# Unit 4: Competitive Markets

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## 1 Allocations

### 1.1 What is an allocation?

- Framework:
  - Look at market for single good $x$ in isolation
  - $1, \ldots, C$ = consumers in market (fixed)
  - $1, \ldots, F$ = firms in market (fixed)
  - All firms and consumers interact through the marketplace, and the outcome is an *allocation*
  - $W_i$ = wealth of consumer $i$ = initial endowment of $m$ for consumer $i$
  - Firms bring 0 $m$ to marketplace (i.e, they don't have 'exogenous' wealth)
- An *allocation* is a list:
  - Who produces what: $q_1^f, \ldots, q_F^f$, where $q_j^f$ = quantity of good $x$ produced by firm $j$, etc.
  - How is it produced: $m_1^f, \ldots, m_F^f$, where $m_j^f$ = amount of good $m$ used by firm $j$ (as costs of production)
  - Who consumes what: $q_1^c, \ldots, q_C^c$,$m_1^c, \ldots, m_C^c$, where $q_i^c, m_i^c$ = amount of $x, m$ consumed by consumer $i$

- A *feasible allocation* is an allocation that can be achieved given the resource and technological constraints of the market; i.e. an allocation that satisfies:

  - Feasibility constraint for good $x$:

$$\sum_{i=1}^{C} q_i^c = \sum_{j=1}^{F} q_j^f, \quad q_i^c, q_j^f \geq 0$$

  - Feasibility constraint for good $m$:

$$\sum_{i=1}^{C} m_i^c + \sum_{j=1}^{F} m_j^f = \sum_{i=1}^{C} W_i$$

$$m_j^f = C_j\left(q_j^f\right) \text{ for every firm } j$$

### 1.2 Example

- 100 consumers, each with $W_i = 2$
- 1 firm w/ SFC = \$100, MC = \$1/unit
- Exercise: Plot the set of feasible symmetric allocations (in which all consumers get the same bundle), as shown in the video lecture

# 2 Aggregating supply and demand

## 2.1 Aggregate demand

- $x_i^D(p)$ = consumer $i$'s demand at price $p$
- $X^D(p) = \sum_{i=1}^{C} x_i^D(p)$ = total market demand at price $p$
- Example:

  - $x_1^D(p) = 100 - p$; $x_2^D(p) = 75 - p$; $x_3^D(p) = 50 - p$; $x_4^D(p) = 25 - p$

– Aggregate demand is given by

$$X^D(p) = \begin{cases} 0 & \text{if } p \geq 100 \\ 100 - p & \text{if } 100 \geq p \geq 75 \\ 175 - 2p & \text{if } 75 \geq p \geq 50 \\ 225 - 3p & \text{if } 50 \geq p \geq 25 \\ 250 - 4p & \text{if } 25 \geq p > 0 \end{cases}$$

## 2.2 Aggregate consumer surplus

- RECALL from Unit 2:
  - Inverse demand function: Can invert $x^D(p)$ to get $p^D(x)$.
  - For rational consumer: $p^D = B'$
  - For all consumers with $x^D(p) > 0$, MB of last unit equals $p$
- Similarly:
  - Can invert $X^D(p)$ to get inverse aggregate demand function $P^D(x)$
  - $P^D(x)$ = price at which total consumer demand is $x$
  - $P^D(x)$ also equals MB of giving additional unit to any unconstrained consumer
- Aggregate consumer surplus = total increase in well-being for all consumers from participating in the market (in $s)
- Mathematically:

$$\begin{aligned} CS_{mkt}(p) &= \sum_{i=1}^{C} CS_i(p) \\ &= \sum_{i=1}^{C} \int_0^{x_i^D(p)} \left(p_i^D(x) - p\right) dx \\ &= \int_0^{X^D(p)} \left(P^D(x) - p\right) dx \end{aligned}$$

## 2.3 Aggregate supply

- $x_j^S(p)$ = supply of firm $j$ at price $p$
- $X^S(p) = \sum_{j=1}^{F} x_j^F(p)$ = total supply at price $p$ of all firms in market
- Example:
  - $x_1^S(p) = p$
  - $x_2^S(p) = \begin{cases} 0 & \text{if } p \leq 5 \\ p & \text{if } p \geq 5 \end{cases}$
  - $x_3^S(p) = \begin{cases} 0 & \text{if } p \leq 10 \\ p & \text{if } p \geq 10 \end{cases}$
  - Aggregate supply is then given by:

$$X^S(p) = \begin{cases} p & \text{if } p \leq 5 \\ 2p & \text{if } 5 \leq p \leq 10 \\ 3p & \text{if } p \geq 10 \end{cases}$$

## 2.4 Aggregate producer surplus

- Parallels discussion of aggregate consumer surplus
- RECALL from Unit 3:
  - Can invert $x_j^S(p)$ to get the inverse supply function $p_j^S(x)$
  - If $x_j^S(p) > 0$ , then $p_j^S(x) = MC_j(x)$
- Similarly:
  - Can invert $X^S(p)$ to get inverse aggregate supply function $P^S(x)$
  - $P^S(x)$ = marginal cost of producing another unit for *any* firm with positive production
- Aggregate producer surplus: total increase in profits for all firms from participating in market (in $s)

- Mathematically:

$$\begin{aligned} PS_{mkt}(p) &= \sum_{j=1}^{F} PS_j(p) \\ &= \sum_{j=1}^{F} \int_0^{x_j^S(p)} \big(p - p_j^S(x)\big)\, dx \\ &= \int_0^{X^S(p)} \big(p - P^S(x)\big)\, dx \end{aligned}$$

# 3 Competitive Markets

## 3.1 Competitive market equilibrium

- Key assumptions:

  1. Every consumer is a PRICE-TAKER
  2. Every firm is a PRICE-TAKER
  3. Market forces *rapidly* drive prices to a market clearing level $p^*$ at which
  $$X^D(p^*) = X^S(p^*)$$

- Price-taker = consumers and firms take prices as given; i.e., they don't believe they can affect prices by buying or selling more or less

- Price taking hypothesis justified if $C$ and $F$ are large. In this case each actor approximately negligible with respect to entire market, so has no market power

- Logic behind the market clearing hypothesis:

  - If $p > p^*$, then there's excess supply $X^S(p) > X^D(p)$, which puts pressure on prices to drop
  - If $p < p^*$, then there's excess demand $X^S(p) < X^D(p)$, which put pressure on prices to increase

- About firm ownership:

- If there are positive profits, they are returned to consumers that own the firms.
- They can use those profits, together with their wealth, to consume good $m$ or to buy good $x$
- Definitions:
  - $\sigma_{i,j}$ = fraction of firm $j$ owned by consumer $i$ ($0 \leq \sigma_{i,j} \leq 1$)
  - $\Pi_j(p)$ = profits of firm $j$ at price $p$
  - $\Pi_i(p) = \sum_{j=1}^{F} \sigma_{i,j} \Pi_j(p)$ = total profit income of consumer $i$
- So $m_i^c(p) = W_i - p x_i^D(p) + \Pi_i(p^*)$

- A Competitive Market Equilibrium (cME) is given by:
  - A price $p^*$ at which $X^S(p^*) = X^D(p^*)$
  - the allocation $\alpha^*$ that $p^*$ induces

- The allocation $\alpha^*$ is given by:
  - For each consumer $i$:
    $q_i^c = x_i^D(p^*)$
    $m_i^c = W_i - p^* x_i^D(p^*) + \Pi_i(p^*)$
  - For each firm $j$:
    $q_j^f = x_j^S(p^*)$
    $m_j^f = C_j(x_j^S(p^*))$

## 3.2 Graphical depiction of cME

- Graphically, market equilibrium is determined by the intersection of the aggregate demand ($X^D(p)$) and aggregate supply curves ($X^S(p)$).
- This is a very important diagram: see video lecture for details!

## 3.3 Computing a cME

- Computing a cME:
  - Step 1: compute $X^D(p)$
  - Step 2: compute $X^S(p)$
  - Step 3: compute equilibrium price $p^*$ by solving $X^D(p) = X^S(p)$
  - Step 4 (if needed): compute equilibrium allocaiton

- Example:
  - 100 identical consumers:
    * $W = 100$ units of $m$
    * $U(x, m) = 20\sqrt{x} + m$
    * $\sigma_{i,j} = \frac{1}{100}$ for every consumer $i$ and firm $j$.
  - 10 identical firms:
    * $c(q) = \frac{1}{2}q^2$
  - To compute $p^*, \alpha^*$:
    * $xX(p) = \frac{100}{p^2}$ for every consumer $i \implies X^D(p) = \frac{10000}{p^2}$ (by horizontal addition)
    * $x_j^S(p) = p$ for all $j \implies X^S(p) = 10p$ (by horizontal addition)
    * $X^D(p) = X^S(p) \implies 10p = \frac{10000}{p^2} \implies p^* = 10$
    * Substituting back we get that: $x_j^S(p^*) = 10$
    * This implies: $\Pi_j(p^*) = p^* x_j^S(p^*) - c(x_j^S(p^*)) = 100 - \frac{100}{2} = 50$, for each firm $j$
    * This also implies that each consumer gets \$5 in total profits (since firm ownership distributed symmetrically)
    * Substituting back we get that: $x_i^D(p^*) = 1$ unit and $m_i^c(p^*) = W - p^* x_i^D(p^*) + \Pi_i(p^*) = 100 - 10 + 5 = \$95$

## 3.4 Existence of cME

- Basic question: Does a market equilibrium price exists in every market?
- No, a CME doesn't always exist. See video lecture for an example

- Remark 1: Equilibrium exists if either:

    1. Crossing conditions are satisfied for consumer and cost functions exhibit DRS wihtout SFCs , or
    2. $MB \to 0$ as $x \to \infty$ for consumer, and cost function exhibit CRS without SFCs

- Remark 2: Problems with existence if either:

    - SFCs, or
    - Increasing returns in production

## 3.5 Example

- Consider and example in which firms have CRS costs
- In particular: Suppose 10 firms with $c_j(q) = jq$.
- QUESTION: Given this information, what is $p^*$ (regardless of the consumers' preferences)?
- For every firm,

$$x_j^S(p) = \begin{cases} 0 & \text{if } p < j \\ \text{anything} & \text{if } p \geq j \end{cases}$$

- $X^S(p) = x_1^S(p)$. Note: consumers won't buy from firms with higher MCs
- It follows that, in the cME, $p^* = 1 = MC$ of lowest cost firm

## 3.6 Comparative Statics

- Basic question: How does equilibrium change when some parameter changes?
- Let $a$ be a parameter of interest.
- We can write $x_i^D(p, a) \implies X^D(p, a)$ and $x_j^S(p, a) \implies X^S(p, a)$.

- Solve $X^D(p,a) = X^S(p,a)$ to get cME $p^*(a), \alpha^*(a)$ as a function of the parameter
- Can study how price and allocation changes as $a$ changes
- Basic formula:
  - At equilibrium, $X^D(p^*(a),a) = X^S(p^*(a),a)$

$$\implies \frac{\partial X^D}{\partial p}\frac{\partial p^*}{\partial a} + \frac{\partial X^D}{\partial a} = \frac{\partial X^S}{\partial p}\frac{\partial p^*}{\partial a} + \frac{\partial X^S}{\partial a}$$

$$\implies \frac{\partial p^*}{\partial a} = \frac{\frac{\partial X^S}{\partial a} - \frac{\partial X^D}{\partial a}}{\frac{\partial X^D}{\partial p} - \frac{\partial X^S}{\partial p}}$$

  - Forumla only works at interior solution where everything is continuous & differentiable
  - Laws of Demand and Supply $\Rightarrow$ the term $\frac{\partial X^D}{\partial p} - \frac{\partial X^S}{\partial p}$ in the denominator is negative
  - $\Rightarrow sign \frac{\partial p^*}{\partial a} = -sign\left\{\frac{\partial X^S}{\partial a} - \frac{\partial X^D}{\partial a}\right\}$
- Example
  - 10 consumers: $x_i^D(p) = 10a - p \implies X^D(p) = 100a - 10p$
  - 10 firms: $x_j^S(p) = 10ap \implies X^S(p) = 100ap$
  - $X^D(p) = X^S(p) \implies p^*(a) = \frac{100a}{100a+10} < 1$ (for $a > 0$)
  - $sign \frac{\partial p^*}{\partial a} = -sign\left\{\frac{\partial X^S}{\partial a} - \frac{\partial X^D}{\partial a}\right\} = -sign\{100p - 100\} > 0$ ; i.e., increases in $a$ lead to increases in the equilibrium price $p^*$

## 3.7 Example: Role of wealth distribution

- Consider a pure exchange economy:
  - Consumers endowed with $\bar{x}_i$ of good $x$ ($\bar{x}_{tot} = \sum_i \bar{x}_i$), and 0 units of good $m$
  - No production

- Consumers free to trade $x$

- Consumer's problem:

$$\max_{x \geq 0} B(x) - p(x - \bar{x}_i)$$

$$\sim \max_{x \geq 0} B(x) - px - p\bar{x}_i$$

- $p\bar{x}_i$ constant. Can think of it as \$ value of endowment
- $p\bar{x}_i$ constant $\Longrightarrow$ $x^D(p)$ independent of $\bar{x}_i$
- Equilibrium $p^*$ and total consumer surplus independent of initial distribution.
- But the well-being of each individual consumer does depend on the initial distribution.
- REMARK: This result depends on the assumption of quasi-linear preferences (and does not extend more generally).

# 4 Social surplus

## 4.1 Basics

- Recall: allocation $\alpha = \{x_i^c, m_i^c; x_j^f, m_j^c\}_{\text{for all } i, \text{ for all } j}$
- Definition of Social Surplus (SS):

$$SS(\alpha) = \sum_{i=1}^{C} B_i(x_i^c) - \sum_{j=1}^{F} c_j(x_j^f)$$

- Assumption:
  - All firm owners are included among consumers
  - Very general: can model "pure firm owners" as consuming having $B(x) = 0$, and thus consuming 0 of good $x$.

- RESULT: For any feasible allocation $\alpha$, $SS(\alpha) = \sum_{i=1}^{C} U_i(\alpha_i) + \mathit{constant}$ (with the constant independent of the allocation $\alpha$ being considered).
- Proof:

$$\begin{aligned}\sum_{i=1}^{C} U_i(\alpha_i) &= \sum_{i=1}^{C} B_i(x_i^C) + m_i^C \\ &= \sum_i B_i(x_i^C) + \sum_i W_i - \sum_{j=1}^{F} c_j(x_j^F) \text{ (by feasibility constraint)} \\ &= SS(\alpha) + \sum_i W_i,\end{aligned}$$

- REMARK: Constant not arbitrary, it equals $-\sum_i W_i$
- REMARK: This measure of social surplus is independent of how good $m$ is distributed in the allocation!
- REMARK: Result requires the quasilinear preference assumption, and does not hold generally.
  - Why? With quasilinear preferences there is transferable utility: transfer \$1 of utility from one person to another by just transfering 1 unit of good $m$
- RESULT: For any allocation $\alpha^*$ generated by a competitive equilibrium with free trade, and equilibrium price $p^*$, we have:

$$SS(\alpha^*) = CS_{mkt}(p^*) + PS_{mkt}(p^*)$$

- Proof:

$$
\begin{aligned}
SS(\alpha^*) &= \sum_{i=1}^{C} B_i(x_i^*) - \sum_{j=1}^{F} c_j(x_j^*) \\
&= \sum_{i=1}^{C} B_i(x_i^*) - \sum_{j=1}^{F} c_j(x_j^*) \overbrace{-p^* \sum_i x_i^{D*} + p^* \sum_j x_j^{S*}}^{=0} \\
&= \sum_i CS_i(p^*) + \sum_j PS_j(p^*) \\
&= CS_{mkt}(p^*) + PS_{mkt}(p^*) \qquad \square
\end{aligned}
$$

- Graphical representation of SS:
  - Equals area between $X^D(.)$ and $X^S(.)$ curves, between $x = 0$ and $x = X^*$.
  - See video lecture for more details.

- REMARKS:
  1. $SS = CS_{mkt} + PS_{mkt} = \sum_i U_i^{EU} +$ consatn, if and only if $DU = EU + const$ (i.e., if consumers are rational).
  2. SS measures total "utility pie" and is, silent on distributional issues

- WARKING: $SS = CS_{mkt} + PS_{mkt}$ under free trade, but not generally. Will see many examples of this later on.

# 5 Final remarks

- The following key concepts should be committed to memory, since they are critical for the rest of the course:
  1. $X^D(p) = \sum_i x_i^D(p)$, $X^S(p) = \sum_j x_j^S(p)$
  2. cME is $p^*, \alpha^*$ s.t.
     (a) everyone maximizes taking price $p^*$ as given

(b) $p^*$ clears the market: $X^D(p^*) = X^S(p^*)$

3. $CS_{mkt}(p^*) = \sum_i CS_i(p^*),\ PS_{mkt}(p^*) = \sum_j PS_j(p^*)$
4. $SS(\alpha) = \sum_i B_i(x_i^c) - \sum_j c_j(x_j^f) = \sum_i U_i(\alpha_i) + constant$
5. $SS(\alpha^*) = CS_{mkt}(p^*) + PS_{mkt}(p^*)$