

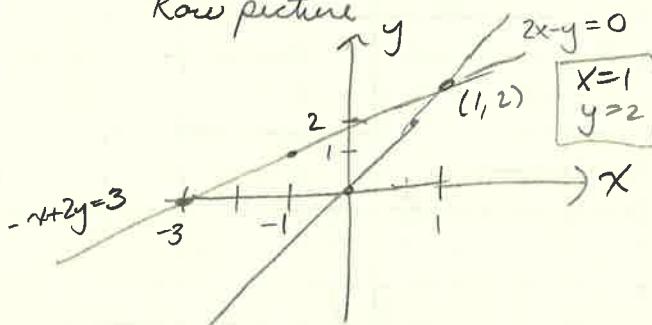
n linear equations, n unknowns

Row picture
Column picture
Matrix form

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_b$$

Row picture

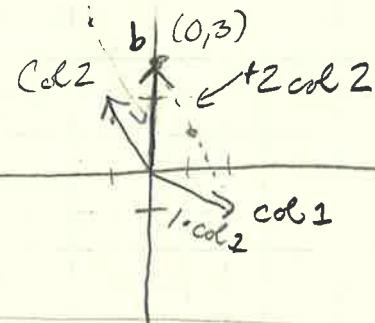


Column picture

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Find the linear combination of the columns to produce $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Take $x=1$ and $y=2$



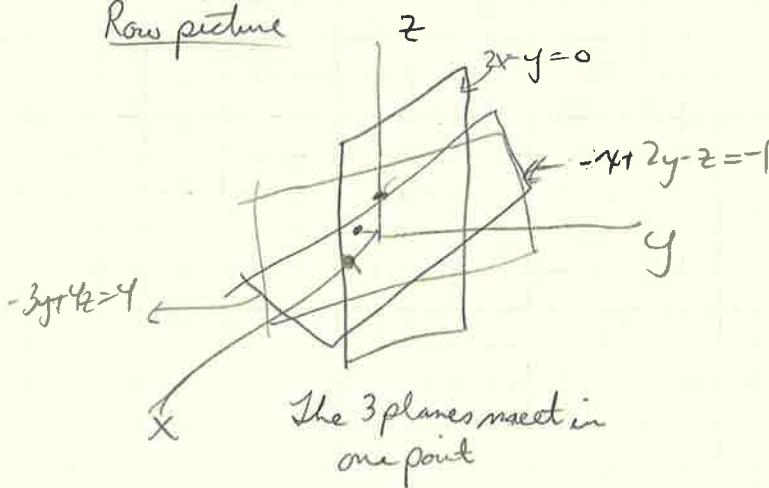
Take all of the combinations

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \text{ which fills the plane}$$

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row picture

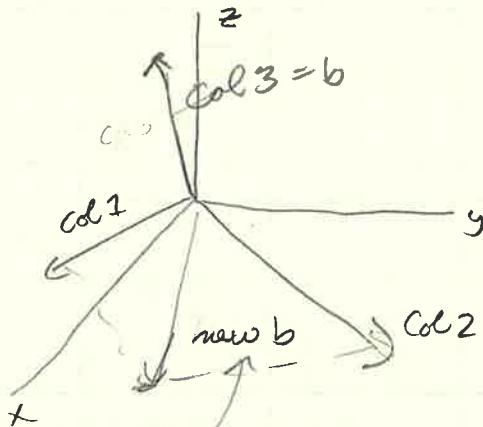


Column picture

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Linear combination
of 3-30 vectors

$$x=0, y=0, z=1$$



$$\text{Suppose } b = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$x \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \Rightarrow x=1, y=1, z=0$$

Can I solve $A\vec{x} = b$ for every b ?

\Leftrightarrow Do the linear combinations of the columns fill 3-D space?
 For this A , the answer is yes.

$$Ax = b$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Ax is a combination of the columns of A

Elimination \leftarrow Success
failure

Back-Substitution

Elimination Matrices

Matrix Multiplication

Lecture 2

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{array} \right] \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 - 0R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{array} \right] \xrightarrow{\substack{R_3 - 2R_2 \\ (3,2)}} \left[\begin{array}{ccc|c} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{array} \right]$$

first pivot Second pivot third pivot

Second pivot

third pivot

U (upper triangle)

Elimination failure (not getting 3 pivots)

Back Substitution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array} \right] \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 - 0R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array} \right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array} \right]$$

A b U C

$$Ux = C:$$

$$\begin{aligned} x + 2y + z &= 2 & x &= 2 \\ 2y - 2z &= 6 & y &= 1 \\ 5z &= -10 & z &= -2 \end{aligned}$$

Elimination Matrices

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} 3 \\ 4 \\ 5 \end{array} \right] = 3 \cdot \text{col 1} + 4 \cdot \text{col 2} + 5 \cdot \text{col 3}$$

Matrix \times column = column

$$\left[\begin{array}{ccc} 1 & 2 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right] = 1 \cdot \text{row 1} + 2 \cdot \text{row 2} + 7 \cdot \text{row 3}$$

Row \times matrix = row

$$E_{21} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{array} \right] = \left[\begin{array}{c} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{array} \right]$$

Step 1:
Subtract 3 row 1 from row 2
and gives new row 2

$$E_{32} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array} \right] \left[\begin{array}{c} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{array} \right] = \left[\begin{array}{c} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{array} \right]$$

Step 2:
Subtract 2 row 2 from
row 3 to get new row 3

$$E_{32}(E_{21}A) = U \Rightarrow \text{associativity}$$

$$(E_{32}E_{21})A = U$$

Permutation matrix \Rightarrow Exchange rows

$$\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = \left[\begin{array}{cc} c & d \\ a & b \end{array} \right]$$

$\uparrow P$

Multiply on left
does row operations

Exchange columns

$$\begin{bmatrix} ab \\ cd \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Multiply on right
does column operations $AB \neq BA$ in general (not commutative in general)Inverses

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{-1} \quad E \quad I$

add 3 row 1 to row 2

Lecture 3

Matrix Multiplication (4 ways!)

Inverse of A , AB , A^T Gauss-Jordan / find A^{-1}

$$\text{row 3} \begin{bmatrix} a_{31} & a_{32} & \dots & a_{3n} \end{bmatrix} \begin{bmatrix} b_{14} & \dots & b_{n4} \\ b_{24} \\ \vdots \\ b_{n4} \end{bmatrix} = \begin{bmatrix} c_{34} \\ \vdots \end{bmatrix}$$

$A_{m \times n} \quad B_{n \times p} \quad C = AB_{m \times p}$

$$c_{34} = (\text{row 3 of } A) \cdot (\text{col 4 of } B)$$

$$= a_{31}b_{14} + a_{32}b_{24} + \dots = \sum_{k=1}^n a_{3k}b_{k4}$$

$$\begin{bmatrix} & & & \end{bmatrix} \begin{bmatrix} & & & \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ A \text{ col 1} & A \text{ col 2} & \dots & A \text{ col } p \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$A_{m \times n} \quad B_{n \times p} \quad C_{m \times p}$

Columns of C are combinations
of columns of A

$$\begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} & & \end{bmatrix} = \begin{bmatrix} \text{row 1 of } B \\ \vdots \\ \text{row } m \text{ of } B \end{bmatrix}$$

$A_{m \times n} \quad B_{n \times p} \quad C_{m \times p}$

rows of C are combinations
of rows of B

Column of A \times row of B
($m \times 1$) \times ($1 \times p$)

4th way

$AB = \text{Sum of (columns of } A) \times (\text{rows of } B)$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 8 & 0 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Block Multiplication

$$\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array} = \begin{array}{c|c} \cancel{A_1 B_1} & \cancel{A_1 B_2} \\ \hline \cancel{A_3 B_1} & \cancel{A_3 B_2} \end{array} \quad \begin{array}{l} A_1 B_1 + A_2 B_3 \\ A_1 B_2 + A_2 B_4 \end{array}$$

Inverses (square matrices)

$$A^{-1} A = I = AA^{-1} \quad \text{called invertible, nonsingular}$$

if this exists

Singular case: no inverse

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{can't form } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ as a linear combination of columns of } A$$

no inverse because we can find a vector $x \neq 0$ with $Ax = 0$

$$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$$

but $x \neq 0$

Back to invertible case

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Ax \text{ column } j A^{-1} = \text{column } j \text{ of } I$$

Gauss-Jordan (solve 2 eqns at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 9 \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 7 & -3 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$$

$A \quad I \qquad I \quad A^{-1}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E[AI] = [EA \ EI] = [I \ A^{-1}] \quad \text{because } EA = I$$

\hookrightarrow Product of E's for the elimination steps

Lecture 4

Inverse of AB , A^T

Product of Elimination Matrices
 $A = LU$ (no row exchanges)

$$AA^{-1} = I = A^{-1}A$$

If AB are both invertible, $ABB^{-1}A^{-1} = AIA^{-1} = I$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

$$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

$$(AA^{-1})^T = (I)^T$$

$$(A^{-1})^T A^T = I$$

\hookrightarrow This is the inverse of $A^T \Rightarrow (A^T)^{-1} = (A^{-1})^T$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} A \\ 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} L & E_2^{-1} \\ 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} U \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$L \quad D \quad U$

$$E_{32} E_{31} E_{21} A_{3 \times 3} = U \text{ (no row exchanges)}$$

$$A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_L U$$

$$\begin{matrix} E_{31} = I \\ E_{32} & E_{21} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \quad EA = U$$

Inverses

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L \quad A = LU$$

$$\underline{A = LU}$$

If no row exchanges, the multipliers go directly into L.

How many operations on an $n \times n$ matrix A?

Take $n=100$

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}_{100 \times 100} \rightarrow \begin{bmatrix} \square & \square & \square \\ 0 & \square & \square \\ \vdots & \square & \square \end{bmatrix} \text{ like } 99 \times 99 \rightarrow \begin{bmatrix} 0 & \square & \square \\ 0 & 0 & \square \\ 0 & 0 & \square \end{bmatrix} \text{ like } 98 \times 98$$

(about 100^2)

(about 99^2 ops)

$$\# \text{ of ops} \sim n^2 + (n-1)^2 + \dots + 3^2 + 2^2 + 1^2 \approx \frac{1}{3} n^3 \text{ on A}$$

Cost for every RHS is n^2

$$\begin{matrix} \text{Permutations} & 3 \times 3 \\ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \end{matrix}$$

switch rows 1+2

$$P_{123} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{132} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{Group of 6 matrices}$$

$$\text{cycle} \quad P_{12}^{-1} = P_{12} \quad P^{-1} = P^T \text{ for permutation matrices}$$

$$\# 4 \times 4 \text{ permutation matrices} = 24$$

Section 2.7 $PA = LU$

Section 3.1 Vector Spaces and Subspaces

Permutations P : execute row exchanges

$$A = LU = \begin{bmatrix} 1 & 0 \\ \cancel{1} & 1 \end{bmatrix} \begin{bmatrix} 0 & \star \\ 0 & 1 \end{bmatrix} \text{ (no row exchanges)}$$

becomes $PA = LU$ (with row exchanges)

for any invertible A

P = identity matrix with reordered rows.

$n!$ permutation matrices of size n .

$$P^{-1} = P^T \Leftrightarrow P^T P = I$$

Transposes

$$R^T \rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \end{bmatrix}$$

$$(A^T)_{ij} = A_{ji}$$

$R^T R$ is always symmetric

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 12 \end{bmatrix}$$

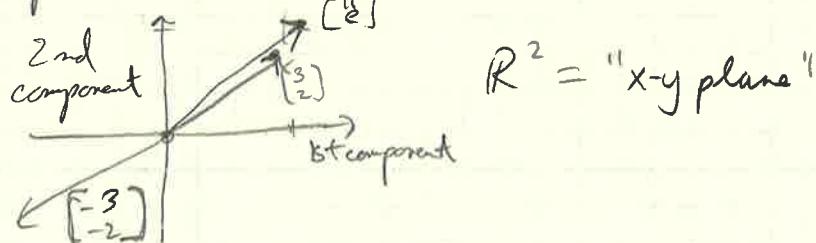
Symmetric Matrices : $A^T = A$

$$\text{Ex. } \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$$

$$(R^T R)^T = R^T R^{T^T} = R^T R$$

Vector Spaces

Examples: \mathbb{R}^2 = all 2-dim real vectors such as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ e \end{bmatrix}, \dots$

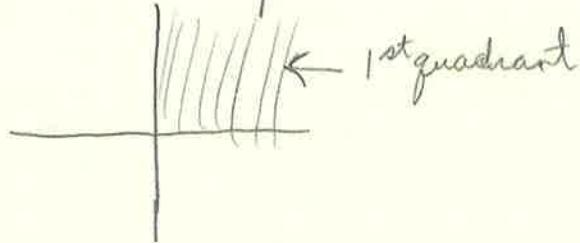


\mathbb{R}^3 = all vectors with 3 real components

$$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

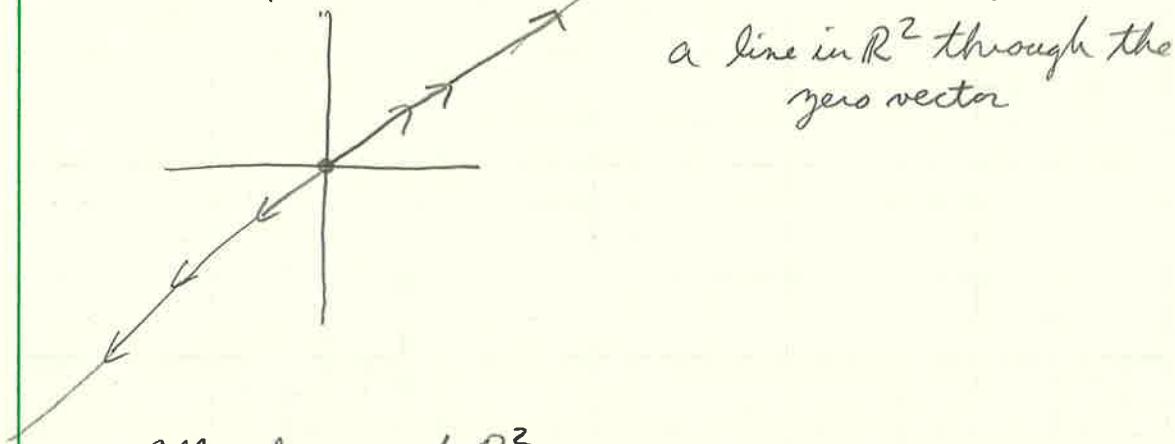
\mathbb{R}^n = all vectors with n real components

not a vector space



Can add and stay in the 1st quadrant, but scalar multiplication may bring us out of the space

A vector space inside \mathbb{R}^2 called a subspace of \mathbb{R}^2



All subspaces of \mathbb{R}^2

① All of \mathbb{R}^2

② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, L

③ zero vector only, { }
Z

All subspaces of \mathbb{R}^3

① All of \mathbb{R}^3

② any line through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

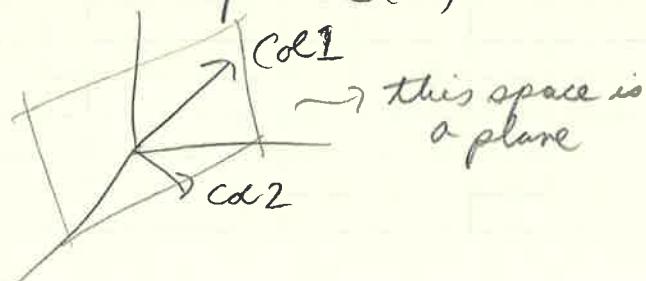
③ any plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

④ zero vector only

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

columns in \mathbb{R}^3

all their linear combinations form a subspace
called the column space C(A)



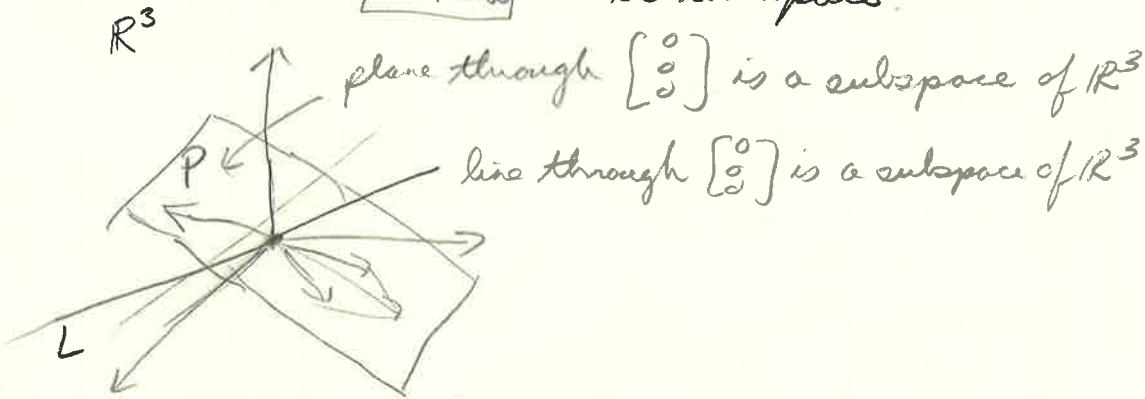
→ this space is
a plane

Vector Spaces and Subspaces

Column space of A : Solving $AX=b$
 Null space of A

Vector Space requirements: $v+w$ and cw are in the space

all combinations $[cv+dw]$ are in the space.



2 subspaces: P and L

$P \cup L =$ all vectors in P or L or both

This ~~(is not)~~ a subspace

$P \cap L =$ all vectors in both P and L

This is a vector space

Subspaces S and T : $S \cap T$ is a subspace

$C(A)$: Column Space of A (is a subspace of \mathbb{R}^4 here)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \quad C(A) = \text{all linear combinations of the columns}$$

Does $AX=b$ have a solution for every b ? No

$$AX = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \text{For some RHS } b \text{ this can be solved.}$$

Which b allow this system to be solved? $b=0$ always

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ works} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ works}$$

Exactly when $b \in C(A)$

In this case $C(A)$ is a 2-D subspace of \mathbb{R}^4

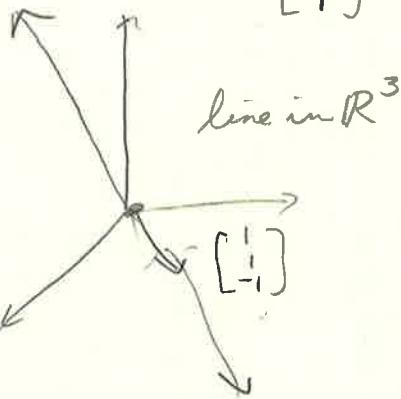
$N(A)$: Nullspace of A = all x 's that solve $Ax=0$ (in \mathbb{R}^3 here)

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$ contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} c \\ -c \\ -c \end{bmatrix}$

$O \in N(A)$ always

$N(A)$ contains $c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for any c



Check that solutions to $Ax=0$ always give a subspace

If $Av=0$ and $Aw=0$ then $A(v+w)=0$

$$A(v+w) = Av + Aw = 0 + 0 = 0 \quad \checkmark$$

If $Av=0$ then $A(cv)=0$

$$A(cv) = cAv = c0 = 0 \quad \checkmark$$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Do the solutions form a subspace? No
No 0-vector to start

$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution

$x = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is a solution

Computing the nullspace ($Ax=0$)
Pivot Variables - free variables
Special Solutions - rref(A) = R

Lecture 7

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

echelon form

↑ ↑
2 pivot columns

The rank of A = number of pivots = 2 in this case

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ ↑
free columns

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
 $"$ $2x_3 + 4x_4 = 0$
 $x_3 = -2x_4$
 $"$ 16
 $x_1 + 2 = 0$

$$c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$$

$$\text{Choose } x_2 = 0, x_4 = 1 \Rightarrow x_1 + 2x_3 + 2 = 0$$

$$2x_3 + 4 = 0$$

$$x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

$$x_3 = -2$$

$$x_1 - 4 + 2 = 0 \Rightarrow x_1 = 2$$

$$d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

$$c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

Rank $r = 2 = \# \text{ of pivot variables}$

$n - r = 4 - 2 = 2$ free variables

R = reduced row echelon form has zeros above and below

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot rows and columns

$$\begin{array}{rcl} x_1 + 2x_2 & -2x_4 = 0 \\ x_3 + 2x_4 = 0 & & Rx = 0 \end{array}$$

$$I \rightarrow \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \quad \boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}} \quad F \quad F$$

pivot cols free cols

Signs are flipped for free variables

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$\begin{matrix} r \\ r \end{matrix}$ pivot rows
 $\begin{matrix} r \\ r \end{matrix}$ pivot cols $n-r$ free cols

$$Rx = 0$$

nullspace matrix N
 (columns are special solutions)

$$RN = 0$$

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$Rx = 0$$

$$[I \ F] \begin{bmatrix} x_{\text{pivot}} \\ x_{\text{free}} \end{bmatrix} = 0$$

$$x_{\text{pivot}} = -Fx_{\text{free}}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

Rank $r=2$
again!
 $n-r=3-2=1$

set free var to 1

$$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = -1$$

$$2x_2 + 2x_3 = 0 \Rightarrow x_2 = -1$$

$$C \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = N(A^T) = \text{a line in } \mathbb{R}^3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

$$-C \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = C \begin{bmatrix} -F \\ I \end{bmatrix}$$

Complete Solution of $Ax=b$

Rank r

$r=m$: Solution exists $r=n$: Solution is unique

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 &= b_3 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array} \right]$$

Augmented matrix $[A \ b]$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right]$$

$$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \text{ is OK}$$

Lecture 8

Solvability: Condition on b

$Ax = b$ is solvable iff $b \in C(A)$

If a combination of the rows of A gives a zero row, then the same combination of the entries of b must give 0.

To find the complete solution to $Ax = b$

① $x_{\text{particular}}$: Set all free variables to zero. Solve $Ax = b$ for pivot variables
 $x_2 = 0, x_4 = 0$

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \Rightarrow x_3 = 3/2 \text{ and } x_1 = -2 \end{aligned}$$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

② $x_{\text{nullspace}} (x_n)$

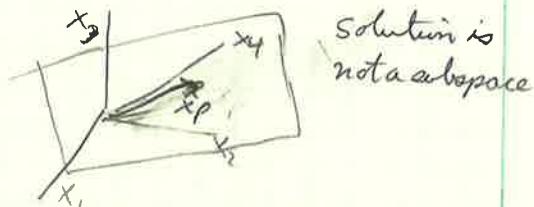
③ Complete solution $x = x_p + x_n$

$$\begin{array}{l} Ax_p = b \\ + Ax_n = 0 \end{array}$$

$$A(x_p + x_n) = b$$

$$x = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Plot all solutions x in \mathbb{R}^4



M by n matrix A of rank r (know $r \leq m, r \leq n$)

Full column rank means $r=n$: no free variables

$N(A) = \{0\}$, Solution to $Ax = b$: $x = x_p$ unique if it exists

$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ either 0 or 1 solutions.

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } b = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix} \Rightarrow x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Full row rank means $r=m$: m pivots

Can solve $Ax = b$ for every b Exists

Left with $n-r$ free variables ($n-m$)

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \text{ rank } r=2, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \boxed{-F-}$$

$r = m = n$ (invertible)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad R = I \quad N(A) = \{0\}$$

b can be anything

$$\begin{array}{l} r = m = n \\ R = I \\ 1 \text{ solution} \\ \text{to } Ax = b \end{array}$$

$$\begin{array}{l} r = m < n \\ R = \begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 \text{ or } 1 \text{ solution} \\ \text{to } Ax = b \end{array}$$

$$\begin{array}{l} r = m < n \\ R = \begin{bmatrix} I \\ F \end{bmatrix} \\ \text{columns mixed} \\ \infty \text{ solutions to } Ax = b \end{array}$$

$$r < m, r < n$$

$$R = \begin{bmatrix} I \\ F \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} 0 \text{ or } \infty \text{ solutions to } Ax = b \\ \text{columns mixed} \end{array}$$

Linear independence
Spanning a space
Basis and dimension

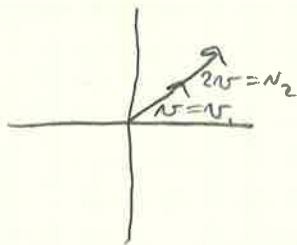
Lecture 9

Suppose A is m by n with $m < n$, then there are nonzero solutions to $Ax = 0$.
More unknowns than equations

Reason: There will be at least one free variable !! ($n-m$ free variables)

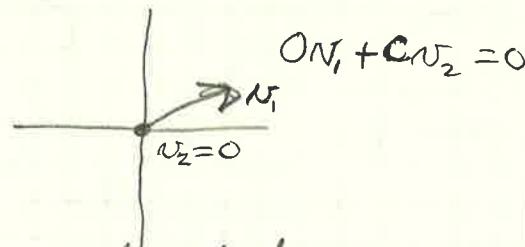
Independence Vectors x_1, x_2, \dots, x_m are linearly independent if no combination gives the zero vector except for the zero combination (all $c_i = 0$)

$$c_1 x_1 + c_2 x_2 + \dots + c_m x_m \neq 0 \quad \text{for all } c_i \neq 0$$



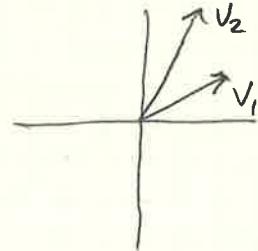
$$2v_1 - v_2 = 0$$

dependent

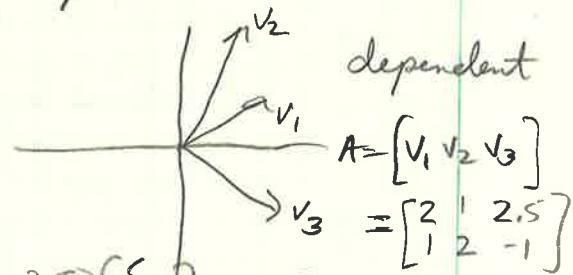


$$0v_1 + cv_2 = 0$$

If any of the v_i are 0, then they are dependent



independent



dependent

$$A = [v_1 \ v_2 \ v_3]$$

$$= \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix}$$

dependence $\Leftrightarrow N(A)$ is more than $\{0\}$

$$\begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Repeat when v_1, \dots, v_n are columns of A . They are independent if $N(A)$ is $\{0\}$. They are dependent if $Ac = 0$ for some nonzero $c \in N(A)$.

rank $= n$ and $N(A) = \{0\}$ for independence
rank $< n$ and yes free variables \Leftrightarrow dependence

Spanning a Space

Vectors v_1, \dots, v_d span a space means that the space consists of all combinations of those vectors.

A Basis for a vector space is a sequence of vectors v_1, v_2, \dots, v_d with 2 properties:

- ① They are independent
- ② They span the space

Example:

Space is \mathbb{R}^3 :

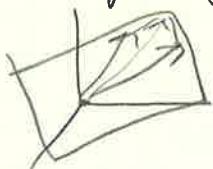
One basis is: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \text{ iff } c_1 = c_2 = c_3 = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I, N(I) = \{0\}$$

Another basis $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$ matrix must be square and invertible

\mathbb{R}^n : n vectors give a basis if the $n \times n$ matrix with those columns is invertible

are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ a basis for any space? They are independent and span a plane
so yes



Bases are not unique, but they all have the same number of vectors.

Given a space, every basis for the space has the same number of vectors.
The number is the dimension of the space.

Examples: $C(A)$

$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{cols} & \text{free} & \end{bmatrix}$ $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \in N(A) \Rightarrow$ the columns span but are not independent
rank $A = 2$

1) Correct error in Lecture 9

2) Four fundamental subspaces (for matrix A)

Example space is \mathbb{R}^3

(from lecture 9)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

standard basis

Another basis

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$$

, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$ is Not invertible

(two equal rows)

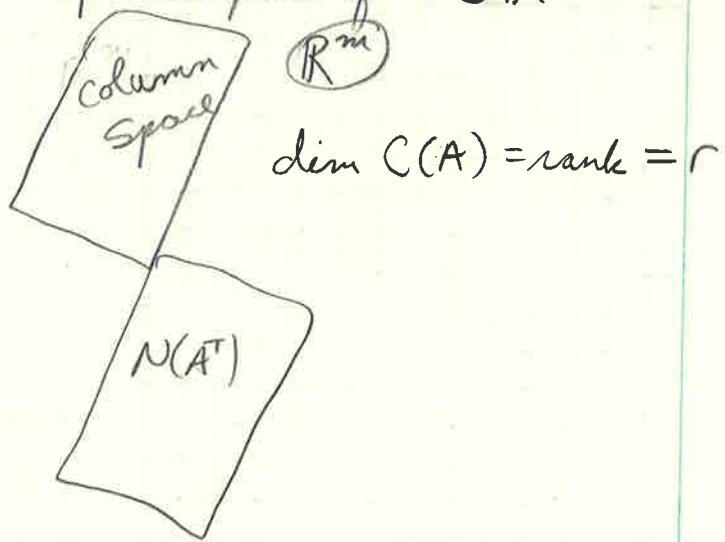
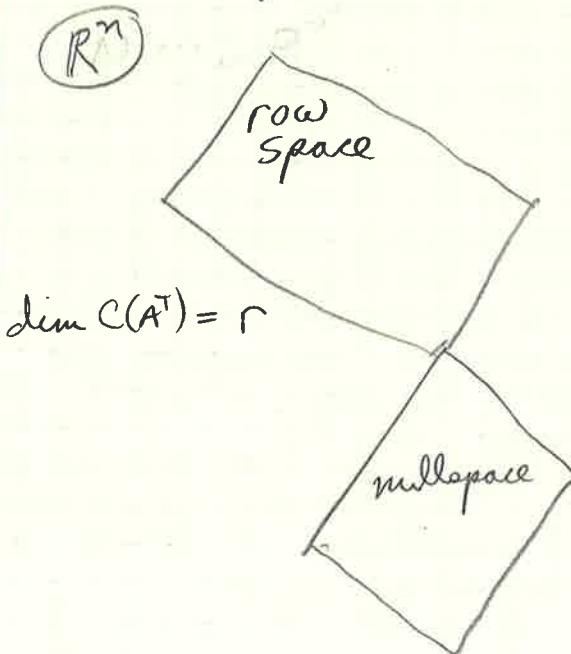
4 Fundamental Subspaces

Column space $C(A) \in \mathbb{R}^m$ A is $m \times n$

Nullspace $N(A) \in \mathbb{R}^n$

Rowspace = all combinations of rows = all combinations of columns of A^T
 $= C(A^T) \in \mathbb{R}^n$

Nullspace of $A^T = N(A^T) =$ left nullspace of $A \in \mathbb{R}^m$



$$\dim C(A^T) = r$$

	$C(A)$	$C(A^T)$	$N(A)$	$N(A^T)$
basis	pivot cols	see below	special solutions for each free variable	See below
dimension	r	r	$n-r$ ↑ number of free variables in A	$m-r$ ↑ # of free variables in A^T

$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ F
 $\rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = R$

$$C(R) \neq C(A)$$

different column spaces

Basis of row space for A
and R is first r rows
of R (not of A)

some row space

$$\underline{N(A^T)} : A^T y = 0 \Rightarrow y \in N(A^T)$$

$$\begin{bmatrix} \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad \text{transpose } y^T A = 0^T$$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\text{ref } \begin{bmatrix} A_{m \times n} & I_{n \times n} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m \times n} & E_{n \times n} \end{bmatrix}$$

$$\Leftrightarrow E \begin{bmatrix} A_{m \times n} & I_{n \times n} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m \times n} & E_{n \times n} \end{bmatrix}$$

$$\Rightarrow EA = R$$

In Chapter 2, R was I and E was A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{array} \right] = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

\uparrow Basis for $N(A^T)$

New vector space!

M : All 3×3 matrices!! $A+B$, cA (not AB for now)

Subspaces of M { all upper triangular
all symmetric matrices
all diagonal matrices } D : $\dim D=3$

{ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \}$ is a basis for D

Lecture 11

- Bases of new vector spaces
- Rank one matrices
- Small world graphs

$M = \text{all } 3 \times 3 \text{ matrices}$ $\dim M = 9$

Subspaces $\left\{ \begin{array}{l} \text{symmetric } 3 \times 3 = S, \dim S = 6 \\ \text{upper triangular } 3 \times 3 = U, \dim U = 6 \\ \text{diagonal } 3 \times 3 \end{array} \right.$

Basis for $M = \text{all } 3 \times 3 \text{ matrices}$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \quad \dim M = 9$$

Basis for S is not in basis for M

$S \cap U = \text{symmetric and upper triangular} = \text{diagonals}$, $\dim(S \cap U) = 3$

$S \cup U$ is not a subspace

$S + U = \text{any element of } S + \text{any element of } U = \text{all } 3 \times 3's$
 $\dim(S + U) = 9$

$$\dim S + \dim U = \dim(S \cap U) + \dim(S + U)$$

$$\frac{d^2y}{dx^2} + y = 0 \Rightarrow y = \cos x, \sin x$$

complete solution: $y = C_1 \cos x + C_2 \sin x$

$\{\sin x, \cos x\}$ is a basis

$$\dim(\text{solution space}) = 2$$

Rank one matrices

$$A_{2 \times 3} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} \quad \text{Basis for } C(A^T) = [1 \ 4 \ 5] \quad \dim C(A) = \text{rank} = \dim C(A^T) = 1 = r$$

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \ 4 \ 5] \quad \text{Rank 1 matrices : } A = uv^T$$

u, v are column vectors

$M = \text{all } 5 \times 17 \text{ matrices}$

Subset of rank 4 matrices, is this a subspace? no

$$\text{rank}(A+B) \leq \text{rank } A + \text{rank } B$$

In \mathbb{R}^4 , $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, $S = \text{all } v \in \mathbb{R}^4 \text{ with } v_1 + v_2 + v_3 + v_4 = 0$

S is a subspace since $C[v_1 + v_2 + v_3 + v_4] = 0$ and

$$\text{for } v, w \in S \quad v_1 + v_2 + v_3 + v_4 + w_1 + w_2 + w_3 + w_4 = 0$$

$\dim S = 3$, $S = \text{null space of } A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}_{n \times n=4 \times 4} \quad (Av=0)$

$$\text{rank } A = 1 = r$$

$$\dim N(A) = n - r = 4 - 1 = 3$$

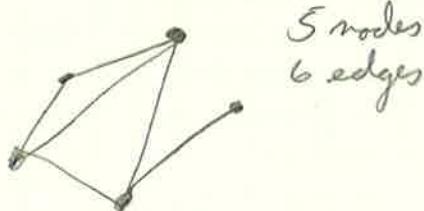
Basis for $S = N(A)$

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$C(A) = \mathbb{R}^1$$

$$N(A^T) = \{0\} \quad \dim N(A^T) = 0$$

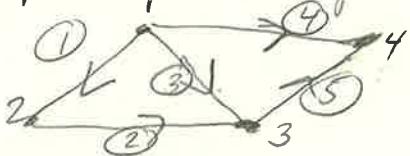
Graph = {nodes, edges}



Lecture 12

- Graphs & Networks
- Incidence Matrices
- Kirchhoff's Laws

Graph : Nodes, Edges



$n=4$ nodes
 $m=5$ edges

Incidence Matrix

$$A = \begin{bmatrix} \text{nodes} & 1 & 2 & 3 & 4 \\ & -1 & 1 & 0 & 0 \\ & 0 & -1 & 1 & 0 \\ & -1 & 0 & 1 & 0 \\ & -1 & 0 & 0 & 1 \\ & 0 & 0 & -1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{edge} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \quad \left. \begin{array}{l} \text{loop} \end{array} \right\}$$

$$Ax = 0$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = N(A) \quad \dim N(A) = 1$$

$$x = (x_1, x_2, x_3, x_4)$$

Potentials at nodes
 $e = Ax \downarrow$

$x_2 - x_1$, etc. Potential differences across edges

$$y = Ce \downarrow \quad (\text{Ohm's law})$$

currents on edges
 y_1, y_2, y_3, y_4, y_5

Set $x_4 = 0$ to ground a node. rank = 3

$$A^T y = 0 \quad \text{to get } N(A^T), \quad \dim N(A^T) = 5-3=2$$

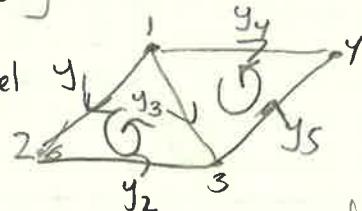
$$A^T y = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-y_1 - y_3 - y_4 = 0 \quad \text{current into node 1}$$

$$y_1 - y_2 = 0 \quad \text{current into node 2}$$

$$y_2 + y_3 - y_5 = 0 \quad \text{node 3}$$

$$y_4 + y_5 = 0 \quad \text{node 4}$$



around big loop

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \text{sum of basis vectors}$$

$$\text{Basis for } N(A^T) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$A^T y = f \quad \uparrow \text{current source}$$

Kirchhoff's current law.

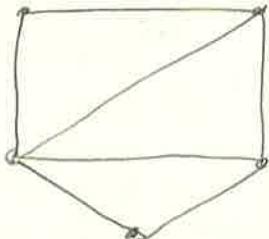
Tree: Graph with no loops

$$\dim N(A^T) = m - r \quad r = n - 1$$

$$\# \text{ loops} = \# \text{ edges} - (\# \text{ nodes} - 1)$$

$$\# \text{ nodes} - \# \text{ edges} + \# \text{ loops} = 1 \quad \text{Euler's formula}$$

$$5 - 7 + 3 = 1$$



$$A^T C A x = f \quad \text{for equilibrium}$$

Lecture 13 : review for exam

13.1

(Q1) Let $U_{5 \times 3}$ be in rref with $r=3$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow N(U) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$B = \begin{bmatrix} U \\ 2U \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} U \\ 0 \end{bmatrix} \quad \text{by block operations}$$

$$C = \begin{bmatrix} u & u \\ u & 0 \end{bmatrix}_{10 \times 6} \xrightarrow{} \begin{bmatrix} u & u \\ 0 & -u \end{bmatrix} \xrightarrow{} \begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix} \xrightarrow{} \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$$

$$\text{rank } C = 6, \dim N(C^T) = 10 - 6 = 4$$

$$(Q2) Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A is 3×3 , rank $r=1$ because $\dim N(A) = 2$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$Ax = b$ can be solved if
 b has the form $b = c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q) If $N(A) = \{0\}$, then $N(A^T) = \{0\}$

Q) The space of invertible matrices is NOT a subspace.

Q) If $B^2 = 0 \Rightarrow B = 0$? False

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Q) If $A_{n \times n}$ has independent columns, is $Ax=b$ always solvable?
Yes

Q) $B = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{3 \times 4} \underbrace{\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$ dont multiply them

basis for $N(B) \subseteq \mathbb{R}^4$

If $B = CD$, and C is invertible $\Rightarrow N(CD) = N(D)$

basis for $N(B) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Complete solution to $Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$x_p + x_n = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_g + c \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Since 1st column of $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Q) If $m=n \Rightarrow$ then $C(A) = C(A^T)$? False

Q) A and $-A$ share the same 4 subspaces? True

Q) If A, B have the same 4 subspaces, then $A = cB$ for some $c \neq 0$?
False. For example, $A+B$ are any invertible 6×6 matrices.

Q) If we exchange two rows of A , which subspaces stay the same?
 $C(A^T), N(A)$

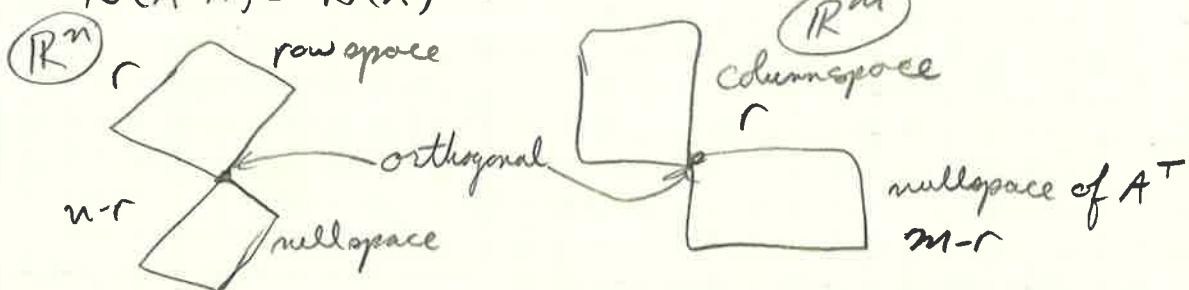
Q) $v = \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix}$ can't be in nullspace and be a row of A. Why not?

$$A \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

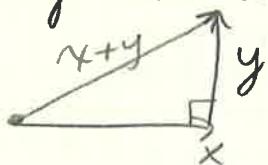
$$N(A) \cap C(A^T) = \emptyset$$

Lecture 14

- Orthogonal vectors and subspaces
- nullspace \perp row space
- $N(A^T A) = N(A)$



Orthogonal vectors



Pythagoras $x^T y = 0$

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2 \quad (\text{only for right triangles})$$

$$x^T x + y^T y = (x+y)^T (x+y)$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$\|x\|^2 = 14, \|y\|^2 = 5, \|x+y\|^2 = 19$$

~~$$x^T x + y^T y = x^T x + y^T y + x^T y + y^T x$$~~

Subspace S is orthogonal to subspace T means:
every vector in S is orthogonal to every vector in T.

row space is orthogonal to nullspace
Why?

$$x \in N(A) \Rightarrow Ax=0$$

$$\begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c_1(\text{row 1})^T x + c_2(\text{row 2})^T x + \dots + c_n(\text{row } n)^T x = 0$$

$$c_1(\text{row 1})^T x = 0, c_2(\text{row 2})^T x = 0, \dots, c_n(\text{row } n)^T x = 0$$

$$c_1(\text{row 1})^T x + c_2(\text{row 2})^T x + \dots + c_n(\text{row } n)^T x = 0$$

$$A = \begin{bmatrix} 1 & 25 \\ 2 & 4/10 \end{bmatrix}, \dim(C(A^T)) = 1, \dim N(A) = 2$$

$n=3, r=1 \quad N(A)$ is a plane \perp to $\left[\begin{smallmatrix} 1 \\ 2 \\ 5 \end{smallmatrix} \right]$

Nullspace and rowspace are orthogonal complements in \mathbb{R}^n ,
i.e., nullspace contains all vectors \perp rowspace.

Coming: "solve" $Ax=b$ when there is no solution
 $n > m$ $\rightarrow A^T A \hat{x} = A^T b$

$A^T A$ is symmetric

$$\underset{n \times m}{\underset{m \times n}{(A^T A)^T}} = A^T A^T = A^T A$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{rank } r = 2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

$$N(A^T A) = N(A)$$

$$\text{rank}(A^T A) = \text{rank}(A)$$

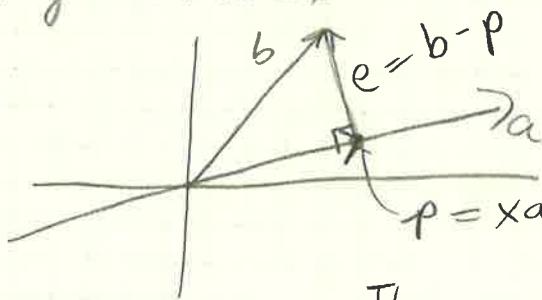
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix} \text{ is not invertible}$$

$A^T A$ is invertible exactly if A has independent columns

Lecture 15

- Projections
- Least Squares
- Projection Matrix

15.1



$$a^T(b-xa)=0$$

$$xa^T a = a^T b$$

$$X = \frac{a^T b}{a^T a} \Rightarrow p = a \frac{a^T b}{a^T a}$$

$$p = a \frac{a^T b}{a^T a} = P b \quad \underset{\text{projection matrix}}{\uparrow} \quad \Rightarrow \quad P = a \frac{a^T}{a^T a}$$

$C(P) = \text{line through } a$

$$\text{rank}(P) = 1$$

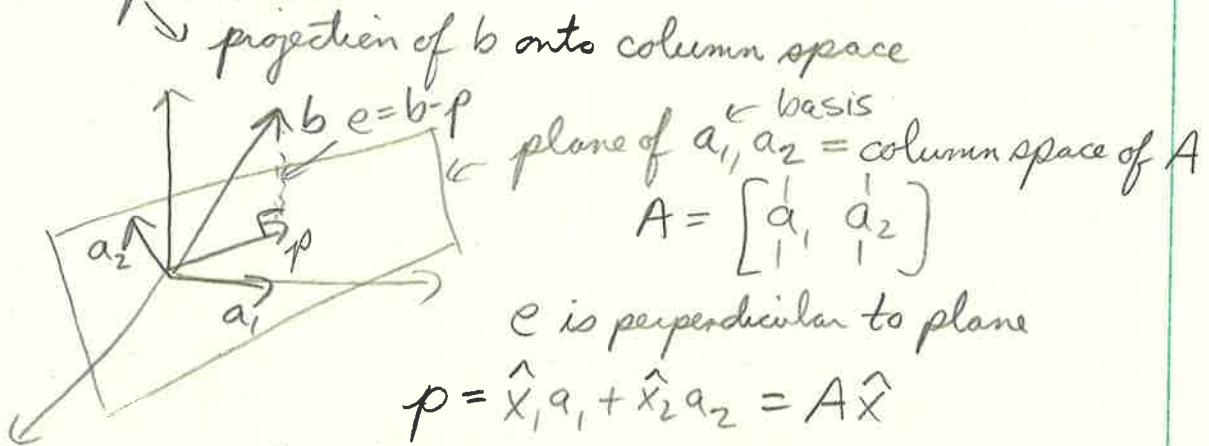
$P^T = P$
$P^2 = P$

Two properties
of a projection matrix

Why project?

Because $Ax=b$ may have no solution.

Solve $A\hat{x}=p$ instead



$$p = A \hat{x}, \text{ find } \hat{x}$$

Key: $b - A \hat{x} \perp$ plane

$$a_1^T(b - A \hat{x}) = 0 \text{ and } a_2^T(b - A \hat{x}) = 0$$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A \hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow A^T(b - A \hat{x}) = 0$$

$$e \in N(A^T) \Leftrightarrow e \perp C(A)$$

$$A^T A \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$\hat{p} = A \hat{x} = A(A^T A)^{-1} A^T b = Pb$$

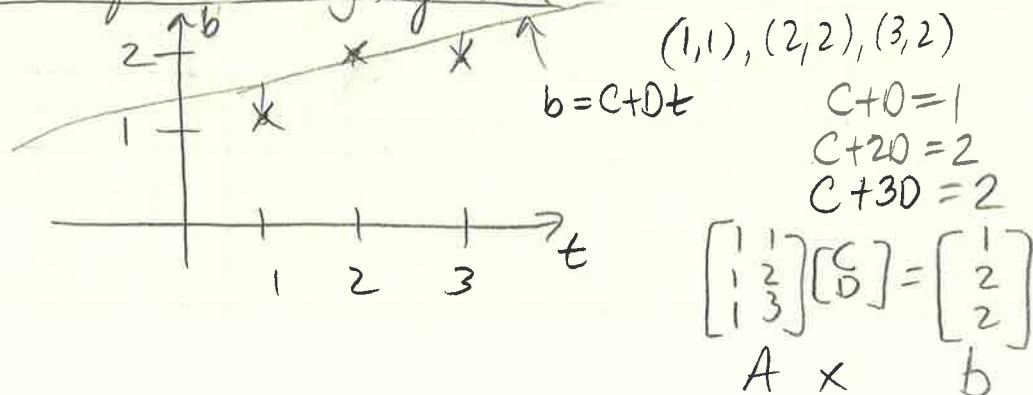
$$P = A(A^T A)^{-1} A^T \quad (A \text{ is not a square matrix})$$

If A is square, $C(A) = \mathbb{R}^n$ and $P = I$ and $b \in C(A)$

$$P^T = P$$

$$P^2 = A(A^T A)^{-1} A^T / A(A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P$$

Least Squares Fitting by a line



Lecture 16

- Projections

- Least Squares and best straight line

Projection matrix

$$P = A(A^T A)^{-1} A^T$$

$$p = Pb$$

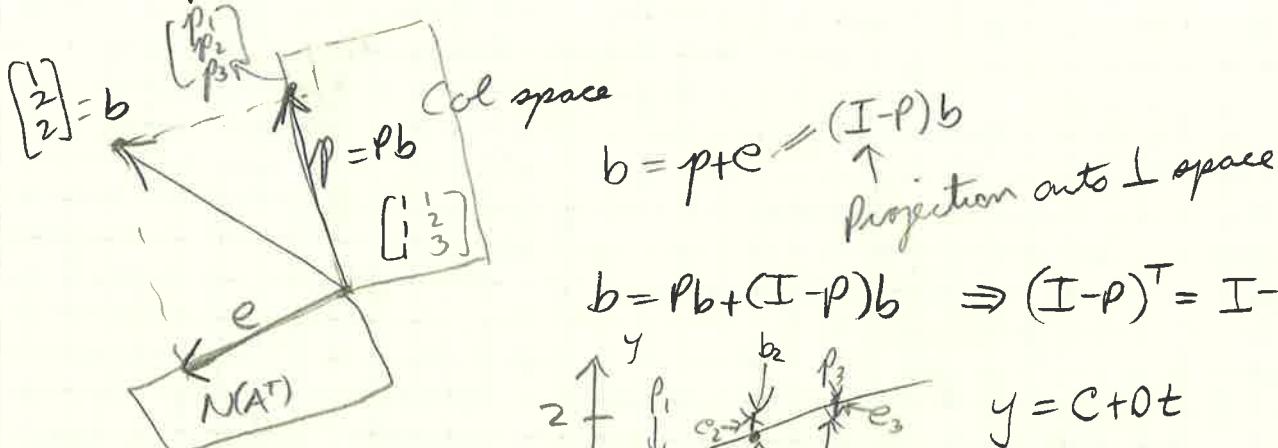
If b in column space, $Pb = b$

$$b = Ax$$

(If $b \perp$ column space, $Pb = 0$)

$$(b \in N(A^T))$$

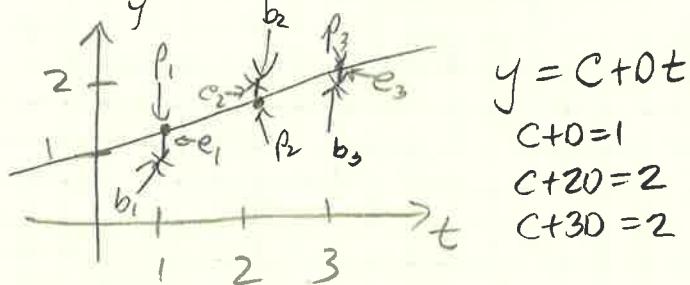
$$p = A(A^T A)^{-1} A^T A x = Ax = b$$



$$b = p + e \quad (I-P)b$$

projection onto \perp space

$$b = Pb + (I - P)b \Rightarrow (I - P)^T = I - P$$



$$Ax = b$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\text{Want to minimize } \|Ax - b\|^2 = \|e\|^2$$

$$= e_1^2 + e_2^2 + e_3^2$$

Find $\hat{x} = \begin{bmatrix} \hat{C} \\ 0 \end{bmatrix}, P : A^T A \hat{x} = A^T b$ (Normal equations)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} = A^T A \quad A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\begin{array}{r|l} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array} \Leftrightarrow \begin{array}{l} 3C + 60 = 5 \\ 6C + 14D = 11 \end{array}$$

$$e_1^2 + e_2^2 + e_3^2 = (C+0-1)^2 + (C+20-2)^2 + (C+30-2)^2 = \|e\|^2$$

 $\frac{\partial \|e\|^2}{\partial C} = 0$ and $\frac{\partial \|e\|^2}{\partial D} = 0$ give the normal equations

$$\left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 6 & 14 & 11 \\ 0 & 2 & 1 \end{array} \right] \Rightarrow 20=1 \Leftrightarrow 0=\frac{1}{2}$$

$$3C + 3 = 5 \Rightarrow C = \frac{2}{3}$$

Best line is $y = \frac{2}{3} + \frac{1}{2}t$

$$p_1 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6} \Rightarrow e_1 = \frac{1}{6}$$

$$p_2 = \frac{2}{3} + 1 = \frac{5}{3} \Rightarrow e_2 = -\frac{1}{3}$$

$$p_3 = \frac{2}{3} + \frac{3}{2} = \frac{13}{6} \Rightarrow e_3 = -\frac{1}{6} \quad (p+e=b)$$

$$b = p+e$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{6} \\ \frac{5}{3} \\ \frac{13}{6} \end{pmatrix} + \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{pmatrix}$$

$$p^T e = 0$$

$e \perp$ to column space

$$e^T [1] = 0, e^T \left[\begin{matrix} 1 \\ 2 \end{matrix} \right] = 0$$

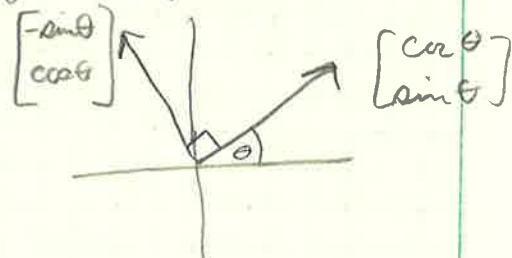
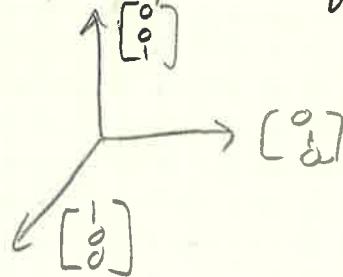
If A has independent columns, then $A^T A$ is invertible

Prof: Suppose $A^T A x = 0$, we want to show that x must be 0

Idea: $\underset{\text{ind col of } A}{x^T A^T A x = 0} = (Ax)^T A x \stackrel{\text{square}}{=} Ax = 0$
 $\Rightarrow x = 0$

Columns are definitely independent if they are perpendicular unit vectors

orthonormal vectors



Lecture 17

- Orthogonal basis q_1, \dots, q_n
- Orthogonal matrix Q : square
- Gram-Schmidt $A \rightarrow Q$

Orthonormal vectors

$$q_i^T q_j = \begin{cases} 0, & i \neq j \\ 1, & i=j \end{cases}$$

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \quad Q^T Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = I$$

If Q is square, then $Q^T Q = I$ tells us that $Q^T = Q^{-1}$ Examples

Permutation matrix $Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \quad Q = \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

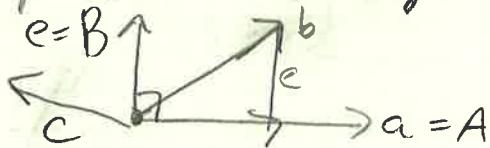
Q has orthonormal columns, project onto its column space

$$P = Q(Q^T Q)^{-1} Q^T = Q Q^T (= I \text{ if } Q \text{ is square})$$

$$P^2 = Q Q^T Q Q^T = Q Q^T = P$$

$$A^T A \hat{x} = A^T b$$

$$\text{Now } A = Q \Rightarrow Q^T Q \hat{x} = Q^T b \Leftrightarrow \hat{x} = Q^T b \Rightarrow \hat{x}_i = q_i^T b$$

Gram-Schmidt(independent) Vectors $a, b \rightarrow$ get orthogonal vectors $A, B \rightarrow$ orthonormal vectors

$$B = b - \frac{A^T b}{A^T A} A$$

$$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|}$$

Check $A \perp B$

$$A^T B = A^T \left(b - \frac{A^T b}{A^T A} A \right) = 0$$

Independent vectors $a, b, c \rightarrow$ orthogonal vectors A, B, C
 \rightarrow orthogonal vectors $g_1 = \frac{A}{\|A\|}, g_2 = \frac{B}{\|B\|}, g_3 = \frac{C}{\|C\|}$

$$A = \tilde{a}$$

$$B = b - \frac{A^T b}{A^T A} A$$

$$C = c - \frac{A^T C}{A^T A} A - \frac{B^T C}{B^T B} B$$

comp in A
dir
 comp in B
dir

$$C \perp A$$

$$C \perp B$$

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

A

$$Q = \begin{bmatrix} 1 & 1 \\ q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \quad C(A) = C(Q)$$

$$A = QR \quad (\text{Gram-Schmidt})$$

$$\begin{bmatrix} 1 & 1 \\ a & b \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ q_1 & q_2 \end{bmatrix} \begin{bmatrix} a^T q_1 & k \\ a^T q_2 & k \end{bmatrix}$$

Lecture 18

18.1

- determinants $\det A = |A|$

- Properties 1, 2, 3, 4-10

- \pm signs

Properties ① $\det I = 1$

② exchange rows: reverse the sign of det

$\det P = \pm 1$, even or odd exchanges of P-permutation matrix

$$\textcircled{1} \quad \left| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right| = 1 \quad , \quad \textcircled{2} \quad \left| \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right| = -1$$

$$\left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = ad - bc$$

Property ③a

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

③b

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\det(A+B) \neq \det A + \det B$$

det is linear for each row

④ 2 equal rows $\Rightarrow \det = 0$ Exchange those rows \rightarrow same matrix⑤ Subtract $l \times \text{row } i$ from row $k \rightarrow$ determinant doesn't change

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \stackrel{\text{③b}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

$$\stackrel{\text{③c}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[0]{\text{④}}$$

⑥ Row of zeros $\Rightarrow \det A = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} \stackrel{\text{③b}}{=} \begin{vmatrix} t \cdot a & t \cdot b \\ c & d \end{vmatrix} \Big|_{t=0} = 0 \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

⑦ $U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & d_3 & * \\ \vdots & \vdots & \vdots & d_n \end{bmatrix}$, $\det U = d_1 d_2 \dots d_n$
(upper triangular) (product of pivots)

$$\det U = d_n d_{n-1} \dots d_2 d_1 \quad \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| \quad \text{after 5, 3a, 1, 2 elimination}$$

⑧ $\det A = 0$ exactly when A is singular \rightarrow row of zero $\Leftrightarrow \det A \neq 0$ when A is invertible $\rightarrow U \xrightarrow{\text{un}} D \rightarrow d_1 d_2 \dots d_n$

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| \stackrel{\text{⑤}}{\rightarrow} \left| \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix} \right| = ad - \frac{c}{a}ab = ad - bc$$

$$\textcircled{9} \quad \det AB = (\det A)(\det B)$$

$$A^{-1}A = I \Rightarrow \det A^{-1} = \frac{1}{\det A}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \text{ and } \det A \det A^{-1} = 6 \left(\frac{1}{6}\right) = 1$$

Proved for diagonal matrices

$$\det A^2 = (\det A)^2$$

$$\det 2A = 2^n (\det A) \quad (\text{factor 2 from every row})$$

$$\textcircled{10} \quad \det A^T = \det A$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

\Rightarrow properties about rows hold
equally for columns.

Proof, using 1-9

$$|A^T| = |A|$$

$$|U^T L^T| = |LU|$$

↓ elimination (in almost every case)

$$|U^T| |L^T| = |L| |U| \rightarrow \begin{matrix} \text{lower triangular with 1's on diagonal} \\ \text{upper } \dots \quad \dots \quad \dots \quad \dots \end{matrix}$$

$$\det L^T = 1, \det L = 1 \Rightarrow |U^T| = |U| \quad \square$$

Lecture 19

19.1

- Formula for $\det A$ ($n!$ terms)
- Cofactor formula
- Tridiagonal Matrices

$$\textcircled{1} \quad \det I = 1$$

$$\textcircled{2} \quad \text{Sign reverse with row exchange}$$

$$\textcircled{3} \quad \det \text{ is linear in each row separately}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \stackrel{\textcircled{3}}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \cancel{\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}$$

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad - \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} = ad - bc$$

column of zeros

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \dots$$

one row exchange one row exchange

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \dots$$

determinants with columns of all zeros
" 0

↙ 2 exchanges ↙ 2 exchanges ↘ 1 exchange

$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The Big Formula

$$\det A = \sum_{n! \text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \dots a_{n\omega} \xrightarrow{\text{sign of permutation of columns}}$$

$(\alpha, \beta, \gamma, \dots, \omega)$ = permutation of $(1, 2, \dots, n)$

Example

$$\det A = \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{\substack{\text{column permutations} \\ 1 : (4, 3, 2, 1) \rightarrow +1 \\ 1 : (3, 2, 1, 4) \rightarrow -1}} \Rightarrow \det A = 0$$

Cofactors 3×3 (in parentheses)

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Cofactor of $a_{ij} = \pm \det \left(\begin{matrix} n-1 \text{ matrix} \\ \text{with row } i, \text{ column } j \text{ erased} \end{matrix} \right) = C_{ij}$

+ when $i+j$ is even

- when $i+j$ is odd

Cofactor formula (along row 1)

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex : Tridiagonal matrix

$$|A_1| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}, |A_2| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$$|A_4| = 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot |A_3| - 1 \cdot 1 \cdot |A_2| = -1$$

$\begin{matrix} " \\ A_3 \\ 0 \end{matrix}$

$$|A_n| = |A_{n-1}| - |A_{n-2}| \Rightarrow |A_5| = -1 - (-1) = 0$$

$$|A_6| = 0 - (-1) = 1, |A_7| = 1 \quad \text{they have period 6} \Rightarrow |A_6| = |A_1| = 1$$

Lecture 20

1. Formula for A^{-1} 2. Cramer's Rule for $x = A^{-1}b$ 3. $|\det A| = \text{volume of box}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T$$

matrix of cofactors (products of $n-1$ entries)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}$$

products of n entries

$$\text{Check } AA^{-1} = AC^T = (\det A)I$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{ni} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \dots & C_{n1} \\ C_{12} & \dots & C_{n2} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} = \det A & & 0 \\ 0 & \det A & \dots \\ 0 & \dots & \sum_{i=1}^n a_{ni} C_{ni} = \det A \end{bmatrix}$$

$$a_{i1} C_{ji} + a_{i2} C_{j2} + \dots + a_{in} C_{jn} = 0, i \neq j$$

Why?

When multiplying like this, it's like the matrix has 2 equal columns.

$$Ax = b$$

$$x = A^{-1}b = \frac{1}{\det A} C^T b$$

Cramer's Rule

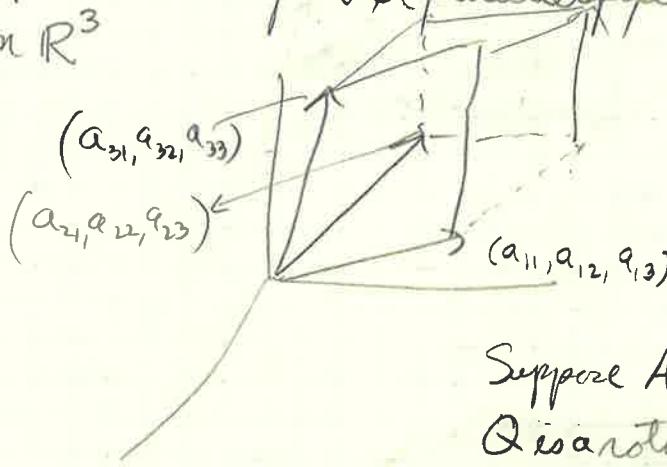
$$\begin{aligned} x_1 &= \frac{\det B_1}{\det A} \\ \vdots \\ x_j &= \frac{\det B_j}{\det A} \end{aligned}$$

$$B_1 = \begin{bmatrix} 1 & & & \\ b & & & \\ & \ddots & & \\ & & n-1 & \\ & & & \text{columns} \\ & & & \text{of } A \end{bmatrix} = A \text{ with column 1 replaced by } b$$

$$\det B_1 = b_1 C_{11} + b_2 C_{21} + \dots + b_n C_{n1}$$

$B_j = A$ with column j replaced by b

$|\det A| = \text{volume of a box (parallelepiped)}$
in \mathbb{R}^3



$A = I \Rightarrow$ the box is the unit cube
and $\det I = 1$

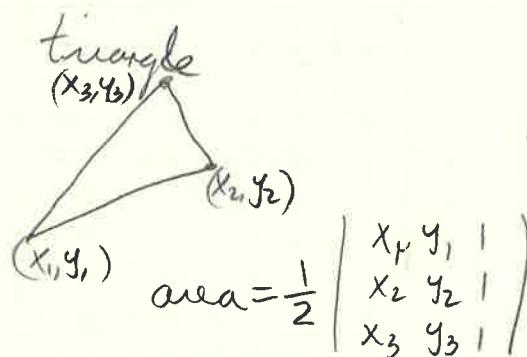
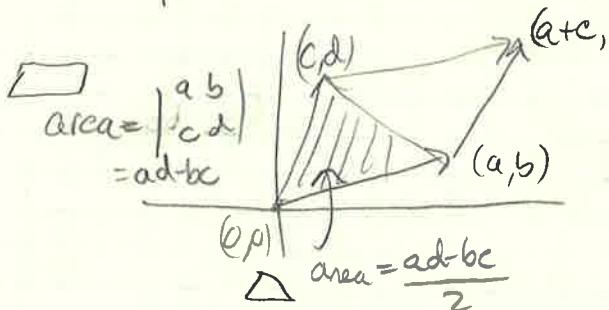
Suppose $A = Q$ orthogonal matrix
 Q is a rotated unit cube (volume = 1)

$$\det(Q^T Q) = \det I = 1 \Rightarrow \det Q = \pm 1$$

double row 1 \Rightarrow 2 identical boxes next to each other $\stackrel{\text{since } \det Q^T = \det Q}{\Rightarrow}$ volume doubles
and the determinant also doubles

$|\det A| = \text{volume of box}$ has det properties ①, ②, ③a, ③b
works too

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



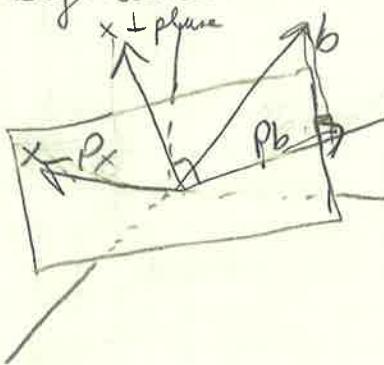
Lecture 21

- Eigenvalues - Eigenvectors
- $\det [A - \lambda I] = 0$
- Trace = $\lambda_1 + \lambda_2 + \dots + \lambda_n$

Ax parallel to x are the eigenvectors

$$Ax = \lambda x \quad \lambda's \text{ are eigenvalues}$$

If A is singular, i.e. $Ax=0$ for some $x \neq 0$, then $\lambda=0$ is an eigenvalue.



What are the eigenvalues and eigenvectors for a projection matrix?

Any x in the plane is an eigenvector:

$$Px = x \Rightarrow \lambda = 1$$

Any $x \perp$ to the plane is an eigenvector.

$$Px = 0x \Rightarrow \lambda = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = x \Rightarrow \lambda = 1$$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -x \Rightarrow \lambda = -1$$

The x's are \perp

Fact: Sum of the eigenvalues equals $\text{tr}(A)$.

How to solve $Ax = \lambda x$:

Rewrite: $(A - \lambda I)x = 0 \Rightarrow A - \lambda I$ must be singular for $x \neq 0$
 $\Rightarrow \det(A - \lambda I) = 0$ (Characteristic eqn.)

Find $n - \lambda$'s first:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$

$$(3-\lambda)^2 - 1 = 0 \Leftrightarrow \lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda-4)(\lambda-2) = 0 \Rightarrow \lambda_1 = 4, \lambda_2 = 2$$

$$(A - 4I)x_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 4$$

$$(A - 2I)x_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda_2 = 2$$

How are $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ related?

Same eigenvectors, $\lambda_2 = 3 + \lambda_1$, $\lambda_{22} = 3 + \lambda_{11}$.

If we add kI to a matrix, its eigenvectors don't change but its eigenvalues increase by k .

If $Ax = \lambda x \Rightarrow (A + kI)x = \lambda x + kx = (\lambda + k)x$ (Great)

Not so great

If $Ax = \lambda x$, B has eigenvalues α . We have no reason to believe that x is an eigenvector of B .

$Ax = \lambda x$, $By = \alpha y \Rightarrow A+B$ or AB can't be combined to find eigenvectors given $x+y$.

Example

90° rotation $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\text{tr } Q = 0 = \lambda_1 + \lambda_2$
 $\det Q = 1 \Rightarrow \lambda_1, \lambda_2$

$$\det(\alpha - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i, \lambda_2 = -i$$

If λ is a complex eigenvalue $\Rightarrow \bar{\lambda}$ and i are a complex conjugate pair

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ eigenvalues of a Δ matrix are on the diagonal
 $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 3$

$$(A - \lambda I)x = 0 \Leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}x = 0 \Rightarrow x_1 = [1]_0, \lambda_1 = 3$$

repeated eigenvalues \Rightarrow no 2nd independent eigenvector

Lecture 22

- Diagonalizing a matrix $S^{-1}AS = \Lambda$
- Powers of A / equation $u_{k+1} = Au_k$
- $A - \lambda I$ singular $\Leftrightarrow Ax = \lambda x$

Suppose n independent eigenvectors of A . Put them in the columns of matrix S .

$$\begin{aligned} AS &= A \begin{bmatrix} | & | & | \\ x_1 & x_2 & \dots & x_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ x_1 & x_2 & \dots & x_n \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} = S\Lambda \end{aligned}$$

eigenvalue matrix
eigenvector matrix

$$AS = S\Lambda$$

$$\begin{aligned} n \text{ independent eigenvectors} &\Rightarrow S^{-1} \text{ exists} \Rightarrow S^{-1}AS = \Lambda \\ \Leftrightarrow A &= S\Lambda S^{-1} \end{aligned}$$

Ex: If $Ax = \lambda x$, $A^2x = \lambda Ax = \lambda^2 x$, squared eigenvalues, same e-vectors

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1} \quad \text{Then, since } \Lambda \text{ is diagonal, the e-vals get squared.}$$

$$\Rightarrow A^k = \lambda^k x \Leftrightarrow A^k = S\Lambda^k S^{-1}$$

Theorem

$A^k \rightarrow 0$ as $k \rightarrow \infty$ if all $|\lambda_i| < 1$. (Needs n -indep eigenvectors)

A is sure to have n -independent eigenvectors (and be diagonalizable) if all λ 's are distinct (no repeated eigenvalues).

Repeated eigenvalues \Rightarrow may or may not have n -independent eigenvectors

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 2 \quad (\text{algebraic multiplicity} = 2)$$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{geometric multiplicity} = 1)$$

Equation Start with given u_0 . solve $u_{k+1} = Au_k$

$$u_1 = Au_0, u_2 = A^2u_0 = A^3u_0 \Rightarrow u_k = A^k u_0$$

To really solve, write $u_k = c_1 x_1 + c_2 x_2 + \dots + c_n x_n = Sc$

$$Au_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_n \lambda_n x_n$$

$$\Rightarrow A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \dots + c_n \lambda_n^{100} x_n (= u_{100}) \\ = SA^{100}c$$

Fibonacci example: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots, F_{100} = ?$

$$F_{k+2} = F_{k+1} + F_k \quad (2^{\text{nd}} \text{ order})$$

$$\text{Trick } u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \Rightarrow F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1}$$

$$\Rightarrow u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_A \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Au_k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = 0$$

$$\Leftrightarrow \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618.. \\ \lambda_2 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.618..$$

same recursion
as in original
 $F_{k+2} - F_{k+1} - F_k = 0$

$$F_{100} \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

Fibonacci numbers grow at an approximate rate of λ ,

$$F_{100} = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2$$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$(A - \lambda I)x = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}x = 0 \Rightarrow x = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ gives } c_1 \text{ and } c_2$$

Lecture 23

- Differential Eqs $\frac{du}{dt} = Au$
- Exponential e^{At} of a matrix

Example

$$\begin{aligned}\frac{du_1}{dt} &= u_1 + 2u_2 \Rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}, u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{du_2}{dt} &= u_1 - 2u_2 \\ &\downarrow \\ \lambda_1 &= 0 \text{ (singular matrix)} \\ \lambda_2 &= -3\end{aligned}$$

$$\lambda_1 = 0: Ax_1 = 0x_1 \Rightarrow x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3: (A + 3I)x_2 = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ ($c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$ for different eqn)

Check: $\frac{du}{dt} = Au$, plug in $e^{\lambda_1 t} x_1$ $\Rightarrow \lambda_1 e^{\lambda_1 t} x_1 = A e^{\lambda_1 t} x_1$
 $\Rightarrow u(t) = c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow c_1 = \frac{1}{3}, c_2 = \frac{1}{3}$
 $u(t) = \frac{1}{3} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$

Steady state: $u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

① When do we get stability, $u(t) = 0$? Need $e^{\lambda t} \rightarrow 0$, $\operatorname{Re} \lambda < 0$
 $|e^{(-3+6i)t}| = e^{-3t}, |e^{6it}| = 1$

② Steady state, $u(\infty) = c$: $\lambda_1 = 0$ and other $\operatorname{Re} \lambda < 0$

③ Blow up if $\operatorname{Re} \lambda_i > 0$ for any i

2×2 stability: $\operatorname{Re} \lambda_1 < 0, \operatorname{Re} \lambda_2 < 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}: \operatorname{tr} A = a+d = \lambda_1 + \lambda_2 < 0 \\ \det A = \lambda_1 \lambda_2 > 0$$

trace < 0
but still blow up
 $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \lambda_1 = 1, \lambda_2 = -2$

$$\frac{du}{dt} = Au \quad \text{Set } u = Sv$$

↑ eigenvector matrix

$$S \frac{dv}{dt} = ASv \Leftrightarrow \frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

$$\frac{dv_1}{dt} = \lambda_1 v_1$$

⋮

$$\frac{dv_n}{dt} = \lambda_n v_n$$

$$\Rightarrow v(t) = e^{\Lambda t} v(0)$$

$$u(t) = Se^{\Lambda t} S^{-1} u(0) = e^{\Lambda t} u(0)$$

$$e^{\Lambda t} = Se^{\Lambda t} S^{-1}$$

$$\text{Matrix exponential } e^{\Lambda t} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots + \frac{(At)^n}{n!} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \dots$$

all $|\lambda_i| < 1$; eigenvalues of At

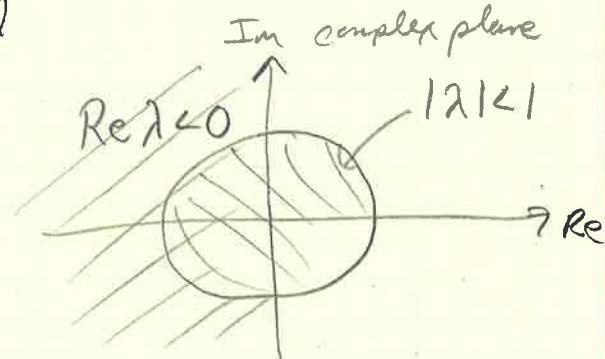
for good approximation when t is small

$$e^{At} = S S^{-1} + S A S^{-1} t + \frac{S A^2 S^{-1} t^2}{2!} + \dots + \frac{S A^n S^{-1} t^n}{n!} + \dots = S e^{\Lambda t} S^{-1}$$

only if A can be diagonalized

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & 0 \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$



$$y'' + b y' + k y = 0$$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \Rightarrow$$

1 - 2nd order eqn \rightarrow 2x2 1st order system

$$u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow 5^{\text{th}} \text{ order to } 5 \times 5 \text{ 1st order}$$

Lecture 24

• Markov Matrices

Steady State $\Rightarrow \lambda = 1$

• Fourier Series and Projections

Markov Matrices

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

① all entries ≥ 0

② all columns sum to 1

1. $\lambda=1$ is an eigenvalue
2. All other $|X_i| < 1$
3. All components of $X_1 \geq 0$

$$A - I\mathbb{I} = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$$

$$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n$$

as $k \rightarrow \infty$, $u(\infty) = c_1 x_1$ since $\lambda_1 = 1$
is the steady state

This is singular: row 2+row 3+row 1 = 0
or all columns sum to 0.

$A - I$ is singular \Leftrightarrow the rows are dependent $\Leftrightarrow (1, 1, 1) \in N(A^T)$
 $\Leftrightarrow X_1 \in N(A)$

eigenvalues of A
eigenvalues of A^T

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \det(A - \lambda I)^T$$

$$= \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$$

$$\begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix} \begin{bmatrix} .6 \\ 33 \\ -.7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_{k+1} = A u_k, A \text{ is Markov}$$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k}$$

↑ Stay in cal ↑ Mass → Cal
 ↓ Cal → Mass ↓ Stay in Mass

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = .7 \quad (\text{trace} = 1.7)$$

$$(A - I)X_1 = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Steady state:
 $u(\infty) = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$(A - .7I)X_2 = \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (-1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, c_1 = \frac{1000}{3}, c_2 = \frac{2000}{3}$$

Projections with orthonormal basis

g_1, \dots, g_n (orthonormal)

$$\text{Any } v = x_1 g_1 + x_2 g_2 + \dots + x_n g_n \rightarrow \begin{bmatrix} g_1 & g_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_n \end{bmatrix} = v \Leftrightarrow Qx = v$$

$$g_1^T v = x_1 g_1^T g_1 + 0 + 0 + \dots + 0 \Rightarrow x_1 = g_1^T v \Leftrightarrow x = Q^T v$$

Fourier Series

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$ Fourier series
 basis = $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$ (orthogonal functions)

for vectors: $v^T w = v_1 w_1 + \dots + v_n w_n$ $f(x) = f(x+2\pi)$

for functions: $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$ periodic

$$\int_0^{2\pi} \sin x \cos x dx = \frac{1}{2} \sin^2 x \Big|_0^{2\pi} = 0$$

$$a_1: \int_0^{2\pi} f(x) \cos x dx = \int_0^{2\pi} a_0 \cos x dx + \int_0^{2\pi} a_1 \cos^2 x dx + \int_0^{2\pi} b_1 \cos x \sin x dx + \dots$$

$$\Leftrightarrow a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x dx$$

Lecture 24b

Review for Quiz 2

$$\textcircled{1} \quad Q = [g_1 \dots g_n]$$

Projections - Least Squares
 Gram-Schmidt

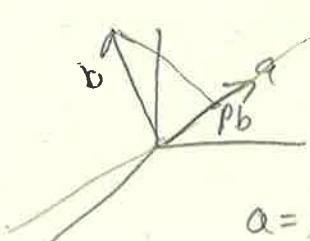
$$Q^T Q = I$$

\textcircled{2} $\det A$
 properties 1-3
 Big formula
 (n^2 terms, \pm)
 Cofactors / A^{-1}

\textcircled{3} Eigenvalues
 $Ax = \lambda x$
 $\det(A - \lambda I) = 0$
 Diagonalize $S^{-1}AS = \Lambda$
 Powers A^k

$$\text{(1)} \quad a = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad P = A(A^T A)^{-1} A^T$$

$$= \frac{aa^T}{a^T a} = \frac{1}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$



$$\text{rank } P = 1 \quad \lambda = 0, 0, \frac{2}{3} = 1$$

$$C(P) = \left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

a = eigenvector for $\lambda = 1 \Leftrightarrow Pa = a$

Solve $u_{k+1} = P u_k$, $u_0 = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$

$$u_1 = P u_0 = a \frac{a^T u_0}{a^T a} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \frac{1}{9} \cdot 27 = \overset{\hat{x}}{3a} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

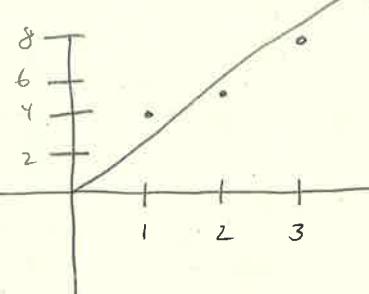
$$u_k = P^k u_0 = P u_0 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$u_0 = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + c_3 \lambda_3^k x_3$$

Q2) $t=1, y=4$
 $t=2, y=5$
 $t=3, y=8$

Fit line through origin



$$y = Dt$$

$$1 \cdot D = 4$$

$$2 \cdot D = 5$$

$$3 \cdot D = 8$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

$$A \hat{x} = b$$

Best D

$$A^T A \hat{D} = A^T b$$

$$14 \hat{D} = 38 \Rightarrow \hat{D} = \frac{38}{14}$$

Projecting b onto $C(A)$ (line)

Q3) $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \text{plane } C(A)$
 Find 2 orthogonal vectors

$$B + a_1 : B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{a_1^T a_2}{a_1^T a_1} a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{6}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Q4) 4×4 matrix $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. What are conditions on λ_i for matrix to be invertible?

a) Invertible \Leftrightarrow no zero eigenvalues

$$(b) \det A^{-1} = \left(\frac{1}{\lambda_1} \right) \left(\frac{1}{\lambda_2} \right) \left(\frac{1}{\lambda_3} \right) \left(\frac{1}{\lambda_4} \right)$$

$$(c) \operatorname{tr}(A + I) = (\lambda_1 + 1) + (\lambda_2 + 1) + (\lambda_3 + 1) + (\lambda_4 + 1) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4$$

Q5) $A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $D_n := \det A_n$

\downarrow
 D_{n-1}

Use cofactors:

$$D_n = \frac{1}{2} D_{n-1} + \frac{-1}{2} D_{n-2} = D_{n-1} - D_{n-2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \uparrow \\ D_{n-2} \end{matrix}$$

$$Solve D_n = D_{n-1} - D_{n-2}$$

$$D_1 = 1 \quad D_1 = [1]$$

$$D_2 = 0$$

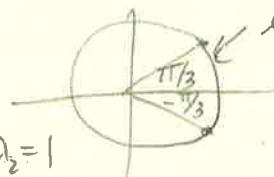
$$D_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0 \quad \lambda = e^{\pm i\pi/3}$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow \lambda_1 = \frac{1+i\sqrt{3}}{2}, \lambda_2 = \frac{1-i\sqrt{3}}{2}$$

$$|\lambda| = \frac{1}{4} + \frac{3}{4} = 1$$



unit circle $|\lambda|=1$

$$\text{Given } \lambda_1^6 = \lambda_2^6 = 1 \Rightarrow A^6 \text{ has } \lambda_1 = \lambda_2 = 1$$

$$\Rightarrow A^6 = I$$

Q6)

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} = A_4^T$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \quad (\text{singular})$$

$$P_3 = A_3(A_3^T A_3)^{-1} A_3^T$$

$$|A_3 - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = -\lambda^3 + 5\lambda = 0 \Leftrightarrow \lambda(-\lambda^2 + 5) = 0$$

$$\lambda_1 = 0, \lambda_2 = \sqrt{5}, \lambda_3 = -\sqrt{5}$$

Find P_4 : $P_4 = I$ if $C(A_4) = \mathbb{R}^4$

$$\det A_4 = -1 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{vmatrix} = -1 \cdot 1 \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = 9 \Rightarrow P_4 = I$$

Probably A_n are singular, n even
 { singular, n odd }

Lecture 25:

- Symmetric Matrices (real)
- Eigenvalues / Eigenvectors
- Start: Positive definite matrices

$$A = A^T \text{ (real)}$$

① The eigenvalues are real.

② The eigenvectors are perpendicular

can be chosen

leads to orthonormal eigenvectors

\Rightarrow columns of Q

Usual case: $A = S \Lambda S^{-1}$

Symmetric case: $A = Q \Lambda Q^{-1} = Q \Lambda Q^T$ - spectral theorem

Why real eigenvalues?

$$Ax = \lambda x \xrightarrow{\text{always}} \bar{A}\bar{x} = \bar{\lambda}\bar{x} \quad \text{Complex conjugate}$$

$$= A\bar{x} = \bar{\lambda}\bar{x} \quad \text{for real matrix}$$

$$\Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda} \Leftrightarrow \bar{x}^T A = \bar{x}^T \bar{\lambda}$$

$$\bar{x}^T A x = \bar{\lambda} \bar{x}^T x \quad \text{and} \quad \bar{x}^T A x = \bar{x}^T \bar{\lambda} x$$

$$\Rightarrow \lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x \Leftrightarrow \lambda = \bar{\lambda} \quad \text{provided } \bar{x}^T x \neq 0$$

$\Leftrightarrow \lambda$ is real

$$\bar{x}^T x = [\bar{x}_1 \bar{x}_2 \dots \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots + \bar{x}_n x_n$$

$$= \|x\|^2 = \text{length squared}$$

Good matrices: Real λ 's and 1×1 's

$$A = A^T \text{ if real}$$

$$A = \bar{A}^T \text{ if complex (conjugate transpose)}$$

$$A = Q \Lambda Q^T \quad (A = A^T)$$

$$= [q_1 \dots q_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots + \lambda_n q_n q_n^T$$

projection matrix

- Every symmetric matrix is a combo of mutually \perp projection matrices.
- Signs of pivots (for $A = A^T$) are same as the signs of the λ 's
 $\Rightarrow \# \text{ positive pivots} = \# \text{ positive } \lambda \text{'s}$
- product of pivots ($A = A^T$) = product of λ 's.

Positive definite symmetric matrix

- All λ 's are positive
- All pivots are positive

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{pivots are } 5, \frac{11}{5}$$

since $\det = 11$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 11 = 0 \Rightarrow \lambda = 4 \pm \sqrt{5}$$

- All sub-determinants are positive

Lecture 26

26.1

Complex vectors, matrices

inner products
Discrete Fourier (FAST) transform = FFT $O(n \log_2 n)$
Fourier matrix F_n

length

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n \quad z^T z \text{ is no good} \quad [1 \ i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

$$\bar{z}^T z = [\bar{z}_1 \ \bar{z}_2 \ \dots \ \bar{z}_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \Rightarrow [1 \ -i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$z^H = \bar{z}^T \quad (\text{Hermitian})$$

Inner product is $z^H z = |z_1|^2 + \dots + |z_n|^2$

Symmetric : $A^T = A$ no good if A is complex

Hermitian $A^H = A$ if A is complex

$$A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} = \bar{A}^T = A^H$$

Perpendicular

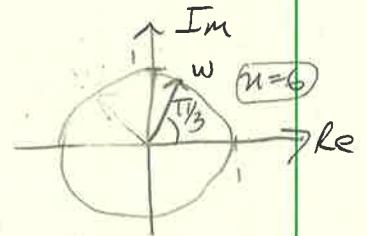
$$q_1, q_2, \dots, q_n \Rightarrow q_i^H q_j = \begin{cases} 0, & i \neq j \\ 1, & i=j \end{cases} \quad Q = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}$$

$Q^H Q = I$ unitary (orthogonal complex matrix)

$n \times n$ Fourier matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2n-1} & \dots & w^{(2n-1)^2} \end{bmatrix} \quad (F_n)_{ij} = w^{ij} \quad i, j = 0, \dots, n-1$$

$$w^n = 1 \quad w = e^{i \frac{2\pi}{n}}$$



$$\underline{n=4} \quad w^4 = 1 \Leftrightarrow w = e^{i \frac{2\pi}{4}} = e^{i \frac{\pi}{2}} = i$$

$$w = i, w^2 = i^2 = -1, w^3 = -i, w^4 = 1$$

$$F_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \quad a^H b = [1 \ -i \ -1 \ i] \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

$$= 1 - i - 1 + i = 0$$

All columns are orthonormal

$$F_4^H F_4 = I$$

$$w_n = e^{i \frac{2\pi}{n}} \Rightarrow w_{64}^2 = w_{32} = e^{i \frac{2\pi}{32}}$$

$$[F_{64}] = [I \ D \ F_{32} \ O] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \leftarrow \text{Permutation matrix } = P$$

$\hookrightarrow 64^2$ calcs

$\hookrightarrow [2(32^2) + 32]$ calcs

$$D = \begin{bmatrix} w & w^2 & & \\ & w^4 & & \\ & & \ddots & \\ & & & w^{31} \end{bmatrix}$$

$$[F_{64}] = [ID \ F_{16}^H C \ F_{16} \ O] \begin{bmatrix} P & & \\ & P & \\ & & P_{64} \end{bmatrix}$$

$$2[2(16)^2 + 16] + 32 \rightarrow 6 \times 32$$

$$\log_2 64 \cdot \frac{64}{2}$$

$$\text{calcs} = \frac{1}{2} n \log_2 n$$

$$n = 1024 = 2^{10}, n^2 > 10^6$$

$$\frac{1}{2} n \log_2 n = \frac{1}{2} (1024)(10) = 5 \cdot 1024$$

Lecture 27

- Positive Definite Matrix (Tests)
- Tests for Minimum ($x^T A x > 0$)
- Ellipsoids in \mathbb{R}^n

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

① $\lambda_1 > 0, \lambda_2 > 0$
 ② det of principal minors > 0
 ③ pivots $a > 0, \frac{ac-b}{a} > 0$
 ④ $x^T A x > 0$ (new)
 $\forall x \neq 0$

complete tests
for positive
definiteness

Examples

$$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

positive semidefinite : $\lambda = 0, 20$,
eigenvalues ≥ 0

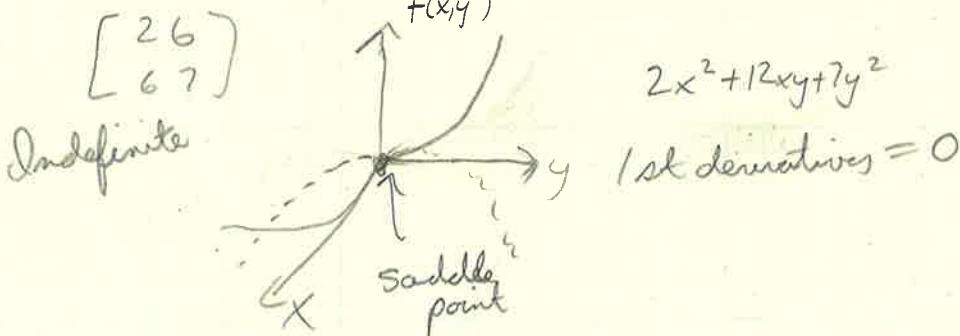
Pivots: 2

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 18x_2^2$$

quadratic form

$$ax^2 + 2bxxy + cy^2 > 0$$

Graph of $f(x,y) = \vec{x}^T A \vec{x} = ax^2 + 2bxy + cy^2$

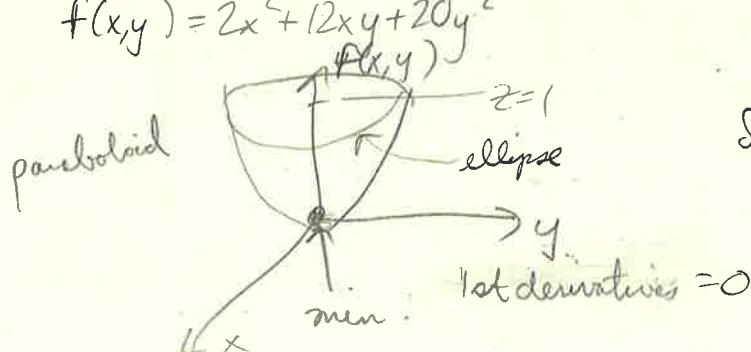


$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \Rightarrow 2x^2 + 12x_1x_2 + 20x_2^2 \quad (\text{positive definite})$$

$$\det = 4, \text{trace} = 22 \Rightarrow \lambda_1, \lambda_2 > 0$$

$x^T A x > 0$ except at $x=0$

$$f(x,y) = 2x^2 + 12xy + 20y^2$$



Calculus: Min $\sim \frac{d^2u}{dx^2} > 0$
 $\sim \frac{du}{dx} = 0$

In linear algebra:
 MIN: $f(x_1, x_2, \dots, x_n)$
 Matrix of 2nd derivatives
 is positive definite

$$f(x,y) = 2x^2 + 12xy + 20y^2$$

$$= 2(x+3y)^2 + 2y^2 > 0 \quad \forall x, y \neq 0 \text{ since it is all squares}$$

pivot multiplier pivots

If we cut this at $z=1$, the curve is an ellipse

$$\begin{bmatrix} A \\ 2 & 6 \\ 6 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} I \\ 2 & 6 \\ 0 & 2 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix of 2nd derivatives (2x2)

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \quad f_{xy} = f_{yx}$$

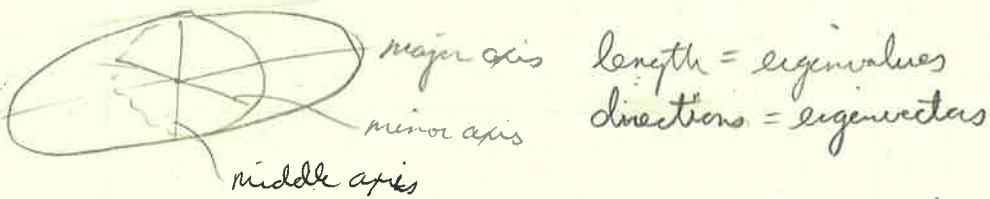
3x3 example

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \begin{array}{l} \text{dets are } 2, 3, 4 \\ \text{pivots } 2, \frac{3}{2}, \frac{4}{3} \\ \text{product of pivots} = \det \\ \text{eigenvalues are all } > 0 \quad 2-\sqrt{2}, 2, 2+\sqrt{2} \end{array}$$

$$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$$

Cut through at $f(x_1, x_2, x_3) = 1$ to get an ellipsoid

$$A = Q \Lambda Q^T$$



major axis length = eigenvalues

minor axis directions = eigenvectors

middle axis

If two eigenvalues are equal, the minor and middle axes have equal length

Lecture 28:

28.1

- $A^T A$ is positive definite!
- Similar Matrices A, B $B = M^{-1} A M$
- Jordan Form

If A, B are positive definite, what about $A+B$? Yes

$$x^T (A+B)x = x^T Ax + x^T Bx > 0$$

Now, $A_{mn} : A^T A$ (rank $A^T A = n$)

$$x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 > 0, \forall x \neq 0$$

square, symmetric, pos def

A and B are similar (not necessarily symmetric), $n \times n$
means: for some M , $B = M^{-1}AM$

Example: A is similar to Λ

$$S^{-1}AS = \Lambda$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-15}{2} \\ 1 & 6 \end{bmatrix} \quad \checkmark B$$

Similar matrices have the same eigenvalues and same # of eigenvectors
(but needs more to complete the picture)

$\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$ are in the family.

$$Ax = \lambda x \quad (B = M^{-1}AM)$$

$$AMM^{-1}x = \lambda x \Leftrightarrow M^{-1}AMM^{-1}x = \lambda M^{-1}x$$

$\Leftrightarrow BM^{-1}x = \lambda M^{-1}x \Rightarrow \lambda$ is an eigenvalue of B

eigenvector of B is M^{-1} (eigenvector of A)

BAD CASE $\lambda_1 = \lambda_2 = 4$.

One family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ which is not similar to $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$
(only member)

Big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ (all others)

$$M^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = 4M^{-1}IM = 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ is (nondiagonalizable) called the Jordan Form

More members of family (same trace, same det)

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} a & \# \\ \# & 8-a \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \text{ all } \lambda = 0, \text{ rank} = 2, \dim \text{Nullspace} = 2 = \# \text{ of indep eigenvectors}$$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\lambda = 0, 0, 0, 0$, rank 2, ... similar to $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\lambda = 0, 0, 0, 0$, rank = 2, has 2 eigenvectors but not (different Jordan blocks) similar to $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Jordan block (has 1 eigenvector only)

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & \\ 0 & & & \lambda_i \end{bmatrix}$$

Every square matrix A is similar to a Jordan matrix J

$$J = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_2 \end{bmatrix} \quad \# \text{ blocks} = \# \text{ of eigenvectors}$$

Good Case: J is Λ

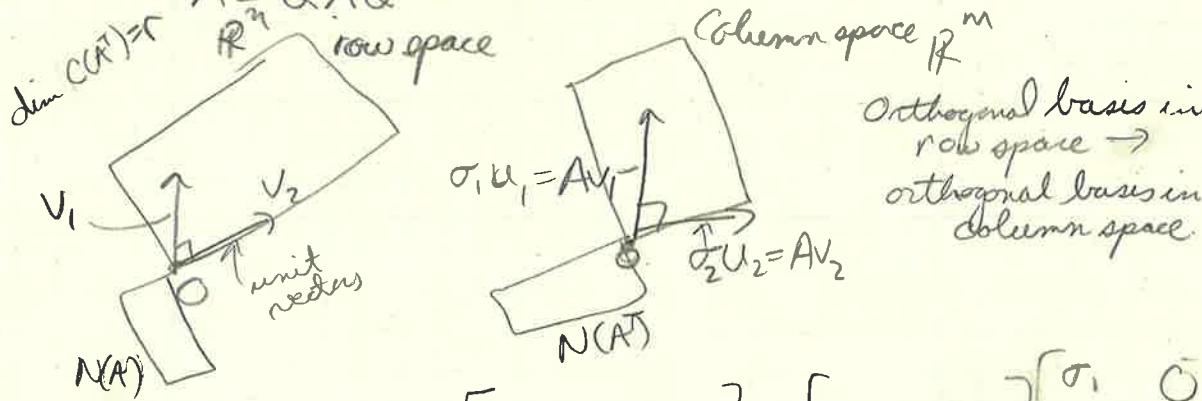
Lecture 29:

- Singular Value Decomposition (SVD) $A = U \Sigma V^T$

Σ - diagonal, U, V - orthogonal

Symmetric positive definite:

$$A = Q \Lambda Q^T$$



$$A \begin{bmatrix} V_1 & V_2 & \dots & V_m \end{bmatrix} = \begin{bmatrix} U_1 & U_2 & \dots & U_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n & 0 \end{bmatrix}$$

$$AV = U\Sigma$$

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

v_1, v_2 in row space = \mathbb{R}^2
 u_1, u_2 in col space = \mathbb{R}^2
 $\sigma_1 > 0, \sigma_2 > 0$

$$Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2$$

$$AV = U\Sigma \Rightarrow A = U\Sigma V^{-1} = U\Sigma V^T$$

$$A^T A = V\Sigma^T U^T U\Sigma V^T$$

$$= V\Sigma^T \Sigma V^T = V\Sigma^2 V^T = V \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^T$$

V are the eigenvectors of $A^T A$, Σ^2 are the eigenvalues

$$AA^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma^2 U^T$$

U are the eigenvectors of AA^T , Σ^2 are the eigenvalues

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \text{ eigenvectors } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 32 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 18 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sigma}{\sqrt{32}} & 0 \\ 0 & \frac{\sigma}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} V^T$$

To finish
eigenvalue

Find U

$$AA^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma^2 U^T$$

Sym positive
semidefinite

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$N(A)$

$$v_2^T v_1 = 0 \Rightarrow v_2 = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$$

$$C(A) = \left\{ k \begin{bmatrix} 4 \\ 8 \end{bmatrix} \mid k \in \mathbb{R} \right\}$$

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u_2^T u_1 = 0 \Rightarrow u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{4}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \end{bmatrix}$$

A U S V^T

$$\begin{aligned} A^T A &= \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix} \end{aligned}$$

$$\lambda_1 = 0, \lambda_2 = 125$$

dim

$r \{v_1, \dots, v_r\}$ - orthonormal basis for the row space

$r \{u_1, \dots, u_r\}$ - " " " " col "

$n-r \{v_{r+1}, \dots, v_n\}$ - " " " " null " of A

$n-r \{u_{r+1}, \dots, u_m\}$ - " " " " null " of A^T

$$\text{and } Av_i = \sigma_i u_i$$

Lecture 30:

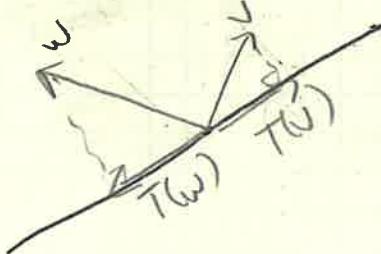
30.1

Linear Transformations T
 without coordinates: normative
 with coordinates: matrix

$$\begin{aligned} T(v+w) &= T(v) + T(w) \\ T(cv) &= cT(v) \end{aligned} \quad \left. \begin{array}{l} T(cv+dw) = cT(v) + dT(w) \\ T(0) = 0 \end{array} \right\}$$

Example 1: Projection

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



(Non)Example 2: Shift whole plane by v_0

$$T(v) = v + v_0$$

This is not linear

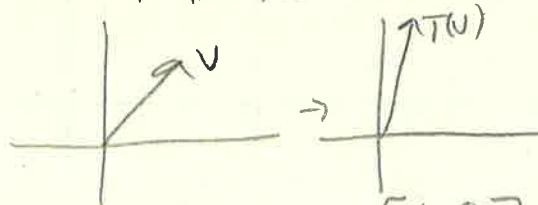
(Non)Example 3: $T(v) = \|v\|$

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^1$$

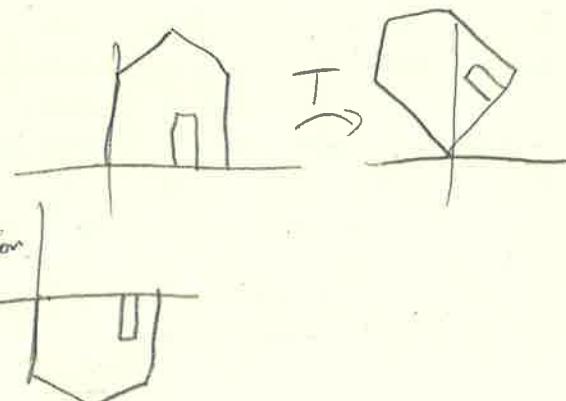
This is not linear

Example 4: Rotation by 45°

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{\text{reflection}}$$



Example 5: Matrix A

$$T(v) = Av$$

$$A(v+w) = Av + Aw, A(cv) = cAv \quad \checkmark \text{ linear}$$

Start: $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

Example: $T(v) = Av$, $A \in \mathbb{R}^{2 \times 3}$
 $v \in \mathbb{R}^3$

Information needed to know $T(v)$ for all inputs

$$T(v_1), T(v_2), \dots, T(v_n) \text{ for any basis } \{v_1, \dots, v_n\}$$

$$\text{Every } v = c_1v_1 + \dots + c_nv_n \Rightarrow T(v) = c_1T(v_1) + \dots + c_nT(v_n)$$

Coordinates come from a basis. The coordinates of v are the c 's.

$$T(v) = c_1v_1 + \dots + c_nv_n$$

$$v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

coordinates for standard basis

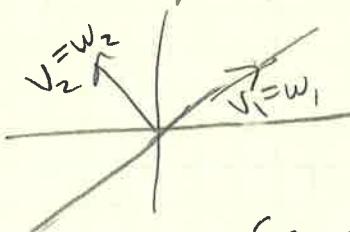
Construct matrix A that represents the linear transformation T

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Choose a basis v_1, \dots, v_n for the inputs in \mathbb{R}^n

Choose a basis w_1, \dots, w_m for the outputs in \mathbb{R}^m .

Want matrix A

Projection Example



eigenvector basis leads to
diagonal matrix

$$v = c_1v_1 + c_2v_2$$

$$T(v) = c_1T(v_1) + c_2T(v_2)$$

$$= c_1v_1 + c_2 \cdot 0 = c_1v_1$$

$$(c_1, c_2) \rightarrow (c_1, 0)$$

$$Ac = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

input coords output coords

Projecting onto 45° line, use standard basis $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$

$$\text{matrix is } \frac{aa^T}{a^T a} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Rule to find A. Given bases v_1, \dots, v_n & w_1, \dots, w_m

1st column of A: Write $T(v_1) = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m$

2nd column of A: $T(v_2) = a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m$

$$A \begin{pmatrix} \text{input} \\ \text{coords} \end{pmatrix} = \begin{pmatrix} \text{output} \\ \text{coords} \end{pmatrix}$$

$$T = \frac{d}{dx} \quad \text{Input space: } C_1 + C_2x + C_3x^2, \text{ basis } = \{1, x, x^2\}$$

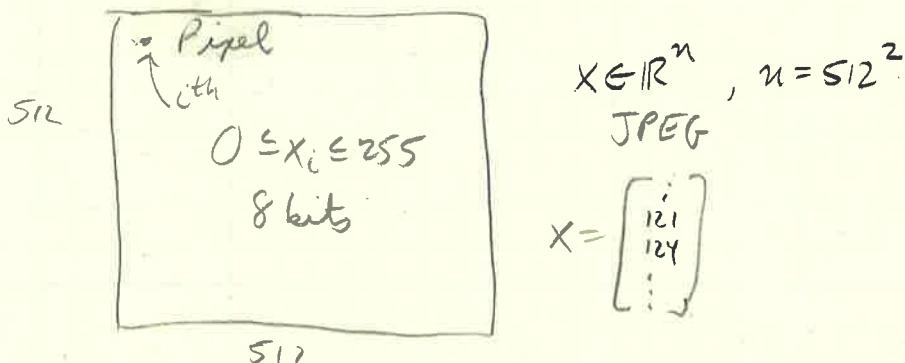
linear Output = $C_2 + 2C_3x$, basis = $\{1, x\}$

$$A \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} C_2 \\ 2C_3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Lecture 31

- Change of Basis
- Compression of Images
- Transformation \hookrightarrow Matrix

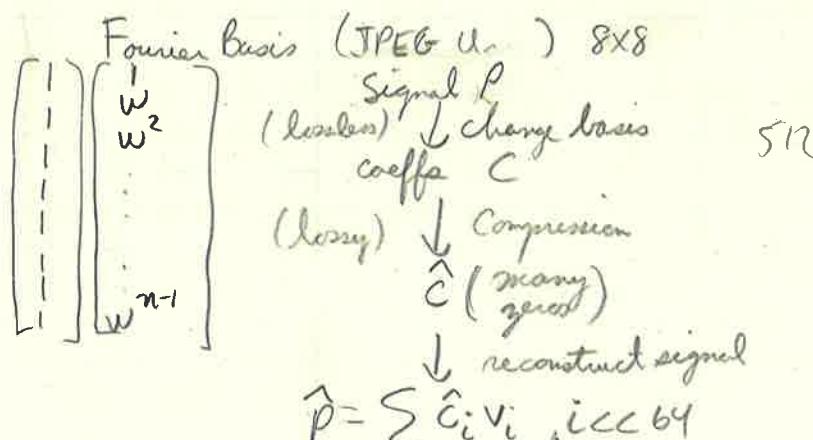
31.1



Standard basis	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$	Better Basis
----------------	--	--------------

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$$

64 pixels



Video

Sequence of images - highly correlated.

Wavelets in \mathbb{R}^8

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8$ ↗ orthogonal

standard basis

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_8 \end{bmatrix}$$

$$P = c_1 w_1 + c_2 w_2 + \dots + c_8 w_8$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ w_1 & w_2 & \dots & w_8 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}$$

W

$$P = WC$$

$$C = W^{-1}P$$

Good basis

① Fast in multiplication and inverse
FFT and FWT

② Few is enough

Change of Basis

Columns of W = new basis vectors

$$\begin{bmatrix} x \\ \vdots \\ x \end{bmatrix}_{\substack{\text{old} \\ \text{basis}}} \rightarrow \begin{bmatrix} c \\ \vdots \\ c \end{bmatrix}_{\substack{\text{new} \\ \text{basis}}} \quad x = WC$$

T with respect to v_1, \dots, v_8 it has a matrix A
with respect to w_1, \dots, w_8 it has a matrix B

What's the relation between A and B ? A is similar to B

$$B = M^{-1} A M$$

What is A ? Using basis v_1, \dots, v_8

Know T completely from $T(v_1), \dots, T(v_8)$ because every $x = c_1 v_1 + \dots + c_8 v_8$

$$\Rightarrow T(x) = c_1 T(v_1) + \dots + c_8 T(v_8)$$

$$\text{Write } T(v_1) = a_{11} v_1 + a_{21} v_2 + \dots + a_{81} v_8$$

$$T(v_2) = a_{12} v_1 + a_{22} v_2 + \dots + a_{82} v_8$$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{81} \\ \vdots & \vdots & & \vdots \\ a_{18} & a_{28} & \dots & a_{88} \end{bmatrix}$$

Eigenvector basis

$T(v_i) = \lambda_i v_i$, what is A ?

$$A = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & 0 & \ddots & ? \\ 0 & 0 & & 0 \\ 0 & \ddots & & \lambda_N \end{bmatrix}$$

1st input is v_1 , its output is $\lambda_1 v_1$,
2nd input is v_2 , " " " $\lambda_2 v_2$

Quiz 3 Review (Lecture 32)

32.1

Linear Transformations on final, not on quiz

6.1-2 \vec{x} and x
 6.3 $du/dt = Au \quad u = e^{At}$

6.4 $A = A^T \Rightarrow A = Q \Lambda Q^T$

6.5 Positive definite

6.6 Similar $B = M^{-1} A M \Rightarrow B^k = M^{-1} A^k M$ but same λ 's

6.7 $A = U \Sigma V^T$ (SVD)

$$\frac{du}{dt} = Au = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} u \quad \text{general solution: } u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

A is singular $\Rightarrow \lambda_1 = 0$ and $A^T = -A \Rightarrow \lambda_2 = ki, \lambda_3 = -ki$
for some $k > 0$

$$(A - \lambda I)x = \begin{bmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 - 2\lambda = 0 \Leftrightarrow \lambda(\lambda^2 + 2) = 0$$

$\lambda = 0, \lambda = \pm \sqrt{2}i$

$$u(t) = c_1 x_1 + c_2 e^{\sqrt{2}it} x_2 + c_3 e^{-\sqrt{2}it} x_3$$

Periodic

$$\sqrt{2}i T = 2\pi i \Rightarrow T = \pi\sqrt{2}$$

(Eigenvectors of skew-symmetric matrices are orthogonal)

Orthogonal eigenvectors happen when $AA^T = A^TA$

Special cases are $\left\{ \begin{array}{l} \text{symmetric} \\ \text{skewsymmetric} \\ \text{orthogonal} \end{array} \right.$

Find e^{At} , where $u(t) = e^{At} u(0)$ (if $A = S \Lambda S^{-1}$)

$$e^{At} = S e^{\Lambda t} S^{-1}, e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_3 t} \end{bmatrix}$$

Q) 3×3 matrix.

$$\lambda_1 = 0 \quad \lambda_2 = c \quad \lambda_3 = 2$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

a) diagonalizable? yes, the eigenvectors are independent
 \Rightarrow all c

b) symmetric? which c ? all $c \in \mathbb{R}$

c) positive definite? which c ? Not p.d. since $\lambda_1 = 0$
 $\text{If } c \geq 0 \Rightarrow \text{positive semi-definite}$

d) Markov matrix? No, since not all eigenvalues ≤ 1 .

e) $P = \frac{A}{2}$ a projection matrix? eigenvalues are 0 and 1 only.
 $P^2 = P$ so $\lambda^2 = \lambda$ so $\lambda = 0 \text{ or } 1$.

Need $C=0$ or $C=2$

SVD every $A = (\text{orthogonal})(\text{diag})(\text{orthogonal}) = U \Sigma V^T$ rank = r
 $\downarrow \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$

$$A^T A = (V \Sigma^T U^T)(U \Sigma V^T) = V (\Sigma^T \Sigma) V^T$$

Symmetric Diagonal

$U \Sigma V^T \Rightarrow V$ is eigenvector matrix for $A^T A$

σ_i^2 = eigenvalues of $A^T A$

$$A A^T = (U \Sigma V^T)(V \Sigma^T U^T) = U (\Sigma \Sigma^T) U^T$$

U is eigenvector matrix for $A A^T$

Instead use $A V_i = \sigma_i u_i$ to get signs correct (connection to earlier lecture)

$$AV = U \Sigma$$

Suppose $A = [u_1, u_2] \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} [v_1, v_2]^T \Rightarrow A$ is singular

$$\text{rank } A = 1, \dim N(A) = 1, N(A) = V_2$$

Q) Given A is symmetric and orthogonal $\sim |\lambda| = 1 \Rightarrow$ why $Qx = \lambda x$
① eigenvalues can be 1 and -1

$$(Qx)^T (Qx) = |\lambda| \|x\|^2$$

$$x^T Q^T Q x = \|x\|^2$$

$$\Rightarrow |\lambda| = 1$$

- T or F: A is sure to be pos def: False
- Will have repeated eigenvalues? True
- Diagonalizable? True
- Non-singular? True

Show that $\frac{1}{2}(A+I)$ is a projection matrix

$$\frac{1}{2}(A+I) \cdot \frac{1}{2}(A+I) = \frac{1}{4}(A^2 + 2A + I) \stackrel{?}{=} \frac{1}{2}(A+I)$$

$$\text{What is } A^2 \dots? \quad A = A^T = A^{-1} \Rightarrow A^2 = AA^{-1} = I$$

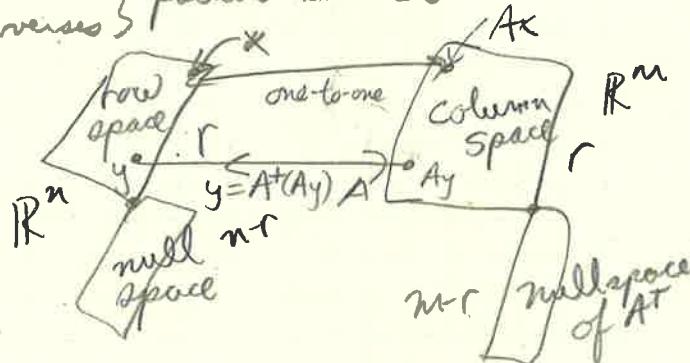
$$\Rightarrow \frac{1}{4}(A^2 + 2A + I) = \frac{1}{4}(2I + 2A) = \frac{1}{2}(A+I) \checkmark$$

eigenvalues of $A+I$ are 2 and 0 $\Rightarrow \frac{1}{2}(A+I)$ has eigenvalues 0 and 1.

Lecture 33

33.1

4 subspaces
 Left-inverses } pseudo-inverses
 Right-inverses }



2-sided inverse $\Rightarrow AA^{-1} = I = A^{-1}A$
 $r = m = n$ (full rank)

left inverse

full column rank $r = n < m$, nullspace = {0} (independent cols)

0 or 1 solutions to $Ax = b$

$A^T A$ has full rank $n \Leftrightarrow \underbrace{(A^T A)^{-1} A^T}_{{A^{-1}}_{\text{left}}} A = I$

$$\underbrace{A^{-1}_{\text{left}}}_{{n \times n}} A = I_{n \times n}$$

$$AA^{-1}_{\text{left}} = A(A^T A)^{-1} A^T = P \text{ projection onto } C(A)$$

right-inverse:

full row-rank $r = m < n$. $N(A^T) = \{0\} \Leftrightarrow$ indep. rows
 ∞ solns to $Ax = b$; $n-m$ free variables

$$A, A^T(AA^T)^{-1} = I$$

$$A^{-1} \text{ right } = I \quad A^{-1} \text{ right } A = A^T(AA^T)^{-1}A = \text{the projection onto the row space}$$

pseudo-inverses (A^+)

For $x, y \in C(A^T)$, $x+y \Rightarrow Ax, Ay \in C(A)$ and $Ax \neq Ay$

Proof:

$$\text{Suppose } Ax = Ay \Leftrightarrow A(x-y) = 0$$

$$\Rightarrow x-y \in N(A) \text{ and also in } C(A^T) \Leftrightarrow x-y = 0$$

since $N(A) \cap C(A^T) = \{0\}$

How to find pseudoinverse A^+

① Start from SVD : $A = U\Sigma V^T$

projection
onto colspace

$$\Sigma\Sigma^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{m \times m}^{r \times r}$$

projection
onto rowspace

$$\Sigma^+\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{n \times n}^{r \times r}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & 0 \\ 0 & & 0 \end{bmatrix}_{m \times n}$$

m rows
 n columns

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & 0 \\ 0 & & 0 \end{bmatrix}_{n \times m}$$

$$A^+ = V\Sigma^+U^T$$

Lecture 34
Course Review

Q) Given

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ has no soln } \Rightarrow \text{rank } r < m$$

$$Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has 1 soln } \Rightarrow N(A) = \{0\} \Rightarrow r = n$$

$m=3 > r=n \Leftrightarrow$ independent columns.

One such A :

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ works or } A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ works}$$

- ~) T or F $\text{for } A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ if A were square $\Rightarrow (\det A)(\det A^T)$
- (F) 1. $\det A^T A = \det A A^T$? True if A were square
- (T) 2. $A^T A$ is invertible? Only if $r=n$ (\Leftrightarrow independent cols in A)
- (F) 3. $A A^T$ is positive definite?

$$A A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^T y = c$ at least 1 soln for every c . (when full row rank)
 $n \times m$ in fact ∞ " " " "

$$\dim N(A^T) = m - r > 0$$

Q) $A = \begin{bmatrix} 1 & 1 & 1 \\ v_1 & v_2 & v_3 \end{bmatrix}$ (a) Solve $Ax = v_1 - v_2 + v_3 \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(b) Suppose $v_1 - v_2 + v_3 = 0 \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in N(A)$. So, solns are never unique

(c) Suppose v_1, v_2, v_3 are orthonormal, then the combination
 $\underline{\Omega} v_1 + \underline{\Omega} v_2$ is closest to v_3

Q) $A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 4 & 4 \end{bmatrix}$ $\text{col } 1 + \text{col } 2 = 2(\text{col } 3) \Rightarrow \lambda_1 = 0$
 $\lambda = 1$ because Markov
 $\lambda = .2$ because $\text{tr}(A) = 0.8$

$$u_n = A^k u(0) \quad , \quad u(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

$$u_n = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + c_3 \lambda_3^k x_3$$

// // //
 0 1 -2

$$u_{\infty} = c_2 x_2 \Rightarrow (A - I)x_2 = \begin{bmatrix} -1.8 & 0.4 & 0.3 \\ 0.4 & -1.8 & 0.3 \\ 0.4 & 0.4 & -26 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_{12} = 1 \cdot \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \text{ since there are 10 at the start } x_2$$

a) $\frac{2 \times 2}{?}$

① Projection onto line through $a = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$. $P = \frac{aa^T}{a^Ta} = \begin{bmatrix} \quad \end{bmatrix}$

$$② \lambda_1=0, x_1=\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \lambda_2=3, x_2=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A = S \Lambda S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

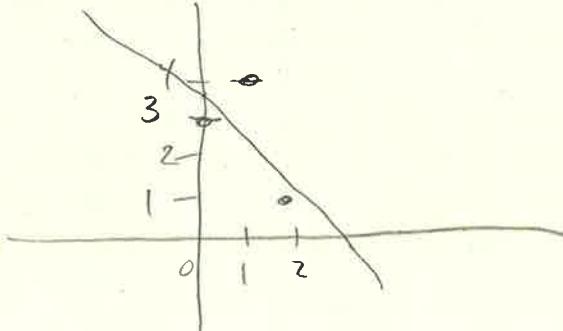
③ $A \neq B^T B$ for any B . $B^T B$ is always symmetric $\Rightarrow A \neq A^T$

④ Orthogonal e-vectors, but not symmetric are } skew-symmetric
orthogonal

Q) Least Squares

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathbb{R}^2} \quad \begin{bmatrix} \hat{c} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1 \end{bmatrix}$$

① P (Projection of b onto $C(A)$) is $\frac{1}{\sqrt{3}} \times \text{col } 1 - \frac{1}{\sqrt{2}} \times \text{col } 2$



② Find $b \neq 0$ so that $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$b \perp CCA$$

$$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$