10-basis-point (0.1% or 0.001) increase in the bond's yield leads to a decrease of approximately 0.4% in the bond price. If Vasicek's model is used with $a = 0.1$,

$$\hat{D} = B(0, 4) = \frac{(1 - e^{-0.1\times 4})}{0.1} = 3.30$$

This means that a 10-basis-point increase in the short rate leads to a decrease in the bond price that is approximately 0.33%. The change in the bond price from a certain movement in the short rate is less than that from the same movement in its yield because of the impact of mean reversion.

When Q is the price of a coupon-bearing bond that provides a cash flow c_i at time T_i,

$$\hat{D} = -\frac{1}{Q}\frac{\partial Q}{\partial r} = -\frac{1}{Q}\sum_{i=1}^{n} c_i \frac{\partial P(t, T_i)}{\partial r} = \sum_{i=1}^{n} \frac{c_i P(t, T_i)}{Q}\hat{D}_i$$

where $\hat{D}_i = B(t, T_i)$ is the $\hat{D}$ for $P(t, T_i)$. This shows that the $\hat{D}$ for a coupon-bearing bond can be calculated as a weighted average of the $\hat{D}$'s for the underlying zero-coupon bonds, similarly to the way the usual duration measure D is calculated (see Table 4.6). This analysis can be extended to cover convexity measures (see Problem 31.11).

The Bond Price Process

The expected growth rate of $P(t, T)$ in the traditional risk-neutral world at time t is $r(t)$ because $P(t, T)$ is the price of a traded security that provides no income. Since $P(t, T)$ is a function of $r(t)$, the coefficient of $dz(t)$ in the process for $P(t, T)$ can be calculated from Itô's lemma as $\sigma\,\partial P(t, T)/\partial r(t)$ for Vasicek and $\sigma\sqrt{r(t)}\,\partial P(t, T)/\partial r(t)$ for CIR. Substituting from equation (31.9), the processes for $P(t, T)$ in a risk-neutral world are therefore

$$\text{Vasicek:} \quad d\,P(t, T) = r(t)\,P(t, T)\,dt - \sigma B_{\text{vas}}(t, T)\,P(t, T)\,dz(t) \qquad \textbf{(31.11)}$$

$$\text{CIR:} \quad d\,P(t, T) = r(t)\,P(t, T)\,dt - \sigma\sqrt{r(t)}\,B_{\text{cir}}(t, T)\,P(t, T)\,dz(t) \qquad \textbf{(31.12)}$$

where the subscripts to B indicate the model it applies to.

To compare the term structure of interest rates given by Vasicek and CIR for a particular value of r, it makes sense to use the same a and b. However, the Vasicek σ

should be chosen to be approximately equal to the CIR σ times $\sqrt{r(t)}$. For example, if r is 4% and σ is 0.01 in Vasicek, an appropriate value for σ in CIR would be $0.01/\sqrt{0.04} = 0.05$. Software for experimenting with the models can be found at www-2.rotman.utoronto.ca/~hull/VasicekCIR.

31.3 REAL-WORLD VS. RISK-NEUTRAL PROCESSES

Bonds have a positive market price of risk (i.e., positive systematic risk), so investors require an extra return over the risk-free rate for investing in bonds. Interest rates are negatively related to bond prices and therefore have negative market prices of risk.

Suppose that λ is the (negative) market price of risk of the short rate, r. If the risk-neutral process for r is

$$dr = m(r)\,dt + s(r)\,dz$$

the real-world process, from Chapter 28, is

$$dr = [m(r) + \lambda s(r)]\,dt + s(r)\,dz$$

In the case of Vasicek's model, the risk-neutral process is

$$dr = a(b - r)\,dt + \sigma\,dz$$

so that the real-world process is

$$dr = [a(b - r) + \lambda\sigma]\,dt + \sigma\,dz$$

Assuming λ is constant, this process is

$$dr = [a(b^* - r)]\,dt + \sigma\,dz \qquad \textbf{(31.13)}$$

where $b^* = b + \lambda\sigma/a$. The real-world process is therefore the same as the risk-neutral process except that the reversion level is lower (because λ is negative).

In the case of CIR, the risk-neutral process is

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

It is convenient to assume that $\lambda = \kappa\sqrt{r}$, where κ is a (negative) constant, so that the real-world process is

$$dr = [a(b - r) + \kappa\sigma r]\,dt + \sigma\sqrt{r}\,dz$$

In this case,

$$dr = a^*(b^* - r)\,dt + \sigma\sqrt{r}\,dz$$

where $a^* = a - \kappa\sigma$ and $b^* = ab/a^*$. The real-world process is therefore the same as the risk-neutral process except that the reversion rate is higher and the reversion level is lower.

Next, consider bonds. Equation (31.11) gives the risk-neutral process in Vasicek's model for a zero-coupon bond. When the market price of risk of the short rate is λ, the real-world process is

$$d\,P(t,T) = [r(t) - \lambda\sigma B_{\text{vas}}(t,T)]\,P(t,T)\,dt - \sigma B_{\text{vas}}(t,T)\,P(t,T)\,dz(t)$$

Equation (31.12) gives the risk-neutral process in the CIR model for a zero-coupon bond. When the market price of risk is $\kappa\sqrt{r}$, the real-world process is

$$d\,P(t,T) = [r(t) - \kappa\sigma B_{\text{cir}}(t,T)r(t)]\,P(t,T)\,dt - \sigma\sqrt{r(t)}B_{\text{cir}}(t,T)\,P(t,T)\,dz(t)$$

31.4 ESTIMATING PARAMETERS

We will illustrate the estimation of parameters with Vasicek's model. The discrete version of the model in the real world is, from equation (31.13),

$$\Delta r = a(b^* - r)\Delta t + \sigma\epsilon\sqrt{\Delta t}$$

where ϵ is a random sample from a standard normal distribution. Define r_i as the rate on day i. Daily data on 3-month Treasury rates in the United States between January 4, 1982, and August 23, 2016, together with a worksheet for the analysis in this section, is on www-2.rotman.utoronto.ca/~hull/VasicekCIR. When we use this data to regress $r_{i+1} - r_i$ against r_i, we obtain

$$r_{i+1} - r_i = 0.00000915 - 0.000545 r_i$$

with a standard error of 0.000754.

We have about 250 observations per year, so that $\Delta t = 1/250$. Because $0.00000915 = 0.00229/250$, $0.000545 = 0.136/250$, and $0.000754 = 0.0119/\sqrt{250}$, the regression result is equivalent to

$$\Delta r = (0.00229 - 0.136r)\Delta t + 0.0119\epsilon\sqrt{\Delta t}$$

or

$$\Delta r = 0.136(0.0168 - r)\Delta t + 0.0119\epsilon\sqrt{\Delta t}$$

indicating that the best-fit parameters are $a = 0.136$, $b^* = 0.0168$ (or 1.68%), and $\sigma = 0.0119$ (or 1.19%). Problem 31.15 shows that the same results are obtained using maximum-likelihood methods.

So far we have estimated the parameters for Vasicek's model in the real world. If the market price of risk is λ, the previous section shows that in the risk-neutral world the parameters are $a = 0.136$, $b = 0.168 - 0.0119\lambda/a$, and $\sigma = 0.0119$. For a trial value of λ, we can use equations (31.7), (31.8), and (31.10) to estimate zero-coupon rates as a function of maturity. Solver can then be used to determine the value of λ that minimizes the sum of squared errors between the zero-coupon rates given by the model and those in the market. This best-fit value of λ turns out to be -0.175.[4] Table 31.1 compares the zero-coupon rates given by the model when this best-fit value of λ is used with those in the market. Problem 31.16 uses the same data for the CIR model.

The two-step estimation procedure we have used is a simple procedure and there has been no attempt to fit the term structure of interest rates at times other than today. In practice, researchers use more sophisticated econometric procedures. The parameters we have estimated for Vasicek's model are reasonable, but this will not always be the case.[5]

[4] Research such as R. Stanton, "A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk," *Journal of Finance*, 52, 5 (December 1997): 1973–2002, has tried to estimate the market price of risk from the average slope of the term structure of interest rates at the short end. As explained in J. C. Hull, A. Sokol, and A. White, "Short-Rate Joint-Measure Models," *Risk*, 15, 3 (2015): 59–63, this tends to produce values for the market price of risk that are much too negative (typically -1.0 or less).

[5] As Problem 31.16 shows, the results using the same data for the CIR model are not as reasonable.

Table 31.1 Best fit of model rates to those in the market, August 23, 2016.

Maturity (years)	*Model rate (%)*	*Market Rate (%)*
0.5	0.40	0.45
1.0	0.49	0.58
2.0	0.65	0.74
3.0	0.80	0.86
5.0	1.06	1.15
7.0	1.27	1.40
10.0	1.52	1.55
20.0	2.02	1.88
30.0	2.26	2.24

The period of time over which parameters are estimated and the current shape of the term structure can have a big effect on the results obtained. Some judgment is necessary to determine the most appropriate data to use when a model such as Vasicek or CIR is estimated in practice.

31.5 MORE SOPHISTICATED MODELS

The Vasicek and CIR models are simple one-factor Markov models of the short rate. A two-factor Markov model of the short rate that is an extension of Vasicek is[6]

$$dr = (u - ar)\,dt + \sigma_1\,dz_1$$
$$du = -bu\,dt + \sigma_2\,dz_2$$

In this model, the reversion level $(= u/a)$ is not constant. It follows a stochastic process. However, bond prices are analytic:

$$P(t, T) = A(t, T)e^{-B(t, T)r - C(t, T)u}$$

where $B(t, T)$ is the same as in the Vasicek model and

$$C(t, T) = \frac{1}{a(a - b)}e^{-a(T-t)} - \frac{1}{b(a - b)}e^{-b(T-t)} + \frac{1}{ab} \qquad \textbf{(31.14)}$$

The function $A(t, T)$ is more complex (see Technical Note 14 on the author's website).

Another two-factor model which has analytic bond prices and involves similar processes to CIR was developed by Longstaff and Schwartz.[7] Other multifactor models are sometimes used in practice to describe the real-world evolution of the term structure. In Chapter 33, we will describe how general risk-neutral models can be developed in terms of forward rates.

[6] See J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, 2, 2 (Winter 1994): 37–48.

[7] See F. A. Longstaff and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model," *Journal of Finance*, 47, 4 (September 1992): 1259–82.