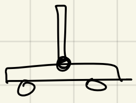
카트위 역진자 LQR 예제.



카트 질량 : $M = 0.4kg$

막대길이 : $2l = 0.44m$

$$x = \begin{bmatrix} p \\ v \\ \theta \\ \omega \end{bmatrix}$$

p : 카트위치

$v = \dot{p}$

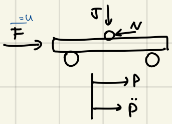
θ : 막대각도 (위쪽이 0)

$\omega = \dot{\theta}$

입력 : u : 카트에 수평작용힘

역학 System 세우기 (Free body diagram)

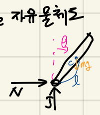
카트자유물체도



$\underbrace{F}_{=u} - N = \underbrace{M\ddot{p}}_{카트위치}$

J. N : Pole이 카트에 가하는힘

Pole 자유물체도



$\Sigma M = I\dot{\omega}$

J.N : 카트로 Pole에 가하는힘

$m\ddot{x}_c = N$

$m\ddot{y}_c = J - mg$

$I\dot{\omega} = l\sin\theta J - l\cos\theta N$: 시계방향이 ⊕. C중기준의 모멘트. ⇒ 막대무게중심기준 $I = \frac{4}{3}ml^2$

$x_c = \overset{카트중심위치}{p} + l\sin\theta$, $y_c = l\cos\theta$: pole 무게중심 위치

$\dot{x}_c = v + l\omega\cos\theta$

$\dot{y}_c = -l\omega\sin\theta$

$\ddot{x}_c = \ddot{p} + l\dot{\omega}\cos\theta - l\omega^2\sin\theta$

$\ddot{y}_c = -l\dot{\omega}\sin\theta - l\omega^2\cos\theta$

선형화 과정)

1. cart, pole로 분리하여 운동방정식 설립
2. 운동방정식을 state variable x에 대해 표현
3. 커플링된 $\ddot{p}, \ddot{\theta}$를 역행렬로 이용해 디커플링
4. 평형점 ($x=0, u=0 \to \dot{x}=0$)에 대해 선형화
   - 4-1. 평형점에 대한 자코비안 (A행렬, B행렬) 설립
   - 4-2. 자코비안 요소 직접 계산하여 구하기

⊕ Continuous to discrete.

이제 위식들을 $x, \dot{x}$에 대한식으로 바꿔야함

from pole

$N = m\ddot{x}_c$

$J = m\ddot{y}_c + mg$

$I\dot{\omega} = l\sin\theta(m\ddot{y}_c + mg) - l\cos\theta(m\ddot{x}_c)$

$= l\sin\theta(-ml\dot{\omega}\sin\theta - ml\omega^2\cos\theta + mg) - lm\cos\theta(\ddot{p} + l\dot{\omega}\cos\theta - l\omega^2\sin\theta)$

$= -ml^2\dot{\omega}\sin^2\theta + l\sin\theta mg - lm\cos\theta\ddot{p} - lm\cos^2\theta\dot{\omega}$

$= -ml^2\dot{\omega} + l\sin\theta mg - lm\cos\theta\ddot{p}$

$\Rightarrow (I + ml^2)\dot{\omega} + lm\cos\theta\ddot{p} - l\sin\theta mg = 0$. 하나도출.

from cart

$F - N = M\ddot{x} \to u - N = M\ddot{x}$

$\to u - m\ddot{x}_c = M\ddot{p}$

$u - m(\ddot{p} + l\dot{\omega}\cos\theta - l\omega^2\sin\theta) = M\ddot{p}$

$\ddot{p}(M+m) + lm\dot{\omega}\cos\theta - lm\omega^2\sin\theta = u$.

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ \ddot{p} \\ \omega \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{p} \\ \dfrac{u - lm\dot{\omega}\cos\theta + lm\omega^2\sin\theta}{M+m} \\ \theta \\ \dfrac{l\sin\theta mg - lm\cos\theta\ddot{p}}{I + ml^2} \end{bmatrix}$$

x에 대해 : 비선형 → **선형화 필요.**

① x가 모두 0이면 그리고 $u=0$이면 $p, v, \theta, \omega = 0$ 그러면 $\dot{x} = 0$됨 : 평형점 (사실 p는 다른 어떤값이어도됨)

② 선형화는 ~~$\theta=0, \omega=0$ 으로 진행함~~ 모든 x 요소를 0으로 근사함. ⊕ u도 0으로근사 $\theta=0$, $\omega=0$, $v=0$을 위하여, p는 어디든상관없으니 0으로 근사.

$\ddot{p}, \ddot{\theta}$ 커플링풀기 (역행렬 이용) : 선형화하기위해 필요.

둘은 서로 커플링되어있기에 선형화를위해 디커플링필요 ↓ 일단 커플링 풀기

$(I + ml^2)\dot{\omega} + lm\cos\theta\ddot{p} - l\sin\theta mg = 0$

$\ddot{p}(M+m) + lm\dot{\omega}\cos\theta - lm\omega^2\sin\theta = u$

→

$\ddot{p}(lm\cos\theta) + \dot{\omega}(I + ml^2) = l\sin\theta mg$

$\ddot{p}(M+m) + \dot{\omega}(lm\cos\theta) = u + lm\omega^2\sin\theta$

$$\begin{bmatrix} lm\cos\theta & I+ml^2 \\ M+m & lm\cos\theta \end{bmatrix}\begin{bmatrix} \ddot{p} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} l\sin\theta mg \\ u + lm\omega^2\sin\theta \end{bmatrix} \Rightarrow \begin{bmatrix} \ddot{p} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} lm\cos\theta & I+ml^2 \\ M+m & lm\cos\theta \end{bmatrix}^{-1}\begin{bmatrix} l\sin\theta mg \\ u + lm\omega^2\sin\theta \end{bmatrix}$$

$$= \frac{1}{l^2m^2\cos^2\theta - (I+ml^2)(M+m)}\begin{bmatrix} lm\cos\theta & -(I+ml^2) \\ -(M+m) & lm\cos\theta \end{bmatrix}\begin{bmatrix} l\sin\theta mg \\ u + lm\omega^2\sin\theta \end{bmatrix}$$

$$= \underbrace{\frac{1}{l^2m^2\cos^2\theta - (I+ml^2)(M+m)}}_{\triangleq D(x)} \begin{bmatrix} \overbrace{l^2m^2g\sin\theta\cos\theta - (I+ml^2)(u + lm\omega^2\sin\theta)}^{\triangleq Q(x)} \\ \underbrace{-(M+m)l\sin\theta mg + lm\cos\theta(u + lm\omega^2\sin\theta)}_{\triangleq Z(x)} \end{bmatrix}$$

본격 시스템 선형화 to 평형점 ($x=0, u=0$)

선형화 for ~~$\theta \approx 0, \omega \approx 0$~~ (모든 $x=0$으로근사함) u도 0으로근사함

~~그러면 θ, ω도 0 아닌가?~~

~~: 원점이 아닌 미세적인값으로 보는가.~~

~~0에 대한 테일러 전개를 하는거임~~

$$\Rightarrow \delta\dot{x} = \begin{bmatrix} \delta v \\ \delta\ddot{p} \\ \delta\omega \\ \delta\ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial\dot{p}}{\partial p}\Big|_{x=0,u=0} \overset{=0}{} & \frac{\partial\dot{p}}{\partial v}\Big|_{x=0,u=0} \overset{=1}{} & \frac{\partial\dot{p}}{\partial\theta}\Big|_{x=0,u=0} \overset{=0}{} & \frac{\partial\dot{p}}{\partial\omega}\Big|_{x=0,u=0} \overset{=0}{} \\ \frac{\partial\ddot{p}}{\partial p}\Big|_{x=0,u=0} \overset{=0}{} & \frac{\partial\ddot{p}}{\partial v}\Big|_{x=0,u=0} \overset{=0}{} & \frac{\partial\ddot{p}}{\partial\theta}\Big|_{x=0,u=0} & \frac{\partial\ddot{p}}{\partial\omega}\Big|_{x=0,u=0} \\ \frac{\partial\omega}{\partial p}\Big|_{x=0,u=0} & \frac{\partial\omega}{\partial v}\Big|_{x=0,u=0} & \frac{\partial\omega}{\partial\theta}\Big|_{x=0,u=0} & \frac{\partial\omega}{\partial\omega}\Big|_{x=0,u=0} \\ \frac{\partial\ddot{\theta}}{\partial p}\Big|_{x=0,u=0} & \frac{\partial\ddot{\theta}}{\partial v}\Big|_{x=0,u=0} & \frac{\partial\ddot{\theta}}{\partial\theta}\Big|_{x=0,u=0} & \frac{\partial\ddot{\theta}}{\partial\omega}\Big|_{x=0,u=0} \end{bmatrix}}_{A_{continuous}} \begin{bmatrix} \delta x \\ \delta v \\ \delta\theta \\ \delta\omega \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial\dot{p}}{\partial u}\Big|_{u=0,x=0} \\ \frac{\partial\ddot{p}}{\partial u}\Big|_{u=0,x=0} \\ \frac{\partial\omega}{\partial u}\Big|_{u=0,x=0} \\ \frac{\partial\ddot{\theta}}{\partial u}\Big|_{u=0,x=0} \end{bmatrix}}_{B_{continuous}} \delta u$$

선형화 자코비안행렬은 평형점 대입까지가 완성 (ex. θ로 미분했다고 θ만 평형점 대입이 아닌 다 대입해야함)

1차 테일러 전개 form (with 기준점 $x=a$ for function $f(x)$)

$g(x) = f(a) + f'(a)(x-a)$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{3mg}{7M+4m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3(m+M)g}{l(4m+7M)} & 0 \end{bmatrix}}_{A_{continuous}} \begin{bmatrix} \delta x \\ \delta v \\ \delta\theta \\ \delta\omega \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{7}{7M+4m} \\ 0 \\ -\frac{3}{l(7M+4m)} \end{bmatrix}}_{B_{continuous}} \delta u$$

$$\frac{\partial\ddot{p}}{\partial\theta}\Big|_{x=0,u=0} = \frac{\partial\left(\frac{Q(x)}{D(x)}\right)}{\partial\theta}\Bigg|_{x=0,u=0} = \frac{Q'(0)D(0) - \overset{0}{Q(0)}\overset{0}{D'(0)}}{D^2(0)} = \frac{l^2m^2g(l^2m^2 - (I+ml^2)(M+m))}{\{l^2m^2 - (I+ml^2)(M+m)\}^2}$$

$$= \frac{l^2m^2g}{l^2m^2 - (I+ml^2)(M+m)} = \frac{l^2m^2g}{l^2m^2 - \frac{7}{3}ml^2(M+m)} = \frac{m^2g}{-\frac{4}{3}m^2 - \frac{7}{3}mM} = -\frac{3mg}{7M+4m}$$

$$\frac{\partial\ddot{\theta}}{\partial\theta}\Big|_{x=0,u=0} = \frac{\partial\left(\frac{Z(x)}{D(x)}\right)}{\partial\theta}\Bigg|_{x=0,u=0} = \frac{Z'(0)D(0) - \overset{0}{D'(0)}Z(0)}{D^2(0)} = \frac{-(M+m)lmg(l^2m^2 - (I+ml^2)(M+m))}{\{l^2m^2 - (I+ml^2)(M+m)\}^2}$$

$$= \frac{-(M+m)lmg}{l^2m(-\frac{4}{3}m - \frac{7}{3}M)} = \frac{3(m+M)g}{l(4m+7M)}$$

$Z(x) = -(M+m)l\sin\theta mg + lm\cos\theta(u + lm\omega^2\sin\theta)$

$\frac{dZ(x)}{d\theta} = -(M+m)l\cos\theta mg - lm\sin\theta(u + lm\omega^2\sin\theta) + lm\cos\theta(lm\omega^2\cos\theta)$

$\frac{dZ(x)}{d\theta}\Big|_{x=0} = -(M+m)lmg$

$$\frac{\partial\ddot{p}}{\partial u}\Big|_{x=0,u=0} = \frac{\partial\left(\frac{Q(x)}{D(x)}\right)}{\partial u}\Bigg|_{x=0,u=0} = \frac{Q'(0)D(0) - Q(0)D'(0)}{D^2(0)} = \frac{-(I+ml^2)\{l^2m^2 - (I+ml^2)(M+m)\}}{\{l^2m^2 - (I+ml^2)(M+m)\}^2}$$

$$= \frac{-\frac{7}{3}ml^2}{l^2m(-\frac{4}{3}m - \frac{7}{3}M)} = \frac{7}{7M+4m}$$

$$\frac{\partial\ddot{\theta}}{\partial u}\Big|_{x=0,u=0} = \frac{\partial\left(\frac{Z(x)}{D(x)}\right)}{\partial u}\Bigg|_{x=0,u=0} = \frac{Z'(0)D(0) - \overset{0}{D'(0)}Z(0)}{D^2(0)} = \frac{lm\{l^2m^2 - (I+ml^2)(M+m)\}}{\{l^2m^2 - (I+ml^2)(M+m)\}^2}$$

$$= \frac{lm}{l^2m(-\frac{4}{3}m - \frac{7}{3}M)} = \frac{-3}{l(7M+4m)}$$

$$\delta\dot{x} = \begin{bmatrix} \delta v \\ \delta\ddot{p} \\ \delta\omega \\ \delta\ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{3mg}{7M+4m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3(m+M)g}{l(4m+7M)} & 0 \end{bmatrix}}_{A_{continuous}} \begin{bmatrix} \delta x \\ \delta v \\ \delta\theta \\ \delta\omega \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{7}{7M+4m} \\ 0 \\ -\frac{3}{l(7M+4m)} \end{bmatrix}}_{B_{continuous}} \delta u$$

∴ 선형화 완료

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continuous to discrete

(forward euler form : approx)

for Continuous to discrete, if time step is small enough, forward euler, $A_d \approx I + A_c dt$, $B_d \approx B_c dt$

← 중요함.

$$\to \delta x_{k+1} = \underbrace{\begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 1 & -\frac{3mg}{7M+4m}dt & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & \frac{3(m+M)g}{l(4m+7M)}dt & 1 \end{bmatrix}}_{A_{discrete}} \begin{bmatrix} \delta x \\ \delta v \\ \delta\theta \\ \delta\omega \end{bmatrix}_k + \underbrace{\begin{bmatrix} 0 \\ \frac{7dt}{7M+4m} \\ 0 \\ -\frac{3dt}{l(7M+4m)} \end{bmatrix}}_{B_{discrete}} \delta u_k$$