Title: Notes on a topic

Author name

Season year

Brief description of the notes. Some filler content provided, which comes from a combination of my own notes and those of Trevor Leslie (Illinois Institute of Technology).

Contents

1 Topological concepts in a metric space

1.1 Sequences of functions

We restrict our attention to real-valued functions; nearly all of theses statements and proofs continue to hold (with minor alterations) if $\mathbb{R}$ is replaced with any complete metric space.

Definition 1.1 (Pointwise convergence)**.** *Let X be any set, and let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions defined on X. If for each $x \in E$, the limit $\lim_{n\to\infty} f_n(x)$ exists, then we can defined the (pointwise) limit function*

$$f(x) = \lim_{n\to\infty} f_n(x),$$

for each $x \in E$. In this case, we say that $(f_n)_{n=1}^{\infty}$ converges pointwise to f on E.

Example 1.2. *For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Then for $x \in [0,1)$, we have $\lim_{n\to\infty} f_n(x) = 0$, while $\lim_{n\to\infty} f_n(1) = 1$, so the pointwise limit funciton is given by*

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in [0,1). \end{cases}$$

Clearly the limit function is not continuous at $x = 1$, even though all the f_n's are continuous everywhere.

Definition 1.3 (Uniform convergence)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces; let $f_n : X \to Y$ be functions. We say that $(f_n)_{n=1}^{\infty}$ converges uniformly to a function $f : X \to Y$ on $E \subset X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d_Y(f_n(x), f(x)) < \varepsilon$ for all $x \in E$.*

It is fairly clear that uniform convergence implies pointwise convergence, and that the converse is not true.

Proposition 1.4. *Assume $f_n \to f$ pointwise on E, and put*

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Assume additionally that $M_n < +\infty$ for all $n \in \mathbb{N}$. Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Example 1.5. *If $f_n(x) = x^n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, then $f_n \to 0$ pointwise on $[0,1)$, as we have seen above. The situation with regard to uniform convergence is more subtle. It turns out that*

- *$f_n \to 0$ uniformly on $[0,c]$ for any $c \in (0,1)$, but*
- *(f_n) does not converge uniformly on $[0,1)$.*

Proof. Choose $c \in (0,1)$ and $\varepsilon > 0$. Then $M_n = \sup_{x\in[0,c]} |x^n - 0| = c^n$ tends to zero as $n \to \infty$, so $f_n \to 0$ uniformly on $[0,c]$.

If (f_n) were to converge uniformly on $[0,1)$ to some funciton f, then it would also converge pointwise to that function. It follows that if $f_n \to f$ on $[0,1)$, then $f(x) = 0$ for all $x \in [0,1)$, as we already know $f_n \to 0$ pointwise on $[0,1)$. However, $\widetilde{M}_n = \sup_{x\in[0,1)} |x^n - 0| = 1$, which does not tend to zero as $n \to \infty$. Therefore, (f_n) does not converge uniformly to 0 (or to any function, for that matter) on $[0,1)$. $\square$

1.1.1 Uniform convergence and the vector space $B(X)$

Recall that if X is any set, then $(B(X), \|\cdot\|_u)$ denotes the vector space of all bounded, real-valued functions on X, together with the supremum norm

$$\|f\|_u = \sup_{x\in X} |f(x)|.$$

Recall also that we can make any vector space norm into a metric in the canonical way. Therefore, we can consider $B(C)$ as a metric space, with metric $d_u(f, g) = \|f - g\|_u = \sup_{x\in X} |f(x) - g(x)|$. Unless otherwise stated, we always give $B(X)$ this metric.

Proposition 1.6. *Let (f_n) be a sequence of bounded functions on a set X; let f be another funciton in $B(X)$. Then $f_n \to f$ uniformly on X if and only if $f_n \to f$ in $B(X)$.*

Exercise 1.7. A collection $\mathcal{A}$ of real-valued functions on a set E is said to be *uniformly bounded* on E if there exists $M > 0$ such that $|f(x)| < M$ for all $x \in E$, for all $f \in \mathcal{A}$. Let (f_n) be a sequence of bounded functions which converges uniformly on E to a limit function f. Prove that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded subset of $(B(X), d_u)$.

Proposition 1.8. *Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences of real-valued functions on a set E, which converge uniformly on E to limit functions f and g, respectively.*

(i) $(f_n + g_n)_{n=1}^{\infty}$ converges uniformly to $f + g$ on E.

(ii) If each f_n and each g_n is bounded, show that $(f_n g_n)_{n=1}^{\infty}$ converges uniformly to fg on E.

1.1.2 Uniformly Cauchy sequences and completeness of $B(X)$

Definition 1.9 (Uniformly Cauchy)**.** *The sequence $(f_n)_{n=1}^{\infty}$ of real-valued functions on a set X is said to be uniformly Cauchy on $E \subset X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in E$.*

Clearly, this definition is equivalent to the requirement that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_u(f_n, f_m) = \sup_{x\in E} |f_n(x) - f_m(x)| < \varepsilon$ whenever $m \geq n \geq N$. If $(f_n)_{n=1}^{\infty}$ is a sequence in $B(X)$, this requirement just says that $(f_n)_{n=1}^{\infty}$ is Cauchy in $B(E)$. We record this observtion in the following proposition.

Proposition 1.10. *If $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $B(X)$, then $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy on X.*

Theorem 1.11. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on a set X. Then $(f_n)_{n=1}^{\infty}$ converges uniformly on $E \subset X$ if and only if it is uniformly Cauchy on E.*

Proof. ($\Longrightarrow$) Assume that $(f_n)_{n=1}^{\infty}$ converges uniformly on E, and let f denote the limit function. Choose $\varepsilon > 0$, then choose N large enough to that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon/2$ for all $x \in E$. Then $m \geq n \geq N$ implies that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy on E.

($\Longleftarrow$) Assume that $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy on E. By the completeness of $\mathbb{R}$, $(f_n(x))_{n=1}^{\infty}$ converges to some number; we can therefore define a pointwise limit function $f(x)$. We need to show that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on E. To this end, pick $\varepsilon > 0$. Choose N large enough so that $m \geq n \geq N$ implies $|f_m(x) - f_n(x)| < \varepsilon/2$ for all $x \in E$. Now choose some $n \geq N$ and observe that

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \lim_{m\to\infty} \varepsilon/2 = \varepsilon/2 < \varepsilon,$$

for all $x \in E$. Thus, $f_n \to f$ unifromly on E. $\square$

Theorem 1.12 (Completeness of $B(X)$)**.** *For any set X, the metric space $(B(x), d_u)$ is complete.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $(B(X), d_u)$. Then $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy, so it converges uniformly (and pointwise) to some function f. We need to show that $f \in B(X)$, that is, f is bounded. Choose $N \in \mathbb{N}$ such that $f_N(x) - f(x)| < 1$ for all $x \in X$. Then choose $M > 0$ so that $f_N(x)| \leq M$ for all $x \in X$. Then for all $x \in X$, we have

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M.$$

So f is bounded, as claimed. $\square$

1.1.3 Uniform limit theorem and completeness of $BC(X)$

Theorem 1.13 (Uniform limit theorem, version 1)**.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on the metric space (X, d). Assume $f : E \to \mathbb{R}$ is a function such that $f_n \to f$ uniformly on $E \subset X$. Then f is continuous.*

Proof. Choose $\varepsilon > 0$ and $x \in E$. We need to show $\exists\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$ and $y \in E$. In light of the inequality,

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$

(which is valid for any $N \in \mathbb{N}$), we break up the task into three parts, making each of the three terms above less than $\varepsilon/3$.

Choose N large enough so that $|f(z) - f_N(z)| < \varepsilon/3$ for all $z \in E$. THen, for this same N, choose $\delta > 0$ small enough so that $d(x, y) < \delta$ and $y \in E$ together imply that $|f_N(x) - f_N(y)| < \varepsilon/3$. Then for $y \in E$ such that $d(x, y) < \delta$, we have

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

Definition 1.14. *Let (X, d) be a metric space. The space $(BC(X), d_u)$ is complete.*

Proof. To prove the statement, it suffices to show that $BC(X)$ is a closed subset of the complete metric space $B(X)$. Let f be a limit point of $BC(X)$ with respect to $B(X)$. Then there exists a sequence $(f_n)_{n=1}^{\infty}$ in $BC(X)$ that converges in $B(X)$ to f. Since $f_n \to f$ uniformly and each f_n is continuous, we have by the uniform limit theorem that f is continuous as well. This implies that $f \in BC(X)$, as needed. $\square$

This is a special case of the theorem, but it serves as a nice introduction. We now state the result in its entirety.

Theorem 1.15 (Uniform limit theorem, version 2)**.** *Let (X, d) be a metric space. Let E be a subset of X; let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on E which converge uniformly to another function $f : E \to \mathbb{R}$. Let x be a limit point of E, and assume that for each $n \in \mathbb{N}$, the limit $\lim_{t\to x} f_n(t)$ exists and is equal to some number A_n. Then the sequence (A_n) of numbers converges, and*

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n.$$

That is,

$$\lim_{t\to x}\left(\lim_{n\to\infty} f_n(t)\right) = \lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right).$$

Proof. (Step 1) We show that $(A_n)_{n=1}^{\infty}$ is a Cauchy sequence of numbers (and therefore converges in $\mathbb{R}$ as $n \to \infty$, since $\mathbb{R}$ is complete). To this end, choose $\varepsilon > 0$. Since $(f_n)_{n=1}^{\infty}$ converges uniformly, it is uniformly Cauchy. Choose $N \in \mathbb{N}$ large enough so that $m \geq n \geq N$ implies that $|f_m(t) - f_n(t)| < \frac{\varepsilon}{3}$ for all $t \in X$. Given such $m, n \in \mathbb{N}$, choose $t_0 \in X$ such that $|f_n(t_0) - A_n|$ and $|f_m(t_0) - A_m|$ are both less than $\frac{\varepsilon}{3}$ (this is possible by choosing t_0 close enough to x). Then

$$|A_n - A_m| \leq |A_n - f_n(t_0)| + |f_n(t_0) - f_m(t_0)| + |f_m(t_0) - A_m| < \varepsilon.$$

Let A denote the limit in $\mathbb{R}$ of the A_n's.

(Step 2) We prove the desired equality of limits. To this end, choose $\varepsilon > 0$, then choose $n \in \mathbb{N}$ such that $|f_n(t) - f(t)| < \frac{\varepsilon}{3}$ for all $t \in X$, and such that $|A_n - A| < \frac{\varepsilon}{3}$. For this n, let $\delta > 0$ be such that $d(t, x) < \delta$ implies $|f_n(t) - A_n| < \frac{\varepsilon}{3}$. Then $d(t, x) < \delta$ implies

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| < \varepsilon.$$

This completes the proof. □

This page is left blank intentionally.