Lecture Notes for Linear Algebra

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preface

Before we begin, I should warn you that I assume a few things from the reader. These notes are intended for someone who has already grappled with the problem of constructing proofs. I assume you know the difference between $\Rightarrow$ and $\Leftrightarrow$. I assume the phrase "iff" is known to you. I assume you are ready and willing to do a proof by induction, strong or weak. I assume you know what $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{Z}$ denote. I assume you know what a subset of a set is. I assume you know how to prove two sets are equal. I assume you are familar with basic set operations such as union and intersection. More importantly, I assume you have started to appreciate that mathematics is more than just calculations. Calculations without context, without theory, are doomed to failure. At a minimum theory and proper mathematics allows you to communicate analytical concepts to other like-educated individuals.

Some of the most seemingly basic objects in mathematics are insidiously complex. We've been taught they're simple since our childhood, but as adults, mathematical adults, we find the actual definitions of such objects as $\mathbb{R}$ or $\mathbb{C}$ are rather involved. I will not attempt to provide foundational arguments to build numbers from basic set theory. I believe it is possible, I think it's well-thought-out mathematics, but we take the existence of the real numbers as a given truth for these notes. We assume that $\mathbb{R}$ exists and that the real numbers possess all their usual properties. In fact, I assume $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{Z}$ all exist complete with their standard properties. In short, I assume we have numbers to work with. We leave the rigorization of numbers to a different course.

These notes are offered for the Spring 2017 semester at Liberty University. These are a major revision of my older linear algebra notes. They reflect the restructuring of the course which I intend for this semester. These are a work in progress, I will update these as I complete edits of the later parts (which are currently missing)

Just a bit more advice before I get to the good part. How to study? I have a few points:

- spend several days on the homework. Try it by yourself to begin. Later, compare with your study group. Leave yourself time to ask questions.

- come to class, take notes, think about what you need to know to solve problems.

- assemble a list of definitions, try to gain an inuitive picture of each concept, be able to give examples and counter-examples

- learn the notation, a significant part of this course is learning to deal with new notation.

- methods of proof, how do we prove things in linear algebra? There are a few standard proofs, know them.

- method of computation, I show you tools, learn to use them.

- it's not impossible. You can do it. Moreover, doing it the right way will make the courses which follow this easier. Mathematical thinking is something that takes time for most of us to master. You began the process in Math 200, now we continue that process.

style guide

I use a few standard conventions throughout these notes. They were prepared with LaTeX which automatically numbers sections and the hyperref package provides links within the pdf copy from the Table of Contents as well as other references made within the body of the text.

I use color and some boxes to set apart some points for convenient reference. In particular,

1. definitions are in green.
2. remarks are in red.
3. theorems, propositions, lemmas and corollaries are in blue.
4. proofs start with a **Proof:** and are concluded with a $\square$.

However, I do make some definitions within the body of the text. As a rule, I try to put what I am defining in **bold**. Doubtless, I have failed to live up to my legalism somewhere. If you keep a list of these transgressions to give me at the end of the course it would be worthwhile for all involved.

The symbol $\square$ indicates that a proof is complete. The symbol $\triangledown$ indicates part of a proof is done, but it continues.

ERRORS: there may be errors. I make every effort to eliminate them, but, I do make mistakes. When you find one, simply send me an email about your concern anytime, day, night, even the weekend. I usually see it fairly soon and I can confirm or deny the error. Most of the examples are not new, so previous generations of students have combed through them for mistakes. However, I am adding some examples in this edit, so, mistakes may happen. Keep in mind, I'm trying to be a bit more abstract in the initial arc of the course. Mostly this involves replacing $\mathbb{R}$ with R. I may have missed an $\mathbb{R}$ somewhere, don't think too hard about it. If in doubt email. Please. Thanks!

reading guide

A number of excellent texts have helped me gain deeper insight into linear algebra. Let me discuss a few of them here.

1. Charles W. Curtis' *Linear Algebra: An Introductory Approach (Undergraduate Texts in Mathematics)*, 4-th Edition is our required text for the Spring 2016 Semester. I'm hopeful this text brings a fresh perspective. I'm trying to update the notes this semester so they better reflect this course. My goal is to rewrite the notes as to assimilate the wisdom of Curtis' text while preserving the calculational core of my notes. As an organizational convenience I have grouped all the major matrix computations at the start of the course as will be tested on our Quiz 1. Then we start using Curtis more regularly. At the moment, I only have notes up to Quiz 1 in this document.
2. Damiano and Little's *A Course in Linear Algebra* published by Dover. I chose this as the required text in Spring 2015 as it is a well-written book, inexpensive and has solutions in the back to many exercises. The notation is fairly close to the notation used in these notes. One noted exception would be my $[T]_{\alpha,\beta}$ is replaced with $[T]_{\alpha}^{\beta}$. In fact, the notation of Damiano and Little is common in other literature I've read in higher math. I also liked the

appearance of some diagrammatics for understanding Jordan forms. The section on minimal and characteristic polynomials is lucid.

3. Berberian's *Linear Algebra* published by Dover. This book is a joy. The exercises are challenging for this level and there were no solutions in the back of the text. This book is full of things I would like to cover, but, don't quite have time to do.

4. Takahashi and Inoue's *The Manga Guide to Linear Algebra*. Hillarious. Fun. Probably a better algorithm for Gaussian elimnation than is given in my notes.

5. Axler *Linear Algebra Done Right*. If our course was a bit more pure, I might use this. Very nicely written. This is an honest to goodness linear algebra text, it is actually just about the study of linear transformations on vector spaces. Many texts called "linear algebra" are really about half-matrix theory. Admittedly, such is the state of our course. But, I have no regrets, it's not as if I'm teaching matrix techinques that the students already know before this course. Ideally, I will openly admit, it would be better to have two courses. First, a course on matrices and applications. Second, a course like that outlined in this book.

6. Hefferon's *Linear Algebra*: this text has nice gentle introductions to many topics as well as an appendix on proof techniques. The emphasis is linear algebra and the matrix topics are delayed to a later part of the text. Furthermore, the term linear transformation as supplanted by homomorphism and there are a few other, in my view, non-standard terminologies. All in all, very strong, but we treat matrix topics much earlier in these notes. Many theorems in this set of notes were inspired from Hefferon's excellent text. Also, it should be noted the solution manual to Hefferon, like the text, is freely available as a pdf.

7. Anton and Rorres' *Linear Algebra: Applications Version* or Lay's Linear Algebra, or Larson and Edwards Linear Algebra, or... standard linear algebra text. Written with non-math majors in mind. Many theorems in my notes borrowed from these texts.

8. Insel, Spence and Friedberg's *Elementary Linear Algebra*. This text is a little light on applications in comparison to similar texts, however, the theory of Gaussian elimination and other basic algorithms are extremely clear. This text focus on column vectors for the most part.

9. Insel, Spence and Friedberg's *Linear Algebra*. It begins with the definition of a vector space essentially. Then all the basic and important theorems are given. Theory is well presented in this text and it has been invaluable to me as I've studied the theory of adjoints, the problem of simultaneous diagonalization and of course the Jordan and rational cannonical forms.

10. Strang's *Linear Algebra*. If geometric intuition is what you seek and/or are energized by then you should read this in paralell to these notes. This text introduces the dot product early on and gives geometric proofs where most others use an algebraic approach. We'll take the algebraic approach whenever possible in this course. We relegate geometry to the place of motivational side comments. This is due to the lack of prerequisite geometry on the part of a significant portion of the students who use these notes.

11. my advanced calculus notes. I review linear algebra and discuss multilinear algebra in some depth. I've heard from some students that they understood linear in much greater depth after the experience of my notes. Ask if interested, I'm always editing these.

12. Olver and Shakiban *Applied Linear Algebra*. For serious applications and an introduction to modeling this text is excellent for an engineering, science or applied math student. This book is somewhat advanced, but not as sophisticated as those further down this list.

13. Sadun's *Applied Linear Algebra: The Decoupling Principle* this is a second book in linear algebra. It presents much of the theory in terms of a unifying theme; decoupling. Probably this book is very useful to the student who wishes deeper understanding of linear system theory. Includes some Fourier analysis as well as a Chapter on Green's functions.

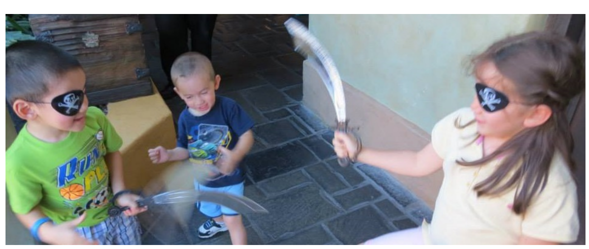
14. Curtis' *Abstract Linear Algebra*. Great supplement for a clean presentation of theorems. Written for math students without apology. His treatment of the wedge product as an abstract algebraic system is .

15. Roman's *Advanced Linear Algebra*. Treats all the usual topics as well as the generalization to modules. Some infinite dimensional topics are discussed. This has excellent insight into topics beyond this course.

16. Dummit and Foote *Abstract Algebra*. Part III contains a good introduction to the theory of modules. A module is roughly speaking a vector space over a ring. I believe many graduate programs include this material in their core algebra sequence. If you are interested in going to math graduate school, studying this book puts you ahead of the game a bit. Understanding Dummit and Foote by graduation is a nontrivial, but worthwhile, goal.

And now, a picture of Hannah in a shark,

I once told linear algebra that Hannah was them and my test was the shark. A wise student prayed that they all be *shark killers*. I pray the same for you this semester. I've heard from a certain student this picture and comment is unsettling. Therefore, I add this to ease the mood:



As you can see, Hannah survived to fight new monsters.

Contents

Chapter 1

foundations

In this chapter we settle some basic notational issues. There are not many examples in this chapter and the main task the reader is assigned here is to read and learn the definitions and notations. I begin Lecture in Chapter 2, but, I would like you to read through this so we understand each other.

1.1 sets and multisets

A set is a collection of objects. The set with no elements is called the **empty-set** and is denoted $\emptyset$. If we write $x \in A$ then this is read "x is an element of A". In your previous course you learned that $\{a, a, b\} = \{a, b\}$. In other words, there is no allowance for repeats of the same object. In linear algebra, we often find it more convenient to use what is known as a **multiset**. In other instances we'll make use of an **ordered set** or even an **ordered mulitset**. To summarize:

1. a **set** is a collection of objects with no repeated elements in the collection.
2. a **multiset** is a collection of objects. Repeats are possible.
3. an **ordered set** is a collection of objects with no repeated elements in which the collection has a specific ordering.
4. an **ordered multiset** is a collection of objects which has an ordering and possibly has repeated elements.

Notice, every set is a multiset and every ordered set is an ordered multiset. In the remainder of this course, we make the slight abuse of langauge and agree to call an ordinary set a **set with no repeated elements** and a multiset will simply be called in sequel a **set**. This simplifies our langauge and will help us to think better[1].

Let us denote sets by capital letters in as much as is possible. Often the lower-case letter of the same symbol will denote an element; $a \in A$ is to mean that the object a is in the set A. We can abbreviate $a_1 \in A$ and $a_2 \in A$ by simply writing $a_1, a_2 \in A$, this is a standard notation. The **union** of two sets A and B is denoted[2] $A \cup B = \{x | x \in A \ or \ x \in B\}$. The **intersection** of two sets is

[1]there is some substructure to describe here, multisets and ordered sets can be constructed from sets. However, that adds little to our discussion and so I choose to describe multisets, ordered sets and soon Cartesian products formally. Formally, means I describe there structure without regard to its explicit concrete realization.

[2]note that $S = \{x \in R \ : \ x \text{ meets condition } P\} = \{x \in R \mid x \text{ meets condition } P\}$. Some authors use : whereas I prefer to use | in the set-builder notation.

denoted $A \cap B = \{x | x \in A \ and \ x \in B\}$. It sometimes convenient to use unions or intersections of several sets:

$$\bigcup_{\alpha \in \Lambda} U_\alpha = \{x \mid \text{there exists } \alpha \in \Lambda \text{ with } x \in U_\alpha\}$$

$$\bigcap_{\alpha \in \Lambda} U_\alpha = \{x \mid \text{for all } \alpha \in \Lambda \text{ we have } x \in U_\alpha\}$$

we say Λ is the **index set** in the definitions above. If Λ is a finite set then the union/intersection is said to be a finite union/interection. If Λ is a countable set then the union/intersection is said to be a countable union/interection[3].

Suppose A and B are both sets then we say A is a **subset** of B and write $A \subseteq B$ iff $a \in A$ implies $a \in B$ for all $a \in A$. If $A \subseteq B$ then we also say B is a **superset** of A. If $A \subseteq B$ then we say $A \subset B$ iff $A \neq B$ and $A \neq \emptyset$. Recall, for sets A, B we define $A = B$ iff $a \in A$ implies $a \in B$ for all $a \in A$ and conversely $b \in B$ implies $b \in A$ for all $b \in B$. This is equivalent to insisting $A = B$ iff $A \subseteq B$ and $B \subseteq A$. Note, if we deal with ordered sets equality is measured by checking that both sets contain the same elements in the same order. The **difference** of two sets A and B is denoted $A - B$ and is defined by $A - B = \{a \in A \mid \text{such that } a \notin B\}$[4].

A **Cartesian product** of two sets A, B is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. We denote,

$$A \times B = \{(a, b) \mid a \in A, \ b \in B\}$$

Likewise, we define

$$A \times B \times C = \{(a, b, c) \mid a \in A, \ b \in B, \ c \in C\}$$

We make no distinction between $A \times (B \times C)$ and $(A \times B) \times C$. This means we are using the obvious one-one correspondence $(a, (b, c)) \leftrightarrow ((a, b), c)$. If $A_1, A_2, \dots A_n$ are sets then we define $A_1 \times A_2 \times \cdots \times A_n$ to be the set of ordered n-tuples:

$$\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid a_i \in A_i \text{ for all } i \in \mathbb{N}_n\}$$

Notice, I define $\mathbb{N} = \{1, 2, \dots\}$ as the set of **natural numbers** whereas $\mathbb{N}_n$ is the set of natural numbers upto and including $n \in \mathbb{N}$; $\mathbb{N}_n = \{1, \dots, n\}$. If we take the Cartesian product of a set A with itself n-times then it is customary to denote the set of all n-tuples from A as A^n:

$$\underbrace{A \times \cdots \times A}_{n-copies} = A^n.$$

Real numbers can be constructed from set theory and about a semester of mathematics. We will accept the following as **axioms**[5]

[3] recall the term **countable** simply means there exists a bijection to the natural numbers. The cardinality of such a set is said to be $\aleph_o$

[4] other texts somtimes use $A - B = A \setminus B$

[5] an axiom is a basic belief which cannot be further reduced in the conversation at hand. If you'd like to see a construction of the real numbers from other math, see Ramanujan and Thomas' *Intermediate Analysis* which has the construction both from the so-called Dedekind cut technique and the Cauchy-class construction. Also, I've been informed, Terry Tao's Analysis I text has a very readable exposition of the construction from the Cauchy viewpoint.

Definition 1.1.1. *real numbers*

The set of real numbers is denoted $\mathbb{R}$ and is defined by the following axioms:

(A1) addition commutes; $a + b = b + a$ for all $a, b \in \mathbb{R}$.

(A2) addition is associative; $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$.

(A3) zero is additive identity; $a + 0 = 0 + a = a$ for all $a \in \mathbb{R}$.

(A4) additive inverses; for each $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ and $a + (-a) = 0$.

(A5) multiplication commutes; $ab = ba$ for all $a, b \in \mathbb{R}$.

(A6) multiplication is associative; $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$.

(A7) one is multiplicative identity; $a1 = a$ for all $a \in \mathbb{R}$.

(A8) multiplicative inverses for nonzero elements;
for each $a \neq 0 \in \mathbb{R}$ there exists $\frac{1}{a} \in \mathbb{R}$ and $a\frac{1}{a} = 1$.

(A9) distributive properties; $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in \mathbb{R}$.

(A10) totally ordered field; for $a, b \in \mathbb{R}$:

(i) antisymmetry; if $a \leq b$ and $b \leq a$ then $a = b$.

(ii) transitivity; if $a \leq b$ and $b \leq c$ then $a \leq c$.

(iii) totality; $a \leq b$ or $b \leq a$

(A11) least upper bound property: every nonempty subset of $\mathbb{R}$ that has an upper bound, has a least upper bound. This makes the real numbers **complete**.

Modulo A11 and some math jargon this should all be old news. An **upper bound** for a set $S \subseteq \mathbb{R}$ is a number $M \in \mathbb{R}$ such that $M > s$ for all $s \in S$. Similarly a lower bound on S is a number $m \in \mathbb{R}$ such that $m < s$ for all $s \in S$. If a set S is bounded above and below then the set is said to be **bounded**. For example, the open set (a, b) is bounded above by b and it is bounded below by a. In contrast, rays such as $(0, \infty)$ are not bounded above. Closed intervals contain their least upper bound and greatest lower bound. The bounds for an open interval are outside the set.

We often make use of the following standard sets:

- natural numbers (positive integers); $\mathbb{N} = \{1, 2, 3, \dots\}$.
- natural numbers up to the number n; $\mathbb{N}_n = \{1, 2, 3, \dots, n-1, n\}$.
- integers; $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Note, $\mathbb{Z}_{>0} = \mathbb{N}$.
- non-negative integers; $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$.
- negative integers; $\mathbb{Z}_{<0} = \{-1, -2, -3, \dots\} = -\mathbb{N}$.
- rational numbers; $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z},\ q \neq 0\}$.

- irrational numbers; $\mathbb{J} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$.
- open interval from a to b; $(a, b) = \{x | a < x < b\}$.
- half-open interval; $(a, b] = \{x \mid a < x \leq b\}$ or $[a, b) = \{x \mid a \leq x < b\}$.
- closed interval; $[a, b] = \{x \mid a \leq x \leq b\}$.

We define $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. I refer to $\mathbb{R}^2$ as "R-two" in conversational mathematics. Likewise, "R-three" is defined by $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$. We are ultimately interested in studying "R-n" where $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}$ for $i = 1, 2, \dots, n\}$. In this course if we consider $\mathbb{R}^m$ it is assumed from the context that $m \in \mathbb{N}$.

In terms of cartesian products you can imagine the x-axis as the number line then if we paste another numberline at each x value the union of all such lines constucts the plane; this is the picture behind $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Another interesting cartesian product is the **unit-square**; $[0, 1]^2 = [0, 1] \times [0, 1] = \{(x, y) \mid 0 \leq x \leq 1, \ \ 0 \leq y \leq 1\}$. Sometimes a rectangle in the plane with it's edges included can be written as $[x_1, x_2] \times [y_1, y_2]$. If we want to remove the edges use $(x_1, x_2) \times (y_1, y_2)$.

Moving to three dimensions we can construct the **unit-cube** as $[0, 1]^3$. A generic rectangular solid can sometimes be represented as $[x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$ or if we delete the edges: $(x_1, x_2) \times (y_1, y_2) \times (z_1, z_2)$.

1.2 complex arithmetic

The purpose of this section is to show how to do arithmetic in $\mathbb{C}$. Let us begin with a reminder of the basics: multiply by the usual distributive rules, however, $i^2 = -1$,

$$\begin{aligned} (a + ib)(c + id) &= a(c + id) + ib(c + id) \\ &= ac + iad + ibc + i^2bd \\ &= ac - bd + i(ad + bc). \end{aligned}$$

It follows, for $z, w, v \in \mathbb{C}$,

$$zw = wz, \qquad (z + w)v = zv + wv.$$

If $z = x + iy$ then the **real part** of z is defined by $\text{Re}(z) = x$ whereas the **imaginary part** of z is defined by $\text{Im}(z) = y$. We say z is **real** if $\text{Im}(z) = 0$ or equivalently, $\text{Re}(z) = z$. It is often useful to use **complex conjugation** to understand the algebraic structure of $\mathbb{C}$:

$$\boxed{\overline{x + iy} = x - iy}$$

Given this notation we can write slick formulas for the real and imaginary parts since $z + \bar{z} = 2x$ and $z - \bar{z} = 2iy$ yield $\text{Re}(z) = \frac{1}{2}(z + \bar{z})$ and $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$. Furthermore, the conjugation operation is used to define the Euclidean length for $z = x + iy$ by $\boxed{|z| = \sqrt{z\bar{z}}}$. Hence,

$$|z|^2 = z\overline{z} = x^2 + y^2.$$

and, since $\overline{zw} = \overline{z}\,\overline{w}$ it follows that $|zw| = |z||w|$; that is, the length of a product of complex numbers is simply the product of the lengths of the numbers.

Example 1.2.1. *Let* $z = 3+i$ *and* $w = 1-i$. *Notice* $zw = (3+i)(1-i) = 3+i-3i-i^2 = 4-2i$. *Notice,* $|z| = \sqrt{9+1} = \sqrt{10}$ *and* $|w| = \sqrt{1+1} = \sqrt{2}$ *and observe* $|zw| = \sqrt{16+4} = \sqrt{20}$. *This confirms* $|zw| = |z||w|$.

The length of a complex number is important to forming the reciprocal. If $z \neq 0$ then $|z| \neq 0$ hence $|z|^2 = z\bar{z} = x^2 + y^2 \neq 0$ and we derive:

$$\frac{1}{z} = \frac{\bar{z}}{x^2+y^2} \quad \Rightarrow \quad \boxed{\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}}$$

I boxed the three most important formulas in this section for your convenience.

Example 1.2.2. $\dfrac{4}{1+i} = 4\left(\dfrac{1}{1+i}\right) = 4\left(\dfrac{1-i}{2}\right) = 2-2i$.

It is useful to know the usual algebra of fractions works for complex numbers. For example,

$$\frac{z}{w} \cdot \frac{\eta}{\zeta} = \frac{z\eta}{w\zeta} \qquad \& \qquad \frac{1}{\frac{1}{z}} = z$$

Example 1.2.3. *Let* $z = 3+7i$ *and* $w = 2-i$ *and find the Cartesian form of* $\frac{z}{z^2+wz}$. *Observe* $\frac{z}{z^2+wz} = \frac{z}{z(z+w)} = \frac{1}{z+w}$ *thus calculate:* $z+w = 5+6i$ *and*

$$\frac{1}{z+w} = \frac{1}{5+6i} = \frac{5-6i}{25+36} = \frac{5}{61} - \frac{6}{61}i.$$

The idea of the Example above is to avoid calculation when we can.

Example 1.2.4. *To solve* $(3+i)z = 4-i$ *we need to divide by* $3+i$ *to obtain* $z = \frac{4-i}{3+i}$. *To place this answer in Cartesian form we need to eliminate the* i *in the denominator. In particular, calculate:*

$$z = \left[\frac{4-i}{3+i}\right]\left[\frac{3-i}{3-i}\right] = \frac{13+i}{10}.$$

For calculational purposes, it's actually better to leave it as the fraction. In any event, I think this should suffice[6].

1.3 modular arithmetic

Perhaps you have studied **congruence** modulo n in a previous course. For $n \in \mathbb{N}$, and integers x, y, we say $x \approxeq y \text{ mod}(n)$ iff $x - y = nk$ for some $k \in \mathbb{Z}$. In this view, we work with integers with a modified idea of equality. In order to obtain an operation where congruence essentially becomes equality it is necessary to trade integers for sets of integers called **equivalence classes**. In particular, we trade x for $x + n\mathbb{Z} = \{x + nk \mid k \in \mathbb{Z}\}$. In fact, if $x + n\mathbb{Z} = y + n\mathbb{Z}$ then it follows that $x = y + nk$ for some $k \in \mathbb{Z}$. In other words, $x + n\mathbb{Z} = y + n\mathbb{Z}$ if and only if $x \approxeq y \text{ mod}(n)$. Furthermore, we can prove[7] the following:

$$(x+n\mathbb{Z}) + (y+n\mathbb{Z}) = x+y+n\mathbb{Z} \qquad \& \qquad (x+n\mathbb{Z})(y+n\mathbb{Z}) = xy+n\mathbb{Z}$$

[6] please ask me if you need further examples, the main point here is to show you how to work with complex fractions. Again, to summarize $i^2 = -1$ and calculate as usual.

[7] not in this course, we're just going to use these, in number theory or Math 200 I'd be inclined to prove such things carefully. For instance, see §4 of my number theory handout

The coset $x + n\mathbb{Z}$ has **representative** x and the results above show we can define an operation on cosets by set addition and multiplication simply by acting on the representatives modulo n. The notation we will use this semester is less cumbersome. Instead of $x + n\mathbb{Z}$ we write $\bar{x}$ and we define the operation of addition and multiplication in $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\}$ as follows

$$\overline{x} + \overline{y} = \overline{x+y}, \qquad \overline{x} \cdot \overline{y} = \overline{xy}, \qquad \overline{x} = \overline{x+nk}$$

for any $k \in \mathbb{Z}$. We should note for any given choice of n,

$$\overline{0} \cdot \overline{x} = \overline{0 \cdot x} = \overline{0} \qquad \& \qquad \overline{1} \cdot \overline{x} = \overline{1 \cdot x} = \overline{x}$$

In particular, $\overline{n} = 0$ and a useful way of thinking about $\mathbb{Z}/n\mathbb{Z}$ is that the number n behaves like zero. This idea is much like the cyclicity of time measurement. In terms of an hour of the day, if we shift by some multiple of 24 hours then we have added zero to the time. For example, if it is noon and I go precisely 3 days into the future then it is still noon because adding 3 days is adding a multiple of 24 hours. In other words, military time is done modulo 24 hour periods.

The modest goal of this section is simply to exhibit how to actually carry out the operations on $\overline{x}$ and $\overline{y}$ for a given n.

Example 1.3.1. *Consider $\mathbb{Z}/3\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}\}$. Addition:*

$$\overline{1} + \overline{1} = \overline{2}, \qquad \overline{1} + \overline{2} = \overline{3} = \overline{0}, \qquad \overline{2} + \overline{2} = \overline{4} = \overline{1}.$$

Multiplication: (since we know multiplication by $\overline{0}$ gives $\overline{0}$ and $\overline{1}x = x$ there is only one interesting product to try)

$$\overline{2} \cdot \overline{2} = \overline{4} = \overline{1}$$

In other words, $1/\overline{2} = \overline{2}$ or $\overline{2}^{-1} = \overline{2}$.

Example 1.3.2. *Consider $\mathbb{Z}/6\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. Addition: well, a tabular form is convenient;*

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{5}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$

&

$\cdot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{0}$	$\overline{2}$	$\overline{4}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{2}$	$\overline{0}$	$\overline{4}$	$\overline{2}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

In the table above we have $\overline{5} \cdot \overline{5} = \overline{1}$ hence $\overline{5}^{-1} = \overline{5}$. Also, $\overline{1} \cdot \overline{1} = \overline{1}$ thus $\overline{1}^{-1} = \overline{1}$. However, the remaining numbers in $\mathbb{Z}/6\mathbb{Z}$ do not have reciprocals. In other words, we are only able to divide by $\overline{1}$ and $\overline{5}$ in $\mathbb{Z}/6\mathbb{Z}$.

I hope there is no error in the example above or the one below, but, if you find one, please let me know. Notice, we don't need to include certain rows and columns in the table since we know $\overline{0} + x = x$ and $\overline{1}x = x$ for all x. I could have written less in the previous example.

Example 1.3.3. *Consider $\mathbb{Z}/7\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$. Addition: well, a tabular form is convenient;*

$+$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{0}$
$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{0}$	$\overline{1}$
$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{5}$	$\overline{6}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{6}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$

&

$\cdot$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{1}$	$\overline{3}$	$\overline{5}$
$\overline{3}$	$\overline{6}$	$\overline{2}$	$\overline{5}$	$\overline{1}$	$\overline{4}$
$\overline{4}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{6}$	$\overline{3}$
$\overline{5}$	$\overline{3}$	$\overline{1}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

In the table above we have each number appearing once in each row and column. This pattern will occur for $\mathbb{Z}/p\mathbb{Z}$ *when we look a table with everything except* $\overline{1}$ *and* $\overline{0}$ *provided* p *is prime. Also, every nonzero number will have a multiplicative inverse:*

$$\overline{2}^{-1} = \overline{4}, \quad \overline{3}^{-1} = \overline{5}, \quad \overline{6}^{-1} = \overline{6}.$$

We'll do some row-reduction over $\mathbb{Z}/p\mathbb{Z}$ for p-prime since it allows us to divide by any nonzero element. Let me conclude with a short calculation based on the Example with $n = 7$.

Example 1.3.4. *Solve* $\overline{3}x + \overline{6} = \overline{4}$ *in* $\mathbb{Z}/7\mathbb{Z}$*. Note,* $\overline{3}x = \overline{4} - \overline{6} = \overline{4-6} = \overline{-2} = \overline{5}$*. Therefore, using that* $1/\overline{3} = \overline{5}$ *we find* $x = \overline{5}/\overline{3} = (\overline{5})(\overline{5}) = \overline{25} = \overline{4}$*.*

1.4 finite sums

In this section we introduce a nice notation for finite sums[8] of arbitrary size. Most of these statements are "for all $n \in \mathbb{N}$" thus proof by mathematical induction is the appropriate proof tool. I offer a few sample arguments and leave the rest to the reader. Let's begin by giving a precise definition for the finite sum $A_1 + A_2 + \cdots + A_n$:

Definition 1.4.1.

Let A_i for $i = 1, 2, \dots n$ be objects which allow addition. We recursively define:

$$\sum_{i=1}^{n+1} A_i = A_{n+1} + \sum_{i=1}^{n} A_i$$

for each $n \geq 1$ and $\sum_{i=1}^{1} A_i = A_1$.

The "summation notation" or "sigma" notation allows us to write sums precisely. In $\sum_{i=1}^{n} A_i$ the index i is called the **dummy index of summation**. One dummy is just a good as the next, it follows that $\sum_{i=1}^{n} A_i = \sum_{j=1}^{n} A_j$. This relabeling is sometimes called *switching dummy variables*, or *switching the index of summation from* i *to* j. The terms which are summed in the sum are called **summands**. For the sake of specificity I will assume real summands for the remainder of this section. It should be noted the arguments given here generalize with little to no work for a wide variety of other spaces where addition and multiplication by numbers is well-defined[9].

[8] the results of this section apply to objects which allow addition and multiplication by numbers, it is quite general
[9] in the later part of this course we learn such spaces are called vector spaces

Proposition 1.4.2.

Let $A_i, B_i \in \mathbb{R}$ for each $i \in \mathbb{N}$ and suppose $c \in \mathbb{R}$ then for each $n \in \mathbb{N}$,

$$(1.)\ \sum_{i=1}^{n}(A_i + B_i) = \sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i$$

$$(2.)\ \sum_{i=1}^{n} cA_i = c\sum_{i=1}^{n} A_i.$$

Proof: Let's begin with (1.). Notice the claim is trivially true for $n = 1$. Inductively assume that (1.) is true for $n \in \mathbb{N}$. Consider, the following calculations are justified either from the recursive definition of the finite sum or the induction hypothesis:

$$\begin{aligned}\sum_{i=1}^{n+1}(A_i + B_i) &= \sum_{i=1}^{n}(A_i + B_i) + A_{n+1} + B_{n+1} \\ &= \left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i\right) + A_{n+1} + B_{n+1} \\ &= \left(\sum_{i=1}^{n} A_i\right) + A_{n+1} + \left(\sum_{i=1}^{n} B_i\right) + B_{n+1} \\ &= \sum_{i=1}^{n+1} A_i + \sum_{i=1}^{n+1} B_i.\end{aligned}$$

Thus (1.) is true for $n + 1$ and hence by proof by mathematical induction (PMI) we find (1.) is true for all $n \in \mathbb{N}$. The proof of (2.) is similar. $\square$

Proposition 1.4.3.

Let $A_i, B_{ij} \in \mathbb{R}$ for $i, j \in \mathbb{N}$ and suppose $c \in \mathbb{R}$ then for each $n \in \mathbb{N}$,

$$(1.)\ \sum_{i=1}^{n}\left(\sum_{j=1}^{n} B_{ij}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} B_{ij}\right).$$

$$(2.)\ \sum_{i=1}^{n}\sum_{j=1}^{n} A_i B_{ij} = \sum_{i=1}^{n} A_i \sum_{j=1}^{n} B_{ij}$$

Proof: The proof of (1.) proceeds by induction on n. If $n = 1$ then there is only one possible term, namely B_{11} and the sums trivially agree. Consider the $n = 2$ case as we prepare for the induction step,

$$\sum_{i=1}^{2}\sum_{j=1}^{2} B_{ij} = \sum_{i=1}^{2}[B_{i1} + B_{i2}] = [B_{11} + B_{12}] + [B_{21} + B_{22}]$$

On the other hand,

$$\sum_{j=1}^{2}\sum_{i=1}^{2} B_{ij} = \sum_{j=1}^{2}[B_{1j} + B_{2j}] = [B_{11} + B_{21}] + [B_{11} + B_{21}].$$

The sums in opposite order produce the same terms overall, however the ordering of the terms may differ[10]. Fortunately, real number-addition commutes.

Assume inductively that (1.) is true for some $n > 1$. Using the definition of sum throughout and the induction hypothesis in transitioning from the 3-rd to the 4-th line:

$$\begin{aligned}
\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} B_{ij} &= \sum_{i=1}^{n+1}\left[B_{i,n+1} + \sum_{j=1}^{n} B_{ij}\right] \\
&= \sum_{i=1}^{n+1} B_{i,n+1} + \sum_{i=1}^{n+1}\sum_{j=1}^{n} B_{ij} \\
&= \sum_{i=1}^{n+1} B_{i,n+1} + \sum_{j=1}^{n} B_{n+1,j} + \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} \\
&= \sum_{i=1}^{n+1} B_{i,n+1} + \sum_{j=1}^{n} B_{n+1,j} + \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ij} \\
&= \sum_{i=1}^{n+1} B_{i,n+1} + \sum_{j=1}^{n}\left[B_{n+1,j} + \sum_{i=1}^{n} B_{ij}\right] \\
&= \sum_{i=1}^{n+1} B_{i,n+1} + \sum_{j=1}^{n}\sum_{i=1}^{n+1} B_{ij} \\
&= \sum_{j=1}^{n+1}\sum_{i=1}^{n+1} B_{ij}
\end{aligned}$$

Thus n implies $n+1$ for (1.) therefore by proof by mathematical induction we find (1.) is true for all $n \in \mathbb{N}$. In short, we can swap the order of finite sums. The proof of (2.) involves similar induction arguments. $\square$

From (1.) of the above proposition we find that multiple summations may be listed in any order. Moreover, a notation which indicates multiple sums is unambiguous:

$$\sum_{i,j=1}^{n} A_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}.$$

If we have more than two summations nested the same result holds. Therefore, define:

$$\sum_{i_1,\dots i_k=1}^{n} A_{i_1\dots i_k} = \sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} A_{i_1\dots i_k}.$$

Remark 1.4.4.

The purpose of this section is primarily notational. I want you to realize what is behind the notation and it is likely I assign some homework based on utilizing the recursive definition given here. I usually refer to the results of this section as " properties of finite sums".

[10] reordering terms in the infinite series case can get you into trouble if you don't have absolute convergence. Riemann showed a conditionally convergent series can be reordered to force it to converge to any value you might choose.

1.5 functions

Remark 1.5.1.

You can skip this section for now, we'll need some of these ideas later in the course and this material is not really linear algebra so I wish to confine it to this Chapter.

Suppose A and B are sets, we say $f : A \to B$ is a **function** if for each $a \in A$ the function f assigns a single element $f(a) \in B$. Moreover, if $f : A \to B$ is a function we say it is a **B-valued function of an A-variable** and we say $A = dom(f)$ whereas $B = codomain(f)$. For example, if $f : \mathbb{R}^2 \to [0, 1]$ then f is real-valued function of $\mathbb{R}^2$. On the other hand, if $f : \mathbb{C} \to \mathbb{R}^2$ then we'd say f is a vector-valued function of a complex variable. The term **mapping** will be used interchangeably with function in these notes. Suppose $f : U \to V$ and $U \subseteq S$ and $V \subseteq T$ then we may consisely express the same data via the notation $f : U \subseteq S \to V \subseteq T$.

Definition 1.5.2.

Suppose $f : U \to V$. We define the **image of U_1 under f** as follows:

$$f(U_1) = \{ \ y \in V \mid \text{there exists } x \in U_1 \text{ with } f(x) = y\}.$$

The **range** of f is $f(U)$. The **inverse image of V_1 under f** is defined as follows:

$$f^{-1}(V_1) = \{ \ x \in U \mid f(x) \in V_1\}.$$

The inverse image of a single point in the codomain is called a **fiber**. Suppose $f : U \to V$. We say f **is surjective** or **onto** V_1 iff there exists $U_1 \subseteq U$ such that $f(U_1) = V_1$. If a function is onto its codomain then the **function is surjective.** If $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in U_1 \subseteq U$ then we say **f is injective on** U_1 or $1 - 1$ on U_1. If a function is injective on its domain then we say the **function is injective**. If a function is both injective and surjective then the function is called a **bijection** or a **1-1 correspondence**.

Example 1.5.3. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $f(x, y) = x$ for each $(x, y) \in \mathbb{R}^2$. The function is not injective since $f(1, 2) = 1$ and $f(1, 3) = 1$ and yet $(1, 2) \neq (1, 3)$. Notice that the fibers of f are simply vertical lines:*

$$f^{-1}(x_o) = \{(x, y) \in dom(f) \mid f(x, y) = x_o\} = \{(x_o, y) \mid y \in \mathbb{R}\} = \{x_o\} \times \mathbb{R}$$

Example 1.5.4. *Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = \sqrt{x^2} + 1$ for each $x \in \mathbb{R}$. This function is not surjective because $0 \notin f(\mathbb{R})$. In contrast, if we construct $g : \mathbb{R} \to [1, \infty)$ with $g(x) = f(x)$ for each $x \in \mathbb{R}$ then can argue that g is surjective. Neither f nor g is injective, the fiber of x_o is $\{-x_o, x_o\}$ for each $x_o \neq 0$. At all points except zero these maps are said to be* **two-to-one**. *This is an abbreviation of the observation that two points in the domain map to the same point in the range.*

Example 1.5.5. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $f(x, y, z) = (x^2 + y^2, z)$ for each $(x, y, z) \in \mathbb{R}^3$. You can easily see that $range(f) = [0, \infty] \times \mathbb{R}$. Suppose $R^2 \in [0, \infty)$ and $z_o \in \mathbb{R}$ then*

$$f^{-1}(\{(R^2, z_o)\}) = S_1(R) \times \{z_o\}$$

where $S_1(R)$ denotes a circle of radius R. This result is a simple consequence of the observation that $f(x, y, z) = (R^2, z_o)$ implies $x^2 + y^2 = R^2$ and $z = z_o$.

Function composition is one important way to construct new functions. If $f : U \to V$ and $g : V \to W$ then $g \circ f : U \to W$ is the composite of g with f. We also create new functions by extending or restricting domains of given functions. In particular:

Definition 1.5.6.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. If $R \subset U$ then we define the **restriction of f to R** to be the mapping $f|_R : R \to V$ where $f|_R(x) = f(x)$ for all $x \in R$. If $U \subseteq S$ and $V \subset T$ then we say a mapping $g : S \to T$ is an **extension of f** iff $g|_{dom(f)} = f$.

When I say $g|_{dom(f)} = f$ this means that these functions have matching domains and they agree at each point in that domain; $g|_{dom(f)}(x) = f(x)$ for all $x \in dom(f)$. Once a particular subset is chosen the restriction to that subset is a unique function. Of course there are usually many susbets of $dom(f)$ so you can imagine many different restictions of a given function. The concept of extension is more vague, once you pick the enlarged domain and codomain it is not even necessarily the case that another extension to that same pair of sets will be the same mapping. To obtain uniqueness for extensions one needs to add more stucture. This is one reason that complex variables are interesting, there are cases where the structure of the complex theory forces the extension of a complex-valued function on a one-dimensional subset of $\mathbb{C}$ of a complex variable to be unique. This is very surprising. An even stronger result is available for a special type of function called a linear transformation. We'll see that a linear transformation is uniquely defined by its values on a basis. This means that a linear transformation is uniquely extended from a zero-dimensional subset of a vector space[11].

Definition 1.5.7.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping, if there exists a mapping $g : f(U) \to U$ such that $f \circ g = Id_{f(U)}$ and $g \circ f = Id_U$ then g is the inverse mapping of f and we denote $g = f^{-1}$.

If a mapping is injective then it can be shown that the inverse mapping is well defined. We define $f^{-1}(y) = x$ iff $f(x) = y$ and the value x must be a single value if the function is one-one. When a function is not one-one then there may be more than one point which maps to a particular point in the range.

Notice that the inverse image of a set is well-defined even if there is no inverse mapping. Moreover, it can be shown that the fibers of a mapping are disjoint and their union covers the domain of the mapping:

$$f(y) \neq f(z) \quad \Rightarrow \quad f^{-1}\{y\} \cap f^{-1}\{z\} = \emptyset \qquad\qquad \bigcup_{y \,\in\, range(f)} f^{-1}\{y\} = dom(f).$$

This means that the nonempty fibers of a mapping *partition* the domain.

Example 1.5.8. *Consider $f(x,y) = x^2 + y^2$ this describes a mapping from $\mathbb{R}^2$ to $\mathbb{R}$. Observe that $f^{-1}\{R^2\} = \{x^2 + y^2 = R^2 \mid (x,y) \in \mathbb{R}^2\}$. In words, the nonempty fibers of f are concentric circles about the origin and the origin itself.*

Technically, the emptyset is always a fiber. It is the fiber over points in the codomain which are not found in the range. In the example above, $f^{-1}(-\infty, 0) = \emptyset$. Perhaps, even from our limited

[11] technically, we don't know what this word "dimension" means just yet. Or linear transformation, or vector space, all in good time...

array of examples, you can begin to appreciate there is a unending array of possible shapes, curves, volumes and higher-dimensional objects which can appear as fibers. In contrast, as we will prove later in this course, the inverse image of any linear transformation is essentially[12] a line, plane or n-volume containing the origin.

Definition 1.5.9.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. A **cross section** of the fiber partiition is a subset $S \subseteq U$ for which $S \cap f^{-1}\{v\}$ contains a single element for every $v \in f(U)$.

How do we construct a cross section for a particular mapping? For particular examples the details of the formula for the mapping usually suggests some obvious choice. However, in general if you accept the **axiom of choice** then you can be comforted in the existence of a cross section even in the case that there are infinitely many fibers for the mapping. In this course, we'll see later that the problem of constructing a cross-section for a linear mapping is connected to the problem of finding a representative for each point in the quotient space of the mapping.

Example 1.5.10. *An easy cross-section for $f(x, y) = x^2 + y^2$ is given by any ray eminating from the origin. Notice that, if $ab \neq 0$ then $S = \{t(a, b) \mid t \in [0, \infty)\}$ interects the a circle of radius $R^2 = t^2(a^2 + b^2)$ at the point (ta, tb)*

Proposition 1.5.11.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. The restriction of f to a cross section S of U is an injective function. The mapping $\tilde{f} : U \to f(U)$ is a surjection. The mapping $\tilde{f}|_S : S \to f(U)$ is a bijection.

The proposition above tells us that we can take any mapping and cut down the domain and/or codomain to give the modfied function the structure of an injection, surjection or even a bijection.

Example 1.5.12. *Continuing with our example, $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y) = x^2 + y^2$ is neither surjective or injective. However, just to make a choice, $S = \{(t, 0) \mid t \in [0, \infty)\}$ then clearly $\tilde{f} : S \to [0, \infty)$ defined by $\tilde{f}(x, y) = f(x, y)$ for all $(x, y) \in S$ is a bijection.*

Definition 1.5.13.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping then we say a mapping g is a **local inverse** of f iff there exits $S \subseteq U$ such that $g = (f|_S)^{-1}$.

Usually we can find local inverses for functions in calculus. For example, $f(x) = \sin(x)$ is not 1-1 therefore it is not invertible. However, it does have a local inverse $g(y) = \sin^{-1}(y)$. If we were more pedantic we wouldn't write $\sin^{-1}(y)$. Instead we would write $g(y) = \left(\sin|_{[-\pi/2,\ \pi/2]}\right)^{-1}(y)$ since the inverse sine is actually just a local inverse. To construct a local inverse for some mapping we must locate some subset of the domain upon which the mapping is injective. Then relative to that subset we can reverse the mapping. I mention this concept in passing so you may appreciate its absense from this course. In linear algebra, the existence of a local inverse for a linear transformation will imply the existence of a global inverse. The case we study in this course is very special. We provide the bedrock on which other courses form arguments. Calculus linearizes problems locally, so, to understand local problems we must first understand linear problems. That is our task this semester, to unravel the structure of linear transformations as deeply as we dare.

[12] up to an isomorphism which is roughly speaking a change of notation

Chapter 2

matrix theory

Matrix calculation is often taught in the highschool algebra course. In this chapter we discuss basic matrix calculations. Perhaps more importantly, we introduce the concept of **matrix notation**. The concept that a whole array of numbers (a matrix) can be thought of as a single object is an absurdly useful economy of notation.

In the next Chapter we'll see that m-equations[1] in n-variables can be expressed as a single matrix equation[2]; $Ax = b$. However, matrices are so much more than a notation for equations. Matrices take on a life of their own in some sense. We can add, substract, sometimes multiply, sometimes divide. You can glue them together or rip them apart. You can change their rows to columns and their columns to rows. All of this without so much as a thought about which equations they represent. It turns out they represent much more than simple linear equations.

Motivation for the matrix operations given in this Chapter stem from the interplay between linear transformations and the matrices which they represent. I hope the reader will be patient as we defer discussion of such motivations to a slightly later point. Pegagogically I think this is very reasonable as the motivations do not bring much intuition to the calculations of this Chapter.

2.1 matrices

An array of objects is a collection of objects where we can keep track of which row and column each object resides. A finite sequence has the form $\{a_1, a_2, \dots, a_n\}$. There is a bijective correspondence between finite sequences[3] in a set S and functions from $\mathbb{N}_n$ to S. In particular, given $\{a_1, a_2, \dots, a_n\}$ we define $a(j) = a_j$ for each $j \in \mathbb{N}_n$. Likewise, if $a : \mathbb{N}_n \to S$ is a function then $\{a(1), a(2), \cdots, a(n)\}$ is a finite, ordered list in S; that is, a finite sequence in S. An $m \times n$ **array** of objects in S is likewise in bijective correspondence between functions from $\mathbb{N}_m \times \mathbb{N}_n$ to S. In particular, given $a : \mathbb{N}_m \times \mathbb{N}_n \to S$ we may construct an array as follows:

$$\begin{bmatrix} a(1,1) & a(1,2) & \cdots & a(1,n) \\ a(2,1) & a(2,2) & \cdots & a(2,n) \\ \vdots & \vdots & \cdots & \vdots \\ a(m,1) & a(m,2) & \cdots & a(m,n) \end{bmatrix}$$

[1]linear equations! If the equations are non-linear then we'll see the problem is radically different

[2]A is an $m \times n$ **matrix** and x is an $n \times 1$ **column vector** so the product Ax is a *matrix-column-product*.

[3]you studied the more subtle topic of infinite sequences in second semester calculus, there the sequences are functions from the positive integers to the real numbers typically

For the foundationalist, you might wonder how we construct an array. There are various answers to this question. One, we could adopt the viewpoint that an $m \times n$ array is simply a notation for a function from $\mathbb{N}_m \times \mathbb{N}_n$. Alternatively, you could view an array as a vector of vectors. The second view is sometimes used as a basis for the syntax used to manipulate matrices in Computer Algebra Systems[4] (CASs). Anyway, the construction of arrays from basic principles is really just something we assume in this course so I've probably already said too much about this substructure. Definition 2.1.3 captures the essential feature of an array we wish to exploit.

Definition 2.1.1.

An $m \times n$ matrix is an array of objects with m rows and n columns. The elements in the array are called entries or components. If A is an $m \times n$ matrix then A_{ij} denotes the object in the i-th row and the j-th column. We denote:

$$A = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

The label i is a **row index** and $1 \leq i \leq m$. The index j is a **column index** and $1 \leq j \leq n$.

$$\text{row}_i(A) = [A_{i1}, \dots, A_{in}] \qquad \& \quad \text{col}_j(A) = \begin{bmatrix} A_{1j} \\ \vdots \\ A_{rj} \end{bmatrix}$$

Generally, is S is a set then $S^{m \times n}$ is the set of $m \times n$ arrays of objects from S. If a matrix has the same number of rows and columns then it is called a **square matrix**.

The set $m \times n$ of matrices with real number entries is denoted $\mathbb{R}^{m \times n}$. The set of $m \times n$ matrices with complex entries is $\mathbb{C}^{m \times n}$. An $m \times n$ matrix can be seen as a **concatenation** of rows or columns:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \cdots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} = [\text{col}_1(A)| \cdots |\text{col}_n(A)] = \begin{bmatrix} \text{row}_1(A) \\ \hline \vdots \\ \hline \text{row}_m(A) \end{bmatrix}$$

To concentate to matrices is to join them together to make a larger matrix. The horizontal and vertical lines simply point to where the matrices have been glued together.

It is important to distinguish between the matrix A and the[5] i, j-th component A_{ij}. It is simply **not true** that $A = A_{ij}$. However, $A = [A_{ij}]$ as the brackets denote the array of all components.

Example 2.1.2. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *and* $B = \begin{bmatrix} i & 10 \\ 0 & 3+i \\ 11 & 12 \end{bmatrix}$ *then* $A \in \mathbb{R}^{3 \times 3}$ *and* $B \in \mathbb{C}^{3 \times 2}$. *If* $M = [A|B]$ *and* $N = [B|A]$ *then* $M, N \in \mathbb{C}^{3 \times 5}$. *Notice,* $M_{25} = 3 + i$ *whereas* $N_{25} = 6$.

In the example above we observed that the $2, 5$ components of M and N differ. It follows $M \neq N$. Let us pause to remind what is required for two arrays to be identitical:

[4]not to be confused with the ever more interesting ROUSs

[5]It is understood in this course that $i, j, k, l, m, n, p, q, r, s$ are in $\mathbb{N}$. I will not belabor this point. Please ask if in doubt.

Definition 2.1.3.

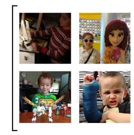
If $A, B \in S^{m\times n}$ then $A = B$ if and only if $A_{ij} = B_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

It is simple to see that $A = B$ is equivalent[6] to $\text{row}_i(A) = \text{row}_i(B)$ for all $i \in \mathbb{N}_m$. Likewise, $A = B$ is equivalent to $\text{col}_j(A) = \text{col}_j(B)$ for all $j \in \mathbb{N}_n$. We can measure the verity of a matrix equation at the level of components, rows, or columns. We use all three notations throughout this study.

Example 2.1.4. *Matrices may contain things other than numbers. For instance, if $f, g, h : \mathbb{R} \to \mathbb{R}$ are functions then $A = \begin{bmatrix} f & g \\ h & f \end{bmatrix}$ is a matrix of functions.*

But, we could look at more interesting matrices.

Example 2.1.5. *For example, this is a matrix of my kids: $K =$*



.

Our typical examples involve matrices with numbers as components. What is a *number*? That question is more philosophical than mathematical. I generally think of a number as an object which I can add, subtract and multiply.

Remark 2.1.6. *an overview of abstract algebraic terminology*

A **group** is a set paired with an operation which is associative, unital and is closed under inverses. If the operation of the group is commutative then the group is said to be **abelian**. A set R is called a **ring** if it has a pair of operations known as addition and multiplication. In particular, it is assumed that R paired with addition forms an abelian group. If R has a **unity** it is also assumed there exists $1 \in R$ for which $1x = x$ for each $x \in R$. Finally, multiplication of the ring must satisfy the following: for all $a, b, c, x, y \in R$,

$$a(x + y) = ax + ay \quad \& \quad (a + b)x = ax + bx.$$

In short, a ring is a place where you can do arithmetic as we usually practice. If $ab = ba$ for all $a, b \in R$ then R is a **commutative ring**. The study of commutative rings occupies a large part of the abstract algebra sequence at many universities. Even commutative rings are a bit more perilous than you might expect. For example, there are rings for which $ab = ac$ with $a \neq 0$ does not imply $b = c$. For example, $\mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ has $\bar{2}\,\bar{2} = \bar{0}$ and $\bar{2}\,\bar{0} = \bar{0}$. The number $\bar{2}$ is a **zero-divisor** in $\mathbb{Z}/4\mathbb{Z}$. If $r \in R$ has $s \in R$ for which $rs = 1$ then r is said to be a **unit** with multiplicative inverse s usually denoted $r^{-1} = s$. **Zero-divisors** are nonzero elements $a, b \in R$ for which $ab = 0$. A commutative ring with no zero divisors is called an **integral domain**. The quintessential example of an integral domain is $\mathbb{Z}$. If every nonzero element of a commutative ring is a unit then we say that ring is a **field**. It is a fun exercise to prove that no unit is a zero divisor (use proof by contradiction). It follows that every field is an integral domain. In the finite case, the converse is also true. Every finite integral domain is a field. This is the sort of claim we will prove in the study of abstract algebra course sequence. I mention it here for your informational edification. I'll leave you with a few claims whose proof I leave to another course; $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are fields. If p is a prime then $\mathbb{Z}/p\mathbb{Z}$ is also a field. The distinction between $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and the finite field $\mathbb{Z}/p\mathbb{Z}$ is quite clearly seen by the concept of **characteristic**. If $1 + 1 + \cdots + 1 \neq 0$ then the field has **characteristic zero**. In constrast, if we add p-fold copies of 1 in $\mathbb{Z}/p\mathbb{Z}$ then $1+1+\cdots+1 = p1 = 0$. We say the characteristic of $\mathbb{Z}/p\mathbb{Z}$ is p. However, if n is composite then $\mathbb{Z}/n\mathbb{Z}$ is not a field as all divisors of n produce zero divisors.

[6] two rows are equal iff the given pair of rows have the same components in the same order. Likewise, equality of columns is defined by equality of matching components.

Row and column matrices deserve further discussion. First, we need to define transposition[7] of a matrix: once more let S be a set,

Definition 2.1.7.

If $A \in S^{m\times n}$ then $A^T \in S^{n\times m}$ is defined by $(A^T)_{ji} = A_{ij}$ for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$. We say A^T is the **transpose** of A.

If we think through this definition in terms of rows and columns we can identify that transposition converts columns to rows and rows to columns.

Example 2.1.8. *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$ *then* $A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \end{bmatrix}$. *Observe*

$$row_1(A^T) = [A_{11}, A_{21}, A_{31}] = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix}^T = (col_1(A))^T.$$

Likewise, $row_2(A^T) = (col_2(A))^T$.

Proposition 2.1.9.

Let $A \in S^{m\times n}$ then (i.) $(A^T)^T = A$. Furthermore, for each $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$,

$$(ii.)\ \ \mathrm{row}_j(A^T) = (\mathrm{col}_j(A))^T \qquad \& \qquad (iii.)\ \mathrm{col}_i(A^T) = (\mathrm{row}_i(A))^T.$$

Proof: To prove (i.) simply note that $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$ for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$. Notice, $(\mathrm{col}_j(A))_i = A_{ij}$ whereas $(\mathrm{row}_i(A))_j = A_{ij}$. Thus consider,

$$(\mathrm{row}_j(A^T))_i = (A^T)_{ji} = A_{ij} = (\mathrm{col}_j(A))_i = ((\mathrm{col}_j(A))^T)_i$$

the last step is simply that the i-th component of a row vector is the i-th component of the transpose of the row vector. I leave the proof of (iii.) to the reader. $\square$

In principle one can use column vectors for everything or row vectors for everything. I choose a subtle **convention** that allows us to use both[8]

Definition 2.1.10. *hidden column notation.*

We denote $(v_1, v_2, \dots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and we write $S^n = S^{n\times 1}$.

If I want to denote a real row vector then we will just write $[v_1, v_2, \dots, v_n]$. This convention means we view points as column vectors. This is just a notational choice.

[7] some authors prefer the notation ${}^tA$ in the place of A^T

[8] On the one hand it is nice to write vectors as rows since the typesetting is easier. However, once you start talking about matrix multiplication then it is natural to write the vector to the right of the matrix and we will soon see that the vector should be written as a column vector for that to be reasonable.

2.2 matrix addition and scalar multiplication

Often we only need to consider numbers in fields, but, I'll prove a few things just assuming a ring structure for the elements. This generality costs us nothing and helps the reader get in the habit of thinking abstractly. Let us collect the essential algebraic features of a commutative ring R with identity. There is an additive identity $0 \in R$ such that $x + 0 = x$ for each $x \in R$. There is also a multiplicative identity $1 \in R$ such that $1x = x$ for each $x \in R$. For each $x \in R$ there exists $-x \in R$ such that $x + (-x) = 0$. We also have $(-1)(x) = -x$ and $0(x) = 0$ for each $x \in R$. Furthermore, for each for $a, b, c \in R$,

(1.) associativity of addition: $a + (b + c) = (a + b) + c$,

(2.) commutativity of addition: $a + b = b + a$,

(3.) left and right distibutivity: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$,

(4.) associativty of multiplication; $a(bc) = (ab)c$

Some mathematicians include the existence of $1 \in R$ as part of the definition of a ring, but, I do not assume that here or in the abstract algebra courses. That said, in the remainder of this section **please assume R is a commutive ring with identity** 1. Pragmatically, this means I allow $R = \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$. However, we could also have R be the set of continuous funtions on $\mathbb{R}$. There are many sets of things which have the structure of a commutative ring with identity. In all those many cases we can construct a matrix of such objects. The arguments in this Chapter demonstrate that a common algebra is shared for this multitude of examples. Abstraction allows us to carry many loads at once. We cut away all the irrelevant features of R and focus just on the arithmetic properties above. Those suffice to develop the matrix algebra.

Definition 2.2.1. *Let R be a commutative ring with $1 \in R$.*

If $A, B \in R^{m\times n}$ then the **sum** of A, B is $A + B$ and the **scalar multiple** of A by c is cA. these are defined as follows:

$$(A + B)_{ij} = A_{ij} + B_{ij} \qquad \& \qquad (cA)_{ij} = cA_{ij}$$

for all $(i, j) \in \mathbb{N}_m \times \mathbb{N}_n$.

In the special case of row or column vectors we understand the Definition above to reduce to:

$$(x + y)_i = x_i + y_i \qquad \& \qquad (cx)_i = cx_i$$

for $c \in R$ and $x, y \in R^n = R^{n\times 1}$ or for $x, y \in R^{1\times n}$.
Let us pause to consider a few computational examples[9]

Example 2.2.2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$.

Example 2.2.3. *Let $A = \begin{bmatrix} -2 & 4 \\ 10 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} x & x^2 \\ 7+y & 2z^2 \end{bmatrix}$. Can we solve $A = B$? Notice, the equality $A = B$ gives four equations we must solve concurrently:*

$$-2 = x, \qquad 4 = x^2, \qquad 10 = 7 + y, \qquad 4 = 2z^2.$$

We find two solutions $x = -2, y = 3$ and $z = \pm\sqrt{2}$

[9]you might take a moment to notice my examples tend to fit into one of the following three types: question and answer, discussion-discovery or show-case a theorem or definition. If you're looking to identify the problem type so you can solve it when I ask you it again, you need to adjust your thinking here...

Example 2.2.4. *Let $A, B \in \mathbb{R}^{m\times n}$ be defined by $A_{ij} = 3i + 5j$ and $B_{ij} = i^2$ for all i, j. Then we can calculate $(A + 7B)_{ij} = 3i + 5j + 7i^2$ for all i, j.*

Example 2.2.5. *Over $R = \mathbb{Z}/6\mathbb{Z}$ we calculate*

$$\bar{3}\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{3} & \bar{4} \end{bmatrix} = \begin{bmatrix} \bar{3}\cdot\bar{1} & \bar{3}\cdot\bar{2} \\ \bar{3}\cdot\bar{3} & \bar{3}\cdot\bar{4} \end{bmatrix} = \begin{bmatrix} \bar{3} & \bar{6} \\ \bar{9} & \bar{12} \end{bmatrix} = \begin{bmatrix} \bar{3} & \bar{0} \\ \bar{3} & \bar{0} \end{bmatrix}$$

Definition 2.2.1 says we define matrix addition and scalar multiplication **component-wise**. We show next how index notation provides us an elegant formalism to easily prove facts about the algebra of matrices. Notice how each arithmetic property rings induces a similar matrix property:

Proposition 2.2.6. *linearity of matrix addition and scalar multiplication*

Let R be a commutative ring with unity. If $A, B, C \in R^{m\times n}$ and $c_1, c_2 \in R$ then

1. $(A + B) + C = A + (B + C)$,
2. $A + B = B + A$,
3. $c_1(A + B) = c_1A + c_2B$,
4. $(c_1 + c_2)A = c_1A + c_2A$,
5. $(c_1c_2)A = c_1(c_2A)$,
6. $1A = A$,

Proof: Nearly all of these properties are proved by breaking the statement down to components then appealing to a ring property. I supply proofs of (1.) and (5.) and leave (2.),(3.), (4.) and (6.) to the reader.

Proof of (1.): assume A, B, C are given as in the statement of the Theorem. Observe that

$$\begin{aligned} ((A + B) + C)_{ij} &= (A + B)_{ij} + C_{ij} && \text{defn. of matrix add.} \\ &= (A_{ij} + B_{ij}) + C_{ij} && \text{defn. of matrix add.} \\ &= A_{ij} + (B_{ij} + C_{ij}) && \text{assoc. of ring addition} \\ &= A_{ij} + (B + C)_{ij} && \text{defn. of matrix add.} \\ &= (A + (B + C))_{ij} && \text{defn. of matrix add.} \end{aligned}$$

for all i, j. Therefore $(A + B) + C = A + (B + C)$. $\square$

Proof of (5.): assume c_1, c_2, A are given as in the statement of the Theorem. Observe that

$$\begin{aligned} ((c_1c_2)A)_{ij} &= (c_1c_2)A_{ij} && \text{defn. scalar multiplication.} \\ &= c_1(c_2A_{ij}) && \text{assoc. of ring multiplication} \\ &= (c_1(c_2A))_{ij} && \text{defn. scalar multiplication.} \end{aligned}$$

for all i, j. Therefore $(c_1c_2)A = c_1(c_2A)$. $\square$

The proofs of the other items are similar, we consider the i, j-th component of the identity and then apply the definition of the appropriate matrix operation's definition. This reduces the problem to a statement about arithmetic in a ring so we can use the ring properties at the level of components. After applying the crucial fact about rings, we then reverse the steps. Since the calculation works for arbitrary i, j it follows the the matrix equation holds true. This Proposition provides a foundation for later work where we may find it convenient to prove a statement without resorting to a proof by components. Which method of proof is best depends on the question. However, I can't see another way of proving most of 2.2.6.

Definition 2.2.7.

The **zero matrix** in $R^{m\times n}$ is denoted 0 and defined by $0_{ij} = 0$ for all $(i, j) \in \mathbb{N}_m \times \mathbb{N}_n$. The additive inverse of $A \in R^{m\times n}$ is the matrix $-A$ such that $A + (-A) = 0$. The components of the additive inverse matrix are given by $(-A)_{ij} = -A_{ij}$ for all $(i, j) \in \mathbb{N}_m \times \mathbb{N}_n$.

The zero matrix joins a long list of other objects which are all denoted by 0. Usually the meaning of 0 is clear from the context, the size of the zero matrix is chosen as to be consistent with the equation in which it is found.

Example 2.2.8. *Solve the following matrix equation,*

$$0 = \begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x-1 & y-2 \\ z-3 & w-4 \end{bmatrix}$$

The definition of matrix equality means this single matrix equation reduces to 4 scalar equations: $0 = x - 1, 0 = y - 2, 0 = z - 3, 0 = w - 4$. *The solution is* $x = 1, y = 2, z = 3, w = 4$.

Theorem 2.2.9.

If $A \in R^{m\times n}$ then

1. $0 \cdot A = 0$, (scalar multiplication by 0 produces the zero matrix)
2. $A + 0 = 0 + A = A$.

Proof: To prove (1.). Let $A \in R^{m\times n}$ and consider by definition of scalar multiplication of a matrix:

$$(0 \cdot A)_{ij} = 0(A_{ij}) = 0$$

for all i, j. Thus $0 \cdot A = 0$. To see (2.), observe by the definiition of matrix addition and the zero-matrix:

$$(0 + A)_{ij} = 0_{ij} + A_{ij} = 0 + A_{ij} = A_{ij}$$

and as the above holds for all i, j we find $0 + A = A$. $\square$

2.2.1 standard column and matrix bases

The notation introduced in this subsection is near to my heart. It frees you to calculate a multitude of stupidly general claims with a minimum of writing.

Definition 2.2.10.

The symbol $\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$ is called the **Kronecker delta**.

For example, $\delta_{22} = 1$ while $\delta_{12} = 0$.

Definition 2.2.11.

Let $e_i \in R^{n\times 1}$ be defined by $(e_i)_j = \delta_{ij}$. The size of the vector e_i is determined by context. We call e_i the i-th standard basis vector.

Example 2.2.12. *Let me expand on what I mean by "context" in the definition above:*
In R we have $e_1 = (1) = 1$ (by convention we drop the brackets in this case)
In R^2 we have $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
In R^3 we have $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.
In R^4 we have $e_1 = (1, 0, 0, 0)$ and $e_2 = (0, 1, 0, 0)$ and $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$.

Example 2.2.13. *Any vector in R^n can be written as a sum of these basic vectors. For example,*

$$\begin{aligned} v &= (1, 2, 3) = (1, 0, 0) + (0, 2, 0) + (0, 0, 3) \\ &= 1(1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1) \\ &= e_1 + 2e_2 + 3e_3. \end{aligned}$$

We say that v is a **finite linear combination** *of e_1, e_2 and e_3.*

The concept of a finite linear combination is very important[10].

Definition 2.2.14.

A **finite R-linear combination** of objects $A_1, A_2, \dots, A_k$ is a finite sum

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k = \sum_{i=1}^{k} c_iA_i$$

where the **coefficients** $c_i \in R$ for each i. If $c_1 = 0, c_2 = 0, \dots, c_k = 0$ then we say the linear combination is **trivial**. We also say that $\{0\}$ is formed by an **empty sum**. That is, a linear combination of $\emptyset$ is just $\{0\}$.

The statement about the emptyset $\emptyset$ helps theorems we state in future sections to be generally true. We will look at linear combinations of vectors, matrices and even functions in this course. The proposition below generalizes the calculation from Example 2.2.13.

Proposition 2.2.15.

Every vector in R^n is a linear combination of $e_1, e_2, \dots, e_n$.

Proof: Let $v = (v_1, v_2, \dots, v_n) \in R^n$. By the definition of vector addition and zero in R:

$$\begin{aligned} v &= (v_1 + 0, 0 + v_2, \dots, 0 + v_n) \\ &= (v_1, 0, \dots, 0) + (0, v_2, \dots, v_n) \\ &= (v_1, 0, \dots, 0) + (0, v_2, \dots, 0) + \cdots + (0, 0, \dots, v_n) \\ &= (v_1 \cdot 1, v_1 \cdot 0, \dots, v_1 \cdot 0) + (v_2 \cdot 0, v_2 \cdot 1, \dots, v_2 \cdot 0) + \cdots + (v_n \cdot 0, \dots, v_n \cdot 1) \end{aligned}$$

In the last step I rewrote each zero to emphasize that the each entry of the k-th summand has a v_k factor. Continue by applying the definition of scalar multiplication to each vector in the sum above we find,

$$\begin{aligned} v &= v_1(1, 0, \dots, 0) + v_2(0, 1, \dots, 0) + \cdots + v_n(0, 0, \dots, 1) \\ &= v_1e_1 + v_2e_2 + \cdots + v_ne_n. \end{aligned}$$

[10]since we only consider finite linear combinations in this course we generally omit the term *finite*

Therefore, every vector in R^n is a linear combination of $e_1, e_2, \dots, e_n$. For each $v \in R^n$ we have $v = \sum_{i=1}^{n} v_n e_n$. $\square$

We can define a standard basis for matrices of arbitrary size in much the same manner.

Definition 2.2.16.

The ij-th **standard basis matrix** for $R^{m\times n}$ is denoted E_{ij} for $1 \le i \le m$ and $1 \le j \le n$. The matrix E_{ij} is zero in all entries except for the (i,j)-th slot where it has a 1. In other words, we define $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

Proposition 2.2.17.

Every matrix in $R^{m\times n}$ is a linear combination of the E_{ij} where $1 \le i \le m$ and $1 \le j \le n$.

Proof: Let $A \in R^{m\times n}$ then

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \\ &= A_{11}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + A_{12}\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + A_{mn}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= A_{11}E_{11} + A_{12}E_{12} + \cdots + A_{mn}E_{mn}. \end{aligned}$$

The calculation above follows from repeated mn-applications of the definition of matrix addition and another mn-applications of the definition of scalar multiplication of a matrix. We can restate the final result in a more precise langauge,

$$A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}.$$

As we claimed, any matrix can be written as a linear combination of the E_{ij}. $\square$

Alternate Proof: Let $A \in \mathbb{R}^{m\times n}$ then let $B = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}$ and calculate[11],

$$B_{kl} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}\right)_{kl} = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}(E_{ij})_{kl} = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\delta_{ik}\delta_{jl} = A_{kl}.$$

Thus $B_{kl} = A_{kl}$ for all $(k,l) \in \mathbb{N}_m \times \mathbb{N}_n$ hence $A = B$. $\square$

The term "basis" has a technical meaning which we will discuss at length in due time. For now, just think of it as part of the names of e_i and E_{ij}. These are the basic building blocks for matrix theory.

[11] here I have to repeatedly apply the definition of matrix addition and scalar multiplication. I will probably add a homework problem where you get to prove this follows from the definition by a simple induction argument

2.3 matrix multiplication

I don't seek to motivate the definition below. Well, not yet anyway. Rest assure there are many good reasons to multiply matrices in this way. We shall discover them as the course unfolds.

Definition 2.3.1. *Let R be a commutative ring.*

Let $A \in R^{m \times n}$ and $B \in R^{n \times p}$ then we say A and B are **multipliable** or **compatible** or **conformable**. The product of A and B is denoted by juxtaposition AB and $AB \in R^{m \times p}$ is defined by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

for each $1 \leq i \leq m$ and $1 \leq j \leq p$. In the case $m = p = 1$ the indices i, j are omitted in the equation since the matrix product is simply a number which needs no index.

This definition is very nice for general proofs and we will need to know it for proofs. However, for explicit numerical examples, I usually think of matrix multiplication in terms of *dot-products*.

Definition 2.3.2.

Let $v, w \in R^n \cup R^{1 \times n}$ then the **dot-product** of v and w is the number defined below:

$$v \bullet w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k.$$

In a later chapter we study the geometric[12] content of the dot-product[13] when R is the field $\mathbb{R}$.

Proposition 2.3.3. *dot-product as a row-column multiplication:*

Let $v, w \in R^n$ then $v \bullet w = v^T w$.

Proof: Since v^T is an $1 \times n$ matrix and w is an $n \times 1$ matrix the definition of matrix multiplication indicates $v^T w$ should be a 1×1 matrix which is a number. Note in this case the outside indices ij are absent in the boxed equation so the equation reduces to

$$v^T w = {v^T}_1 w_1 + {v^T}_2 w_2 + \cdots + {v^T}_n w_n = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = v \cdot w. \qquad \square$$

Proposition 2.3.4.

Let $A \in R^{m \times n}$ and $B \in R^{n \times p}$ then

$$AB = \begin{bmatrix} row_1(A) \cdot col_1(B) & row_1(A) \cdot col_2(B) & \cdots & row_1(A) \cdot col_p(B) \\ row_2(A) \cdot col_1(B) & row_2(A) \cdot col_2(B) & \cdots & row_2(A) \cdot col_p(B) \\ \vdots & \vdots & \cdots & \vdots \\ row_m(A) \cdot col_1(B) & row_m(A) \cdot col_2(B) & \cdots & row_m(A) \cdot col_p(B) \end{bmatrix}$$

[12] In fact, there is some analog of the dot-product for complex numbers and quaternions. Many interesting *matrix groups* arise as *isometries* for these *inner products*. I hope to cover some of that material in a short course on Lie groups I'm holding this Spring 2016 Semester.

[13] The definition I give above is a bit unusual as it allows us to take the dot-product of row and column vectors. This is mostly a convenience of notation as to avoid writing a multitude of transposes in the Proposition below.

Proof: The formula above claims $(AB)_{ij} = row_i(A) \cdot col_j(B)$ for all i, j. Recall that $(row_i(A))_k = A_{ik}$ and $(col_j(B))_k = B_{kj}$ thus

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} (row_i(A))_k (col_j(B))_k$$

Hence, using definition of the dot-product, $(AB)_{ij} = row_i(A) \cdot col_j(B)$. This argument holds for all i, j therefore the Proposition is true. $\square$

Example 2.3.5. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $v = \begin{bmatrix} x \\ y \end{bmatrix}$ *then we may calculate the product* Av *as follows:*

$$Av = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}.$$

Notice that the product of an $n \times k$ matrix with a $k \times 1$ vector yields another vector of size $k \times 1$. In the example above we observed the pattern $(2 \times 2)(2 \times 1) \to (2 \times 1)$.

Example 2.3.6. *The product of a* 3×2 *and* 2×3 *is a* 3×3

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} [1,0][4,7]^T & [1,0][5,8]^T & [1,0][6,9]^T \\ [0,1][4,7]^T & [0,1][5,8]^T & [0,1][6,9]^T \\ [0,0][4,7]^T & [0,0][5,8]^T & [0,0][6,9]^T \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2.3.7. *The product of a* 3×1 *and* 1×3 *is a* 3×3

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 & 5 \cdot 1 & 6 \cdot 1 \\ 4 \cdot 2 & 5 \cdot 2 & 6 \cdot 2 \\ 4 \cdot 3 & 5 \cdot 3 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

Example 2.3.8. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *and* $v = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ *calculate* Av.

$$Av = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} (1,2,3) \cdot (1,0,-3) \\ (4,5,6) \cdot (1,0,-3) \\ (7,8,9) \cdot (1,0,-3) \end{bmatrix} = \begin{bmatrix} -8 \\ -14 \\ -20 \end{bmatrix}.$$

Example 2.3.9. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. *We calculate*

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} [1,2][5,7]^T & [1,2][6,8]^T \\ [3,4][5,7]^T & [3,4][6,8]^T \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Notice the product of square matrices is square. For numbers $a, b \in \mathbb{R}$ *it we know the product of* a *and* b *is commutative* ($ab = ba$). *Let's calculate the product of* A *and* B *in the opposite order,*

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} [5,6][1,3]^T & [5,6][2,4]^T \\ [7,8][1,3]^T & [7,8][2,4]^T \end{bmatrix} = \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Clearly $AB \neq BA$ *thus matrix multiplication is* **noncommutative** *or* **not commutative**.

Remark 2.3.10. *commutators*

The **commutator** of two square matrices A, B is given by $[A, B] = AB - BA$. If $[A, B] \neq 0$ then clearly $AB \neq BA$. There are many interesting properties of the commutator. It has deep physical significance in quantum mechanics. It is also the quintessential example of a **Lie Bracket**. It turns out that if the commutator of two observables is zero then they can be measured simultaneously to arbitrary precision. However, if the commutator of two observables is nonzero (such as is the case with position and momentum) then they cannot be simulaltaneously measured with arbitrary precision. The more precisely you know position, the less you know momentum and vice-versa. This is Heisenberg's **uncertainty principle** of quantum mechanics.

Properties of matrix multiplication are given in the theorem below. To summarize, matrix math works as you would expect with the exception that matrix multiplication is not commutative. We must be careful about the order of letters in matrix expressions.

Theorem 2.3.11.

If $A, B, C \in R^{m\times n}$, $X, Y \in R^{n\times p}$, $Z \in R^{p\times q}$ and $c_1, c_2 \in R$ then

1. $(AX)Z = A(XZ)$,
2. $(c_1A)X = c_1(AX) = A(c_1X) = (AX)c_1$,
3. $A(X + Y) = AX + AY$,
4. $A(c_1X + c_2Y) = c_1AX + c_2AY$,
5. $(A + B)X = AX + BX$,

Proof: I leave the proofs of (1.), (2.), (4.) and (5.) to the reader. Proof of (3.): assume A, X, Y are given as in the statement of the Theorem. Observe that

$$\begin{array}{lll} ((A(X+Y))_{ij} & = \sum_k A_{ik}(X+Y)_{kj} & \text{defn. matrix multiplication,} \\ & = \sum_k A_{ik}(X_{kj}+Y_{kj}) & \text{defn. matrix addition,} \\ & = \sum_k (A_{ik}X_{kj}+A_{ik}Y_{kj}) & \text{dist. prop. of rings,} \\ & = \sum_k A_{ik}X_{kj} + \sum_k A_{ik}Y_{kj}) & \text{prop. of finite sum,} \\ & = (AX)_{ij} + (AY)_{ij} & \text{defn. matrix multiplication}(\times\ 2), \\ & = (AX+AY)_{ij} & \text{defn. matrix addition,} \end{array}$$

for all i, j. Therefore $A(X + Y) = AX + AY$. $\square$

The proofs of the other items are similar, I invite the reader to try to prove them in a style much like the proof I offer above.

We began our study of transpose in Proposition 2.1.9. Let us continue it:

Proposition 2.3.12. *Let R be a commutative ring with identity.*

1. $(A^T)^T = A$ for all $A \in R^{m\times n}$,
2. $(AB)^T = B^T A^T$ for all $A \in R^{m\times n}$ and $B \in R^{n\times p}$ (socks-shoes),
3. $(cA)^T = cA^T$ for all $A \in R^{m\times n}$ and $c \in R$,
4. $(A + B)^T = A^T + B^T$ for all $A, B \in R^{m\times n}$.

Proof: We proved (1.) for Proposition 2.1.9. Proof of (2.) is left to the reader. Proof of (3.) and (4.) is simple enough,

$$((A + cB)^T)_{ij} = (A + cB)_{ji} = A_{ji} + cB_{ji} = (A^T)_{ij} + ((cB)^T)_{ij}$$

for all i, j. Set $A = 0$ to obtain (3.) and set $c = 1$ to obtain (4.). □

2.3.1 multiplication of row or column concatenations

Proposition 2.3.4 is not the only way to calculate the matrix product. In this subsection we find several new ways to decompose a product which are ideal to reveal such row or column patterns. In some sense, this section is just a special case of the later section on block-multiplication. However, you could probably just as well say block multiplication is a simple outgrowth of what we study here. In any event, we need this material to properly understand the method to calculate A^{-1} and the final proposition of this section is absolutely critical to properly understand the structure of the solution set for $Ax = b$.

Example 2.3.13. *The product of a 2×2 and 2×1 is a 2×1. Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and let $v = \left[\begin{smallmatrix} 5 \\ 7 \end{smallmatrix}\right]$,*

$$Av = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} [1,2][5,7]^T \\ [3,4][5,7]^T \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Likewise, define $w = \left[\begin{smallmatrix} 6 \\ 8 \end{smallmatrix}\right]$ and calculate

$$Aw = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} [1,2][6,8]^T \\ [3,4][6,8]^T \end{bmatrix} = \begin{bmatrix} 22 \\ 50 \end{bmatrix}$$

Something interesting to observe here, recall that in Example 2.3.9 we calculated $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$. But these are the same numbers we just found from the two matrix-vector products calculated above. We identify that B is just the **concatenation** *of the vectors v and w; $B = [v|w] = \left[\begin{array}{c|c} 5 & 6 \\ 7 & 8 \end{array}\right]$. Observe that:*

$$AB = A[v|w] = [Av|Aw].$$

The term **concatenate** is sometimes replaced with the word **adjoin**. I think of the process as gluing matrices together. This is an important operation since it allows us to lump together many solutions into a single matrix of solutions. (I will elaborate on that in detail in a future section)

Proposition 2.3.14. *the concatenation proposition for columns*

Let $A \in R^{m\times n}$ and $B \in R^{n\times p}$ then we can understand the matrix multiplication of A and B as the concatenation of several matrix-vector products,

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)]$$

Proof: see the Problem Set. You should be able to follow the same general strategy as the Proof of Proposition 2.3.4. Show that the i, j-th entry of the L.H.S. is equal to the matching entry on the R.H.S. Good hunting. $\square$

There are actually many many different ways to perform the calculation of matrix multiplication. Proposition 2.3.14 essentially parses the problem into a bunch of (matrix)(column vector) calculations. You could go the other direction and view AB as a bunch of (row vector)(matrix) products glued together. In particular,

Proposition 2.3.15. *the concatenation proposition for rows*

Let $A \in R^{m\times n}$ and $B \in R^{n\times p}$ then we can understand the matrix multiplication of A and B as the concatenation of several matrix-vector products,

$$AB = \begin{bmatrix} row_1(A) \\ row_2(A) \\ \vdots \\ row_m(A) \end{bmatrix} B = \begin{bmatrix} row_1(A)B \\ row_2(A)B \\ \vdots \\ row_m(A)B \end{bmatrix}.$$

Proof: let $R_i = \text{row}_i(A)$ hence $A^T = [R_1^T|R_2^T|\cdots|R_m^T]$. Use Proposition 2.3.14 to calculate:

$$B^TA^T = B^T[R_1^T|R_2^T|\cdots|R_m^T] = [B^TR_1^T|B^TR_2^T|\cdots|B^TR_m^T] \quad \star$$

But, $(B^TA^T)^T = (A^T)^T(B^T)^T = AB$. Thus, taking the transpose of $\star$ yields

$$AB = [B^TR_1^T|B^TR_2^T|\cdots|B^TR_m^T]^T = \left[\begin{array}{c} (B^TR_1^T)^T \\ \hline (B^TR_2^T)^T \\ \hline \vdots \\ \hline (B^TR_m^T)^T \end{array}\right] = \left[\begin{array}{c} R_1B \\ \hline R_2B \\ \hline \vdots \\ \hline R_mB \end{array}\right].$$

where we used $(B^TR_i^T)^T = (R_i^T)^T(B^T)^T = R_iB$ for each i in the last step. $\square$

There are stranger ways to calculate the product. You can also assemble the product by adding together a bunch of outer-products of the rows of A with the columns of B. The dot-product of two vectors is an example of an inner product and we saw $v \cdot w = v^Tw$. The outer-product of two vectors goes the other direction: given $v \in R^n$ and $w \in R^m$ we find $vw^T \in R^{n\times m}$.

Proposition 2.3.16. *matrix multiplication as sum of outer products.*

Let $A \in R^{m\times n}$ and $B \in R^{n\times p}$ then

$$AB = col_1(A)row_1(B) + col_2(A)row_2(B) + \cdots + col_n(A)row_n(B).$$

Proof: consider the i,j-th component of AB, by definition we have

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$$

but note that $(col_k(A)row_k(B))_{ij} = col_k(A)_i row_k(B)_j = A_{ik}B_{kj}$ for each $k = 1, 2, \dots, n$ and the proposition follows. $\square$

A corollary is a result which falls immediately from a given result. Take the case $B = v \in R^{n\times 1}$ to prove the following:

Corollary 2.3.17. *matrix-column product is linear combination of columns.*

Let $A \in R^{m\times n}$ and $v \in R^n$ then

$$Av = v_1 col_1(A) + v_2 col_2(A) + \cdots + v_n col_n(A).$$

Some texts use the result above as the foundational definition for matrix multiplication. We took a different approach in these notes, largely because I wish for students to gain better grasp of index calculation. If you'd like to know more about the other approach, I can recommend some reading.

2.3.2 all your base are belong to us (e_i and E_{ij} that is)

Example 2.3.18. *Suppose $A \in R^{m\times n}$ and $e_i \in R^n$ is a standard basis vector,*

$$(Ae_i)_j = \sum_{k=1}^{n} A_{jk}(e_i)k = \sum_{k=1}^{n} A_{jk}\delta_{ik} = A_{ji}$$

Thus, $\boxed{[Ae_i] = col_i(A)}$. We find that multiplication of a matrix A by the standard basis e_i yields the $i - th$ column of A.

Example 2.3.19. *Suppose $A \in R^{m\times n}$ and $e_i \in R^{m\times 1}$ is a standard basis vector,*

$$({e_i}^T A)_j = \sum_{k=1}^{n} (e_i)_k A_{kj} = \sum_{k=1}^{n} \delta_{ik}A_{kj} = A_{ij}$$

Thus, $\boxed{[{e_i}^T A] = row_i(A)}$. We find multiplication of a matrix A by the transpose of standard basis e_i yields the $i - th$ row of A.

Example 2.3.20. *Again, suppose $e_i, e_j \in R^n$ are standard basis vectors. The product ${e_i}^T e_j$ of the $1 \times n$ and $n \times 1$ matrices is just a 1×1 matrix which is just a number. In particular consider,*

$${e_i}^T e_j = \sum_{k=1}^{n} ({e_i}^T)_k (e_j)_k = \sum_{k=1}^{n} \delta_{ik}\delta_{jk} = \delta_{ij}$$

The product is zero unless the vectors are identical.

Example 2.3.21. *Suppose $e_i \in R^{m\times 1}$ and $e_j \in R^n$. The product of the $m \times 1$ matrix e_i and the $1 \times n$ matrix ${e_j}^T$ is an $m \times n$ matrix. In particular,*

$$(e_i {e_j}^T)_{kl} = (e_i)_k (e_j^T)_l = \delta_{ik}\delta_{jl} = (E_{ij})_{kl}.$$

Thus the standard basis matrices are constructed from the standard basis vectors; $E_{ij} = e_i {e_j}^T$.

Example 2.3.22. *What about the matrix E_{ij}? What can we say about multiplication by E_{ij} on the right of an arbitrary matrix? Let $A \in \mathbb{R}^{m\times n}$ and consider,*

$$(AE_{ij})_{kl} = \sum_{p=1}^{n} A_{kp}(E_{ij})_{pl} = \sum_{p=1}^{n} A_{kp}\delta_{ip}\delta_{jl} = A_{ki}\delta_{jl}$$

Notice the matrix above has zero entries unless $j = l$ which means that the matrix is mostly zero except for the j-th column. We can select the j-th column by multiplying the above by e_j, using Examples 2.3.20 and 2.3.18,

$$(AE_{ij}e_j)_k = (Ae_i{e_j}^T e_j)_k = (Ae_i\delta_{jj})_k = (Ae_i)_k = (col_i(A))_k$$

This means,

$$AE_{ij} = \begin{array}{c} \begin{array}{cccccc} & & & \text{column } j & & \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & A_{1i} & \cdots & 0 \\ 0 & 0 & \cdots & A_{2i} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_{mi} & \cdots & 0 \end{bmatrix} \end{array}$$

Right multiplication of matrix A by E_{ij} moves the i-th column of A to the j-th column of AE_{ij} and all other entries are zero. It turns out that left multiplication by E_{ij} moves the j-th row of A to the i-th row and sets all other entries to zero.

Example 2.3.23. *Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ consider multiplication by E_{12},*

$$AE_{12} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{cc} 0 & 1 \\ \hline 0 & 3 \end{array}\right] = \left[\begin{array}{c|c} 0 & col_1(A) \end{array}\right]$$

Which agrees with our general abstract calculation in the previous example. Next consider,

$$E_{12}A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c} row_2(A) \\ \hline 0 \end{array}\right].$$

Example 2.3.24. *Calculate the product of E_{ij} and E_{kl}.*

$$(E_{ij}E_{kl})_{mn} = \sum_p (E_{ij})_{mp}(E_{kl})_{pn} = \sum_p \delta_{im}\delta_{jp}\delta_{kp}\delta_{ln} = \delta_{im}\delta_{jk}\delta_{ln}$$

For example,

$$(E_{12}E_{34})_{mn} = \delta_{1m}\delta_{23}\delta_{4n} = 0.$$

In order for the product to be nontrivial we must have $j = k$,

$$(E_{12}E_{24})_{mn} = \delta_{1m}\delta_{22}\delta_{4n} = \delta_{1m}\delta_{4n} = (E_{14})_{mn}.$$

We can make the same identification in the general calculation,

$$(E_{ij}E_{kl})_{mn} = \delta_{jk}(E_{il})_{mn}.$$

Since the above holds for all m, n,

$$\boxed{E_{ij}E_{kl} = \delta_{jk}E_{il}}$$

this is at times a very nice formula to know about.

Remark 2.3.25.

The proofs in these examples are much longer if written without the benefit of index notation. It usually takes most students a little time to master the idea of index notation. There are a few homeworks assigned which require this sort of thinking, I do expect all students of Math 321 to gain proficiency in index calculation.

Example 2.3.26. *Let $A \in R^{m\times n}$ and suppose $e_i \in R^{m\times 1}$ and $e_j \in R^n$. Consider,*

$$(e_i)^T Ae_j = \sum_{k=1}^{m}((e_i)^T)_k(Ae_j)_k = \sum_{k=1}^{m}\delta_{ik}(Ae_j)_k = (Ae_j)_i = A_{ij}$$

This is a useful observation. If we wish to select the (i,j)-entry of the matrix A then we can use the following simple formula,

$$\boxed{A_{ij} = (e_i)^T Ae_j}$$

This is analogous to the idea of using dot-products to select particular components of vectors in analytic geometry; (reverting to calculus III notation for a moment) recall that to find v_1 of $\vec{v}$ we learned that the dot product by $\hat{i} =< 1,0,0 >$ selects the first components $v_1 = \vec{v}\cdot\hat{i}$. The following theorem is simply a summary of our results for this subsection.

Theorem 2.3.27.

Let $A \in R^{m\times n}$ and $v \in R^n$ if $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and $(e_i)_j = \delta_{ij}$ then,

$v = \sum_{i=1}^{n} v_n e_n$	$A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}$	$e_i{}^T A = row_i(A)$	$Ae_i = col_i(A)$
$A_{ij} = (e_i)^T Ae_j$	$E_{ij}E_{kl} = \delta_{jk}E_{il}$	$E_{ij} = e_i e_j{}^T$	$e_i{}^T e_j = \delta_{ij}$

2.4 matrix algebra

In this subsection we discover the matrix analog of the number 1, the formulation of the multiplicative inverse and raising a matrix to a power.

2.4.1 identity and inverse matrices

We begin by studying the 2×2 case.

Example 2.4.1. *Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We calculate*

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Likewise calculate,

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the matrix A was arbitrary we conclude that $IA = AI$ for all $A \in R^{2\times 2}$.

Definition 2.4.2.

The identity matrix in $R^{n\times n}$ is the $n \times n$ square matrix I which has components $I_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. The notation I_n is sometimes used if the size of the identity matrix needs emphasis, otherwise the size of the matrix I is to be understood from the context.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

You might wonder, do other square matrices D satisfy $AD = DA$ for each square matrix A ? That sounds like an excellent homework problem[14]. For now, let's see how Example 2.4.1 generalizes:

Proposition 2.4.3.

If $X \in R^{n\times p}$ then $XI_p = X$ and $I_nX = X$.

Proof: I omit the p in I_p to reduce clutter below. Consider the i, j component of XI,

$$\begin{aligned} (XI)_{ij} &= \sum_{k=1}^{p} X_{ik} I_{kj} && \text{defn. matrix multiplication} \\ &= \sum_{k=1}^{p} X_{ik} \delta_{kj} && \text{defn. of } I \\ &= X_{ij} \end{aligned}$$

The last step follows from the fact that all other terms in the sum are made zero by the Kronecker delta. Finally, observe the calculation above holds for all i, j hence $XI = X$. The proof of $IX = X$ is left to the reader. $\square$

Before we define the inverse of a matrix it is wise to prove the following:

Proposition 2.4.4. *Let R be a commutative ring.*

Suppose $A \in R^{n\times n}$. If $B, C \in R^{n\times n}$ satisfy $AB = BA = I$ and $AC = CA = I$ then $B = C$.

Proof: suppose $A, B, C \in R^{n\times n}$ and $AB = BA = I$ and $AC = CA = I$ thus $AB = AC$. Multiply B on the left of $AB = AC$ to obtain $BAB = BAC$ hence $IB = IC \;\Rightarrow\; B = C$. $\square$

The identity matrix plays the role of the multiplicative identity for matrix multiplication. If $AB = I$ then we **do not write** $B = I/A$, instead, the following notation is customary:

Definition 2.4.5.

Let $A \in R^{n\times n}$. If there exists $B \in R^{n\times n}$ such that $AB = I$ and $BA = I$ then we say that A is **invertible** and $A^{-1} = B$. Invertible matrices are also called **nonsingular**. If a matrix has no inverse then it is called a **noninvertible** or **singular** matrix.

[14]see § 2.3.2 for notation which is helpful to characterize D for which $AD = DA$ for all A

We'll discuss how and when it is possible to calculate A^{-1} for a given square matrix A, however, we need to develop a few tools before we're ready for that problem. This much I can share now:

Example 2.4.6. *Consider the problem of inverting a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We seek $B = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$ such that $AB = I$ and $BA = I$. The resulting algebra would lead you to conclude $x = d/t, y = -b/t, z = -c/t, w = a/t$ where $t = ad - bc$.*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

It's not hard to show this formula works,

$$\begin{aligned} \tfrac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} &= \tfrac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -bc+da \end{bmatrix} \\ &= \tfrac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Proof that $BA = I$ is similar.

the quantity $ad-bc$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is known as the **determinant** of A. In particular, we denote $\det(A) = ad - bc$. If $R = \mathbb{R}$ then the significance of the determinant is that it provides the signed-area of the paralellogram with sides $\langle a, b\rangle$, $\langle c, d\rangle$. The sign tells us if $\langle a, b\rangle$ is rotated clockwise (CW) or counterclockwise (CCW) to reach $\langle c, d\rangle$. If you study it carefully, you'll find positive determinant indicates the second row is obtained from the first by a CCW rotation. Determinants are discussed in more detail later in this course. In fact, this formula is generalized to n-th order matrices. However, the formula is so complicated that only a truly silly student[15] would try to implement it for anything beyond the 2×2 case. Computationally, we find an efficient algorithm for finding inverses larger matrices in § 3.7.

Example 2.4.7. *One interesting application of 2×2 matrices is that they can be used to generate rotations in the plane. In particular, a counterclockwise* **rotation** *by angle θ in the plane can be represented by a matrix $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Calculate via the 2×2 inverse formula with $a = d = \cos\theta$ and $b = -c = \sin\theta$*

$$(\,R(\theta)\,)^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = R(-\theta)$$

We observe the inverse matrix corresponds to a rotation by angle $-\theta$; $R(\theta)^{-1} = R(-\theta)$. Notice that $R(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ thus $R(\theta)R(-\theta) = R(0) = I$. Rotations are very special invertible matrices, we shall see them again.

Noninvertible martices challenge our intuition. For example, if A^{-1} does not exist for A it is possible to have $Av = Aw$ and yet $v \neq w$. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible and we observe $Ae_1 = e_1 = Ae_2$ and obviously $e_1 \neq e_2$. Invertible matrices allow some of our usual thinking:

[15] his name is Minh

Theorem 2.4.8.

If $A, B \in R^{n\times n}$ are invertible, $X, Y \in R^{m\times n}$, $Z, W \in R^{n\times m}$ and nonzero $c \in R$ then

1. $(AB)^{-1} = B^{-1}A^{-1}$,
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$,
3. $XA = YA$ implies $X = Y$,
4. $AZ = AW$ implies $Z = W$,
5. $(A^T)^{-1} = (A^{-1})^T$.
6. $(A^{-1})^{-1} = A$.

Proof: To prove (1.) simply notice that

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$$

also[16] note $B^{-1}A^{-1}(AB) = I$. The proof of (2.) follows from the calculation below,

$$(\tfrac{1}{c}A^{-1})cA = \tfrac{1}{c}cA^{-1}A = A^{-1}A = I.$$

and note $cA(\frac{1}{c}A^{-1}) = I$ by nearly the same calculation. To prove (3.) assume that $XA = YA$ and multiply both sides by A^{-1} on the right to obtain $XAA^{-1} = YAA^{-1}$ which reveals $XI = YI$ or simply $X = Y$. To prove (4.) multiply by A^{-1} on the left. Finally, consider $AA^{-1} = I$ and $A^{-1}A = I$ implies by the socks-shoes identity for the transpose that $(A^{-1})^TA^T = I^T = I$ and $A^T(A^{-1})^T = I^T = I$ therefore $(A^T)^{-1} = (A^{-1})^T$. Finally, (6.) is immediate from the definition. $\square$

Remark 2.4.9.

The proofs just given were all matrix arguments. These contrast the component level proofs needed for 2.2.6. We could give component level proofs for the Theorem above but that is not necessary and those arguments would only obscure the point. I hope you gain your own sense of which type of argument is most appropriate as the course progresses.

The importance of inductive arguments in linear algebra ought not be overlooked.

Proposition 2.4.10.

If $A_1, A_2, \dots, A_k \in R^{n\times n}$ are invertible then $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}$.

Proof: follows from induction on k. In particular, $k = 1$ is trivial. Assume inductively the proposition is true for some k with $k \geq 2$,

$$(\underbrace{A_1A_2\cdots A_k}_{B}A_{k+1})^{-1} = (BA_{k+1})^{-1} = A_{k+1}^{-1}B^{-1}$$

by Theorem 2.4.8 part (1.). Applying the induction hypothesis to B yields

$$(A_1A_2\cdots A_{k+1})^{-1} = A_{k+1}^{-1}A_k^{-1}\cdots A_1^{-1} \qquad \square.$$

[16]My apologies to the reader who already knows that $AB = I$ implies $BA = I$ for square matrices A, B. We have yet to learn that. We shall soon, but, for now these proofs have a bit extra.

2.4.2 matrix powers

The power of a matrix is defined in the natural way. Notice we need for A to be square in order for the product AA to be defined.

Definition 2.4.11.

Let $A \in R^{n\times n}$. We define $A^0 = I$, $A^1 = A$ and $A^m = AA^{m-1}$ for all $m \geq 1$. If A is invertible then $A^{-p} = (A^{-1})^p$.

As you would expect, $A^3 = AA^2 = AAA$.

Proposition 2.4.12. *laws of exponents*

Consider nonzero $A, B \in R^{n\times n}$ and

1. $A^pA^q = A^{p+q}$ for all $p, q \in \mathbb{N} \cup \{0\}$,
2. $(A^p)^q = A^{pq}$ for all $p, q \in \mathbb{N} \cup \{0\}$,
3. if A^{-1} exists then the $A^pA^q = A^{p+q}$ and $(A^p)^q = A^{pq}$ for $p, q \in \mathbb{Z}$.

Proof: we prove (1.) by induction on q. Fix $p \in \mathbb{N} \cup \{0\}$. If $q = 0$ then $A^q = A^0 = I$ thus $A^pA^q = A^pI = A^p = A^{p+0} = A^{p+q}$ thus (1.) is true for $q = 0$. Suppose inductively that (1.) is true for some $q \in \mathbb{N}$. Consider,

$$\begin{aligned} A^pA^{q+1} &= A^pA^qA && \text{by definition of matrix power} \\ &= A^{p+q}A && \text{by induction hypothesis} \\ &= A^{p+q+1} && \text{by definition of matrix power} \end{aligned}$$

thus (1.) holds for $q+1$ and we conclude (1.) is true for all $q \in \mathbb{N}\cup\{0\}$ for arbitrary $p \in \mathbb{N}\cup\{0\}$. I leave (2.) for the reader, it can be shown by a similar inductive argument. To prove $A^pA^q = A^{p+q}$ fix $p \in \mathbb{Z}$ and notice the claim is true for $q = 0$. Our argument for $q \in \mathbb{N}$ still is valid when we take $p \in \mathbb{Z}$ (it was non-negative in our argument for (1.)). Hence, consider $q \in \mathbb{Z}$ with $q \leq 0$. Let $q = -r$ and observe $r \geq 0$. We intend to prove $A^pA^{-r} = A^{p+(-r)}$ for all $r \in \mathbb{N}\cup\{0\}$ by induction on r. Note $A^pA^{-r} = A^{p+(-r)}$ is true for $r = 0$. Suppose inductively $A^pA^{-r} = A^{p+(-r)}$ for some $r \in \mathbb{N}$. Consider,

$$\begin{aligned} A^pA^{-(r+1)} &= A^p(A^{-1})^{r+1} && \text{by definition of matrix power} \\ &= A^p(A^{-1})^rA^{-1} && \text{by definition of matrix power} \\ &= A^pA^{-r}A^{-1} && \text{by definition of matrix power} \\ &= A^{p+(-r)}A^{-1} && \text{by induction hypothesis} \\ &= A^{-(r-p)}A^{-1} && \text{arithmetic} \\ &= (A^{-1})^{r-p}A^{-1} && \text{definition of matrix power} \\ &= (A^{-1})^{r-p+1} && \text{definition of matrix power} \\ &= A^{p-(r+1)} && \text{definition of matrix power} \end{aligned}$$

hence the claim is true for $r + 1$ and it follows by induction it is true for $r \in \mathbb{N} \cup \{0\}$. Hence, $A^pA^q = A^{p+q}$ is true for all $p, q \in \mathbb{Z}$. I leave proof of the other half of (3.) to the reader, the argument should be similar. $\square$

You should notice that $(AB)^p \neq A^pB^p$ for matrices. Instead,

$$(AB)^2 = ABAB, \qquad (AB)^3 = ABABAB, etc...$$

This means the binomial theorem will not hold for matrices. For example,

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = AA + AB + BA + BB$$

hence $(A+B)^2 \neq A^2 + 2AB + B^2$ as the matrix product is not generally commutative. However, in the special case that $AB = BA$ and we can prove that $(AB)^p = A^pB^p$ and the binomial theorem holds true as well. I may have assigned you the proof of the binomial theorem in homework.

2.4.3 symmetric and antisymmetric matrices

Definition 2.4.13.

Let $A \in R^{n\times n}$. We say A is **symmetric** iff $A^T = A$. We say A is **antisymmetric** iff $A^T = -A$.

At the level of components, $A^T = A$ gives $A_{ij} = A_{ji}$ for all i, j. Whereas, $A^T = -A$ gives $A_{ij} = -A_{ji}$ for all i, j. Both symmetric and antisymmetric matrices appear in common physical applications. For example, the intertia tensor which describes the rotational motion of a body is represented by a symmetric 3×3 matrix. The Faraday tensor is represented by an antisymetric 4×4 matrix. The Faraday tensor includes both the electric and magnetic fields. Physics aside, we'll see later that symmetric matrices play an important role in multivariate Taylor series and the Spectral Theorem makes symmetric matrices especially simple to analyze in general. We might study the Spectral Theorem towards the end of this course.

Example 2.4.14. *Examples and non-examples of symmetric and antisymmetric matrices:*

$$\underbrace{I,\ O,\ E_{ii},\ \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}}_{symmetric} \qquad \underbrace{O,\ \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}}_{antisymmetric} \qquad \underbrace{[1,2],\ E_{i,i+1},\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{neither}$$

Proposition 2.4.15.

Let $A \in \mathbb{R}^{m\times n}$ then A^TA is symmetric.

Proof: Proposition 2.3.12 yields $(A^TA)^T = A^T(A^T)^T = A^TA$. Thus A^TA is symmetric. $\square$

Proposition 2.4.16.

If A is symmetric then A^k is symmetric for all $k \in \mathbb{N}$.

Proof: Suppose $A^T = A$. Proceed inductively. Clearly $k = 1$ holds true since $A^1 = A$. Assume inductively that A^k is symmetric.

$$\begin{aligned} (A^{k+1})^T &= (AA^k)^T && \text{defn. of matrix exponents,} \\ &= (A^k)^TA^T && \text{socks-shoes prop. of transpose,} \\ &= A^kA && \text{using inducition hypothesis.} \\ &= A^{k+1} && \text{defn. of matrix exponents,} \end{aligned}$$

thus by proof by mathematical induction A^k is symmetric for all $k \in \mathbb{N}$. $\square$

2.4.4 triangular and diagonal matrices

Definition 2.4.17.

Let $A \in R^{m\times n}$. If $A_{ij} = 0$ for all i, j such that $i \neq j$ then A is called a **diagonal** matrix. If A has components $A_{ij} = 0$ for all i, j such that $i \leq j$ then we call A a **upper triangular** matrix. If A has components $A_{ij} = 0$ for all i, j such that $i \geq j$ then we call A a **lower triangular** matrix. If the diagonal of a matrix is zero then the matrix is **hollow**.

Example 2.4.18. *Let me illustrate a generic example of each case for* 3×3 *matrices:*

$$\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

As you can see the diagonal matrix only has nontrivial entries on the diagonal, and the names lower triangular and upper triangular are likewise natural.

If an upper triangular matrix has zeros on the diagonal then it is said to be **strictly upper triangular**. Likewise, if a lower triangular matrix has zeros on the diagonal then it is said to be **strictly lower triangular**. Obviously and matrix can be written as a sum of a diagonal and strictly upper and strictly lower matrix,

$$\begin{aligned} A &= \sum_{i,j} A_{ij} E_{ij} \\ &= \sum_{i} A_{ii} E_{ii} + \sum_{i<j} A_{ij} E_{ij} + \sum_{i>j} A_{ij} E_{ij} \end{aligned}$$

There is an algorithm called LU-factorization which for many matrices[17] A finds a lower triangular matrix L and an upper triangular matrix U such that $A = LU$. It is one of several factorization schemes which is calculationally advantageous for large systems. There are many many ways to solve a system, but some are faster methods. Algorithmics is the study of which method is optimal.

Example 2.4.19. *In the* 2×2 *case it is simple to verify the product of upper(lower) triangular matrices is once more (upper)lower triangular:*

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} ax & 0 \\ bx + cy & cz \end{bmatrix} \qquad \& \qquad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ax & ay + bz \\ 0 & cz \end{bmatrix}$$

Generally, the 2×2 case is surprisingly insightful. This is such a case:

Proposition 2.4.20.

Let $A, B \in R^{n\times n}$.

1. If A, B are diagonal then AB is diagonal.
2. If A, B are upper triangular then AB is upper triangular.
3. If A, B are lower triangular then AB is lower triangular.

[17] An LU decomposition exists iff the principal minors are all positive. However, a PLU (permutation, lower, upper) factorization always exists. I will discuss this in Math 221.

Proof of (1.): Suppose A and B are diagonal. It follows there exist a_i, b_j such that $A = \sum_i a_i E_{ii}$ and $B = \sum_j b_j E_{jj}$. Calculate,

$$AB = \sum_i a_i E_{ii} \sum_j b_j E_{jj} = \sum_i \sum_j a_i b_j E_{ii} E_{jj} = \sum_i \sum_j a_i b_j \delta_{ij} E_{ij} = \sum_i a_i b_i E_{ii}$$

thus the product matrix AB is also diagonal and we find that the diagonal of the product AB is just the product of the corresponding diagonals of A and B.
Proof of (2.): Suppose A and B are upper triangular. It follows there exist A_{ij}, B_{ij} such that[18] $A = \sum_{i \leq j} A_{ij} E_{ij}$ and $B = \sum_{k \leq l} B_{kl} E_{kl}$. Calculate,

$$AB = \sum_{i \leq j} A_{ij} E_{ij} \sum_{k \leq l} B_{kl} E_{kl} = \sum_{i \leq j} \sum_{k \leq l} A_{ij} B_{kl} E_{ij} E_{kl} = \sum_{i \leq j} \sum_{k \leq l} A_{ij} B_{kl} \delta_{jk} E_{il} = \sum_{i \leq j} \sum_{j \leq l} A_{ij} B_{jl} E_{il}.$$

Notice that every term in the sum above has $i \leq j$ and $j \leq l$ hence $i \leq l$. It follows the product is upper triangular since it is a sum of upper triangular matrices. The proof of (3.) is similar. $\square$.

I hope you can appreciate these arguments are superior to component level calculations with explicit listing of components and .... The notations e_i and E_{ij} are extremely helpful on many such questions. Futhermore, a proof captured in the notation of this section will more clearly show the root cause for the truth of the identity in question. What is easily lost in several pages of brute-force can be elegantly seen in a couple lines of carefully crafted index calculation.

2.4.5 nilpotent matrices

Definition 2.4.21.

Let $N \in R^{n \times n}$ be nonzero then N is **nilpotent** of degree k if k is the first positive integer for which $N^k = 0$.

Nilpotent matrices are easy to find. Here is an important example:

Example 2.4.22. *Let* $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *hence* $N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $N^3 = 0$ *thus* N *is nilpotent of degree* 3*. If* $N \in R^{n \times n}$ *and* $N_{i,i+1} = 1$ *for* $i = 1, \dots, n-1$ *and* $N_{ij} = 0$ *otherwise then we can show through similar calculation that* $N^{n-1} = E_{1n}$ *and* $N^n = 0$.

A fun question to ponder: which of the matrix units are nilpotent? Moving on, another interesting aspect of a nilpotent matrix is that if we modify the identity matrix by N then it is still invertible. For example:

Example 2.4.23. *Suppose* N *is nilpotent of degree* 2 *then* $I + N$ *has inverse* $I - N$ *as is easily seen by* $(I+N)(I-N) = I + N - N - N^2 = I$ *and* $(I-N)(I+N) = I - N + N - N^2 = I$ *hence* $(I+N)^{-1} = I - N$.

The inverse in the example above is not hard to guess. Try out the next case, can you find $(I+N)^{-1}$ for N nilpotent of degree 3. As a formal intuition, you might think about the geometric series.

[18] the notation $\sum_{i \leq j}$ indicates we sum over all pairs i, j for which $i \leq j$. For example, if $n = 3$ then we sum over $(i, j) = (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)$.

2.5 block matrices

If you look at most undergraduate linear algbera texts they will not bother to even attempt much of a proof that block-multiplication holds in general. I will foolishly attempt it here. However, I'm going to cheat a little and employ uber-sneaky physics notation.

The Einstein summation convention states that if an index is repeated then it is assumed to be summed over it's values. This means that the letters used for particular indices are reserved. If i, j, k are used to denote components of a spatial vector then you cannot use them for a spacetime vector at the same time. A typical notation in physics would be that v^j is a vector in xyz-space whereas v^μ is a vector in $txyz$-spacetime. A spacetime vector could be written as a sum of space components and a time component; $v = v^\mu e_\mu = v^0 e_0 + v^1 e_1 + v^2 e_2 + v^3 e_3 = v^0 e_0 + v^j e_j$. This is not the sort of langauge we tend to use in mathematics. For us notation is usually not reserved. Anyway, cultural commentary aside, if we were to use Einstein-type notation in linear algebra then we would likely omit sums as follows:

$$v = \sum_i v_i e_i \quad \longrightarrow \quad v = v_i e_i$$

$$A = \sum_{ij} A_{ij} E_{ij} \quad \longrightarrow \quad A = A_{ij} E_{ij}$$

We wish to partition a matrices A and B into 4 parts, use indices M, N which split into subindices m, μ and n, ν respectively. In this notation there are 4 different types of pairs possible:

$$A = [A_{MN}] = \left[\begin{array}{c|c} A_{mn} & A_{m\nu} \\ \hline A_{\mu n} & A_{\mu\nu} \end{array}\right] \qquad B = [B_{NJ}] = \left[\begin{array}{c|c} B_{nj} & B_{n\gamma} \\ \hline B_{\nu j} & B_{\nu\gamma} \end{array}\right]$$

Then the sum over M, N breaks into 2 cases,

$$A_{MN} B_{NJ} = A_{Mn} B_{nJ} + A_{M\nu} B_{\nu J}$$

But, then there are 4 different types of M, J pairs,

$$[AB]_{mj} = A_{mN} B_{Nj} = A_{mn} B_{nj} + A_{m\nu} B_{\nu j}$$

$$[AB]_{m\gamma} = A_{mN} B_{N\gamma} = A_{mn} B_{n\gamma} + A_{m\nu} B_{\nu\gamma}$$

$$[AB]_{\mu j} = A_{\mu N} B_{Nj} = A_{\mu n} B_{nj} + A_{\mu\nu} B_{\nu j}$$

$$[AB]_{\mu\gamma} = A_{\mu N} B_{N\gamma} = A_{\mu n} B_{n\gamma} + A_{\mu\nu} B_{\nu\gamma}$$

Let me summarize,

$$\left[\begin{array}{c|c} A_{mn} & A_{m\nu} \\ \hline A_{\mu n} & A_{\mu\nu} \end{array}\right] \left[\begin{array}{c|c} B_{nj} & B_{n\gamma} \\ \hline B_{\nu j} & B_{\nu\gamma} \end{array}\right] = \left[\begin{array}{c|c} [A_{mn}][B_{nj}] + [A_{m\nu}][B_{\nu j}] & [A_{mn}][B_{n\gamma}] + [A_{m\nu}][B_{\nu\gamma}] \\ \hline [A_{\mu n}][B_{nj}] + [A_{\mu\nu}][B_{\nu j}] & [A_{\mu n}][B_{n\gamma}] + [A_{\mu\nu}][B_{\nu\gamma}] \end{array}\right]$$

Let me again summarize, but this time I'll drop the annoying indices:

Theorem 2.5.1. *block multiplication.*

Suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ such that both A and B are partitioned as follows:

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad \text{and} \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

where A_{11} is an $m_1 \times n_1$ block, A_{12} is an $m_1 \times n_2$ block, A_{21} is an $m_2 \times n_1$ block and A_{22} is an $m_2 \times n_2$ block. Likewise, $B_{n_k p_k}$ is an $n_k \times p_k$ block for $k = 1, 2$. We insist that $m_1 + m_2 = m$ and $n_1 + n_2 = n$. If the partitions are compatible as decribed above then we may multiply A and B by multiplying the blocks as if they were scalars and we were computing the product of 2×2 matrices:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right].$$

To give a careful proof we'd just need to write out many sums and define the partition with care from the outset of the proof. In any event, notice that once you have this partition you can apply it twice to build block-multiplication rules for matrices with more blocks. The basic idea remains the same: you can parse two matrices into matching partitions then the matrix multiplication follows a pattern which is as if the blocks were scalars. However, the blocks are not scalars so the multiplication of the blocks is nonabelian. For example,

$$AB = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline A_{31} & A_{32} \end{array}\right]\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \\ \hline A_{31}B_{11} + A_{32}B_{21} & A_{31}B_{12} + A_{32}B_{22} \end{array}\right].$$

where if the partitions of A and B are compatible it follows that the block-multiplications on the RHS are all well-defined.

Example 2.5.2. *Let* $R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ *and* $B(\gamma) = \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix}$. *Furthermore construct* 4×4 *matrices* Λ_1 *and* Λ_2 *as follows:*

$$\Lambda_1 = \left[\begin{array}{c|c} B(\gamma_1) & 0 \\ \hline 0 & R(\theta_1) \end{array}\right] \qquad \Lambda_2 = \left[\begin{array}{c|c} B(\gamma_2) & 0 \\ \hline 0 & R(\theta_2) \end{array}\right]$$

Multiply Λ_1 *and* Λ_2 *via block multiplication:*

$$\begin{aligned} \Lambda_1\Lambda_2 &= \left[\begin{array}{c|c} B(\gamma_1) & 0 \\ \hline 0 & R(\theta_1) \end{array}\right]\left[\begin{array}{c|c} B(\gamma_2) & 0 \\ \hline 0 & R(\theta_2) \end{array}\right] \\ &= \left[\begin{array}{c|c} B(\gamma_1)B(\gamma_2) + 0 & 0 + 0 \\ \hline 0 + 0 & 0 + R(\theta_1)R(\theta_2) \end{array}\right] \\ &= \left[\begin{array}{c|c} B(\gamma_1 + \gamma_2) & 0 \\ \hline 0 & R(\theta_1 + \theta_2) \end{array}\right]. \end{aligned}$$

The last calculation is actually a few lines in detail, if you know the adding angles formulas for cosine, sine, cosh and sinh it's easy. If $\theta = 0$ *and* $\gamma \neq 0$ *then* Λ *would represent a* **velocity boost** *on spacetime. Since it mixes time and the first coordinate the velocity is along the* x*-coordinate. On the other hand, if* $\theta \neq 0$ *and* $\gamma = 0$ *then* Λ *gives a* **rotation** *in the* yz *spatial coordinates in space* *time. If both parameters are nonzero then we can say that* Λ *is a* **Lorentz transformation** *on spacetime. Of course there is more to say here, perhaps we could offer a course in special relativity if enough students were interested in concert.*

Example 2.5.3. Problem: *Suppose* M *is a square matrix with submatrices* $A, B, C, 0$ *where* A, C *are square. What conditions should we insist on for* $M = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right]$ *to be invertible.*

Solution: *I propose we partition the potential inverse matrix* $M^{-1} = \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right]$. *We seek to find conditions on* A, B, C *such that there exist* D, E, F, G *and* $MM^{-1} = I$. *Each block of the equation* $MM^{-1} = I$ *gives us a separate submatrix equation:*

$$MM^{-1} = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right] = \left[\begin{array}{c|c} AD+BF & AE+BG \\ \hline 0D+CF & 0E+CG \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

We must solve simultaneously the following:

$$(1.)\ AD + BF = I, \qquad (2.)\ AE + BG = 0, \qquad (3.)\ CF = 0, \qquad (4.)\ CG = I$$

If C^{-1} *exists then* $G = C^{-1}$ *from (4.). Moreover,* (3.) *then yields* $F = C^{-1}0 = 0$. *Our problem thus reduces to (1.) and (2.) which after substituting* $F = 0$ *and* $G = C^{-1}$ *yield*

$$(1.)\ AD = I, \qquad (2.)\ AE + BC^{-1} = 0.$$

Equation (1.) *says* $D = A^{-1}$. *Finally, let's solve (2.) for* E,

$$E = -A^{-1}BC^{-1}.$$

Let's summarize the calculation we just worked through. IF A, C *are invertible then the matrix* $M = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right]$ *is invertible with inverse*

$$\boxed{M^{-1} = \left[\begin{array}{c|c} A^{-1} & -A^{-1}BC^{-1} \\ \hline 0 & C^{-1} \end{array}\right].}$$

Consider the case that M *is a* 2×2 *matrix and* $A, B, C \in \mathbb{R}$. *Then the condition of invertibility reduces to the simple conditions* $A, C \neq 0$ *and* $-A^{-1}BC^{-1} = \frac{-B}{AC}$ *we find the formula:*

$$M^{-1} = \left[\begin{array}{c|c} \frac{1}{A} & \frac{-B}{AC} \\ \hline 0 & \frac{1}{C} \end{array}\right] = \frac{1}{AC}\left[\begin{array}{c|c} C & -B \\ \hline 0 & A \end{array}\right].$$

This is of course the formula for the 2×2 *matrix in this special case where* $M_{21} = 0$.

Of course the real utility of formulas like those in the last example is that they work for partitions of arbitrary size. If we can find a block of zeros somewhere in the matrix then we may reduce the size of the problem. The time for a computer calculation is largely based on some power of the size of the matrix. For example, if the calculation in question takes n^2 steps then parsing the matrix into 3 nonzero blocks which are $n/2 \times n/2$ would result in something like $[n/2]^2 + [n/2]^2 + [n/2]^2 = \frac{3}{4}n^2$ steps. If the calculation took on order n^3 computer operations (flops) then my toy example of 3 blocks would reduce to something like $[n/2]^3 + [n/2]^3 + [n/2]^3 = \frac{3}{8}n^2$ flops. A savings of more than 60% of computer time. If the calculation was typically order n^4 for an $n \times n$ matrix then the saving is even more dramatic. If the calculation is a determinant then the cofactor formula depends on the factorial of the size of the matrix. Try to compare 10!+10! verses say 20!. Hope your calculator has a big display:

$$10! = 3628800 \quad \Rightarrow \quad 10! + 10! = 7257600 \qquad \text{or} \qquad 20! = 2432902008176640000.$$

Perhaps you can start to appreciate why numerical linear algebra software packages often use algorithms which make use of block matrices to streamline large matrix calculations. If you are very interested in this sort of topic you might strike up a conversation with Dr. Van Voorhis. I suspect he knows useful things about this type of mathematical inquiry.

Finally, I would comment that breaking a matrix into blocks is basically the bread and butter of quantum mechanics. One attempts to find a basis of state vectors which makes the Hamiltonian into a block-diagonal matrix. Each block corresponds to a certain set of statevectors sharing a common energy. The goal of representation theory in physics is basically to break down matrices into blocks with nice physical meanings. On the other hand, abstract algebraists also use blocks to rip apart a matrix into it's most basic form. For linear algebraists, the so-called Jordan form is full of blocks. Wherever reduction of a linear system into smaller subsystems is of interest there will be blocks.

Chapter 3

systems of linear equations

In this Chapter we discuss how m-linear equations in n-variables can be expressed as a single matrix equation[1]; $Ax = b$ Furthermore, we'll see how the technique of substitution can be translated into series of **elementary row operations** on the **augmented coefficient matrix** $[A|b]$. The end result of this **Gauss-Jordan elimination** produces the **reduced-row-echelon-form** which is denoted $\text{rref}[A|b]$. Beautifully, all possible solutions to the equation $Ax = b$ are simply read directly by inspection of $\text{rref}[A|b]$.

We also spend some time studying the structure of row reduction as seen through the construction of elementary matrices. These elementary matrices allow us to prove a number of interesting theorems about inverse matrices. We show how and when it is possible to calculate the inverse of an $n \times n$ matrix. These elementary matrices are used in the next chapter to help derive the linear correspondence (aka the CCP).

3.1 the row reduction technique for linear systems

We now focus our attention on the case $R = \mathbb{F}$. Throughout this section we suppose $\mathbb{F}$ is a field. Our primary interest is in the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. However, I do include some discussion in the case of a finite field. Field structure is essential since we need to divide by elements of the matrices in the calculations of this section. Remember, a field allows for us to divide by any nonzero element.

First, in the case of nonlinear equations over $\mathbb{R}$ the structure possible is staggering. Think of surfaces you've seen in text books. Nearly every one of those appears as the solution set of some nonlinear equation. Even something as simple as the quadratic equation $ax^2 + bx + c = 0$ has three cases for $a \neq 0$. In particular, we have the case of distinct real roots, a repeated real root or no real solutions. It's not too complicated because we've studied it in some depth. Contrast that with the linear equation $ax = b$ for $a \neq 0$; there is **one** solution given by $x = b/a$. We're primarily interested in the solution of **systems** of linear equations so let's move on to that now[2].

Example 3.1.1. *Consider the following system of 2 equations and 2 unknowns,*

$$\begin{array}{c} x + y = 2 \\ x - y = 0 \end{array}.$$

[1] A is an $m \times n$ **matrix** and x is an $n \times 1$ **column vector** so the product Ax is a *matrix-column-product.*

[2] against my better judgement I have postponed the formal definitions until after a few examples, if this is as disturbing to you as it is to me then by all means skip ahead and read the definition of linear system, solution and such

Adding equations reveals $2x = 2$ hence $x = 1$. Then we find $1 + y = 2$ thus $y = 1$. Hence the solution $(1, 1)$ is **unique**.

Example 3.1.2. *Consider the following system of 2 equations and 2 unknowns,*

$$\begin{aligned} x + y &= 2 \\ 3x + 3y &= 6 \end{aligned}.$$

We can multiply the second equation by $1/3$ to see that it is equivalent to $x + y = 2$ thus our two equations are in fact the same equation. There are infinitely many equations of the form (x, y) where $x + y = 2$. In other words, the solutions have the form $(x, 2 - x)$ for all $x \in \mathbb{R}$.

Both of the examples thus far were **consistent**.

Example 3.1.3. *Consider the following system of 2 equations and 2 unknowns,*

$$\begin{aligned} x + y &= 2 \\ x + y &= 3 \end{aligned}.$$

These equations are inconsistent. Notice substracting the second equation yields that $0 = 1$. This system has no solutions, it is **inconsistent**

It is remarkable that these three simple examples reveal the general structure of solutions to linear systems over an infinite field[3]. Either we get a unique solution, infinitely many solutions or no solution at all. For our examples above, these cases correspond to the possible graphs for a pair of lines in the plane. A pair of lines may intersect at a point (unique solution), be the same line (infinitely many solutions) or be paralell (inconsistent). I recommend you don't try to visualize the solutions of the examples which follow below.

Essentially the same techniques work in fields you have less experience. The main trouble is learning to find multiplicative inverses and simplify representatives.

Example 3.1.4. *To solve $\bar{3}x + \bar{2}y = \bar{2}$ and $\bar{3}x - y = 0$ for $x, y \in \mathbb{Z}/5\mathbb{Z}$ we can solve the second equation for $y = \bar{3}x$ and substitute to obtain $\bar{3}x + \bar{2}\,\bar{3}x = \bar{2}$. But, $\bar{2}\,\bar{3} = \bar{6} = \bar{1}$ thus $\bar{4}x = \bar{2}$. Note $\bar{4}\,\bar{4} = \overline{16} = \bar{1}$ thus $1/\bar{4} = \bar{4}$. Thus, $\bar{4}\,\bar{4}x = \bar{4}\,\bar{2}$ yields $x = \bar{8} = \bar{3}$. Now return to the y-equation to deduce $y = \bar{3}\,\bar{3} = \bar{9} = \bar{4}$. Thus, the solution is $(\bar{3}, \bar{4})$*

In the study of complex variables the multiplicative inverse of nonzero $z = x + iy$ is given by $\frac{x-iy}{x^2+y^2}$.

Example 3.1.5. *Solve $(2 + i)x - 5y = 0$ and $x + (4 - i)y = i$. Note $\frac{1}{2+i} = \frac{2-i}{5}$ therefore multiply the first equation by $\frac{1}{2+i}$ to obtain $x - 5\left(\frac{2-i}{5}\right) y = 0$ which simplifies to $x + (i - 2)y = 0$. Subtract $x + (i - 2)y = 0$ from $x + (4 - i)y = i$ to eliminate x and deduce $[(4 - i) - (i - 2)]y = i$. Thus, $(6 - 2i)y = i$ and we find $y = \frac{i}{6-2i} = \frac{i(6+2i)}{36+4} = \frac{-2+6i}{40} = \frac{-1+3i}{20}$. From the second given equation we have $x = i + (i - 4)y$ hence*

$$x = i + (i - 4)y = i + (i - 4)\left(\frac{-1 + 3i}{20}\right) = i + \frac{1 - 13i}{20} = \frac{1 + 7i}{20}$$

In summary, we find the solution $\left(\frac{1+7i}{20}, \frac{-1+3i}{20}\right)$.

[3] in constrast, the finite field $\mathbb{Z}/3\mathbb{Z}$ has 3 elements, and the system $x - y = 0$ and $\bar{2}x - \bar{2}y = 0$ has solutions $(\bar{1}, \bar{1}), (\bar{2}, \bar{2}), (0, 0)$. Generally, we'll find either no solutions, or 3^k solutions for $k = 0, 1, \ldots$ for a linear system over $\mathbb{Z}/3\mathbb{Z}$. The reasoning behind this claim is not yet clear in our discussion to this point.

Linear equations over a field are not difficult to solve. Basically, we can eliminate variables by various forms of substitution and ultimately solve for a particular variable. Once that is accomplished, we back-substitute our way through the equations until the solution(s) are found. However, the cases of inconsistent and non-consistent cases are not so simple in general. That said, the basic insights used in the examples thus far motivate and guide the creation of the **Gauss-Jordan algorithm**.

3.2 augmented coefficient matrix and elementary row operations

We now introduce some notation which will help bring organization to our method of solving linear systems: we assume $\mathbb{F}$ is a field in what follows:

Definition 3.2.1. *system of m-linear equations in n-unknowns*

Let $x_1, x_2, \dots, x_n$ be n variables and suppose $b_i, A_{ij} \in \mathbb{F}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ then

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \quad &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m \end{aligned}$$

is called a **system of linear equations**. If $b_i = 0$ for $1 \leq i \leq m$ then we say the system is **homogeneous**. The **solution set** is the set of all $(x_1, x_2, \dots, x_n) \in \mathbb{F}^n$ which satisfy all the equations in the system simultaneously.

Remark 3.2.2.

We use variables $x_1, x_2, \dots, x_n$ mainly for general theoretical statements. In particular problems and especially for applications we tend to defer to the notation x, y, z etc...

Definition 3.2.3.

The augmented coefficient matrix is an array of numbers which provides an abbreviated notation for a system of linear equations.

$$\left[\begin{array}{c} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \;\; \vdots \quad \vdots \quad \vdots \quad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m \end{array}\right] \quad \text{replaced by} \quad \left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1n} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} & b_m \end{array}\right].$$

We say $A = [A_{ij}]$ is the **coefficient matrix** of the system and $b = [b_i]$ is the inhomogenous term. When $b = 0$ the system is **homogeneous**.

The vertical bar is optional, I include it to draw attention to the distinction between the matrix of coefficients A_{ij} and the nonhomogeneous terms b_i. Notice in the examples I borrow from my brother Bill the vertical bar in replaced with :. It is just a book-keeping device and we should think of it like the commas in some row vectors, or the bars sometimes given in block-matrices, they're merely decorative.

Remark 3.2.4.

In the examples below I use the notation $L \xrightarrow{r_2 - r_1} R$ to express that $\text{row}_2(R) = \text{row}_2(L) - \text{row}_1(L)$ and the other rows remain unaltered. In constrast, $L \xrightarrow{r_1 + 2r_2} R$ would indicate that $\text{row}_1(R) = \text{row}_1(L) + 2\text{row}_2(L)$ and $\text{row}_i(R) = \text{row}_i(L)$ for $i \neq 1$. Also, I use the notation $L \xrightarrow{\alpha r_j} R$ to indicate $\text{row}_i(R) = \alpha \text{row}_i(L)$ whereas $\text{row}_j(R) = \text{row}_j(L)$ for $i \neq j$. We also will eventually use the notation $\xrightarrow{r_i \leftrightarrow r_j}$ to indicate we have swapped the i-th and j-th rows while leaving all other rows unaltered.

Let's revisit Examples 3.1.1, 3.1.2 and 3.1.3 to illustrate the Gauss-Jordan method for each.

Example 3.2.5. *The system $x + y = 2$ and $x - y = 0$ has augmented coefficient matrix:*

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right] \xrightarrow{r_2/(-2)} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

The last augmented matrix represents the equations $x = 1$ and $y = 1$. Rather than adding and subtracting equations we added and subtracted rows in the matrix. Incidentally, the last step is called the **backward pass** *whereas the first couple steps are called the* **forward pass***. Gauss is credited with figuring out the forward pass then Jordan added the backward pass. Calculators can accomplish these via the commands ref (Gauss' row echelon form) and $rref$ (Jordan's reduced row echelon form). In particular (I should caution, row echelon forms(ref) is not unique),*

$$ref \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \qquad rref \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Example 3.2.6. *The system $x + y = 2$ and $3x + 3y = 6$ has augmented coefficient matrix:*

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 3 & 6 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The nonzero row in the last augmented matrix represents the equation $x + y = 2$. In this case we cannot make a backwards pass so our ref and $rref$ are the same.

Example 3.2.7. *The system $x + y = 3$ and $x + y = 2$ has augmented coefficient matrix:*

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right] \xrightarrow{r_1 + r_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right] \xrightarrow{-r_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

The last row indicates that $0x + 0y = 1$ which means that there is no solution since $0 \neq 1$. Generally, when the bottom row of the $rref(A|b)$ is zeros with a 1 in the far right column then the system $Ax = b$ is inconsistent because there is no solution to the equation.

3.2.1 elementary row operations

Let me attempt a general definition for the sake of logical completeness:

Definition 3.2.8. Elementary Row operations: *Let $A \in \mathbb{F}^{m\times n}$ we define*

	Effect on the linear system:		Effect on the matrix:
Type I	Interchange equation i and equation j (List the equations in a different order.)	$\Longleftrightarrow$	Swap Row i and Row j
Type II	Multiply both sides of equation i by a non-zero scalar c	$\Longleftrightarrow$	Multiply Row i by c where $c \neq 0$
Type III	Multiply both sides of equation i by c and add to equation j	$\Longleftrightarrow$	Add c times Row i to Row j where c is any scalar

If we can get matrix A from matrix B by performing a series of elementary row operations, then A and B are called **row equivalent matrices**.

Of course, there are also corresponding *elementary column operations*. If we can get matrix A from matrix B by performing a series of elementary column operations, we call A and B **column equivalent matrices**. Both of these *equivalences* are in fact equivalence relations (proof?). While both row and column operations are important, we will (for now) focus on row operations since they correspond to steps used when solving linear systems.

Example 3.2.9. *Here we illustrate* **Type I** *operation where we swap rows 1 and 3:*

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcrcr} -x & & & + & 4z & = & 7 \\ & & 5y & - & 6z & = & 0 \\ 2x & - & y & + & 3z & = & -1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} \underset{\sim}{\overset{R1\leftrightarrow R3}{}} \begin{bmatrix} -1 & 0 & 4 & : & 7 \\ 0 & 5 & -6 & : & 0 \\ 2 & -1 & 3 & : & -1 \end{bmatrix}$$

Example 3.2.10. *Now we illustrate* **Type II** *operation where we scale row 3 by -2:*

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ 2x & & & + & -8z & = & -14 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} \underset{\sim}{\overset{-2\times R3}{}} \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ 2 & 0 & -8 & : & -14 \end{bmatrix}$$

Example 3.2.11. *Next illustrate the* **Type III** *operation where we add 3 times row 3 to row 2:*

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ -3x & & 5y & + & 6z & = & 21 \\ -x & & & + & 4z & = & 7 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} \underset{\sim}{\overset{3\times R3+R2}{}} \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ -3 & 5 & 6 & : & 21 \\ -1 & 0 & 4 & : & 7 \end{bmatrix}$$

It is important to notice several things about these operations. First, they are all reversible (that's why we want $c \neq 0$ in type II operations) — in fact the inverse of a type X operation is another type X operation. Next, these operations don't effect the set of solutions for the system — that is — row equivalent matrices represent systems with the same set of solutions. Finally, these are *row* operations — columns *never* interact with each other. This last point is quite important as it will allow us to check our work and later allow us to find bases for subspaces associated with matrices (see "Linear Correspondence" between columns aka the "CCP").

3.3 Gauss-Jordan Algorithm

Now let's come back to trying to solve a linear system. Doing operations blindly probably won't get us anywhere. Instead we will choose our operations carefully so that we head towards some shape of equations which will let us read off the set of solutions. Thus the next few defintions.

Definition 3.3.1. *A matrix is in* **Row Echelon Form** *(or REF) if...*

- Each non-zero row is above all zero rows – that is – zero rows are "pushed" to the bottom.
- The leading entry of a row is *strictly* to the right of the leading entries of the rows above. (The leftmost non-zero entry of a row is called the "leading entry".)

If in addition...

- Each leading entry is "1". (*Note:* Some textbooks say this is a requirement of REF.)
- Only zeros appear above (& below) a leading entry of a row.

then a matrix is in **reduced row echelon form** (or RREF).

The matrix A (below) is in REF but is not reduced. The matrix B is in RREF.

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 1 & 5 \\ 0 & 0 & 0 & -3 & 3 & 2 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The **Gauss-Jordan Elimination** is an "algorithm" which given a matrix returns a row equivalent matrix in reduced row echelon form (RREF). Let us give a precise account of this algorithm:

Definition 3.3.2. *A matrix is in* **Row Echelon Form** *(or REF) if...*

Given a matrix over a field $\mathbb{F}$. We first perform a **forward pass**:

1. Determine the leftmost non-zero column. This is a **pivot column** and the topmost entry is a **pivot position**. If "0" is in this pivot position, swap (an unignored) row with the topmost row (use a Type I operation) so that there is a non-zero entry in the pivot position.
2. Add appropriate multiples of the topmost (unignored) row to the rows beneath it so that only "0" appears below the pivot (use several Type III operations).
3. Ignore the topmost (unignored) row. If any non-zero rows remain, go to step 1.

The forward pass is now complete. Now our matrix in row echelon form (this may not agree with a particular textbook's sense)[4]. Now let's finish Gauss-Jordan Elimination by performing a **backward pass**:

1. If necessary, scale the rightmost unfinished pivot to 1 (use a Type II operation).
2. Add appropriate multiples of the current pivot's row to rows above it so that only 0 appears above the current pivot (using several Type III operations).
3. The current pivot is now "finished". If any unfinished pivots remain, go to step 4.

It should be fairly obvious that the entire Gauss-Jordan algorithm will terminate in finitely many steps. Also, only elementary row operations have been used. So we end up with a row equivalent matrix. A tedious, wordy, and unenlightening proof would show us that the resulting matrix is in reduced row echelon form (RREF).

Example 3.3.3. *Let's solve the system* $\begin{array}{rcrcr} x & + & 2y & = & 1 \\ 3x & + & 4y & = & -1 \end{array}$

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 3 & 4 & : & -1 \end{bmatrix} \underset{\sim}{\overset{-3\times R1+R2}{}} \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$$

The first non-zero column is just the first column. So the upper left hand corner is a pivot position.This position already has a non-zero entry so no swap is needed. The type III operation "-3 times row 1 added to row 2" clears the only position below the pivot, so after one operation we have finished with this pivot and can ignore row 1.

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$$

Among the (unignored parts of) columns the leftmost non-zero column is the second column. So the "-2" sits in a pivot position. Since it's non-zero, no swap is needed. Also, there's nothing below it, so no type III operations are necessary. Thus we're done with this row and we can ignore it.

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$$

Nothing's left so we're done with the forward pass. $\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$ *is in row echelon form. Next, we need to take the rightmost pivot (the "-2") and scale it to 1 then clear everything above it.*

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix} \underset{\sim}{\overset{-1/2\times R2}{}} \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & 1 & : & 2 \end{bmatrix} \underset{\sim}{\overset{-2\times R2+R1}{}} \begin{bmatrix} 1 & 0 & : & -3 \\ 0 & 1 & : & 2 \end{bmatrix}$$

This "finishes" that pivot. The next rightmost pivot is the 1 in the upper left hand corner. But it's already scaled to 1 and has nothing above it, so it's finished as well. That takes care of all of the pivots so the backward pass is complete leaving our matrix in reduced row echelon form. Finally, let translate the RREF matrix back into a system of equations. The (new equivalent) system is $\begin{array}{rcr} x & & = & -3 \\ & y & = & 2 \end{array}$. *So the only solution for this system is $x = -3$ and $y = 2$.*

[4] Sometimes the forward pass alone is referred to as "Gaussian Elimination". However, we should be careful since the term "Gaussian Elimination" more commonly refers to *both* the forward and backward passes.

Note: One can also solve a system quite easily once (just) the forward pass is complete. This is done using "back substitution". Notice that the system after the forward pass was $\begin{array}{rcrcr} x & + & 2y & = & 1 \\ & & -2y & = & -4 \end{array}$. *So we have* $-2y=-4$ *thus* $y=2$. *Substituting this back into the first equation we get* $x+2(2)=1$ *so* $x=-3$.

Example 3.3.4. *Let's solve the system* $\begin{array}{rcrcrcr} 2x & - & y & + & z & = & 0 \\ 4x & - & 2y & + & 2z & = & -1 \\ x & & & + & z & = & 1 \end{array}$

$$\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 4 & -2 & 2 & : & 1 \\ 1 & 0 & 1 & : & -1 \end{bmatrix} \overset{-2\times R1+R2}{\sim} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 1 & 0 & 1 & : & -1 \end{bmatrix} \overset{-1/2\times R1+R3}{\sim} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 0 & 1/2 & 1/2 & : & -1 \end{bmatrix} \overset{\text{Ignore } R1}{\sim}$$

$$\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 0 & 1/2 & 1/2 & : & -1 \end{bmatrix} \overset{R2\leftrightarrow R3}{\sim} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \overset{\text{Ignore } R2}{\sim} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \overset{\text{Ignore } R3}{\sim}$$

which leaves us with nothing. So the forward pass is complete and $\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$ *is in REF. Now for the backwards pass:*

$$\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \overset{1\times R3+R2}{\sim} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \overset{2\times R2}{\sim} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \overset{1\times R2+R1}{\sim}$$

$$\begin{bmatrix} 2 & 0 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \overset{1/2\times R1}{\sim} \begin{bmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

This finishes the backward pass and our matrix is now in RREF. Our new system of equations is $\begin{array}{rcrcrcr} x & & & + & z & = & 0 \\ & & y & + & z & = & 0 \\ & & & & 0 & = & 1 \end{array}$. *Of course* $0\neq 1$, *so this is an* **inconsistent** *system — it has* **no solutions**.

Note: If our only goal was to solve this system, we could have stopped after the very first operation (row number 2 already said "$0=1$*").*

Example 3.3.5. *Let's solve the system* $\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 3 \\ 4x & + & 5y & + & 6z & = & 9 \\ 7x & + & 8y & + & 9z & = & 15 \end{array}$

$$\begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 4 & 5 & 6 & : & 9 \\ 7 & 8 & 9 & : & 15 \end{bmatrix} \overset{-4\times R1+R2}{\sim} \begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 7 & 8 & 9 & : & 15 \end{bmatrix} \overset{-7\times R1+R3}{\sim} \begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 0 & -6 & -12 & : & -6 \end{bmatrix} \overset{-2\times R2+R3}{\sim}$$

$$\begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \overset{-1/3\times R2}{\sim} \begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \overset{-2\times R2+R1}{\sim} \begin{bmatrix} 1 & 0 & -1 & : & 1 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Now our matrix is in RREF. The new system of equations is $\begin{array}{ccccc} x & & - & z & = & 1 \\ & y & + & 2z & = & 1 \\ & & & 0 & = & 0 \end{array}$. *This is new — we don't have an equation of the form "$z = ...$" This is because z does not lie in a pivot column. So we can make z a "free variable." Let's relabel it $z = t$. Then we have $x - t = 1$, $y + 2t = 1$, and $z = t$. So $x = 1 + t$, $y = 1 - 2t$, and $z = t$ is a solution for any choice of t. In particular, $x = y = 1$ and $z = 0$ is a solution. But so is $x = 2$, $y = -1$, $z = 1$. In fact, there are infinitely many solutions.*

If you've worked a few row reductions then the following is not too surprising. Another place to read the proof of uniqueness of the rref is the short article by Thomas Yuster: *The Reduced Row Echelon Form of a Matrix is Unique: A Simple Proof* from Vol. 57, No. 2 of the Mathematics Magazine of March 1984.

Theorem 3.3.6.

The reduced row echelon form of an $m \times n$ matrix over a field $\mathbb{F}$ is unique.

Proof: Let $A \in \mathbb{F}^{m \times n}$ have reduced row echelon forms B and C. We proceed by induction on n. Observe for $n = 1$ it is clear that $B = C$. Suppose $n > 1$ and form A' by deleting the n-th column of A. By the induction hypothesis there exists a unique $\text{rref}(A')$. Notice any row reduction which reduces A also reduces A'. Therefore, $\text{col}_i(B) = \text{col}_i(C)$ for $i = 1, \dots, n-1$. Suppose the n-th column of B is a pivot column and the n-th column of C is not a pivot column. It follows $Cx = 0$ has solutions for which x_n is free. On the other hand, $Bx = 0$ necessarily has $x_n = 0$ since $\text{col}_n(B)$ is a pivot column. Thus the solution sets of $Bx = 0$ and $Cx = 0$ differ. But, this is impossible since $Ax = 0$ shares the same homogeneous solution set as does any row equivalent matrix. It follows the n-th column of B and C must either both be pivot columns or non-pivot columns. As the first $n - 1$ columns of B and C coincide it follows that $B = C$ since the n-th column is forced to be the same sort of column (either pivot or non-pivot) by the structure of the initial $n - 1$ columns. Hence, by induction, we find uniqueness of the reduced row echelon form. $\square$

3.3.1 examples of row reduction

In this subsection I provide a number of worked and unworked examples. It is very important for students of this course to be able to calculate row reductions with optimal speed. The only way to attain this status is through practice.

Example 3.3.7. *For each of the following matrices perform Gauss-Jordan elimination (carefully following the algorithm given earlier in this section). I have given the result of the performing the forward pass to help you verify that you are on the right track. [Note: Each matrix has a unique RREF. However, REF is not unique. So if you do not follow my algorithm and use your own random assortment of operations, you will almost certainly get different REFs along the way to the RREF.] Once you have completed row reduction, identify the pivots and pivot columns. Finally, interpret your matrices as a system or collection of systems of equations and note the corresponding solutions.*

1. $\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \Longrightarrow \textit{forward pass} \Longrightarrow \begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -4 & 0 & 4 \\ 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 4 & -8 & 1 & 6 \end{bmatrix} \Longrightarrow \textit{forward pass} \Longrightarrow \begin{bmatrix} 2 & -4 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Longrightarrow \textit{forward pass} \Longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & -2 \\ 1 & -3 & -2 \end{bmatrix} \Longrightarrow \textit{forward pass} \Longrightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Note: I do not follow the strict algorithm introduced earlier in this section in the examples which follow. Instead, I try to illustrate how it is convenient to calculate in practice. When I ask you for a row-reduction I do not insist you follow the algorithm of Gauss-Jordan, however, we all agree on the end rref result.

Example 3.3.8. *The equations $x + 2y - 3z = 1$, $2x + 4y = 7$ and $-x + 3y + 2z = 0$ can be solved by row operations on the matrix $[A|b]$ below: Given $[A|b] = \left[\begin{smallmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{smallmatrix}\right]$ calculate rref(A).*

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 6 & 5 \\ -1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{r_1 + r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 6 & 5 \\ 0 & 5 & -1 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{array}\right] = ref([A|b])$$

that completes the forward pass. We begin the backwards pass,

$$ref(A) = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{array}\right] \xrightarrow{\frac{1}{6}r_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{r_2 + r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & 0 & 11/6 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{r_1 + 3r_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 21/6 \\ 0 & 5 & 0 & 11/6 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{\frac{1}{5}r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 21/6 \\ 0 & 1 & 0 & 11/30 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{r_1 - 2r_2} \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 83/30 \\ 0 & 1 & 0 & 11/30 \\ 0 & 0 & 1 & 5/6 \end{array}\right] = rref(A)}$$

Thus, we've found the system of equations $x + 2y - 3z = 1$, $2x + 4y = 7$ and $-x + 3y + 2z = 0$ has solution $x = 83/30, y = 11/30$ and $z = 5/6$.

Example 3.3.9. *Given* $A = \left[\begin{smallmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{smallmatrix}\right]$ *calculate* $rref(A)$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow{r_3 - 2r_1}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -5 \end{bmatrix} \begin{array}{c} \xrightarrow{3r_3} \\ \xrightarrow{5r_2} \end{array} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -15 \\ 0 & 0 & -15 \end{bmatrix} \begin{array}{c} \xrightarrow{r_3 - r_2} \\ \xrightarrow{\frac{-1}{15}r_2} \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \boxed{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = rref(A)}$$

Note it is customary to read multiple row operations from top to bottom if more than one is listed between two of the matrices. The multiple arrow notation should be used with caution as it has great potential to confuse. Also, you might notice that I did not strictly-speaking follow Gauss-Jordan in the operations $3r_3 \to r_3$ *and* $5r_2 \to r_2$. *It is sometimes convenient to modify the algorithm slightly in order to avoid fractions.*

Example 3.3.10. *Find the rref of the matrix A given below:*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 + r_1}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & -2 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{r_3 + 2r_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \begin{array}{c} \xrightarrow{4r_1} \\ \xrightarrow{2r_2} \end{array} \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \begin{array}{c} \xrightarrow{r_2 - r_3} \\ \xrightarrow{r_1 - r_3} \end{array}$$

$$\begin{bmatrix} 4 & 4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \xrightarrow{r_1 - 2r_2} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \begin{array}{c} \xrightarrow{r_1/4} \\ \xrightarrow{r_2/2} \\ \xrightarrow{r_3/4} \end{array}$$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 3/4 & 1 \end{bmatrix} = rref(A)}$$

Example 3.3.11.

$$[A|I] \ = \ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] \ \begin{array}{c} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - 4r_1} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 4 & 4 & -4 & 0 & 1 \end{array}\right] \ \xrightarrow{r_3 - 2r_2} \ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 0 & -2 & 1 \end{array}\right] \ \begin{array}{c} \xrightarrow{r_2/2} \\ \xrightarrow{r_3/4} \end{array}$$

$$\boxed{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right] = rref[A|I]}$$

Example 3.3.12. *easy examples are sometimes disquieting, let $r \in \mathbb{R}$,*

$$v = \left[\begin{array}{ccc} 2 & -4 & 2r \end{array}\right] \xrightarrow{\frac{1}{2}r_1} \boxed{\left[\begin{array}{ccc} 1 & -2 & r \end{array}\right] = rref(v)}$$

Example 3.3.13. *here's another next to useless example,*

$$v = \left[\begin{array}{c} 0 \\ 1 \\ 3 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{c} 1 \\ 0 \\ 3 \end{array}\right] \xrightarrow{r_3 - 3r_1} \boxed{\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] = \ rref(v)}$$

Example 3.3.14.

$$A \ = \ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 3 & 2 & 0 & 0 \end{array}\right] \xrightarrow{r_4 - 3r_1} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & -3 & 0 \end{array}\right] \xrightarrow{r_4 - r_2}$$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{r_4 + r_3} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - r_4}$$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_2/2} \\ \xrightarrow{r_3/3} \\ \xrightarrow{r_1 - r_3} \end{array} \boxed{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = rref(A)}$$

I should remind you that there are numerous online resources to help you become efficient in your row reduction. Notice, I follow the spirit of the algorithm, but, not necessarily the exact steps:

Example 3.3.15. *In $\mathbb{Z}/2\mathbb{Z}$ we calculate:*

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[r_3+r_1]{r_2+r_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{r_1+r_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{r_1+r_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

then swap rows 2 and 3 to obtain

$$\boxed{rref(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}.$$

Example 3.3.16. *In this example we calculate in $\mathbb{Z}/5\mathbb{Z}$:*

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 0 & 3 \\ 3 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 4 & 0 & 3 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-2r_1} \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & -7 & 1 & -6 \\ 0 & -11 & 1 & -8 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -1 & 1 & 2 \end{bmatrix} \xrightarrow{2r_2} \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & -1 & 1 & 2 \end{bmatrix} \xrightarrow[r_3+r_2]{r_1+r_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{2r_3}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[r_2-2r_3]{r_1-2r_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} = rref(A)}.$$

I hope these examples suffice. One last advice, you should think of the Gauss-Jordan algorithm as a sort of road-map. It's ok to take detours to avoid fractions and such but the end goal should remain in sight. If you lose sight of that it's easy to go in circles. Incidentally, I would strongly recommend you find a way to check your calculations with technology.

3.4 on the structure of solution sets

Surprisingly Examples 3.2.5,3.2.6 and 3.2.7 illustrate all the possible types of solutions for a linear system. In this section I interpret the calculations of the last section as they correspond to solving systems of equations.

Example 3.4.1. *Solve the following system of real linear equations*[5] *if possible,*

$$\begin{aligned} &x + 2y - 3z = 1 \\ &2x + 4y = 7 \\ &-x + 3y + 2z = 0 \end{aligned}$$

We solve by doing Gaussian elimination on the augmented coefficient matrix (see Example 3.3.8 for details of the Gaussian elimination),

$$rref \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 83/30 \\ 0 & 1 & 0 & 11/30 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \Rightarrow \boxed{\begin{array}{c} x = 83/30 \\ y = 11/30 \\ z = 5/6 \end{array}}$$

[5] in other words, I suppose x, y, z are real variables. I keep it real in this subsection as to allow for the natural geometric remarks

Remark 3.4.2.

The geometric interpretation of the last example is interesting. The equation of a plane with normal vector $< a, b, c >$ is $ax + by + cz = d$. Each of the equations in the system of Example 3.3.8 has a solution set which is in one-one correspondence with a particular plane in $\mathbb{R}^3$. The intersection of those three planes is the single point $(83/30, 11/30, 5/6)$.

Example 3.4.3. *Solve the following system of real linear equations if possible,*

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= 0 \\ 2x - 2y &= -3 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 3.3.9 for details of the Gaussian elimination)

$$rref\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

which shows the system has $\boxed{\textit{no solutions}}$ *since row two in the rref corresponds to the equation* $0\,x + 0\,y = 1$. *The given equations are inconsistent.*

Remark 3.4.4.

The geometric interpretation of the last example is also interesting. The equation of a line in the xy-plane is is $ax + by = c$, hence the solution set of a particular equation corresponds to a line. To have a solution to all three equations at once that would mean that there is an intersection point which lies on all three lines. In the preceding example there is no such point.

Example 3.4.5. *Solve the following system of real linear equations if possible,*

$$\begin{aligned} x - y + z &= 0 \\ 3x - 3y &= 0 \\ 2x - 2y - 3z &= 0 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 4.4.7 for details of the Gaussian elimination)

$$rref\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & -3 & 0 & 0 \\ 2 & -2 & -3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\begin{array}{c} x - y = 0 \\ z = 0 \end{array}}$$

The row of zeros indicates that we will not find a unique solution. We have a choice to make, either x or y can be stated as a function of the other. Typically in linear algebra we will solve for the variables that correspond to the pivot columns in terms of the non-pivot column variables. In this problem the pivot columns are the first column which corresponds to the variable x and the third column which corresponds the variable z. The variables x, z are called **dependent variables** *while y is called a* **free** *variable*[6]. *The solution set is* $\boxed{\{(y, y, 0) \mid y \in \mathbb{R}\}}$; *in other words, $x = y, y = y$ and $z = 0$ for all $y \in \mathbb{R}$.*

[6] the choice of free and depedendent variables is suggested by the pivot positions, however, there may also be other reasonable choices

You might object to the last example. You might ask why is y the free variable and not x. This is roughly equivalent to asking the question why is y the dependent variable and x the independent variable in the usual calculus. However, the roles are reversed. In the preceding example the variable x depends on y. Physically there may be a reason to distinguish the roles of one variable over another. There may be a clear cause-effect relationship which the mathematics fails to capture. For example, velocity of a ball in flight depends on time, but does time depend on the ball's velocty ? I'm guessing no. So time would seem to play the role of independent variable. However, when we write equations such as $v = v_o - gt$ we can just as well write $t = \frac{v-v_o}{-g}$; the algebra alone does not reveal which variable should be taken as "independent". Hence, a choice must be made. In the case of infinitely many solutions, we customarily **choose** the pivot variables as the "dependent" or "basic" variables and the non-pivot variables as the "free" variables. Sometimes the word *parameter* is used instead of variable, it is synonomous.

Example 3.4.6. *The real system below is solved by Gaussian elimination on the augmented coefficient matrix (see Example 4.4.7 for details of the Gaussian elimination)*

$$\begin{array}{l} x = 0 \\ 0x + 0y + 0z = 0 \\ 3x = 0 \end{array} \quad \Rightarrow \quad rref\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

thus $\boxed{\{(0, y, z) \mid \ y, z \in \mathbb{R}\}}$ *is the solution set. The variables y and z are free.*

Example 3.4.7. *Once more, I state a real system and calculate its solution by Gaussian elimination on the augmented coefficient matrix (details of the row reduction are in Example 3.3.10)*

$$\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 - x_2 + x_3 = 1 \\ -x_1 + x_3 + x_4 = 1 \end{array} \quad \Rightarrow \quad rref\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 3/4 & 1 \end{array}\right].$$

We find solutions of the form $x_1 = 0$, $x_2 = -x_4/2$, $x_3 = 1 - 3x_4/4$ where $x_4 \in \mathbb{R}$ is free. The solution set is a subset of $\mathbb{R}^4$, namely[7] $\boxed{\{(0, -2s, 1 - 3s, 4s) \mid s \in \mathbb{R}\}}$.

Remark 3.4.8.

The geometric interpretation of the last example is difficult to visualize. Equations of the form $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$ represent volumes in $\mathbb{R}^4$, they're called *h*yperplanes. The solution is parametrized by a single free variable, this means it is a line. We deduce that the three hyperplanes corresponding to the given system intersect along a line. Geometrically solving two equations and two unknowns isn't too hard with some graph paper and a little patience you can find the solution from the intersection of the two lines. When we have more equations and unknowns the geometric solutions are harder to grasp. Analytic geometry plays a secondary role in this course so if you have not had calculus III then don't worry too much. I should tell you what you need to know in these notes.

Example 3.4.9. *Again, I state a system and calculate its solution by Gaussian elimination on the augmented coefficient matrix (details of the row reduction are found in Example 3.3.11)*

$$\begin{array}{l} x_1 + x_4 = 0 \\ 2x_1 + 2x_2 + x_5 = 0 \\ 4x_1 + 4x_2 + 4x_3 = 1 \end{array} \quad \Rightarrow \quad rref\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right]$$

[7] I used $s = 4x_4$ to get rid of the annoying fractions

Consequently, x_4, x_5 are free and solutions are of the form

$$\boxed{\begin{aligned} x_1 &= -x_4 \\ x_2 &= x_4 - \tfrac{1}{2}x_5 \\ x_3 &= \tfrac{1}{4} + \tfrac{1}{2}x_5 \end{aligned}}$$

for all $x_4, x_5 \in \mathbb{R}$. Thus $\{(-a,\ a - b/2,\ 1/4 + b/2,\ a,\ b) \mid a, b \in \mathbb{R}\}$ is the solution set.

Example 3.4.10. *See Example 3.3.14 for details of the Gaussian elimination.*

$$\begin{array}{l} x_1 + x_3 = 0 \\ 2x_2 = 0 \\ 3x_3 = 1 \\ 3x_1 + 2x_2 = 0 \end{array} \quad \Rightarrow \quad rref\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 3 & 2 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore,there are $\boxed{\text{no solutions}}$.

Example 3.4.11. *See Example 4.4.7 for details of the Gaussian elimination.*

$$\begin{array}{l} x_1 + x_3 = 0 \\ 2x_2 = 0 \\ 3x_3 + x_4 = 0 \\ 3x_1 + 2x_2 = 0 \end{array} \quad \Rightarrow \quad rref\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the unique solution is $\boxed{x_1 = x_2 = x_3 = x_4 = 0}$. The solution set is $\{(0, 0, 0, 0)\}$.

Theorem 3.4.12.

Given a system of m linear equations and n unknowns over an infinite field, the solution set falls into one of the following cases:

(i.) the solution set is empty.

(ii.) the solution set has only one element.

(iii.) the solution set is infinite and is parametrized by $(n-k)$-parameters where k is the number of pivot columns in the reduced row echelon form of the augmented coefficient matrix for the system.

Proof: Consider the augmented coefficient matrix $[A|b] \in \mathbb{F}^{m\times(n+1)}$ for the given system of m-linear equations in n-unknowns over the infinite field $\mathbb{F}$ (Theorem 3.3.6 assures us it exists and is unique). Apply the Gauss-Jordan Algorithm to compute $rref[A|b]$ and consider the possible cases:

If $rref[A|b]$ contains a row of zeros with a 1 in the last column then the system is inconsistent and we find no solutions thus the solution set is empty. This brings us to case **(i.)**.

Suppose $rref[A|b]$ does not contain a row of zeros with a 1 in the far right position. Then there are less than $n+1$ pivot columns and we may break into two possible subcases:

(a.) Suppose there are n pivot columns, let c_i for $i = 1, 2, \dots m$ be the entries in the rightmost column. We find $x_1 = c_1, x_2 = c_2, \dots x_n = c_m$. Consequently the solution set is $\{(c_1, c_2, \dots, c_m)\}$ which we identify as case **(ii.)**.

(b.) If $rref[A|b]$ has $k < n$ pivot columns then there are $(n+1-k)$-non-pivot positions. Since the last column corresponds to b it follows there are $n-k \geq 1$ free variables. Examining $rref[A|b]$ we find the k-pivot variables can be written as affine linear combinations of the k-free variables. In short, the solution set is parametrized by the $(n-k)$-free variables and since $n-k \geq 1$ and each free variable takes as many values as $\mathbb{F}$ we find the cardinality of the solution set is infinite.

Naturally, the last case considered provides case **(iii.)** and the proof is complete $\square$

In the case of a finite field we find a very similar theorem. The proof is nearly the same so we omit all but the most interesting detail.

Theorem 3.4.13.

Given a system of m linear equations and n unknowns over a finite field with P elements , the solution set falls into one of the following cases:

(i.) the solution set is empty.

(ii.) the solution set has P^{n-k} solutions which are parametrized by $n-k$-parameters where k is the number of pivot columns in the reduced row echelon form of the augmented coefficient matrix for the system.

Proof: If there are $k = n$ pivot columns then we find a unique solution and this is consistent with the formula $P^{n-n} = P^0 = 1$. On the other hand, in the case the system is consistent and there are $k < n$ pivot columns there are $n-k$ free variables. Each free variable ranges over the P elements of $\mathbb{F}$ hence there are P^{n-k} possible solutions. $\square$

Example 3.4.14. *To give an easy and possibly interesting example, consider $x-y-z=1$ in $\mathbb{Z}/2\mathbb{Z}$ the finite field with 2 elements. Observe y,z serve as parameters of the solution set as $x = 1+y+z$:*

$$\textit{solution set} = \{(1+y+z, y, z) \mid y, z \in \mathbb{Z}/3\mathbb{Z}\}$$

To be explicit, the solution set has $2^2 = 4$ solutions:

$$\textit{solution set} = \{(1,0,0), (0,0,1), (0,1,0), (1,1,1).\}$$

3.5 elementary matrices

Since we already know about matrix multiplication, it is interesting to note that an elementary row operation performed on A can be accomplished by multiplying A on the *left* by a square matrix (called an elementary matrix)[8]. Likewise, multiplying A on the *right* by an elementary matrix performs a column operation. We assume $\mathbb{F}$ is a field in what follows.

[8]in a compressed treatment of linear algebra I might avoid discussion of these, but, as we will see shortly, these elementary matrices allow for concrete proofs of many important aspects of row reduction, the structure of inverse matrices, even the product identity for determinants. Skipping these is not wise if we care about why things work

Recall, $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{F}^n$ where the non-zero entry is located in the i-position[9]. For example, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{F}^2$. We saw in Subsection 2.3.2 that we can construct the identity matrix and the standard basis in $\mathbb{F}^{n\times n}$ from the standard basis vectors $e_i \in \mathbb{F}^n$:

$$I = [e_1|e_2|\cdots|e_n] \qquad \& \qquad E_{ij} = e_i e_j^T = [0, \dots, 0, e_i, 0, \dots, 0]$$

The matrix E_{ij} has a 1 in the (i,j)-position and 0's elsewhere. An illustration from the 2×2 case:

Example 3.5.1. *In* $\mathbb{F}^{2\times 2}$, $\quad E_{21} = e_2 e_1^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

In Example 2.3.18 we learned multilpication by e_j on the right of A produces the j-th column of A. Likewise, in Example 2.3.19 we saw multiplication of e_i^T on the left of A allows us to select the i-th row of A:

$$Ae_j = \mathrm{col}_j(A) \qquad \& \qquad e_i^T A = \mathrm{row}_i(A).$$

Again, we illustrate these identities in the 2×2 case:

Example 3.5.2. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. *Thus,*

$$A\mathbf{e}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \& \qquad \mathbf{e}_2^T A = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix}.$$

Recall you proved Proposition 2.3.14 which allows us to view matrix multiplication as

$$A[\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_\ell] = [A\mathbf{v}_1 \ A\mathbf{v}_2 \ \cdots \ A\mathbf{v}_\ell]$$

(i.e. done column-by-column) and Proposition 2.3.15 allows us to view matrix multiplication as:

$$\begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_k \end{bmatrix} A = \begin{bmatrix} \mathbf{w}_1 A \\ \vdots \\ \mathbf{w}_k A \end{bmatrix}$$

(i.e. done row-by-row). Therefore,

$$AE_{ij} = A[0 \ \cdots \ 0 \ \mathbf{e}_i \ 0 \ \cdots \ 0] = [0 \ \cdots \ 0 \ A\mathbf{e}_i \ 0 \ \cdots \ 0]$$

is merely the i^{th} column of A slapped into the j^{th} column of the zero matrix. Likewise, $E_{ij}A$ is the j^{th} row of A slapped into the i^{th} row of the zero matrix. Illustrate via the 2×2 case once more:

Example 3.5.3. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. *Then,*

$$AE_{21} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} \qquad \& \qquad E_{21}A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}.$$

[9]in $\mathbb{R}^2$ we sometimes call $\mathbf{e}_1 = \mathbf{i}$ and $\mathbf{e}_2 = \mathbf{j}$ and in $\mathbb{R}^3$ sometimes we say $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, $\mathbf{e}_3 = \mathbf{k}$. However, in my multivariate calculus notes I use the notation $e_i = \widehat{x}_i$ to denote unit-vectors in the direction of the i-th Cartesian coordinate, or $\widehat{x}, \widehat{y}, \widehat{z}$ for the usual three-dimensional applications.

Proposition 3.5.4. *Elementary Matrix of Type I:*

> If $E = I_m - E_{ii} - E_{jj} + E_{ij} + E_{ji} \in \mathbb{F}^{m\times m}$ and $A \in \mathbb{F}^{m\times n}$ then EA is A with the i-th and j-th row of A swapped.

Proof: let $E = I_m - E_{ii} - E_{jj} + E_{ij} + E_{ji}$. Let's track through how the various parts of E act on A by left-multiplication:

(1.) $E_{ii}A$ would be the i^{th} row of A left in place with all other rows zeroed out. Likewise $E_{jj}A$ provides be the j^{th} row of A left in place with all other rows zeroed out

(2.) by (1.) we find $(I_m - E_{ii} - E_{jj})A = A - E_{ii}A - E_{jj}A$ wipes out rows i and j

(3.) $E_{ij}A$ would be the j^{th} row of A moved to the i^{th} row with all other rows zeroed out. Thus, by adding in $(E_{ij} + E_{ji})A$, we put rows i and j back but interchanging their locations in $E = I_m - E_{ii} - E_{jj} + E_{ij} + E_{ji}$.

In summary, EA swaps rows i and j. In particular, $E = EI_m$ is the identity matrix with rows i and j swapped (this gives an explicit formula for E). $\square$

It is interesting to note that $E^T = I_m^T - E_{ii}^T - E_{jj}^T + E_{ij}^T + E_{ji}^T = I_m - E_{ii} - E_{jj} + E_{ji} + E_{ij} = E$. Also, since swapping twice undoes the swap, $E^{-1} = E$ (E is its own inverse). It is also worth noting that if $B \in \mathbb{R}^{k\times m}$ (i.e. compatibly sized), then BE is B with *columns* i and j swapped. An example seems in order at this juncture:

Example 3.5.5. *Echoing Example 3.2.9 we see how to obtain an elementary matrix E for Type I operation formed by swapping rows 1 and 3 (so $E = I_3 - E_{11} - E_{33} + E_{13} + E_{31}$).*

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 4 & : & 7 \\ 0 & 5 & -6 & : & 0 \\ 2 & -1 & 3 & : & -1 \end{bmatrix}$$

Proposition 3.5.6. *Elementary Matrix of Type II: for $c \neq 0$ in $\mathbb{F}$:*

> If $E = I_m - E_{ii} + cE_{ii} \in \mathbb{F}^{m\times m}$ and $A \in \mathbb{F}^{m\times n}$ then EA is A with the i-th row of A multiplied by c.

Proof: Let $E = I_m - E_{ii} + cE_{ii}$. Observe $(I_m - E_{ii})A$ gives A with the i-th row zeroed out. Note $cE_{ii}A$ provides a matrix with zero all rows except the i-th row. In the i-th row of $cE_{ii}A$ we find $c\text{row}_i(A)$. Hence, EA gives A with the i-th row multiplied by c and the remaining rows unaltered. $\square$

Notice that $E^{-1} = I_m - E_{ii} + c^{-1}E_{ii}$ (to undo scaling row i by c we should scale row i by $1/c$). So E^{-1} corresponds to a type II operation. Again, EA scales row i of A by c and BE scales column i of B by c. An example may help the reader here:

Example 3.5.7. *Echoing Example 3.2.10 we find E for a Type II operation of scaling row 3 by -2 (so $E = I_3 - E_{33} + (-2)E_{33}$).*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ 2 & 0 & -8 & : & -14 \end{bmatrix}$$

Proposition 3.5.8. *Elementary Matrix of Type III: for $s \in \mathbb{F}$ and $i \neq j$:*

If $E = I_m + sE_{ji} \in \mathbb{F}^{m\times m}$ and $A \in \mathbb{F}^{m\times n}$ then EA has $\text{row}_k(EA) = \text{row}_k(A)$ for $k \neq j$ and $\text{row}_j(EA) = \text{row}_j(A) + s\,\text{row}_i(A)$.

Proof: Let $E = I_m + sE_{ji}$. Again, recall left multiplication by E_{ji} (note the subscripts) will copy row i into row j's place. So $E = I_m + sE_{ji}$ will add s times row i to j. $\square$

To undo adding s times row j to row i we should subtract s times row j from row i. Therefore, $E^{-1} = I_m - sE_{ji}$, so yet again the inverse of an elementary operation is an elementary operation of the same type. Also, EA adds s times row i to row j while BE adds s times column j to column i. For example:

Example 3.5.9. *Echoing Example 3.2.11 we calculate a Type III operation formed by adding 3 times row 3 to row 2 (so $E = I_3 + 3E_{23}$).*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ -3 & 5 & 6 & : & 21 \\ -1 & 0 & 4 & : & 7 \end{bmatrix}$$

Notice, the discussion in this section shows:

Proposition 3.5.10.

Each elementary matrix is invertible with inverse matrix of the same type.

Proposition 3.5.11.

Let $A \in R^{m\times n}$ then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $rref(A) = E_k \cdots E_2E_1A$.

Proof: Gauss Jordan elimination consists of a sequence of k elementary row operations. Each row operation can be implemented by multiplying the corresponding elementary matrix on the left. Hence successively left-multiplying A by each elementary matrix corresponding to a row operation in the sequence will produce $rref(A)$. $\square$

Example 3.5.12. *Let $A = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}$. We calculate:*

$$E_{r_2+3r_1\to r_2}A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3a+1 & 3b+2 & 3c+3 \\ u & m & e \end{bmatrix}$$

$$E_{7r_2\to r_2}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ 7 & 14 & 21 \\ u & m & e \end{bmatrix}$$

$$E_{r_2\to r_3}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ u & m & e \\ 1 & 2 & 3 \end{bmatrix}.$$

Example 3.5.13. *Let's see what happens if we multiply the elementary matrices on the right instead.*

$$AE_{r_2+3r_1\to r_2} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3b & b & c \\ 1+6 & 2 & 3 \\ u+3m & m & e \end{bmatrix}$$

$$AE_{7r_2\to r_2} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 7b & c \\ 1 & 14 & 3 \\ u & 7m & e \end{bmatrix}$$

$$AE_{r_2\to r_3} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c & b \\ 1 & 3 & 2 \\ u & e & m \end{bmatrix}$$

Curious, they generate column operations[10]*, these are* **elementary column operations**. *In our notation the row operations are more important.*

3.6 theory of invertible matrices

We already learned the basic results about inverses in our larger discussion of matrix algebra. Notably, for invertible matrices over a field $\mathbb{F}$ and nonzero $c \in \mathbb{F}$,

$$(AB)^{-1} = B^{-1}A^{-1}, \ \& \ (A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}, \ \& \ (cA)^{-1} = \frac{1}{c}A^{-1}.$$

These results and several others were given by Theorem 2.4.8. In this section we study further theorems for invertible matrices. Throughout this section we assume A is a square $n \times n$ matrix over a field $\mathbb{F}$.

Lemma 3.6.1.

Let $A \in \mathbb{F}^{n\times n}$. The solution of $Ax = 0$ is unique if and only if $rref(A) = I$.

Proof: Suppose $Ax = 0$ has a unique solution. Since $A(0) = 0$ the unique solution is in fact the zero solution. Consequently, Theorem 3.4.12 or 3.4.13 provide that $rref[A|0] = [I|0]$. Furthermore, by Theorem 3.5.11, there exist a product of elementary matrices $E = E_k\cdots E_2E_1$ for which $E[A|0] = [I|0]$. Hence, $[EA|0] = [I|0]$ and we find $EA = I$ which implies $rref(A) = I$ as the rref of A is unique and I is certainly a rref which is obtained from A via elementary row operations.

Conversely, suppose $rref(A) = I$ then there exists by Theorem 3.5.11, there exist a product of elementary matrices $E = E_k\cdots E_2E_1$ for which $EA = I$ thus $E[A|0] = [EA|0] = [I|0]$. Noting $[I|0]$ is in reduced row echelon form we find $rref[A|0] = [I|0]$ as $[A|0]$ is row equivalent to $[I|0]$ under the sequence of row operations implicit within E. Again we use the uniqueness of rref as given by Theorem 3.3.6. $\square$

[10] careful, they're not quite the same column operation, for example $r_2 + 3r_1 \to r_2$ is replaced with $c_1 + 3c_2 \to c_1$. For Type X elementary row op. with elementary matrix E we'll find AE^T is related to A by a type X column op. Column operations are used in the study of the Smith Normal Form in module theory (Math 422 sometimes).

Lemma 3.6.2.

Let $A \in \mathbb{F}^{n\times n}$, $rref(A) = I$ if and only if A is a product of elementary matrices.

Proof: Suppose $rref(A) = I$ then by Theorem 3.5.11, there exist a product of elementary matrices $E = E_k \cdots E_2 E_1$ for which $EA = I$. Hence, by Theorem 2.4.10,

$$A = E^{-1}I = (E_k \cdots E_2 E_1)^{-1} = E_1^{-1}E_2^{-1}\cdots E_k^{-1}$$

and as the inverse of each elementary matrix is itself elementary this shows A is the product of elementary matrices.

Conversely, suppose $A = E_1 E_2 \dots E_j$ where $E_1, E_2, \dots, E_j$ are elementary matrices. Thus,

$$E_j^{-1}\dots E_2^{-1}E_1^{-1}A = E_j^{-1}\dots E_2^{-1}E_1^{-1}E_1E_2\dots E_j = I$$

where we have made use of $E_i^{-1}E_i = I$ for $i = 1, 2, \dots, j$. Thus A is row equivalent to I and by uniqueness of rref we deduce $rref(A) = I$. $\square$

Given the two lemmas above the following is immediate:

Corollary 3.6.3.

Let $A \in \mathbb{F}^{n\times n}$, then the the following are equivalent:

(a.) $Ax = 0$ if and only if $x = 0$

(b.) $rref(A) = I$

(c.) A is a product of elementary matrices.

Theorem 3.6.4.

Let $A \in \mathbb{F}^{n\times n}$, A is invertible if and only if A is a product of elementary matrices.

Proof: Suppose A^{-1} exists then $Ax = 0$ provides $A^{-1}Ax = A^{-1}0$ thus $x = 0$. Hence, by Corollary 3.6.3, we deduce A is the product of elementary matrices.

Conversely, suppose $A = E_k \cdots E_2 E_1$ where $E_1, E_2, \dots, E_k$ are elementary matrices. Note each of E_1, $E_2, \dots, E_k$ is invertible hence their product is invertible by Theorem 2.4.10. In particular,

$$A^{-1} = (E_1E_2\cdots E_k)^{-1} = E_k^{-1}\cdots E_2^{-1}E_1^{-1}$$

illustrates that A^{-1} is also formed by a product of elementary matrices as the inverse of each elementary matrix is itself elementary. $\square$

If you haven't proved Theorem 2.4.10 for yourself, then it's probably a good exercise to verify the formula above for A^{-1} satisfies both conditions needed according to Definition 2.4.5. That said, we now learn that only half of the Definition needs to be checked:

Proposition 3.6.5.

Let $A \in \mathbb{R}^{n\times n}$.

1. If $BA = I$ then $AB = I$.
2. If $AB = I$ then $BA = I$.

Proof of (1.): Suppose $BA = I$. If $Ax = 0$ then $BAx = B0$ hence $Ix = 0$. We have shown that $Ax = 0$ only has the trivial solution. Therefore, Theorem 3.6.4 shows us that A^{-1} exists. Multiply $BA = I$ on the right by A^{-1} to find $BAA^{-1} = IA^{-1}$ hence $B = A^{-1}$ and by definition it follows $AB = I$.
Proof of (2.): Suppose $AB = I$. If $Bx = 0$ then $ABx = A0$ hence $Ix = 0$. We have shown that $Bx = 0$ only has the trivial solution. Therefore, Theorem 3.6.4 shows us that B^{-1} exists. Multiply $AB = I$ on the right by B^{-1} to find $ABB^{-1} = IB^{-1}$ hence $A = B^{-1}$ and by definition it follows $BA = I$. $\square$

Proposition 3.6.5 shows that we don't need to check both conditions $AB = I$ and $BA = I$. If either holds the other condition automatically follows. In retrospect, we can simplify a number of proofs in our early study of invertible matrices. Notice, this is special to matrices. It is not usually the case the left inverses imply right inverses and vice-versa.

We can also use inverse matrices to solve nonhomogeneous linear systems:

Theorem 3.6.6. *Let $A \in \mathbb{F}^{n\times n}$.*

If A is invertible then for each $b \in \mathbb{F}^n$ the system of equations $Ax = b$ has a unique solution. Conversely, if $Ax = b$ has a unique solution for each $b \in \mathbb{F}^n$ then A^{-1} exists.

Proof: suppose A is invertible. Consider $Ax = b$. Observe $x_o = A^{-1}b$ is a solution since $Ax_o = A(A^{-1}b) = Ib = b$. Moreover, it is the only solution as $Ax = b$ implies $A^{-1}Ax = A^{-1}b$ hence $x = A^{-1}b$. Thus, A invertible implies $Ax = b$ has a unique solution for each $b \in \mathbb{F}^n$.

Conversely, suppose $Ax = b$ has a unique solution for each $b \in \mathbb{F}^n$. Hence there exist $y_1, y_2, \dots, y_n \in \mathbb{F}^n$ for which $Ay_1 = e_1$, $Ay_2 = e_2, \dots Ay_n = e_n$. Calculate,

$$A[y_1|y_2|\dots|y_n] = [Ay_1|Ay_2|\cdots|Ay_n] = [e_1|e_2|\cdots|e_n] = I$$

thus $A^{-1} = [y_1|y_2|\dots|y_n]$ by Proposition 3.6.5. $\square$

Let use agree to use Proposition 3.6.5 without explicit acknowledgement in the remainder of this course. It suffices to either show $AB = I$ or $BA = I$ in order to prove $B = A^{-1}$.

3.7 calculation of inverse matrix

We prepare for the calculation of inverse matrices by discussing how Gauss-Jordan can handle solving multiple systems at once, if these systems share the same coefficient matrix. We begin with an example courtesy of William Cook of Appalachian State University in Boone, N.C.

Example 3.7.1. *Suppose we wanted to solve both* $\begin{array}{rcrcr} x & + & 2y & = & 3 \\ 4x & + & 5y & = & 6 \\ 7x & + & 8y & = & 9 \end{array}$ *and also* $\begin{array}{rcrcr} x & + & 2y & = & 3 \\ 4x & + & 5y & = & 9 \\ 7x & + & 8y & = & 15 \end{array}$.
These lead to the following augmented matrices: $\begin{bmatrix} 1 & 2 & : & 3 \\ 4 & 5 & : & 6 \\ 7 & 8 & : & 9 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 2 & : & 3 \\ 4 & 5 & : & 9 \\ 7 & 8 & : & 15 \end{bmatrix}$. *We can combine them together and get* $\begin{bmatrix} 1 & 2 & : & 3 & 3 \\ 4 & 5 & : & 6 & 9 \\ 7 & 8 & : & 9 & 15 \end{bmatrix}$ *which we already know has the RREF of* $\begin{bmatrix} 1 & 0 & : & -1 & 1 \\ 0 & 1 & : & 2 & 1 \\ 0 & 0 & : & 0 & 0 \end{bmatrix}$ *(from Example 3.3.5 — only the :'s have moved). This corresponds to the augmented matrices* $\begin{bmatrix} 1 & 0 & : & -1 \\ 0 & 1 & : & 2 \\ 0 & 0 & : & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 & : & 1 \\ 0 & 1 & : & 1 \\ 0 & 0 & : & 0 \end{bmatrix}$. *These in turn tell us that the first system's solution is* $x = -1$, $y = 2$ *and the second system's solution is* $x = 1$ *and* $y = 1$.

This works because we aren't mixing columns together (all operations are row operations). Also, notice that the same matrix can be interpreted in a number of ways. Before we had a single system in 3 variables and now we have 2 systems in 2 variables. It's easy to extend to multiple systems:

Proposition 3.7.2.

Let $A \in \mathbb{F}^{m\times n}$. Vectors $v_1, v_2, \dots, v_k$ are solutions of $Av = b_i$ for $i = 1, 2, \dots k$ iff $V = [v_1|v_2|\cdots|v_k]$ solves $AV = B$ where $B = [b_1|b_2|\cdots|b_k]$.

Proof: Let $A \in \mathbb{F}^{m\times n}$ and suppose $Av_i = b_i$ for $i = 1, 2, \dots k$. Let $V = [v_1|v_2|\cdots|v_k]$ and use the concatenation Proposition 2.3.14,

$$AV = A[v_1|v_2|\cdots|v_k] = [Av_1|Av_2|\cdots|Av_k] = [b_1|b_2|\cdots|b_k] = B.$$

Conversely, suppose $AV = B$ where $V = [v_1|v_2|\cdots|v_k]$ and $B = [b_1|b_2|\cdots|b_k]$ then by Proposition 2.3.14 $AV = B$ implies $Av_i = b_i$ for each $i = 1, 2, \dots k$. $\square$

Example 3.7.3. *Solve the systems given below,*

$$\begin{array}{l} x+y+z=1 \\ x-y+z=0 \\ -x+z=1 \end{array} \qquad \textit{and} \qquad \begin{array}{l} x+y+z=1 \\ x-y+z=1 \\ -x+z=1 \end{array}$$

The systems above share the same coefficient matrix, however $b_1 = [1,0,1]^T$ *whereas* $b_2 = [1,1,1]^T$. *We can solve both at once by making an extended augmented coefficient matrix* $[A|b_1|b_2]$

$$[A|b_1|b_2] = \left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{array}\right] \qquad rref[A|b_1|b_2] = \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 3/4 & 1 \end{array}\right]$$

We use Proposition 3.7.2 to conclude that

$$\begin{array}{l} x+y+z=1 \\ x-y+z=0 \\ -x+z=1 \end{array} \qquad \textit{has solution } x = -1/4, y = 1/2, z = 3/4$$

$$\begin{array}{l} x+y+z=1 \\ x-y+z=1 \\ -x+z=1 \end{array} \qquad \textit{has solution } x = 0, y = 0, z = 1.$$

I hope the reader will forgive me for treating the 3×3 case in what follows. The extension to $n \times n$ should be easy to deduce:

PROBLEM: how should we calculate A^{-1} for a 3×3 matrix ?

Consider that the Proposition 3.7.2 gives us another way to look at the problem (also see the proof of Theorem 3.6.6),

$$AA^{-1} = I \;\Leftrightarrow\; A[v_1|v_2|v_3] = I_3 = [e_1|e_2|e_3]$$

Where $v_i = col_i(A^{-1})$ and $e_1 = [1\ 0\ 0]^T, e_2 = [0\ 1\ 0]^T, e_3 = [0\ 0\ 1]^T$. We observe that the problem of finding A^{-1} for a 3×3 matrix amounts to solving three separate systems:

$$Av_1 = e_1,\; Av_2 = e_2,\; Av_3 = e_3$$

when we find the solutions then we can construct $A^{-1} = [v_1|v_2|v_3]$. Think about this, if A^{-1} exists then it is unique thus the solutions v_1, v_2, v_3 are likewise unique. Consequently, by Theorem **??**,

$$rref[A|e_1] = [I|v_1],\; rref[A|e_2] = [I|v_2],\; rref[A|e_3] = [I|v_3].$$

Each of the systems above required the same sequence of elementary row operations to cause $A \mapsto I$. We can just as well do them at the same time in one big matrix calculation:

$$rref[A|e_1|e_2|e_3] = [I|v_1|v_2|v_3]$$

While this discuss was done for $n = 3$ we can just as well do the same for $n > 3$. This provides the proof for the first sentence of the theorem below. Theorem **??** together with the discussion above proves the second sentence.

Theorem 3.7.4.

$A \in \mathbb{F}^{n\times n}$ is invertible if and only if $rref[A|I_n] = [I_n|A^{-1}]$. Moreover, if $rref[A|I_n] \neq [I|B]$ for some $B \in \mathbb{F}^{n\times n}$ then A is **not** invertible.

This is perhaps the most pragmatic theorem so far stated in these notes. This theorem tells us how and when we can find an inverse for a square matrix.

Example 3.7.5. *Recall that in Example 3.3.11 we worked out the details of*

$$rref\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right]$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 4 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 0 & -1/2 & 1/4 \end{bmatrix}.$$

Example 3.7.6. *I omit the details of the Gaussian elimination,*

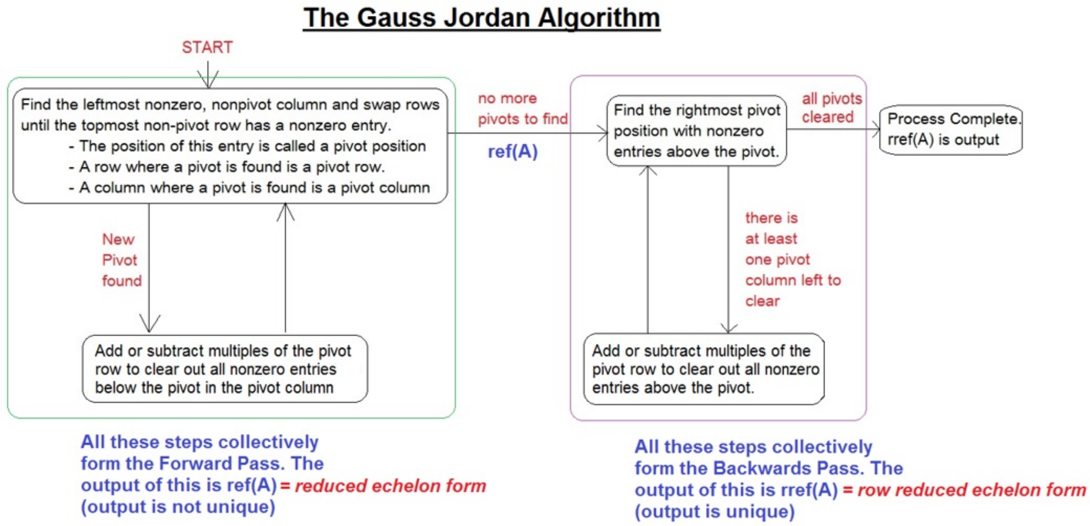
$$rref\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

Thus,

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

3.8 conclusions

I made this flow-chart to illustrate the idea of the Gauss-Jordan elimination perhaps this helps:



The theorem that follows here collects the many things we have learned about an $n \times n$ invertible matrices and corresponding linear systems:

Theorem 3.8.1.

Let A be an $n \times n$ matrix over a field $\mathbb{F}$ then the following are equivalent:

(a.) A is invertible,

(b.) $rref[A] = I$,

(c.) $Ax = 0$ iff $x = 0$,

(d.) A is the product of elementary matrices,

(e.) there exists $B \in \mathbb{R}^{\ n\times n}$ such that $AB = I$,

(f.) there exists $B \in \mathbb{R}^{\ n\times n}$ such that $BA = I$,

(g.) for each $b \in \mathbb{F}^m$ there exists $x \in \mathbb{F}^n$ for which $rref[A|b] = [I|x]$,

(h.) $Ax = b$ is consistent for every $b \in \mathbb{F}^m$,

(i.) $Ax = b$ has exactly one solution for each $b \in \mathbb{F}^m$,

(j.) A^T is invertible.

We also learned that for a system of m-linear-equations in n-unknowns we can deduce the solution set by studying the reduced row echelon form of the augmented coefficient matrix.

Chapter 4

spans, LI, and the CCP

Theoretically, the results in this chapter provide the archtypical example of dimension theory. Moreover, even as we study more abstract material after this chapter, it is still often the case we fall back on the calculational schemes set-forth here.

In this chapter we study careful definitions of generating sets and linear independence for sets of column vectors over a field $\mathbb{F}$. Since both the question of spanning and the question of linear independence amount to particular systems of equations we find our questions eventually reduce to an appropriate Gauss-Jordan calculation.

We discover the Column Correspondence Property (CCP or sometimes the *linear correspondence*) provides an amazingly simple calculational method to understand spanning and linear independence questions. The CCP provides the fastest method to read data off the row-reduced matrix. It might be tempting to skip to the CCP since it is the take-away calculation of this chapter, but, I think you'll understand it more if you go with the flow and work your way up to the sneakiest method. My apologies to you finite field fans, it seems I am mostly keeping it real in this chapter, but, this is not a necessary restriction.

4.1 matrix vector multiplication notation for systems

Let us begin with a simple example.

Example 4.1.1. *Consider the following generic system of two equations and three unknowns,*

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \end{aligned}$$

in matrix form this system of equations is $Av = b$ where

$$Av = \underbrace{\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v} = \begin{bmatrix} (a,b,c)\cdot(x,y,z) \\ (e,f,g)\cdot(x,y,z) \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ ex+fy+gz \end{bmatrix} = \underbrace{\begin{bmatrix} d \\ h \end{bmatrix}}_{b}$$

Definition 4.1.2.

Let $x_1, x_2, \dots, x_n$ be n variables and suppose $b_i, A_{ij} \in \mathbb{F}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. The system of linear equations

$$\begin{aligned}
&A_{11}x_1 + A_{12}x_2 + \cdots + A_{1k}x_k = b_1 \\
&A_{21}x_1 + A_{22}x_2 + \cdots + A_{2k}x_k = b_2 \\
&\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\
&A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nk}x_k = b_n
\end{aligned}$$

has **coefficient matrix** $A = [A_{ij}] \in \mathbb{F}^{m \times n}$ and **inhomogeneous term** $b = [b_i] \in \mathbb{F}^n$. A vector $x \in \mathbb{F}^n$ for which $Ax = b$ is called a **vector solution** to the matrix form of the system. Also, the solution set is $\{x \in \mathbb{F}^n \mid Ax = b\}$.

The vector format gives us an efficient way to express a solution to the system. Moreover, we know Gauss-Jordan elimination on the augmented coefficient matrix is a reliable algorthim to solve any such system.

Example 4.1.3. *We found that the system in Example 3.4.1,*

$$\begin{aligned}
&x + 2y - 3z = 1 \\
&2x + 4y = 7 \\
&-x + 3y + 2z = 0
\end{aligned}$$

has the unique solution $x = 83/30, y = 11/30$ *and* $z = 5/6$. *This means the matrix equation* $Av = b$ *where*

$$Av = \underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 0 \\ -1 & 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}}_{b} \quad \textit{has vector solution} \quad \boxed{v = \begin{bmatrix} 83/30 \\ 11/30 \\ 5/6 \end{bmatrix}}.$$

Example 4.1.4. *We can rewrite the following system of linear equations*

$$\begin{aligned}
&x_1 + x_4 = 0 \\
&2x_1 + 2x_2 + x_5 = 0 \\
&4x_1 + 4x_2 + 4x_3 = 1
\end{aligned}$$

in matrix form this system of equations is $Av = b$ *where*

$$Av = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b}.$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 3.3.11 for details of the Gaussian elimination)

$$rref \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right].$$

Consequently, x_4, x_5 are free and solutions are of the form

$$x_1 = -x_4$$
$$x_2 = x_4 - \tfrac{1}{2}x_5$$
$$x_3 = \tfrac{1}{4} + \tfrac{1}{2}x_5$$

for all $x_4, x_5 \in \mathbb{R}$. The vector form of the solution is as follows:

$$\boxed{v = \begin{bmatrix} -x_4 \\ x_4 - \frac{1}{2}x_5 \\ \frac{1}{4} + \frac{1}{2}x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{bmatrix}.}$$

Remark 4.1.5.

You might ask the question: what is the geometry of the solution set above ? Let $S = Sol_{[A|b]} \subset \mathbb{R}^5$, we see S is formed by tracing out all possible linear combinations of the vectors $v_1 = (-1, 1, 0, 1, 0)$ and $v_2 = (0, -\frac{1}{2}, \frac{1}{2}, 0, 1)$ based from the point $p_o = (0, 0, \frac{1}{4}, 0, 0)$. In other words, this is a two-dimensional plane containing the vectors v_1, v_2 and the point p_o. This plane is placed in a 5-dimensional space, this means that at any point on the plane you could go in three different directions away from the plane.

Corollary 2.3.17 is worth revisiting at this time.

Example 4.1.6. *Let $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \end{bmatrix}$ and $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then we may calculate the product Av as follows:*

$$Av = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ a \end{bmatrix} + y \begin{bmatrix} 1 \\ b \end{bmatrix} + z \begin{bmatrix} 1 \\ c \end{bmatrix} = \begin{bmatrix} x + y + z \\ ax + by + cz \end{bmatrix}.$$

The example above shows us we can interpret $Av = b$ as a vector equation. In particular, a useful Corollary to Corollary 2.3.17 is given below (I'm not calling it a corollary):

Proposition 4.1.7.

If $A = [A_1|A_2|\cdots|A_n] \in \mathbb{F}^{m\times n}$ and $b \in \mathbb{F}^m$ then the matrix equation $Ax = b$ has the same set of solutions as the vector equation

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = b.$$

The calculations of the next subsection are built on this Proposition. We should pause to prove some low-hanging items about systems from this viewpoint:

Theorem 4.1.8. *Superposition of solutions:*

Let $A \in \mathbb{F}^{m \times n}$. Let $c_1, c_2 \in \mathbb{F}$ and suppose there exist $x_1, x_2 \in \mathbb{F}^n$ for which $Ax_1 = b_1$ and $Ax_2 = b_2$ then $x = c_1x_1 + c_2x_2$ is a solution of $Ax = c_1b_1 + c_2b_2$. In particular, if $Ax_1 = 0$ and $Ax_2 = 0$ then $c_1x_2 + c_2x_2$ is a solution to $Ax = 0$.

Proof: with x_1, x_2 as in the Theorem we note:

$$A(c_1x_1 + c_2x_2) = c_1Ax_1 + c_2Ax_2 = c_1b_1 + c_2b_2.$$

The homogeneous case follows from setting $b_1 = b_2 = 0$. $\square$

Example 4.1.9. *If we have two nonhomogeneous solutions of the same linear system then it is easy to generate a homogeneous solution. To see this, suppose $Ax_1 = b$ and $Ax_2 = b$. Notice $A(x_2 - x_1) = Ax_2 - Ax_1 = b - b = 0$ thus $x_2 - x_1$ is a solution of $Ax = 0$.*

The example above is interesting for physical systems. We can subject a given linear system to two known forces and from the difference in the response functions it is possible to determine the intrinsic character of the system in the absense of external force. In a linear system, the net-response is a superposition of the responses to each source driving the system.

4.2 linear combinations and spanning

Proposition 2.2.15 showed that linear combinations of the standard basis will generate any vector in $\mathbb{F}^n$. A natural generalization of that question is given below:

PROBLEM: Given vectors $v_1, v_2, \dots, v_k \in \mathbb{F}^n$ and a vector $b \in \mathbb{F}^n$ do there exist constants $c_1, c_2, \dots, c_k \in \mathbb{F}$ such that $c_1v_1 + c_2v_2 + \cdots + c_kv_k = b$? If so, how to find $c_1, \dots, c_k$?

We have all the tools we need to solve such problems. Ultimately, the CCP gives us the most efficient solution, However, I think it is best for us to work our way with less optimal methods before we learn the fastest method. For now, we just use Proposition 4.1.7 or common sense.

Example 4.2.1. Problem: *given that $v = (2, -1, 3)$, $w = (1, 1, 1)$ and $b = (4, 1, 5)$ find values for x, y such that $xv + yw = b$* **(if possible).**

Solution: *using our column notation we find $xv + yw = b$ gives*

$$x\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2x + y \\ -x + y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$$

We are faced with solving the system of equations $2x + y = 4$, $-x + y = 1$ and $3x + y = 5$. As we discussed in depth last chapter we can efficiently solve this type of problem in general by Gaussian elimination on the corresponding augmented coefficient matrix. In this problem, you can calculate that

$$rref\left[\begin{array}{cc|c} 2 & 1 & 4 \\ -1 & 1 & 1 \\ 3 & 1 & 5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

hence $x = 1$ and $y = 2$. Indeed, it is easy to check that $v + 2w = b$.

The set of all linear combinations of several vectors is called the *span* of those vectors. To be precise

Definition 4.2.2.

Let $S = \{v_1, v_2, \dots, v_k\} \subset \mathbb{F}^n$ be a finite set of n-vectors then $\text{span}(S)$ is defined to be the set of all linear combinations formed from vectors in S:

$$\text{span}\{v_1, v_2, \dots, v_k\} = \left\{ \sum_{i=1}^{k} c_i v_i \mid c_i \in \mathbb{F} \text{ for } i = 1, 2, \dots, k \right\}$$

Let $W = \text{span}(S)$. We say that S is a **generating set** or **spanning set** for W.

If we have one vector then it has a span which could be a line. With two vectors we might generate a plane. With three vectors we might generate a volume. With four vectors we might generate a hypervolume or 4-volume. We'll return to these geometric musings in § 4.3 and explain why I have used the word "might" rather than an affirmative "will" in these claims. For now, we return to the question of how to decide if a given vector is in the span of another set of vectors.

Example 4.2.3. Problem: *Let $b_1 = (1, 1, 0), b_2 = (0, 1, 1)$ and $b_3 = (0, 1, -1)$. Is*[1] *$e_3 \in span\{b_1, b_2, b_3\}$?*

Solution: *Find the explicit linear combination of b_1, b_2, b_3 that produces e_3. We seek to find $x, y, z \in \mathbb{R}$ such that $xb_1 + yb_2 + zb_3 = e_3$,*

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ x+y+z \\ y-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Following essentially the same arguments as the last example we find this question of solving the system formed by gluing the given vectors into a matrix and doing row reduction. In particular, we can solve the vector equation above by solving the corresponding system below:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{r_3 - r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{-r_3/2} \\ \xrightarrow{r_2 - r_3} \\ \xrightarrow{r_1 - r_3} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

Therefore, $x = 0, y = \frac{1}{2}$ and $z = -\frac{1}{2}$. We find that $\boxed{e_3 = \frac{1}{2}b_2 - \frac{1}{2}b_3}$ thus $e_3 \in span\{b_1, b_2, b_3\}$.

The power of the matrix technique is shown in the next example.

[1] challenge: once you understand this example for e_3 try answering it for other vectors or for an arbitrary vector $v = (v_1, v_2, v_3)$. How would you calculate $x, y, z \in \mathbb{R}$ such that $v = xb_1 + yb_2 + zb_3$?

Example 4.2.4. Problem: *Let* $b_1 = (1,2,3,4)$*,* $b_2 = (0,1,0,1)$ *and* $b_3 = (0,0,1,1)$*.*
Is $w = (1,1,4,4) \in span\{b_1, b_2, b_3\}$*?*

Solution: *Following the same method as the last example we seek to find* x_1, x_2 *and* x_3 *such that* $x_1b_1 + x_2b_2 + x_3b_3 = w$ *by solving the aug. coeff. matrix as is our custom:*

$$[b_1|b_2|b_3|w] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 4 \\ 4 & 1 & 1 & 4 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - 3r_1} \\ \xrightarrow{r_4 - 4r_1} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_4 - r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_4 - r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = rref[b_1|b_2|b_3|w]$$

We find $x_1 = 1, x_2 = -1, x_3 = 1$ *thus* $\boxed{w = b_1 - b_2 + b_3}$*. Therefore,* $w \in span\{b_1, b_2, b_3\}$*.*

Pragmatically, if the question is sufficiently simple you may not need to use the augmented coefficient matrix to solve the question. I use them here to illustrate the method.

Example 4.2.5. Problem: *Let* $b_1 = (1,1,0)$ *and* $b_2 = (0,1,1)$*.*
Is $e_2 \in span\{b_1, b_2\}$*?*

Solution: *Attempt to find the explicit linear combination of* b_1, b_2 *that produces* e_2*. We seek to find* $x, y \in \mathbb{R}$ *such that* $xb_1 + yb_2 = e_3$*,*

$$x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ x+y \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We don't really need to consult the augmented matrix to solve this problem. Clearly $x = 0$ *and* $y = 0$ *is found from the first and third components of the vector equation above. But, the second component yields* $x + y = 1$ *thus* $0 + 0 = 1$*. It follows that this system is inconsistent and we may conclude that* $w \notin span\{b_1, b_2\}$*. For the sake of curiousity let's see how the augmented solution matrix looks in this case: omitting details of the row reduction,*

$$rref\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

note the last row again confirms that this is an inconsistent system.

4.2.1 solving several spanning questions simultaneously

If we are given $B = \{b_1, b_2, \dots, b_k\} \subset \mathbb{F}^n$ and $T = \{w_1, w_2, \dots, w_r\} \subset \mathbb{F}^n$ and we wish to determine if $T \subset span(B)$ then we can answer the question by examining if $[b_1|b_2|\cdots|b_k]x = w_j$ has a solution for each $j = 1, 2, \dots r$. Or we could make use of Proposition 3.7.2 and solve it in one sweeping matrix calculation;

$$rref[b_1|b_2|\cdots|b_k|w_1|w_2|\cdots|w_r]$$

If there is a row with zeros in the first k-columns and a nonzero entry in the last r-columns then this means that at least one vector w_k is not in the span of B(moreover, the vector not in the span corresponds to the nonzero entrie(s)). Otherwise, each vector is in the span of B and we can read the precise linear combination from the matrix. I will illustrate this in the example that follows.

Example 4.2.6. *Let $W = span\{e_1 + e_2, e_2 + e_3, e_1 - e_3\}$ and suppose $T = \{e_1, e_2, e_3 - e_1\}$. Is $T \subseteq W$? If not, which vectors in T are not in W? Consider, $[e_1 + e_2|e_2 + e_3|e_1 - e_3||e_1|e_2|e_3 - e_1] =$*

$$= \left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 - r_2} \left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 + r_3} \\ \xrightarrow{r_1 - r_3} \end{array} \left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Let me summarize the calculation:

$$rref[e_1 + e_2|e_2 + e_3|e_1 - e_3||e_1|e_2|e_3 - e_1] = \left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

We deduce that e_1 and e_2 are not in W. However, $e_3 - e_1 \in W$ and we can read from the matrix $-(e_1 + e_2) + (e_2 + e_3) = e_3 - e_1$. I added the double vertical bar for book-keeping purposes, as usual the vertical bars are just to aid the reader in parsing the matrix.

In short, if we wish to settle if several vectors are in the span of a given generating set then we can do one sweeping row reduction to determine what is in the span. This is a great reduction in labor from what you might naively expect.

4.3 linear independence

In the previous sections we have only considered questions based on a fixed spanning set[2]. We asked if $b \in span\{v_1, v_2, \ldots, v_n\}$ and we even asked if it was possible for all b. What we haven't thought about yet is the following:

PROBLEM: Given vectors $v_1, v_2, \ldots, v_k$ and a vector $b = c_1v_1 + c_2v_2 + \cdots + c_kv_k$ for some constants c_j is it possible that b can be written as a linear combination of some subset of $\{v_1, v_2, \ldots, v_k\}$? If so, how should we determine which vectors can be taken away from the spanning set? How should we decide which vectors to keep and which are redundant?

The span of a set of vectors is simply all possible finite linear combinations of vectors from the set. If you think about it, we don't need a particular vector in the generating set if that vector can be written as a linear combination of other vectors in the generating set. To solve the problem stated above we need to remove linear dependencies of the generating set.

[2]sometimes I call it the spanning set, other times the generating set. It turns out that a given space may be generated in many different ways. This section begins the quest to unravel that puzzle

Definition 4.3.1.

Let $S \subseteq \mathbb{F}^n$. If there is some vector $v \in S$ which can be written as a finite linear combination of other vectors in S then S is **linearly dependent**. If S is not linearly dependendent then S is said to be **linearly independent** (LI).

Notice, since the empty set $\emptyset$ has no vectors it cannot be linearly dependent. It follows that $\emptyset$ is LI. We have $\text{span}(\emptyset) = \{0\}$ and $\emptyset$ is LI. This observation is useful later.

Example 4.3.2. *Let $v = [1\ 2\ 3]^T$ and $w = [2\ 4\ 6]^T$ note $w = 2v$ thus $\{v, w\}$ is linearly dependent.*

I often quote the following proposition as the defintion of linear independence, it is an equivalent statement and as such can be used as the definition(but not by us, I already made the definition above). If this was our definition then our definition would become a proposition. Math always has a certain amount of this sort of ambiguity.

Proposition 4.3.3.

Let $v_1, v_2, \dots, v_k \in \mathbb{R}^n$. The set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly independent iff

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \ \Rightarrow \ c_1 = c_2 = \cdots = c_k = 0.$$

Proof: ($\Rightarrow$) Suppose $\{v_1, v_2, \dots, v_k\}$ is linearly independent. Assume that there exist constants $c_1, c_2, \dots, c_k$ such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

and at least one constant, say c_j, is nonzero. Then we can divide by c_j to obtain

$$\tfrac{c_1}{c_j}v_1 + \tfrac{c_2}{c_j}v_2 + \cdots + v_j + \cdots + \tfrac{c_k}{c_j}v_k = 0$$

solve for v_j, (we mean for $\widehat{v_j}$ to denote the deletion of v_j from the list)

$$v_j = -\tfrac{c_1}{c_j}v_1 - \tfrac{c_2}{c_j}v_2 - \cdots - \widehat{v_j} - \cdots - \tfrac{c_k}{c_j}v_k$$

but this means that v_j linearly depends on the other vectors hence $\{v_1, v_2, \dots, v_k\}$ is linearly dependent. This is a contradiction, therefore $c_j = 0$. Note j was arbitrary so we may conclude $c_j = 0$ for all j. Therefore, $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \ \Rightarrow \ c_1 = c_2 = \cdots = c_k = 0$.

Proof: ($\Leftarrow$) Assume that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \ \Rightarrow \ c_1 = c_2 = \cdots = c_k = 0.$$

If $v_j = b_1v_1 + b_2v_2 + \cdots + \widehat{b_jv_j} + \cdots + b_kv_k$ then $b_1v_1 + b_2v_2 + \cdots + b_jv_j + \cdots + b_kv_k = 0$ where $b_j = -1$, this is a contradiction. Therefore, for each j, v_j is not a linear combination of the other vectors. Consequently, $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Example 4.3.4. *Let $v = [1\ 2\ 3]^T$ and $w = [1\ 0\ 0]^T$. Let's prove these are linearly independent. Assume that $c_1v + c_2w = 0$, this yields*

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

thus $c_1 + c_2 = 0$ and $2c_1 = 0$ and $3c_1 = 0$. We find $c_1 = c_2 = 0$ thus v, w are linearly independent. Alternatively, you could explain why there does not exist any $k \in \mathbb{R}$ such that $v = kw$

Think about this, if the set of vectors $\{v_1, v_2, \dots, v_k\} \subset \mathbb{F}^n$ is linearly independent then the equation $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ has the unique solution $c_1 = 0, c_2 = 0, \dots, c_k = 0$. Notice we can reformulate the problem as a matrix equation:

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \;\Leftrightarrow\; [v_1|v_2|\cdots|v_k][c_1\ c_2\ \cdots\ c_k]^T = 0$$

The matrix $[v_1|v_2|\cdots|v_k]$ is an $n \times k$. This is great. We can use the matrix techniques we already developed to probe for linear independence of a set of vectors.

Proposition 4.3.5.

Let $\{v_1, v_2, \dots, v_k\}$ be a set of vectors in $\mathbb{F}^n$.

1. If $rref[v_1|v_2|\cdots|v_k]$ has less than k pivot columns then the set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly dependent.
2. If $rref[v_1|v_2|\cdots|v_k]$ has k pivot columns then the set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Proof: Denote $V = [v_1|v_2|\cdots|v_k]$ and $c = [c_1, c_2, \dots, c_k]^T$. If V contains a linearly independent set of vectors then we must find that $Vc = 0$ implies $c = 0$. Consider $Vc = 0$, this is equivalent to using Gaussian elimination on the augmented coefficent matrix $[V|0]$. We know this system is consistent since $c = 0$ is a solution. Thus Theorem 3.4.12 tells us that there is either a unique solution or infinitely many solutions.

Clearly if the solution is unique then $c = 0$ is the only solution and hence the implication $Av = 0$ implies $c = 0$ holds true and we find the vectors are linearly independent. We find

$$rref[v_1|v_2|\cdots|v_k] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \left[\frac{I_k}{0}\right]$$

where there are n-rows in the matrix above. If $n = k$ then there would be no zero row.
If there are infinitely many solutions then there will be free variables in the solution of $Vc = 0$. If we set the free variables to 1 we then find that $Vc = 0$ does not imply $c = 0$ since at least the free variables are nonzero. Thus the vectors are linearly dependent in this case, proving (2.). $\square$

Before I get to the examples let me glean one more fairly obvious statement from the proof above:

Corollary 4.3.6.

If $\{v_1, v_2, \dots, v_k\}$ is a set of vectors in $\mathbb{F}^n$ and $k > n$ then the vectors are linearly dependent.

Proof: Proposition 4.3.5 tells us that the set is linearly independent if there are k pivot columns in $[v_1|\cdots|v_k]$. However, that is impossible since $k > n$ this means that there will be at least one column of zeros in rref$[v_1|\cdots|v_k]$. Therefore the vectors are linearly dependent. $\square$

This Proposition is obvious but useful. We may have at most 2 linearly independent vectors in $\mathbb{R}^2$, 3 in $\mathbb{R}^3$, 4 in $\mathbb{R}^4$, and so forth...

Example 4.3.7. *Determine if v_1, v_2, v_3 (given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.*

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

We seek to use the Proposition 4.3.5. Consider then,

$$[v_1|v_2|v_3] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{array}{c} \xrightarrow{r_2 - r_1} \\ \xrightarrow{r_3 - r_1} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \begin{array}{c} \xrightarrow{r_1 + 2r_2} \\ \xrightarrow{r_3 - 2r_2} \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we find that,

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

hence the variable c_3 is free in the solution of $Vc = 0$. We find solutions of the form $c_1 = -c_3$ and $c_2 = -c_3$. This means that

$$-c_3 v_1 - c_3 v_2 + c_3 v_3 = 0$$

for any value of c_3. I suggest $c_3 = 1$ is easy to plug in,

$$-v_1 - v_2 + v_3 = 0 \quad \text{or we could write} \quad v_3 = v_1 + v_2$$

We see clearly that v_3 is a linear combination of v_1, v_2.

Example 4.3.8. *Determine if v_1, v_2, v_3, v_4 (given below) are linearly independent or dependent.*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We seek to use the Proposition 4.3.5. Omitting details we find,

$$rref[v_1|v_2|v_3|v_4] = rref \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case no variables are free, the only solution is $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$ hence the set of vectors $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

Let's pause to reflect on the geometric meaning of the examples above.

Remark 4.3.9.

For two vectors the term "linearly dependent" can be taken quite literally: two vectors are linearly dependent if they point along the same line. For three vectors they are linearly dependent if they point along the same line or possibly lay in the same plane. When we get to four vectors we can say they are linearly dependent if they reside in the same volume, plane or line. I don't find the geometric method terribly successful for dimensions higher than two. However, it is neat to think about the geometric meaning of certain calculations in dimensions higher than 3. We can't even draw it but we can eulicidate all sorts of information with the mathematics of linear algebra.

Example 4.3.10. *Determine if v_1, v_2, v_3 (given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \end{bmatrix}$$

We seek to use the Proposition 4.3.5. Consider $[v_1|v_2|v_3] =$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & -3 \end{bmatrix} \xrightarrow{r_4 - 3r_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -9 & -9 \end{bmatrix} \begin{matrix} \xrightarrow{r_1 - 3r_2} \\ \xrightarrow{r_3 - 2r_2} \\ \xrightarrow{r_4 + 9r_2} \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = rref[V].$$

Hence the variable c_3 is free in the solution of $Vc = 0$. We find solutions of the form $c_1 = c_3$ and $c_2 = -c_3$. This means that

$$c_3 v_1 - c_3 v_2 + c_3 v_3 = 0$$

for any value of c_3. I suggest $c_3 = 1$ is easy to plug in,

$$v_1 - v_2 + v_3 = 0 \quad \textit{or we could write} \quad v_3 = v_2 - v_1$$

We see clearly that v_3 is a linear combination of v_1, v_2.

Example 4.3.11. *Determine if v_1, v_2, v_3, v_4 (given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.*

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

We seek to use the Proposition 4.3.5. Consider $[v_1|v_2|v_3|v_4] =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref[v_1|v_2|v_3|v_4].$$

Hence the variables c_3 and c_4 are free in the solution of $Vc = 0$. We find solutions of the form $c_1 = -c_3 + c_4$ and $c_2 = -c_3 - c_4$. This means that

$$(c_4 - c_3)v_1 - (c_3 + c_4)v_2 + c_3 v_3 + c_4 v_4 = 0$$

for any value of c_3 or c_4. I suggest $c_3 = 1, c_4 = 0$ is easy to plug in,

$$-v_1 - v_2 + v_3 = 0 \quad \textit{or we could write} \quad v_3 = v_2 + v_1$$

Likewise select $c_3 = 0, c_4 = 1$ to find

$$v_1 - v_2 + v_4 = 0 \quad \textit{or we could write} \quad v_4 = v_2 - v_1$$

We find that v_3 and v_4 are linear combinations of v_1 and v_2.

4.4 column correspondence property (CCP)

Recall that we used Proposition 4.3.5 in Examples 4.3.7, 4.3.8, 4.3.10 and 4.3.11 to ascertain the linear independence of certain sets of vectors. If you pay particular attention to those examples you may have picked up on a pattern. The columns of the $rref[v_1|v_2|\cdots|v_k]$ depend on each other in the same way that the vectors $v_1, v_2, \dots v_k$ depend on each other. These provide examples of the so-called **"column correspondence property"**. In a nutshell, the property says you can read the linear dependencies right off the $rref[v_1|v_2|\cdots|v_k]$.

Proposition 4.4.1. *Column Correspondence Property (CCP)*

Let $A = [col_1(A)|\cdots|col_n(A)] \in \mathbb{F}^{m\times n}$ and $R = rref[A] = [col_1(R)|\cdots|col_n(R)]$. There exist constants $c_1, c_2, \dots c_k$ such that $c_1col_1(A)+c_2col_2(A)+\cdots+c_kcol_k(A) = 0$ if and only if $c_1col_1(R)+c_2col_2(R)+\cdots+c_kcol_k(R) = 0$. If $col_j(rref[A])$ is a linear combination of other columns of $rref[A]$ then $col_j(A)$ is likewise the same linear combination of columns of A.

We prepare for the proof of the Proposition by establishing a sick[3] Lemma.

Lemma 4.4.2.

Let $A \in \mathbb{F}^{m\times n}$ then there exists an invertible matrix E such that $col_i(rref(A)) = Ecol_i(A)$ for all $i = 1, 2, \dots n$.

Proof of Lemma: Recall that there exist elementary matrices $E_1, E_2, \dots E_r$ whose product is E such that $EA = \text{rref}(A)$. Recall the concatenation proposition: $X[b_1|b_2|\cdots|b_k] = [Xb_1|Xb_2|\cdots|Xb_k]$. We can unravel the Gaussian elimination in the same way,

$$\begin{aligned} EA &= E[col_1(A)|col_2(A)|\cdots|col_n(A)] \\ &= [Ecol_1(A)|Ecol_2(A)|\cdots|Ecol_n(A)] \end{aligned}$$

Observe $EA = rref(A)$ thus $col_i(rref(A)) = Ecol_i(A)$ for all i. $\square$

Proof of Proposition: Suppose that there exist constants $c_1, c_2, \dots, c_k$ such that $c_1col_1(A) + c_2col_2(A)+\cdots+c_kcol_k(A) = 0$. By the Lemma we know there exists E such that $col_j(rref(A)) = Ecol_j(A)$. Multiply linear combination by E to find:

$$c_1Ecol_1(A) + c_2Ecol_2(A) + \cdots + c_kEcol_k(A) = 0$$

which yields

$$c_1col_1(rref(A)) + c_2col_2(rref(A)) + \cdots + c_kcol_k(rref(A)) = 0.$$

Likewise, if we are given a linear combination of columns of $rref(A)$ we can multiply by E^{-1} to recover the same linear combination of columns of A. $\square$

Example 4.4.3. *I will likely use the abbreviation "CCP" for column correspondence property. We could have deduced all the linear dependencies via the CCP in Examples 4.3.7,4.3.10 and 4.3.11. We found in 4.3.7 that*

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

[3] Sorry, is this so 2009 now ?

Obviously $col_3(R) = col_1(R) + col_2(R)$ *hence by CCP* $v_3 = v_1 + v_2$.
We found in 4.3.10 that

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By inspection, $col_3(R) = col_2(R) - col_1(R)$ *hence by CCP* $v_3 = v_2 - v_1$.
We found in 4.3.11 that

$$rref[v_1|v_2|v_3|v_4] = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection, $col_3(R) = col_1(R) + col_2(R)$ *hence by CCP* $v_3 = v_1 + v_2$. *Likewise by inspection,* $col_4(R) = col_2(R) - col_1(R)$ *hence by CCP* $v_4 = v_2 - v_1$.

You should notice that the CCP saves us the trouble of expressing how the constants c_i are related. If we are only interested in how the vectors are related the CCP gets straight to the point quicker. We should pause and notice another pattern here while were thinking about these things.

Proposition 4.4.4.

The non-pivot columns of a matrix can be written as linear combinations of the pivot columns and the pivot columns of the matrix are linearly independent.

Proof: Let A be a matrix. Notice the Proposition is clearly true for $rref(A)$. Hence, using Lemma 4.4.2 we find the same is true for the matrix A. $\square$

Proposition 4.4.5.

The rows of a matrix A can be written as linear combinations of the transposes of pivot columns of A^T. Furthermore, the set of all rows of A which are transposes of pivot columns of A^T is linearly independent.

Proof: Let A be a matrix and A^T its transpose. Apply Proposition 4.4.1 to A^T to find pivot columns which we denote by $col_{i_j}(A^T)$ for $j = 1, 2, \dots k$. The set of pivot columns for A^T are linearly independent and their span covers each column of A^T. Suppose,

$$c_1 row_{i_1}(A) + c_2 row_{i_2}(A) + \cdots + c_k row_{i_k}(A) = 0.$$

Take the transpose of the equation above, use Proposition 2.3.12 to simplify:

$$c_1(row_{i_1}(A))^T + c_2(row_{i_2}(A))^T + \cdots + c_k(row_{i_k}(A))^T = 0.$$

Recall $(row_j(A))^T = col_j(A^T)$ thus,

$$c_1 col_{i_1}(A^T) + c_2 col_{i_2}(A^T) + \cdots + c_k col_{i_k}(A^T) = 0.$$

hence $c_1 = c_2 = \cdots = c_k = 0$ as the pivot columns of A^T are linearly independent. This shows the corresponding rows of A are likewise linearly independent. The proof that each row of A is obtained from a span of $\{row_{i_1}(A), row_{i_2}(A), \dots, row_{i_k}(A)\}$ is similar. $\square$

Proposition 4.4.6.

If a particular column of a matrix is all zeros then it will be unchanged by the Gaussian elimination. Additionally, if we know $rref(A) = B$ then $rref[A|0] = [B|0]$ where 0 denotes one or more columns of zeros.

Proof: left to reader. $\square$

Example 4.4.7. *Use Example 3.3.9 and Proposition 4.4.6 to calculate,*

$$rref\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Similarly, use Example 3.3.13 and Proposition 4.4.6 to calculate:

$$rref\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 4.4.8.

Suppose that $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{m\times p}$ then the first n columns of $rref[A]$ and $rref[A|B]$ are identical.

Proof: The forward pass of the elimination proceeds from the leftmost-column to the rightmost-column. The matrices A and $[A|B]$ have the same n-leftmost columns thus the n-leftmost columns are identical after the forward pass is complete. The backwards pass acts on column at a time just clearing out above the pivots. Since the $ref(A)$ and $ref[A|B]$ have identical n-leftmost columns the backwards pass modifies those columns in the same way. Thus the n-leftmost columns of A and $[A|B]$ will be identical. $\square$

4.5 applications

We explore a few fun simple examples in this section. Of course, the applications of linear algebra are vast and I make no attempt to even survey them here. If you look at my older notes you'll find some circuit and chemistry and even traffic flow problems. I don't want to distract you from the math so I removed them.

Example 4.5.1. *Find a polynomial $P(x)$ whose graph $y = P(x)$ fits through the points $(0, -2.7)$, $(2, -4.5)$ and $(1, 0.97)$. We expect a quadratic polynomial will do nicely here: let A, B, C be the coefficients so $P(x) = Ax^2 + Bx + C$. Plug in the data,*

$$\begin{aligned} P(0) &= C = -2.7 \\ P(2) &= 4A + 2B + C = -4.5 \\ P(1) &= A + B + C = 0.97 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} A & B & C & \\ \hline 0 & 0 & 1 & -2.7 \\ 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 0.97 \end{array}\right]$$

I put in the A, B, C labels just to emphasize the form of the augmented matrix. We can then perform Gaussian elimination on the matrix (I omit the details) to solve the system,

$$rref \left[\begin{array}{ccc|c} 0 & 0 & 1 & -2.7 \\ 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 0.97 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4.52 \\ 0 & 1 & 0 & 8.14 \\ 0 & 0 & 1 & -2.7 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} A = -4.52 \\ B = 8.14 \\ C = -2.7 \end{array}$$

The requested polynomial is $\boxed{P(x) = -4.52x^2 + 8.14x - 2.7}$.

Example 4.5.2. *Find which cubic polynomial $Q(x)$ have a graph $y = Q(x)$ which fits through the points $(0, -2.7)$, $(2, -4.5)$ and $(1, 0.97)$. Let A, B, C, D be the coefficients of $Q(x) = Ax^3 + Bx^2 + Cx + D$. Plug in the data,*

$$\begin{aligned} Q(0) &= D = -2.7 \\ Q(2) &= 8A + 4B + 2C + D = -4.5 \\ Q(1) &= A + B + C + D = 0.97 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} A & B & C & D & \\ \hline 0 & 0 & 0 & 1 & -2.7 \\ 8 & 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 1 & 0.97 \end{array}\right]$$

I put in the A, B, C, D labels just to emphasize the form of the augmented matrix. We can then perform Gaussian elimination on the matrix (I omit the details) to solve the system,

$$rref \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -2.7 \\ 8 & 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 1 & 0.97 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & -0.5 & 0 & -4.07 \\ 0 & 1 & 1.5 & 0 & 7.69 \\ 0 & 0 & 0 & 1 & -2.7 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} A = -4.07 + 0.5C \\ B = 7.69 - 1.5C \\ C = C \\ D = -2.7 \end{array}$$

It turns out there is a whole family of cubic polynomials which will do nicely. For each $C \in \mathbb{R}$ the polynomial is $\boxed{Q_C(x) = (C - 4.07)x^3 + (7.69 - 1.5C)x^2 + Cx - 2.7}$ *fits the given points. We asked a question and found that it had infinitely many answers. Notice the choice $C = 4.07$ gets us back to the last example, in that case $Q_C(x)$ is not really a cubic polynomial.*

Definition 4.5.3.

Let $P \in \mathbb{R}^{\ n \times n}$ with $P_{ij} \geq 0$ for all i, j. If the sum of the entries in any column of P is one then we say P is a stochastic matrix.

Example 4.5.4. Stochastic Matrix: *A medical researcher*[4] *is studying the spread of a virus in 1000 lab. mice. During any given week it's estimated that there is an 80% probability that a mouse will overcome the virus, and during the same week there is an 10% likelyhood a healthy mouse will become infected. Suppose 100 mice are infected to start, (a.) how many sick next week? (b.) how many sick in 2 weeks ? (c.) after many many weeks what is the steady state solution?*

$$I_k = \text{infected mice at beginning of week } k \qquad P = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}$$
$$N_k = \text{noninfected mice at beginning of week } k$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = [I_k, N_k]$ *by the probability transition matrix* P *given above. Notice we are given that* $X_1 = [100, 900]^T$. *Calculate then,*

$$X_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 900 \end{bmatrix} = \begin{bmatrix} 110 \\ 890 \end{bmatrix}$$

After one week there are 110 infected mice Continuing to the next week,

$$X_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 110 \\ 890 \end{bmatrix} = \begin{bmatrix} 111 \\ 889 \end{bmatrix}$$

After two weeks we have 111 mice infected. What happens as $k \to \infty$*? Generally we have* $X_k = PX_{k-1}$. *Note that as* k *gets large there is little difference between* k *and* $k-1$, *in the limit they both tend to infinity. We define the steady-state solution to be* $X^* = \lim_{k\to\infty} X_k$. *Taking the limit of* $X_k = PX_{k-1}$ *as* $k \to \infty$ *we obtain the requirement* $X^* = PX^*$. *In other words, the steady state solution is found from solving* $(P - I)X^* = 0$. *For the example considered here we find,*

$$(P - I)X^* = \begin{bmatrix} -0.8 & 0.1 \\ 0.8 & -0.1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \qquad v = 8u \qquad X^* = \begin{bmatrix} u \\ 8u \end{bmatrix}$$

However, by conservation of mice, $u + v = 1000$ *hence* $9u = 1000$ *and* $u = 111.\bar{1}$ *thus the steady state can be shown to be* $X^* = [111.\bar{1}, 888.\bar{8}]$

Example 4.5.5. Diagonal matrices are nice: *Suppose that demand for doorknobs halves every week while the demand for yo-yos it cut to 1/3 of the previous week's demand every week due to an amazingly bad advertising campaign*[5]. *At the beginning there is demand for* 2 *doorknobs and* 5 *yo-yos.*

$$D_k = \text{demand for doorknobs at beginning of week } k \qquad P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$
$$Y_k = \text{demand for yo-yos at beginning of week } k$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = [D_k, Y_k]$ *by the transition matrix* P *given above. Notice we are given that* $X_1 = [2, 5]^T$. *Calculate then,*

$$X_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/3 \end{bmatrix}$$

[4] this example and most of the other applied examples in these notes are borrowed from my undergraduate linear algebra course taught from Larson's text by Dr. Terry Anderson of Appalachian State University

[5] insert your own more interesting set of quantities that doubles/halves or triples during some regular interval of time

Notice that we can actually calculate the k-th state vector as follows:

$$X_k = P^k X_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}^k \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k} & 0 \\ 0 & 3^{-k} \end{bmatrix}^k \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k+1} \\ 5(3^{-k}) \end{bmatrix}$$

Therefore, assuming this silly model holds for 100 weeks, we can calculate the 100*-the step in the process easily,*

$$X_{100} = P^{100} X_1 = \begin{bmatrix} 2^{-101} \\ 5(3^{-100}) \end{bmatrix}$$

Notice that for this example the analogue of X^* is the zero vector since as $k \to \infty$ we find X_k has components which both go to zero.

Example 4.5.6. Naive encryption: *in Example 3.7.6 we found observed that the matrix A has inverse matrix A^{-1} where:*

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & 2 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

We use the alphabet code

$$A = 1,\ B = 2,\ C = 3,\ \dots,\ Y = 25,\ Z = 26$$

and a space is encoded by 0*. The words are parsed into row vectors of length 3 then we multiply them by A on the right; $[decoded]A = [coded]$. Suppose we are given the string, already encoded by A*

$$[9,-1,-9], [38,-19,-19], [28,-9,-19], [-80,25,41], [-64,21,31], [-7,4,7].$$

Find the hidden message by undoing the multiplication by A. Simply multiply by A^{-1} on the right,

$$[9,-1,-9]A^{-1},\ [38,-19,-19]A^{-1},\ [28,-9,-19]A^{-1},$$
$$[-80,25,41]A^{-1},\ [-64,21,31]A^{-1},\ [-7,4,7]A^{-1}$$

This yields,

$$[19,19,0], [9,19,0], [3,1,14], [3,5,12], [12,5,4]$$

which reads C*LASS IS CANCELLED* [6].

If you enjoy this feel free to peruse my Math 121 notes, I have additional examples of this naive encryption. I say it's naive since real encryption has much greater sophistication by this time.

Example 4.5.7. Complex Numbers: *matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ multiply like complex numbers. For example, consider $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ observe $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$. This matrix plays the role of $i = \sqrt{-1}$ where $i^2 = -1$. Consider,*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} ax - by & -(ay + bx) \\ ay + bx & ax - by \end{bmatrix}$$

Recall, $(a + ib)(x + iy) = ax - by + i(ay + bx)$. These 2×2 matrices form a model of the complex number system. Many algebraic systems permit a representaion via some matrix model.[7]

[6] Larson's pg. 100-102 # 22

[7] Minh Nguyen, Bailu Zhang and Spencer Leslie worked with me to study the calculus over semisimple algebras. In that work, one important concept is the matrix formulation of the given algebra. I may have an open project which extends that work, ask if interested

Chapter 5

determinants

In this chapter we motivate the determinant of a matrix as a simple criteria to judge the invertibility of a given square matrix. Once the definition is settled we prove a series of useful proposition to simplify the computations of determinants. We explain the determinant arises from the abstraction of length, area and volume to n-volume[1]. In addition, the determinant serves to construct Cramer's Rule which gives us a formula to solve systems with a unique solution. Then, a formula for the inverse of a matrix itself is obtained via the transpose of matrix of cofactors rescaled by division of the determinant. Finally, we pause to again give a long list of equivalent conditions for invertibility or singularity of an $n \times n$ matrix. The determinant finds an important place on that list as there are many problems one can ask which are shockingly simple to answer with determinants and yet confound in the other approaches.

I should warn you there are some difficult calculations in this Chapter. However, the good news is these are primarily to justify the various properties of the determinant. I probably will not present these in lecture because the method used to prove them is not generally of interest in this course. Index manipulation and even the elementary matrix arguments are a means to an end in this chapter. That said, I do hope you read them so you can appreciate the nature of the tool when you use it. For example, when you solve a problem using $\det(AB) = \det(A)\det(B)$ you should realize that is a nontrivial algebraic step. That move carries with it the full force of the arguments we see in this chapter.

5.1 on the definition of the determinant

In this section I begin by studying the problem of invertibility of an arbitrary 2×2 matrix and we discover the necessity that a particular constant be nonzero. However, we find the number associated with the criterion of invertibility is not unique. In the next subsection, we tie geometry to the problem and discover the problem of a square matrix being invertible is naturally connected to the problem of the vectors which form its columns providing the edge-vectors of some n-dimensional cube. Ok, that's a cheat, the cube can be squished so it's actually an n-piped, the n-dimensional generalization of a parallelogram. Finally, having had enough motivation for one day, we state the full definition in glorious combinatorial or tensorial splendor. This is the most complicated object we study this semester. To calculate a determinant of an $n \times n$ matrix it generally requires on the

[1] a good slogan for the determinant is just this: the determinant gives the volume. Or more precisely, the determinant of a matrix is the volume subtended by the convex hull of its columns.

order of $n!$ steps, ouch. So, keep in mind, the later sections in this chapter to give computational short-cuts which we must harness.

5.1.1 criteria of invertibility

We have studied a variety of techniques to ascertain the invertibility of a given matrix. Recall, if A is an $n \times n$ invertible matrix then $Ax = b$ has a unique solution $x = A^{-1}b$. Alternatively, $\text{rref}(A) = I$. We now seek some explicit formula in terms of the components of A. Ideally this formula will **determine** if A is invertible or not.

The base case $n = 1$ has $A = a \in \mathbb{R}$ as we identify $\mathbb{R}^{1\times 1}$ with $\mathbb{R}$. The equation $ax = b$ has solution $x = b/a$ provided $a \neq 0$. Thus, the simple criteria in the $n = 1$ case is merely that $\boxed{a \neq 0}$.

The $n = 2$ case has $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We learned that the formula for the 2×2 inverse is:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The necessary and sufficient condition for invertibility here is just that $ad - bc \neq 0$. Of course, we could just as well insist $k(ad - bc) \neq 0$ for any nonzero scalar k. What makes $k = 1$ so special? This is motivated in our next subsection which borrows from your background in vector analysis[2]

5.1.2 determinants and geometry

For the sake of discussion, suppose we set $k = 1$ and define the **determinant** of a 2×2 matrix via the formula:

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Why should we prefer this definition to other possible choices, why choose $k = 1$? To answer this question we ought to examine the geometric signifcance of the formula. I'll argue that

$$\det[v_1|v_2] = \pm\text{Area}\,[\mathcal{P}(v_1, v_2)]$$

where $\mathcal{P}(v_1, v_2)$ denotes the parallelogram with sides v_1 and v_2 which is explicitly given as the point-set $\mathcal{P}(v_1, v_2) = \{c_1v_1 + c_2v_2 \mid c_1, c_2 \geq 0,\ 0 \leq c_1 + c_2 \leq 1\}$. Furthermore, the $\pm$ depends on the orientation of v_1 and v_2 in the following sense: suppose $v_1, v_2 \neq 0$ then

(i.) if the direction of v_2 is obtained by a CCW[3] rotation of v_1 then $\det[v_1|v_2] = \text{Area}\,[\mathcal{P}(v_1, v_2)]$

(ii.) if the direction of v_2 is obtained by a CW[4] of v_1 then $\det[v_1|v_2] = -\text{Area}\,[\mathcal{P}(v_1, v_2)]$.

To confirm my claims above, let's look at the particularly simple case of a rectangle with side-vectors $v_1 = (l, 0)$ and $v_2 = (0, w)$. These give us a rectangle with length l and width w. Notice

$$\det[v_1|v_2] = det\begin{bmatrix} l & 0 \\ 0 & w \end{bmatrix} = lw \qquad \text{whereas} \qquad \det[v_2|v_1] = det\begin{bmatrix} 0 & l \\ w & 0 \end{bmatrix} = -lw$$

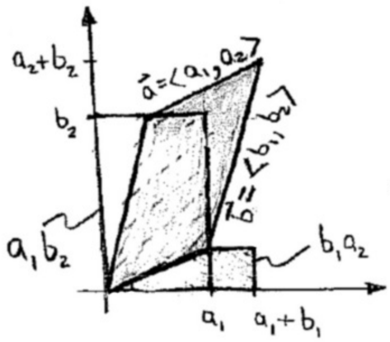
[2] I am aware some of you have been robbed of taking multivariate calculus and have instead spent time wallowing in psychobabble, my apologies, I hope my comments are reasonably self-contained here, feel free to strike up a conversation about vectors and geometry in office hours if you want to know more about what you missed in not taking multivariate calculus

[3] counter-clock-wise

[4] clock-wise rotation

Naturally, to go from $v_1 = (l, 0)$ to $v_2 = (0, w)$ we need a CCW rotation of 90^o. In contrast, to go from v_2 to v_1 we require a CW-rotation of 90^o. This is the reason for the $-$ in the left determinant above.

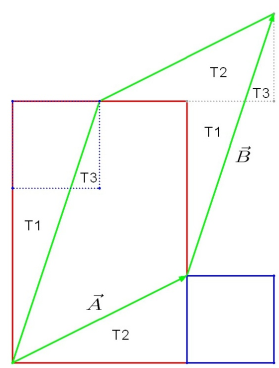
Interestingly this works for parallellograms with sides (a_1, a_2) and (b_1, b_2). Assuming the direction of (b_1, b_2) is obtained by a CCW rotation of (a_1, a_2) we can picture the parallelogram as follows:



$det[\vec{a}|\vec{b}] = det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1b_2 - b_1a_2.$

Note, the diagram $\Rightarrow$ $\underline{Area = a_1b_2 - b_1a_2}$.

Maybe you can see it better in the diagram below: the point is that triangles $T1$ and $T2$ match nicely but the $T3$ is included in the red rectangle but is excluded from the green parallelogram. Therefore, the area of the red rectangle A_1B_2 less the area of the blue square A_2B_1 is precisely the area of the green parallelogram.



Again, my diagram places $\vec{B}$ with a direction which is obtained from $\vec{A}$ by a CCW-rotation. Finally, note that $det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ corresponds to the fact that the unit-square has area 1. For $n = 3$ we should expect $det(I_3) = 1$ since $I = [e_1|e_2|e_3]$ gives the edges of the unit-cube which has volume 1. Notice, this means the sides $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ are somehow in the right **order**.

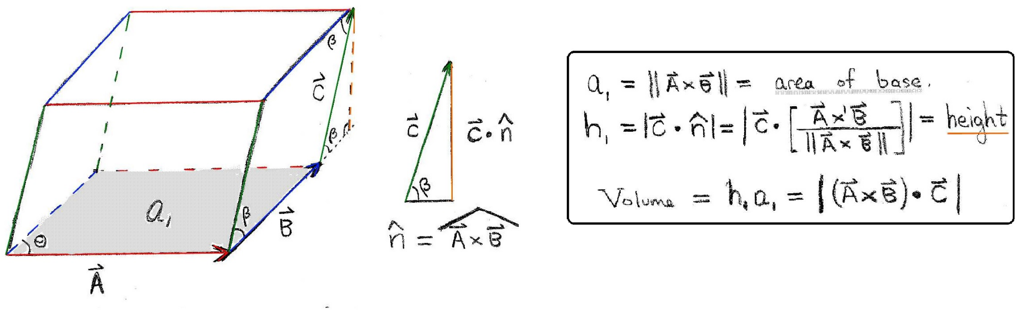
At this point we make use of some vector geometry. In the study of vectors in three dimensions we study the **dot** and **cross-products**. In particular:

$$\vec{B} \times \vec{C} = \langle B_2C_3 - B_3C_2, B_3C_1 - B_1C_3, B_1C_2 - B_2C_1\rangle \quad \& \quad \vec{A} \bullet \vec{V} = A_1V_1 + A_2V_2 + A_3V_3.$$

These are combined to form the **triple-product**:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1)$$

which calculates the **oriented volume** of the parallel-piped with edge-vectors $\vec{A}, \vec{B}, \vec{C}$.



Changing notation a bit,

$$\text{Volume}\,[\mathcal{P}(v_1, v_2, v_3)] = \pm v_1 \cdot (v_2 \times v_3)$$

where $\mathcal{P}(v_1, v_2, v_3)$ denotes the parallel-piped with sides v_1, v_2, v_3 which is explicitly given as the point-set $\mathcal{P}(v_1, v_2, v_3) = \{c_1v_1 + c_2v_2 + c_3v_3 \mid c_1, c_2, c_3 \geq 0,\ 0 \leq c_1 + c_2 + c_3 \leq 1\}$. The sign of this triple product is related to the **right-hand-rule**.

(i.) If $v_2 \times v_3$ points in same general direction as v_2 then $v_1 \cdot (v_2 \times v_3) > 0$ and we say $\{v_1, v_2, v_3\}$ is a right-handed set of vectors,

(ii.) If $v_2 \times v_3$ points opposite the general direction of v_2 then $v_1 \cdot (v_2 \times v_3) < 0$ and we say $\{v_1, v_2, v_3\}$ is a left-handed set of vectors,

The order in which we list the edge-vectors is crucial. If we switch the order of any two vector then it changes the sign of the triple-product. This corresponds to the fact that the determinant switches signs when we swap any two columns. Hence, we define:

$$det \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \vec{A} \cdot (\vec{B} \times \vec{C}).$$

We should expect these observations generalize. In particular, supposing we define the **convex hull** of the vectors $v_1, \dots, v_n$ to form the n-piped:

$$\mathcal{P}(v_1, \dots, v_n) = \{c_1v_1 + \cdots + c_nv_n \mid c_1, \dots, c_n \geq 0,\ 0 \leq c_1 + \cdots + c_n \leq 1\}$$

then $\det[v_1|\dots|v_n] = \pm\text{Volume}\,[\mathcal{P}(v_1, \dots, v_n)]$ where we expect

$$\det[e_1|\dots|e_n] = \text{Volume}\,[\mathcal{P}(e_1, \dots, e_n)] = 1$$

since the unit-n-cube has n-volume 1. Moreover, on the basis of our experience with $n = 2, 3$ we expect that interchanging any two columns ought to produces a change of sign in the determinant. So, if we can find a formula producing these features then we should be pleased to call it the **determinant**[5]. Furthermore, if a set of vectors $\{v_1, \dots, v_n\}$ is in **standard orientation** if

$$\det[v_1|\cdots|v_n] > 0$$

for example, the standard basis $e_1, \dots, e_n$ has the standard orientation. Anyway, we should go on as I think I have said more than enough to emphasize the interconnection between the determinant and the problem of calculating generalized volume.

[5]it can be shown the determinant is the unique alternating n-linear map which sends the standard n-basis to 1. I leave that for a different course. To better understand the mechanics of the formulas here we probably would do well to study the **wedge** product. For a linear algebra quick take on the wedge product you can look at Curtis' *Abstract Linear Algebra* text, which is a joy. Or, take Math 332 where we study wedge products in some detail

5.1.3 definition of determinant

The precise definition of the determinant is intrinsically combinatorial. A permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ is a bijection. Every permutation can be written as a product of an even or odd composition of transpositions. The $sgn(\sigma) = 1$ if σ is formed from an even product of transpositions. The $sgn(\sigma) = -1$ if σ is formed from an odd product of transpositions. The sum below is over all possible permutations,

$$\boxed{det(A) = \sum_{\sigma} sgn(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}}$$

this provides an explicit definition of the determinant. For example, in the $n = 2$ case we have $\sigma_o(x) = x$ or $\sigma_1(1) = 2, \sigma_1(2) = 1$. The sum over all permutations has just two terms in the $n = 2$ case,

$$det(A) = sgn(\sigma_o) A_{1\sigma_o(1)} A_{2\sigma_o(2)} + sgn(\sigma_1) A_{1\sigma_1(1)} A_{2\sigma_1(2)} = A_{11}A_{22} - A_{12}A_{21}$$

In the notation $A_{11} = a, A_{12} = b, A_{21} = c, A_{22} = d$ the formula above says $det(A) = ad - bc$.

Another way of writing the sum above is as a sum over many indicices which range over $1, 2, \dots, n$. For example, in the $n = 2$ case we use ϵ_{ij} to denote the **completely antisymmetric symbol** where $\epsilon_{12} = 1$ and $\epsilon_{21} = -1$ whereas $\epsilon_{11} = \epsilon_{22} = 0$. Hence,

$$det(A) = A_{11}A_{22} - A_{12}A_{21} = \epsilon_{12}A_{11}A_{22} + \epsilon_{21}A_{12}A_{21}.$$

We generalize this pattern in what follows:

Definition 5.1.1.

Let $\epsilon_{i_1 i_2 \dots i_n}$ be defined to be the completely antisymmetric symbol in n-indices. We define $\epsilon_{12\dots n} = 1$ then all other values are generated by demanding the interchange of any two indices is antisymmetric. This is also known as the **Levi-Civita** symbol. In view of this notation, we define the **determinant** of $A \in \mathbb{R}^{n \times n}$ as follows:

$$det(A) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1 i_1} A_{2 i_2} \cdots A_{n i_n}.$$

Direct implementation of the formula above is straightforward, but, tedious.

Example 5.1.2. *I prefer this definition. I can actually calculate it faster, for example the $n = 3$ case is pretty quick:*

$$\begin{aligned} det(A) = \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{12}A_{23}A_{31} + \epsilon_{312}A_{13}A_{21}A_{32} \\ + \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32} \end{aligned}$$

In principle there are 27 terms above but only these 6 are nontrivial because if any index is repeated the ϵ_{ijk} is zero. The only nontrivial terms are $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$. Thus,

$$\begin{aligned} det(A) = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} \\ - A_{13}A_{22}A_{31} - A_{12}A_{21}A_{33} - A_{11}A_{23}A_{32} \end{aligned}$$

There is a cute way to remember this formula by crossing diagonals in the matrix twice written.

Cute-tricks aside, we more often find it convenient to use Laplace's expansion by minor formulae to actually calculate explicit determinants. I'll postpone proof of the equivalence with the definition until Section 5.2 where you can see the considerable effort which is required to connect the formulas.[6] These formulas show you how to calculate determinants of $n \times n$ matrices as an alternating sum of $(n-1) \times (n-1)$ matrix determinants. I'll begin with the 2×2 case,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Then the 3×3 formula is:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

and finally the 4×4 determinant is given by

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = a \cdot \det \begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \cdot \det \begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix}$$
$$+ c \cdot \det \begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d \cdot \det \begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$

5.2 cofactor expansion for the determinant

The Levi-Civita definition of the determinant of an $n \times n$ matrix A is:

$$det(A) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1 i_1} A_{2 i_2} \cdots A_{n i_n}.$$

This is our definition for the determinant. All other facts flow from that source. In some other texts, the **cofactor expansion** of the determinant is given as the definition. I already recorded the standard cofactor expansions for determinants up to order 4 in the first section of this chapter. The aim of this section is to describe the general cofactor expansions and to prove they give another equivalent characterization of the determinant.

Definition 5.2.1.

Let $A = [A_{ij}] \in \mathbb{R}^{\,n \times n}$. The minor of A_{ij} is denoted M_{ij} which is defined to be the determinant of the $\mathbb{R}^{(n-1)\times(n-1)}$ matrix formed by deleting the i-th column and the j-th row of A. The (i,j)-th co-factor of A is $C_{ij} = (-1)^{i+j} M_{ij}$.

[6] those are probably the most difficult calculations contained in these notes.

Theorem 5.2.2.

The determinant of $A \in \mathbb{R}^{n \times n}$ can be calculated from a sum of cofactors either along any row or column;

1. $det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in}$ (i-th row expansion)
2. $det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj}$ (j-th column expansion)

Proof: I'll attempt to sketch a proof of (2.) directly from the general definition. Let's try to identify A_{1i_1} with A_{1j} then A_{2i_2} with A_{2j} and so forth, keep in mind that j is a fixed but arbitrary index, it is not summed over.

$$det(A) = \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n}$$

$$= \sum_{i_2,\dots,i_n} \epsilon_{j,i_2,\dots,i_n} A_{1j}A_{2i_2}\cdots A_{ni_n} + \sum_{i_1\neq j,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n}$$

$$\begin{aligned} &= \sum_{i_2,\dots,i_n} \epsilon_{j,i_2,\dots,i_n} A_{1j}A_{2i_2}\cdots A_{ni_n} + \sum_{i_1\neq j,i_3,\dots,i_n} \epsilon_{i_1,j,\dots,i_n} A_{1i_1}A_{2j}\cdots A_{ni_n} \\ &\quad + \cdots + \sum_{i_1\neq j,i_2\neq j,\dots,i_{n-1}\neq j} \epsilon_{i_1,i_2,\dots,i_{n-1},j} A_{1i_1}\cdots A_{n-1,i_{n-1}}A_{nj} \\ &\quad + \sum_{i_1\neq j,\dots,i_n\neq j} \epsilon_{i_1,\dots,i_n} A_{1i_1}A_{1i_2}\cdots A_{ni_n} \end{aligned}$$

Consider the summand. If all the indices $i_1, i_2, \dots i_n \neq j$ then there must be at least one repeated index in each list of such indices. Consequently the last sum vanishes since $\epsilon_{i_1,\dots,i_n}$ is zero if any two indices are repeated. We can pull out A_{1j} from the first sum, then A_{2j} from the second sum, and so forth until we eventually pull out A_{nj} out of the last sum.

$$\begin{aligned} det(A) = A_{1j}\Big(\sum_{i_2,\dots,i_n} \epsilon_{j,i_2,\dots,i_n} A_{2i_2}\cdots A_{ni_n}\Big) + A_{2j}\Big(\sum_{i_1\neq j,\dots,i_n} \epsilon_{i_1,j,\dots,i_n} A_{1i_1}\cdots A_{ni_n}\Big) + \cdots \\ + A_{nj}\Big(\sum_{i_1\neq j,i_2\neq j,\dots,j\neq i_{n-1}} \epsilon_{i_1,i_2,\dots,j} A_{1i_1}A_{2i_2}\cdots A_{n-1,i_{n-1}}\Big) \end{aligned}$$

The terms appear different, but in fact there is a hidden symmetry. If any index in the summations above takes the value j then the Levi-Civita symbol with have two j's and hence those terms are zero. Consequently we can just as well take all the sums over all values **except** j. In other words, each sum is a completely antisymmetric sum of products of $n-1$ terms taken from all columns except j. For example, the first term has an antisymmetrized sum of a product of $n-1$ terms not including column j or row 1.Reordering the indices in the Levi-Civita symbol generates a sign of $(-1)^{1+j}$ thus the first term is simply $A_{1j}C_{1j}$. Likewise the next summand is $A_{2j}C_{2j}$ and so forth until we reach the last term which is $A_{nj}C_{nj}$. In other words,

$$\boxed{det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj}}$$

The proof of (1.) is probably similar. We will soon learn that $det(A^T) = det(A)$ thus (2.) $\implies$ (1.). since the j-th row of A^T is the j-th columns of A.

All that remains is to show why $det(A) = det(A^T)$. Recall $(A^T)_{ij} = A_{ji}$ for all i, j, thus

$$det(A^T) = \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} (A^T)_{1i_1}(A^T)_{2i_2}\cdots(A^T)_{ni_n}$$

$$= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{i_11}A_{i_22}\cdots A_{i_nn}$$

$$= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n} = det(A)$$

to make the last step one need only see that both sums contain all the same terms just written in a different order. Let me illustrate explicitly how this works in the $n = 3$ case,

$$\begin{aligned} det(A^T) &= \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{21}A_{32}A_{13} + \epsilon_{312}A_{31}A_{12}A_{23} \\ &\quad + \epsilon_{321}A_{31}A_{22}A_{13} + \epsilon_{213}A_{21}A_{12}A_{33} + \epsilon_{132}A_{11}A_{32}A_{23} \end{aligned}$$

The I write the entries so the column indices go $1, 2, 3$

$$\begin{aligned} det(A^T) &= \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{13}A_{21}A_{32} + \epsilon_{312}A_{12}A_{23}A_{31} \\ &\quad + \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32} \end{aligned}$$

But, the indices of the Levi-Civita symbol are not in the right order yet. Fortunately, we have identities such as $\epsilon_{231} = \epsilon_{312}$ which allow us to reorder the indices without introducing any new signs,

$$\begin{aligned} det(A^T) &= \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{312}A_{13}A_{21}A_{32} + \epsilon_{231}A_{12}A_{23}A_{31} \\ &\quad + \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32} \end{aligned}$$

But, these are precisely the terms in $det(A)$ just written in a different order (see Example 5.1.2). Thus $det(A^T) = det(A)$. I leave the details of how to reorder the order n sum to the reader. $\square$

Remark 5.2.3.

Lay's text circumnavigates many of the difficulties I face in this chapter by using the co-factor definition as the definition of the determinant. One place you can also find a serious treatment of determinants is in *L*inear Algebra by Insel, Spence and Friedberg where you'll find the proof of the co-factor expansion is somewhat involved. However, the heart of the proof involves multilinearity. Multilinearity is practically manifest with our Levi-Civita definition. Anywho, a better definition for the determinant is as follows: **the determinant is the alternating, n-multilinear, real valued map such that** $det(I) = 1$. It can be shown this uniquely defines the determinant. All these other things like permutations and the Levi-Civita symbol are just notation.

Example 5.2.4. *I suppose it's about time for an example. Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

I usually calculate by expanding across the top row out of habit,

$$\begin{aligned} det(A) &= 1det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \\ &= 1(45-48) - 2(36-42) + 3(32-35) \\ &= -3 + 12 - 9 \\ &= 0. \end{aligned}$$

Now, we could also calculate by expanding along the middle row,

$$\begin{aligned} det(A) &= -4det\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 5det\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} - 6det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \\ &= -4(18-24) + 5(9-21) - 6(8-14) \\ &= 24 - 60 + 36 \\ &= 0. \end{aligned}$$

Many other choices are possible, for example expan along the right column,

$$\begin{aligned} det(A) &= 3det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} - 6det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} + 9det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \\ &= 3(32-35) - 6(8-14) + 9(5-8) \\ &= -9 + 36 - 27 \\ &= 0. \end{aligned}$$

which is best? Certain matrices might have a row or column of zeros, then it's easiest to expand along that row or column. Calculation completed, let's pause to appreciate the geometric significance. Our calculations show that the parallel piped spanned by $(1,2,3),(4,5,6),(7,8,9)$ *is flat, it's actually just a two-dimensional parallelogram.*

Example 5.2.5. *I choose the the column/row for the co-factor expansion to make life easy each time:*

$$\begin{aligned} det\begin{bmatrix} 0 & 1 & 0 & 2 \\ 13 & 71 & 5 & \pi \\ 0 & 3 & 0 & 4 \\ -2 & e & 0 & G \end{bmatrix} &= -5det\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ -2 & e & G \end{bmatrix} \\ &= -5(-2)det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= 10(4-6) \\ &= -20. \end{aligned}$$

Example 5.2.6. *Let's look at an example where we can exploit the co-factor expansion to greatly reduce the difficulty of the calculation. Let*

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 5 & 0 & 0 \\ 6 & 7 & 8 & 0 & 0 \\ 0 & 9 & 3 & 4 & 0 \\ -1 & -2 & -3 & 0 & 1 \end{bmatrix}$$

Begin by expanding down the 4-th column,

$$det(A) = (-1)^{4+4}M_{44} = 4det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 \\ 6 & 7 & 8 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

Next expand along the 2-row of the remaining determinant,

$$det(A) = (4)(5(-1)^{2+3}M_{23}) = -20det\begin{bmatrix} 1 & 2 & 4 \\ 6 & 7 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

Finish with the trick for 3×3 determinants, it helps me to write out

$$\left[\begin{array}{ccc|cc} 1 & 2 & 4 & 1 & 2 \\ 6 & 7 & 0 & 6 & 7 \\ -1 & -2 & 1 & -1 & -2 \end{array}\right]$$

then calculate the products of the three down diagonals and the three upward diagonals. Subtract the up-diagonals from the down-diagonals.

$$det(A) = -20(7 + 0 - 48 - (-28) - (0) - (12)) = -20(-25) = 500.$$

It is fun to note this is the 5-volume of the 5-piped region in $\mathbb{R}^5$ which has the columns of A as edges from a common vertex.

I will abstain from further geometric commentary for the most part in what follows. However, one last comment, it would be interesting to understand the geometric interpretation of the cofactor expansion. Note that it relates n-volumes to $(n-1)$-volumes.

5.3 properties of determinants

In this section we learn the most important properties of the determinant. A sequence of results born of elementary matrix arguments allows us to confirm that the motivating concept for the determinant is in fact true for arbitrary order; that is, Propositon 5.3.5 proves $\det(A) \neq 0$ iff A^{-1} exists. It is important that you appreciate how the results of this section are accumulated through a series of small steps, each building on the last. However, it is even more important that you learn how the results of this section can be applies to a variety of matrix problems. Your exercises will help you in that direction naturally.

The properties given in the proposition below are often useful to greatly reduce the difficulty of a determinant calculation.

Proposition 5.3.1.

Let $A \in \mathbb{R}^{\,n\times n}$,

1. $det(A^T) = det(A)$,
2. If there exists j such that $row_j(A) = 0$ then $det(A) = 0$,
3. $det[A_1|A_2|\cdots|aA_k + bB_k|\cdots A_n] = adet[A_1|\cdots|A_k|\cdots|A_n] + bdet[A_1|\cdots|B_k|\cdots|A_n]$,
4. $det(kA) = k^n det(A)$
5. if $B = \{A : r_k \leftrightarrow r_j\}$ then $det(B) = -det(A)$,
6. if $B = \{A : r_k + ar_j \rightarrow r_k\}$ then $det(B) = det(A)$,
7. if $row_i(A) = krow_j(A)$ for $i \neq j$ then $det(A) = 0$
8. each row property also holds for columns

where I mean to denote $r_k \leftrightarrow r_j$ as the row interchange and $r_k + ar_j \rightarrow r_k$ as a row addition and I assume $k < j$.

Proof: we already proved (1.) in the proof of the cofactor expansion Theorem 5.2.2. The proof of (2.) follows immediately from the cofactor expansion if we expand along the zero row or column. The proof of (3.) is not hard given our Levi-Civita defintion, let

$$C = [A_1|A_2|\cdots|aA_k + bB_k|\cdots|A_n]$$

Calculate from the definition,

$$\begin{aligned}
det(C) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} C_{1i_1}\cdots C_{ki_k}\cdots C_{ni_n} \\
&= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}\cdots(aA_{ki_k} + bB_{ki_k})\cdots A_{ni_n} \\
&= a\Big(\sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}\cdots A_{ki_k}\cdots A_{ni_n}\Big) \\
&\qquad + b\Big(\sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}\cdots B_{ki_k}\cdots A_{ni_n}\Big) \\
&= a\, det[A_1|A_2|\cdots|A_k|\cdots|A_n] + b\, det[A_1|A_2|\cdots|B_k|\cdots|A_n].
\end{aligned}$$

by the way,the property above is called multilinearity. The proof of (4.) is similar,

$$\begin{aligned}
det(kA) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} kA_{1i_1}kA_{2i_2}\cdots kA_{ni_n} \\
&= k^n \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n} \\
&= k^n\, det(A)
\end{aligned}$$

Let B be as in (5.), this means that $col_k(B) = col_j(A)$ and vice-versa,

$$\begin{aligned} det(B) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_k,\dots,i_j,\dots,i_n} A_{1i_1} \cdots A_{ji_k} \cdots A_{ki_j} \cdots A_{ni_n} \\ &= \sum_{i_1,i_2,\dots,i_n} -\epsilon_{i_1,\dots,i_j,\dots,i_k,\dots,i_n} A_{1i_1} \cdots A_{ji_k} \cdots A_{ki_j} \cdots A_{ni_n} \\ &= -det(A) \end{aligned}$$

where the minus sign came from interchanging the indices i_j and i_k.

To prove (6.) let us define B as in the Proposition: let $row_k(B) = row_k(A) + arow_j(A)$ and $row_i(B) = row_i(A)$ for $i \neq k$. This means that $B_{kl} = A_{kl} + aA_{jl}$ and $B_{il} = A_{il}$ for each l. Consequently,

$$\begin{aligned} det(B) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_k,\dots,i_n} A_{1i_1} \cdots (A_{ki_k} + aA_{ji_k}) \cdots A_{ni_n} \\ &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_n} A_{1i_1} \cdots A_{ki_k} \cdots A_{ni_n} \\ &\quad + a\Bigg(\sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_j,\dots,i_k,\dots,i_n} A_{1i_1} \cdots A_{j,i_j} \cdots A_{ji_k} \cdots A_{ni_n} \Bigg) \\ &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_n} A_{1i_1} \cdots A_{ki_k} \cdots A_{ni_n} \\ &= det(A). \end{aligned}$$

The term in parenthesis vanishes because it has the sum of an antisymmetric tensor in i_j, i_k against a symmetric tensor in i_j, i_k. Here is the pattern, suppose $S_{ij} = S_{ji}$ and $T_{ij} = -T_{ji}$ for all i, j then consider

$$\begin{aligned} \sum_i \sum_j S_{ij} T_{ij} &= \sum_j \sum_i S_{ji} T_{ji} && \text{switched indices} \\ &= \sum_j \sum_i -S_{ij} T_{ij} && \text{used sym. and antisym.} \\ &= -\sum_i \sum_j S_{ij} T_{ij} && \text{interchanged sums.} \end{aligned}$$

thus we have $\sum S_{ij} T_{ij} = -\sum S_{ij} T_{ij}$ which indicates the sum is zero. We can use the same argument on the pair of indices i_j, i_k in the expression since $A_{ji_j} A_{ji_k}$ is symmetric in i_j, i_k whereas the Levi-Civita symbol is antisymmetric in i_j, i_k.

We get (7.) as an easy consequence of (2.) and (6.), just subtract one row from the other so that we get a row of zeros. Finally, we obtain column-based versions of (2.) (5.), (7.) or (8.) by noting that row operations on A^T the transpose correspond to column operations on A and in view of (1.) we find column-versions of (2.) (5.) (7.) and (8.). $\square$

Proposition 5.3.2.

The determinant of a diagonal matrix is the product of the diagonal entries.

Proof: Use multilinearity on each row,

$$det\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1\, det\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \cdots = d_1 d_2 \cdots d_n det\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Thus $det(D) = d_1 d_2 \cdots d_n$ as claimed. $\square$

Proposition 5.3.3.

Let L be a lower triangular square matric and U be an upper triangular square matrix.

1. $det(L) = L_{11}L_{22}\cdots L_{nn}$
2. $det(U) = U_{11}U_{22}\cdots U_{nn}$

Proof: I'll illustrate the proof of (2.) for the 3×3 case. We use the co-factor expansion across the first column of the matrix to begin,

$$det\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = A_{11} det\begin{bmatrix} U_{22} & U_{23} \\ 0 & U_{33} \end{bmatrix} = U_{11}U_{22}U_{33}$$

The proof of the $n \times n$ case is essentially the same. For (1.) use the co-factor expansion across the top row of L, to get $det(L) = L_{11}C_{11}$. Not the submatrix for calculating C_{11} is again has a row of zeros across the top. We calculate $C_{11} = L_{22}C_{22}$. This continues all the way down the diagonal. We find $det(L) = L_{11}L_{22}\cdots L_{nn}$. $\square$

Proposition 5.3.4.

Let $A \in \mathbb{R}^{\,n\times n}$ and $k \neq 0 \in \mathbb{R}$,

1. $det(E_{r_i \leftrightarrow r_j}) = -1$,
2. $det(E_{kr_i \rightarrow r_i}) = k$,
3. $det(E_{r_i + br_j \rightarrow r_i}) = 1$,
4. for B square and E elementary,
$$det(EB) = det(E)det(B) \qquad \& \qquad det(BE) = det(B)det(E),$$
5. if $E_1, E_2, \dots, E_k$ are elementary then $det(E_1E_2\cdots E_k) = det(E_1)det(E_2)\cdots det(E_k)$

Proof: Proposition 5.6.2 shows us that $det(I) = 1$ since $I^{-1} = I$ (there are many easier ways to show that). Note then that $E_{r_i \leftrightarrow r_j}$ is a row-swap of the identity matrix thus by Proposition 5.3.1

we find $det(E_{r_i \leftrightarrow r_j}) = -1$. To prove (2.) we use multilinearity from Proposition 5.3.1. For (3.) we use multilinearity again to show that:

$$det(E_{r_i + br_j \to r_i}) = det(I) + bdet(E_{ij})$$

Again $det(I) = 1$ and since the unit matrix E_{ij} has a row of zeros we know by Proposition 5.3.1 $det(E_{ij}) = 0$.

To prove (4.) we use Proposition 5.3.1 multiple times in the arguments below. Let $B \in \mathbb{R}^{\ n \times n}$ and suppose E is an elementary matrix. If E is multiplication of a row by k then $det(E) = k$ from (2.). Also EB is the matrix B with some row multiplied by k. Use multilinearity to see that $det(EB) = kdet(B)$. Thus $det(EB) = det(E)det(B)$. If E is a row interchange then EB is B with a row swap thus $det(EB) = -det(B)$ and $det(E) = -1$ thus we again find $det(EB) = det(E)det(B)$. Finally, if E is a row addition then EB is B with a row addition and $det(EB) = det(B)$ and $det(E) = 1$ hence $det(EB) = det(E)det(B)$. Notice that (5.) follows by repeated application of (4.). $\square$

Proposition 5.3.5.

A square matrix A is invertible iff $det(A) \neq 0$.

Proof: recall there exist elementary matrices $E_1, E_2, \dots, E_k$ such that $rref(A) = E_1 E_2 \cdots E_k A$. Thus $det(rref(A)) = det(E_1)det(E_2) \cdots det(E_k)det(A)$. Either $det(rref(A)) = 0$ and $det(A) = 0$ or they are both nonzero.

Suppose A is invertible. Then $Ax = 0$ has a unique solution and thus $rref(A) = I$ hence $det(rref(A)) = 1 \neq 0$ implying $det(A) \neq 0$.

Conversely, suppose $det(A) \neq 0$, then $det(rref(A)) \neq 0$. But this means that $rref(A)$ does not have a row of zeros. It follows $rref(A) = I$. Therefore $A^{-1} = E_1 E_2 \cdots E_k$. $\square$

Consider this, if A^{-1} is not invertible then $Ax = 0$ has a nontrivial solution. It follows that at least one column of A can be written as a linear combination of the remaining columns. Therefore, by (3.) and the column version of (7.) of Proposition 5.3.4 we conclude $\det(A) = 0$.

Proposition 5.3.6.

If $A, B \in \mathbb{F}^{n \times n}$ then $det(AB) = det(A)det(B)$.

Proof: Suppose B is invertible. Then there exist elementary matrices $E_1, E_2, \dots, E_k$ for which $B = E_1 \cdots E_k$. Thus, by repeated application of parts (4.) of Proposition 5.3.4 we calculate:

$$\begin{aligned} det(AB) &= det(AE_1 \cdots E_{k-1}E_k) \\ &= det(AE_1 \cdots E_{k-1})det(E_k) \qquad \text{continuing in this fashion,} \\ &= det(A)det(E_1) \cdots det(E_{k-1})det(E_k) \end{aligned}$$

Thus, as $det(E_1)det(E_2) \cdots det(E_k) = det(E_1 E_2 \cdots E_k) = det(B)$ by (5.) of Proposition 5.3.4 we find $det(AB) = det(A)det(B)$.

Suppose B is singular. Thus, there exists $x \neq 0$ for which $Bx = 0$ hence $ABx = 0$ and we find AB is not invertible. Hence $det(AB) = 0$ and $det(B) = 0$ and we are once more able to conclude $det(AB) = det(A)det(B)$. $\square$

Proposition 5.3.7.

If $A \in \mathbb{R}^{\ n\times n}$ is invertible then $det(A^{-1}) = \frac{1}{det(A)}$.

Proof: If A is invertible then there exists $A^{-1} \in \mathbb{R}^{\ n\times n}$ such that $AA^{-1} = I$. Apply Proposition 5.3.6 to see that

$$det(AA^{-1}) = det(A)det(A^{-1}) = det(I) \ \Rightarrow \ det(A)det(A^{-1}) = 1.$$

Thus, $det(A^{-1}) = 1/det(A)$ $\square$

Many of the properties we used to prove $det(AB) = det(A)det(B)$ are easy to derive if you were simply given the assumption $det(AB) = det(A)det(B)$. When you look at what went into the proof of Proposition 5.3.6 it's not surprising that $det(AB) = det(A)det(B)$ is a powerful formula to know.

Proposition 5.3.8.

If A is block-diagonal with square blocks $A_1, A_2, \dots, A_k$ then

$$det(A) = det(A_1)det(A_2)\cdots det(A_k).$$

Proof: for a 2×2 matrix this is clearly true since a block diagonal matrix is simply a diagonal matrix. In the 3×3 nondiagonal case we have a 2×2 block A_1 paired with a single diagonal entry A_2. Simply apply the cofactor expansion on the row of the diagonal entry to find that $det(A) = A_2 det(A_1) = det(A_2)det(A_1)$. For a 4×4 we have more cases but similar arguments apply. I leave the general proof to the reader. $\square$

Example 5.3.9. *If* $M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$ *is a block matrix where* A, B *are square blocks then* $det(M) = det(A)det(B)$.

5.4 examples of determinants

In the preceding section we saw the derivation of determinant properties requires some effort. Thankfully, the use of the properties to solve problems typically takes much less effort.

Example 5.4.1. *Notice that row* 2 *is twice row* 1*,*

$$det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0.$$

Example 5.4.2. *To calculate this one we make a single column swap to get a diagonal matrix. The determinant of a diagonal matrix is the product of the diagonals, thus:*

$$det \begin{bmatrix} 0 & 6 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = -det \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = -48.$$

Example 5.4.3. *Find the values of λ such that the matrix $A - \lambda I$ is singular given that*

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The matrix $A - \lambda I$ is singular iff $det(A - \lambda I) = 0$,

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} 1-\lambda & 0 & 2 & 3 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} \\ &= (3-\lambda) det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 1 & \lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda) det \begin{bmatrix} 1-\lambda & 0 \\ 1 & \lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)(1-\lambda)(-\lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3) \end{aligned}$$

Thus we need $\lambda = 0, 1, 2$ or 3 in order that $A - \lambda I$ be a noninvertible matrix. These values are called the **eigenvalues** *of A. We will have much more to say about that later.*

Example 5.4.4. *Suppose we are given the LU-factorization of a particular matrix (borrowed from the text by Spence, Insel and Friedberg see Example 2 on pg. 154-155.)*

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = LU$$

The LU-factorization is pretty easy to find, we do not study it directly in these notes[7]*. It is an important topic if you delve into serious numerical work where you need to write your own code and so forth. Note that L, U are triangular so we can calculate the determinant with ease:*

$$det(A) = det(L)det(U) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 = 4.$$

From a numerical perspective, the LU-factorization is a superior method for calculating $det(A)$ as compared to the co-factor expansion. It has much better "convergence" properties. Incidentally, you might read Insel Spence and Friedberg's Elementary Linear Algebra for more discussion of algorithmics.

[7] there are many additional techniques of matrix theory concerning various special ways to factor a matrix. I can recommend some reading past this course if you are interested.

Example 5.4.5. *Recall that the columns in A are linearly independent iff $Ax = 0$ has only the $x = 0$ solution. We also found that the existence of A^{-1} was equivalent to that claim in the case A was square since $Ax = 0$ implies $A^{-1}Ax = A^{-1}0 = 0$ hence $x = 0$. In Proposition 5.3.5 we proved $det(A) \neq 0$ iff A^{-1} exists. Thus the following check for $A \in \mathbb{R}^{\ n\times n}$ is nice to know:*

$$\boxed{\textit{columns of } A \textit{ are linearly independent} \quad \Leftrightarrow \quad det(A) \neq 0.}$$

Observe that this criteria is only useful if we wish to examine the linear independence of preciely n-vectors in $\mathbb{R}^n$. For example, $(1, 1, 1), (1, 0, 1), (2, 1, 2) \in \mathbb{R}^3$ have

$$det \left[\begin{array}{c|c|c} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{array} \right] = 0.$$

Therefore, $\{(1, 1, 1), (1, 0, 1), (2, 1, 2)\}$ form a linearly dependent set of vectors.

A natural curiousity, what about less than n-vectors? Is there some formula for that? Is there some formula we can plug say k-vectors into to ascertain the LI of those k-vectors? The answer is given by the **wedge product**. In short, if $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$ then $\{v_1, v_2, \dots, v_k\}$ is LI. This ties in with determinants at order $k = n$ by the beautiful formula: for n-vectors in $\mathbb{R}^n$,

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n = \det[v_1|v_2|\cdots|v_n] e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

The wedge product is an algebraic structure which can be built over any finite dimensional vector space. The external direct sum of all possible wedge products of vectors in V gives $\Omega(V)$ the $2^{\dim(V)}$-dimensional **exterior algebra** of V. For example, $V = \mathbb{R}^2$ has $\Omega(V) = \text{span}\{1, e_1, e_2, e_1 \wedge e_2\}$. If you'd like to know more about this algebra and how it extends and clarifies calculus III to calculus on n-dimensional space then you might read my advanced calculus Lecture notes. Another nice place to read more about these things from a purely linear-algebraic perspective is the text *Abstract Linear Algebra* by Morton L. Curtis.

5.5 Cramer's Rule

The numerical methods crowd seem to think this is a loathsome brute. It is an incredibly clumsy way to calculate the solution of a system of equations $Ax = b$. Moreover, Cramer's rule fails in the case $det(A) = 0$ so it's not nearly as general as our other methods. However, it does help calculate the variation of parameters formulas in differential equations so it is still of theoretical interest at a minimum. Students sometimes like it because it gives you a *formula* to find the solution. Students sometimes incorrectly jump to the conclusion that a formula is easier than say a *method*. It is certainly wrong here, the method of Gaussian elimination beats Cramer's rule by just about every objective criteria in so far as concrete numerical examples are concerned.

Proposition 5.5.1.

If $Ax = b$ is a linear system of equations with $x = [x_1 \ x_2 \ \cdots \ x_n]^T$ and $A \in \mathbb{F}^{n\times n}$ such that $det(A) \neq 0$ then we find solutions

$$x_1 = \frac{det(A_1)}{det(A)}, \quad x_2 = \frac{det(A_2)}{det(A)}, \quad \dots, \quad x_n = \frac{det(A_n)}{det(A)}$$

where we define A_k to be the $n \times n$ matrix obtained by replacing the k-th column of A by the inhomogeneous term b.

Proof: Since $det(A) \neq 0$ we know that $Ax = b$ has a unique solution. Suppose $x_j = \frac{det(A_j)}{det(A)}$ where $A_j = [col_1(A)|\cdots|col_{j-1}(A)|b|col_{j+1}(A)|\cdots|col_n(A)]$. We seek to show $x = [x_j]$ is a solution to $Ax = b$. Notice that the n-vector equations

$$Ae_1 = col_1(A), \dots, Ae_{j-1} = col_{j-1}(A), Ae_{j+1} = col_{j+1}(A), \dots, Ae_n = col_n(A), Ax = b$$

can be summarized as a single matrix equation:

$$A[e_1|\dots|e_{j-1}|x|e_{j+1}|\cdots|e_n] = \underbrace{[col_1(A)|\cdots|col_{j-1}(A)|b|col_{j+1}(A)|\cdots|col_n(A)]}_{\text{this is precisely } A_j} = A_j$$

Notice that if we expand on the j-th column it's obvious that

$$det[e_1|\dots|e_{j-1}|x|e_{j+1}|\cdots|e_n] = x_j$$

Returning to our matrix equation, take the determinant of both sides and use that the product of the determinants is the determinant of the product to obtain:

$$det(A)x_j = det(A_j)$$

Since $det(A) \neq 0$ it follows that $x_j = \frac{det(A_j)}{det(A)}$ for all j. $\square$

This is the proof that is given in Lay's text. The construction of the matrix equation is not really an obvious step in my estimation. Whoever came up with this proof originally realized that he would need to use the determinant product identity to overcome the subtlety in the proof. Once you realize that then it's natural to look for that matrix equation. This is a clever proof[8]

Example 5.5.2. *Solve $Ax = b$ given that*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

where $x = [x_1 \ x_2]^T$. Apply Cramer's rule, note $det(A) = 2$,

$$x_1 = \frac{1}{2} det \begin{bmatrix} 1 & 3 \\ 5 & 8 \end{bmatrix} = \frac{1}{2}(8 - 15) = \frac{-7}{2}.$$

and,

$$x_2 = \frac{1}{2} det \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} = \frac{1}{2}(5 - 2) = \frac{3}{2}.$$

The original system of equations would be $x_1 + 3x_2 = 1$ and $2x_1 + 8x_2 = 5$. As a quick check we can substitute in our answers $x_1 = -7/2$ and $x_2 = 3/2$ and see if they work.

Please note: the following two examples are for breadth of exposition.

[8] as seen from my humble vantage point naturally

Example 5.5.3. *An nonhomogeneous system of linear, constant coefficient ordinary differential equations can be written as a matrix differential equation:*

$$\frac{dx}{dt} = Ax + f$$

It turns out we'll be able to solve the homogeneous system $dx/dt = Ax$ via something called the matrix exponential. Long story short, we'll find n-solutions which we can concatenate into one big matrix solution X. To solve the given nonhomogeneous problem one makes the ansatz that $x = Xv$ is a solution for some yet unknown vector of functions. Then calculus leads to the problem of solving

$$X\frac{dv}{dt} = f$$

where X is matrix of functions, dv/dt and f are vectors of functions. X is invertible so we expect to find a unique solution dv/dt. Cramer's rule says,

$$\left(\frac{dv}{dt}\right)_i = \frac{1}{det(X)} det[\vec{x}_1|\cdots|g|\cdots|\vec{x}_n] = \frac{W_i[f]}{det(X)} \quad \textit{defining } W_i \textit{ in the obvious way}$$

For each i we integrate the equation above,

$$v_i(t) = \int \frac{W_i[f]dt}{det(X)}.$$

The general solution is thus,

$$x = Xv = X\left[\int \frac{W_i[f]dt}{det(X)}\right].$$

The first component of this formula justifies n-th order variation of parameters. For example in the $n = 2$ case you may have learned that $y_p = y_1v_1 + y_2v_2$ solves $ay'' + by' + cy = g$ if

$$v_1 = \int \frac{-gy_2dt}{a(y_1y_2' - y_2y_1')} \qquad v_2 = \int \frac{gy_1dt}{a(y_1y_2' - y_2y_1')}$$

These come from the general result above. Notice that these formulas need $y_1y_2' - y_2y_1' \neq 0$. This is precisely the **Wronskian** *$W[y_1, y_2] = y_1y_2' - y_2y_1'$ of the fundamental solutions y_1, y_2. It turns out that the Wronskian is nonzero for fundamental solutions thus the formulas above are entirely general.*

The example that follows is borrowed from my 2013 Advanced Calculus notes. Here I used Cramer's Rule to solve for differentials of the dependent variables.

Example 5.5.4. *Suppose $x+y+z+w = 3$ and $x^2-2xyz+w^3 = 5$.* **Calculate partial derivatives of z and w with respect to the independent variables x, y.** *Solution: we begin by calculation of the differentials of both equations:*

$$\begin{aligned}&dx + dy + dz + dw = 0\\&(2x - 2yz)dx - 2xzdy - 2xydz + 3w^2dw = 0\end{aligned}$$

We can solve for (dz, dw). In this calculation we can treat the differentials as formal variables.

$$\begin{aligned}&dz + dw = -dx - dy\\&-2xydz + 3w^2dw = -(2x - 2yz)dx + 2xzdy\end{aligned}$$

I find matrix notation is often helpful,

$$\left[\begin{array}{cc} 1 & 1 \\ -2xy & 3w^2 \end{array}\right]\left[\begin{array}{c} dz \\ dw \end{array}\right] = \left[\begin{array}{c} -dx-dy \\ -(2x-2yz)dx+2xzdy \end{array}\right]$$

Use Cramer's rule, multiplication by inverse, substitution, adding/subtracting equations etc... whatever technique of solving linear equations you prefer. Our goal is to solve for dz and dw in terms of dx and dy. I'll use Cramer's rule this time:

$$dz = \frac{det\left[\begin{array}{c|c} -dx-dy & 1 \\ -(2x-2yz)dx+2xzdy & 3w^2 \end{array}\right]}{det\left[\begin{array}{cc} 1 & 1 \\ -2xy & 3w^2 \end{array}\right]} = \frac{3w^2(-dx-dy)+(2x-2yz)dx-2xzdy}{3w^2+2xy}$$

Collecting terms,

$$dz = \left(\frac{-3w^2+2x-2yz}{3w^2+2xy}\right)dx + \left(\frac{-3w^2-2xz}{3w^2+2xy}\right)dy$$

From the expression above we can read various implicit derivatives,

$$\boxed{\left(\frac{\partial z}{\partial x}\right)_y = \frac{-3w^2+2x-2yz}{3w^2+2xy} \qquad \& \qquad \left(\frac{\partial z}{\partial y}\right)_x = \frac{-3w^2-2xz}{3w^2+2xy}}$$

The notation above indicates that z is understood to be a function of independent variables x, y. $\left(\frac{\partial z}{\partial x}\right)_y$ means we take the derivative of z with respect to x while holding y fixed. The appearance of the dependent variable w can be removed by using the equations $G(x, y, z, w) = (3, 5)$. Similar ambiguities exist for implicit differentiation in calculus I. Apply Cramer's rule once more to solve for dw:

$$dw = \frac{det\left[\begin{array}{c|c} 1 & -dx-dy \\ -2xy & -(2x-2yz)dx+2xzdy \end{array}\right]}{det\left[\begin{array}{cc} 1 & 1 \\ -2xy & 3w^2 \end{array}\right]} = \frac{-(2x-2yz)dx+2xzdy-2xy(dx+dy)}{3w^2+2xy}$$

Collecting terms,

$$dw = \left(\frac{-2x+2yz-2xy}{3w^2+2xy}\right)dx + \left(\frac{2xzdy-2xydy}{3w^2+2xy}\right)dy$$

We can read the following from the differential above:

$$\boxed{\left(\frac{\partial w}{\partial x}\right)_y = \frac{-2x+2yz-2xy}{3w^2+2xy} \qquad \& \qquad \left(\frac{\partial w}{\partial y}\right)_x = \frac{2xzdy-2xydy}{3w^2+2xy}.}$$

5.6 adjoint matrix

In this section we derive a general formula for the inverse of an $n \times n$ matrix. We already saw this formula in the 2×2 case and I work it out for the 3×3 case later in this section. As with Cramer's Rule, the results of this section are not to replace our earlier row-reduction based algorithms. Instead, these simply give us another tool, another view to answer questions concerning inverses.

Definition 5.6.1.

Let $A \in \mathbb{R}^{\,n\times n}$ the the matrix of cofactors is called the **adjoint** of A. It is denoted $adj(A)$ and is defined by and $adj(A)_{ij} = C_{ij}$ where C_{ij} is the (i,j)-th cofactor.

I'll keep it simple here, lets look at the 2×2 case:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has cofactors $C_{11} = (-1)^{1+1}det(d) = d$, $C_{12} = (-1)^{1+2}det(c) = -c$, $C_{21} = (-1)^{2+1}det(b) = -b$ and $C_{22} = (-1)^{2+2}det(a) = a$. Collecting these results,

$$adj(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

This is interesting. Recall we found a formula for the inverse of A (if it exists). The formula was

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice that $det(A) = ad - bc$ thus in the 2×2 case the relation between the inverse and the adjoint is rather simple:

$$\boxed{A^{-1} = \frac{1}{det(A)}adj(A)^T}$$

In fact, this is true for all n,

Proposition 5.6.2.

If $A \in \mathbb{R}^{n\times n}$ is invertible then $A^{-1} = \frac{1}{det(A)}adj(A)^T$.

Proof I: Calculate the product of A and $adj(A)^T$,

$$Aadj(A)^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The (i,j)-th component of the product above is

$$(Aadj(A)^T)_{ij} = A_{i1}C_{j1} + A_{i2}C_{j2} + \cdots + A_{in}C_{jn}.$$

Suppose that $i = j$ then the sum above is precisely the i-th row co-factor expansion for $det(A)$:

$$(Aadj(A)^T)_{ij} = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} = det(A)$$

If $i \neq j$ then the sum vanishes. Why ? Consider, if $i \neq j$, you're looking at a cofactor expansion of the determinant of A with the j-th row replaced by the i-th row. Since this matrix has a repeated row, its determinant is zero[9] $\square$

[9]adding this used to be worth bonus points, sadly for you students, Bill took your points by adding this nice observation.

Proof II: To find the inverse of A we need only apply Cramer's rule to solve the equations implicit within $AA^{-1} = I$. Let $A^{-1} = [v_1|v_2|\cdots|v_n]$ we need to solve

$$Av_1 = e_1, \qquad Av_2 = e_2, \qquad \dots \qquad Av_n = e_n$$

Cramer's rule gives us $(v_1)_j = \frac{C_{1j}}{det(A)}$ where $C_{1j} = (-1)^{1+j}M_{ij}$ is the cofactor formed from deleting the first row and j-th column. Apply Cramer's rule to deduce the j-component of the i-th column in the inverse $(v_i)_j = \frac{C_{ij}}{det(A)}$. Therefore, $col_i(A^{-1})_j = (A^{-1})_{ji} = \frac{C_{ij}}{det(A)}$. By definition $adj(A) = [C_{ij}]$ hence $adj(A)^T_{ij} = C_{ji}$ and it follows that $A^{-1} = \frac{1}{det(A)}adj(A)^T$. $\square$

Example 5.6.3. *Let's calculate the general formula for the inverse of a 3×3 matrix. Assume it exists for the time being. (the criteria for the inverse existing is staring us in the face everywhere here). Let*

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Calculate the cofactors,

$$C_{11} = det\left[\begin{smallmatrix} e & f \\ h & i \end{smallmatrix}\right] = ei - fh,$$

$$C_{12} = -det\left[\begin{smallmatrix} d & f \\ g & i \end{smallmatrix}\right] = fg - di,$$

$$C_{13} = det\left[\begin{smallmatrix} d & e \\ g & h \end{smallmatrix}\right] = dh - eg,$$

$$C_{21} = -det\left[\begin{smallmatrix} b & c \\ h & i \end{smallmatrix}\right] = ch - bi,$$

$$C_{22} = det\left[\begin{smallmatrix} a & c \\ g & i \end{smallmatrix}\right] = ai - cg,$$

$$C_{23} = -det\left[\begin{smallmatrix} a & b \\ g & h \end{smallmatrix}\right] = bg - ah,$$

$$C_{31} = det\left[\begin{smallmatrix} b & c \\ e & f \end{smallmatrix}\right] = bf - ce,$$

$$C_{32} = -det\left[\begin{smallmatrix} a & c \\ d & f \end{smallmatrix}\right] = cd - af,$$

$$C_{33} = det\left[\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right] = ae - bd.$$

Hence the transpose of the adjoint is

$$adj(A)^T = \left[\begin{array}{c|c|c} ei - fh & fg - di & dh - eg \\ \hline ch - bi & ai - cg & bg - ah \\ \hline bf - ce & cd - af & ae - bd \end{array}\right]$$

Thus, using the $A^{-1} = det(A)adj(A)^T$

$$\boxed{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{aei + bfg + cdh - gec - hfa - idb}\left[\begin{array}{c|c|c} ei - fh & ch - bi & bf - ce \\ \hline fg - di & ai - cg & cd - af \\ \hline dh - eg & bg - ah & ae - bd \end{array}\right]}$$

You should notice that are previous method for finding A^{-1} is far superior to this method. It required much less calculation. Let's check my formula in the case $A = 3I$, this means $a = e = i = 3$ and the others are zero.

$$I^{-1} = \frac{1}{27}\left[\begin{array}{c|c|c} 9 & 0 & 0 \\ \hline 0 & 9 & 0 \\ \hline 0 & 0 & 9 \end{array}\right] = \frac{1}{3}I$$

This checks, $(3I)(\frac{1}{3}I) = \frac{3}{3}II = I$. I do not recommend that you memorize this formula to calculate inverses for 3×3 matrices.

5.7 applications

The determinant is a convenient mnemonic to create expressions which are antisymmetric. The key property is that if we switch a row or column it creates a minus sign. This means that if any two rows are repeated then the determinant is zero. Notice this is why the cross product of two vectors is naturally phrased in terms of a determinant. The antisymmetry of the determinant insures the formula for the cross-product will have the desired antisymmetry. In this section we examine a few more applications for the determinant.

Example 5.7.1. *The Pauli's exclusion principle in quantum mechanics states that the wave function of a system of fermions is antisymmetric. Given N-electron wavefunctions $\chi_1, \chi_2, \dots, \chi_N$ the following is known as the* **Slater Determinant**

$$\Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = det\begin{bmatrix} \chi_1(\vec{r}_1) & \chi_2(\vec{r}_1) & \cdots & \chi_N(\vec{r}_1) \\ \chi_1(\vec{r}_2) & \chi_2(\vec{r}_2) & \cdots & \chi_N(\vec{r}_2) \\ \vdots & \vdots & \dots & \vdots \\ \chi_1(\vec{r}_N) & \chi_2(\vec{r}_N) & \cdots & \chi_N(\vec{r}_N) \end{bmatrix}$$

Notice that $\Psi(\vec{r}_1, \vec{r}_1, \dots, \vec{r}_N) = 0$ and generally if any two of the position vectors $\vec{r}_i = \vec{r}_j$ then the total wavefunction $\Psi = 0$. In quantum mechanics the wavefunction's modulus squared gives the probability density of finding the system in a particular circumstance. In this example, the fact that any repeated entry gives zero means that no two electrons can share the same position. This is characteristic of particles with half-integer **spin**, *such particles are called* **fermions**. *In contrast,* **bosons** *are particles with integer spin and they can occupy the same space. For example, light is made of photons which have spin 1 and in a laser one finds many waves of light traveling in the same space.*

Example 5.7.2. *Here's an application to differential equations. Suppose you want a second order linear ODE $L[y] =$ for which a given pair of functions y_1, y_2 are solutions. A simple way to express the desired equation is $L[y] = 0$ where*

$$L[y] = det\begin{bmatrix} y & y' & y'' \\ y_1 & y_1' & y_1'' \\ y_2 & y_2' & y_2'' \end{bmatrix}$$

Observe $L[y_1] = 0$ and $L[y_2] = 0$ are immediately clear as setting $y = y_1$ or $y = y_2$ gives a repeated row.

Example 5.7.3. *This is an example of a Vandermonde determinant. Note the following curious formula:*

$$det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} = 0$$

Let's reduce this by row-operations[10]

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} \begin{array}{c} \xrightarrow{r_2 - r_1 \to r_2} \\ \xrightarrow{r_3 - r_1 \to r_3} \end{array} \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x - x_1 & y - y_1 \end{bmatrix}$$

Notice that the row operations above could be implemented by multiply on the left by $E_{r_2-r_1\to r_2}$ and $E_{r_3-r_1\to r_3}$. These are invertible matrices and thus $det(E_{r_2-r_1\to r_2}) = k_1$ and $det(E_{r_3-r_1\to r_3}) = k_2$ for some pair of nonzero constants k_1, k_2. If X is the given matrix and Y is the reduced matrix above then $Y = E_{r_3-r_1\to r_3}E_{r_2-r_1\to r_2}X$ thus,

$$\begin{aligned} 0 = det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} &= k_1k_2 det \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x - x_1 & y - y_1 \end{bmatrix} \\ &= k_1k_2\big[(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)\big] \end{aligned}$$

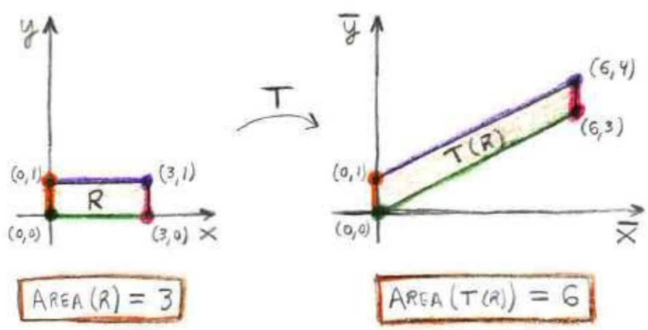
Divide by k_1k_2 and rearrange to find:

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1) \qquad \Rightarrow \qquad \boxed{y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

The boxed equation is the famous two-point formula for a line.

Example 5.7.4. *Let us consider a linear transformation*[11] $T([x, y]^T) = [2x, x + y]^T$*. Furthermore, let's see how a rectangle R with corners $(0, 0), (3, 0), (3, 1), (0, 1)$. Since this linear transformation is invertible it follows that the image of a line is again a line. Therefore, if we find the image of the corners under the mapping T then we can just connect the dots in the image to see what $T(R)$ resembles. Our goal here is to see what a linear transformation does to a rectangle.*

$$\begin{array}{ccc} T([0, 0]^T) = [0, 0]^T & \& & T([3, 0]^T) = [6, 3]^T \\ T([3, 1]^T) = [6, 4]^T & \& & T([0, 1]^T) = [0, 1]^T \end{array}$$



[10] of course we could calculate it straight from the co-factor expansion, I merely wish to illustrate how we can use row operations to simplify a determinant

[11] we study the definition and structure of linear transformations in a future chapter, but, I like to give this example here, my apologies for the asyncronisity of this example, I don't plan to test you on linear transformations until a bit later

As you can see from the picture we have a paralellogram with base 6 *and height* 1 *thus* $Area(T(R)) = 6$. *In constrast,* $Area(R) = 3$. *You can calculate that* $det(T) = 2$. *Curious,* $Area(T(R)) = det(T)Area(R)$. *This can be derived in general, it's not too hard given our definition of* n*-volume and the wonderful identities we've learned for matrix multiplication and determinants.*

The examples that follow illustrate how determinants arise in the study of infinitesimal areas and volumes in multivariate calculus.

Example 5.7.5. *The infinitesimal area element for polar coordinate is calculated from the Jacobian:*

$$dS = det \begin{bmatrix} r\sin(\theta) & -r\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} drd\theta = (r\sin^2(\theta) + r\cos^2(\theta))drd\theta = rdrd\theta$$

Example 5.7.6. *The infinitesimal volume element for cylindrical coordinate is calculated from the Jacobian:*

$$dV = det \begin{bmatrix} r\sin(\theta) & -r\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} drd\theta dz = (r\sin^2(\theta) + r\cos^2(\theta))drd\theta dz = rdrd\theta dz$$

Jacobians are needed to change variables in multiple integrals. The Jacobian[12] is a determinant which measures how a tiny volume is rescaled under a change of coordinates. Each row in the matrix making up the Jacobian is a tangent vector which points along the direction in which a coordinate increases when the other two coordinates are fixed.

5.8 conclusions

The theorem which follows collects ideas somewhat comprehensively for our course thus far, I believe I was inspired by Anton's text to make such a verbose theorem. We should pay special attention to the fact that the above comments below only for a square matrix. For rectangular matrices we have to think about the number of pivot columns verses the number of non-pivot columns. The CCP helps us analyze such systems of equations where the number of equations and the numbers of variables differ. There is still something more to learn, but, I defer the conclusion of this story[13] until we introduce the concept of a subspace.

[12] see pages 206-208 of Spence Insel and Friedberg or perhaps my advanced calculus notes where I develop differentiation from a linear algebraic viewpoint.

[13] in Theorem 7.9.1

Theorem 5.8.1. *Characterizations of an invertible matrix*

Let A be a real $n \times n$ matrix then the following are equivalent:

(a.) A is invertible,

(b.) $Ax = 0$ iff $x = 0$,

(c.) A is the product of elementary matrices,

(d.) there exists $B \in \mathbb{R}^{\ n \times n}$ such that $AB = I$,

(e.) there exists $B \in \mathbb{R}^{\ n \times n}$ such that $BA = I$,

(f.) $rref[A] = I$,

(g.) for each $b \in \mathbb{F}^n$ there exists $x \in \mathbb{F}^n$ such that $rref[A|b] = [I|x]$,

(h.) for each $b \in \mathbb{F}^n$ there exists $x \in \mathbb{F}^n$ such that $Ax = b$,

(i.) A^T is invertible,

(j.) $det(A) \neq 0$,

(k.) the columns of A are linearly independent,

(m.) the rows of A are linearly independent,

Next, the list of equivalent statements for a singular $n \times n$ matrix:

Theorem 5.8.2. *Characterizations of a non-invertible matrix*

Let A be a real $n \times n$ matrix then the following are equivalent:

(a.) A is not invertible,

(b.) $Ax = 0$ has at least one nontrivial solution.,

(c.) there exists $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent,

(d.) $det(A) = 0$,

It turns out this theorem is also useful. We shall see it is fundamental to the theory of eigenvectors.

Chapter 6

Vector Spaces

Up to this point the topics we have discussed loosely fit into the category of matrix theory. The concept of a matrix is milienia old. If I trust my source, and I think I do, the Chinese even had an analog of Gaussian elimination about 2000 years ago. The modern notation likely stems from the work of Cauchy in the 19-th century. Cauchy's prolific work colors much of the notation we still use. The concept of coordinate geometry as introduced by Descartes and Fermat around 1644 is what ultimately led to the concept of a vector space.[1]. Grassmann, Hamilton, and many many others worked out volumous work detailing possible transformations on what we now call $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4,$. Argand(complex numbers) and Hamilton(quaternions) had more than what we would call a vector space. They had a linear structure plus some rule for multiplication of vectors. A vector space with a multiplication is called an *algebra* in the modern terminology.

Honestly, I think once the concept of the Cartesian plane was discovered the concept of a vector space almost certainly must follow. That said, it took a while for the definition I state in the next section to appear. Giuseppe Peano gave the modern definition for a vector space in 1888[2]. In addition he put forth some of the ideas concerning linear transformations. Peano is also responsible for the modern notations for intersection and unions of sets[3]. He made great contributions to proof by induction and the construction of the natural numbers from basic set theory.

I should mention the work of Hilbert, Lebesque, Fourier, Banach and others were greatly influential in the formation of infinite dimensional vector spaces. Our focus is on the finite dimensional case.[4]

Let me summarize what a vector space is before we define it properly. In short, a vector space over a field $\mathbb{F}$ is simply a set which allows you to add its elements and multiply by the numbers in $\mathbb{F}$. A field is a set with addition and multiplication defined such that every nonzero element has a multiplicative inverse. Typical examples, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Z}/p\mathbb{Z}$ where p is prime.

Vector spaces are found throughout modern mathematics. Moreover, the theory we cover in this chapter is applicable to a myriad of problems with real world content. This is the beauty of linear algebra: it simultaneously illustrates the power of application and abstraction in mathematics.

[1] Bourbaki 1969, ch. *"Algebre lineaire et algebre multilineaire", pp. 78-91.*

[2] Peano, Giuseppe (1888), *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle Operazioni della Logica Deduttiva, Turin*

[3] see Pg 87 of A Transition to Advanced Mathematics: A Survey Course By William Johnston

[4] this history is flawed, one-sided and far too short. You should read a few more books if you're interested.

6.1 definition and examples

Axioms are not derived from a more basic logic. They are the starting point. Their validity is ultimately judged by their use. However, this definition is naturally motivated by the structure of vector addition and scalar multiplication in $\mathbb{R}^n$ (or $\mathbb{F}^n$ if that is where your intuition rests)

Definition 6.1.1.

A vector space V over a field $\mathbb{F}$ is a nonempty set V together with a function $+ : V \times V \to V$ called **vector addition** and another function $\cdot : \mathbb{F} \times V \to V$ called **scalar multiplication**. We require that the operations of vector addition and scalar multiplication satisfy the following 10 axioms: for all $x, y, z \in V$ and $a, b \in \mathbb{F}$,

(A1.) $x + y = y + x$ for all $x, y \in V$,

(A2.) $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$,

(A3.) there exists $0 \in V$ such that $x + 0 = x = 0 + x$ for all $x \in V$,

(A4.) for each $x \in V$ there exists $-x \in V$ such that $x + (-x) = 0 = (-x) + x$,

(A5.) $1 \cdot x = x$ for all $x \in V$,

(A6.) $(ab) \cdot x = a \cdot (b \cdot x)$ for all $x \in V$ and $a, b \in \mathbb{F}$,

(A7.) $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $x, y \in V$ and $a \in \mathbb{F}$,

(A8.) $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $x \in V$ and $a, b \in \mathbb{F}$,

(A9.) If $x, y \in V$ then $x + y$ is a single element in V,
(we say V is closed with respect to addition)

(A10.) If $x \in V$ and $c \in \mathbb{F}$ then $c \cdot x$ is a single element in V.
(we say V is closed with respect to scalar multiplication)

We call 0 in axiom 3 the **zero vector** and the vector $-x$ is called the **additive inverse** of x. We will sometimes omit the $\cdot$ and instead denote scalar multiplication by juxtaposition; $a \cdot x = ax$.

Axioms (9.) and (10.) are admittably redundant given that those automatically follow from the statements that $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ are functions. I've listed them so that you are less likely to forget they must be checked.

The terminology "vector" does not necessarily indicate an explicit geometric interpretation in this general context. Sometimes I'll insert the word "abstract" to emphasize this distinction. We'll see that matrices, polynomials and functions in general can be thought of as abstract vectors.

Example 6.1.2. Real Matrices form real vector spaces: $\mathbb{R}$ *is a vector space if we identify addition of real numbers as the vector addition and multiplication of real numbers as the scalar multiplication. Likewise,* $\mathbb{R}^n$ *forms a vector space over* $\mathbb{R}$ *with respect to the usual vector addition and scalar multiplication:*

$$(x + y)_i = x_i + y_i \qquad \& \qquad (cx)_i = cx_i$$

for each $i \in \mathbb{N}_n$ *and* $x, y \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$. *In fact, even* $\mathbb{R}^n$ *is just the* $n \times 1$ *case of* $\mathbb{R}^{m\times n}$. *Indeed,* $\mathbb{R}^{m\times n}$ *forms a vector space over* $\mathbb{R}$ *with addition and scalar multiplication of matrices defined as we studied in previous chapters:*

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \& \qquad (cA)_{ij} = cA_{ij}$$

for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$ *and* $A, B \in \mathbb{R}^{m\times n}$ *and* $c \in \mathbb{R}$.

In the previous example, I introduced the standard interpretation of $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ as **real vector spaces**. To say V is a **real** vector space is just another way of saying V is a vector space with the field of scalars being the real numbers. Proof that Axioms 1-10 are met was already given in part in Proposition 2.2.6 and Theorem 2.2.9 (if we prove something for an arbitrary commutative ring then this naturally includes the case $R = \mathbb{R}$). I should mention, we can also view $\mathbb{R}, \mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ as vector spaces over the rational numbers $\mathbb{Q}$. However, even $\mathbb{R}$ is infinite dimensional over the rational numbers. In contrast, $\mathbb{R}$ is one-dimensional over $\mathbb{R}$. For now, I use the term **dimensional** as an intutitive term, we shall soon give it a rigorous meaning. The next example should not be surprising in view of Example 6.1.2.

Example 6.1.3. Complex matrices form complex vector spaces: $\mathbb{C}$ *is a vector space if we identify addition of complex numbers as the vector addition and multiplication of complex numbers as the scalar multiplication. Likewise,* $\mathbb{C}^n$ *forms a vector space over* $\mathbb{C}$ *with respect to the usual vector addition and scalar multiplication:*

$$(x+y)_i = x_i + y_i \qquad \& \qquad (cx)_i = cx_i$$

for each $i \in \mathbb{N}_n$ *and* $x, y \in \mathbb{C}^n$ *and* $c \in \mathbb{C}$. *In fact, even* $\mathbb{C}^n$ *is just the* $n \times 1$ *case of* $\mathbb{C}^{m\times n}$. *Indeed,* $\mathbb{C}^{m\times n}$ *forms a vector space over* $\mathbb{C}$ *with addition and scalar multiplication of matrices defined as we studied in previous chapters:*

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \& \qquad (cA)_{ij} = cA_{ij}$$

for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$ *and* $A, B \in \mathbb{C}^{m\times n}$ *and* $c \in \mathbb{C}$.

Generalizing a bit:

Example 6.1.4. Matrices over a field $\mathbb{F}$ form vector spaces over $\mathbb{F}$: *in particular,* $\mathbb{F}^{m\times n}$ *forms a vector space over* $\mathbb{F}$ *with addition and scalar multiplication of matrices defined as we studied in previous chapters:*

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \& \qquad (cA)_{ij} = cA_{ij}$$

for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$ *and* $A, B \in \mathbb{F}^{m\times n}$ *and* $c \in \mathbb{F}$. *We understand* $\mathbb{F}$ *and* $\mathbb{F}^n$ *as sub-cases.*

If a given point-set permits the assignment of a vector space structure (meaning we can define addition and scalar multiplication which adhere to Axioms 1-10) then it may be possible to assign a different vector space structure to the set as well. In Example 6.1.4 we discussed $\mathbb{C}^{m\times n}$ as a complex vector space. In constrast, in the example below we give $\mathbb{C}^{m\times n}$ a real vector space structure:

Example 6.1.5. *Let* $V = \mathbb{C}^{m\times n}$ *be a vector space over* $\mathbb{R}$ *where addition and scalar multiplication are defined by:*

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \& \qquad (cA)_{ij} = cA_{ij}$$

for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$ *and* $A, B \in \mathbb{C}^{m\times n}$ *and* $c \in \mathbb{R}$.

I think the next example will seem a bit different.

Example 6.1.6. *Let S be a set and denote*[5] *the set of all functions from S to $\mathbb{R}$ by $\mathcal{F}(S,\mathbb{R})$. Let $f, g \in \mathcal{F}(S,\mathbb{R})$ and suppose $c \in \mathbb{R}$, define addition and scalar multiplication of functions by*

$$(f+g)(x) \equiv f(x)+g(x) \qquad \& \qquad (cf)(x) = cf(x)$$

for all $x \in S$. In short, we define addition and scalar multiplication by the natural "point-wise" rules. This is an example of a function space. Notice that no particular structure is needed for the domain. The vector space structure is inherited from the codomain of the functions. I invite the reader to check Axioms 1-10 for this point-set. For example, define $z(x) = 0$ for all $x \in S$ then we can prove $z+f = f+z = f$ for each $f \in \mathcal{F}(S,\mathbb{R})$. This shows $z : S \to \mathbb{R}$ serves as the **zero-vector** *for $\mathcal{F}(S,\mathbb{R})$.*

In the interest of confusing the students, we often write $z = 0$ for the zero-function of the last example. The notation 0 really means just about nothing. Or, perhaps it means everything. For example, 0 is used to denote

$$0,\ [0,0],\ [0,0,0],\ \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in appropriate contexts. Let us move on to a less trivial discussion:

Example 6.1.7. *Let S be a set and let W be a vector space over $\mathbb{F}$ and let $V = \mathcal{F}(\mathcal{S},\mathcal{W})$ denotes the set of functions from S to W. If we define addition and scalar multiplication of functions in V in the same fashion as Example 6.1.6 then once more we have V as a vector space over $\mathbb{F}$. For example, functions from $\mathbb{R}$ to $\mathbb{C}^2$ are naturally viewed as a complex vector space. Or, functions from $\{a,b,c\}$ to $\mathbb{Z}/11\mathbb{Z}$ naturally form a vector space over $\mathbb{Z}/11\mathbb{Z}$.*

Example 6.1.8. *Let $P_2(\mathbb{R}) = \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$, the set of all real polynomials up to quadratic order. Define addition and scalar multiplication by the usual operations on polynomials. Notice that if $ax^2+bx+c, dx^2+ex+f \in P_2(\mathbb{R})$ then*

$$(ax^2+bx+c)+(dx^2+ex+f) = (a+d)x^2+(b+e)x+(c+f) \in P_2(\mathbb{R})$$

thus $+ : P_2(\mathbb{R}) \times P_2(\mathbb{R}) \to P_2(\mathbb{R})$ (it is a binary operation on $P_2(\mathbb{R})$). Similarly,

$$d(ax^2+bx+c) = dax^2+dbx+dc \in P_2(\mathbb{R})$$

thus scalar multiplication maps $\mathbb{R} \times P_2(\mathbb{R}) \to P_2(\mathbb{R})$ as it ought. Verification of the other 8 axioms is straightfoward. We denote the set of polynomials of order n or less via $P_n(\mathbb{R}) = \{a_nx^n + \cdots + a_1x + a_o | a_i \in \mathbb{R}\}$. Naturally, $P_n(\mathbb{R})$ also forms a vector space. Finally, if we take the set of all polynomials $\mathbb{R}[x]$ it forms a real vector space. Notice,

$$\mathbb{R} \subset P_1(\mathbb{R}) \subset P_2(\mathbb{R}) \subset P_3(\mathbb{R}) \subset P_4(\mathbb{R}) \subset \cdots \subset \mathbb{R}[x]$$

where $\mathbb{R}$ is naturally identified with the set of constant real polynomials $P_0(\mathbb{R})$

[5] another popular notation for the set of functions from a set A to a set B is simple B^A. That is, $\mathcal{F}(A,B) = B^A$. In particular, this is in some sense consistent with the notation $\mathbb{R}^3$ as in we can view triples of real numbers as functions from $\{1,2,3\}$ to $\mathbb{R}$.

Real polynomials in a different variable are denoted in the natural fashion; for example, $\mathbb{R}[t]$ is the set of real polynomials in the variable t. Furthermore, once the context is clear we are free to drop the $\mathbb{R}$ notation from $P_n(\mathbb{R})$ and simply write P_n. We can generalize the last example by replacing $\mathbb{R}$ with an arbitrary field $\mathbb{F}$:

Example 6.1.9. *We denote the set of polynomials with coefficients in $\mathbb{F}$ of order n or less via $P_n(\mathbb{F}) = \{a_n x^n + \cdots + a_1 x + a_0 | a_i \in \mathbb{F}\}$. Naturally, $P_n(\mathbb{F})$ also forms a vector space over $\mathbb{F}$. The set of all polynomials with coefficients in $\mathbb{F}$ is denoted $\mathbb{F}[x]$ and we can show it forms a vector space over $\mathbb{F}$ with respect to the usual addition and scalar multiplication of polynomials.*

This list of examples is nowhere near comprehensive. We can also mix together examples we've covered thus far to create new vector spaces.

Example 6.1.10. *Let $[P_2(\mathbb{R})]^{2\times 2}$ denote the set of 2×2 matrices of possibly-degenerate quadratic polynomials. Addition and scalar multiplication are defined in the natural manner:*

$$\left[\begin{array}{cc} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{array}\right] + \left[\begin{array}{cc} B_{11}(x) & B_{12}(x) \\ B_{21}(x) & B_{22}(x) \end{array}\right] = \left[\begin{array}{c|c} A_{11}(x)+B_{11}(x) & A_{12}(x)+B_{12}(x) \\ \hline A_{21}(x)+B_{21}(x) & A_{22}(x)+B_{22}(x) \end{array}\right]$$

and $c\left[\begin{array}{cc} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{array}\right] = \left[\begin{array}{cc} cA_{11}(x) & cA_{12}(x) \\ cA_{21}(x) & cA_{22}(x) \end{array}\right]$. *We can similarly define $m\times n$ matrices of possibly degenerate n-th order real polynomials by $[P_n(\mathbb{R})]^{m\times n}$. Furthermore, we can replace $\mathbb{R}$ with $\mathbb{F}$ to obtain even more varied examples.*

Given a pair of vector spaces over a particular field there is a natural way to combine them to make a larger vector space[6]:

Example 6.1.11. *Let V, W be vector spaces over $\mathbb{F}$. The Cartesian product $V\times W$ has a natural vector space structure inherited from V and W: if $(v_1,w_1),(v_2,w_2)\in V\times W$ then we define*

$$(v_1,w_1)+(v_2,w_2) = (v_1+v_2, w_1+w_2) \qquad \& \qquad c\cdot(v_1,w_1) = (c\cdot v_1, c\cdot w_1)$$

where the vector and scalar operations on the L.H.S. of the above equalities are given from the vector space structure of V and W. All the axioms of a vector space for $V\times W$ are eaily verified from the corresponding axioms for V and W.

Example 6.1.12. *Let $V = \mathbb{R}[x,y] = (\mathbb{R}[x])[y]$ then V contains elements such as*

$$1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \cdots$$

more generally, suppose $c_{ij}\neq 0$ for only finitely many i,j then $f(x,y)\in V$ has coeffcients c_{ij} and

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij}x^iy^j$$

If $f(x,y), g(x,y)$ have coefficients c_{ij} and b_{ij} respectively then $f(x,y)+g(x,y)$ is defined to have coefficients $c_{ij}+b_{ij}$. Notice $f(x,y)+g(x,y)$ once more has only finitely many nonzero coefficients and is hence in V. Likewise, define $\alpha f(x,y) = \alpha\left(\sum_{i,j=0}^{\infty} a_{ij}x^iy^j\right) = \sum_{i,j=0}^{\infty}\alpha a_{ij}x^iy^j$. It follows that V is a real vector space. There are many ways to extend this example.

The theorem that follows is full of seemingly obvious facts. I show how each of these facts follow from the vector space axioms.

---

[6]this is roughly like adding the vector spaces, in contrast, much later we study $V\otimes W$ which is like multiplying the spaces. The tensor product $\otimes$ also gives us a larger new space from a given pair of vector spaces

Theorem 6.1.13.

Let $\mathbb{F}$ be a field and V a vector space over $\mathbb{F}$ with zero vector 0 and let $c \in \mathbb{F}$,

1. $0 \cdot x = 0$ for all $x \in V$,
2. $c \cdot 0 = 0$ for all $c \in \mathbb{F}$,
3. $(-1) \cdot x = -x$ for all $x \in V$,
4. if $cx = 0$ then $c = 0$ or $x = 0$.

Lemma 6.1.14. Law of Cancellation:

Let a, x, y be vectors in a vector space V. If $x + a = y + a$ then $x = y$.

Proof of Lemma: Suppose $x + a = y + a$. By A4 there exists $-a$ such that $a + (-a) = 0$. Thus $x+a = y+a$ implies $(x+a)+(-a) = (y+a)+(-a)$. By A2 we find $x+(a+(-a)) = y+(a+(-a))$ which gives $x+0 = y+0$. Continuing we use A3 to obtain $x+0 = 0$ and $y+0 = y$ and consequently $x = y$. $\square$.

We now seek to prove (1.). Consider:

$$\begin{aligned} 0 \cdot x + 0 &= 0 \cdot x && \text{by A3} \\ &= (0+0) \cdot x && \text{defn. of zero scalar} \\ &= 0 \cdot x + 0 \cdot x && \text{by A8} \end{aligned}$$

Finally, apply the cancellation lemma to conclude $0 \cdot x = 0$. Note x was arbitrary thus (1.) has been shown true. $\square$

We now prove (2.). Suppose $c \in \mathbb{F}$.

$$\begin{aligned} c \cdot 0 + 0 &= c \cdot 0 && \text{by A3} \\ &= c \cdot (0+0) && \text{by A3} \\ &= c \cdot 0 + c \cdot 0 && \text{by A7} \end{aligned}$$

Consquently, by the cancellation lemma we find $c \cdot 0 = 0$ for all $c \in \mathbb{F}$. $\square$

The proof of (3.) is similar. Consider,

$$\begin{aligned} 0 &= 0 \cdot x && \text{by (1.)} \\ &= (1 + (-1)) \cdot x && \text{scalar arithmetic} \\ &= 1 \cdot x + (-1) \cdot x && \text{by A8} \\ &= x + (-1) \cdot x && \text{by A5} \end{aligned}$$

Thus, adding $-x$ to the equation above,

$$(-x) + 0 = (-x) + x + (-1) \cdot x = 0 + (-1) \cdot x$$

thus, using A3 once more, $(-1) \cdot x = -x$ for all $x \in V$. $\square$

To prove (4.), suppose $c \cdot x = 0$. If $c = 0$ then we have that the claim of (4.) is verified. If $c \neq 0$ then $1 = \frac{c}{c} = \frac{1}{c}c$ hence using A5 in the first equality and A6 in the third equality we find:

$$x = 1 \cdot x = \left(\frac{1}{c}c\right) \cdot x = \frac{1}{c} \cdot (c \cdot x) = \frac{1}{c} \cdot 0 = 0$$

where we used (2.) in the final equality. In summary, $c \cdot x = 0$ implies $c = 0$ or $x = 0$. $\square$

Perhaps we should pause to appreciate what was not in the last page or two of proofs. There were no components, no reference to the standard basis. The arguments offered depended only on the definition of the vector space itself. This means the truths we derived above are completely general; they hold for all vector spaces. In what follows past this point we sometimes use Theorem 6.1.13 without explicit reference. That said, I would like you to understand the results of the theorem do require proof and that is why we have taken some effort here to supply that proof.

6.2 subspaces

Definition 6.2.1.

Let V be a vector space over a field $\mathbb{F}$. If $W \subseteq V$ such that W is a vector space over $\mathbb{F}$ with respect to the operations of V restricted to W then we say W is a **subspace** of V and write $W \leq V$.

Example 6.2.2. *Let V be a vector space. Notice that $V \subseteq V$ and obviously V is a vector space with respect to its operations. Therefore $V \leq V$. Likewise, the set containing the zero vector $\{0\} \leq V$. Notice that $0 + 0 = 0$ and $c \cdot 0 = 0$ so Axioms 9 and 10 are satisfied. I leave the other axioms to the reader. The subspace $\{0\}$ is called the* **trivial subspace**.

Example 6.2.3. *Let $L = \{(x, y) \in \mathbb{R}^2 | ax + by = 0\}$. Define addition and scalar multiplication by the natural rules in $\mathbb{R}^2$. Note if $(x, y), (z, w) \in L$ then $(x, y) + (z, w) = (x + z, y + w)$ and $a(x + z) + b(y + w) = ax + by + az + bw = 0 + 0 = 0$ hence $(x, y) + (z, w) \in L$. Likewise, if $c \in \mathbb{R}$ and $(x, y) \in L$ then $ax + by = 0$ implies $acx + bcy = 0$ thus $(cx, cy) = c(x, y) \in L$. We find that L is closed under vector addition and scalar multiplication. The other 8 axioms are naturally inherited from $\mathbb{R}^2$. This makes L a* **subspace** *of $\mathbb{R}^2$.*

Example 6.2.4. *If $V = \mathbb{R}^3$ then*

1. *$\{(0, 0, 0)\}$ is a subspace,*
2. *any line through the origin is a subspace,*
3. *any plane through the origin is a subspace.*

Example 6.2.5. *Let $V = \mathbb{R}$ and let $W = [a, b]$ where $a, b > 0$. Observe $a + b \notin [a, b]$ hence $W \not\leq \mathbb{R}$. Indeed, we can generalize this observation, if $W \neq \{0\}$ is a proper subset of $\mathbb{R}$ then it cannot form a subspace of $\mathbb{R}$. Vector addition and scalar mulltiplication will take us outside W. For example $W = \mathbb{Z}$ has $\sqrt{2}z \notin \mathbb{Z}$ for each $z \in \mathbb{Z}$.*

Example 6.2.6. *Let $W = \{(x, y, z) \mid x + y + z = 1\}$. Is this a subspace of $\mathbb{R}^3$ with the standard*[7] *vector space structure? The answer is no. There are many reasons,*

1. $(0, 0, 0) \notin W$ *thus W has no zero vector, axiom 3 fails. Notice we cannot change the idea of "zero" for the subspace, if $(0, 0, 0)$ is zero for $\mathbb{R}^3$ then it is the only zero for potential subspaces. Why? Because subspaces inherit their structure from the vector space which contains them.*
2. *Observe $(1, 0, 0), (0, 1, 0) \in W$ yet $(1, 0, 0) + (0, 1, 0) = (1, 1, 0)$ is not in W since $1 + 1 + 0 = 2 \neq 1$. Thus W is not closed under vector addition (A9 fails).*
3. *Again $(1, 0, 0) \in W$ yet $2(1, 0, 0) = (2, 0, 0) \notin W$ since $2 + 0 + 0 = 2 \neq 1$. Thus W is not closed under scalar multiplication (A10 fails).*

Of course, one reason is all it takes.

My focus on the last two axioms is not without reason. Let me explain this obsession[8].

Theorem 6.2.7. Subspace Test:

Let V be a vector space over a field $\mathbb{F}$ and suppose $W \subseteq V$ with $W \neq \emptyset$ then $W \leq V$ if and only if the following two conditions hold true

1. if $x, y \in W$ then $x + y \in W$ (W is closed under addition),
2. if $x \in W$ and $c \in \mathbb{F}$ then $c \cdot x \in W$ (W is closed under scalar multiplication).

Proof: $(\Rightarrow)$If $W \leq V$ then W is a vector space with respect to the operations of addition and scalar multiplication thus (1.) and (2.) hold true.

$(\Leftarrow)$ Suppose W is a nonempty set which is closed under vector addition and scalar multiplication of V. We seek to prove W is a vector space with respect to the operations inherited from V. Let $x, y, z \in W$ then as $W \subseteq V$ we have $x, y, z \in V$. Use A1 and A2 for V (which were given to begin with) to find

$$x + y = y + x \qquad \text{and} \qquad (x + y) + z = x + (y + z).$$

Thus A1 and A2 hold for W. By (3.) of Theorem 6.1.13 we know that $(-1) \cdot x = -x$ and $-x \in W$ since we know W is closed under scalar multiplication. Consequently, $x + (-x) = 0 \in W$ since W is closed under addition. It follows A3 is true for W. Then by the arguments just given A4 is true for W. Let $a, b \in \mathbb{F}$ and notice that by A5,A6,A7,A8 for V we find

$$1 \cdot x = x, \quad (ab) \cdot x = a \cdot (b \cdot x), \quad a \cdot (x + y) = a \cdot x + a \cdot y, \quad (a + b) \cdot x = a \cdot x + b \cdot x.$$

Thus A5,A6,A7,A8 likewise hold for W. Finally, we assumed closure of addition and scalar multiplication on W so A9 and A10 are likewise satisfied and we conclude that W is a vector space over $\mathbb{F}$. Thus $W \leq V$. (if you're wondering where we needed W nonempty it was to argue that there exists at least one vector x and consequently the zero vector is in W.) $\square$

[7] yes, there is a non-standard addition which gives this space a vector space structure

[8] notice tha Charles Curtis uses this Theorem as his definition for subspace. It is equivalent in view of the proof below

Remark 6.2.8.

The application of Theorem 6.2.7 is a four-step process

1. check that $W \subset V$
2. check that $0 \in W$ (this is a matter of convenience, and if it fails it's usually blatant)
3. take arbitrary $x, y \in W$ and show $x + y \in W$
4. take arbitrary $x \in W$ and $c \in \mathbb{F}$ and show $cx \in W$

We usually omit comment about (1.) since it is obviously true for examples we encounter.

Example 6.2.9. *The function space $\mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{R}, \mathbb{R})$ has many subspaces.*

1. *continuous functions: $C(\mathbb{R})$*
2. *differentiable functions: $C^1(\mathbb{R})$*
3. *smooth functions: $C^\infty(\mathbb{R})$*
4. *polynomial functions (which are naturally identified with $\mathbb{R}[x]$)*
5. *analytic functions*
6. *solution set of a linear homogeneous ODE with no singular points*

The proof that each of these follows from Theorem 6.2.7. For example, $f(x) = x$ is continuous therefore $C(\mathbb{R}) \neq \emptyset$. Moreover, the sum of continuous functions is continuous and a scalar multiple of a continuous function is continuous. Thus $C(\mathbb{R}) \leq \mathcal{F}(\mathbb{R})$. The arguments for (2.),(3.),(4.),(5.) and (6.) are identical. The solution set example is one of the most important examples for engineering and physics, linear ordinary differential equations. Also, we should note that $\mathbb{R}$ can be replaced with some subset I of real numbers. $\mathcal{F}(I)$ likewise has subspaces $C(I), C^1(I), C^\infty(I)$ etc.

Example 6.2.10. *Let $Ax = 0$ denote a homogeneous system of m-equations in n-unknowns over $\mathbb{R}$. Let W be the solution set of this system; $W = \{x \in \mathbb{R}^n \mid Ax = 0\}$. Observe that $A0 = 0$ hence $0 \in W$ so the solution set is nonempty. Suppose $x, y \in W$ and $c \in \mathbb{R}$,*

$$A(x + cy) = Ax + cAy = 0 + c(0) = 0$$

thus $x + cy \in W$. Closure of addtion for W follows from $c = 1$ and closure of scalar multiplication follows from $x = 0$ in the just completed calculation. Thus $W \leq \mathbb{R}^n$ by the Subspace Test Theorem. $\square$

Sometimes it's easier to check both scalar multiplication and addition at once. It saves some writing. If you don't understand it then don't use the trick I just used, we should understand our work.

The example that follows here illustrates an important point in abstract math. Given a particular point set, there is often more than one way to define a structure on the set. Therefore, it is important to view things as more than mere sets. Instead, think about sets paired with a structure.

Example 6.2.11. *Let*[9] V_p *be the set of all vectors with base point* $p \in \mathbb{R}^n$*,*

$$V_p = \{p + v \mid v \in \mathbb{R}^n\}$$

We define a nonstandard vector addition on V_p*, if* $p + v, p + w \in V_p$ *and* $c \in \mathbb{R}$ *define:*

$$(p + v) \ +_p \ (p + w) = p + v + w \qquad \& \qquad c \cdot_p (p + v) = p + cv.$$

Clearly $+_p : V_p \times V_p \to V_p$ *and* $\cdot_p : \mathbb{R} \times V_p \to V_p$ *are closed and verification of the other axioms is straightforward. Observe* $0_p = p$ *as* $(p + v) +_p (p + 0) = p + v + 0 = p + v$ *hence* $O_p = p + 0 = p$*. Mainly, the vector space axioms for* V_p *follow from the corresponding axioms for* $\mathbb{R}^n$*. Geometrically,* $+_p$ *corresponds to the* **tip-to-tail** *rule we use in physics to add vectors. Consider* S_p *defined below:*

$$S_p = \{p + v \mid v \in W \leq \mathbb{R}^n\}$$

Notice $0_p \in S_p$ *as* $0 \in W$ *and* $0_p = p + 0$*. Furthermore, consider* $p + v, p + w \in S_p$ *and* $c \in \mathbb{R}$

$$(p + v) +_p (p + w) = p + (v + w) \qquad \& \qquad c \cdot_p (p + v) = p + cv$$

note $v + w, cv \in W$ *as* $W \leq \mathbb{R}^n$ *is closed under addition and scalar multiplication. We find* $(p + v) +_p (p + w), c \cdot_p (p + v) \in S_p$ *thus* $S_p \leq V_p$ *by the subspace test Theorem 6.2.7.*

In the previous example, S_p need not be a subspace with respect to the standard vector addition of column vectors. However, with the modified addition based at p it is a subspace. We often say the solution set to $Ax = b$ with $b \neq 0$ is not a subspace. It should be understood that what is meant is that the solution set of $Ax = b$ is not a subspace with respect to the usual vector addition. It is possible to define a different vector addition which gives the solution set of $Ax = b$ a vector space structure. It seems likely there is a homework problem to expand this thought.

Example 6.2.12. *Let* $W = \{A \in \mathbb{R}^{\ n\times n} \mid A^T = A\}$*. This is the set of* **symmetric matrices***, it is nonempty since* $I^T = I$ *(of course there are many other examples, we only need one to show it's nonempty). Let* $A, B \in W$ *and suppose* $c \in \mathbb{R}$ *then*

$$\begin{aligned} (A + B)^T &= A^T + B^T && \textit{prop. of transpose} \\ &= A + B && \textit{since } A, B \in W \end{aligned}$$

thus $A + B \in W$ *and we find* W *is closed under addition. Likewise let* $A \in W$ *and* $c \in \mathbb{R}$*,*

$$\begin{aligned} (cA)^T &= cA^T && \textit{prop. of transpose} \\ &= cA && \textit{since } A, B \in W \end{aligned}$$

thus $cA \in W$ *and we find* W *is closed under scalar multiplication. Therefore, by the subspace test Theorem 6.2.7,* $W \leq \mathbb{R}^{\ n\times n}$*.*

I invite the reader to modify the example above to show the set of antisymmetric matrices also forms a subspace of the vector space of square matrices.

Example 6.2.13. *Let* $W = \{f \in \mathcal{F}(\mathbb{R}) \mid \int_{-1}^{1} f(x)\, dx = 0\}$*. Notice the zero function* $0(x) = 0$ *is in* W *since* $\int_{-1}^{1} 0\, dx = 0$*. Let* $f, g \in W$*, use linearity property of the definite integral to calculate*

$$\int_{-1}^{1} (f(x) + g(x))\, dx = \int_{-1}^{1} f(x)\, dx + \int_{-1}^{1} g(x)\, dx = 0 + 0 = 0$$

[9] it may be better to use the notation (p, v) for $p + v$, this has the advantage of making the base-point p explicit whereas p can be obscured in the more geometrically direct $p + v$ notation. Another choice is to use v_p.

thus $f+g \in W$. Likewise, if $c \in \mathbb{R}$ and $f \in W$ then

$$\int_{-1}^{1} cf(x)\,dx = c\int_{-1}^{1} f(x)\,dx = c(0) = 0$$

thus $cf \in W$ and by subspace test Theorem 6.2.7 $W \leq \mathcal{F}(\mathbb{R})$.

Example 6.2.14. *Here we continue discussion of the product space introduced in Example 6.1.11. Suppose $V = \mathbb{C}$ and $W = P_2$ then $V \times W = \{(a+ib, cx^2+dx+e) \mid a,b,c,d,e \in \mathbb{R}\}$. Let $U = \{(a,b) \mid a,b \in \mathbb{R}\}$. We can easily show $U \leq V \times W$ by the subspace test Theorem 6.2.7 $W \leq \mathcal{F}(\mathbb{R})$. Can you think of other subspaces? Is it possible to have a subspace of $V \times W$ which is not formed from a pair of subspaces from V and W respective?*

Example 6.2.15. *Let W be the set of real-valued functions on $\mathbb{R}$ for which $f(a)=0$ for some fixed value $a \in \mathbb{R}$. If $f,g \in W$ and $c \in \mathbb{R}$ then $(f+cg)(a) = f(a)+cg(a) = 0+c(0) = 0$ thus $f+cg \in W$. Observe W is closed under addition by the case $c=1$ and W is closed under scalar multiplication by the case $f=0$. Furthermore, $f(x)=0$ for all $x \in \mathbb{R}$ defines the zero function which is in W. Hence $W \leq \mathcal{F}(\mathbb{R})$ by subspace test Theorem 6.2.7.*

6.3 spanning sets and subspaces

In a vector space V over a field $\mathbb{F}$ we are free to form $\mathbb{F}$-linear combinations[10] of vectors; we say $v \in V$ is a linear combination of $v_1,\dots,v_k \in V$ if there exist $c_1,\dots,c_k \in \mathbb{F}$ such that $v = c_1v_1+\cdots+c_kv_k$. In other words, $v = \sum_{j=1}^{k} c_jv_j$ is a **finite linear combination** of vectors v_j in V with coefficients x_j in $\mathbb{F}$. The Lemma below is quite useful[11]:

Lemma 6.3.1. *Let V be a vector space over the field $\mathbb{F}$ and let S be a nonempty subset of V,*

> The finite $\mathbb{F}$-linear combination of finite $\mathbb{F}$-linear combinations of vectors from S is once more a finite $\mathbb{F}$-linear combination of vectors from S.

Proof: Suppose V is a vector space over a field $\mathbb{F}$. Let $s_i = \sum_{j=1}^{n_i} c_{ij}t_{ij}$ where $c_{ij} \in \mathbb{F}$ and $t_{ij} \in S$ for $n_i, i \in \mathbb{N}$ with $i = 1,2,\dots,k$. Let $b_1,\dots,b_k \in \mathbb{F}$ and consider by (2.) of Proposition 1.4.3

$$\sum_{i=1}^{k} b_is_i = \sum_{i=1}^{k} b_i\left(\sum_{j=1}^{n_i} c_{ij}t_{ij}\right) = \sum_{i=1}^{k}\sum_{j=1}^{n_i} b_ic_{ij}t_{ij}.$$

Notice, this is a $\mathbb{F}$-linear combination of vectors in S as $b_ic_{ij} \in \mathbb{F}$. $\square$

Definition 6.3.2.

> Let $S = \{v_1,v_2,\dots,v_k\}$ be a finite set of vectors in a vector space V over a field $\mathbb{F}$ then $\text{span}(S)$ is defined to be the set of all $\mathbb{F}$-linear combinations of S:
>
> $$\text{span}\{v_1,v_2,\dots,v_k\} = \left\{\sum_{i=1}^{k} c_iv_i \;\middle|\; c_i \in \mathbb{F} \text{ for } i = 1,2,\dots,k\right\}$$
>
> If $W = \mathit{span}(S)$ then we say that S is a **generating set** for W. We also say S **spans** W in this case. Furthermore, if S is an infinite set then $\text{span}(S)$ is defined to be all possible finite linear combinations from S. Finally, $\text{span}(\emptyset) = \{0\}$

Spans are important because they are subspaces which are presented in a particularly lucid manner.

[10] following Definition 2.2.14 with $R = \mathbb{F}$

[11] Charles Curtis' proof of (4.4) on page 27-28 has an induction-based refinement of proof I offer for the Lemma

Theorem 6.3.3. *span(S) is a subspace.*

Let V be a vector space over a field $\mathbb{F}$ and suppose $S = \{v_1, \dots, v_k\} \subseteq V$ then $\text{span}(S) \leq V$. Furthermore, if $W \leq V$ and $S \subseteq W$ then $\text{span}(S) \subseteq W$; that is, span($S$) is the smallest subspace of V which contains S.

Proof: If $S = \emptyset$ then $\text{span}(\emptyset) = \{0\} \leq V$. Otherwise, $S \neq \emptyset$ hence consider $x, y \in \text{span}(S)$ and $c \in \mathbb{F}$. Apply Lemma 6.3.1 to see the linear combination of linear combinations $x + y$ and cx is once more a linear combination of vectors in S. Thus $x + y, cx \in \text{span}(S)$ and we conclude by the Subspace Test Theorem $\text{span}(S) \leq V$.

Suppose W is any subspace of V which contains $v_1, v_2, \dots, v_k$. By defintion W is closed under scalar multiplication and vector addition thus all linear combinations of $v_1, v_2, \dots, v_k$ must be in W hence $\text{span}(S) \subseteq R$. Finally, it is clear that $v_1, v_2, \dots, v_k \in \text{span}(S)$ since $v_1 = 1v_1 + 0v_2 + \cdots + 0v_k$ and $v_2 = 0v_1 + 1v_2 + \cdots + 0v_k$ and so forth. $\square$

Example 6.3.4. *Proposition 2.2.15 explained how $\mathbb{R}^n$ was spanned by the standard basis; $\mathbb{R}^n = span\{e_i\}_{i=1}^n$. Likewise, Proposition 2.2.17 showed the $m \times n$ matrix units E_{ij} spanned the set of all $m \times n$ matrices; $\mathbb{R}^{m \times n} = span\{E_{ij}\}_{i,j=1}^n$.*

Example 6.3.5. *Let $S = \{1, x, x^2, \dots, x^n\}$ then $span(S) = P_n$. For example,*

$$span\{1, x, x^2\} = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\} = P_2$$

The set of all polynomials is spanned by $\{1, x, x^2, x^3, \dots\}$. We are primarily interested in the span of finite sets however this case is worth mentioning.

Example 6.3.6. *Let $W = \{(s + t, 2s + t, 3s + t) \mid s, t \in \mathbb{R}\}$. Observe,*

$$(s + t, 2s + t, 3s + t) = s(1, 2, 3) + t(1, 1, 1)$$

thus $W = \{s(1, 2, 3) + t(1, 1, 1) \mid s, t \in \mathbb{R}\} = span\{(1, 2, 3), (1, 1, 1)\}$. Therefore, Theorem 6.3.3 gives us $W \leq \mathbb{R}^3$.

The lesson of the last example is that we can show a particular space is a subspace by finding its generating set. Theorem 6.3.3 tells us that any set generated by a span is a subspace. This test is only convenient for subspaces which are defined as some sort of span. In that case we can immediately conclude the subset is in fact a subspace.

Example 6.3.7. *Suppose $a, b, c \in \mathbb{R}$ and $a \neq 0$. Consider the differential equation $ay'' + by' + cy = 0$. There is a theorem in the study of differential equations which states* **every** *solution can be written as a linear combination of a pair of special solutions y_1, y_2; we say $y = c_1y_1 + c_2y_2$ is the "general solution" in the terminology of Differential Equations. In other words, there exist solutions y_1, y_2 such that the solution set S of $ay'' + by' + cy = 0$ is*

$$S = span\{y_1, y_2\}.$$

Since S is a span it is clear that $S \leq \mathcal{F}(\mathbb{R})$.

Example 6.3.8. *Suppose $L = P(D)$ where $D = d/dx$ and P is a polynomial with real coefficients. This makes L a smooth operator on the space of smooth functions. Suppose $deg(P) = n$, a theorem in differential equations states that there exist solutions $y_1, y_2, \dots, y_n$ of $L[y] = 0$ such that* **every** *solution of $L[y] = 0$ can be written in the form $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ for constants $c_1, c_2, \dots, c_n \in \mathbb{R}$. In other words, the solution set S of $L[y] = 0$ is formed from a span:*

$$S = span\{y_1, y_2, \dots, y_n\}.$$

Notice the last example is a subcase of this example. Simply set $L = aD^2 + bD + c$.

Perhaps the examples above were too abstract for you at this point. Let me give a couple specific examples in the same vein.

Example 6.3.9. *Consider $y' = y$. Or, taking t as the independent variable, $\frac{dy}{dt} = y$. Separation of variables (that you are expected to know from calculus II) shows $\frac{dy}{y} = dt$ hence $\ln|y| = t + c$. It follows that $y = \pm e^c e^t$. Note $y = 0$ is also a solution of $y' = y$. In total, we find solutions of the form $y = c_1e^t$. The solution set of this differential equation is a span; $S = span\{e^t\} \leq \mathcal{F}(\mathbb{R})$.*

Example 6.3.10. *Consider $y'' - y = 0$. I invite the reader to verify that $y_1 = \cosh(t)$ and $y_2 = \sinh(t)$ are solutions. The solution set is $S = span\{y_1, y_2\} \leq \mathcal{F}(\mathbb{R})$.*

Example 6.3.11. *Consider $y''+y = 0$. I invite the reader to verify that $y_1 = \cos(t)$ and $y_2 = \sin(t)$ are solutions. The solution set is $S = span\{y_1, y_2\} \leq \mathcal{F}(\mathbb{R})$. Physically, this could represent Newton's equation for a spring with mass $m = 1$ and stiffness $k = 1$, the set of all possible physical motions forms a linear subspace of function space.*

Example 6.3.12. *Consider, $y''' = 0$. Integrate both sides to find $y'' = c_1$. Integrate again to find $y' = c_1t + c_2$. Integrate once more, $y = c_1\frac{1}{2}t^2 + c_2t + c_3$. The general solution of $y''' = 0$ is a subspace S of function space:*

$$S = span\left\{\frac{1}{2}t^2,\ t,\ 1\right\} \leq \mathcal{F}(\mathbb{R})$$

Physically, we often consider the situation $c_1 = -g$.

The analysis in Examples 6.3.9 and 6.3.12 simply derive from combining prerequisite calculus knowledge with linear algebra. In constrast, you'd need to take our Differential Equations course to understand how I found y_1, y_2 as in Examples 6.3.10 and 6.3.11. Actually, given our current way of thinking[12] it would be fairly easy to produce the major theorem on the solution set of constant coefficient differential equations. Perhaps I will assign some homework which develops this idea more completely. It should wait until we study *linear transformations* (aka the next story arc in this course).

Example 6.3.13. Subspaces associated with a given matrix: *Let $A \in \mathbb{R}^{m\times n}$. Define* **column space** *of A as the span of the columns of A:*

$$\boxed{Col(A) = span\{col_j(A) \mid j = 1, 2, \dots, n\}}$$

this is clearly a subspace of $\mathbb{R}^m$ since each column has as many components as there are rows in A. We also define **row space** *as the span of the rows:*

$$\boxed{Row(A) = span\{row_i(A) \mid i = 1, 2, \dots, m\}}$$

[12] ninja way if you prefer this terminology

this is clearly a subspace of $\mathbb{R}^{1\times n}$ *since it is formed as a span of vectors. Since the columns of* A^T *are the rows of* A *and the rows of* A^T *are the columns of* A *we can conclude that* $Col(A^T) = Row(A)$ *and* $Row(A^T) = Col(A)$*. Finally, we define the* **null space** *of* A *by:*

$$\boxed{Null(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}}$$

Observe, we studied this in Example 6.2.10 where we showed $Null(A) \leq \mathbb{R}^n$*. With some effort and the insight of the* $rref(A)$ *we can write the null space as a span of an appropriate set of solutions to* $Ax = 0$*. See Example 6.2.10 for instance.*

I would remind the reader we have the CCP and associated techniques to handle spanning questions for column vectors. In contrast, the following example requires a direct assault[13]:

Example 6.3.14. *Is* $E_{11} \in span\{E_{12} + 2E_{11}, E_{12} - E_{11}\}$*? Assume* $E_{ij} \in \mathbb{R}^{2\times 2}$ *for all* i, j*. We seek to find solutions of*

$$E_{11} = a(E_{12} + 2E_{11}) + b(E_{12} - E_{11})$$

in explicit matrix form the equation above reads:

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= a\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) + b\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2a & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -b & b \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2a - b & a + b \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

thus $1 = 2a - b$ *and* $0 = a + b$*. Substitute* $a = -b$ *to find* $1 = 3a$ *hence* $a = \frac{1}{3}$ *and* $b = \frac{-1}{3}$*. Indeed,*

$$\frac{1}{3}(E_{12} + 2E_{11}) - \tfrac{1}{3}(E_{12} - E_{11}) = \tfrac{2}{3}E_{11} + \tfrac{1}{3}E_{11} = E_{11}.$$

Therefore, $E_{11} \in span\{E_{12} + 2E_{11}, E_{12} - E_{11}\}$*.*

Example 6.3.15. *Find a generating set for the set of symmetric* 2×2 *matrices. That is find a set* S *of matrices such that* $span(S) = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\} = W$*. There are many approaches, but I find it most natural to begin by studying the condition which defines* W*. Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ *and note*

$$A^T = A \ \Leftrightarrow\ \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ \Leftrightarrow\ A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

In summary, we find an equivalent way to express $A \in W$ *is as:*

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = aE_{11} + b(E_{12} + E_{21}) + dE_{22}.$$

Consequently $W = span\{E_{11},\ E_{12} + E_{21},\ E_{22}\}$ *and the set* $\{E_{11},\ E_{12} + E_{21},\ E_{22}\}$ *generates* W*. This is not unique, there are many other sets which also generate* W*. For example, if we took* $\bar{S} = \{E_{11},\ E_{12} + E_{21},\ E_{22}, E_{11} + E_{22}\}$ *then the span of* $\bar{S}$ *would still work out to* W*.*

[13] However, once we have the idea of *coordinates* ironed out then we can use the CCP tricks on the coordinate vectors then push back the results to the world of abstract vectors. For now we'll just confront each question by brute force. For an example such as this, the method used here is as good as our later methods.

6.4 linear independence

We have seen a variety of generating sets in the preceding section. In the last example I noted that if we added an additional vector $E_{11}+E_{22}$ then the same span would be created. The vector $E_{11}+E_{22}$ is **redundant** since we already had E_{11} and E_{22}. In particular, $E_{11}+E_{22}$ is a linear combination of E_{11} and E_{22} so adding it will not change the span. How can we decide if a vector is absolutely necessary for a span? In other words, if we want to span a subspace W then how do we find a **minimal spanning set**? We want a set of vectors which does not have any linear dependences. We say such vectors are linearly independent. Let me be precise[14]:

Definition 6.4.1.

Let $S \subseteq V$ a vector space over a field $\mathbb{F}$. The set of vectors S is Linearly Independent (LI) iff for any $\{v_1, v_2, \dots, v_k\} \subseteq S$ and $c_1, \dots, c_k \in \mathbb{F}$,

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0.$$

Furthermore, to be clear, $\emptyset$ is LI. Moreover, if S is not LI then S is **linearly dependent**.

Example 6.4.2. *Let $v = \cos^2(t)$ and $w = 1+\cos(2t)$. Clearly v, w are linearly dependent since $w - 2v = 0$. We should remember from trigonometry $\cos^2(t) = \frac{1}{2}\big(1+\cos(2t)\big)$.*

It turns out the claim in the Proposition below does not generalize to the study of modules[15] so this is why we should use the Definition above as stated. Furthermore, this definition is oft used for calculations in our study of linear independence.

Proposition 6.4.3.

There exists at least one v_j which can be written as a linear combination of vectors $\{v_1, v_2, v_{j-1}, v_{j+1} \dots, v_k\}$ iff $\{v_1, v_2, \dots, v_k\}$ is a **linearly dependent** set.

Proof: ($\Rightarrow$) Suppose $\{v_1, v_2, \dots, v_k\}$ is linearly independent. Assume that there exist constants $c_1, c_2, \dots, c_k \in \mathbb{F}$ such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

and at least one constant, say c_j, is nonzero. Then we can divide by c_j to obtain

$$\tfrac{c_1}{c_j}v_1 + \tfrac{c_2}{c_j}v_2 + \cdots + v_j + \cdots + \tfrac{c_k}{c_j}v_k = 0$$

solve for v_j, (we mean for $\widehat{v_j}$ to denote the deletion of v_j from the list)

$$v_j = -\tfrac{c_1}{c_j}v_1 - \tfrac{c_2}{c_j}v_2 - \cdots - \widehat{v_j} - \cdots - \tfrac{c_k}{c_j}v_k$$

but this means that v_j linearly depends on the other vectors hence $\{v_1, v_2, \dots, v_k\}$ is linearly dependent. This is a contradiction, therefore $c_j = 0$. Note j was arbitrary so we may conclude $c_j = 0$ for all j. Therefore, $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0$.

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[14] if you have a sense of deja vu here, it is because I uttered many of the same words in the context of $\mathbb{R}^n$. Notice, in constrast, I now consider the abstract case. We cannot use the CCP directly here

[15] A **module** is like a vector space, except, we use scalars from a ring. For example, if we consider pairs from $\mathbb{Z}/6\mathbb{Z}$ we have $2(3,0)+3(0,2) = 0$ hence $\{(3,0),(0,2)\}$ is linearly dependent. But, note we cannot solve for $(3,0)$ as a multiple of $(0,2)$. In short, the solving for a vector criteria is special to using scalars from a field.

Proof: ($\Leftarrow$) Assume that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0.$$

If $v_j = b_1v_1 + b_2v_2 + \cdots + \widehat{b_jv_j} + \cdots + b_kv_k$ then $b_1v_1 + b_2v_2 + \cdots + b_jv_j + \cdots + b_kv_k = 0$ where $b_j = -1$, this is a contradiction. Therefore, for each j, v_j is not a linear combination of the other vectors. Consequently, $\{v_1, v_2, \dots, v_k\}$ is linearly independent. $\square$

What follows could also be used to capture the concept of LI. In short, our ability to equate coefficients for a given set of objects is interchangeable with the LI of the set of objects. This proposition is extremely important to the logical foundation of *coordinate maps*. We soon identify the coefficients $a_1, \dots, a_k$ as coordinates and this proposition basically says they are uniquely given when we use a LI set.

Proposition 6.4.4.

Let V be a vector space over a field $\mathbb{F}$ and $S \subseteq V$. S is a linearly independent set of vectors iff whenever there are constants $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{F}$ such that

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k = b_1v_1 + b_2v_2 + \cdots + b_kv_k$$

then $a_1 = b_1$, $a_2 = b_2, \dots, a_k = b_k$. In other words, we can equate coefficients of a set of vectors iff the set of vectors is a LI set.

Proof: likely homework problem. $\square$

In retrospect, partial fractions in Calculus II is based on the LI of the basic rational functions. The technique of equating coefficients only made sense because the set of functions involved was in fact LI. Likewise, if you've taken Differential Equations, you can see LI of solutions being utilized througout the undetermined coefficients technique. The ubiquity of the role of LI in common mathematical calculation is hard to overstate.

Proposition 6.4.5.

If S is a finite set of vectors which contains the zero vector then S is linearly dependent.

Proof: Let $\{\vec{0}, v_2, \dots v_k\} = S$ and observe that $1 \cdot \vec{0} = 0$ thus S is not linearly independent, that is, S is linearly dependent. $\square$

Proposition 6.4.6.

Let v and w be nonzero vectors.

$$v, w \text{ are linearly dependent } \Leftrightarrow \exists k \neq 0 \in \mathbb{F} \text{ such that } v = kw.$$

Proof: Let $v, w \neq 0$. Suppose $\{v, w\}$ is linearly dependent. Hence we find $c_1, c_2 \in \mathbb{F}$, not both zero, for which $c_1v + c_2w = 0$. Without loss of generality suppose $c_1 \neq 0$ then $v = -\frac{c_2}{c_1}w$ so identify $k = -c_2/c_1$ and $v = kw$. Conversely, if $v = kw$ with $k \neq 0$ then $v - kw = 0$ shows $\{v, w\}$ is linearly dependent. $\square$

Remark 6.4.7.

We should keep in mind that in the abstract context statements such as "v and w go in the same direction" or "u is contained in the plane spanned by v and w" are not statments about ordinary three dimensional geometry. Moreover, you **cannot** write that $u, v, w \in \mathbb{R}^n$ unless you happen to be working with that rather special vector space. I caution the reader against the common mistake of trying to use column calculation techniques for things which are not columns. This is a very popular mistake in past years.

Given a set of vectors in $\mathbb{F}^n$ the question of LI is elegantly answered by the CCP. In examples that follow in this section we leave the comfort zone and study LI in abstract vector spaces. For now we only have brute force at our disposal. In other words, I'll argue directly from the definition without the aid of the CCP from the outset.

Example 6.4.8. *Suppose $f(x) = \cos(x)$ and $g(x) = \sin(x)$ and define $S = \{f, g\}$. Is S linearly independent with respect to the standard vector space structure on $\mathcal{F}(\mathbb{R})$? Let $c_1, c_2 \in \mathbb{R}$ and assume that*

$$c_1 f + c_2 g = 0.$$

It follows that $c_1 f(x) + c_2 g(x) = 0$ for each $x \in \mathbb{R}$. In particular,

$$c_1 \cos(x) + c_2 \sin(x) = 0$$

for each $x \in \mathbb{R}$. Let $x = 0$ and we get $c_1 \cos(0) + c_2 \sin(0) = 0$ thus $c_1 = 0$. Likewise, let $x = \pi/2$ to obtain $c_1 \cos(\pi/2) + c_2 \sin(\pi/2) = 0 + c_2 = 0$ hence $c_2 = 0$. We have shown that $c_1 f + c_2 g = 0$ implies $c_1 = c_2 = 0$ thus $S = \{f, g\}$ is a linearly independent set.

Example 6.4.9. *Let $f_n(t) = t^n$ for $n = 0, 1, 2, \dots$. Suppose $S = \{f_0, f_1, \dots, f_n\}$. Show S is a linearly independent subset of function space. Assume $c_0, c_1, \dots, c_n \in \mathbb{R}$ and*

$$c_0 f_0 + c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0. \qquad \star$$

I usually skip the expression above, but I'm including this extra step to emphasize the distinction between the function and its formula. The $\star$ equation is a function equation, it implies

$$c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0 \qquad \star\star$$

for all $t \in \mathbb{R}$. Evaluate $\star\star$ at $t = 0$ to obtain $c_0 = 0$. Differentiate $\star^2$ and find

$$c_1 + 2c_2 t + \cdots + n c_n t^{n-1} = 0 \qquad \star^3$$

Evaluate $\star^3$ at $t = 0$ to obtain $c_1 = 0$. If we continue to differentiate and evaluate we will similarly obtain $c_2 = 0$, $c_3 = 0$ and so forth all the way up to $c_n = 0$. Therefore, $\star$ implies $c_0 = c_1 = \cdots = c_n = 0$.

Linear dependence in function space is sometimes a source of confusion for students. The idea of evaluation doesn't help in the same way as it just has in the two examples above.

Example 6.4.10. *Let $f(t) = t - 1$ and $g(t) = t + t^2$ is f linearly dependent on g? A common mistake is to say something like $f(1) = 1 - 1 = 0$ so $\{f, g\}$ is linearly independent since it contains zero. Why is this wrong? The reason is that we have confused the value of the function with the function itself. If $f(t) = 0$ for all $t \in \mathbb{R}$ then f is the zero function which is the zero vector in function space. Many functions will be zero at a point but that doesn't make them the zero function.*

To prove linear dependence we must show that there exists $k \in \mathbb{R}$ such that $f = kg$, but this really means that $f(t) = kg(t)$ for all $t \in \mathbb{R}$ in the current context. I leave it to the reader to prove that $\{f, g\}$ is in fact LI. You can evaluate at $t = 1$ and $t = 0$ to obtain equations for c_1, c_2 which have a unique solution of $c_1 = c_2 = 0$.

Example 6.4.11. *Let $f(t) = t^2 - 1$, $g(t) = t^2 + 1$ and $h(t) = 4t^2$. Suppose*

$$c_1(t^2 - 1) + c_2(t^2 + 1) + c_3(4t^2) = 0 \qquad \star$$

A little algebra reveals,

$$(c_1 + c_2 + 4c_3)t^2 - (c_1 - c_2)1 = 0$$

Using linear independence of t^2 and 1 we find

$$c_1 + c_2 + 4c_3 = 0 \qquad \textit{and} \qquad c_1 - c_2 = 0$$

We find infinitely many solutions,

$$c_1 = c_2 \qquad \textit{and} \qquad c_3 = -\frac{1}{4}(c_1 + c_2) = -\frac{1}{2}c_2$$

Therefore, $\star$ allows nontrivial solutions. Take $c_2 = 1$,

$$1(t^2 - 1) + 1(t^2 + 1) - \frac{1}{2}(4t^2) = 0.$$

We can write one of these functions as a linear combination of the other two,

$$f = -g + \frac{1}{2}h.$$

Once we get past the formalities of the particular vector space structure it always comes back to solving systems of linear equations.

6.5 basis, dimension and coordinates

We have seen that linear combinations can generate vector spaces. We have also seen that sometimes we can remove a vector from the generating set and still generate the whole vector space. For example,

$$span\{e_1, e_2, e_1 + e_2\} = \mathbb{R}^2$$

and we can remove any one of these vector and still span $\mathbb{R}^2$,

$$span\{e_1, e_2\} = span\{e_1, e_1 + e_2\} = span\{e_2, e_1 + e_2\} = \mathbb{R}^2$$

However, if we remove another vector then we will not span $\mathbb{R}^2$. A generating set which is just big enough is called a basis. We can remove vectors which are linearly dependent on the remaining vectors without changing the span. Therefore, we should expect that a minimal spanning set is linearly independent. The intuition I outline will be carefully proven in the next several sections. Let me settle a definition to get the conversation started here:

Definition 6.5.1.

A **basis** for a vector space V over a field $\mathbb{F}$ is an **ordered** set of vectors S such that

1. $V = span(S)$,
2. S is linearly independent.

Next, we consider Theorem 5.1 from Charles Curtis' *Linear Algebra: An Introductory Approach*.

Theorem 6.5.2.

Let V be a vector space over a field $\mathbb{F}$ and suppose $W = \text{span}\{v_1, \dots, v_n) \subseteq V$. If $\{w_1, \dots, w_m\} \subseteq W$ and $m > n$ then $\{w_1, \dots, w_m\}$ is linearly dependent.

Proof: is given on page 34-35 of Curtis' text. $\square$

Of course, if $V = \mathbb{F}^n$ then we already know this result. It was Corollary 4.3.6.

Theorem 6.5.3.

If V is a vector space over a field $\mathbb{F}$ and $\{v_1, \dots, v_n\} \subseteq V$ and $\{w_1, \dots, w_m\}$ are linearly independent with $\text{span}\{v_1, \dots, v_n\} = \text{span}\{w_1, \dots, w_m\}$ then $m = n$.

Proof: Suppose $m > n$. Observe $w_i \in \text{span}\{v_1, \dots, v_n\}$ for each $i \in \mathbb{N}_m$ thus $\{w_1, \dots, w_m\} \subseteq \text{span}\{v_1, \dots, v_n\}$. Therefore, by Theorem 6.5.2 $\{w_1, \dots, w_m\}$ is linearly dependent. But, $\{w_1, \dots, w_m\}$ is given to be LI hence we obtain a contradiction. Likewise, if $n < m$ we can argue the LI of $\{v_1, \dots, v_n\}$ is contradicted. It follows $m = n$. $\square$

We are now free to make the following definition[16]:

Definition 6.5.4.

Let V be a vector space over a field $\mathbb{F}$. If $\beta \subseteq V$ is a finite basis and $\#(\beta)$ denotes the number of vectors in β then we define the **dimension** of V by $\dim(V) = \#(\beta)$. In general, if β is a basis for V then $\dim(V)$ is the **cardnality** of β. If V over $\mathbb{F}$ has a finite basis then V is a **finite dimensional** vector space over $\mathbb{F}$.

Example 6.5.5. *I called $\{e_1, e_2, \dots, e_n\}$ the standard basis of $\mathbb{R}^n$. Since $v \in \mathbb{R}^n$ can be written as*

$$v = \sum_i v_i e_i$$

it follows $\mathbb{R}^n = span\{e_i \mid 1 \leq i \leq n\}$. Moreover, linear independence of $\{e_i \mid 1 \leq i \leq n\}$ follows from a simple calculation:

$$0 = \sum_i c_i e_i \;\Rightarrow\; 0 = \left[\sum_i c_i e_i\right]_k = \sum_i c_i \delta_{ik} = c_k$$

hence $c_k = 0$ for all k. Thus $\{e_i \mid 1 \leq i \leq n\}$ is a basis for $\mathbb{R}^n$, we continue to call it the **standard basis** *of $\mathbb{R}^n$. The vectors e_i are also called "unit-vectors".*

[16] at least in the context of vector spaces you now possess a proper definition, however, the term dimension also has meaning in less structured contexts such as a topological space or manifold, but, that is for another course. Also, you will note, I have not shown this definition is reasonable in the infinite case, I leave those details to the eager reader

Example 6.5.6. *Since $A \in \mathbb{R}^{m\times n}$ can be written as*

$$A = \sum_{i,j} A_{ij}E_{ij}$$

it follows $\mathbb{R}^{m\times n} = span\{E_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Moreover, linear independence of $\{E_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ follows from a simple calculation:

$$0 = \sum_{i,j} c_{ij}E_{ij} \ \Rightarrow \ 0 = \left[\sum_{i,j} c_{ij}E_{ij}\right]_{kl} = \sum_{i,j} c_{ij}\delta_{ik}\delta_{jl} = c_{kl}$$

hence $c_{kl} = 0$ for all k, l. Thus $\{E_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis for $\mathbb{R}^{m\times n}$, we continue to call it the **standard basis** *of $\mathbb{R}^{m\times n}$. The matrices E_{ij} are also called "unit-matrices".*

Example 6.5.7. *$\mathbb{R}^n, \mathbb{R}^{m\times n}, P_n(\mathbb{R})$ are examples of finite-dimensional vector spaces. On the other hand, $\mathcal{F}(\mathbb{R}), C^0(\mathbb{R}), C^1(\mathbb{R}), C^\infty(\mathbb{R})$ are infinite-dimensional.*

Example 6.5.8. *If $V = \mathbb{R}$ and we use $\mathbb{Q}$ as the set of scalars then V can be shown to be an infinite dimensional $\mathbb{Q}$-vector space. In contrast, $\mathbb{R}$ is a one-dimensional real vector space with basis $\{1\}$. Or, to be less bizarre, consider $\mathbb{C}$ is a one-complex-dimensional vector space with basis $\{1\}$ whereas $\mathbb{C}$ can also be viewed as a two-real-dimensional vector space with basis $\{1, i\}$.*

A given point set may have different dimension depending on the context. The last example gives a rather exotic example of that idea. What follows is probably easier to understand:

Example 6.5.9. *$\mathbb{C}$ is a one-dimensional vector space over $\mathbb{C}$ with basis $\{1\}$. However, $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$ with basis $\{1, i\}$. Note: $z = x + iy = x(1) + y(i)$ hence $span_{\mathbb{R}}\{1, i\} = \mathbb{C}$. Moveover, if $c_1, c_2 \in \mathbb{R}$ and $c_1(1) + c_2(i) = 0$ then we obtain $c_1 = c_2 = 0$. Hence, $\{1, i\}$ is LI over $\mathbb{R}$ and we have $dim_{\mathbb{R}}(\mathbb{C}) = 2$. On the other hand, since $z = z(1)$ for all $z \in \mathbb{C}$ and $\{1\}$ is linearly independent it follows $dim_{\mathbb{C}}(\mathbb{C}) = 1$ thus $\{1, i\}$ is linearly-dependendent. Indeed, it is obvious that $i = i(1)$ thus 1 and i are linearly dependent over $\mathbb{C}$.*

When multiple fields are in use it is wise to adorn the dimension notation with the field to reduce possible confusions. Usually, we work with just one field so we omit the explicit field dependence[17].

Proposition 6.5.10.

Suppose $B = \{f_1, f_2, \dots, f_n\}$ is a basis for V over a field $\mathbb{F}$. If $v \in V$ with $v = \sum_{i=1}^n x_if_i$ and $v = \sum_{i=1}^n y_if_i$ for constants $x_i, y_i \in \mathbb{F}$. Then $x_i = y_i$ for $i = 1, 2, \dots, n$.

Proof: Suppose $v = x_1f_1 + x_2f_2 + \cdots + x_nf_n$ and $v = y_1f_1 + y_2f_2 + \cdots + y_nf_n$ notice that

$$\begin{aligned} 0 = v - v &= (x_1f_1 + x_2f_2 + \cdots + x_nf_n) - (y_1f_1 + y_2f_2 + \cdots + y_nf_n) \\ &= (x_1 - y_1)f_1 + (x_2 - y_2)f_2 + \cdots + (x_n - y_n)f_n \end{aligned}$$

then by LI of the basis vectors we find $x_i - y_i = 0$ for each i. Thus $x_i = y_i$ for all i. $\square$

Notice that both LI and spanning were necessary for the idea of a coordinate vector (defined below) to make sense.

[17] the logical minimalist will notice this is also a previous proposition, but, I include the proof here

Definition 6.5.11.

Suppose $\beta = \{f_1, f_2, \dots, f_n\}$ is a basis for vector space V over field $\mathbb{F}$. If $v \in V$ has

$$v = v_1 f_1 + v_2 f_2 + \cdots + v_n f_n$$

then $[v]_\beta = [v_1 \ v_2 \ \cdots \ v_n]^T \in \mathbb{R}^n$ is called the **coordinate vector** of v with respect to β.

Technically, the each basis considered in the course is an "ordered basis". This means the set of vectors that forms the basis has an ordering to it. This is more structure than just a plain set since basic set theory does not distinguish $\{1, 2\}$ from $\{2, 1\}$. I should always say "we have an ordered basis" but I will not (and most people do not) say that in this course. Let it be understood that when we list the vectors in a basis they are listed in order and we cannot change that order without changing the basis. For example $v = (1, 2, 3)$ has coordinate vector $[v]_{B_1} = (1, 2, 3)$ with respect to $B_1 = \{e_1, e_2, e_3\}$. On the other hand, if $B_2 = \{e_2, e_1, e_3\}$ then the coordinate vector of v with respect to B_2 is $[v]_{B_2} = (2, 1, 3)$.

Example 6.5.12. *Let $\beta = \{E_{11}, E_{12}, E_{22}, E_{21}\}$. Observe:*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_{11} + bE_{12} + dE_{22} + cE_{21}.$$

Therefore, $[A]_\beta = (a, b, d, c)$.

Example 6.5.13. *Consider $\beta = \{(t+1)^2, t+1, 1\}$ and calculate the coordinate vector of $f(t) = t^2$ with respect to β. I often use an adding zero trick for such a problem:*

$$f(t) = t^2 = (t + 1 - 1)^2 = (t + 1)^2 - 2(t + 1) + 1.$$

From the expression above we can read that $[f(t)]_\beta = (1, -2, 1)$.

Example 6.5.14. *Suppose A in invertible and $Av = b$ has solution $v = (1, 2, 3, 4)$. It follows that A has 4 columns. Define,*

$$\beta = \{col_4(A), col_3(A), col_2(A), col_1(A)\}$$

Given that $(1, 2, 3, 4)$ is a solution of $Av = b$ we know:

$$col_1(A) + 2col_2(A) + 3col_3(A) + 4col_4(A) = b$$

Given the above, we can deduce $[b]_\beta = (4, 3, 2, 1)$.

The three examples above were simple enough that not much calculation was needed. Understanding the definition of basis was probably the hardest part. In general, finding the coordinates of a vector with respect to a given basis is a spanning problem.

Example 6.5.15. *Let $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ find the coordinates of v relative to B_1, B_2 and B_3 where $B_1 = \{e_1, e_2\}$ and $B_2 = \{e_1, e_1 + e_2\}$ and $B_3 = \{e_2, e_1 + e_2\}$. We'll begin with the standard basis, (I hope you could see this without writing it)*

$$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1e_1 + 3e_2$$

thus $[v]_{B_1} = [1\ 3]^T$*. Find coordinates relative to the other two bases is not quite as obvious. Begin with* B_2*. We wish to find* x, y *such that*

$$v = xe_1 + y(e_1 + e_2)$$

we can just use brute-force,

$$v = e_1 + 3e_2 = xe_1 + y(e_1 + e_2) = (x + y)e_1 + ye_2$$

using linear independence of the standard basis we find $1 = x + y$ *and* $y = 3$ *thus* $x = 1 - 3 = -2$ *and we see* $v = -2e_1 + 3(e_1 + e_2)$ *so* $[v]_{B_2} = [-2\ 3]^T$*. This is interesting, the same vector can have different coordinate vectors relative to distinct bases. Finally, let's find coordinates relative to* B_3*. I'll try to be more clever this time: we wish to find* x, y *such that*

$$v = xe_2 + y(e_1 + e_2) \quad \Leftrightarrow \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We can solve this via the augemented coefficient matrix

$$rref \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 3 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right] \quad \Leftrightarrow \quad x = 2,\ y = 1.$$

Thus, $[v]_{B_3} = [2\ 1]^T$*. Notice this is precisely the rightmost column in the rref matrix. Perhaps my approach for* B_3 *is a little like squashing a fly with with a dumptruck. However, once we get to an example with 4-component vectors you may find the matric technique useful.*

Example 6.5.16. *Given that* $B = \{b_1, b_2, b_3, b_4\} = \{e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4\}$ *is a basis for* $\mathbb{R}^4$ *find coordinates for* $v = [1, 2, 3, 4]^T \in \mathbb{R}^4$*. Given the discussion in the preceding example it is clear we can find coordinates* $[x_1, x_2, x_3, x_4]^T$ *such that* $v = \sum_i x_i b_i$ *by calculating* $rref[b_1|b_2|b_3|b_4|v]$ *the rightmost column will be* $[v]_B$*.*

$$rref \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 4 \end{array} \right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \quad \Rightarrow \quad \boxed{[v]_B = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}}$$

The calculation above should be familar. We discussed it at length in the spanning section.

Example 6.5.17. *We can prove that* S *from Example 6.3.15 is linearly independent, thus symmetric* 2×2 *matrices have a* S *as a basis*

$$S = \{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right],\ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right],\ \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \}$$

thus the dimension of the vector space of 2×2 *symmetric matrices is 3. (notice* $\bar{S}$ *from that example is not a basis because it is linearly dependent). While we're thinking about this let's find the coordinates of* $A = \left[\begin{smallmatrix} 1 & 3 \\ 3 & 2 \end{smallmatrix}\right]$ *with respect to* S*. Denote* $[A]_S = [x, y, z]^T$*. We calculate,*

$$\left[\begin{smallmatrix} 1 & 3 \\ 3 & 2 \end{smallmatrix}\right] = x \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + y \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] + z \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \quad \Rightarrow \quad [A]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Remark 6.5.18.

Curvelinear coordinate systems from calculus III are in a certain sense more general than the idea of a coordinate system in linear algebra. If we focus our attention on a single point in space then a curvelinear coordinate system will produce three linearly independent vectors which are tangent to the coordinate curves. However, if we go to a different point then the curvelinear coordinate system will produce three different vectors in general. For example, in spherical coordinates the radial unit vector is $e_\rho = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$ and you can see that different choices for the angles θ, ϕ make e_ρ point in different directions. In contrast, in this course we work with vector spaces. Our coordinate systems have the same basis vectors over the whole space. Vector spaces are examples of $flat$ manifolds since they allow a single global coordinate system. Vector spaces also allow for curvelinear coordinates (which are not coordinates in the sense of linear algebra). However the converse is not true; spaces with nonzero curvature do not allow for global coordinates. I digress, we may have occassion to discuss these matters more cogently in our Advanced Calculus course (Math 332)(join us)

6.5.1 how to calculate a basis for a span of row or column vectors

Given some subspace of $\mathbb{F}^n$ we would like to know how to find a basis for that space. In particular, if $V = span\{v_1, v_2, \dots, v_k\}$ then what is a basis for W? Likewise, given some set of row vectors $W = \{w_1, w_2, \dots w_k\} \subset \mathbb{F}^{1\times n}$ how can we select a basis for $span(W)$. We would like to find answers to these question since most subspaces are characterized either as spans or solution sets(see the next section on $Null(A)$). We already have the tools to answer these questions, we just need to apply them to the tasks at hand.

Proposition 6.5.19.

Let $W = span\{v_1, v_2, \dots, v_k\} \subset \mathbb{F}^n$ then a basis for W can be obtained by selecting the vectors that reside in the pivot columns of $[v_1|v_2|\cdots|v_k]$.

Proof: this is immediately obvious from Proposition 4.4.1. $\square$

The proposition that follows is also follows immediately from Proposition 4.4.1.

Proposition 6.5.20.

Let $A \in \mathbb{F}^{m\times n}$ the pivot columns of A form a basis for $Col(A)$.

Example 6.5.21. *Suppose A is given as below: (I omit the details of the Gaussian elimination)*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Rightarrow \qquad rref[A] = \begin{bmatrix} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Identify that columns 1,2 and 4 are pivot columns. Moreover,

$$Col(A) = span\{col_1(A), col_2(A), col_4(A)\}$$

In particular we can also read how the second column is a linear combination of the basis vectors.

$$\begin{aligned} col_3(A) &= \tfrac{5}{3}col_1(A) + \tfrac{2}{3}col_2(A) \\ &= \tfrac{5}{3}[1,2,0]^T + \tfrac{2}{3}[2,1,0]^T \\ &= [5/3, 10/3, 0]^T + [4/3, 2/3, 0]^T \\ &= [3,4,0]^T \end{aligned}$$

What if we want a basis for $Row(A)$ which consists of rows in A itself?

Proposition 6.5.22.

Let $W = span\{w_1, w_2, \dots, w_k\} \subset \mathbb{F}^{1\times n}$ and construct A by concatenating the row vectors in W into a matrix A:

$$A = \begin{bmatrix} w_1 \\ \hline w_2 \\ \hline \vdots \\ \hline w_k \end{bmatrix}$$

A basis for W is given by the transposes of the pivot columns for A^T.

Proof: this is immediately obvious from Proposition 4.4.5. $\square$

The proposition that follows is also follows immediately from Proposition 4.4.5. Incidentally, it is almost more important to notice what we do **not** say below; we do not say the pivot rows of A form a basis for the row space.

Proposition 6.5.23.

Let $A \in \mathbb{F}^{m\times n}$, the rows which are transposes of the pivot columns of A^T form a basis for $Row(A)$.

Example 6.5.24.

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 0 \\ 4 & 1 & 3 \end{bmatrix} \qquad \Rightarrow \qquad rref[A^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that each column is a pivot column in A^T thus a basis for $Row(A)$ is simply the set of all rows of A; $Row(A) = span\{[1,2,3,4],[2,1,4,1],[0,0,0,3]\}$ and the spanning set is linearly independent.

Example 6.5.25.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 2 \end{bmatrix} \qquad \Rightarrow \qquad A^T = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 0 & 2 \end{bmatrix} \qquad \Rightarrow \qquad rref[A^T] = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We deduce that rows 1 and 3 or A form a basis for $Row(A)$. Notice that $row_2(A) = 2row_1(A)$ and $row_4(A) = row_3(A) + 2row_1(A)$. We can read linear dependendcies of the rows from the corresponding linear dependencies of the columns in the rref of the transpose.

The preceding examples are nice, but what should we do if we want to find both a basis for $Col(A)$ and $Row(A)$ for some given matrix ? Let's pause to think about how elementary row operations modify the row and column space of a matrix. In particular, let A be a matrix and let A' be the result of performing an elementary row operation on A. It is fairly obvious that

$$Row(A) = Row(A').$$

Think about it. If we swap to rows that just switches the order of the vectors in the span that makes $Row(A)$. On the other hand if we replace one row with a nontrivial linear combination of itself and other rows then that will not change the span either. Column space is not so easy though. Notice that elementary row operations can change the column space. For example,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow rref[A] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

has $Col(A) = span\{[1,1]^T\}$ whereas $Col(rref(A)) = span([1,0]^T)$. We cannot hope to use columns of $ref(A)$ (or $rref(A)$) for a basis of $Col(A)$. That's no big problem though because we already have the CCP-principle which helped us pick out a basis for $Col(A)$. Let's collect our thoughts:

Proposition 6.5.26.

Let $A \in \mathbb{F}^{m\times n}$ then a basis for $Col(A)$ is given by the pivot columns in A and a basis for $Row(A)$ is given by the nonzero rows in $ref(A)$.

This means we can find a basis for $Col(A)$ and $Row(A)$ by performing the forward pass on A. We need only calculate the $ref(A)$ as the pivot columns are manifest at the end of the forward pass.

Example 6.5.27.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = ref[A]$$

We deduce that $\{[1,1,1],[0,1,2]\}$ *is a basis for* $Row(A)$ *whereas* $\{[1,1,1]^T,[1,1,2]^T\}$ *is a basis for* $Col(A)$. *Notice that if I wanted to reveal further linear dependencies of the non-pivot columns on the pivot columns of* A *it would be wise to calculate* $rref[A]$ *by making the backwards pass on* $ref[A]$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2 \to r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = rref[A]$$

From which I can read $col_3(A) = 2col_2(A) - col_1(A)$, *a fact which is easy to verify.*

Example 6.5.28.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 8 & 10 \\ 1 & 2 & 4 & 11 \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix} = ref[A]$$

We find that $Row(A)$ *has basis*

$$\{[1,2,3,4],[0,1,5,6],[0,0,1,7]\}$$

and $Col(A)$ *has basis*

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix} \right\}$$

Proposition 6.5.26 was the guide for both examples above.

6.5.2 calculating basis of a solution set

Often a subspace is described as the solution set of some equation $Ax = 0$. How do we find a basis for $Null(A)$? If we can do that we find a basis for subspaces which are described by some equation.

Proposition 6.5.29.

Let $A \in \mathbb{F}^{m\times n}$ and define $W = Null(A)$. A basis for W is obtained from the solution set of $Ax = 0$ by writing the solution as a linear combination where the free variables appear as coefficients in the vector-sum.

Proof: $x \in W$ implies $Ax = 0$. Denote $x = [x_1, x_2, \dots, x_n]^T$. Suppose that $rref[A]$ has r-pivot columns (we must have $0 \leq r \leq n$). There will be $(m - r)$-rows which are zero in $rref(A)$ and $(n - r)$-columns which are not pivot columns. The non-pivot columns correspond to free-variables in the solution. Define $p = n - r$ for convenience. Suppose that $x_{i_1}, x_{i_2}, \dots, x_{i_p}$ are free whereas $x_{j_1}, x_{j_2}, \dots, x_{j_r}$ are functions of the free variables: in particular they are linear combinations of the free variables as prescribed by $rref[A]$. There exist constants b_{ij} such that

$$\begin{array}{ll} x_{j_1} & = b_{11}x_{i_1} + b_{12}x_{i_2} + \cdots + b_{1p}x_{i_p} \\ x_{j_2} & = b_{21}x_{i_1} + b_{22}x_{i_2} + \cdots + b_{2p}x_{i_p} \\ \vdots & \qquad \vdots \qquad \cdots \qquad \vdots \\ x_{j_r} & = b_{r1}x_{i_1} + b_{r2}x_{i_2} + \cdots + b_{rp}x_{i_p} \end{array}$$

For convenience of notation assume that the free variables are put at the end of the list. We have

$$\begin{array}{ll} x_1 & = b_{11}x_{r+1} + b_{12}x_{r+2} + \cdots + b_{1p}x_n \\ x_2 & = b_{21}x_{r+1} + b_{22}x_{r+2} + \cdots + b_{2p}x_n \\ \vdots & \qquad \vdots \qquad \cdots \qquad \vdots \\ x_r & = b_{r1}x_{r+1} + b_{r2}x_{n-p+2} + \cdots + b_{rp}x_n \end{array}$$

and $x_j = x_j$ for $j = r+1, r+2, \dots, r+p = n$ (those are free, we have no conditions on them, they can take any value). We find,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = x_{r+1}\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{r1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{r+2}\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{r2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n\begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{rp} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We define the vectors in the sum above as $v_1, v_2, \dots, v_p$. If any of the vectors, say v_j, was linearly dependent on the others then we would find that the variable x_{r+j} was likewise dependent on the other free variables. This would contradict the fact that the variable x_{r+j} was free. Consequently the vectors $v_1, v_2, \dots, v_p$ are linearly independent. Moreover, they span the null-space by virtue of their construction. $\square$

Didn't follow the proof above? You might do well to sort through the example below before attempting a second reading. The examples are just the proof in action for specific cases.

Example 6.5.30. *Find a basis for the null space of $A = [1, 2, 3, 4]$. This example requires no additional calculation except this; $Ax = 0$ for $x = (x_1, x_2, x_3, x_4)$ yields $x_1 = -2x_2 - 3x_3 - 4x_4$ thus:*

$$x = \begin{bmatrix} -2x_2 - 3x_3 - 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, $\{(-2, 1, 0, 0), (-3, 0, 1, 0), (-4, 0, 0, 1)\}$ forms a basis for $Null(A)$.

Example 6.5.31. *Find a basis for the null space of A given below,*

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 3.3.11 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 \end{bmatrix}$$

Denote $x = [x_1, x_2, x_3, x_4, x_5]^T$ in the equation $Ax = 0$ and identify from the calculation above that x_4 and x_5 are free thus solutions are of the form

$$\begin{aligned} x_1 &= -x_4 \\ x_2 &= x_4 - \tfrac{1}{2}x_5 \\ x_3 &= \tfrac{1}{2}x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

for all $x_4, x_5 \in \mathbb{R}$. We can write these results in vector form to reveal the basis for $Null(A)$,

$$x = \begin{bmatrix} -x_4 \\ x_4 - \frac{1}{2}x_5 \\ \frac{1}{2}x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

It follows that the basis for $Null(A)$ is simply

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Of course, you could multiply the second vector by 2 if you wish to avoid fractions. In fact there is a great deal of freedom in choosing a basis. We simply show one way to do it.

Example 6.5.32. *Find a basis for the null space of A given below,*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Gaussian elimination on the augmented coefficient matrix reveals:

$$rref \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Denote $x = [x_1, x_2, x_3, x_4]^T$ in the equation $Ax = 0$ and identify from the calculation above that x_2, x_3 and x_4 are free thus solutions are of the form

$$\begin{aligned} x_1 &= -x_2 - x_3 - x_4 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

for all $x_2, x_3, x_4 \in \mathbb{R}$. We can write these results in vector form to reveal the basis for $Null(A)$,

$$x = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

It follows that the basis for $Null(A)$ is simply

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The process of finding the basis for the null space is a bit different than the column or row space basis problem. Both the column and row calculations allow us to read the basis of either rref(A) or rref(A^T) on the logical basis of the CCP. There is a matrix-theoretic way to do the same for the null space, but, it comes at a considerable cost to esthetics. See my Question and Answer on the Math Stackexchange[18].

[18] there is a link to math.stackexchange.com/q/1612616/36530 if you look at the pdf in the proper viewer

6.6 further theory of linear dependence: a tale of two maths

I guess there are two paths we can go down, but, in the long-run, we're still on the road we're on. In particular, I present a choose your own adventure format for this section. In Story I we follow Curtis' Lemma 7.1 and Theorems 7.2, 7.3 and 7.4 of Charles W. Curtis' *Linear Algebra: An Introductory Approach*. But, in Story II I give a family of theorems stemming from trace and coordinate-based argument. Both stories have the same ending.

6.6.1 Story I: a coordinate free approach

To begin, I should confess, we already borrowed Theorems 5.1 and 5.3 from Curtis' text in our Theorems 6.5.2 and 6.5.3. Now, I reproduce Theorems from §7 of Charles W. Curtis' *Linear Algebra: An Introductory Approach*.

Lemma 6.6.1. *(Curtis'* LEMMA *(7.1))*

If $\{a_1, \cdots, a_m\}$ is linearly dependent and if $\{a_1, \dots, a_{m-1}\}$ is linearly independent, then a_m is a linear combination of $a_1, \dots, a_{m-1}$.

Proof: proof is found on page 49 of §7 of Curtis' text. □

Theorem 6.6.2. *(Curtis'* THEOREM *(7.2))*

Every finitely generated vector space has a basis.

Proof: proof is found on page 49 of §7 of Curtis' text. □

Theorem 6.6.3. *(Curtis'* THEOREM *(7.3))*

Let $V = \text{span}\{a_1, \dots, a_m\}$ be a finitely generated vector space with generators $\{a_1, \dots, a_m\}$. Then a basis for V can be selected from among the generators $\{a_1, \dots, a_m\}$. In other words, a set of generators for a finitely generated vector space always contains a basis.

Proof: proof is found on page 50 of §7 of Curtis' text. □

Theorem 6.6.4. *(Curtis'* THEOREM *(7.4))*

Let $\{b_1, \dots, b_m\}$ be a linearly independent set of vectors in a finitely generated vector space V. If $\{b_1, \dots, b_q\}$ is not a basis of V, then there exist other vectors $b_{q+1}, \dots, b_m$ in V such that $\{b_1, \dots, b_m\}$ forms a basis for V.

Proof: proof is found on page 50 of §7 of Curtis' text. □

The proofs of the Theorems given in this subsection are not especially difficult, I simply did not have time to reformulate them for the notes this semester. Also, I do think there is something to gain from the proofs I give in Story II. I do give a logically complete defense of the major theory of linear independence and spanning if you work through Story II. I include this incomplete section since the logic here is followed by many authors.

6.6.2 Story II: a coordinate-based approach

Earlier in this Chapter we saw Theorems[19] 6.5.2 and 6.5.3 established the concept of dimension for finitely generated vector spaces over a field $\mathbb{F}$. A different approach to the problem of defining dimension is given by the Theorem below. In this attack we first prove that if a vector space is finitely generated by a linearly independent set then any other linearly independent generating set must likewise have the same cardnality. Once that is established I invoke systematic thievery to *borrow* results from $\mathbb{F}^n$.

Proposition 6.6.5.

Let V be vector space over a field $\mathbb{F}$ and suppose there exists $B = \{b_1, b_2, \dots, b_n\}$ a basis of V. Then any other basis for V also has n-elements. In other words, any two finite linearly independent generating sets for a vector space V have the same number of elements.

Proof: Suppose $B = \{b_1, b_2, \dots, b_n\}$ and $F = \{f_1, f_2, \dots, f_p\}$ are both bases for a vector space V. Since F is a basis it follows $b_k \in span(F)$ for all k so there exist constants $c_{ik} \in \mathbb{F}$ such that

$$b_k = c_{1k}f_1 + c_{2k}f_2 + \cdots + c_{pk}f_p$$

for $k = 1, 2, \dots, n$. Likewise, since $f_j \in span(B)$ there exist constants $d_{lj} \in \mathbb{F}$ such that

$$f_j = d_{1j}b_1 + d_{2j}b_2 + \cdots + d_{nj}b_n$$

for $j = 1, 2, \dots, p$. Substituting we find

$$\begin{aligned} f_j &= d_{1j}b_1 + d_{2j}b_2 + \cdots + d_{nj}b_n \\ &= d_{1j}(c_{11}f_1 + c_{21}f_2 + \cdots + c_{p1}f_p) + \\ &\quad + d_{2j}(c_{12}f_1 + c_{22}f_2 + \cdots + c_{p2}f_p) + \\ &\quad\quad + \cdots + d_{nj}(c_{1n}f_1 + c_{2n}f_2 + \cdots + c_{pn}f_p) \\ &= (d_{1j}c_{11} + d_{2j}c_{12} + \cdots d_{nj}c_{1n})f_1 \\ &\quad (d_{1j}c_{21} + d_{2j}c_{22} + \cdots d_{nj}c_{2n})f_2 + \\ &\quad\quad + \cdots + (d_{1j}c_{p1} + d_{2j}c_{p2} + \cdots d_{nj}c_{pn})f_p \end{aligned}$$

Suppose $j = 1$. We deduce, by the linear independence of F, that

$$d_{11}c_{11} + d_{21}c_{12} + \cdots d_{n1}c_{1n} = 1$$

from comparing coefficients of f_1, whereas for f_2 we find,

$$d_{11}c_{21} + d_{21}c_{22} + \cdots d_{n1}c_{2n} = 0$$

likewise, for f_q with $q \neq 1$,

$$d_{11}c_{q1} + d_{21}c_{q2} + \cdots d_{n1}c_{qn} = 0$$

Notice we can rewrite all of these as

$$\delta_{q1} = c_{q1}d_{11} + c_{q2}d_{21} + \cdots c_{qn}d_{n1}$$

[19] you might notice these are borrowed from §5 of Curtis

Similarly, for arbitrary j we'll find

$$\delta_{qj} = c_{q1}d_{1j} + c_{q2}d_{2j} + \cdots c_{qn}d_{nj}$$

If we define $C = [c_{ij}] \in \mathbb{F}^{p\times n}$ and $D = [d_{ij}] \in \mathbb{F}^{n\times p}$ then we can translate the equation above into the matrix equation that follows:

$$CD = I_p.$$

We can just as well argue that

$$DC = I_n$$

The **trace** of a matrix is the sum of the diagonal entries in the matrix; $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in \mathbb{F}^{n\times n}$. It is not difficult to show that $\text{trace}(AB) = \text{trace}(BA)$ provided the products AB and BA are both defined. Moreover, it is also easily seen $tr(I_p) = p$ and $tr(I_n) = n$. It follows that,

$$tr(CD) = tr(DC) \quad\Rightarrow\quad tr(I_p) = tr(I_n) \quad\Rightarrow\quad p = n.$$

Since the bases were arbitrary this proves any pair of bases have the same number of vectors. □

The proof above is rather different than that we gave for Theorem 6.5.2. One nice feature of this proof is that we get to see the trace in action. The trace has use far beyond this proof. For example, to obtain an invariant over a symmetry group in physics one takes the trace of an expression to form the Lagrangian of a gauge theory.

Proposition 6.6.6.

Suppose V is a vector space with $dim(V) = n$.

1. If S is a set with more than n vectors then S is linearly dependent.
2. If S is a set with less than n vectors then S does not generate V.

Proof of (1.): Suppose $S = \{s_1, s_2, \dots, s_m\}$ has m vectors and $m > n$. Let $B = \{b_1, b_2, \dots, b_n\}$ be a basis of V. Consider the corresponding set of coordinate vectors of the vectors in S, we denote

$$[S]_B = \{[s_1]_B, [s_2]_B, \dots, [s_m]_B\}.$$

The set $[S]_B$ has m vectors in $\mathbb{F}^n$ and $m > n$ therefore by Proposition 4.3.6 we know $[S]_B$ is a linearly dependent set. Therefore at least one, say $[s_j]_B$, vector can be written as a linear combination of the other vectors in $[S]_B$ thus there exist constants c_i with (this is a vector equation)

$$[s_j]_B = c_1[s_1]_B + c_2[s_2]_B + \cdots + \widehat{c_j[s_j]_B} + \cdots + c_m[s_m]_B$$

Also notice that (introducing a new shorthand $B[s_j]$ which is not technically matrix multiplication since b_i are not column vectors generally, they could be chickens for all we know)

$$s_j = B[s_j] = s_{j1}b_1 + s_{j2}b_2 + \cdots + s_{jn}b_n$$

We also know, using the notation $([s_j]_B)_k = s_{jk}$,

$$s_{jk} = c_1 s_{1k} + c_2 s_{2k} + \cdots + \widehat{c_j s_{jk}} + \cdots + c_m s_{mk}$$

for $k = 1, 2, \dots, n$. Plug these into our s_j equation,

$$\begin{aligned} s_j &= (c_1s_{11} + c_2s_{21} + \cdots + \widehat{c_js_{j1}} + \cdots + c_ms_{m1})b_1 + \\ &\quad (c_1s_{12} + c_2s_{22} + \cdots + \widehat{c_js_{j2}} + \cdots + c_ms_{m2})b_2 + \\ &\quad + \cdots + (c_1s_{1n} + c_2s_{2n} + \cdots + \widehat{c_js_{jn}} + \cdots + c_ms_{mn})b_n \\ \\ &= c_1(s_{11}b_1 + s_{12}b_2 + \cdots + s_{1n}b_n) + \\ &\quad c_2(s_{21}b_1 + s_{22}b_2 + \cdots + s_{2n}b_n) + \\ &\quad + \cdots + c_m(s_{m1}b_1 + s_{m2}b_2 + \cdots + s_{mn}b_n) : \quad \text{excluding } c_j(\cdots) \\ \\ &= c_1s_1 + c_2s_2 + \cdots + \widehat{c_js_j} + \cdots + c_ns_n. \end{aligned}$$

Well this is a very nice result, the same linear combination transfers over to the abstract vectors. Clearly s_j linearly depends on the other vectors in S so S is linearly dependent. The heart of the proof was Proposition 4.3.6 and the rest was just battling notation.

Proof of (2.): Use the corresponding result for $\mathbb{F}^n$ which was given by Proposition 4.3.5. Given m abstract vectors if we concantenate their coordinate vectors we will find a matrix $[S]$ in $\mathbb{F}^{n\times m}$ with $m < n$ and as such there will be some choice of the vector b for which $[S]x \neq b$. The abstract vector corresponding to b will not be covered by the span of S. $\square$

Proposition 6.6.7.

Suppose V is a vector space with $dim(V) = n$ and $W \leq V$ then there exists a basis for W and $dim(W) \leq n$.

Proof: If $W = \{0\}$ then the proposition is true since $\emptyset$ is basis and $\dim(W) = 0 \neq n$. Suppose $W \neq 0$ and set S be a finite subset of W. Apply Proposition 6.6.6 to modify S to a basis β_W for W by possibly deleting or adjoining vectors from W. Again, apply Proposition 6.6.6 to see $\#(\beta_W) \leq n$ and this completes the proof. $\square$.

Howard Anton calls the following proposition the "Plus/Minus" Theorem in his linear algebra text.

Proposition 6.6.8.

Let V be a vector space and suppose S is a finite nonempty set of vectors in V.

(1.) S is linearly independent with nonzero vector $v \notin span(S)$ if and only if $S \cup \{v\}$ is a linearly independent set.

(2.) The vector $v \in S$ is a linear combination of other vectors in S if and only if $span(S - \{v\}) = span(S)$.

Proof of (1.): Suppose $S = \{s_1, s_2, \dots, s_k\}$ is LI and $v \notin \text{span}(S)$. Consider,

$$c_1s_1 + c_2s_2 + \cdots + c_ks_k + c_{k+1}v = 0$$

If $c_{k+1} \neq 0$ it follows that v is a linear combination of vectors in S but this is impossible so $c_{k+1} = 0$. Then since S is linear independent

$$c_1s_1 + c_2s_2 + \cdots + c_ks_k = 0 \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_k = 0$$

thus $S \cup \{v\}$ is linearly independent. Conversely, if $S \cup \{v\}$ is LI then one can easily verify S is LI and if $v \in \text{span}(S)$ this gives a clear contradiction to the supposed LI of $S \cup \{v\}$.

Proof of (2.): Suppose $v = s_j$. We are given that there exist constants d_i such that

$$s_j = d_1 s_1 + \cdots + \widehat{d_j s_j} + \cdots + d_k s_k$$

Let $w \in span(S)$ so there exist constants c_i such that

$$w = c_1 s_1 + c_2 s_2 + \cdots + c_j s_j + \cdots + c_k s_k$$

Now we can substitute the linear combination with d_i-coefficients for s_j,

$$\begin{aligned} w &= c_1 s_1 + c_2 s_2 + \cdots + c_j(d_1 s_1 + \cdots + \widehat{d_j s_j} + + \cdots + d_k s_k) + \cdots + c_k s_k \\ &= (c_1 + c_j d_1)s_1 + (c_2 + c_j d_2)s_2 + \cdots + \widehat{c_j d_j s_j} + \cdots + (c_k + c_j d_k)s_k \end{aligned}$$

thus w is a linear combination of vectors in S, but not $v = s_j$, thus $w \in span(S - \{v\})$ and we find $span(S) \subseteq span(S - \{v\})$.

Next, suppose $y \in span(S - \{v\})$ then y is a linear combination of vectors in $S - \{v\}$ hence y is a linear combination of vectors in S and we find $y \in span(S)$ so $span(S - \{v\}) \subseteq span(S)$. (this inclusion is generally true even if v is linearly independent from other vectors in S). We conclude that $span(S) = span(S - \{v\})$.

Conversely, suppose $span(S) = span(S - \{v\})$. If $v \in S$ then $v = 1 \cdot v \in span(S) = span(S - \{v\})$ thus $v \in span(S - \{v\})$ which means v is a linear combination of other vectors in S. $\square$

Proposition 6.6.9.

Let V be an n-dimensional vector space and let S be a set of n-vectors in V. The following are equivalent:

(1.) S is a basis of V,

(2.) S is linearly independent,

(3.) $\text{span}(S) = V$.

Proof: Suppose V is n-dimensional with S a subset of V containing n-vectors. If S is a basis then S is LI and $\text{span}(S) = V$ thus (1.) implies (2.) and (3.). Suppose (2.) is true and $\text{span}(S) \neq V$. Then there exists nonzero $v \notin \text{span}(S)$ hence $S \cup \{v\}$ is a LI set of $(n+1)$-vectors in the n-dimensional vector space V thus by part (1.) of Proposition 6.6.6 we find a contradiction hence $\text{span}(S) = V$. Therefore, (2.) implies (3.). Finally, suppose (3.) is true; $\text{span}(S) = V$. If S is linearly dependent then there exists $v \in S$ such that $\text{span}(S - \{v\}) = \text{span}(S) = V$ by (2.) of Proposition 6.6.8. But, then we have a generating set with at most $n - 1$ vectors for V which contradicts part (2.) of Proposition 6.6.6. Hence S is *LI* which means (3.) implies (2.). The Proposition follows. $\square$

Remark 6.6.10.

Intuitively speaking, linear independence is like injectivity for functions whereas spanning is like the onto property for functions. Suppose A is a finite set. If a function $f : A \to A$ is 1-1 then it is onto. Also if the function is onto then it is 1-1. The finiteness of A is what blurs the concepts. For a vector space, we also have a sort of finiteness in play if $dim(V) = n$. When a set with $dim(V)$-vectors spans (like onto) V then it is automatically linearly independent. When a set with $dim(V)$-vectors is linearly independent (like 1-1) V then it automatically spans V. However, in an infinite dimensional vector space this need not be the case. For example, d/dx is a surjective linear mapping on $\mathbb{R}[x] = span\{1, x, x^2, x^3, \dots\}$ however if $f, g \in \mathbb{R}[x]$ and $df/dx = dg/dx$ we can only conclude that $f = g + c$ thus d/dx is not injective on vector space of polynomials in x. Many theorems we discuss do hold in the infinite dimensional context, but you have to be careful.

Theorem 6.6.11.

Let S be a subset of a finite dimensional vector space V.

(1.) If $span(S) = V$ but S is not a basis then S **contains** a basis which can be constructed by removing redundant vectors.

(2.) If S is linearly independent and $span(S) \neq V$ then S can be **extended** to a basis for V by adjoining vectors outside $span(S)$.

Proof of (1.): If $span(S) = V$ but S is not a basis we find S is linearly dependent. (if S is linearly independent then Proposition 6.6.9 says S is a basis which is a contradiction). Since S is linearly dependent we can write some $v \in S$ as a linear combination of other vectors in S. Furthermore, by Proposition 6.6.6 $span(S) = span(S - \{v\})$. If $S - \{v\}$ is linearly independent then $S - \{v\}$ is a basis. Otherwise $S - \{v\}$ is linearly dependent and we can remove another vector. Continue until the resulting set is linearly independent (we know this happens when there are just $dim(V)$-vectors in the set so this is not an endless loop)

Proof of (2.): If S is linearly independent but $span(S) \neq V$ then there exists $v \in V$ but $v \notin span(S)$. Proposition 6.6.8 shows that $S \cup \{v\}$ is linearly independent. If $span(S \cup \{v\}) = V$ then $S \cup \{v\}$ is a basis. Otherwise there is still some vector outside $span(S \cup \{v\}) = V$ and we can repeat the argument for that vector and so forth until we generate a set which spans V. Again we know this is not an endless loop because V is finite dimensional and once the set is linearly independent and contains $dim(V)$ vectors it must be a basis (see Proposition 6.6.9). $\square$

Remark 6.6.12.

We already saw in the previous sections that we can implement part (1.) of the preceding proposition in $\mathbb{F}^n$ and $\mathbb{F}^{1\times n}$ through matrix calculations. The corresponding abstract calculation can be made by lifting the our CCP-based arguments through the coordinate maps. We've already seen this "lifting" idea come into play in several proof of Proposition 6.6.6. Part (2.) involves making a choice. How do you choose a vector outside the span?

I suppose I should mention that the statement (1.) *every generating set contains a basis* and (2.) *every linearly independent set can be extended to a a basis* are still true for infinite sets and infinite dimensional vector spaces. However, proofs get a little more sophisticated and require use of Zorn's lemma. I can recommend a reference if you're interested. With these results in hand we can prove the existence of bases in general (two different ways). Consider, the empty set is LI hence extend to a basis. Also, V spans itself, thus V contains a basis.

6.7 subspace theorems

The most important theorems about subspaces were already given earlier in this Chapter. In particular, the Subspace Test Theorem 6.2.7, the Span is a Subspace Theorem 6.3.3 and Example 6.3.13. We'll begin by studying subspaces associated to a given matrix. We collect the definitions of row, column and null space for future reference. We also define the dimensions of column and null space as **rank** and **nullity**:

Definition 6.7.1.

Let $A \in \mathbb{F}^{m\times n}$. We define

1. $\text{Col}(A) = span\{col_j(A)|j = 1, 2, \dots, n\}$ and $\text{rank}(A) = dim(\text{Col}(A))$
2. $\text{Row}(A) = span\{row_i(A)|i = 1, 2, \dots, m\}$
3. $\text{Null}(A) = \{x \in \mathbb{F}^n | Ax = 0\}$ and $\nu = \text{nullity}(A) = \dim(\text{Null}(A))$

Observe, column and row space are formed from spans hence Theorem 6.3.3 gives us $\text{Col}(A) \leq \mathbb{F}^m = \mathbb{F}^{m\times 1}$ and $\text{Row}(A) \leq \mathbb{F}^{1\times n}$ for $A \in \mathbb{F}^{m\times n}$. We argued in Example 6.3.13 that $\text{Null}(A) \leq \mathbb{F}^n = \mathbb{F}^{n\times 1}$. It turns out these fundamental subspaces associated to A have dimensions which are nicely related:

Proposition 6.7.2.

Let $A \in \mathbb{R}^{m\times n}$ then $dim(Row(A)) = dim(Col(A))$

Proof: By Proposition 6.5.20 we know the number of vectors in the basis for $Col(A)$ is the number of pivot columns in A. Likewise, Proposition 6.5.26 showed the number of vectors in the basis for $Row(A)$ was the number of nonzero rows in $ref(A)$. But the number of pivot columns is precisely the number of nonzero rows in $ref(A)$ therefore, $dim(Row(A)) = dim(Col(A))$. $\square$

The theorem which follows seems trivial from the view-point of row-reduction, but, it is a central result of this course:

Theorem 6.7.3. Rank-Nullity Theorem for a matrix:

Let $A \in \mathbb{F}^{m\times n}$ then $n = rank(A) + nullity(A)$.

Proof: The proof of Proposition 6.5.29 makes is clear that if a $m \times n$ matrix A has r-pivot columns then there will be $n - r$ vectors in the basis of $Null(A)$. It follows that

$$rank(A) + nullity(A) = r + (n - r) = n. \qquad \square$$

We now turn to the question of how we can produce new subspaces from a given pair.

Theorem 6.7.4.

Let V be a vector space and suppose $U \leq V$ and $W \leq V$ then $U \cap W \leq V$.

Proof: it is tempting to prove this here. But, I leave it for homework. $\square$

Examples of the Theorem in $\mathbb{R}^3$ are fun to think about. For example, one-dimensional subspaces are lines through the origin and two-dimensional subspaces are planes through the origin. The interection of a line and plane is either the line once more or the origin. On the other hand, the intersection of two planes is either the plane once more (if the given planes are identical), or a line. Two planes cannot intersect in just one point in $\mathbb{R}^3$. In contrast, in $\mathbb{R}^4$ the planes $x_1 = x_2 = 0$ and $x_3 = x_4 = 0$ share only $(0, 0, 0, 0)$ in common. Apparently, the calculation of the intersection of two subspaces has many cases to enumerate for an abitrary vector space. That said, we do have a nice theorem which relates the intersection to the sum of two subspaces. The following definition is quite natural:

Definition 6.7.5.

Let V be a vector space and $U \leq V$ and $W \leq V$ then define the **sum** of U and W by

$$U + W = \{x + y \mid x \in U, \ y \in W\}.$$

If you consider the union of two subspaces you'll find the result is only a subspace when one of the subspaces contains the other. For example, the union of the x and y-axes in $\mathbb{R}^2$ is missing the sums such as $(x, 0) + (0, y) = (x, y)$ for $x, y \neq 0$. It turns out the set defined above is the smallest subspace which contains the union of the given subspaces.

Theorem 6.7.6.

Let V be a vector space and suppose $U \leq V$ and $W \leq V$ then $U + W \leq V$. Moreover, no smaller subspace contains $U \cup W$.

Proof: it is tempting to prove this here. But, I leave it for homework. $\square$

Theorem 6.7.7.

Let V be a finite dimensional vector space with subspace W. Then $\dim(W) \leq \dim(V)$ where equality is attained only if $V = W$.

Proof: Let β be a basis for W, if β is also a basis for V then $dim(V) = dim(W)$ and $V = W = \text{span}(\beta)$. Otherwise, if $span(\beta) \neq V$, apply Theorem 6.6.11 to extend β to γ a basis for V. Hence, $\dim(W) = \#(\beta) < \#(\gamma) = \dim(V)$. $\square$

This result which more useful than you might first expect. In particular, suppose $U_1, U_2, \dots$ are subspaces of a finite dimensional vector space V. If $U_j \leq U_{j-1}$ for $j = 2, \dots, k$ with $U_j \neq U_{j-1}$ then we have the following nested-sequence of subsets:

$$U_k \subset U_{k-1} \subset \cdots \subset U_2 \subset U_1 \subset V$$

where $\dim(U_j) < \dim(U_{j-1})$ for each $j = 1, 2 \dots$. Simple counting then reveals we cannot keep descending to smaller subspaces without end. Eventually, we obtain a smallest subspace (it might be $\{0\}$). Conversely, we could think about a sequence of subspaces which gets larger as the sequence progresses. Once again, we cannot continue without end as the dimension of V bounds the dimension of subspaces.

There is a natural relation between the dimensions of the sum and intersection of two subspaces which is related to the counting problem for two sets: if A and B are finite sets then

$$\#(A \cup B) = \#A + \#B - \#(A \cap B).$$

This rule is easy to see in a Venn Diagram.

Theorem 6.7.8.

Let $V(\mathbb{F})$ be a finite-dimensional vector space and suppose $U \leq V$ and $W \leq V$ then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Proof: Given $U \leq V$ and $W \leq V$ we have $U \cap W \leq V$ and $U + W \leq V$ by Theorems 6.7.6 and 6.7.6. I invite the reader to verify $U \cap W \leq U \leq U + W$ and $U \cap W \leq W \leq U + W$ (both of these assertions are simple to obtain from the Subspace Test Theorem). Observe by Proposition 6.6.7 we can find a basis $\beta_{U\cap W} = \{v_1, \dots, v_n\}$ for $U \cap W$. We count $\dim(U \cap W) = n$. Notice $\beta_{U\cap W}$ is a LI subset of U thus by Proposition 6.6.8 we can complete $\beta_{U\cap W}$ to a basis $\beta_U = \{v_1, \dots, v_n, u_1, \dots, u_d\}$ for U by adjoining vectors $u_1, \dots, u_d \in U - U \cap W$. Notice $\dim(U) = n + d$ in our current notation. Similarly, $\beta_{U\cap W}$ is a LI subset of W thus by Proposition 6.6.8 we can complete $\beta_{U\cap W}$ to a basis $\beta_W = \{v_1, \dots, v_n, w_1, \dots, w_e\}$ for W by adjoining vectors $w_1, \dots, w_e \in W - U \cap W$. We count, $\dim(W) = n + e$. We argue that $\beta_{U+V} = \{v_1, \dots, v_n, u_1, \dots, u_d, w_1, \dots, w_e\}$ forms a basis for $U + V$ hence the Theorem follows as

$$\dim(U + V) = n + d + e = (n + d) + (n + e) - n = \dim(U) + \dim(W) - \dim(U \cap W).$$

It remains to show β_{U+V} is a basis for $U + W$. Let $z \in U + W$ then there exist $x \in U$ and $y \in W$ such that $z = x + y$. However, as $\{v_1, \dots, v_n, u_1, \dots, u_d\}$ is basis for U there exist $c_i, b_j \in \mathbb{F}$ such that $x = \sum_{i=1}^{n} c_i v_i + \sum_{j=1}^{d} b_j u_j$. Also, as $\{v_1, \dots, v_n, w_1, \dots, w_e\}$ is a basis for W and $y \in W$ there exist $\alpha_i, \beta_k \in \mathbb{F}$ for which $y = \sum_{i=1}^{n} \alpha_i v_i + \sum_{k=1}^{e} \beta_j w_k$. Thus,

$$z = x + y = \sum_{i=1}^{n} (\alpha_i + c_i) v_i + \sum_{j=1}^{d} b_j u_j + \sum_{k=1}^{e} \beta_j w_k$$

and we find β_{U+V} is a generating set for $U + V$. Finally, we must demonstrate β_{U+V} is LI. Suppose there exist $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$ for which

$$\sum_{i=1}^{n} \alpha_i v_i + \underbrace{\sum_{j=1}^{d} \beta_j u_j}_{x \in U} + \underbrace{\sum_{k=1}^{e} \gamma_k w_k}_{y \in W} = 0 \quad \star$$

Recall, by construction, $u_j \in U - U \cap W$ and $w_k \in W - U \cap W$. Solve for the vector in W,

$$\sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{d} \beta_j u_j = -\sum_{k=1}^{e} \gamma_k w_k = -y \in W$$

But, $\sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{d} \beta_j u_j \in U$ thus $-y \in U$ and $-y \in W$ hence $-y \in U \cap W$! Thus, there exist $\eta_1, \dots, \eta_n$ for which $\sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{d} \beta_j u_j = \sum_{i=1}^{n} \eta_i v_i$ $(\star)$. Hence, by LI of β_U we learn $\beta_j = 0$

for $j \in \mathbb{N}_d$ by comparing coefficients[20] of the LHS and RHS of $\star\star$. To complete the proof we make an entirely similar argument for $-x$ which shows $\gamma_k = 0$ for $k \in \mathbb{N}_e$. Finally, returning to $\star$ we have $\sum_{i=1}^{n} \alpha_i v_i = 0$ and LI of $\beta_{U\cap W} = \{v_1, \dots, v_n\}$ shows $\alpha_i = 0$ for each $i \in \mathbb{N}_n$. This completes the proof. $\square$

You might ask, what about three subspaces[21] ? Threats aside, the problem of studying the decomposition of a vector space into a finite set of subspaces is an interesting and central problem of linear algebra we will devote substantial energy towards in a later part of this course. This is just Chapter 1 of that story.

6.8 general theory of linear systems

We've seen some rather abstract results thus far in this chapter. I thought it might be helpful to tie them back to our fundamental problem; how does dimension theory help us understand the structure of solutions to $Ax = b$? We begin by studying the solution set of $Ax = b$. Then, I include a short subsection surveying similarities in our results and those seen in the usual Differential Equations course. Finally, we investigate the geometry of linear manifolds where we find that the appear naturally as solution sets to inhomogeneous systems of linear equations.

6.8.1 structure of the solution set of an inhomogeneous system

Let $A \in \mathbb{F}^{m\times n}$ we should notice that $\text{Null}(A) \leq \mathbb{F}^n$ is only possible since homogeneous systems of the form $Ax = 0$ have the nice property that linear combinations of solutions is again a solution:

Proposition 6.8.1.

Let $Ax = 0$ denote a homogeneous linear system of m-equations and n-unknowns over $\mathbb{F}$. If v_1 and v_2 are solutions then any linear combination $c_1v_1 + c_2v_2$ is also a solution of $Ax = 0$.

Proof: Let $A \in \mathbb{F}^{n\times n}$ and suppose $Av_1 = 0$ and $Av_2 = 0$ for $v_1, v_2 \in \mathbb{F}^n$. Let $c_1, c_2 \in \mathbb{F}$ and recall Theorem 2.3.11,

$$A(c_1v_1 + c_2v_2) = c_1Av_1 + c_2Av_2 = c_10 + c_20 = 0.$$

Therefore, $c_1v_1 + c_2v_2$ is also a solution of $Ax = 0$. $\square$

We proved this before, but I thought it might help to see it again. In words, a matrix is a homogeneous solution matrix if and only if its columns are homogeneous solutions:

Proposition 6.8.2.

Let $v_1, \dots, v_k \in \mathbb{F}^m$ and $V = [v_1|\cdots|v_k] \in \mathbb{F}^{m\times k}$. Then $AV = 0$ if and only if $Av_i = 0$ for $i = 1, \dots, k$.

Proof: Let $A \in \mathbb{F}^{m\times n}$ and let $V = [v_1|v_2|\cdots|v_k] \in \mathbb{F}^{m\times k}$ where $v_1, \dots, v_k \in \mathbb{F}^m$. Observe,

$$AV = A[v_1|v_2|\cdots|v_k] = [Av_1|Av_2|\cdots|Av_k].$$

Therefore, $AV = 0$ if and only if $Av_1 = 0, \dots, Av_k = 0$. $\square$

A solution matrix of a linear system is a matrix in which each column is itself a solution.

[20] like the wings of an elephant, the coefficients of u_j are set to zero on the RHS of $\star\star$

[21] add evil laugh to properly read this question. Or look at this question on math overflow

Proposition 6.8.3.

Let $A \in \mathbb{F}^{m\times n}$. The system of equations $Ax = b$ is consistent iff $b \in Col(A)$.

Proof: Observe,

$$\begin{aligned} Ax = b \ &\Leftrightarrow\ \textstyle\sum_{i,j} A_{ij}x_j e_i = b \\ &\Leftrightarrow\ \textstyle\sum_j x_j \sum_i A_{ij}e_i = b \\ &\Leftrightarrow\ \textstyle\sum_j x_j col_j(A) = b \\ &\Leftrightarrow\ b \in Col(A) \end{aligned}$$

Therefore, the existence of a solution to $Ax = b$ is interchangeable with the statement $b \in Col(A)$. They both amount to saying that b is a linear combination of columns of A. $\square$

Another way to look at the Proposition above is that a system of equations is only consistent if the rank of the coefficient matrix and augmented coefficient matrix are identical; $Ax = b$ consistent iff $\text{rank}(A) = \text{rank}[A|b]$. See Theorem 8.8 in §8 of Curtis.

Proposition 6.8.4.

Let $A \in \mathbb{F}^{m\times n}$ and suppose the system of equations $Ax = b$ is consistent. We find $x \in \mathbb{F}^n$ is a solution of the system if and only if it can be written in the form

$$x = x_h + x_p = c_1v_1 + c_2v_2 + \cdots + c_\nu v_\nu + x_p$$

where $Ax_h = 0$, $\{v_j\}_{j=1}^{\nu}$ are a basis for $Null(A)$, and $Ax_p = b$. We call x_h the homogeneous solution and x_p is the nonhomogeneous solution.

Proof: Suppose $Ax = b$ is consistent then $b \in Col(A)$ therefore there exists $x_p \in \mathbb{R}^n$ such that $Ax_p = b$. Let x be any solution. We have $Ax = b$ thus observe

$$A(x - x_p) = Ax - Ax_p = Ax - b = 0 \ \Rightarrow\ (x - x_p) \in Null(A).$$

Define $x_h = x - x_p$ it follows that there exist constants c_i such that $x_h = c_1v_1 + c_2v_2 + \cdots + c_\nu v_\nu$ since the vectors v_i span the null space.

Conversely, suppose $x = x_p + x_h$ where $x_h = c_1v_1 + c_2v_2 + \cdots + c_\nu v_\nu \in Null(A)$ then it is clear that

$$Ax = A(x_p + x_h) = Ax_p + Ax_h = b + 0 = b$$

thus $x = x_p + x_h$ is a solution. $\square$

Example 6.8.5. *Consider the system of equations $x + y + z = 1, x + z = 1$. In matrix notation,*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ \Rightarrow\ rref[A|b] = rref\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that $x = 1 - y - z$ is a solution for any choice of $y, z \in \mathbb{R}$.

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - y - z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We recognize that $v_p = (1, 0, 0)$ while $v_h = y(-1, 1, 0) + z(-1, 0, 1)$ and $\{(-1, 1, 0),\ (-1, 0, 1)\}$ is a basis for the null space of A. We call y, z parameters in the solution.

We will see that null spaces play a central part in the study of eigenvectors in a later part of this course. In fact, about half of the eigenvector calculation is finding a basis for the null space of a certain matrix. So, don't be too disappointed if I don't have too many examples here. You'll work more than enough soon enough.

The following proposition simply summarizes what we just calculated:

Proposition 6.8.6.

Let $A \in \mathbb{F}^{m\times n}$. If the system of equations $Ax = b$ is consistent then the general solution has as many parameters as the $dim(Null(A))$.

6.8.2 linear algebra in differential equations

A very similar story is told in differential equations. In Math 334 we spend some time unraveling the solution of $L[y] = g$ where $L = P(D)$ is an n-th order polynomial in the differentiation operator with constant coefficients. In total we learn that $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$ is the solution where y_j are the homogeneous solutions which satisfy $L[y_j] = 0$ for each $j = 1, 2, \dots, n$ and, in contrast, y_p is the so-called "particular solution" which satisfies $L[y_p] = g$. On the one hand, the results in DEqns are very different because the solutions are functions which live in the infinite-dimensional function space. However, on the other hand, $L[y] = g$ is a finite dimensional problem thanks to the fortunate fact that $Null(L) = \{f \in \mathcal{F}(\mathbb{R}) | L(f) = 0\} = span\{y_1, y_2, \dots, y_n\}$. For this reason there are n-parameters in the general solution which we typically denote by $c_1, c_2, \dots, c_n$ in the Math 334 course. The particular solution is not found by row reduction on a matrix in DEqns[22]. Instead, we either use the annihilator method, power series techniques, or most generally the method of variation of parameters will calculate y_p. The analogy to the linear system $Av = b$ is striking:

1. $Av = b$ has solution $v = c_1v_1 + c_2v_2 + \cdots + c_kv_n + v_p$ where $v_j \in Null(A)$ and $Av_p = b$.
2. $L[y] = g$ has solution $v = c_1y_1 + c_2y_2 + \cdots + c_ky_n + y_p$ where $y_j \in Null(L)$ and $L[y_p] = b$.

The reason the DEqn $L[y] = g$ possesses such an elegant solution stems from the linearity of L. If you study nonlinear DEqns the structure is not so easily described.

6.8.3 linear manifolds

A linear manifold is a subset of a vector space which is generally not a vector space with respect to the vector space operations of the ambient vector space. However, the idea presented in Example 6.2.11 allows us to give a linear manifold the structure of a vector space. Basically, the idea is just the usual tip-to-tail geometric vector addition where we base the addition on some point in the linear manifold. In contrast, subspace vector addition is based on tip-to-tail vector addition based from 0.

[22] ok, to be fair you could use coordinate vectors of the next chapter to convert $y_1, y_2, \dots y_n$ to coordinate vectors and if you worked in a sufficiently large finite dimensional subspace of function space perhaps you could do a row reduction to find g, but this is not the typical calculation.

Definition 6.8.7.

Let V be a vector space over a field $\mathbb{F}$. If $W \leq V$ is subspace then $p+W = \{p+w \mid w \in W\}$ is a **linear manifold** of V. We say the vectors in W are **tangent** to $p+W$. In this context, W is also called the **directing space**.

A linear manifold not generally a subspace. However:

Proposition 6.8.8.

If $W \leq V$ and $p \in W$ then $p + W = W$.

Proof: assume $p \in W$ and $W \leq V$. If $x \in W$ then $x - p \in W$ and we note $x = p + (x-p) \in p + W$ thus $W \subseteq p + W$. Conversely, if $y \in p + W$ then there exists $w \in W$ such that $y = p + w$. But, $p, w \in W$ hence $p + w \in W$ and hence $y \in W$. Thus, $p + W \subseteq W$. Therefore, $p + W = W$. $\square$

Proposition 6.8.9.

If $Ax = b$ denotes a system of m-equations in n-unknowns over $\mathbb{F}$ and x_p is a particular solution of the system then the solution set is a linear manifold of the form $x_p + \text{Null}(A)$.

Proof: this is immediate from Proposition 6.8.4. $\square$

Conversely, we may inquire: does every linear manifold in $\mathbb{F}^n$ appear as the solution set for a system of linear equations ? Let me give an example to help motivate our final proposition:

Example 6.8.10. *Let $V = \mathbb{R}^{2\times 2}$ and consider the linear manifold $p+W$ with directing space $W =$ span$\{E_{11} - E_{22}, E_{12} + E_{21}\}$ and base-point $p = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. A typical point A in V has coordinates x_1, x_2, x_3, x_4 w.r.t. the basis $\beta = \{E_{11} - E_{22}, E_{12} + E_{21}, E_{12}, E_{22}\}$; thus $A = \begin{bmatrix} x_1 & x_2 + x_3 \\ x_2 & -x_1 + x_4 \end{bmatrix}$ for $A \in V$. Therefore, $A \in p + W$ has:*

$$\begin{bmatrix} x_1 & x_2 + x_3 \\ x_2 & -x_1 + x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for some $s, t \in \mathbb{R}$. Thus

$$x_1 = 1 + t, \quad x_2 + x_3 = 2 + s, \quad x_2 = 3 + s, \quad -x_1 + x_4 = 4 - t$$

solving for s, t yields:

$$x_1 - 1 = t = 4 + x_1 - x_4, \qquad \& \qquad x_2 + x_3 - 2 = s = x_2 - 3.$$

Thus, the linear manifold $p + W$ is the solution set of the system of equations:

$$x_4 = 5 \qquad \& \qquad x_3 = -1.$$

You can verify that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_\beta = (1, 3, -1, 5)$. Moreover, by construction, as the coordinates x_1, x_2 range over all real values we fill out directing space W.

Proposition 6.8.11.

Let V be a finite-dimensional vector space over $\mathbb{F}$. If $W \leq V$ and $p \in V$ then $p + W$ is the solution set for a linear system of equations.

Proof: Let $\beta_W = \{w_1, \dots, w_k\}$ serve as a basis for $W \leq V$. If β_W is not a basis for V then adjoin additional vectors $v_1, \dots, v_{n-k}$ such that $\beta = \{w_1, \dots, w_k, v_1, \dots, v_{n-k}\}$ is a basis for V. Let $[p]_\beta = (a_1, \dots, a_k, b_1, \dots, b_{n-k})$. If $v \in V$ then

$$v = x_1 w_1 + \cdots x_k w_k + y_1 v_1 + \cdots + y_{n-k} v_{n-k}$$

Thus $v \in p + W$ has

$$x_1 w_1 + \cdots x_k w_k + y_1 v_1 + \cdots + y_{n-k} v_{n-k} = (a_1 + t_1) w_1 + \cdots (a_k + t_k) w_k + b_1 v_1 + \cdots + b_{n-k} v_{n-k}$$

for $t_1, \dots, t_k \in \mathbb{F}$. Naturally $x_1, \dots, x_k$ are free to take any value in $\mathbb{F}$ whereas,

$$y_1 = b_1, \quad y_2 = b_2, \ \dots \ , \ y_{n-k} = b_{n-k}.$$

The solution to the $n-k$-equations above is precisely $p + W$. $\square$

We've shown that a k-dimensional linear manifold in an n-dimensional vector space can either be viewed as the solution of $n - k$ linear equations or as the translate of a span. Both viewpoints are essential.

Chapter 7

linear transformations

It would be wise to review §1.5. I do expect you be conversant in images and inverse images of sets under a function. Some of you have thought more about this than others. Of course, we review this as we go, but, you would likely profit from some preparatory reading.

The theorems on dimension also find further illumination in this chapter. We study **isomorphisms**. Roughly speaking, two vector spaces which are isomorphic are just the same set with different notation in so far as the vector space structure is concerned. Don't view this sentence as a license to trade column vectors for matrices or functions. We're not there yet. You can do that after this course, once you understand the abuse of language properly. Sort of like how certain musicians can say forbidden words since they have earned the rights through their life experience.

We also study the problem of coordinate change. Since the choice of basis is not unique the problem of comparing different pictures of vectors or transformations for abstract vector spaces requires some effort. We begin by translating our earlier work on coordinate vectors into a mapping-centered notation. Once you understand the notation properly, we can draw pictures to solve problems. This idea of **diagrammatic argument** is an important and valuable technique of modern mathematics. Modern mathematics is less concerned with equations and more concerned with functions and sets.

A theme you may not yet appreciate is **linearization**. Given a complicated set of equations we can approximate them by a simpler set of linear equations. This is the idea of Newton's Method for root-finding. Ultimately, this approximation paired with the nontrivial contraction-mapping technique provides proof of the implicit and inverse function theorems. In short these theorems say linearization works as well as you would naively hope. On the other hand, given a mapping which twists and contorts one shape into another globally may allow a rather simple description locally. The basic idea is to replace a globally nonlinear function with a local linearization. The linearization is built from a linear transformation. Much can be gleaned from the local linearization for a wide swath of problems. I really can't overstate the use of linear transformations. If you understand them then you understand more things than you know.

7.1 definition and basic theory

Definition 7.1.1.

Let V, W be vector spaces over a field $\mathbb{F}$. If a function $T : V \to W$ satisfies

1. $T(x+y) = T(x) + T(y)$ for all $x, y \in V$; T is **additive** or T **preserves addition**
2. $T(cx) = cT(x)$ for all $x \in V$ and $c \in \mathbb{F}$; T is **preserves scalar mulitplication**

then we say T is a **linear transformation** from V to W. The set of all linear transformations from V to W is denoted $\mathcal{L}(V, W)$. Also, $\mathcal{L}(V, V) = \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ is called a linear transformation **on** V.

I have used the terminology that (2.) is **homogeneity** of T, but, technically, T is homogeneous degree one. More generally, $T(cx) = c^kT(x)$ for all $c \in \mathbb{F}$ and $x \in V$ would make T homogeneous of degree k. In the interest of readability, let us agree that homogeneous means homogeneous of degree one. We have little use of higher homogeneity in these notes. I should also mention, if

$$T(cx + y) = cT(x) + T(y)$$

for all $x, y \in V$ and $c \in T$ then it follows from $c = 1$ that T preserves addition and $y = 0$ that T preserves scalar multiplication. Thus, much like the subspace test arguments, we can combine our analysis into the simple check; does $T(cx + y) = cT(x) + T(y)$. I should also mention, other popular notations,

$$\mathcal{L}(V) = \mathrm{End}(V) \qquad \& \qquad \mathcal{L}(V, W) = \mathrm{Hom}_{\mathbb{F}}(V, W)$$

where $\mathrm{End}(V)$ is read the **endomorphisms** of V.

Example 7.1.2. *Let $L : \mathbb{R} \to \mathbb{R}$ be defined by $L(x) = mx + b$ where $m, b \in \mathbb{R}$ and $b \neq 0$. This is often called a* **linear function** *in basic courses. However, this is unfortunate terminology as:*

$$L(x + y) = m(x + y) + b = mx + b + my + b - b = L(x) + L(y) - b.$$

Thus L is not additive hence it is not a linear transformation. It is certainly true that $y = L(x)$ gives a line with slope m and y-intercept b. An accurate term for L is that it is an **affine** *function.*

Example 7.1.3. *Let $f(x, y) = x^2 + y^2$ define a function from $\mathbb{R}^2$ to $\mathbb{R}$. Observe,*

$$f(c(x, y)) = f(cx, cy) = (cx)^2 + (cy)^2 = c^2(x^2 + y^2) = c^2 f(x, y).$$

Clearly f does not preserve scalar multiplication hence f is not linear. (however, f is homogeneous degree 2)

Example 7.1.4. *Suppose $f(t, s) = (\sqrt{t}, s^2 + t)$ note that $f(1, 1) = (1, 2)$ and $f(4, 4) = (2, 20)$. Note that $(4, 4) = 4(1, 1)$ thus we should see $f(4, 4) = f(4(1, 1)) = 4f(1, 1)$ but that fails to be true so f is not a linear transformation.*

Now that we have a few examples of how not to be a linear transformation, let's take a look at some positive examples.

Example 7.1.5. *Let $M \in \mathbb{F}^{m\times n}$ and define $T(x) = Mx$. Notice, for $c \in \mathbb{F}$ and $x, y \in \mathbb{F}^n$*

$$T(cx + y) = M(cx + y) = cMx + My = cT(x) + T(y).$$

Thus T preserves both addition and scalar multiplication and so $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

I return to the Example above as Proposition 7.2.2. The take-away message here is simple; if a function on column vectors has a formula given by matrix multiplication then it is a linear transformation. This is a nice result, I use it below:

Example 7.1.6. *Let $T(x,y,z) = (x+2y+3z, 4x+5y+6z)$ for all $(x,y,z) \in \mathbb{R}^3$. Notice,*

$$T(x,y,z) = x\begin{bmatrix} 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix} + z\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Thus T is a linear transformation since its formula is given by matrix multiplication.

In a certain technical sense, every linear transformation on a finite dimensional vector space is a matrix multiplication in disguise. That said, there are many examples which can be given without mention of a matrix.

Example 7.1.7. *Define $T : \mathbb{F}^{m\times n} \to \mathbb{F}^{n\times m}$ by $T(A) = A^T$. This is clearly a linear transformation since*

$$T(cA+B) = (cA+B)^T = cA^T + B^T = cT(A) + T(B)$$

for all $A, B \in \mathbb{F}^{m\times n}$ and $c \in \mathbb{F}$.

Example 7.1.8. *Let V, W be a vector spaces over a field $\mathbb{F}$ and $T : V \to W$ defined by $T(x) = 0$ for all $x \in V$. This is a linear transformation known as the* **trivial transformation**

$$T(x+y) = 0 = 0 + 0 = T(x) + T(y)$$

and

$$T(cx) = 0 = c0 = cT(x)$$

for all $c \in \mathbb{F}$ and $x, y \in V$.

Example 7.1.9. *The identity function on a vector space*[1] *$V(\mathbb{F})$ is also a linear transformation. Let $Id : V \to V$ satisfy $T(x) = x$ for each $x \in V$. Observe that*

$$Id(x+cy) = x + cy = x + c\cdot y = Id(x) + c\cdot Id(y)$$

for all $x, y \in V$ and $c \in \mathbb{F}$.

Example 7.1.10. *Define $T : C^0(\mathbb{R}) \to \mathbb{R}$ by $L(f) = \int_0^1 f(x)dx$. Notice that L is well-defined since all continuous functions are integrable and the value of a definite integral is a number. Furthermore,*

$$T(f+cg) = \int_0^1 (f+cg)(x)dx = \int_0^1 [\ f(x) + cg(x)\ ]dx = \int_0^1 f(x)dx + c\int_0^1 g(x)dx = T(f) + cT(g)$$

for all $f, g \in C^0(\mathbb{R})$ and $c \in \mathbb{R}$. The definite integral is a linear transformation.

Example 7.1.11. *Let $T : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ be defined by $T(f)(x) = f'(x)$ for each $f \in P_3$. We know from calculus that*

$$T(f+g)(x) = (f+g)'(x) = f'(x) + g'(x) = T(f)(x) + T(g)(x)$$

and

$$T(cf)(x) = (cf)'(x) = cf'(x) = cT(f)(x).$$

The equations above hold for all $x \in \mathbb{R}$ thus we find function equations $T(f+g) = T(f) + T(g)$ and $T(cf) = cT(f)$ for all $f, g \in C^1(\mathbb{R})$ and $c \in \mathbb{R}$.

[1] it is long overdue notation at this point; $V(\mathbb{F})$ denotes a vector space V over the field $\mathbb{F}$

Example 7.1.12. *Let $a \in \mathbb{R}$. The evaluation mapping $\phi_a : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ is defined by $\phi_a(f) = f(a)$. This is a linear transformation as $(f + cg)(a) = f(a) + cg(a)$ by definition of function addition and scalar multiplication. (we could also replace $\mathbb{R}$ with $\mathbb{F}$ to obtain further examples)*

Let us begin by pointing out two important facts which follow from a linear transformation's preservation of addition and scalar multiplication. We assume V, W are vector spaces over a field $\mathbb{F}$ in the remainder of this section.

Proposition 7.1.13.

Let $L : V \to W$ be a linear transformation,

1. $L(0) = 0$
2. $L(c_1v_1 + c_2v_2 + \cdots c_nv_n) = c_1L(v_1) + c_2L(v_2) + \cdots + c_nL(v_n)$ for all $v_i \in V$ and $c_i \in \mathbb{F}$.

Proof: to prove of (1.) let $x \in V$ and notice that $x - x = 0$ thus

$$L(0) = L(x - x) = L(x) + L(-1x) = L(x) - L(x) = 0.$$

To prove (2.) we use induction on n. Notice the proposition is true for n=1,2 by definition of linear transformation. Assume inductively $L(c_1v_1 + c_2v_2 + \cdots c_nv_n) = c_1L(v_1) + c_2L(v_2) + \cdots + c_nL(v_n)$ for all $v_i \in V$ and $c_i \in \mathbb{F}$ where $i = 1, 2, \dots, n$. Let $v_1, v_2, \dots, v_n, v_{n+1} \in V$ and $c_1, c_2, \dots c_n, c_{n+1} \in \mathbb{F}$ and consider, $L(c_1v_1 + c_2v_2 + \cdots c_nv_n + c_{n+1}v_{n+1}) =$

$$\begin{aligned} &= L(c_1v_1 + c_2v_2 + \cdots c_nv_n) + c_{n+1}L(v_{n+1}) && \text{by linearity of } L \\ &= c_1L(v_1) + c_2L(v_2) + \cdots + c_nL(v_n) + c_{n+1}L(v_{n+1}) && \text{by the induction hypothesis.} \end{aligned}$$

Hence the proposition is true for $n + 1$ and we conclude by the principle of mathematical induction that (2.) is true for all $n \in \mathbb{N}$. $\square$

Proposition 7.1.14.

Let $L \in L(V, W)$. If S is linearly dependent then $L(S)$ is linearly dependent.

Proof: Suppose there exists $c_1, \dots, c_k \in \mathbb{F}$ for which $v = \sum_{i=1}^{k} c_iv_i$ is a linear dependence in S. Calculate,

$$L(v) = L\left(\sum_{i=1}^{k} c_iv_i\right) = \sum_{i=1}^{k} c_iL(v_i)$$

which, noting $L(v), L(v_i) \in L(S)$ for all $i \in \mathbb{N}_k$, shows $L(S)$ has a linear dependence. Therefore, $L(S)$ is linearly dependent. $\square$

Just as the column and null space of a matrix are important to understand the nature of the matrix we likewise study the kernel and image of a linear transformation: (Theorem 7.1.18 shows these are subspaces)

Definition 7.1.15.

Let V, W be vector spaces over a field $\mathbb{F}$ and $T \in \mathcal{L}(V, W)$ then

$$\text{Ker}(T) = \{v \in V \mid T(v) = 0\} \qquad \& \qquad \text{Image}(T) = \text{Range}(T) = \{T(v) \mid v \in V\}$$

Example 7.1.16. *Let $T(f(x)) = f'(x)$ for $f(x) \in P_2(\mathbb{R})$ then clearly T is a linear transformation and we express the action of this map as $T(ax^2 + bx + c) = 2ax + b$. Hence,*

$$Ker(T) = \{ax^2 + bx + c \in P_2(\mathbb{R}) \mid 2ax + b = 0\} = \{c \mid c \in \mathbb{R}\} = \mathbb{R}.$$

Also, $Range(T) = span\{1, x\}$ since $T(ax^2 + bx + c) = 2ax + b$ allows arbitrary $2a, b \in \mathbb{R}$.

Notice, the statement $\text{Ker}(T) = \{0\}$ is equivalent to the statement that the only solution to the equation $T(x) = 0$ is $x = 0$.

Theorem 7.1.17. *linear map is injective iff only zero maps to zero.*

$T : V \to W$ is an injective linear transformation iff $\text{Ker}(T) = \{0\}$.

Proof: this is a biconditional statement. I'll prove the converse direction to begin.
$(\Leftarrow)$ Suppose $T(x) = 0$ iff $x = 0$ to begin. Let $x, y \in V$ and suppose $T(x) = T(y)$. By linearity we have $T(x - y) = T(x) - T(y) = 0$ hence $x - y = 0$ therefore $x = y$ and we find T is injective.

$(\Rightarrow)$ Suppose T is injective. Suppose $T(x) = 0$. Note $T(0) = 0$ by linearity of T but then by 1-1 property we have $T(x) = T(0)$ implies $x = 0$ hence the unique solution of $T(x) = 0$ is the zero solution. $\square$

For a linear transformation, the image of a subspace and the inverse image of a subspace are once again subspaces. This is certainly not true for arbitrary functions. In general, a nonlinear function takes linear spaces and twists them into all sorts of nonlinear shapes. For example, $f(x) = (x, x^2)$ takes the line $\mathbb{R}$ and pastes it onto the parabola $y = x^2$ in the range. We also can observe $f^{-1}\{(0, 0)\} = \{0\}$ and yet the mapping is certainly not injective. The theorems we find for linear functions do not usually generalize to functions in general[2]

Theorem 7.1.18.

If $T : V \to W$ is a linear transformation

(1.) and $V_o \leq V$ then $T(V_o) \leq W$,

(2.) and $W_o \leq W$ then $T^{-1}(W_o) \leq V$.

Proof: to prove (1.) suppose $V_o \leq V$. It follows $0 \in V_o$ and hence $T(0) = 0$ implies $0 \in T(V_o)$. Suppose $T(x), T(y) \in T(V_o)$ and $c \in \mathbb{F}$. Since $x, y \in V_o$ and V_o is a subspace we have $cx + y \in V_o$ thus $T(cx + y) \in T(V_o)$ and as

$$T(cx + y) = cT(x) + T(y)$$

[2]although, perhaps it's worth noting that in advanced calculus we learn how to linearize a function at a point. Some of our results here roughly generalize locally through the linearization and what are known as the inverse and implict function theorems

hence $T(V_o) \neq \emptyset$ is closed under addition and scalar multiplication. Therefore, $T(V_o) \leq W$.

To prove (2.) suppose $W_o \leq W$ and observe $0 \in W_o$ and $T(0) = 0$ implies $0 \in T^{-1}(W_o)$. Hence $T^{-1}(W_o) \neq \emptyset$. Suppose $c \in \mathbb{F}$ and $x, y \in T^{-1}(W_o)$, it follows that there exist $x_o, y_o \in W_o$ such that $T(x) = x_o$ and $T(y) = y_o$. Observe, by linearity of T,

$$T(cx + y) = cT(x) + T(y) = cx_o + y_o \in W_o.$$

hence $cx + y \in T^{-1}(W_o)$. Therefore, by the subspace theorem, $T^{-1}(W_o) \leq V$. $\square$

The special cases $V_o = V$ and $W_o = \{0\}$ merit discussion:

Corollary 7.1.19.

If $T : V \to W$ is a linear transformation then $T(V) \leq W$ and $T^{-1}\{0\} \leq V$. In other words, $\text{Ker}(T) \leq V$ and $\text{Range}(T) \leq W$.

Proof: observe $V \leq V$ and $\{0\} \leq W$ hence by Theorem 7.1.18 the Corollary holds true. $\square$

The following definitions of rank and nullity of a linear transformation are naturally connected to our prior use of the terms. In particular, we will soon[3] see that to each linear transformation we can associate a matrix and the null and column space of the associated matrix will have the same nullity and rank as the kernel and image respective.

Definition 7.1.20.

Let V, W be vector spaces. If a mapping $T : V \to W$ is a linear transformation then

$$\dim(\text{Ker}(T)) = \text{nullity}(T) \qquad \& \qquad \dim(\text{Range}(T)) = \text{rank}(T).$$

Thus far in this section we have studied the behaviour of a particular linear transformation. In what follows, we see how to combine given linear transformations to form new linear transformations.

Definition 7.1.21.

Suppose $T : V \to W$ and $S : V \to W$ are linear transformations then we define $T + S, T - S$ and cT for any $c \in \mathbb{F}$ by the rules

$$(T + S)(x) = T(x) + S(x). \quad (T - S)(x) = T(x) - S(x), \quad (cT)(x) = cT(x)$$

for all $x \in V$.

The proof of the proposition below as it is nearly identical to the proof of Proposition 7.1.

Proposition 7.1.22.

If $T, S \in \mathcal{L}(V, W)$ and $c \in \mathbb{F}$ then $T + S, cT \in \mathcal{L}(V, W)$.

[3] Lemma 7.4.15 and Proposition 7.4.16 to be precise

Proof: I'll be greedy and prove both at once: let $x, y \in V$ and $b, c \in \mathbb{F}$,

$$\begin{aligned}
(T+cS)(x+by) &= T(x+by) + (cS)(x+by) && \text{defn. of sum of transformations} \\
&= T(x+by) + cS(x+by) && \text{defn. of scalar mult. of transformations} \\
&= T(x) + bT(y) + c[S(x) + bS(y)] && \text{linearity of } S \text{ and } T \\
&= T(x) + cS(x) + b[T(y) + cS(y)] && \text{vector algebra props.} \\
&= (T+cS)(x) + b(T+cS)(y) && \text{again, defn. of sum and scal. mult. of trans.}
\end{aligned}$$

Let $c = 1$ and $b = 1$ to see $T + S$ is additive. Let $c = 1$ and $x = 0$ to see $T + S$ is homogeneous. Finally, let $T = 0$ to see cS is additive ($b = 1$) and homogeneous ($x = 0$). $\square$

Recall that function space of all functions from V to W is naturally a vector space according to the point-wise addition and scalar multiplication of functions. It follows from the subspace theorem and the proposition above that:

Proposition 7.1.23.

> The set of all linear transformations from V to W forms a vector space with respect to the natural point-wise addition and scalar multiplication of functions; $\mathcal{L}(V, W) \leq \mathcal{F}(V, W)$.

Proof: If $T, S \in L(V, W)$ and $c \in \mathbb{F}$ then $T + S, cT \in L(V, W)$ hence $L(V, W)$ is closed under addition and scalar multiplication. Moreover, the trivial function $T(x) = 0$ for all $x \in V$ is clearly in $L(V, W)$ hence $L(V, W) \neq \emptyset$ and we conclude by the subspace theorem that $L(V, W) \leq \mathcal{F}(V, W)$. $\square$

Function composition in the context of abstract vector spaces is the same as it was in precalculus.

Definition 7.1.24.

> Suppose $T : V \to U$ and $S : U \to W$ are linear transformations then we define $S \circ T : V \to W$ by $(S \circ T)(x) = S(T(x))$ for all $x \in V$.

The composite of linear maps is once more a linear map.

Proposition 7.1.25.

> Suppose $T \in L(V, U)$ and $S \in L(U, W)$ then $S \circ T \in L(V, W)$.

Proof: Let $x, y \in V$ and $c \in \mathbb{F}$,

$$\begin{aligned}
(S \circ T)(x + cy) &= S(T(x + cy)) && \text{defn. of composite} \\
&= S(T(x) + cT(y)) && T \text{ is linear trans.} \\
&= S(T(x)) + cS(T(y)) && S \text{ is linear trans.} \\
&= (S \circ T)(x) + c(S \circ T)(y) && \text{defn. of composite}
\end{aligned}$$

additivity follows from $c = 1$ and homogeneity of $S \circ T$ follows from $x = 0$ thus $S \circ T \in L(V, W)$. $\square$

A vector space V together with a bilinear multiplication $m : V \times V \to V$ is called an **algebra**[4]. For example, we saw before that square matrices form an algebra with respect to addition and matrix multiplication. Notice that $V = L(W, W)$ is likewise naturally an algebra with respect to function

[4] it is somewhat ironic that all too often we often neglect to define an algebra in our modern algebra courses in the US educational system. As students, you ought to demand more. See Dummit and Foote for a precise definition

addition and composition. One of our goals in this course is to understand the interplay between the algebra of transformations and the algebra of matrices.

The theorem below says the inverse of a linear transformation is also a linear transformation.

Theorem 7.1.26.

Suppose $T \in L(V, W)$ has inverse function $S : W \to V$ then $S \in L(W, V)$.

Proof: suppose $T \circ S = Id_W$ and $S \circ T = Id_V$. Suppose $x, y \in W$ hence there exits $a, b \in V$ for which $T(a) = x$ and $T(b) = y$. Also, let $c \in \mathbb{F}$. Consider,

$$\begin{aligned} S(cx + y) &= S(cT(a) + T(b)). \\ &= S(T(ca + b)) : && \text{by linearity of } T \\ &= ca + b : && \text{def. of identity function} \\ &= cS(x) + S(y) : && \text{note } a = S(T(a)) = S(x) \text{ and } b = S(T(b)) = S(y). \end{aligned}$$

Therefore, S is a linear transformation. $\square$

Observe, linearity of the inverse follows automatically from linearity of the map. Furthermore, it is useful for us to characterize the behaviour of LI sets under invertible linear transformations: What about LI of sets? If S is a LI subset of V and $T \in \mathcal{L}(V, W)$ then is $T(S)$ also LI? The answer is clearly no in general. Consider the trivial transformation of Example 7.1.8. On the other extreme we have the following:

Theorem 7.1.27.

If $T : V \to W$ is an injective linear transformation then S is LI implies $T(S)$ is LI. If $L : V \to W$ is any linear transformation and if U is LI in W then $L^{-1}(U)$ is LI in V.

Proof: suppose T is an injective linear transformation from V to W and suppose $S = \{s_1, \dots, s_k\}$ is a LI subset of V. Consider $T(S) = \{T(s_1), \dots, T(s_k)\}$. In particular, suppose there exist $c_1, \dots, c_k \in \mathbb{F}$ such that $c_1T(s_1) + \cdots + c_kT(s_k) = 0$ implies $T(c_1s_1 + \cdots + c_ks_k) = 0$ by Proposition 7.1.13. Thus $c_1s_1 + \cdots + c_ks_k \in \text{Ker}(T) = \{0\}$ by Theorem 7.1.17. Consequently, $c_1s_1 + \cdots + c_ks_k = 0$ hence $c_1 = 0, \dots, c_k = 0$ by LI of S.

Conversely, if $U = \{u_1, \dots, u_k\}$ is LI in W then suppose $x_1, \cdots, x_n \in L^{-1}(U)$. Assume $c_1x_1 + \cdots + c_nx_n = 0$ and observe:

$$L(c_1x_1 + \cdots + c_nx_n) = L(0) \quad \Rightarrow \quad c_1L(x_1) + \cdots + c_nL(x_n) = 0$$

but, by definition of $L^{-1}(U)$ we have $L(x_1), \dots, L(x_n) \in U$ and thus by LI of U we conclude $c_1 = 0, \dots, c_n = 0$. $\square$

7.2 linear transformations of column vectors

In this section we take a break from the abstractness and study geometrically how linear transformations behave on $\mathbb{R}^n$. However, the Theorems are given for $\mathbb{F}^n$ where visualization is not advisable in general. We also make explicit the interplay between matrix multiplication and composition of linear transformations.

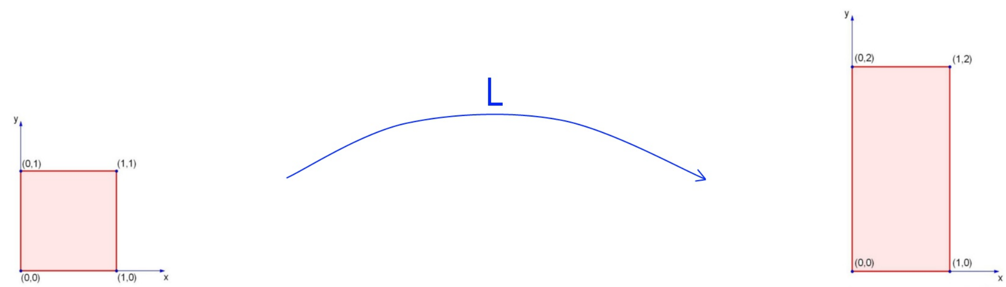
Example 7.2.1. *Let $L(x,y) = (x,2y)$. This is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. Notice*

$$L((x,y)+(z,w)) = L(x+z,y+w) = (x+z,2(y+w)) = (x,2y)+(z,2w) = L(x,y)+L(z,w)$$

and

$$L(c(x,y)) = L(cx,cy) = (cx,2(cy)) = c(x,2y) = cL(x,y)$$

for all $(x,y),(z,w) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Therefore, L is a linear transformation on $\mathbb{R}^2$. Let's examine how this function maps the unit square in the domain: suppose $(x,y) \in [0,1]\times[0,1]$. This means $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Label the Cartesian coordinates of the range by u,v so $L(x,y) = (x,2y) = (u,v)$. We have $x = u$ thus $0 \leq u \leq 1$. Also, $v = 2y$ hence $y = \frac{v}{2}$ hence $0 \leq y \leq 1$ implies $0 \leq \frac{v}{2} \leq 1$ or $0 \leq v \leq 2$.
To summarize: $L([0,1]\times[0,1]) = [0,1]\times[0,2]$. This mapping has stretched out the vertical direction.



The method of analysis we used in the preceding example was a little clumsy, but for general mappings that is more or less the method of attack. You pick some shapes or curves in the domain and see what happens under the mapping. For linear mappings there is an easier way. It turns out that if we map some shape with straight sides then the image will likewise be a shape with flat sides (or faces in higher dimensions). Therefore, to find the image we need only map the corners of the shape then connect the dots. However, I should qualify that it may not be the case the type of shape is preserved. We could have a rectangle in the domain get squished into a line or point in the domain. We would like to understand when such squishing will happen and also when a given mapping will actually cover the whole codomain. For linear mappings there are very satisfying answers to these questions in terms of the theory we have already discussed in previous chapters.

Proposition 7.2.2.

If $A \in \mathbb{F}^{m\times n}$ and $L : \mathbb{F}^n \to \mathbb{F}^m$ is defined by $L(x) = Ax$ for each $x \in \mathbb{F}^n$ then L is a linear transformation.

Proof: Let $A \in \mathbb{F}^{m\times n}$ and define $L : \mathbb{F}^n \to \mathbb{F}^m$ by $L(x) = Ax$ for each $x \in \mathbb{F}^n$. Let $x,y \in \mathbb{F}^n$ and $c \in \mathbb{F}$,

$$L(x+y) = A(x+y) = Ax+Ay = L(x)+L(y)$$

and

$$L(cx) = A(cx) = cAx = cL(x)$$

thus L is a linear transformation. $\square$

Obviously this gives us a nice way to construct examples. The following proposition is really at the heart of all the geometry in this section.

Proposition 7.2.3.

Let $\mathcal{L} = \{p + tv \mid t \in [0,1], p, v \in \mathbb{R}^n \text{ with } v \neq 0\}$ define a line segment from p to $p+v$ in $\mathbb{R}^n$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T(\mathcal{L})$ is a either a line-segment from $T(p)$ to $T(p+v)$ or a point.

Proof: suppose T and $\mathcal{L}$ are as in the proposition. Let $y \in T(\mathcal{L})$ then by definition there exists $x \in \mathcal{L}$ such that $T(x) = y$. But this implies there exists $t \in [0,1]$ such that $x = p + tv$ so $T(p+tv) = y$. Notice that

$$y = T(p+tv) = T(p) + T(tv) = T(p) + tT(v).$$

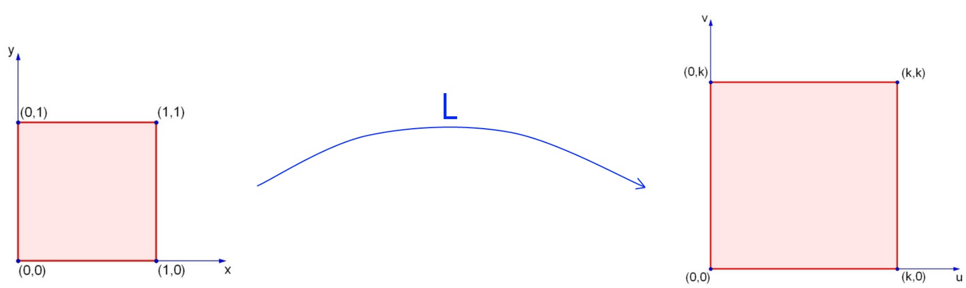
which implies $y \in \{T(p) + sT(v) \mid s \in [0,1]\} = \mathcal{L}_2$. Therefore, $T(\mathcal{L}) \subseteq \mathcal{L}_2$. Conversely, suppose $z \in \mathcal{L}_2$ then $z = T(p) + sT(v)$ for some $s \in [0,1]$ but this yields by linearity of T that $z = T(p+sv)$ hence $z \in T(\mathcal{L})$. Since we have that $T(\mathcal{L}) \subseteq \mathcal{L}_2$ and $\mathcal{L}_2 \subseteq T(\mathcal{L})$ it follows that $T(\mathcal{L}) = \mathcal{L}_2$. Note that $\mathcal{L}_2$ is a line-segment provided that $T(v) \neq 0$, however if $T(v) = 0$ then $\mathcal{L}_2 = \{T(p)\}$ and the proposition follows. $\square$

My choice of mapping the unit square has no particular signficance in the examples below. I merely wanted to keep it simple and draw your eye to the distinction between the examples. In each example we'll map the four corners of the square to see where the transformation takes the unit-square. Those corners are simply $(0,0), (1,0), (1,1), (0,1)$ as we traverse the square in a counter-clockwise direction.

Example 7.2.4. *Let* $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ *for some* $k > 0$*. Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$*. In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}.$$

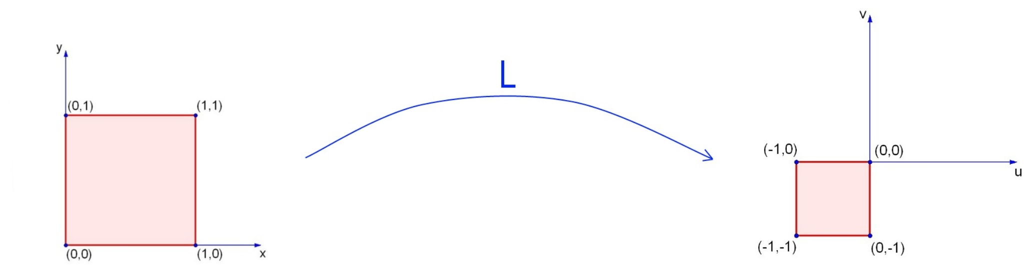
We find $L(0,0) = (0,0),\ L(1,0) = (k,0),\ L(1,1) = (k,k),\ L(0,1) = (0,k)$*. This mapping is called a* **dilation***.*



Example 7.2.5. *Let* $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$*. Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$*. In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

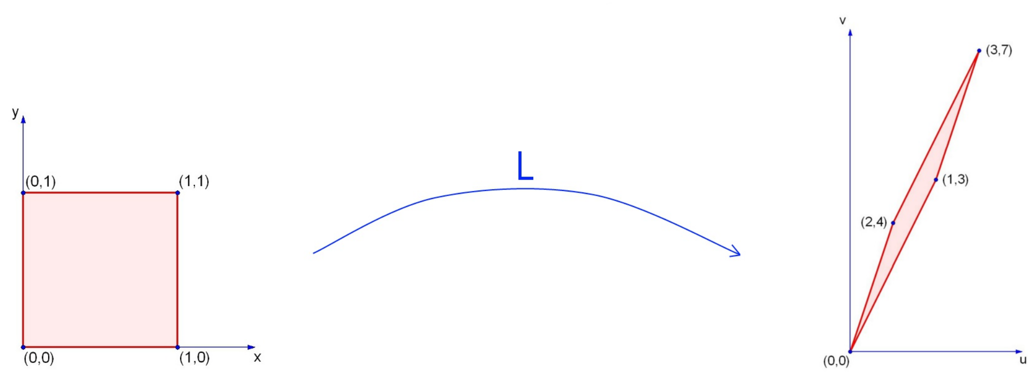
We find $L(0,0) = (0,0),\ L(1,0) = (-1,0),\ L(1,1) = (-1,-1),\ L(0,1) = (0,-1)$*. This mapping is called an* **inversion***.*



Example 7.2.6. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}.$$

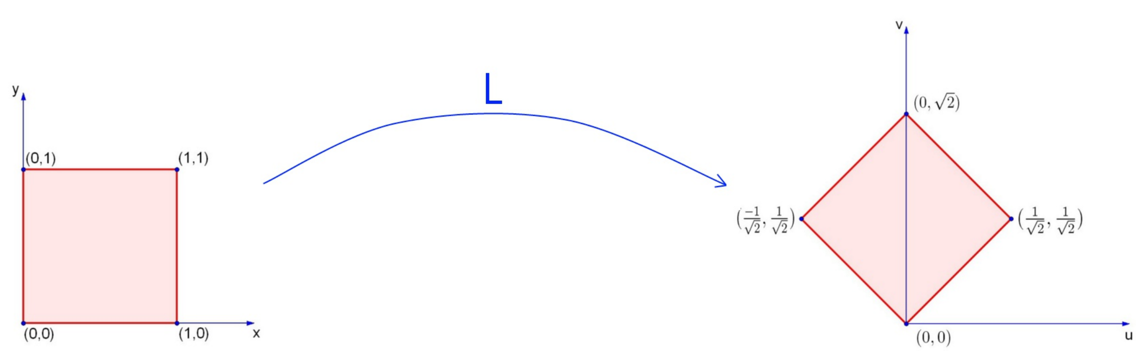
We find $L(0, 0) = (0, 0)$, $L(1, 0) = (1, 3)$, $L(1, 1) = (3, 7)$, $L(0, 1) = (2, 4)$. *This mapping shall remain nameless, it is doubtless a combination of the other named mappings.*



Example 7.2.7. *Let* $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x, y) = A(x, y) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

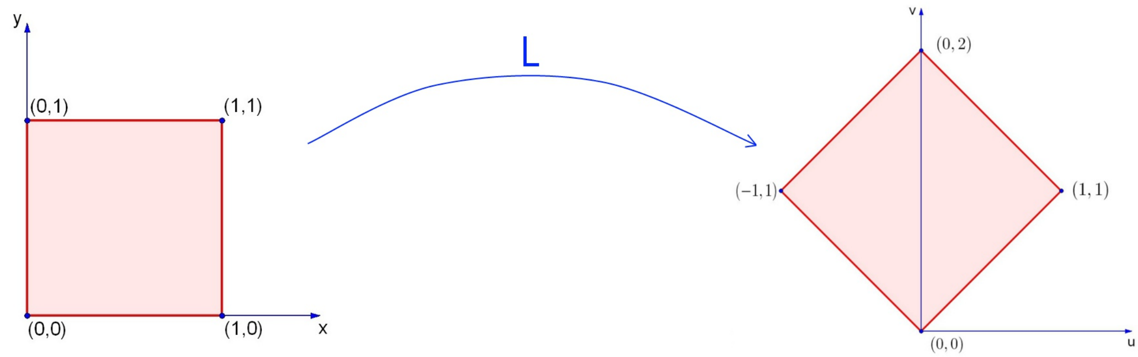
We find $L(0, 0) = (0, 0)$, $L(1, 0) = \frac{1}{\sqrt{2}}(1, 1)$, $L(1, 1) = \frac{1}{\sqrt{2}}(0, 2)$, $L(0, 1) = \frac{1}{\sqrt{2}}(-1, 1)$. *This mapping is a rotation by* $\pi/4$ *radians.*



Example 7.2.8. *Let* $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

We find $L(0,0) = (0,0)$, $L(1,0) = (1,1)$, $L(1,1) = (0,2)$, $L(0,1) = (-1,1)$. *This mapping is a rotation followed by a dilation by* $k = \sqrt{2}$.

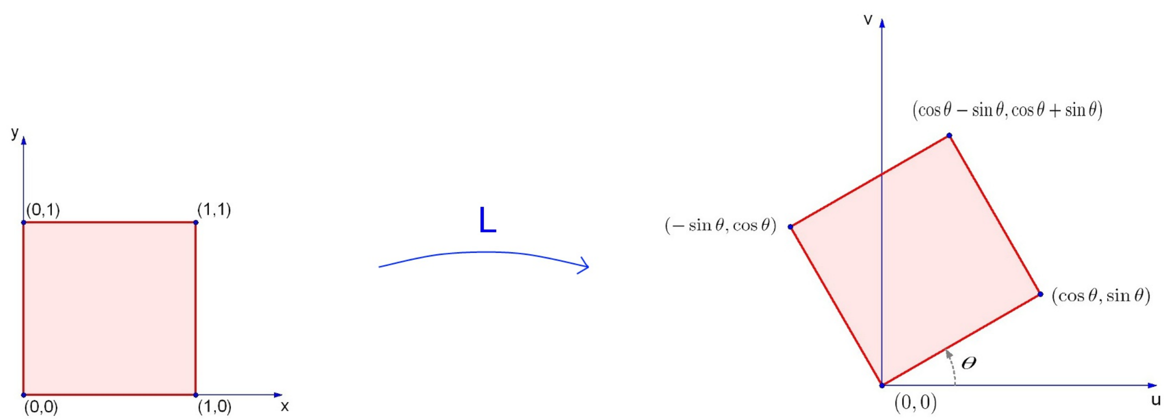


We will come back to discuss rotations a few more times this semester, you'll see they give us interesting and difficult questions later this semester.

Example 7.2.9. *Let* $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}.$$

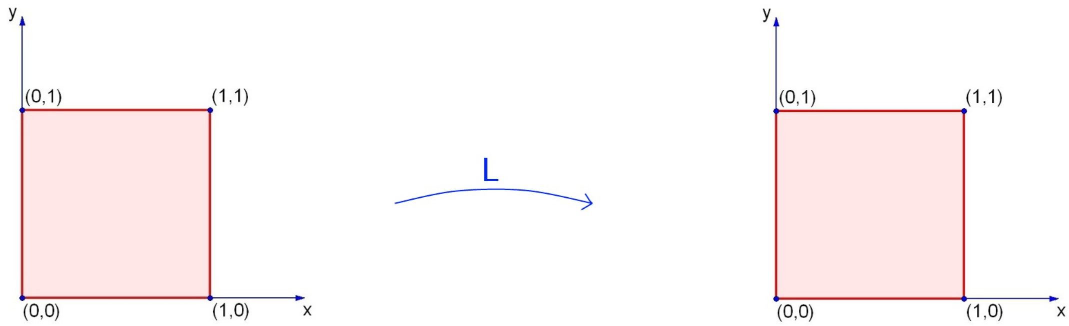
We find $L(0,0) = (0,0)$, $L(1,0) = (\cos(\theta), \sin(\theta))$, $L(1,1) = (\cos(\theta) - \sin(\theta), \cos(\theta) + \sin(\theta))$ $L(0,1) = (\sin(\theta), \cos(\theta))$. *This mapping is a rotation by* θ *in the counter-clockwise direction. Of course you could have derived the matrix* A *from the picture below.*



Example 7.2.10. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We find $L(0,0) = (0,0)$, $L(1,0) = (1,0)$, $L(1,1) = (1,1)$, $L(0,1) = (0,1)$. This mapping is a rotation by zero radians, or you could say it is a dilation by a factor of 1, ... usually we call this the identity mapping because the image is identical to the preimage.



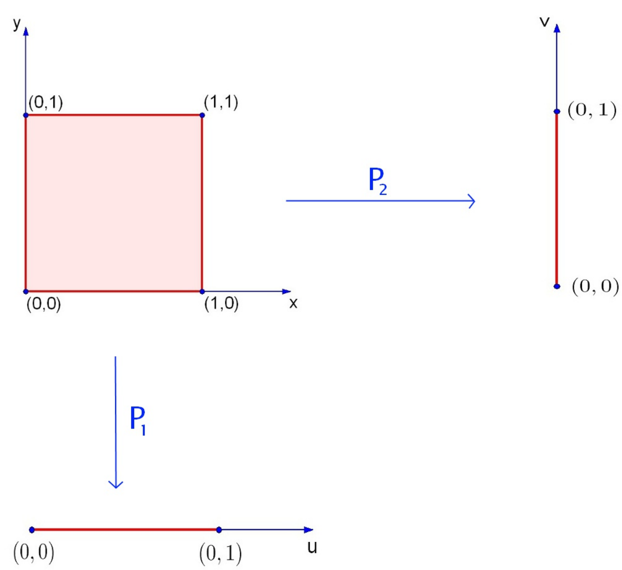
Example 7.2.11. *Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Define $P_1(v) = A_1 v$ for all $v \in \mathbb{R}^2$. In particular this means,*

$$P_1(x,y) = A_1(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

We find $P_1(0,0) = (0,0)$, $P_1(1,0) = (1,0)$, $P_1(1,1) = (1,0)$, $P_1(0,1) = (0,0)$. This mapping is a **projection** *onto the first coordinate.*
Let $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Define $L(v) = A_2 v$ for all $v \in \mathbb{R}^2$. In particular this means,

$$P_2(x,y) = A_2(x,y) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

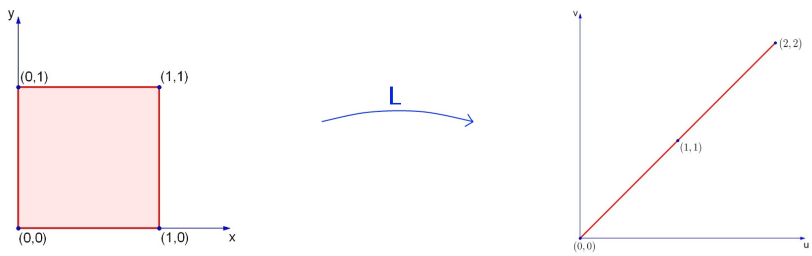
We find $P_2(0,0) = (0,0)$, $P_2(1,0) = (0,0)$, $P_2(1,1) = (0,1)$, $P_2(0,1) = (0,1)$. This mapping is **projection** *onto the second coordinate.*
We can picture both of these mappings at once:



Example 7.2.12. *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$*. Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$*. In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}.$$

We find $L(0,0) = (0,0),\ L(1,0) = (1,1),\ L(1,1) = (2,2),\ L(0,1) = (1,1)$*. This mapping is not a projection, but it does collapse the square to a line-segment.*



A projection has to have the property that if it is applied twice then you obtain the same image as if you applied it only once. If you apply the transformation to the image then you'll obtain a line-segment from $(0,0)$ to $(4,4)$. While it is true the transformation "projects" the plane to a line it is not technically a "projection".

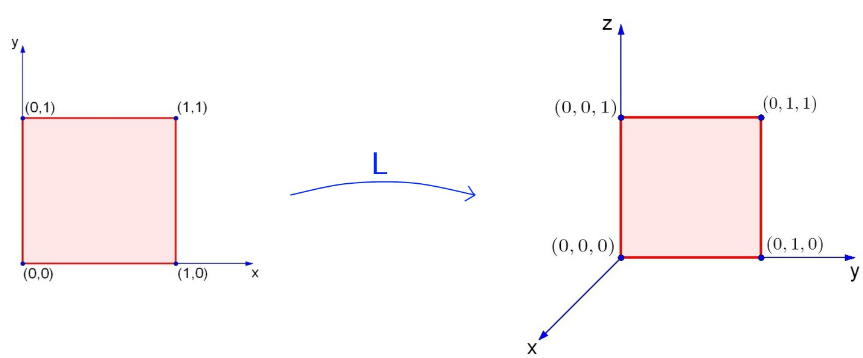
Remark 7.2.13.

The examples here have focused on linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. It turns out that higher dimensional mappings can largely be understood in terms of the geometric operations we've seen in this section.

Example 7.2.14. *Let* $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$*. Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$*. In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}.$$

We find $L(0,0) = (0,0,0),\ L(1,0) = (0,1,0),\ L(1,1) = (0,1,1),\ L(0,1) = (0,0,1)$*. This mapping moves the* xy*-plane to the* yz*-plane. In particular, the horizontal unit square gets mapped to vertical unit square;* $L([0,1] \times [0,1]) = \{0\} \times [0,1] \times [0,1]$*. This mapping certainly is not surjective because no point with* $x \neq 0$ *is covered in the range.*



Example 7.2.15. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^3$. *In particular this means,*

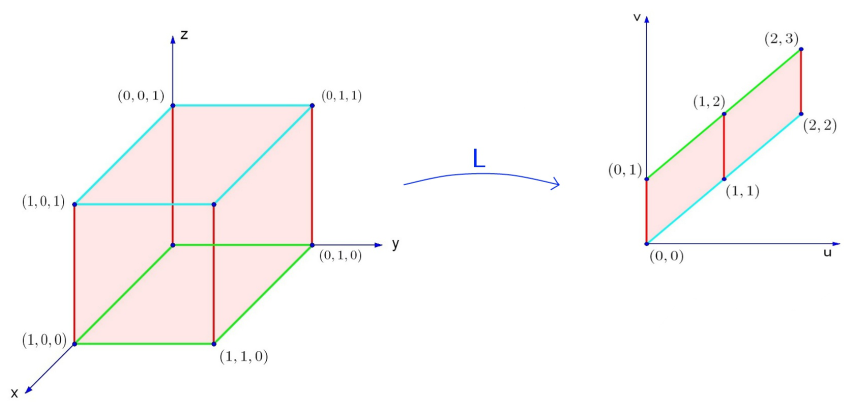
$$L(x,y,z) = A(x,y,z) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x+y+z \end{bmatrix}.$$

Let's study how L *maps the unit cube. We have* $2^3 = 8$ *corners on the unit cube,*

$$L(0,0,0) = (0,0),\ L(1,0,0) = (1,1),\ L(1,1,0) = (2,2),\ L(0,1,0) = (1,1)$$

$$L(0,0,1) = (0,1),\ L(1,0,1) = (1,2),\ L(1,1,1) = (2,3),\ L(0,1,1) = (1,2).$$

This mapping squished the unit cube to a shape in the plane which contains the points $(0,0)$, $(0,1)$, $(1,1)$, $(1,2)$, $(2,2)$, $(2,3)$. *Face by face analysis of the mapping reveals the image is a parallelogram. This mapping is certainly not injective since two different points get mapped to the same point. In particular, I have color-coded the mapping of top and base faces as they map to line segments. The vertical faces map to one of the two parallelograms that comprise the image.*

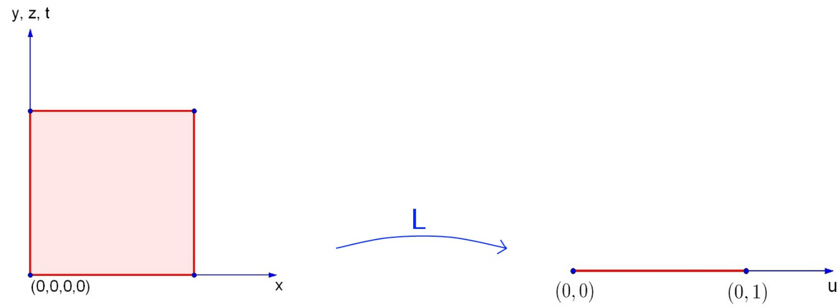


I have used terms like "vertical" or "horizontal" in the standard manner we associate such terms with three dimensional geometry. Visualization and terminology for higher-dimensional examples is not as obvious. However, with a little imagination we can still draw pictures to capture important aspects of mappings.

Example 7.2.16. *Let* $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^4$. *In particular this means,*

$$L(x,y,z,t) = A(x,y,z,t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}.$$

Let's study how L *maps the unit hypercube* $[0,1]^4 \subset \mathbb{R}^4$. *We have* $2^4 = 16$ *corners on the unit hypercube, note* $L(1,a,b,c) = (1,1)$ *whereas* $L(0,a,b,c) = (0,0)$ *for all* $a,b,c \in [0,1]$. *Therefore, the unit hypercube is squished to a line-segment from* $(0,0)$ *to* $(1,1)$. *This mapping is neither surjective nor injective. In the picture below the vertical axis represents the* y,z,t*-directions.*



Obviously we have not even begun to appreciate the wealth of possibilities that exist for linear mappings. Clearly different types of matrices will decribe different types of geometric transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. On the other hand, square matrices describe mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ and these can be thought of as coordinate transformations. A square matrix may give us a way to define new coordinates on $\mathbb{R}^n$. We will return to the concept of linear transformations a number of times in this course. Hopefully you already appreciate that linear algebra is not just about solving equations. It always comes back to that, but there is more here to ponder.

If you are pondering what I am pondering then you probably would like to know if all linear mappings from $\mathbb{F}^n$ to $\mathbb{F}^m$ can be reduced to matrix multiplication? We saw that if a map is defined as a matrix multiplication then it will be linear. A natural question to ask: is the converse true? Given a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$ can we write the transformation as multiplication by a matrix ?

Theorem 7.2.17. *fundamental theorem of linear algebra.*

$L : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation if and only if there exists $A \in \mathbb{F}^{m\times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{F}^n$.

Proof: ($\Leftarrow$) Assume there exists $A \in \mathbb{F}^{m\times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{F}^n$. As we argued before,

$$L(x + cy) = A(x + cy) = Ax + cAy = L(x) + cL(y)$$

for all $x, y \in \mathbb{F}^n$ and $c \in \mathbb{F}$ hence L is a linear transformation.

($\Rightarrow$) Assume $L : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation. Let e_i denote the standard basis in $\mathbb{F}^n$ and let f_j denote the standard basis in $\mathbb{F}^m$. If $x \in \mathbb{F}^n$ then there exist constants $x_i \in \mathbb{F}$ such that $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ and

$$\begin{aligned} L(x) &= L(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\ &= x_1L(e_1) + x_2L(e_2) + \cdots + x_nL(e_n) \end{aligned}$$

where we made use of Proposition 7.1.13. Notice $L(e_i) \in \mathbb{F}^m$ thus there exist constants, say $A_{ji} \in \mathbb{F}$, such that

$$L(e_i) = A_{1i}f_1 + A_{2i}f_2 + \cdots + A_{mi}f_m$$

for each $i = 1, 2, \dots, n$. Let's put it all together,

$$\begin{aligned} L(x) &= \sum_{i=1}^{n} x_i L(e_i) \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} A_{ji} f_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ji} x_i f_j \\ &= Ax. \end{aligned}$$

Notice that $A_{ji} = L(e_i)_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ hence $A \in \mathbb{F}^{m \times n}$ by its construction. $\square$

The fundamental theorem of linear algebra allows us to make the following definition.

Definition 7.2.18.

Let $L : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation, the matrix $A \in \mathbb{F}^{m\times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{F}^n$ is called the **standard matrix** of L. We denote this by $[L] = A$ or more compactly, $[L_A] = A$, we say that L_A is the linear transformation induced by A. Moreover, the components of the matrix A are found from $A_{ji} = (L(e_i)))_j$.

Example 7.2.19. *Given that $L([x, y, z]^T) = [x+2y, 3y+4z, 5x+6z]^T$ for $[x, y, z]^T \in \mathbb{R}^3$ find the the standard matrix of L. We wish to find a 3×3 matrix such that $L(v) = Av$ for all $v = [x, y, z]^T \in \mathbb{R}^3$. Write $L(v)$ then collect terms with each coordinate in the domain,*

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3y+4z \\ 5x+6z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$$

It's not hard to see that,

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad A = [L] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix}$$

Notice that the columns in A are just as you'd expect from the proof of theorem 7.2.17. $[L] = [L(e_1)|L(e_2)|L(e_3)]$. In future examples I will exploit this observation to save writing.

Example 7.2.20. *Suppose that $L((t, x, y, z)) = (t + x + y + z, z - x, 0, 3t - z)$, find $[L]$.*

$$\begin{array}{l} L(e_1) = L((1,0,0,0)) = (1,0,0,3) \\ L(e_2) = L((0,1,0,0)) = (1,-1,0,0) \\ L(e_3) = L((0,0,1,0)) = (1,0,0,0) \\ L(e_4) = L((0,0,0,1)) = (1,1,0,-1) \end{array} \quad \Rightarrow \quad [L] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -1 \end{bmatrix}.$$

I invite the reader to check my answer here and see that $L(v) = [L]v$ for all $v \in \mathbb{R}^4$ as claimed.

Very well, let's return to the concepts of injective and surjectivity of linear mappings. How do our theorems of LI and spanning inform us about the behaviour of linear transformations? The following pair of theorems summarize the sitution nicely.

Theorem 7.2.21. *linear map is injective iff only zero maps to zero.*

$L : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation with standard matrix [L] then

1. L is 1-1 iff the columns of $[L]$ are linearly independent,
2. L is onto $\mathbb{F}^m$ iff the columns of $[L]$ span $\mathbb{F}^m$.

Proof: To prove (1.) use Theorem 7.1.17:

$$L \text{ is 1-1} \;\Leftrightarrow\; \Big\{L(x) = 0 \;\Leftrightarrow\; x = 0\Big\} \;\Leftrightarrow\; \Big\{[L]x = 0 \;\Leftrightarrow\; x = 0.\Big\}$$

and the last equation simply states that if a linear combination of columns of L is zero then the coefficients of that linear equation are zero so (1.) follows.

To prove (2.) recall that Proposition 6.8.3 stated that if $A \in \mathbb{F}^{m\times n}$, $v \in \mathbb{F}^n$ then $Av = b$ is consistent for all $b \in \mathbb{F}^m$ iff the columns of A span $\mathbb{F}^m$. To say L is onto $\mathbb{F}^m$ means that for each $b \in \mathbb{F}^m$ there exists $v \in \mathbb{F}^n$ such that $L(v) = b$. But, this is equivalent to saying that $[L]v = b$ is consistent for each $b \in \mathbb{F}^m$ so (2.) follows. $\square$

Example 7.2.22.
1. *You can verify that the linear mappings in Examples 7.2.4, 7.2.5, 7.2.6, 7.2.7, 7.2.8, 7.2.9 and 7.2.10 wer both 1-1 and onto. You can see the columns of the tranformation matrices were both LI and spanned $\mathbb{R}^2$ in each of these examples.*
2. *In constrast, Examples 7.2.11 and 7.2.12 were neither 1-1 nor onto. Moreover, the columns of transformation's matrix were linearly dependent in each of these cases and they did not span $\mathbb{R}^2$. Instead the span of the columns covered only a particular line in the range.*
3. *In Example 7.2.14 the mapping is injective and the columns of A were indeed LI. However, the columns do not span $\mathbb{R}^3$ and as expected the mapping is not onto $\mathbb{R}^3$.*
4. *In Example 7.2.15 the mapping is not 1-1 and the columns are obviously linearly dependent. On the other hand, the columns of A do span $\mathbb{R}^2$ and the mapping is onto.*
5. *In Example 7.2.16 the mapping is neither 1-1 nor onto and the columns of the matrix are neither linearly independent nor do they span $\mathbb{R}^2$.*

The standard matrix enjoys many natural formulas. The standard matrix of the sum, difference or scalar multiple of linear transformations likewise the sum, difference or scalar multiple of the standard matrices of the respective linear transformations.

Proposition 7.2.23.

Suppose $T : \mathbb{F}^n \to \mathbb{F}^m$ and $S : \mathbb{F}^n \to \mathbb{F}^m$ are linear transformations then

$$(1.)\ [T + S] = [T] + [S], \quad (2.)\ [T - S] = [T] - [S], \quad (3.)\ [cS] = c[S].$$

Proof: Note $(T + cS)(e_j) = T(e_j) + cS(e_j)$ hence $((T + cS)(e_j))_i = (T(e_j))_i + c(S(e_j))_i$ for all i, j hence $[T + cS] = [T] + c[S]$. Set $c = 1$ to obtain (1.). Set $c = -1$ to obtain (2.). Finally, set $T = 0$ to obtain (3.). $\square$

Example 7.2.24. *Suppose $T(x,y) = (x+y, x-y)$ and $S(x,y) = (2x, 3y)$. It's easy to see that*

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } [S] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \Rightarrow \quad [T+S] = [T] + [S] = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Therefore, $(T+S)(x,y) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+y \\ x+2y \end{bmatrix} = (3x+y, x+2y)$. *Naturally this is the same formula that we would obtain through direct addition of the formulas of T and S.*

Matrix multiplication is naturally connected to the problem of composition of linear maps.

Proposition 7.2.25.

> $S : \mathbb{F}^p \to \mathbb{F}^m$ and $T : \mathbb{F}^n \to \mathbb{F}^p$ are linear transformations then $S \circ T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation with standard matrix $[S][T]$; that is, $[S \circ T] = [S][T]$.

Proof: Let us denote $\mathbb{F}^n = span\{e_i \mid i = 1, \dots, n\}$ whereas $\mathbb{F}^p = span\{f_i \mid i = 1, \dots, p\}$ and $\mathbb{F}^m = span\{g_i \mid i = 1, \dots, m\}$. To find the matrix of the composite we need only calculate its action on the standard basis: by definition, $[S \circ T]_{ij} = ((S \circ T)(e_j))_i$, observe

$$\begin{aligned}
(S \circ T)(e_j) &= S(T(e_j)) && : \text{def. of composite} \\
&= S([T]e_j) && : \text{def. of } [T] \\
&= S(\sum_k [T]_{kj} f_k) && : \text{standard basis } \{f_i\} \text{ spans } \mathbb{F}^p \\
&= \sum_k [T]_{kj} S(f_k) && : \text{homogeneity of } S \\
&= \sum_k [T]_{kj} [S] f_k && : \text{def. of } [S] \\
&= \sum_k [T]_{kj} \sum_i [S]_{ik} g_i && : \text{standard basis } \{g_i\} \text{ spans } \mathbb{F}^m \\
&= \sum_k \sum_i [S]_{ik} [T]_{kj} g_i && : \text{by (2.) of Prop. 1.4.3} \\
&= \sum_i \left[\sum_k [S]_{ik} [T]_{kj} \right] g_i && : \text{by (1.) of Prop. 1.4.3} \\
&= \sum_i ([S][T])_{ij} g_i && : \text{def. of matrix multiplication}
\end{aligned}$$

The i-th component of $(S \circ T)(e_j)$ is easily seen from the above expression. In particular, we find $[S \circ T]_{ij} = \sum_k [S]_{jk}[T]_{ki}$ and the proof is complete. $\square$

Think about this: **matrix multiplication was defined to make the above proposition true**. Perhaps you wondered, why don't we just multiply matrices some other way? Well, now you have an answer. If we defined matrix multiplication differently then the result we just proved would not be true. However, as the course progresses, you'll see why it is so important that this result be true. It lies at the heart of many connections between the world of linear transformations and the world of matrices. It says we can trade composition of linear transformations for multiplication of matrices.

7.3 restriction, extension, isomorphism

Another way we can create new linear transformations from a given transformation is by restriction. Recall that the restriction of a given function is simply a new function where part of the domain has been removed. Since linear transformations are only defined on vector spaces we naturally are only permitted restrictions to subspaces of a given vector space.

Definition 7.3.1.

If $T : V \to W$ is a linear transformation and $U \subseteq V$ then we define $T|_U : U \to W$ by $T|_U(x) = T(x)$ for all $x \in U$. We say $T|_U$ is the **restriction of T to U**.

Proposition 7.3.2.

If $T \in L(V, W)$ and $U \leq V$ then $T|_U \in L(U, W)$.

Proof: let $x, y \in U$ and $c \in \mathbb{F}$. Since $U \leq V$ it follows $cx + y \in U$ thus

$$T|_U(cx + y) = T(cx + y) = cT(x) + T(y) = cT|_u(x) + T|_U(y)$$

where I use linearity of T for the middle equality and the definition of $T|_U$ for the outside equalities. Therefore, $T|_U \in L(U, W)$. $\square$

We can create a linear transformation on an infinity of vectors by prescribing its values on the basis alone. This is a fantastic result.

Proposition 7.3.3.

Suppose β is a basis for a vector space V and suppose W is also a vector space. Furthermore, suppose $L : \beta \to W$ is a function. There exists a unique linear extension of L to V.

Proof: to begin, let us understand the final sentence. A linear extension of L to V means a function $T : V \to W$ which is a linear transformation and $T|_\beta = L$. Uniqueness requires that we show if T_1, T_2 are two such extensions then $T_1 = T_2$. With that settled, let us begin the actual proof.

Suppose $\beta = \{v_1, \dots, v_n\}$ if $x \in V$ then there exist **unique** $x_1, \dots, x_n \in \mathbb{F}$ for which $x = \sum_{i=1}^n x_i v_i$. Therefore, define $T : V \to W$ as follows

$$T(x) = T\left(\sum_{i=1}^n x_i v_i\right) = \sum_{i=1}^n x_i L(v_i).$$

Clearly $T|_\beta = L$. I leave proof that $T \in L(V, W)$ to the reader. Suppose T_1, T_2 are two such extensions. Consider, $x = \sum_{i=1}^n x_i v_i$

$$T_1(x) = T_1\left(\sum_{i=1}^n x_i v_i\right) = \sum_{i=1}^n x_i L(v_i).$$

However, the same calculation holds for $T_2(x)$ hence $T_1(x) = T_2(x)$ for all $x \in V$ therefore the extension T is unique. $\square$.

When we make use of the proposition above we typically use it to simplify a definition of a given linear transformation. In practice, we may define a mapping on a basis then **extend linearly**.

We conclude this section by initiating our discussion of **isomorphism**.

Definition 7.3.4.

Vector spaces $V(\mathbb{F})$ and $W(\mathbb{F})$ are **isomorphic** if there exists an invertible linear transformation $\Psi : V \to W$. Furthermore, an invertible linear transformation is called an **isomorphism**. We write $V \approxeq W$ if V and W are isomorphic.

Notice that it suffices to check $\Psi : V \to W$ is linear and invertible. Linearity of Ψ^{-1} follows by Theorem 7.1.26. This is nice as it means we have less work to do when proving some given mapping is an isomorphism.

Theorem 7.3.5.

If $V \approxeq W$ then $\dim(V) = \dim(W)$

Proof: Let $\Psi : V \to W$ be an isomorphism. Invertible linear mappings are injective we know that both Ψ and Ψ^{-1} must preserve LI of sets. In particular, if β is a basis for V then $\Psi(\beta)$ must be a LI set in W. Likewise, if γ is a basis for W then $\Psi^{-1}(\gamma)$ must be a LI set in V. Finally, since bijections preserve cardnality it follows $\dim(V) = \dim(W)$. $\square$

This theorem has a converse. We need a proposition before we prove the other half.

Proposition 7.3.6.

If $T : V \to U$ and $S : U \to W$ are isomorphisms then $S \circ T$ is an isomorphism. Moreover, $\approxeq$ is an equivalence relation on the class of all vector spaces over a given field $\mathbb{F}$.

Proof: let $T \in \mathcal{L}(V, U)$ and $S \in \mathcal{L}(U, W)$ be isomorphisms. Recall Proposition 7.1.25 gives us $S \circ T \in \mathcal{L}(V, W)$ so, by Theorem 7.1.26, all that remains is to prove $S \circ T$ is invertible. Observe that $T^{-1} \circ S^{-1}$ serves as the inverse of $S \circ T$. In particular, calculate:

$$(S \circ T)(T^{-1} \circ S^{-1})(x)) = S(T(T^{-1}(S^{-1}(x)))) = S(S^{-1}(x)) = x.$$

Thus $(S \circ T) \circ (T^{-1} \circ S^{-1}) = Id_W$. Similarly, $(T^{-1} \circ S^{-1}) \circ (S \circ T) = id_V$. Therefore $S \circ T$ is invertible with inverse $T^{-1} \circ S^{-1}$.

The proof that $\approxeq$ is an equivalence relation is not difficult. Begin by noting that $T = Id_V$ gives an isomorphism of V to V hence $V \approxeq V$; that is $\approxeq$ is reflexive. Next, if $T : V \to W$ is an isomorphism then $T^{-1} : W \to V$ is also an isomorphism by Theorem 7.1.26 thus $V \approxeq W$ implies $W \approxeq V$; $\approxeq$ is symmetric. Finally, suppose $V \approxeq U$ and $U \approxeq W$ by $T \in \mathcal{L}(V, U)$ and $S \in \mathcal{L}(U, W)$ isomorphisms. We proved that $S \circ T \in \mathcal{L}(V, W)$ is an isomorphism hence $V \approxeq W$; that is, $\approxeq$ is transitive. Therefore, $\approxeq$ is an equivalence relation on the class of vector spaces over $\mathbb{F}$. $\square$

I included the comment about finite dimension as some of our theorems fail when the dimension is infinite. It is certainly not the case that all infinite dimensional vector spaces are isomorphic.

Theorem 7.3.7.

Let V, W be finite dimensional vector spaces. $V \approxeq W$ iff $\dim(V) = \dim(W)$

Proof: we already proved $\Rightarrow$ in Theorem 7.3.5. Let us work on the converse. Suppose $\dim(V) = \dim(W)$. Let β be a basis for V. In particular, denote $\beta = \{v_1, \dots, v_n\}$. Define $\Phi_\beta : \beta \to \mathbb{F}^n$ by $\Phi_\beta(v_i) = e_i$ and extend linearly. But, if $\gamma = \{w_1, \dots, w_n\}$ is the basis for W (we know they have the same number of elements by our supposition $\dim(V) = \dim(W)$) then we may also define $\Phi_\gamma : W \to \mathbb{R}^n$ by $\Phi_\gamma(w_i) = e_i$ and extend linearly. Clearly Φ_β^{-1} and Φ_γ^{-1} exist and are easily described by $\Phi_\beta^{-1}(e_i) = v_i$ and $\Phi_\gamma^{-1}(e_i) = w_i$ extended linearly. Therefore, Φ_β and Φ_γ are isomorphisms. In particular, we've shown $V \approxeq \mathbb{F}^n$ and $W \approxeq \mathbb{F}^n$. By transitivity of $\approxeq$ we find $V \approxeq W$. $\square$

The proof above leads us naturally to the topic of the next section. In particular, the proof above contains a sketch of why Φ_β is an isomorphism. The examples in the next subsection can be read at any point, you can skip ahead in your first read.

7.3.1 examples of isomorphisms

In your first read of this section, you might just read the examples. I have purposely put the big-picture and extracurricular commentary outside the text of the examples.

The coodinate map is an isomorphism which allows us to trade the abstract for the concrete.

Example 7.3.8. *Let V be a vector space over $\mathbb{R}$ with basis $\beta = \{f_1, \dots, f_n\}$ and define Φ_β by $\Phi_\beta(f_j) = e_j \in \mathbb{R}^n$ extended linearly. In particular,*

$$\Phi_\beta(v_1 f_1 + \cdots + v_n f_n) = v_1 e_1 + \cdots + v_n e_n.$$

This map is a linear bijection and it follows $V \approx \mathbb{R}^n$.

The notation $V(\mathbb{R})$ indicates I intend us to consider $V(\mathbb{R})$ as a vector space over the field $\mathbb{R}$.

Example 7.3.9. *Suppose $V(\mathbb{R}) = \{A \in \mathbb{C}^{2\times 2} \mid A^T = -A\}$ find an isomorphism to $P_n \leq \mathbb{R}[x]$ for appropriate n. Note, $A_{ij} = -A_{ji}$ gives $A_{11} = A_{22} = 0$ and $A_{12} = -A_{21}$. Thus, $A \in V$ has the form:*

$$A = \begin{bmatrix} 0 & a+ib \\ -a-ib & 0 \end{bmatrix}$$

I propose that $\Psi(a+bx) = \begin{bmatrix} 0 & a+ib \\ -a-ib & 0 \end{bmatrix}$ *provides an isomorphism of P_1 to V.*

Example 7.3.10. *Let $V(\mathbb{R}) = (\mathbb{C} \times \mathbb{R})^{2\times 2}$ and $W(\mathbb{R}) = \mathbb{C}^{2\times 3}$. The following is an isomorphism from V to W:*

$$\Psi \begin{bmatrix} (z_1, x_1) & (z_2, x_2) \\ (z_3, x_3) & (z_4, x_4) \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & x_1 + ix_2 & x_3 + ix_4 \end{bmatrix}$$

Example 7.3.11. *Consider $P_2(\mathbb{C}) = \{ax^2 + bx + c \mid a, b, c \in \mathbb{C}\}$ as a complex vector space. Consider the subspace of $P_2(\mathbb{C})$ defined as $V = \{f(x) \in P_2(\mathbb{C}) \mid f(i) = 0\}$. Let's find an isomorphism to $\mathbb{C}^n$ for appropriate n. Let $f(x) = ax^2 + bx + c \in V$ and calculate*

$$f(i) = a(i)^2 + bi + c = -a + bi + c = 0 \quad \Rightarrow \quad c = a - bi$$

Thus, $f(x) = ax^2 + bx + a - bi = a(x^2+1) + b(x-i)$. The isomorphism from V to $\mathbb{C}^2$ is apparent from the calculation above. If $f(x) \in V$ then we can write $f(x) = a(x^2+1) + b(x-i)$ and

$$\Psi(f(x)) = \Psi(a(x^2+1) + b(x-i)) = (a, b).$$

The inverse map is also easy to find: $\Psi^{-1}(a, b) = a(x^2+1) + b(x-i)$

Example 7.3.12. *Let $\Psi(f(x), g(x)) = f(x) + x^{n+1}g(x)$ note this defines an isomorphism of $P_n \times P_n$ and P_{2n+1}. For example, $n = 1$,*

$$\Psi((ax + b, cx + d)) = ax + b + x^2(cx + d) = cx^3 + dx^2 + ax + b.$$

The reason we need $2n+1$ is just counting: $dim(P_n) = n+1$ and $dim(P_n \times P_n) = 2(n+1)$. However, $dim(P_{2n+1}) = (2n + 1) + 1$. Notice, we could take coefficients of P_n in $\mathbb{R}$, $\mathbb{C}$ or some other field $\mathbb{F}$ and this example is still meaningful.

Example 7.3.13. *Let $V = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $W = \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Transposition gives us a natural isomorphism as follows: for each $L \in V$ there exists $A \in \mathbb{R}^{m\times n}$ for which $L = L_A$. However, to $A^T \in \mathbb{R}^{n\times m}$ there naturally corresponds $L_{A^T} : \mathbb{R}^m \to \mathbb{R}^n$. Since V and W are spaces of functions an isomorphism is conveniently given in terms $A \mapsto L_A$ isomorphism of $\mathbb{R}^{m\times n}$ and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$: in particular $\Psi : V \to W$ is given by:*

$$\Psi(L_A) = L_{A^T}.$$

To write this isomorphism without the use of the L_A notation requires a bit more thought. Take off your shoes and socks, but them back on, then write what follows. Let $S \in V$ and $x \in \mathbb{R}^m$,

$$(\Psi(S))(x) = (x^T[S])^T = [S]^T x = L_{[S]^T}(x).$$

Since the above holds for all $x \in \mathbb{R}^m$ it can be written as $\Psi(S) = L_{[S]^T}$.

The interested reader might appreciate the example below shows Theorem 7.6.2 in action.

Warning: the next couple examples make some use of concepts from future sections. You can skip these in your first read

Example 7.3.14. *Let $A = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 0 \end{array}\right]$ find an isomorphism from Null(A) to Col(A). As we recall, the CCP reveals all, we can easily calculate:*

$$rref(A) = \left[\begin{array}{cccc} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Null space is $x \in \mathbb{R}^4$ for which $Ax = 0$ hence $x_1 = -x_2 - 3x_4$ and $x_3 = 2x_4$ with x_2, x_4 free. Thus,

$$x = (-x_2 - 3x_4, x_2, 2x_4, x_4) = x_2(-1, 1, 0, 0) + x_4(-3, 0, 2, 1)$$

and we find $\beta_N = \{(-1, 1, 0, 0), (-3, 0, 2, 1)\}$ is basis for Null(A). On the other hand $\beta_C = \{(1, 2), (1, 3)\}$ forms a basis for the column space by the CCP. Let Ψ : Null(A) $\to$ Col(A) be defined by extending

$$\Psi((-1, 1, 0, 0)) = (1, 2) \qquad \& \qquad \Psi((-3, 0, 2, 1)) = (1, 3)$$

linearly. In particular, if $x \in$ Null(A) then $\Psi(x) = x_2(1, 2) + x_4(1, 3)$. Fun fact, with our choice of basis the matrix $[\Psi]_{\beta_N,\beta_C} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$

The interested reader may also note that whenever we form a linear transformation $T : V \to W$ be mapping the j-th β basis element of V to the j-th γ basis element of W this gives a block-identity matrix in $[T]_{\beta,\gamma}$. If $\#(\beta) = \#(\gamma)$ then, as in the above example, the matrix of T is simply

$[T]_{\beta,\gamma} = I$. However, if $\dim(W) > \dim(V)$ then the other blocks of the matrix are zero as by construction we already mapped all non-trivial parts of V to the first j-dimensions of W. The remaining $\dim(W) - j$ dimensions are untouched by T as we construct it. If T is instead given and our problem is to find bases for V and W for which the matrix is all zero with a identity matrix block in the upper left block then we must choose a basis carefully as described in Section 7.6. Let us return to the considerably easier problem of constructing isomorphisms between given vector spaces. The simplest advice is just, find a basis for each space and map one to the other. I find that is a good approach for many problems. Of course, there are other tools, but first the basics.

Example 7.3.15. *Let $V = P_2$ and $W = \{f(x) \in y \mid f(1) = 0\}$. By the factor theorem of algebra we know $f(x) \in W$ implies $f(x) = (x-1)g(x)$ where $g(x) \in P_2$. Define, $\Psi(f(x)) = g(x)$ where $g(x)(x-1) = f(x)$. We argue that Ψ is an isomorphism. Note $\Psi^{-1}(g(x)) = (x-1)g(x)$ and it is clear that $(x-1)g(x) \in W$ moreover, linearity of Ψ^{-1} is simply seen from the calculation below:*

$$\Psi^{-1}(cg(x)+h(x)) = (x-1)(cg(x)+h(x)) = c(x-1)g(x)+(x-1)g(x) = c\Psi^{-1}(g(x))+\Psi^{-1}(h(x)).$$

Linearity of Ψ follows by Theorem 7.1.26 as $\Psi = (\Psi^{-1})^{-1}$. Thus $V \approxeq W$.

You might note that I found a way around using a basis in the last example. Perhaps it is helpful to see the same example done by the basis mapping technique.

Example 7.3.16. *Let $V = P_2$ and $W = \{f(x) \in y \mid f(1) = 0\}$. Ignoring the fact we* **know** *the factor theorem, let us find a basis the hard way: if $f(x) = ax^3 + bx^2 + cx + d \in W$ then*

$$f(1) = a + b + c + d = 0$$

Thus, $d = -a - b - c$ and

$$f(x) = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$$

We find basis $\beta = \{x^3 - 1, x^2 - 1, x - 1\}$ for W. Define $\phi : W \to P_2$ by linearly extending:

$$\phi(x^3 - 1) = x^2, \qquad \phi(x^2 - 1) = x, \qquad \phi(x - 1) = 1.$$

In this case, a bit of thought reveals:

$$\phi^{-1}(ax^2 + bx + c) = a(x^3 - 1) + b(x^2 - 1) + c(x - 1).$$

Again, these calculations serve to prove $W \approxeq P_2$.

It might be interesting to relate the results of Example 7.3.15 and Example 7.3.16. Examing the formula for $\Psi^{-1}(g(x)) = (x-1)g(x)$ it is evident that we should factor out $(x-1)$ from our ϕ^{-1} formula to connect to the Ψ^{-1} formula,

$$\begin{aligned}
\phi^{-1}(ax^2 + bx + c) &= a(x-1)(x^2 + x + 1) + b(x-1)(x+1) + c(x-1). \\
&= (x-1)[a(x^2 + x + 1) + b(x+1) + c] \\
&= (x-1)[ax^2 + (a+b)x + a + b + c] \\
&= \Psi^{-1}(ax^2 + (a+b)x + a + b + c).
\end{aligned}$$

Evaluating the equation above by Ψ yeilds $(\Psi \circ \phi^{-1})(ax^2 + bx + c) = ax^2 + (a+b)x + a + b + c$. Therefore, if $\gamma = \{x^2, x, 1\}$ then we may easily deduce

$$[\Psi \circ \phi^{-1}]_{\gamma,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Example 7.3.17. *Consider complex numbers $\mathbb{C}$ as a real vector space and let $M_{\mathbb{C}}$ be the set of* **real** *matrices of the form:* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. *Observe that the map* $\Psi(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *is a linear transformation with inverse* $\Psi^{-1}.\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = a+ib$. *Therefore, $\mathbb{C}$ and $M_{\mathbb{C}}$ are isomorphic as* **real** *vector spaces.*

Let me continue past the point of linear isomorphism. In the example above, we can show that $\mathbb{C}$ and $M_{\mathbb{C}}$ are isomorphic as **algebras** over $\mathbb{R}$. In particular, notice

$$(a+ib)(c+id) = ac-bd+i(ad+bc) \quad \& \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}.$$

As you can see the pattern of the multiplication is the same. To be precise,

$$\Psi(\underbrace{(a+ib)(c+id)}_{\text{complex multiplication}}) = \underbrace{\Psi(a+ib)\Psi(c+id)}_{\text{matrix multiplication}}.$$

These special 2×2 matrices form a **representation** of the complex numbers. The term **isomorphism** has wide application in mathematics. In this course, the unqualified term "isomorphism" would be more descriptively termed "linear-isomorphism". An isomorphism of $\mathbb{R}$-algebras is a linear isomorphism which also preserves the multiplication $\star$ of the algebra; $\Psi(v\star w) = \Psi(v)\Psi(w)$. Another related concept, a non-associative algebra on a vector space which is a generalization of the cross-product of vectors in $\mathbb{R}^3$ is known as[5] a Lie Algebra. In short, a Lie Algebra is a vector space paired with a Lie bracket. A *Lie algebra isomorphism* is a linear isomorphism which also preserves the Lie bracket; $\Psi([v,w]) = [\Psi(v),\Psi(w)]$. Not all isomorphisms are linear isomorphisms. For example, in abstract algebra you will study *isomorphisms of groups* which are bijections between groups which preserves the group multiplication. My point is just this, the idea of isomorphism, our current endeavor, is one you will see repeated as you continue your study of mathematics. To quote a certain show: *it has happened before, it will happen again.*

7.4 matrix of linear transformation

I used the notation $[v]_\beta$ in the last chapter since it was sufficient. Now we need to have better notation for the coordinate maps so we can articulate the concepts clearly[6].

Definition 7.4.1.

Let $V(\mathbb{F})$ be a finite dimensional vector space with basis $\beta = \{v_1, v_2, \dots v_n\}$. The coordinate map $\Phi_\beta : V \to \mathbb{F}^n$ is defined by

$$\Phi_\beta(x_1v_1 + x_2v_2 + \cdots + x_nv_n) = x_1e_1 + x_2e_2 + \cdots + x_ne_n = (x_1, x_2, \dots, x_n)$$

for all $v = x_1v_1 + x_2v_2 + \cdots + x_nv_n \in V$.

[5] it is pronounced "Lee", not what Obama does

[6] I hope my Spring 2016 students appreciate that your proof in Mission 4 that $[x+cy]_\beta = [x]_\beta + c[y]_\beta$ for all $x, y \in V(\mathbb{F})$ and $c \in \mathbb{F}$ is precisely the proof that $\Phi_\beta(x) = [x]_\beta$ defines a linear transformation from V to $\mathbb{F}^n$ where $n = \dim(V)$.

We argued in the previous section that Φ_β is an invertible, linear transformation from V to $\mathbb{F}^n$. In other words, Φ_β is an **isomorphism** of V and $\mathbb{F}^n$. It is worthwhile to note the linear extensions of

$$\Phi_\beta(v_i) = e_i \qquad \& \qquad \Phi_\beta^{-1}(e_i) = v_i$$

encapsulate the action of the coordinate map and its inverse. The coordinate map is a machine which converts an abstract basis to the standard basis.

Example 7.4.2. *Let $V = \mathbb{R}^{2\times 2}$ with basis $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ then*

$$\Phi_\beta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d).$$

Example 7.4.3. *Let $V = \mathbb{C}^n$ as a real vector space; that is $V(\mathbb{R}) = \mathbb{C}^n$. Consider the basis $\beta = \{e_1, \dots, e_n, ie_1, \dots, ie_n\}$ of this $2n$-dimensional vector space over $\mathbb{R}$. Observe $v \in \mathbb{C}^n$ has $v = x + iy$ where $x, y \in \mathbb{R}^n$. In particular, if $\overline{a+ib} = a - ib$ and $\bar{v} = (\bar{v_1}, \dots, \bar{v_n})$ then the identity below shows how to construct x, y:*

$$v = \underbrace{\frac{1}{2}(v + \bar{v})}_{Re(v)=x} + \underbrace{\frac{1}{2}(v - \bar{v})}_{iIm(v)=iy}$$

and it's easy to verify $\bar{x} = x$ and $\bar{y} = y$ hence $x, y \in \mathbb{R}^n$ as claimed. The coordinate mapping is simple enough in this notation,

$$\Phi_\beta(x + iy) = (x, y).$$

Here we abuse notation slightly. Technically, I ought to write

$$\Phi_\beta(x + iy) = (x_1, \dots, x_n, y_1, \dots, y_n).$$

Example 7.4.4. *Let $V = P_n$ with $\beta = \{1, (x-1), (x-1)^2, \dots, (x-1)^n\}$. To find the coordinates of an n-th order polynomial in standard form $f(x) = a_n x^n + \cdots + a_1 x + a_o$ requires some calculation. We've all taken calculus II so we know Taylor's Theorem.*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

also, clearly the series truncates for the polynomial in question hence,

$$f(x) = f(1) + f'(1)(x-1) + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n$$

Therefore,

$$\Phi_\beta(f(x)) = \big(f(1), f'(1), \dots, f^{(n)}(1)\big).$$

Example 7.4.5. *Let $V(\mathbb{R}) = \{A = \sum_{i,j=1}^{2} A_{ij}E_{ij} \mid A_{11} + A_{22} = 0,\ A_{12} \in P_1, A_{11}, A_{22}, A_{21} \in \mathbb{C}\}$. If $A \in V$ then we can write:*

$$A = \left[\begin{array}{c|c} a + ib & ct + d \\ \hline x + iy & -a - ib \end{array}\right]$$

A natural choice for basis β is seen

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \right\}$$

The coordinate mapping $\Phi_\beta : V \to \mathbb{R}^6$ follows easily in the notation laid out above,

$$\Phi_\beta(A) = (a, b, c, d, x, y).$$

Now that we have a little experience with coordinates as mappings let us turn to the central problem of this section: *how can we associate a matrix with a given linear transformation $T : V \to W$?*. It turns out we'll generally have to choose a basis for $V(\mathbb{F})$ and $W(\mathbb{F})$ in order to answer this question unambiguously. Therefore, let β once more serve as the basis for V and suppose γ is a basis for W. We assume $\#(\beta), \#(\gamma) < \infty$ throughout this discussion. The answer to the question is actually in the diagram below:

$$\begin{tikzcd} \boxed{V(\mathbb{F})} \arrow[r, "T"] \arrow[d, "\Phi_\beta"] & \boxed{W(\mathbb{F})} \arrow[d, "\Phi_\gamma"] \\ \boxed{\mathbb{F}^n} \arrow[r, "L_{[T]_{\beta,\gamma}}"'] & \boxed{\mathbb{F}^m} \end{tikzcd}$$

The matrix $[T]_{\beta,\gamma}$ induces a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. This means $[T]_{\beta,\gamma} \in \mathbb{F}^{m\times n}$. It is defined by the demand that the diagram above **commutes**. There are several formulas you can read into that comment. To express T explicitly as a combination of matrix multiplication and coordinate maps observe:

$$T = \Phi_\gamma^{-1} \circ L_{[T]_{\beta,\gamma}} \circ \Phi_\beta.$$

On the other hand, we could write

$$L_{[T]_{\beta,\gamma}} = \Phi_\gamma \circ T \circ \Phi_\beta^{-1}$$

if we wish to explain how to calculate $L_{[T]_{\beta,\gamma}}$ in terms of the coordinate maps and T directly. To select the i-th column in $[T]_{\beta,\gamma}$ we simply operate on $e_i \in \mathbb{F}^n$. This reveals,

$$col_i([T]_{\beta,\gamma}) = \Phi_\gamma(T(\Phi_\beta^{-1}(e_i)))$$

However, as we mentioned at the outset of this section, $\Phi_\beta^{-1}(e_i) = v_i$ hence

$$col_i([T]_{\beta,\gamma}) = \Phi_\gamma(T(v_i)) = [T(v_i)]_\gamma$$

where I have reverted to our previous notation for coordinate vectors[7]. Stringing the columns out, we find perhaps the nicest way to look at the matrix of an abstract linear transformation:

$$\boxed{[T]_{\beta,\gamma} = [[T(v_1)]_\gamma | \cdots | [T(v_n)]_\gamma]}$$

Each column is a W-coordinate vector which is found in $\mathbb{F}^m$ and these are given by the n-basis vectors which generate V.

Alternatively, the commuting of the diagram yields:

$$\Phi_\gamma \circ T = L_{[T]_{\beta,\gamma}} \circ \Phi_\beta$$

[7]the mapping notation supplements the $[v]_\beta$ notation, I use both going forward in these notes

If we feed the expression above an arbitrary vector $v \in V$ we obtain:

$$\Phi_\gamma(T(v)) = L_{[T]_{\beta,\gamma}}(\Phi_\beta(v)) \qquad \Rightarrow \qquad \boxed{[T(v)]_\gamma = [T]_{\beta,\gamma}[v]_\beta}$$

In practice, as I work to formulate $[T]_{\beta,\gamma}$ for explicit problems I find the boxed formulas convenient for calculational purposes. On the other hand, I have used each formula on this page for various theoretical purposes. Ideally, you'd like to understand these rather than memorize. I hope you are annoyed I have yet to define $[T]_{\beta,\gamma}$. Let us pick a definition for specificity of future proofs.

Definition 7.4.6.

Let $V(\mathbb{F})$ be a vector space with basis $\beta = \{v_1, \dots, v_n\}$. Let $W(\mathbb{F})$ be a vector space with basis $\gamma = \{w_1, \dots, w_m\}$. If $T : V \to W$ is a linear transformation then we define the **matrix of T with respect to β, γ as** $[T]_{\beta,\gamma} \in \mathbb{F}^{m\times n}$ which is implicitly defined by

$$L_{[T]_{\beta,\gamma}} = \Phi_\gamma \circ T \circ \Phi_\beta^{-1}.$$

The discussion preceding this definition hopefully gives you some idea on what I mean by "implicitly" in the above context. In any event, we pause from our general discussion to illustrate with some explicit examples.

Example 7.4.7. *Let $S : V \to W$ with $V = W = \mathbb{R}^{2\times 2}$ are given bases $\beta = \gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $L(A) = A + A^T$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and calculate,*

$$S(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

Observe,

$$[A]_\beta = (a, b, c, d) \qquad \& \qquad [S(A)]_\gamma = (2a, b+c, b+c, 2d)$$

Moreover, we need a matrix $[S]_{\beta,\gamma}$ such that $[S(A)]_\gamma = [S]_{\beta,\gamma}[A]_\beta$. Tilt head, squint, and see

$$[S]_{\beta,\gamma} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Example 7.4.8. *Let $V(\mathbb{R}) = P_1^{2\times 2}$ be the set of 2×2 matrices with first order polynomials. Define $T(A(x)) = A(2)$ where $T : V \to W$ and $W = \mathbb{R}^{2\times 2}$. Let $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the basis for W. Let β be the basis[8] with coordinate mapping*

$$\Phi_\beta\left(\left[\begin{array}{c|c} a+bx & c+dx \\ \hline e+fx & g+hx \end{array}\right]\right) = (a, b, c, d, e, f, g, h).$$

We calculate for $v = \left[\begin{array}{c|c} a+bx & c+dx \\ \hline e+fx & g+hx \end{array}\right]$ that

$$T(v) = \left[\begin{array}{c|c} a+2b & c+2d \\ \hline e+2f & g+2h \end{array}\right]$$

[8] you should be able to find β in view of the coordinate map formula

Therefore,

$$[T(v)]_\gamma = (a + 2b, c + 2d, e + 2f, g + 2h)$$

and as the coordinate vector $[v]_\beta = (a, b, c, d, e, f, g, h)$ *the formula* $[T(v)]_\gamma = [T]_{\beta,\gamma}[v]_\beta$ *indicates*

$$[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Example 7.4.9. *Suppose* P_3 *is the set of cubic polynomials with real coefficients. Let* $T : P_3 \to P_3$ *be the derivative operator;* $T(f(x)) = f'(x)$. *Give* P_3 *the basis* $\beta = \{1, x, x^2, x^3\}$. *Calculate,*

$$T(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$$

Furthermore, note, setting $v = a + bx + cx^2 + dx^3$

$$[T(v)]_\beta = (b, 2c, 3d, 0) \qquad \& \qquad [v]_\beta = (a, b, c, d) \qquad \Rightarrow \qquad [T]_{\beta,\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The results of Proposition 7.1 and 7.2.25 naturally generalize to our current context.

Proposition 7.4.10.

Let V, W be n, m-dimensional vector spaces over a field $\mathbb{F}$ with bases β, γ respective. Suppose $S, T \in L(V, W)$ then for $c \in \mathbb{F}$ we have $[T \pm S]_{\beta,\gamma}, [T]_{\beta,\gamma}, [S]_{\beta,\gamma}, [cS]_{\beta,\gamma} \in \mathbb{F}^{m\times n}$ and

(1.) $[T + S]_{\beta,\gamma} = [T]_{\beta,\gamma} + [S]_{\beta,\gamma}$, (2.) $[T - S]_{\beta,\gamma} = [T]_{\beta,\gamma} - [S]_{\beta,\gamma}$, (3.) $[cS]_{\beta,\gamma} = c[S]_{\beta,\gamma}$.

Proof: the proof follows immediately from the identity below:

$$\Phi_\gamma \circ (T + cS) \circ \Phi_\beta^{-1} = \Phi_\gamma \circ T \circ \Phi_\beta^{-1} + c\Phi_\gamma \circ S \circ \Phi_\beta^{-1}.$$

This identity is true due to the linearity properties of the coordinate mappings. $\square$

The generalization of Proposition 7.2.25 is a bit more interesting.

Proposition 7.4.11.

Let U, V, W be finite-dimensional vector spaces with bases β, γ, δ respectively. If $S \in L(U, W)$ and $T \in L(V, U)$ then $[S \circ T]_{\gamma,\delta} = [S]_{\beta,\delta}[T]_{\gamma,\beta}$

Proof: Notice that $L_A \circ L_B = L_{AB}$ since $L_A(L_B(v)) = L_A(Bv) = ABv = L_{AB}(v)$ for all v. Hence,

$$\begin{aligned} L_{[S]_{\beta,\delta}[T]_{\gamma,\beta}} &= L_{[S]_{\beta,\delta}} \circ L_{[T]_{\gamma,\beta}} && \text{:set } A = [S]_{\beta,\delta} \text{ and } B = [T]_{\gamma,\beta}, \\ &= (\Phi_\delta \circ S \circ \Phi_\beta^{-1}) \circ (\Phi_\beta \circ T \circ \Phi_\gamma^{-1}) && \text{:defn. of matrix of linear transformation,} \\ &= \Phi_\delta \circ (S \circ T) \circ \Phi_\gamma^{-1} && \text{:properties of function composition,} \\ &= L_{[S\circ T]_{\gamma,\delta}} && \text{:defn. of matrix of linear transformation.} \end{aligned}$$

The mapping $L : \mathbb{F}^{m\times n} \to L(\mathbb{F}^n, \mathbb{F}^m)$ is injective. Thus, $[S \circ T]_{\gamma,\delta} = [S]_{\beta,\delta}[T]_{\gamma,\beta}$ as we claimed. $\square$

If we apply the result above to a linear transformation on a vector space V where the same basis is given to the domain and codomain some nice things occur. For example:

Example 7.4.12. *Continuing Example 7.4.9. Observe that* $T^2(f(x)) = T(T(f(x)) = f''(x)$*. Thus if* $v = a + bx + cx^2 + dx^3$ *then* $T^2 : P_3 \to P_3$ *has* $T^2(v) = 2c + 6dx$ *hence* $[T^2(v)]_\beta = (2c, 6d, 0, 0)$ *and we find* $[T^2]_{\beta,\beta} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$*. You can check that* $[T^2]_{\beta,\beta} = [T]_{\beta,\beta}[T]_{\beta,\beta}$*. Notice, we can easily see that* $[T^3]_{\beta,\beta} \neq 0$ *whereas* $[T^n]_{\beta,\beta} = 0$ *for all* $n \geq 4$*. This makes* $[T]_{\beta,\beta}$ *a* **nilpotent** *matrix of* **index** 4*.*

Example 7.4.13. *Let* V, W *be vector spaces of dimension* n *over* $\mathbb{F}$*. In addition, suppose* $T : V \to W$ *is a linear transformation with inverse* $T^{-1} : W \to V$*. Let* V *have basis* β *whereas* W *has basis* γ*. We know that* $T \circ T^{-1} = Id_W$ *and* $T^{-1} \circ T = Id_V$*. Furthermore, I invite the reader to show that* $[Id_V]_{\beta,\beta} = I \in \mathbb{F}^{n\times n}$ *where* $n = dim(V)$ *and similarly* $[Id_W]_{\gamma,\gamma} = I \in \mathbb{F}^{n\times n}$*. Apply Proposition 7.4.11 to find*

$$[T^{-1} \circ T]_{\beta,\beta} = [T^{-1}]_{\gamma,\beta}[T]_{\beta,\gamma}$$

but, $[T^{-1} \circ T]_{\beta,\beta} = [Id_V]_{\beta,\beta} = I$ *thus* $[T^{-1}]_{\gamma,\beta}[T]_{\beta,\gamma} = I$ *and we conclude* $([T]_{\beta,\gamma})^{-1} = [T^{-1}]_{\gamma,\beta}$*. Phew, that's a relief. Wouldn't it be strange if this weren't true? Moral of story:* **the inverse matrix of the transformation is the matrix of the inverse transformation.**

Lemma 7.4.14.

If $\Psi : V \to W$ is an isomorphism and S is a LI set then $\Psi(S)$ is a LI set with $\#(S) = \#(\Psi(S))$. In particular, $\dim(V_o) = \dim(\Psi(V_o))$ for $V_o \leq V$

Proof: if $\Psi : V \to W$ is a linear injective map then By Theorem 7.1.27 we have S LI implies $\Psi(S)$ is LI. Furthermore, the map $F : S \to \Psi(S)$ defined by $F(x) = \Psi(x)$ for each $x \in S$ is a bijection hence $\#(S) = \#(\Psi(S))$. If β_o is a basis for $V_o \leq V$ then $\Psi(\beta_o)$ is a basis for $\Psi(V_o)$ and as $\#\beta_o = \#(\Psi(\beta_o))$ we find $\dim(V_o) = \dim(\Psi(V_o))$. $\square$

I usually use the Lemma above in the context that Ψ is the coordinate isomorphism and S is the basis for some interesting subspace. Two central applications of this are given next:

Proposition 7.4.15.

Let $T : V \to W$ be a linear transformation where $\dim(V) = n$ and $\dim(W) = m$. Let $\Phi_\beta : V \to \mathbb{F}^n$ and $\Phi_\gamma : W \to \mathbb{F}^m$ be coordinate map isomorphisms. If β, γ are bases for V, W respective then $[T]_{\beta,\gamma}$ satisfies the following

$$(1.)\ \mathrm{Null}([T]_{\beta,\gamma}) = \Phi_\beta(\mathrm{Ker}(T)), \qquad (2.)\ \mathrm{Col}([T]_{\beta,\gamma}) = \Phi_\gamma(\mathrm{Range}(T)).$$

Proof of (1.): Let $v \in \mathrm{Null}([T]_{\beta,\gamma})$ then there exists $x \in V$ for which $v = [x]_\beta$. By definition of nullspace, $[T]_{\beta,\gamma}[x]_\beta = 0$ hence, applying the identity $[T(x)]_\gamma = [T]_{\beta,\gamma}[x]_\beta$ we obtain $[T(x)]_\gamma = 0$ which, by injectivity of Φ_γ, yields $T(x) = 0$. Thus $x \in \mathrm{Ker}(T)$ and it follows that $[x]_\beta \in \Phi_\beta(\mathrm{Ker}(T))$. Therefore, $\mathrm{Null}([T]_{\beta,\gamma}) \subseteq \Phi_\beta(\mathrm{Ker}(T))$.

Conversely, if $[x]_\beta \in \Phi_\beta(\mathrm{Ker}(T))$ then there exists $v \in \mathrm{Ker}(T)$ for which $\Phi_\beta(v) = [x]_\beta$ hence, by injectivity of Φ_β, $x = v$ and $T(x) = 0$. Observe, by linearity of Φ_γ, $[T(x)]_\gamma = 0$. Recall once more, $[T(x)]_\gamma = [T]_{\beta,\gamma}[x]_\beta$. Hence $[T]_{\beta,\gamma}[x]_\beta = 0$ and we conclude $[x]_\beta \in \mathrm{Null}([T]_{\beta,\gamma})$. Consquently,

$\Phi_\beta(\text{Ker}(T)) \subseteq \text{Null}([T]_{\beta,\gamma})$.

Thus $\Phi_\beta(\text{Ker}(T)) = \text{Null}([T]_{\beta,\gamma})$. I leave the proof of (2.) to the reader. $\square$

I should caution that the results above are basis dependendent in the following sense: If β_1, β_2 are bases with coordinate maps $\Phi_{\beta_1}, \Phi_{\beta_2}$ then it is not usually true that $\Phi_{\beta_1}(\text{Ker}(T)) = \Phi_{\beta_2}(\text{Ker}(T))$. It follows that $\text{Null}([T]_{\beta_1,\gamma}) \neq \text{Null}([T]_{\beta_2,\gamma})$ in general. That said, there is something which is common to all the nullspaces (and ranges); dimension. The dimension of the nullspace much match the dimension of the kernel. The dimension of the column space must match the dimension of the range. This result follows immediately from Lemma 7.4.15 and Proposition 7.4.15. See Definition 7.1.20 for rank and nullity of a linear transformation verses Definition 6.7.1 for matrices.

Proposition 7.4.16.

Let $T : V \to W$ be a linear transformation of finite dimensional vector spaces with basis β for V and γ for W then

$$\text{nullity}(T) = \text{nullity}([T]_{\beta,\gamma}) \qquad \& \qquad \text{rank}(T) = \text{rank}([T]_{\beta,\gamma}).$$

You should realize the nullity and rank on the L.H.S. and R.H.S of the above proposition are quite different quantities in concept. It required some effort on our part to connect them, but, now that they are connected, perhaps you appreciate the names.

7.5 coordinate change

Vectors in abstract vector spaces do not generically come with a preferred coordinate system. There are infinitely many different choices for the basis of a given vector space. Naturally, for specific examples, we tend to have a pet-basis, but this is merely a consequence of our calculational habits. We need to find a way to compare coordinate vectors for different choices of basis. Then, the same ambiguity must be faced by the matrix of a transformation. In some sense, if you understand the diagrams then you can write all the required formulas for this section. That said, we will cut the problem for mappings of column vectors a bit more finely. There are nice matrix-theoretic formulas for $\mathbb{R}^n$ that I'd like for you to know when you leave this course[9].

7.5.1 coordinate change of abstract vectors

Let V be a vector space with bases β and $\bar{\beta}$ over the field $\mathbb{F}$. Let $\beta = \{v_1, \dots, v_n\}$ whereas $\bar{\beta} = \{\bar{v}_1, \dots, \bar{v}_n\}$. Let $x \in V$ then there exist **column vectors** $[x]_\beta = (x_1, \dots, x_n)$ and $[x]_{\bar{\beta}} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{F}^n$ such that

$$x = \sum_{i=1}^n x_i v_i \qquad \& \qquad x = \sum_{j=1}^n \bar{x}_i \bar{v}_i$$

Or, in mapping notation, $x = \Phi_\beta^{-1}([x]_\beta)$ and $x = \Phi_{\bar{\beta}}^{-1}([x]_{\bar{\beta}})$. Of course $x = x$ hence

$$\Phi_\beta^{-1}([x]_\beta) = \Phi_{\bar{\beta}}^{-1}([x]_{\bar{\beta}}).$$

Operate by $\Phi_{\bar{\beta}}$ on both sides,

$$[x]_{\bar{\beta}} = \Phi_{\bar{\beta}}(\Phi_\beta^{-1}([x]_\beta)).$$

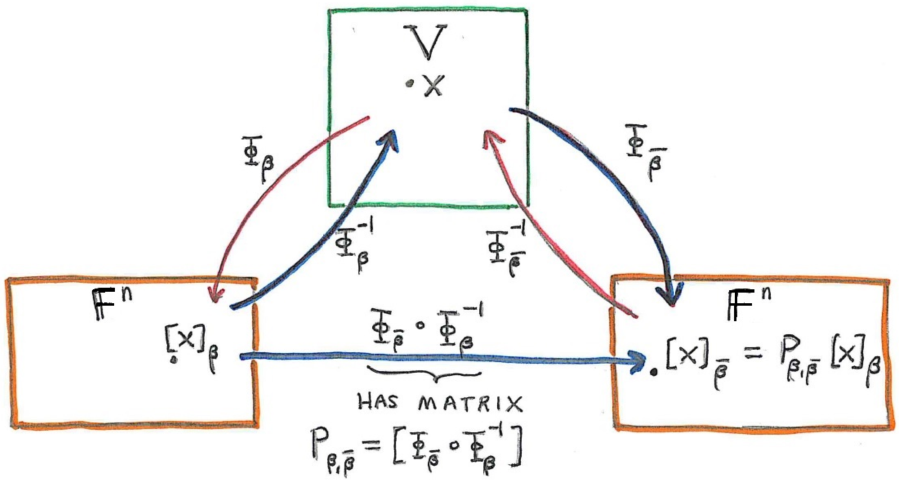
[9] I mean, don't wait until then, nows a perfectly good time to learn them

Observe that $\Phi_{\bar{\beta}} \circ \Phi_{\beta}^{-1} : \mathbb{F}^n \to \mathbb{F}^n$ is a linear transformation, therefore we can calculate its standard matrix. Let us collect our thoughts:

Proposition 7.5.1.

Using the notation developed in this subsection, if $P_{\beta,\bar{\beta}} = [\Phi_{\bar{\beta}} \circ \Phi_{\beta}^{-1}]$ then $[x]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[x]_{\beta}$.

The diagram below contains the essential truth of the above proposition:



Example 7.5.2. *Let $V = \{A \in \mathbb{R}^{2\times 2} | A_{11} + A_{22} = 0\}$. Observe $\beta = \{E_{12}, E_{21}, E_{11} - E_{22}\}$ gives a basis for V. On the other hand, $\bar{\beta} = \{E_{12}+E_{21}, E_{12}-E_{21}, E_{11}-E_{22}\}$ gives another basis. We denote $\beta = \{v_i\}$ and $\bar{\beta} = \{\bar{v}_i\}$. Let's work on finding the change of basis matrix. I can do this directly by our usual matrix theory. To find column i simply multiply by e_i. Or let the transformation act on e_i. The calculations below require a little thinking. I avoid algebra by thinking here.*

$$\Phi_{\bar{\beta}}(\Phi_{\beta}^{-1}(e_1)) = \Phi_{\bar{\beta}}(E_{12}) = \Phi_{\bar{\beta}}\left(\frac{1}{2}[\bar{v}_1 + \bar{v}_2]\right) = (1/2, 1/2, 0).$$
$$\Phi_{\bar{\beta}}(\Phi_{\beta}^{-1}(e_2)) = \Phi_{\bar{\beta}}(E_{21}) = \Phi_{\bar{\beta}}\left(\frac{1}{2}[\bar{v}_1 - \bar{v}_2]\right) = (1/2, -1/2, 0).$$
$$\Phi_{\bar{\beta}}(\Phi_{\beta}^{-1}(e_3)) = \Phi_{\bar{\beta}}(E_{11} - E_{22}) = \Phi_{\bar{\beta}}(\bar{v}_3) = (0, 0, 1).$$

Admittably, if the bases considered were not so easily related we'd have some calculation to work through here. That said, we find:

$$P_{\beta,\bar{\beta}} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's take it for a ride. Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ *clearly* $[A]_\beta = (2, 3, 1)$. *Calculate,*

$$P_{\beta,\bar{\beta}}[A]_\beta = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix} = [A]_{\bar{\beta}}$$

Is this correct? Check,

$$\Phi_{\bar{\beta}}^{-1}(5/2,-1/2,1) = \frac{5}{2}\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\cdot\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = A.$$

Yep. It works.

It is often the case we face coordinate change for mappings from $\mathbb{F}^n \to \mathbb{F}^m$. Or, even more special $m = n$. The formulas we've detailed thus far find streamlined matrix-theoretic forms in that special context. We turn our attention there now.

7.5.2 coordinate change for column vectors

Let β be a basis for $\mathbb{F}^n$. In contrast to the previous subsection, we have a standard basis with which we can compare; in particular, **the** standard basis. Hazzah![10]. Let $\beta = \{v_1, \dots, v_n\}$ and note the **matrix of** β is simply defined by concatenating the basis into an $n \times n$ invertible matrix $[\beta] = [v_1|\cdots|v_n]$. If $x \in \mathbb{F}^n$ then the coordinate vector $[x]_\beta = (y_1, \dots, y_n)$ is the column vector such that

$$x = [\beta][x]_\beta = y_1v_1 + \cdots y_nv_n$$

here I used "y" to avoid some other more annoying notation. It is not written in stone, you could use $([x]_\beta)_i$ in place of y_i. Unfortunately, I cannot use x_i in place of y_i as the notation x_i is already reserved for the Cartesian components of x. Notice, as $[\beta]$ is invertible we can solve for the coordinate vector:

$$[x]_\beta = [\beta]^{-1}x$$

If we had another basis $\bar{\beta}$ then

$$[x]_{\bar{\beta}} = [\bar{\beta}]^{-1}x$$

Naturally, x exists independent of these bases so we find common ground at x:

$$x = [\beta][x]_\beta = [\bar{\beta}][x]_{\bar{\beta}}$$

We find the coordinate vectors are related by:

$$[x]_{\bar{\beta}} = [\bar{\beta}]^{-1}[\beta][x]_\beta$$

Let us summarize are findings in the proposition below:

Proposition 7.5.3.

Using the notation developed in this subsection and the last, if $P_{\beta,\bar{\beta}} = [\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}]$ then $[x]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[x]_\beta$ and a simple formula to calculate the change of basis matrix is given by $P_{\beta,\bar{\beta}} = [\bar{\beta}]^{-1}[\beta]$. We also note for future convenience: $[\bar{\beta}]P_{\beta,\bar{\beta}} = [\beta]$

Example 7.5.4. *Let $\beta = \{(1,1),(1,-1)\}$ and $\gamma = \{(1,0),(1,1)\}$ be bases for $\mathbb{R}^2$. Find $[v]_\beta$ and $[v]_\gamma$ if $v = (2,4)$. Let me frame the problem, we wish to solve:*

$$v = [\beta][v]_\beta \qquad \textit{and} \qquad v = [\gamma][v]_\gamma$$

[10] sorry, we visited Medieval Times over vacation and it hasn't entirely worn off just yet

where I'm using the basis in brackets to denote the matrix formed by concatenating the basis into a single matrix,

$$[\beta] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \textit{and} \qquad [\gamma] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is the 2×2 case so we can calculate the inverse from our handy-dandy formula:

$$[\beta]^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \textit{and} \qquad [\gamma]^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Then multiplication by inverse yields $[v]_\beta = [\beta]^{-1}v$ and $[v]_\gamma = [\gamma]^{-1}v$ thus:

$$[v]_\beta = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad \textit{and} \qquad [v]_\gamma = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

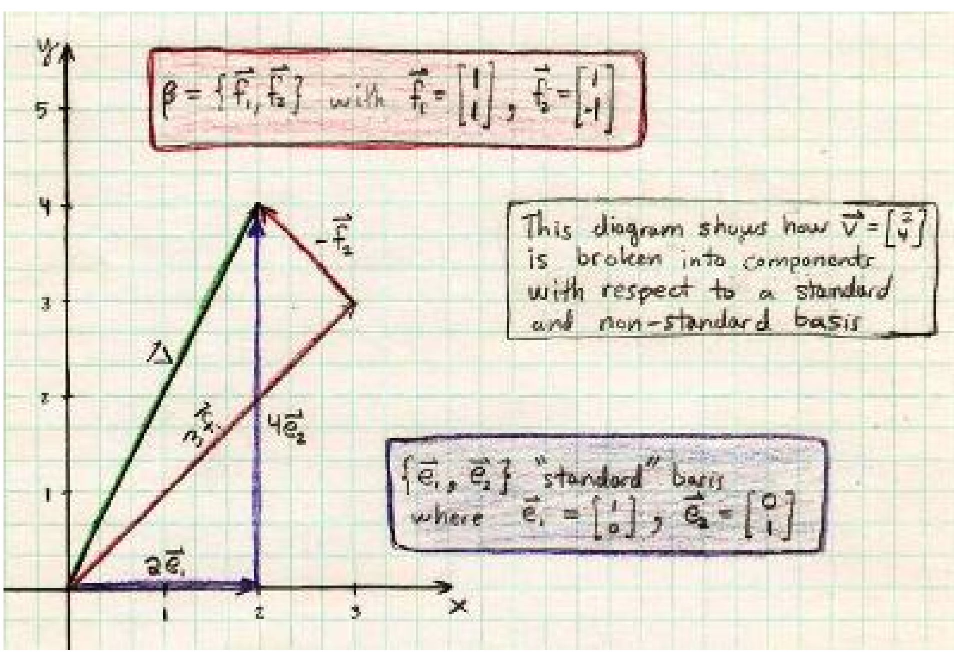
Let's verify the relation of $[v]_\gamma$ and $[v]_\beta$ relative to the change of basis matrix. In particular, we expect that if $P_{\beta,\gamma} = [\gamma]^{-1}[\beta]$ then $[v]_\gamma = P_{\beta,\gamma}[v]_\beta$. Calculate,

$$P_{\beta,\gamma} = [\gamma]^{-1}[\beta] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

As the last great American president said, **trust, but, verify**

$$P_{\beta,\gamma}[v]_\beta = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = [v]_\gamma \;\; \checkmark$$

It might be helpful to some to see a picture of just what we have calculated. Finding different coordinates for a given point (which corresponds to a vector from the origin) is just to prescribe different zig-zag paths from the origin along basis-directions to get to the point. In the picture below I illustrate the standard basis path and the β-basis path.



Now that we've seen an example, let's find $[v]_\beta$ for an arbitrary $v = (x, y)$,

$$[v]_\beta = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x-y) \end{bmatrix}$$

If we denote $[v]_\beta = (\bar{x}, \bar{y})$ then we can understand the calculation above as the relation between the barred and standard coordinates:

$$\bar{x} = \tfrac{1}{2}(x+y) \qquad \bar{y} = \tfrac{1}{2}(x-y)$$

Conversely, we can solve these for x, y to find the inverse transformations:

$$x = \bar{x} + \bar{y} \qquad y = \bar{x} - \bar{y}.$$

Similar calculations are possible with respect to the γ-basis.

7.5.3 coordinate change of abstract linear transformations

In Definition 7.4.6 we saw that if $V(\mathbb{F})$ is a vector space with basis $\beta = \{v_1, \dots, v_n\}$ and $W(\mathbb{F})$ be a vector space with basis $\gamma = \{w_1, \dots, w_m\}$. Then a linear transformation $T : V \to W$ has matrix $[T]_{\beta,\gamma} \in \mathbb{F}^{m\times n}$ defined implicitly by:

$$L_{[T]_{\beta,\gamma}} = \Phi_\gamma \circ T \circ \Phi_\beta^{-1}.$$

If there was another pair of bases $\bar{\beta}$ for V and $\bar{\gamma}$ for W then we would likewise have

$$L_{[T]_{\bar{\beta},\bar{\gamma}}} = \Phi_{\bar{\gamma}} \circ T \circ \Phi_{\bar{\beta}}^{-1}.$$

Solving for T relates the matrices with and without bars:

$$T = \Phi_\gamma^{-1} \circ L_{[T]_{\beta,\gamma}} \circ \Phi_\beta = \Phi_{\bar{\gamma}}^{-1} \circ L_{[T]_{\bar{\beta},\bar{\gamma}}} \circ \Phi_{\bar{\beta}}.$$

From which the proposition below follows:

Proposition 7.5.5.

Using the notation developed in this subsection

$$[T]_{\bar{\beta},\bar{\gamma}} = [\Phi_{\bar{\gamma}} \circ \Phi_\gamma^{-1}][T]_{\beta,\gamma}[\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}].$$

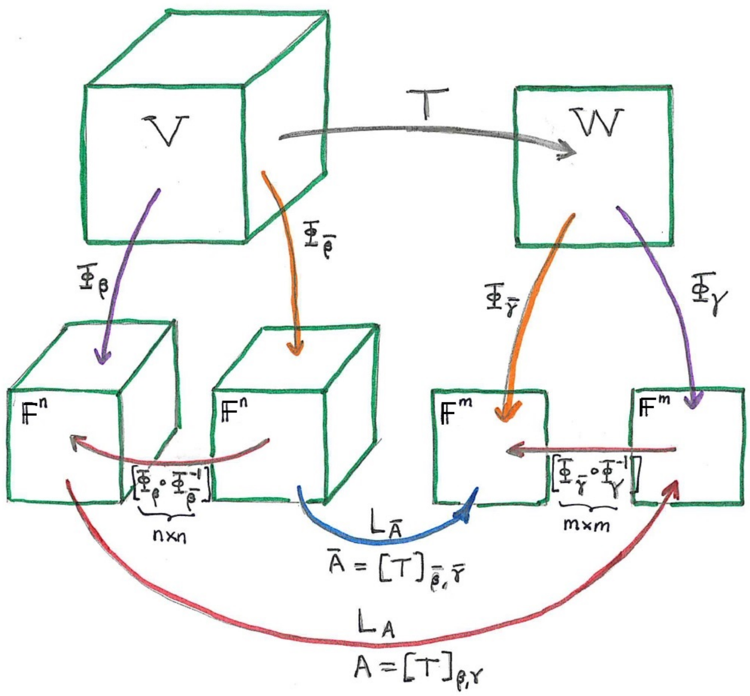
Moreover, recalling $P_{\beta,\bar{\beta}} = [\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}]$ we find:

$$[T]_{\bar{\beta},\bar{\gamma}} = P_{\gamma,\bar{\gamma}}[T]_{\beta,\gamma}(P_{\beta,\bar{\beta}})^{-1}.$$

Suppose $B, A \in \mathbb{F}^{m\times n}$. If there exist invertible matrices $P \in \mathbb{F}^{m\times m}, Q \in \mathbb{F}^{n\times n}$ such that $B = PAQ$ for then we say B and A are **matrix congruent**. The proposition above indicates that the matrices of a given linear tranformation[11] are congruent. In particular, $[T]_{\bar{\beta},\bar{\gamma}}$ is congruent to $[T]_{\beta,\gamma}$.

The picture below can be used to remember the formulas in the proposition above.

[11] of finite dimensional vector spaces



Example 7.5.6. *Let $V = P_2$ and $W = \mathbb{C}$. Define a linear transformation $T : V \to W$ by $T(f) = f(i)$. Thus,*

$$T(ax^2 + bx + c) = ai^2 + bi + c = c - a + ib.$$

Use coordinate maps given below for $\beta = \{x^2, x, 1\}$ and $\gamma = \{1, i\}$:

$$\Phi_\beta(ax^2 + bx + c) = (a, b, c) \qquad \& \qquad \Phi_\gamma(a + ib) = (a, b).$$

Observe $[T(ax^2 + bx + c)]_\gamma = (c - a, b)$ hence $[T]_{\beta,\gamma} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Let us change the bases to

$$\bar\beta = \{(x-2)^2, (x-2), 1\} \qquad \& \qquad \bar\gamma = \{i, 1\}$$

Calculate, if $f(x) = ax^2 + bx + c$ then $f'(x) = 2ax + b$ and $f''(x) = 2a$. Observe, $f(2) = 4a + 2b + c$ and $f'(2) = 4a + b$ and $f''(2) = 2a$ hence, using the Taylor expansion centered at 2,

$$\begin{aligned} f(x) &= f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 \\ &= 4a + 2b + c + (4a + b)(x-2) + a(x-2)^2. \end{aligned}$$

Therefore,

$$\Phi_{\bar\beta}(ax^2 + bx + c) = (a, 4a + b, 4a + 2b + c)$$

But, $\Phi_\beta^{-1}(a, b, c) = ax^2 + bx + c$. Thus,

$$\Phi_{\bar\beta}(\Phi_\beta^{-1}(a, b, c)) = (a, 4a + b, 4a + 2b + c) \qquad \Rightarrow \qquad [\Phi_{\bar\beta} \circ \Phi_\beta^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

Let's work out this calculation in the other direction (it's actually easier and what we need in a bit)

$$\Phi_\beta(a(x-2)^2 + b(x-2) + c) = \Phi_\beta(a(x^2-4x+4) + b(x-2) + c) = (a, -4a+b, 4a-2b+c)$$

But, $\Phi_{\bar{\beta}}^{-1}(a,b,c) = a(x-2)^2 + b(x-2) + c$ *therefore:*

$$\Phi_\beta(\Phi_{\bar{\beta}}^{-1}(a,b,c)) = (4a-2b+c, -4a+b, a) \qquad \Rightarrow \qquad [\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

On the other hand, $\Phi_{\bar{\gamma}}(a+ib) = (b,a)$. *Of course,* $a+ib = \Phi_\gamma^{-1}(a,b)$ *hence* $\Phi_{\bar{\gamma}}(\Phi_\gamma^{-1}(a,b)) = (b,a)$. *It follows that* $[\Phi_{\bar{\gamma}} \circ \Phi_\gamma^{-1}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *We'll use the change of basis proposition to find the matrix w.r.t.* $\bar{\beta}$ *and* $\bar{\gamma}$

$$\begin{aligned} [T]_{\bar{\beta},\bar{\gamma}} &= [\Phi_{\bar{\gamma}} \circ \Phi_\gamma^{-1}][T]_{\beta,\gamma}[\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}]. \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}. \end{aligned}$$

Continuing, we can check this by direct calculation of the matrix. Observe

$$\begin{aligned} T(a(x-2)^2 + b(x-2) + c) &= a(i-2)^2 + b(i-2) + c \\ &= a[-1-4i+4] + b(i-2) + c \\ &= 3a - 2b + c + i(-4a+b) \end{aligned}$$

Thus, $[T(a(x-2)^2 + b(x-2) + c)]_{\bar{\gamma}} = (-4a+b, 3a-2b+c)$ *hence* $[T]_{\bar{\beta},\bar{\gamma}} = \begin{bmatrix} -4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$. *Which agrees nicely with our previous calculation.*

7.5.4 coordinate change of linear transformations of column vectors

We specialize Proposition 7.5.5 in this subsection in the case that $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$. In particular, the result of Proposition 7.5.3 makes life easy; $P_{\beta,\bar{\beta}} = [\bar{\beta}]^{-1}[\beta]$ likewise, $P_{\gamma,\bar{\gamma}} = [\bar{\gamma}]^{-1}[\gamma]$

Proposition 7.5.7.

Using the notation developed in this subsection

$$[T]_{\bar{\beta},\bar{\gamma}} = [\bar{\gamma}]^{-1}[\gamma][T]_{\beta,\gamma}[\beta]^{-1}[\bar{\beta}].$$

The standard matrix $[T]$ is related to the non-standard matrix $[T]_{\bar{\beta},\bar{\gamma}}$ by:

$$[T]_{\bar{\beta},\bar{\gamma}} = [\bar{\gamma}]^{-1}[T][\bar{\beta}].$$

Proof: Proposition 7.5.5 with $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ together with the result of Proposition 7.5.3 give us the first equation. The second equation follows from the observation that for standard bases β and γ we have $[\beta] = I_n$ and $[\gamma] = I_m$. $\square$

Example 7.5.8. *Let $\bar{\beta} = \{(1,0,1),(0,1,1),(4,3,1)\}$. Furthermore, define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by the rule $T(x,y,z) = (2x-2y+2z,\ x-z,\ 2x-3y+2z)$. Find the matrix of T with respect to the basis β. Note first that the standard basis is read from the rule:*

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x-2y+2z \\ x-z \\ 2x-3y+2z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Next, use the proposition with $\bar{\beta} = \bar{\gamma}$ (omitting the details of calculating $[\bar{\beta}]^{-1}$)

$$\begin{aligned}
[\bar{\beta}]^{-1}[T][\bar{\beta}] &= \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & -1/6 \end{bmatrix}\begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & -1/6 \end{bmatrix}\begin{bmatrix} 4 & 0 & 4 \\ 0 & -1 & 3 \\ 4 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Therefore, in the $\bar{\beta}$-coordinates the linear operator T takes on a particularly simple form. In particular, if $\bar{\beta} = \{f_1, f_2, f_3\}$ then[12]

$$\overline{T}(\bar{x},\bar{y},\bar{z}) = 4\bar{x}f_1 - \bar{y}f_2 + \bar{z}f_3$$

This linear transformation acts in a special way in the f_1, f_2 and f_3 directions. The basis we considered here is called an **eigenbasis** *for T. We study eigenvectors and the associated problem of diagonalization later in this course.*

7.6 theory of dimensions for maps

In some sense this material is naturally paired with Section 7.1 and Section 6.6. I had to wait until this point in the presentation because I wanted to tie in some ideas about coordinate change.

This section is yet another encounter with a **classification theorem**. Previously, we learned that vector spaces are classified by their dimension; $V \approxeq W$ iff $\dim(V) = \dim(W)$. In this section, we'll find a nice way to lump together many linear transformations as being essentially the same function with a change of notation. However, the concept of *same* is a slippery one. In this section, **matrix congruence** is the measure of sameness. In contrast, later we study **similarity transformations** or **orthogonal transformations**. The concept that unites these discussions is classification. We seek a standard representative of an equivalence class. The type of equivalence class depends naturally on what is considered the "same". Be careful with this word "same" it might not mean the

[12] some authors just write T, myself included, but, technically $\overline{T} = T \circ \Phi_{\bar{\beta}}^{-1}$, so... as I'm being pretty careful otherwise, it would be bad form to write the prettier, but wrong, T

same thing to you.

The theorem below is to linear transformations what Theorem 6.7.3 is for matrices.

Theorem 7.6.1.

Let V, W be vector spaces of finite dimension over $\mathbb{F}$. In particular, suppose $\dim(V) = n$ and $\dim(W) = m$. If $T : V \to W$ be a linear transformation then

$$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Range}(T)).$$

Proof: I'll give two proofs. The first is based on coordinates and Theorem 7.1.27 which includes the result that an injective linear transformation maps LI sets to LI sets.

Proof 1: Let β, γ be bases for V, W respectively. Define $A = [T]_{\beta,\gamma}$. Observe $A \in \mathbb{F}^{m\times n}$. Apply Theorem 6.7.3 to find

$$n = \dim(\text{Null}(A)) + \dim(\text{Col}(A)).$$

We found in Lemma 7.4.15 that the basis for $\text{Ker}(T)$ is obtained by mapping the basis β_N for $\text{Null}([T]_{\beta,\gamma})$ to V by Φ_β^{-1}. That is, $\Phi_\beta^{-1}(\beta_N) = \beta_K$ serves as a basis for $\text{Ker}(T) \leq V$. On the other hand, Lemma 7.4.15 also stated the basis for the column space $\beta_C \subset \mathbb{F}^m$ is mapped to a basis for $\text{Range}(T)$ in W. In particular, we define $\beta_R = \Phi_\gamma^{-1}(\beta_C)$ and it serves as a basis for $\text{Range}(T) \leq W$. Lemma 7.4.15 also proved $\#(\beta_N) = \#(\beta_K)$ and $\#(\beta_C) = \#(\beta_R)$. Thus,

$$\dim(V) = n = \dim(\text{Null}(A)) + \dim(\text{Col}(A)) = \dim(\text{Ker}(T)) + \dim(\text{Range}(T)). \qquad \square$$

Proof 2: Note $\text{Ker}(T) \leq V$ therefore we may select a basis $\beta_K = \{v_1, \dots, v_k\}$ for $\text{Ker}(T)$ by Proposition 6.6.7. By the basis extension theorem (think $W = \text{Ker}(T)$ and apply Theorem 6.6.11) we can adjoin the set of vectors $\beta_{not\ K} = \{v_{k+1}, \dots, v_n\}$ to make $\beta = \beta_K \cup \beta_{not\ K}$ a basis for V. Suppose $x = \sum_{i=1}^n x_i v_i \in V$ and calculate by linearity of T,

$$T(x) = \sum_{i=1}^k x_i T(v_i) + \sum_{i=k+1}^n x_i T(v_i) = \sum_{i=k+1}^n x_i T(v_i),$$

where $v_1, \dots, v_k \in \text{Ker}(T)$ gives $T(v_1) = \cdots = T(v_k) = 0$. Observe, it follows that the set of $n-k$ vectors $\gamma = \{T(v_{k+1}), \dots, T(v_n)\}$ serves as a spanning set for $\text{Range}(T)$. Moreover, we may argue that γ is a LI set: suppose

$$c_{k+1}T(v_{k+1}) + \cdots + c_n T(v_n) = 0$$

by linearity of T it follows:

$$T(c_{k+1}v_{k+1} + \cdots + c_n v_n) = 0$$

hence $c_{k+1}v_{k+1} + \cdots + c_n v_n \in \text{Ker}(T)$. However, by construction, $\beta_{not\ K} = \{v_{k+1}, \dots, v_n\}$ are not in the kernel thus

$$c_{k+1}v_{k+1} + \cdots + c_n v_n = 0.$$

Next, as $\beta_{not\ K} \subseteq \beta$ the LI of β implies the LI of $\beta_{not\ K}$ hence we conclude $c_{k+1} = 0, \dots, c_n = 0$. Therefore, γ is a basis for $\text{Range}(T)$. Finally, as $\dim(V) = n = n-k+k$ and $\dim(\text{Ker}(T)) = k$ and $\dim(\text{Range}(T)) = n-k$ we conclude

$$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Range}(T)). \qquad \square$$

Proof of the theorem that follows below is essentially contained in the proof of Theorem 7.6.1. However, for the sake of completeness, I include a separate proof.

Theorem 7.6.2.

Let V, W be vector spaces of finite dimension over $\mathbb{F}$. If $T : V \to W$ be a linear transformation with $\text{rank}(T) = \dim(T(V)) = p$. Then, there exist bases β for V and γ for W such that:

$$[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right]$$

where, as is our standard notation, $[T(v)]_\gamma = [T]_{\beta,\gamma}[v]_\beta$ for all $v \in V$.

Proof: Let $\dim(V) = n$ and $\dim(W) = m$ for convenience of exposition. By Theorem 7.6.1) we have $\dim(Ker(T)) = n - p$. Let $\{v_{p+1}, \dots, v_n\}$ form a basis for $Ker(T) \leq V$. Extend the basis for $Ker(T)$ to a basis $\beta = \{v_1, \dots, v_p, v_{p+1}, \dots, v_n\}$ for V. Observe, by construction, $\{T(v_1), \dots, T(v_p)\}$ is linearly independent. Define,

$$w_1 = T(v_1), \ \dots, \ w_p = T(v_p)$$

Clearly $\{w_1, \dots, w_p\}$ forms a basis for the image $T(V)$. Next, extend $\{w_1, \dots, w_p\}$ to a basis $\gamma = \{w_1, \dots, w_p, w_{p+1}, \dots, w_m\}$ for W. Observe:

$$[T(v_j)]_\gamma = [T]_{\beta,\gamma}[v_j]_\beta = [T]_{\beta,\gamma}e_j = Col_j([T]_{\beta,\gamma})$$

Furthermore, for $j = 1, \dots, p$, by construction $T(v_j) = w_j$ and hence $[T(v_j)]_\gamma = [w_j]_\gamma = \bar{e}_j \in \mathbb{R}^m$. On the other hand, for $j = p+1, \dots, n$ we have $T(v_j) = 0$ hence $[T(v_j)]_\gamma = [0]_\gamma = 0 \in \mathbb{R}^m$. Thus,

$$[T]_{\beta,\gamma} = [e_1|\cdots|e_p|0|\cdots|0]$$

and it follows that $[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right]$. $\square$.

The claim of the theorem just proved says the following: there exists a choice of coordinates which makes a given linear transformation a projection onto the range. In terms of matrix congruence, this theorem reveals the canonical form for matrices which are equivalent under matrix congruence $A \mapsto QAP^{-1}$. However, the proof above does not reveal too much about how to find such coordinates. We next investigate a calculational method to find β, γ for which the theorem is realized.

Suppose $T \in L(V, W)$ where $dim(V) = n$ and $dim(W) = m$. Furthermore, suppose $\beta' = \{v_1^{'}, \dots, v_n^{'}\}$ and $\gamma' = \{w_1^{'}, \dots, w_m^{'}\}$ are bases for V and W respective. We define $[T]_{\beta'\gamma'}$ as usual:

$$[T]_{\beta'\gamma'} = [[T(v_1^{'})]_{\gamma'}|\cdots|[T(v_n^{'})]_{\gamma'}]$$

There exists a product of elementary $m \times m$ matrices E_1 for which

$$R_1 = \text{rref}([T]_{\beta'\gamma'}) = E_1[T]_{\beta'\gamma'}$$

Let p be the number of pivot columns in R_1. Observe that the last $(m-p)$ rows in R_1 are zero. Therefore, the last $(m-p)$ columns in R_1^T are zero. Gauss-Jordan elimination on R_1 is accomplished by multiplication by E_2 which is formed from a product of $n \times n$ elementary matrices.

$$R_2 = \text{rref}(R_1^T) = E_2 R_1^T$$

Notice that the trivial rightmost $(m-p)$ columns stay trivial under the Gauss-Jordan elimination. Moreover, the nonzero pivot rows in R_1 become p-pivot columns in R_1^T which reduce to $e_1, \dots, e_p$ standard basis vectors in $\mathbb{R}^n$ for R_2 (the leading ones are moved to the top rows with row-swaps if necessary). In total, we find: (the subscripts indicate the size of the blocks)

$$E_2 R_1^T = [e_1|\cdots|e_p|0|\cdots|0] = \left[\begin{array}{c|c} I_p & 0_{p\times(m-p)} \\ \hline 0_{(n-p)\times p} & 0_{(n-p)\times(m-p)} \end{array}\right]$$

Therefore,

$$E_2(E_1[T]_{\beta'\gamma'})^T = \left[\begin{array}{c|c} I_p & 0_{p\times(m-p)} \\ \hline 0_{(n-p)\times p} & 0_{(n-p)\times(m-p)} \end{array}\right]$$

Transposition of the above equation yields the following:

$$E_1[T]_{\beta'\gamma'}E_2^T = \left[\begin{array}{c|c} I_p & 0_{p\times(n-p)} \\ \hline 0_{(m-p)\times p} & 0_{(m-p)\times(n-p)} \end{array}\right]$$

If β, γ are bases for V and W respective then we relate the matrix $[T]_{\beta,\gamma}$ to $[T]_{\beta'\gamma'}$ as follows:

$$[T]_{\beta,\gamma} = [\Phi_{\beta'} \circ \Phi_\beta^{-1}][T]_{\beta'\gamma'}[\Phi_\gamma \circ \Phi_{\gamma'}^{-1}].$$

Therefore, we ought to define β by imposing $[\Phi_{\beta'} \circ \Phi_\beta^{-1}] = E_1$ and γ by $[\Phi_\gamma \circ \Phi_{\gamma'}^{-1}] = E_2^T$. Using $L_A(v) = Av$ notation for E_1, E_2^T,

$$L_{E_1} = \Phi_{\beta'} \circ \Phi_\beta^{-1} \qquad \& \qquad L_{E_2^T} = \Phi_\gamma \circ \Phi_{\gamma'}^{-1}$$

Thus,

$$\Phi_\beta^{-1} = \Phi_{\beta'}^{-1} \circ L_{E_1} \qquad \& \qquad \Phi_\gamma^{-1} = \left(L_{E_2^T} \circ \Phi_{\gamma'}\right)^{-1} = \Phi_{\gamma'}^{-1} \circ L_{E_2^T}^{-1}.$$

and we construct β and γ explicitly by:

$$\beta = \{(\Phi_{\beta'}^{-1} \circ L_{E_1})(e_j)\}_{j=1}^n \qquad \gamma = \{(\Phi_{\gamma'}^{-1} \circ L_{E_2^T}^{-1})(e_j)\}_{j=1}^m.$$

Note the formulas above merely use the elementary matrices and the given pair of bases. The discussion of this page shows that β and γ so constructed will give $[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right]$.

Continuing, to implement the calculation outlined in the previous page we would like an efficient method to calculate E_1 and E_2. We can to do this much as we did for computation of the inverse. I illustrate the idea below[13]:

Example 7.6.3. *Let* $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. *If we adjoin the identity matrix to right the matrix which is constructed in the Gauss-Joran elimination is the product of elementary matrices P for which $rref(A) = PA$.*

$$rref[A|I_4] = rref\left[\begin{array}{ccc|cccc} 1 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 5 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 & -3/2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right]$$

[13]see Example 2.7 on page 244 of Hefferon's Linear Algebra for a slightly different take built on explicit computation of the product of the elementary matrices needed for the reduction

We can read P for which $rref(A) = PA$ from the result above, it is simply

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1/2 \\ 1 & 0 & 1/2 & -3/2 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

Next, consider row reduction on the transpose of the reduced matrix. This corresponds to column operations on the reduced matrix.

$$rref[(rref(A))^T|I_3] = rref\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Let $Q = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ *and define R by:*

$$R^T = Q[rref(A)]^T = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, $R = (Q[rref(A)]^T)^T = rref(A)Q^T$ hence $R = PAQ^T$. In total,

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1/2 \\ 1 & 0 & 1/2 & -3/2 \\ 0 & 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

There is nothing terribly special about this example. We could follow the same procedure for a general matrix to find the explicit change of basis matrices which show the matrix congruence of A to $\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$ where $p = \text{rank}(A)$. From a coordinate change perspective, in the finite-dimensional case, this means we can always change coordinates on a linear transformation to make the formula for the transformation a simple projection onto the first p-coordinates; $T(y_1, \dots, y_p, y_{p+1}, \dots, y_n) = (y_1, \dots, y_p, 0, \dots, 0) \in \mathbb{R}^m$. The richness we saw in Examples 7.2.4 - 7.2.12 is still here, it's just hidden in the coordinate change. I should mention, this can be viewed as an application of the **Singular Value Decomposition (SVD)** which is a useful Theorem you might argue I should cover. Essentially, the Singular Value Decomposition says any linear transformation can be understood as a combination of a generalized rotation and scaling. You might find this You Tube video by Professor Pavel Grinfeld a useful jumping-off point for further study.

Later in this course we'll study other problems where different types of coordinate change are allowed. When there is less freedom to modify domain and codomain coordiantes it turns out the *cannonical forms* of the object are greater in variety and structure. Just to jump ahead a bit, if we force $m = n$ and change coordinates in domain and codomain simultaneously then the **real Jordan form** captures a representative of each equivalence class of matrix up to a **similarity transformation**. On the other hand, Sylvester's Law of Inertia reveals the cannonical form for the matrix of a quadratic form is simply a diagonal matrix with $Diag(D) = (-1, \dots, -1, 1, \dots, 1, 0, \dots, 0)$. Quadratic forms are non-linear functions which happen to have an associated matrix. The coordinate change for the matrix of a quadratic form is quite different than what we've studied thus far.

In any event, this is just a foreshadowing comment, we will return to this discussion once we study eigenvectors and quadratic forms later in this course.

Let us conclude this section with a beautifully simple result:

Theorem 7.6.4.

Let V be vector spaces of finite dimension over $\mathbb{F}$. If $T : V \to V$ is a linear transformation then the following are equivalent:

(1.) T is injective

(2.) T is surjective

(3.) T is an isomorphism

Proof: left to reader. $\square$

Notice, we can only use this Theorem to circumvent work if we already know the function in question is indeed a linear transformation on a given vector space of finite dimension. But, if that is settled, this Theorem gives us license to claim injectivity and surjectivity after verification of either one. This should remind you of the case of maps on finite sets; if $T : S \to S$ is a function and $\#S < \infty$ then T is surjective iff T is injective. While $V(\mathbb{F})$ generally has infinitely many vectors, the basis construction brings a finiteness which is a large part of why finite-dimensional linear algbera has such simple structure.

7.7 structure of subspaces

I will begin this section by following an elegant construction[14] I found in Morton L. Curtis' *Abstract Linear Algebra* pages 28-30. The results we encounter in this section prove useful in later chapters where we study eigenvectors.

Recall the construction in Example 6.1.11, this is known as the **external direct sum**. If V, W are vector spaces over $\mathbb{R}$ then $V \times W$ is given the following vector space structure:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \qquad \& \qquad c(v, w) = (cv, cw).$$

In the vector space $V \times W$ the vector $(0_V, 0_W) = 0_{V\times W}$. Although, usually we just write $(0, 0) = 0$. Furthermore, if $\beta_V = \{v_1, \dots, v_n\}$ and $\beta_W = \{w_1, \dots, w_m\}$ then a basis for $V \times W$ is simply:

$$\beta = \{(v_i, 0) | i \in \mathbb{N}_n\} \cup \{(0, w_j) | j \in \mathbb{N}_m\}$$

I invite the reader to check LI of β. To see how β spans, please consider the calculation below:

$$\begin{aligned} (x, y) &= (x, 0) + (0, y) \\ &= (x_1 v_1 + \cdots + x_n v_n, 0) + (0, y_1 w_1 + \cdots y_m w_m) \\ &= x_1(v_1, 0) + \cdots + x_n(v_n, 0) + y_1(0, w_1) + \cdots y_m(0, w_m) \end{aligned}$$

Thus β is a basis for $V \times W$ and we can count $\#(\beta) = n+m$ hence $\dim(V \times W) = \dim(V) + \dim(W)$. This result generalizes to an s-fold cartesian product of vector spaces over $\mathbb{F}$:

[14] I don't use his notation that $A \oplus B = A \times B$, I reserve $A \oplus B$ to denote internal direct sums.

Proposition 7.7.1.

If $W_1, W_2, \dots, W_s$ are vector spaces over $\mathbb{F}$ with bases $\beta_1, \beta_2, \dots, \beta_s$ respective then $W_1 \times W_2 \times \cdots \times W_s$ has basis

$$(\beta_1 \times \{0\} \times \cdots \times \{0\}) \cup (\{0\} \times \beta_2 \times \cdots \times \{0\}) \cup \cdots \cup (\{0\} \times \{0\} \times \cdots \times \beta_s)$$

hence $\dim(W_1 \times W_2 \times \cdots \times W_s) = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_s)$.

Proof: left to reader. $\square$

The Example below illustrates the claim of the Proposition above:

Example 7.7.2. *Find a basis for $P_2(\mathbb{R}) \times \mathbb{R}^{2\times 2} \times \mathbb{C}$. Recall the monomial basis $\{1, x, x^2\}$ for $P_2(\mathbb{R})$ and the unit-matrix basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ for $\mathbb{R}^{2\times 2}$ and $\{1, i\}$ serves as the basis for $\mathbb{C}$ as a real vector space. Hence*

$$\beta = \{(1,0,0), (x,0,0), (x^2,0,0), (0,E_{11},0), (0,E_{12},0), (0,E_{21},0), (0,E_{22},0), (0,0,1), (0,0,i)\}$$

serves as a basis for the nine dimensional real vector space $P_2(\mathbb{R}) \times \mathbb{R}^{2\times 2} \times \mathbb{C}$.

The question to ask is when is it possible to find an isomorphism between a given vector space V and some set of subspaces of V whose sum forms V. It turns out there are several ways to understand such a structure and we devote the next page or so of the notes towards exploring a number of equivalent characterizations.

Notice the notation $W_1 + W_2 = \{x_1 + x_2 \mid x_1 \in W_1, x_2 \in W_2\}$ generalizes to:

$$W_1 + W_2 + \cdots + W_k = \{x_1 + x_2 + \cdots + x_k \mid x_i \in W_i \text{ for each } i = 1, 2, \dots, k\}$$

where $W_1, W_2, \dots, W_k$ are subspaces of some vector space.

Definition 7.7.3.

If $W_1, W_2, \dots, W_k \leq V$ and $V = W_1 + W_2 + \cdots W_k$ then we say V is the **internal direct sum** of the subspaces $W_1, W_2, \dots, W_k$ iff for each $x \in V$ there exist unique $x_i \in W_i$ for $i = 1, 2, \dots, k$ such that $x = x_1 + x_2 + \cdots + x_k$. When the above criteria is met we denote this by writing

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

I follow some arguments I found in Chapter 10 of Dummit and Foote's *Abstract Algebra*. I found these are a bit easier than what I've seen in undergraduate linear texts, so, I share them here:

Theorem 7.7.4.

If $W_1, W_2, \dots, W_k \leq V$ and $V = W_1 + W_2 + \cdots W_k$ is finite dimensional over $\mathbb{F}$ then the following are equivalent:

(1.) the map $\pi : W_1 \times W_2 \times \cdots W_k \to V$ defined by $\pi(x_1, x_2, \dots, x_k) = x_1 + x_2 + \cdots + x_k$ is an isomorphism,

(2.) $W_j \cap (W_1 + \cdots + W_{j-1} + W_{j+1} + \cdots + W_k) = \{0\}$ for each $j = 1, 2, \dots, k$,

(3.) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ or, to be precise, for each $x \in V$ there exist unique $x_i \in W_i$ for $i = 1, 2, \dots, k$ for which $x = x_1 + x_2 + \cdots + x_k$,

(4.) if β_i is a basis for W_i for $i = 1, 2 \dots, k$ then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a basis for V,

(5.) if $x_i \in W_i$ for $i = 1, 2, \dots, k$ and $x_1 + x_2 + \cdots + x_k = 0$ then $x_1 = 0$, $x_2 = 0, \cdots, x_k = 0$.

Proof: see my Lectures of 3-8-17 and 3-10-17. $\square$

The case $k = 2$ is most interesting since condition (2.) simply reads that $W_1 \cap W_2 = \{0\}$. If $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$ then we say that W_1, W_2 are **complementary subspaces**.

Example 7.7.5. *If $\mathcal{F}(\mathbb{R})$ is the set of functions on $\mathbb{R}$ then since we have the identity:*

$$f(x) = \frac{1}{2}\bigg(f(x) + f(-x)\bigg) + \frac{1}{2}\bigg(f(x) - f(-x)\bigg)$$

for all $x \in \mathbb{R}$. For example, recall $\cosh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ hence $e^x = \cosh(x) + \sinh(x)$. We note:

$$f_{even}(x) = \frac{1}{2}\left(f(x) + f(-x)\right) \qquad \& \qquad f_{odd}(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

satisfy $f_{even}(-x) = f_{even}(x)$ and $f_{odd}(-x) = -f_{odd}(x)$ for each $x \in \mathbb{R}$. You can easily verify the set of even functions $\mathcal{F}_e(\mathbb{R})$ and the set of odd functions $\mathcal{F}_o(\mathbb{R})$ are subspaces of $\mathcal{F}(\mathbb{R})$. Since $f = f_{even} + f_{odd}$ for any function f we find $\mathcal{F}(\mathbb{R}) = \mathcal{F}_e(\mathbb{R}) + \mathcal{F}_o(\mathbb{R})$. Moreover, it is easy to verify that $\mathcal{F}_e(\mathbb{R}) \cap \mathcal{F}_o(\mathbb{R}) = \{0\}$. Therefore, the subspaces of even and odd functions form complementary subspaces of the space of functions on $\mathbb{R}$; $\mathcal{F}(\mathbb{R}) = \mathcal{F}_e(\mathbb{R}) \oplus \mathcal{F}_o(\mathbb{R})$.

Example 7.7.6. *Let A be an $n \times n$ matrix over $\mathbb{F}$ then notice that:*

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

it follows that $\mathbb{F}^{n \times n}$ is the direct sum of the complementary subspaces of symmetric and antisymmetric matrices. I leave the details as a homework problem. I gave you the essential hint here.

A convenient notation for spans of a single element v in V a vector space over $\mathbb{R}$ is simply $v\mathbb{R}$. I utilize this notation in the examples below.

Example 7.7.7. *The cartesian plane $\mathbb{R}^2 = e_1\mathbb{R} \oplus e_2\mathbb{R}$.*

Example 7.7.8. *The complex numbers $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$. We could discuss how extending $i^2 = -1$ linearly gives this an algebraic structure. We have a whole course in the major to dig into this example.*

Example 7.7.9. *The hyperbolic numbers $\mathcal{H} = \mathbb{R} \oplus j\mathbb{R}$. We could discuss how extending $j^2 = 1$ linearly gives this an algebraic structure. This is less known, but it naturally describes problems with some hyperbolic symmetry.*

Example 7.7.10. *The dual numbers $\mathcal{N} = \mathbb{R} \oplus \epsilon\mathbb{R}$. We could discuss how extending $\epsilon^2 = 0$ linearly gives this an algebraic structure.*

The algebraic comments above are mostly for breadth. We focus on linear algebra[15] in these notes.

Naturally we should consider extending the discussion to more than two subspaces.

Example 7.7.11. *Quaternions. $\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$ where $i^2 = j^2 = k^2 = -1$. Our notation for vectors in most calculus texts has a historical basis in Hamilton's quarternions.*

Unfortunately, trivial pairwise intersections do not generally suffice to give direct sum decompositions for three or more subspaces. The next example illustrates this subtlety.

Example 7.7.12. *Let $W_1 = (1,1)\mathbb{R}$ and $W_2 = (1,0)\mathbb{R}$ and $W_3 = (1,1)\mathbb{R}$. It is not hard to verify $W_1 + W_2 + W_3 = \mathbb{R}^2$ and $W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \{0\}$. However, it is certainly not possible to find an isomorphism of $\mathbb{R}^2$ and the three dimensional vector space $W_1 \times W_2 \times W_3$.*

We now explain the interesting relation between the direct sum decomposition of a vector space V and the block-structure for a matrix of a linear transformation on V.

Definition 7.7.13.

If $T : V \to V$ is a linear transformation on the vector space V over $\mathbb{F}$ and $W \leq V$ is a subspace of V for which $T(W) \subseteq W$ then we say W is an T-**invariant** subspace of V. We let $T_W : W \to W$ denote the map defined by $T_W(x) = T(x)$ for each $x \in W$.

Notice that $T|_W : W \to V$ whereas $T_W : W \to W$. The map T_W is only well-defined if the subspace W is T-invariant. In particular, T-invariance of W gives us that $T(x)$ is in W, that is, the map $T_W : W \to W$ is *into* W. To show a map $f : A \to B$ is well-defined we need several things. First, we need that each element a in A produces a single output $f(a)$. Second, we need that each output $f(a)$ is actually in the codomain B. Usually there is no need to check the codomain, but, T_W is just such a problem. Later, when we study quotient vector spaces, we'll find problems where it is not immediately obvious the value $f(a)$ is in fact a single object in the codomain. More on that later.

Something very interesting happens when $V = W_1 \oplus W_2 \oplus \cdots \oplus W_s$ and each W_j is T-invariant.

[15] a vector space paired with a multiplication is called an algebra. The rules $i^2 = -1, j^2 = 1$ and $\epsilon^2 = 0$ all serve to define non-isomorphic algebraic structures on $\mathbb{R}^2$. These are isomorphic as vector spaces.

Theorem 7.7.14.

Suppose $T : V \to V$ is a linear transformation and $W_j \leq V$ are such that $T(W_j) \leq W_j$ for each $j = 1, \dots, s$. Also, suppose $V = W_1 \oplus \cdots \oplus W_s$ where $\dim(W_j) = d_j$ and $d_1 + \cdots + d_s = n = \dim(V)$. Then there exists a basis $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_s$ for V formed by concatenating β_j basis for W_j for $j = 1, 2, \dots, s$ for which

$$[T]_{\beta,\beta} = \left[\begin{array}{c|c|c|c} M_1 & 0 & \cdots & 0 \\ \hline 0 & M_2 & \cdots & 0 \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline 0 & 0 & \cdots & M_s \end{array}\right] = \text{diag}(M_1, M_2, \dots, M_s).$$

and $M_j = [T_{W_j}]_{\beta_j,\beta_j}$ for $j = 1, 2, \dots, s$.

Proof: let β_j be a basis for W_j then $\#(\beta_j) = d_j$ for $j = 1, 2, \dots, s$. Let us denote $\beta_j = \{v_{j,1}, v_{j,2}, \dots, v_{j,d_j}\}$ for $j = 1, 2, \dots, s$. Furthermore, by 7.7.4 we have that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_s$ is a basis for V. If $v_{j,i} \in \beta_j$ then

$$T(v_{j,i}) \in W_j \quad \Rightarrow \quad T(v_{j,i}) = c_1 v_{j,1} + \cdots + c_{d_j} v_{j,d_j}$$

thus the column vector in $[T]_{\beta,\beta}$ corresponding to the basis vector $v_{j,i}$ only is nonzero in rows corresponding to the β_j part of the basis. It follows that $[T]_{\beta,\beta}$ is block-diagonal where M_j is the $d_j \times d_j$ matrix over $\mathbb{F}$ which is the matrix of T_{W_j} with respect to the β_j matrix; that is $M_j = [T_{W_j}]_{\beta_j,\beta_j}$. $\square$

A block diagonal matrix allows multiplication where the blocks behave as if they were numbers. See Section 2.5 where we studied how block-multiplication works. I should mention, we can *add* matrices of different sizes following the pattern above:

Definition 7.7.15.

If $M_j \in \mathbb{F}^{d_j \times d_j}$ for $j = 1, 2, \dots, s$ then we define $M_1 \oplus M_2 \oplus \cdots \oplus M_s \in \mathbb{F}^{n \times n}$ where $n = d_1 + d_2 + \cdots + d_s$ and

$$M_1 \oplus M_2 \oplus \cdots \oplus M_s = \left[\begin{array}{c|c|c|c} M_1 & 0 & \cdots & 0 \\ \hline 0 & M_2 & \cdots & 0 \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline 0 & 0 & \cdots & M_s \end{array}\right].$$

Certainly this is not standard addition as $A \oplus B \neq B \oplus A$. But, it is a fun new way to make new matrices from old. To be clear, $A \oplus B$ is the **direct sum** of A and B. With this langauge, Theorem 7.7.14 is formulated as follows: when V is a direct sum decomposition of T-invariant subspaces then there exists a matrix for which the matrix of T is likewise formed by a direct sum of submatrices. If $V = W_1 \oplus \cdots \oplus W_s$ and T is W_j-invariant for each $j = 1, \dots, s$ then if β_j is basis for W_j and $\beta = \beta_1 \cup \cdots \cup \beta_s$ then

$$[T]_{\beta,\beta} = [T_{W_1}]_{\beta_1,\beta_1} \oplus [T_{W_2}]_{\beta_2,\beta_2} \oplus \cdots \oplus [T_{W_s}]_{\beta_s,\beta_s}.$$

Let me conclude with the pair of examples which I began our in-class discussion on 3-8-17.

Example 7.7.16. *Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be defined as $T = d/dx$. In particular $T(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$. Thus, for the basis $\beta = \{1, x, x^2, x^3\}$ we find matrix:*

$$[T]_{\beta,\beta} = \big[[T(1)]_\beta|[T(x)]_\beta|[T(x^2)]_\beta|[T(x^3)]_\beta\big] = \big[[0]_\beta|[1]_\beta|[2x]_\beta|[3x^2]_\beta\big] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

Notice, Range(T) = span$\{1, x, x^2\}$ is an invariant subspace of T however the remaining x^3 vector has $T(x^3) = 3x^2$ so Range(T) + span(x^3) is not a direct sum decomposition.

Example 7.7.17. *Let $S = d^2/dx^2$ on $P_3(\mathbb{R})$. Calculate,*

$$S(1) = 0, \quad S(x) = 0, \quad S(x^2) = 2, \quad S(x^3) = 6x$$

Thus (by inspection) we find invariant subspaces W_1 = span$\{1, x^2\}$ and W_2 = span$\{x, x^3\}$. Let $\gamma_1 = \{1, x^2\}$ and $\gamma_2 = \{x, x^3\}$ and $\gamma = \gamma_1 \cup \gamma_2 = \{1, x^2, x, x^3\}$ and we find

$$[S]_{\gamma,\gamma} = \big[[S(1)]_\gamma|[S(x^2)]_\gamma|[S(x)]_\gamma|[S(x^3)]_\gamma\big] = \big[[0]_\gamma|[2]_\gamma|[0]_\gamma|[6x]_\gamma\big] = \left[\begin{array}{cc|cc} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have $[S]_{\gamma,\gamma} = [S_{W_1}]_{\gamma_1,\gamma_1} \oplus [S_{W_2}]_{\gamma_2,\gamma_2} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}$

7.8 similarity and determinants for linear transformations

Thus far this chapter has been mainly matrix theoretic. However, the determinant is also defined and of interest for abstract linear transformations. Suppose V is an n-dimensional vector space over $\mathbb{R}$ and consider $T : V \to V$ a linear transformation. If β, γ are finite bases for V then we can calculate $[T]_{\beta,\beta}$ and $[T]_{\gamma,\gamma}$. Note these are both $n \times n$ matrices as the domain and codomain are both n-dimensional. Furthermore, applying Proposition 7.5.5 we have:

$$[T]_{\gamma,\gamma} = [\Phi_\gamma \circ \Phi_\beta^{-1}][T]_{\beta,\beta}[\Phi_\beta \circ \Phi_\gamma^{-1}]$$

If we set $P = [\Phi_\beta \circ \Phi_\gamma^{-1}]$ then the equation above simply reduces to:

$$[T]_{\gamma,\gamma} = P^{-1}[T]_{\beta,\beta}P.$$

I've mentioned this concept in passing before, but for future reference we should give a precise definition:

Definition 7.8.1.

Let $A, B \in \mathbb{R}^{n\times n}$ then we say A and B are **similar matrices** if there exists $P \in \mathbb{R}^{n\times n}$ such that $B = P^{-1}AP$.

In invite the reader to verify that matrix similarity is an equivalence relation. Furthermore, you might contrast this idea of sameness with that of **matrix congruence**. To say A, B are matrix congruent it sufficed to find P, Q such that $B = P^{-1}AQ$. Here $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ and we needed only that $A, B \in \mathbb{R}^{m\times n}$. Matrix congruence was defined for rectangular matrices whereas similarity is only for square matrices. The idea is this, two congruent matrices represent the same linear transformation $T : V \to W$. There is some choice of bases for V and W which change the formula of T from A to B or vice-versa. Moreover, Theorem 7.6.2 revealed the cannonical form relative to matrix congruence classes was simply an identity matrix as big as the rank of the transformation padded with zeros. To understand the difference between congruence and similarity it is important to notice that congruence is based on adjusting both the basis in the domain and **separately** the basis in the codomain. In contrast, similarity is is related to changing the basis in the domain and codomain in the same exact fashion. This means it is a stronger condition for two matrices to be similar. The analog for Theorem 7.6.2 is what is known as the real Jordan form and it provides the concluding thought of this course. The criteria which will guide us to find the Jordan form is simply this: any two similar matrices should have the exact same Jordan form. With a few conventional choices made, this gives us a cannonical representative of each equivalence class of similar matrices. It is worthwhile to note the following:

Proposition 7.8.2.

Let $A, B, C \in \mathbb{R}^{\,n\times n}$.

1. A is similar to A.
2. If A is similar to B then B is similar to A.
3. If A is similar to B and B is similar to C then A similar to C.
4. If A and B are similar then $det(A) = det(B)$
5. If A and B are similar then $tr(A) = tr(B)$
6. If A and B are similar then $rank(A) = rank(B)$ and $nullity(A) = nullity(B)$

Given the proposition above we can make the following definitions without ambiguity.

Definition 7.8.3.

Let $T : V \to V$ be a linear transformation on a finite-dimensional vector space V and let β be any basis of V,

1. $det(T) = det([T]_{\beta,\beta})$.
2. $tr(T) = tr([T]_{\beta,\beta})$
3. $rank(T) = rank([T]_{\beta,\beta})$.

Example 7.8.4. *Consider $D : P_2 \times P_2$ defined by $D[f(x)] = df/dx$ note that $D[ax^2+bx+c] = 2ax+b$ implies that in the $\beta = \{x^2, x, 1\}$ coordinates we find:*

$$[D]_{\beta,\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus $det(D) = 0$.

Example 7.8.5. *Consider $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $L(A) = A^T$. Observe:*

$$L(E_{11}) = E_{11},\ L(E_{12}) = E_{21},\ L(E_{21}) = E_{12},\ L(E_{122}) = E_{22}.$$

Therefore, if $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ then

$$[L]_{\beta,\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Swapping columns 2 and 3 brings $[L]_{\beta,\beta}$ to the identity matrix. Hence, $det(L) = -1$.

Example 7.8.6. *Consider $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ defined by $L(A) = A^T$*

$$L(E_{11}) = E_{11},\ L(E_{22}) = E_{22},\ L(E_{33}) = E_{33}$$

these explain the first three columns in $[L]_{\beta,\beta}$. Next,

$$L(E_{12}) = E_{21},\ L(E_{13}) = E_{31},\ L(E_{21}) = E_{12},\ L(E_{23}) = E_{32},\ L(E_{31}) = E_{13},\ L(E_{32}) = E_{23}.$$

Let us order β so the diagonals come first: $\beta = \{E_{11}, E_{22}, E_{33}, E_{12}, E_{21}, E_{23}, E_{32}, E_{13}, E_{31}\}$. Thus,

$$[L]_{\beta,\beta} = \left[\begin{array}{ccc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Three column swaps modify the above to the identity. Thus, $det(L) = -1$.

7.9 conclusions

The theorem which follows collects ideas somewhat comprehensively for our course thus far.

Theorem 7.9.1.

Let A be a real $n \times n$ matrix then the following are equivalent:

(a.) A is invertible,

(b.) $rref[A|0] = [I|0]$ where $0 \in \mathbb{R}^n$,

(c.) $Ax = 0$ iff $x = 0$,

(d.) A is the product of elementary matrices,

(e.) there exists $B \in \mathbb{R}^{\ n\times n}$ such that $AB = I$,

(f.) there exists $B \in \mathbb{R}^{\ n\times n}$ such that $BA = I$,

(g.) $rref[A] = I$,

(h.) $rref[A|b] = [I|x]$ for an $x \in \mathbb{R}^n$,

(i.) $Ax = b$ is consistent for every $b \in \mathbb{R}^n$,

(j.) $Ax = b$ has exactly one solution for every $b \in \mathbb{R}^n$,

(k.) A^T is invertible,

(l.) $det(A) \neq 0$,

(m.) Cramer's rule yields solution of $Ax = b$ for every $b \in \mathbb{R}^n$.

(n.) $Col(A) = \mathbb{R}^{\ n\times 1}$,

(o.) $Row(A) = \mathbb{R}^{\ 1\times n}$,

(p.) $rank(A) = n$,

(q.) $Null(A) = \{0\}$,

(r.) $\nu = 0$ for A where $\nu = dim(Null(A))$,

(s.) the columns of A are linearly independent,

(t.) the rows of A are linearly independent,

This list is continued on the next page.

Continuing, Theorem 7.9.1 which is a big list of equivalent statements to A^{-1} exists for a given square matrix A:

Let A be a real $n \times n$ matrix then the following are equivalent:

(u.) the induced linear operator L_A is onto; $L_A(\mathbb{R}^n) = \mathbb{R}^n$.

(v.) the induced linear operator L_A is 1-1

(w.) the induced linear operator L_A is an isomorphism.

(x.) the kernel of the induced linear operator is trivial; $ker(L_A) = \{0\}$.

We should pay special attention to the fact that the above comments hold only for a square matrix. If we consider a rectangular matrix then the connection between the concepts in the theorem are governed by the rank-nullity theorem.

Next, the list of equivalent statements for a singular $n \times n$ matrix:

Theorem 7.9.2.

Let A be a real $n \times n$ matrix then the following are equivalent:

(a.) A is not invertible,

(b.) $Ax = 0$ has at least one nontrivial solution.,

(c.) there exists $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent,

(d.) $det(A) = 0$,

(e.) $Null(A) \neq \{0\}$,

(f.) there are $1 \leq \nu = dim(Null(A))$ parameters in the general solution to $Ax = 0$,

(g.) the induced linear operator L_A is **not** onto; $L_A(\mathbb{R}^n) \neq \mathbb{R}^n$.

(h.) the induced linear operator L_A is **not** 1-1

(i.) the induced linear operator L_A is **not** an isomorphism.

(j.) the kernel of the induced linear operator is **non**trivial; $ker(L_A) \neq \{0\}$.

It turns out this theorem is also useful. We shall see it is fundamental to the theory of eigenvectors.

Chapter 8

Jordan form and diagonalization

We study the problem of diagonalization in this Chapter. In particular, this is an extension of our study of the direct sum decomposition for an endomorphism. Given $T : V \to V$ we ask when it is possible to obtain a basis β for which $[T]_{\beta,\beta}$ is a diagonal matrix. If such a basis β can be found then we say T is a **diagonalizable linear transformation**. Similarly, if there exists P such that PAP^{-1} is a diagonal matrix then we say A is a **diagonalizable matrix**. We explain that T is diagonalizable if and only if its matrix is diagonalizable.

For $T \in \text{End}(V)$ the characteristic equation has roots called **eigenvalues**. For an n-dimensional vector space V over $\mathbb{F}$, we find eigenvalues appear as solutions to an n-th order polynomial equation over $\mathbb{F}$. We begin by focusing on the case that all the eigenvalues are found in the field $\mathbb{F}$. In the case $\mathbb{F} = \mathbb{C}$ this is no limitation, however, if $\mathbb{F} = \mathbb{R}$ we can find T which have no real eigenvalues. Furthermore, even in the case that we have eigenvalues in $\mathbb{F}$ it need not be the case that T is diagonalizable. We need to be able to find a basis of eigenvectors, that is, an **eigenbasis**. An eigenbasis for T exists if and only if T is diagonalizable. We discuss several ways to find such an eigenbasis.

Since an eigenbasis for an endomorphism cannot always be found, we explain how **generalized eigenvectors** must also be considered. This leads to a direct sum decomposition of the vector space into generalized eigenspaces. Furthermore, we describe how a **Jordan basis** can be constructed by collecting **chains of generalized eigenvectors for** T. A basis β made of such chains will give the matrix $[T]_{\beta,\beta} = J_1 \oplus J_2 \oplus \cdots \oplus J_s$ where J_i is a **Jordan block**. This is the **Jordan form** of T which is unique up to the ordering of the blocks. This Jordan form contains all the essential data about the linear transformation T. The Jordan form allows us to solve a host of general applied problems. It also provides a label for equivalence classes of similar matrices; any two (complex) matrices are similar if and only if they have the same Jordan form. In short, the Jordan form is a natural generalization of a diagonal matrix[1].

For $\mathbb{F} = \mathbb{R}$ we either have to restrict our attention to matrices which have all their eigenvalues (this can be packed into the statement that the characteristic polynomial **splits**) or we can extend our concept of Jordan form past what I described already. Once more the choice of base field for the vector space cannot be ignored. For example, if a given matrix has characteristic equation $t^2+1=0$

[1]not entirely natural in the sense that many fields $\mathbb{F}$ are not algebraically closed, so, we have endomorphisms which are missing eigenvalues. It happens there is something called the **rational cannonical form** which makes sense for any endomorphism. I'll probably assign a challenge homework about it, but, I am convinced it is more important to understand the Jordan form in your first tour through this material.

then over $\mathbb{F} = \mathbb{R}$ we find no eigenvalues, yet, over $\mathbb{F} = \mathbb{C}$ we find $t = \pm i$. We describe a systematic procedure for replacing a real linear transformation on $V(\mathbb{R})$ with a corresponding complexified transformation on $V_{\mathbb{C}}$. Essentially the idea is to replace real scalars with complex scalars and hence allow for complex eigenvalues and vectors. We give an explicit and general construction. This leads us to a different standard form for the matrix of the transformation. In particular, we find a Jordan form of the complexified problem and then pass back to the original real problem as to obtain the so-called **real Jordan form**. This includes things like rotations as blocks. With this extended concept of the form to deal with complex eigenvalues we can say that any two real matrices are similar if and only if they share the same real Jordan form up to reordering of the blocks.

The question if A and B are similar matrices in $\mathbb{R}^{\, n\times n}$ is actually the same question as if L_A and L_B are the same linear transformation on $\mathbb{R}^n$ written in different coordinate systems. The task of deciding if A and B are similar is formiddable (ignoring the Jordan form technology I outlined here, with Jordan forms, it's a relatively simple calculation)

There are many applications of eigenvectors and generalized eigenvectors. Too many to count. I merely offer a few in the body of this Chapter. In a later chapter I show how the exponential of a matrix can be used to assemble to solution to an arbitrary system of constant coefficient n-differential equations in n-unknowns. Then, a bit later, we study another application of eigenvectors to the diagonalization of a quadratic form. That application must wait until we've thought some about geometry and orthogonality. I will likely explore other interesting directions such as Markov chains, or the Gerschgorin disk, simultaneous diagonalizablility in the homework.

8.1 eigenvectors and diagonalization

We assume V is a finite dimensional vector space over $\mathbb{F}$ in this section.

Definition 8.1.1.

Let $T : V \to V$ be a linear transformation then we say T is **diagonalizable** if there exists a basis β of V for which $[T]_{\beta,\beta}$ is a diagonal matrix.

One goal of this Section is to explain why diagonalizability is tied to the existence of eigenvectors:

Definition 8.1.2.

Let $T : V \to V$ be a linear transformation then we say $v \neq 0$ is an **eigenvector** with **eigenvalue** λ for T if $T(v) = \lambda v$. An **eigenbasis** for $T : V \to V$ is a basis for V where each basis vector is an eigenvectors of T. Also, define $\mathcal{E}_\lambda = \text{Ker}(T - \lambda \cdot Id)$ to be the **λ-eigenspace** of T

The definition above also is reasonable in the context of infinite dimensional V.

Example 8.1.3. *For $T[f(x)] = f'(x)$ for each $f(x) \in C^1(\mathbb{R})$ we find eigenfunctions from solving the differential equation $T[y] = \lambda y$ which is just $\frac{dy}{dx} = \lambda y$. The solutions are easily found by separation of variables: $y = Ce^{\lambda x}$ for any λ. Hence, we have many eigenfunctions for the differentiation operator on the space of once continuously differentiable functions. In fact, every $\lambda \in \mathbb{R}$ serves as an eigenvalue. This contrasts starkly what we will find in the finite dimensional context.*

Example 8.1.4. *Let $T : V \to V$ be a linear transformation and $x \in Ker(T)$ a nonzero vector then $T(x) = 0 = 0 \cdot x$ hence x is an eigenvector with $\lambda = 0$. If T is injective then T has no eigenvectors with eigenvalue zero. In contrast, the zero operator $T = 0$ has $T(x) = 0 = 0 \cdot x$ for each $x \in V$.*

Example 8.1.5. *Let $Id : V \to V$ be the identity transformation defined by $Id(x) = x = 1 \cdot x$ for each $x \in V$. Observe each vector in V is an eigenvector with $\lambda = 1$ for the identity map on V.*

Example 8.1.6. *Some linear transformations have no eigenvectors. Consider $T(v) = R_\theta v$ where $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\theta \neq n\pi$ for $n \in \mathbb{Z}$. It is geometrically clear that $v \neq 0$ and $T(v)$ are not colinear hence there does not exist $\lambda \in \mathbb{R}$ for which $T(v) = \lambda v$.*

The following result is our main motivation for studying eigenvalues and vectors.

Proposition 8.1.7.

Let $T : V \to V$ be a linear transformation. If there exists a basis β for V with $[T]_{\beta,\beta}$ diagonal then β is an eigenbasis. Conversely, if β is an eigenbasis then $[T]_{\beta,\beta}$ is diagonal.

Proof: If $\beta = \{v_1, \dots, v_n\}$ is a basis for V and $[T]_{\beta,\beta} = [\lambda_1 e_1 | \lambda_2 e_2 | \cdots | \lambda_n e_n]$ then $[T(v_j)]_\beta = \lambda_j e_j$ hence $T(v_j) = \lambda_j v_j$ with $v_j \neq 0$ for $j = 1, 2, \dots, n$. Therefore, β is an eigenbasis.

Conversely, if $\beta = \{v_1, \dots, v_n\}$ has $T(v_j) = \lambda_j v_j$ where $\lambda_j \in \mathbb{F}$ for $j = 1, \dots, n$ then we find:

$$[T]_{\beta,\beta} = [[\lambda_1 v_1]_\beta | [\lambda_2 v_2]_\beta | \cdots | [\lambda_n v_n]_\beta] = [\lambda_1 e_1 | \lambda_2 e_2 | \cdots | \lambda_n e_n] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \quad \square$$

We also define eigenvectors and values and diagonalizability for a matrix over $\mathbb{F}$:

Definition 8.1.8.

Let $A \in \mathbb{F}^{n \times n}$ then we say A **diagonalizable** if there exists an invertible matrix P for which PAP^{-1} is a diagonal matrix.

In other words, A is diagonalizable if and only if A is **similar** to a diagonal matrix. Recall, two matrices A, B are similar iff there exists and invertible matrix P for which $B = PAP^{-1}$.

Definition 8.1.9.

Let $A \in \mathbb{F}^{n \times n}$. We say $0 \neq v \in \mathbb{F}^n$ is an **eigenvector** of A with **eigenvalue** $\lambda \in \mathbb{F}$ if $(A - \lambda I)v = 0$. Also, define $\mathcal{E}_\lambda = \text{Null}(A - \lambda I)$ to be the λ-**eigenspace** of A.

The following result shows that the definition of eigevector for matrix and transformation are naturally related:

Proposition 8.1.10.

Let $A \in \mathbb{F}^{n \times n}$ and $L_A(x) = Ax$ for each $x \in \mathbb{F}^n$. Also, suppose $T : V \to V$ is an endomorpism of an n-dimensional vector space V over $\mathbb{F}$.

(1.) $x \in \mathbb{F}^n$ is an eigenvector of A with eigenvalue λ if and only if x is an eigenvector of L_A with eigenvalue λ,

(2.) $v \in V$ is an eigenvector of T with eigenvalue λ if and only if $[v]_\beta$ is an eigenvector of $[T]_{\beta,\beta}$ with eigenvalue λ.

Proof: to see why (1.) is true simply note that:

$$(A-\lambda I)x = 0 \quad \Leftrightarrow \quad Ax = \lambda x \quad \Leftrightarrow \quad L_A(x) = \lambda x.$$

Next, to prove (2.) consider that (by properties of the coordinate map Φ_β for basis β)

$$T(v) = \lambda v \quad \Leftrightarrow \quad [T(v)]_\beta = [\lambda v]_\beta \quad \Leftrightarrow \quad [T]_{\beta,\beta}[v]_\beta = \lambda [v]_\beta \quad \Leftrightarrow \quad ([T]_{\beta,\beta} - \lambda I)[v]_\beta = 0. \;\square$$

If we combine the results of Proposition 8.1.7 and 8.1.11 we obtain:

Proposition 8.1.11.

Let $A \in \mathbb{F}^{n\times n}$ is diagonalizable if and only if there exists eigenbasis $\{v_1, \dots, v_n\} \subset \mathbb{F}^n$ for A where $(A - \lambda_j)v_j = 0$ for $j = 1, \dots, n$. Furthermore, for A diagonalizable and β its eigenbasis, we have $[\beta]A[\beta]^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n)$.

Proof: Suppose A is diagonalizable then there exists $P \in \mathbb{F}^{n\times n}$ for which $PAP^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n)$. In view of Proposition 7.5.7 we see that L_A has matrix $[L_A]_{\beta,\beta} = PAP^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n)$ with respect to basis $\beta = \{\text{Col}_1(P), \dots, \text{Col}_n(P)\}$. Hence by Proposition 8.1.7 we have that β is an eigenbasis. Conversely, if there is an eigenbasis β of A then by (1.) of Proposition 8.1.11 we have β is an eigenbasis of L_A and thus, using the coordinate change Proposition 7.5.7 once more, $[L_A]_{\beta,\beta} = [\beta]A[\beta]^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n)$. $\square$

The proof above outlines a procedure for diagonalizing a matrix:

(i.) given A find an eigenbasis $\beta = \{v_1, \dots, v_n\}$ for A,

(ii.) construct $[\beta] = [v_1|\cdots|v_n]$ and calculate $[\beta]^{-1}$,

(iii.) construct $D = [\beta]A[\beta]^{-1}$ where D is a diagonal matrix whose diagonal entries are precisely the eigenvalues $\lambda_1, \dots, \lambda_n$ for $v_1, \dots, v_n$ respective.

If we know how to diagonalize A then we can easily diagonalize A^2. Consider, if $D = PAP^{-1}$ then $A = P^{-1}DP$ hence

$$A^2 = (P^{-1}DP)(P^{-1}DP) = P^{-1}D^2P$$

thus $D^2 = PA^2P^{-1}$. But,

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \Rightarrow \quad D^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix}$$

Consequently we find A^2 is diagonalizable with eigenvalues which are squares of the eigenvalues of A. Moreover, the same eigenbasis diagonalizes both A and A^2. This calculation can be generalized to A^k for any $k \in \mathbb{N}$, or, for invertible A for any $k \in \mathbb{Z}$:

Proposition 8.1.12.

If $A \in \mathbb{F}^{n\times n}$ is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$ then A^k is diagonalizable with eigenvalues $\lambda_1^k, \dots, \lambda_n^k$ for $k \in \mathbb{N}$ (if A^{-1} exists the results holds for $k \in \mathbb{Z}$).

Proof: homework for the reader. There are numerous ways to solve this proof given the theorems we've discovered thus far. $\square$

Clearly we would like to find when a given λ is in fact and eigenvalue. Fortunately, this is easily addressed via the theory of determinants. Let us adopt the notation $c\text{Id}_V = c$. For example, $Id_V = 1$ in this streamilined notation (I use this in part (b.) of the Theorem below).

Theorem 8.1.13. *Equations* CHARACTERIZING *eigenvalues of matrix and operators:*

(a.) If $A \in \mathbb{F}^{n\times n}$ then $\lambda \in \mathbb{F}$ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$,

(b.) If $T : V \to V$ is a linear transformation on a finite dimensional vector space $V(\mathbb{F})$ then $\lambda \in \mathbb{F}$ is an eigenvalue for T if and only if $\det(T - \lambda) = 0$.

Proof: Observe, $\det(A-\lambda I) = 0$ if and only if $(A-\lambda I)^{-1}$ does not exist. However, nonexistence of the inverse of $A - \lambda I$ is equivalent to the existence of multiple solutions to the $(A - \lambda I)v = 0$ equation. In short, $\det(A - \lambda I) = 0$ if and only if there exists a nonzero solution v with $(A - \lambda I)v = 0$. Therefore, $\det(A - \lambda I) = 0$ if and only λ is an eigenvalue of A.

Suppose $\det(T - \lambda) = 0$. Hence, for any basis β of $V(\mathbb{F})$ we have:

$$\det(T - \lambda) = \det([T - \lambda]_{\beta,\beta}) = \det([T]_{\beta,\beta} - \lambda I) = 0$$

thus by (a.) we find $\det(T - \lambda) = 0$ if and only if λ is an eigenvalue of $[T]_{\beta,\beta}$. Hence by (2.) of Proposition 8.1.11 we find $\det(T - \lambda) = 0$ if and only if λ is an eigenvalue of T. $\square$

Definition 8.1.14.

Let $A \in \mathbb{F}^{n\times n}$ we define the **characteristic polynomial** of A via:

$$p(x) = \det(A - xI)$$

and the **characteristic equation** of A is $\det(A - xI) = 0$. Likewise, the characteristic polynomial of $T : V \to V$ is $p(x) = \det(T - x)$ and the characteristic equation for T is $\det(T - x) = 0$.

Solutions in $\mathbb{F}$ to the characteristic equations are eigenvalues for A or T. Furthermore, we can calculate that $\deg(p(x)) = n$ when $\dim(V) = n$. Therefore, by the theory of polynomial algebra we find the following general guidelines:

Proposition 8.1.15.

If $A \in \mathbb{F}^{n\times n}$ then $p(x) = \det(A - xI)$ has at most n distinct zeros.
If $A \in \mathbb{C}^{n\times n}$ then, allowing repeats, A has n-eigenvalues.

Revisiting Example 8.1.6 we can explain algebraically rather than geometrically:

Example 8.1.16. *Let* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *hence*

$$det(A - xI) = det \begin{bmatrix} \cos\theta - x & -\sin\theta \\ \sin\theta & \cos\theta - x \end{bmatrix} = (\cos\theta - x)^2 + \sin^2\theta = (x - \cos\theta)^2 + \sin^2\theta$$

Thus $\lambda = \cos\theta \pm i\sin\theta$ are eigenvalues of A. If $A \in \mathbb{R}^{2\times 2}$ then $\lambda \in \mathbb{R}$ only if $\theta = n\pi$ for some $n \in \mathbb{Z}$. If $A \in \mathbb{C}^{2\times 2}$ then $\lambda = e^{i\theta} = \cos\theta + i\sin\theta$ are eigenvalues of A. Given the theory we've studied in this section we find A is diagonalizable via a complex similarity transformation, but, A is not diagonlizable via purely real matrices (unless $\theta = n\pi$ for $n \in \mathbb{Z}$ in which case $A = \pm I$).

We may use λ in the place of x for the characteristic equation. I typically use λ.

Example 8.1.17. *Let $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ find the e-values and e-vectors of A. Demonstrate how A is diagonalized and use the diagonalization to derive a formula for A^n.*

$$det(A - \lambda I) = det\begin{bmatrix} 3-\lambda & 1 \\ 3 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) - 3 = \lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$$

We find $\lambda_1 = 0$ and $\lambda_2 = 4$. Thus[2]

$$\mathcal{E}_0 = Null(A) = Null\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = span\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$$
$$\mathcal{E}_4 = Null(A - 4I) = Null\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} = span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

We can diagonalize A via the eigenbasis $\beta = \{(1,1),(1,-3)\}$ for which $[\beta] = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$ and, by the 2×2 formula for the inverse, $[\beta]^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$. Calculate, $[\beta]A[\beta]^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = D$. We can calculate the n-th power of A: solve for $A = [\beta]D[\beta]^{-1}$ hence[3]

$$A^n = [\beta]D^n[\beta]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 4^n & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 4^n & 0 \\ 4^n & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3(4^n) & 4^n \\ 3(4^n) & 4^n \end{bmatrix} = 4^{n-1}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}.$$

For this example you probably could have found the formula above with a little experimentation. Notice the method of the example generalizes to larger matrices without much more work.

I should introduce some terminology to aid our study of eigenvectors.

Definition 8.1.18.

A polynomial $f(x) \in \mathbb{F}[x]$ is **split** over $\mathbb{F}$ if $f(x)$ factors into linear factors (possibly repeated) over $\mathbb{F}$; that is, for $f(x) = a_n x^n + \cdots + c_1 x + c_0$ there exist $r_1, \dots, r_n$ for which

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

To say $f(x)$ is split in $\mathbb{F}[x]$ is to say that all the zeros of $f(x)$ are found in $\mathbb{F}$.

The definition above is not minimal, I inserted the logical result of the **factor theorem** of high-school algebra into the definition for pedagogical clarity. Since the zeros of the characteristic polynomial are the eigenvalues we have the following result:

[2] omitting details of this standard calculation, remember we learned how to calculate the basis for null spaces early in our study

[3] since $A^n = AA\cdots AA = [\beta]D[\beta]^{-1}[\beta]D[\beta]^{-1}\cdots[\beta]D[\beta]^{-1}[\beta]D[\beta]^{-1} = [\beta]DD\cdots DD[\beta]^{-1} = [\beta]D^n[\beta]^{-1}$.

Proposition 8.1.19.

If $p(x) = \det(A - xI) = c_n x^n + \cdots + c_1 x + c_0$ is the characteristic polynomial for $A \in \mathbb{F}^{n\times n}$ is split over $\mathbb{F}$ if and only if there are n-eigenvalues (possibly repeated) for A in $\mathbb{F}$. Moreover, if $f(x)$ is split then $c_0 = \lambda_1\lambda_2\cdots\lambda_n$. The same theorem holds for $T \in \text{End}(V(\mathbb{F}))$.

Proof: the equivalence of $p(x)$ being split and A having n-eigenvalues (possibly repeated) follows immediately from the theory of polynomial factoring and Proposition 8.1.13. Suppose $p(\lambda_j) = 0$ for $j = 1, \dots, n$ then noting the leading term in $p(x)$ is $c_n = (-1)^n$ we have:

$$\begin{aligned} p(x) &= (-1)^n(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \\ &= (-1)^n x^n + \cdots + (-1)^n(-1)^n\lambda_1\lambda_2\cdots\lambda_n \\ &= (-1)^n x^n + \cdots + \lambda_1\lambda_2\cdots\lambda_n \end{aligned}$$

Thus the constant term in a split characteristic polynomial is just the product of the eigenvalues (allowing repeats). I leave the proof for $T \in \text{End}(V(\mathbb{F}))$ to the reader. $\square$

In fact, you can prove the $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$ when $det(A - xI)$ is split. However, we defer proof of this claim for the moment.

Example 8.1.20. *Let* $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ *find the e-values and e-vectors of A.*

$$\begin{aligned} 0 = det(A - \lambda I) = det \begin{bmatrix} -\lambda & 0 & -4 \\ 2 & 4-\lambda & 2 \\ 2 & 0 & 6-\lambda \end{bmatrix} \\ &= (4-\lambda)\big[-\lambda(6-\lambda)+8\big] \\ &= (4-\lambda)\big[\lambda^2 - 6\lambda + 8\big] \\ &= -(\lambda-4)(\lambda-4)(\lambda-2) \end{aligned}$$

Thus we have a repeated e-value of $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = 2$. Observe $det(A) = 32 = (4)^2(2)$ as a check on our calculation. Let's find the eigenvector $u_3 = (u, v, w)$ such that $(A - 2I)u_3 = 0$, we find the general solution by row reduction

$$rref \left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{c} u + 2w = 0 \\ v - w = 0 \end{array} \quad \Rightarrow \quad \boxed{u_3 = w \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

Thus, the $\lambda = 2$ eigenspace is $\mathcal{E}_2 = span\{(-2, 1, 1)\}$. Next find the e-vectors with e-value 4. Let $u_1 = (u, v, w)$ satisfy $(A - 4I)u_1 = 0$. Calculate,

$$rref \left[\begin{array}{ccc|c} -4 & 0 & -4 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad u + w = 0$$

Notice this case has two free variables, we can use v, w as parameters in the solution,

$$u_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -w \\ v \\ w \end{bmatrix} = v \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \boxed{u_1 = v \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_2 = w \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

In fact, the $\lambda = 4$ eigenspace is $\mathcal{E}_4 = span\{(0, 1, 0), (-1, 0, 1)\}$.

Notice, all we did in the Example above was to find eigenvalues and calculate the nullspace of two matrices. I hope the reader identifies the details as the routine calculation of the null-space basis. Of course, finding eigenvalues is a new thing.

Definition 8.1.21.

If λ is an eigenvalue for $A \in \mathbb{F}^{n\times n}$ (or endomorphism T on $V(\mathbb{F})$) then

(1.) **algebraic multiplicity** of λ is the largest power r for which $(x-\lambda)^r$ appears as a factor in $\det(A - xI)$ (or $\det(T - x)$)

(2.) **geometric multiplicity** of λ is $\dim(\mathcal{E}_\lambda) = \dim(\text{Null}(A - \lambda I))$ (or $\dim(\text{Ker}(T - \lambda))$

Observe the algebraic and geometric multiplicity of $\lambda = 4$ were both 2 in Example 8.1.20.

Example 8.1.22. *Let* $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$ *find the e-values and e-vectors of* A.

$$det(A - \lambda I) = det\left[\begin{array}{cc} 1-\lambda & 1 \\ 0 & 1-\lambda \end{array}\right] = (1-\lambda)(1-\lambda) = 0$$

Hence we have a repeated e-value of $\lambda_1 = 1$. *Find all e-vectors for* $\lambda_1 = 1$, *let* $u_1 = [u, v]^T$,

$$(A - I)u_1 = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{c} u \\ v \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \quad \Rightarrow \quad v = 0 \quad \Rightarrow \quad \mathcal{E}_1 = span\{(1,0)\}.$$

The **algebraic multiplicity** *of* $\lambda = 1$ *is two whereas the* **geometric multiplicity** *is* $dim(\mathcal{E}_1) = 1$.

Theorem 8.1.13 assures us that for each solution of the characteristic equation we get an eigenvalue. This means each eigenvalue has at least one eigenvector; $\dim(\mathcal{E}_\lambda) \geq 1$. In fact, we can argue that the geometric multiplicity is at most the algebraic multiplicity:

Theorem 8.1.23.

If T an endomorphism on $V(\mathbb{F})$ has eigenvalue $\lambda \in \mathbb{F}$ with algebraic multiplicity m then $1 \leq \dim(\mathcal{E}_\lambda) \leq m$. Likewise, the geometric multiplicity of an eigenvalue of A is bounded above by the algebraic multiplicity of the eigenvalue of A.

Proof: let $T : V \to V$ be linear with eigenvalue $\lambda \in \mathbb{F}$ and algebraic multiplicity m then by definition of algebraic multiplicity we have $\det(T - x) = (x-\lambda)^m g(x)$ where $g(\lambda) \neq 0$. Let $\mathcal{E}_\lambda$ have basis $\{v_1, \dots, v_p\}$ which means the geometric multiplicity $\dim(\mathcal{E}_\lambda) = p$. Extend the basis for the λ-eigenspace to $\beta = \{v_1, \dots, v_n\}$ for all of V. Notice:

$$[T]_{\beta,\beta} = [[T(v_1)]_\beta|\cdots|[T(v_p)]_\beta|[T(v_{p+1}]_\beta|\cdots|[T(v_n)]_\beta] = [[\lambda v_1]_\beta|\cdots|[\lambda v_p]_\beta|[T(v_{p+1}]_\beta|\cdots|[T(v_n)]_\beta]$$

but $[v_i]_\beta = e_i$ for $i = 1, 2, \dots, p$. Hence, (use B and C to label the last $n - p$ columns)

$$[T]_{\beta,\beta} = \left[\begin{array}{cc} \lambda I_p & B \\ 0 & C \end{array}\right] \quad \Rightarrow \quad \det(T - x) = \det\left([T]_{\beta,\beta} - x[Id]_{\beta,\beta}\right) = \det\left[\begin{array}{cc} \lambda I_p - xI_p & B \\ 0 & C - xI_{n-p} \end{array}\right]$$

Thankfully, a block-matrix determinant identity allows us the following simplification:

$$\det(T - x) = \det(\lambda I_p - xI_p)\det(C - xI_{n-p}) = (x-\lambda)^p(-1)^p\det(C - xI_{n-p}).$$

Thus $(x-\lambda)^p$ is a factor of $(x-\lambda)^m g(x)$ where $g(\lambda) \neq 0$. It follows that $p \leq m$. Therefore, $1 \leq p \leq m$ which is to say $1 \leq \dim(\mathcal{E}_\lambda) \leq m$. $\square$

Example 8.1.24. *Let* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and find the e-values and e-vectors of the matrix. Observe that* $det(A-\lambda I) = \lambda^2+1$ *hence the eigevalues are* $\lambda = \pm i$. *Find* $u_1 = (u,v)$ *such that* $(A-iI)u_1 = 0$

$$0 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -iu+v \\ -u-iv \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} -iu+v=0 \\ -u-iv=0 \end{matrix} \quad \Rightarrow \quad v = iu \quad \Rightarrow \quad \mathcal{E}_i = span\{(1,i)\}.$$

Since A *is real if we complex conjugate the equation* $Au_1 = iu_1$ *it yields* $A\bar{u}_1 = -i\bar{u}_1$. *Hence, we find* $\mathcal{E}_{-i} = span\{(1,-i)\}$.

If a real matrix A has eigenvalue $\lambda = \alpha + i\beta$ with $u = a+ib$ eigenvector then $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue with eigenvector $\bar{u} = a - ib$. However, if A is not real then no such simplification is expected:

Example 8.1.25. *Let* $A = \begin{bmatrix} 2i & 0 \\ 0 & 3 \end{bmatrix}$ *has eigenvalues* $\lambda_1 = 2i$ *and* $\lambda_2 = 3$ *over* $\mathbb{C}$. *Notice,* $A(1,0) = 2i(1,0)$ *whereas* $A(0,1) = 3(0,1)$ *thus* $\mathcal{E}_{2i} = span\{(1,0)\}$ *and* $\mathcal{E}_3 = span\{(0,1)\}$.

We may enumerate eigenvalues and refer to the enumeration as labels. This is probably a more common notation than the one I have thus far used.

Example 8.1.26. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *find the e-values and e-vectors of* A *over* $\mathbb{C}$.

$$det(A-\lambda I) = \begin{bmatrix} 1-\lambda & 1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)\big[(1-\lambda)^2+1\big] = 0.$$

Hence $\lambda_1 = 3$, $\lambda_2 = 1+i$ *and* $\lambda_3 = 1-i$. *Begin with* $\lambda_1 = 3$, *find* $u_1 = (u,v,w)$ *such that* $(A-3I)u_1 = 0$:

$$rref\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathcal{E}_1 = span\{(0,0,1)\}.$$

Next find e-vector with $\lambda_2 = 1+i$. *We wish to find* $u_2 = (u,v,w)$ *such that* $(A-(1+i)I)u_2 = 0$:

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -1-i & 0 \end{array}\right] \xrightarrow[\frac{1}{-1-i}r_3 \to r_3]{r_2 + ir_1 \to r_2} \left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

One more row-swap and a rescaling of row 1 and it's clear that

$$rref\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -1-i & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{matrix} u+iv=0 \\ w=0 \end{matrix} \quad \Rightarrow \quad \mathcal{E}_2 = span\{(-i,1,0)\}$$

Then, by complex conjugation we derive $\mathcal{E}_3 = span\{(i,1,0)\}$ *for* $\lambda_3 = 1-i$. *If we form the basis* $\beta = \{(0,0,1),(1,i,0),(1,-i,0)\}$ *then we can calculate* $[\beta]A[\beta]^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$. *In contrast, if we view* A *as a* 3×3 *real matrix and only consider real similarity transformations then we find that* A *is not diagonalizable over* $\mathbb{R}$.

I hope the examples we consider in this section give you a good picture of the many cases we must consider. There are two main things we have to consider as we attempt to diagonalize a real matrix. First, we need to have all real eigenvalues. Second, we need to find enough eigenvectors for each real eigenvalue. Both of these problems lead to interesting constructions. It turns out that complex eigenvalues for a real matrix lead us to find a similarity transformation suggested by the complexification which yields a rotation and dilation. On the other hand, when we find the geometric multiplicity is smaller than the algebraic multiplicity then the complex numbers are no help. We need to develop some machinery for polynomials of operators then we can describe the fix to the missing eigenvectors. When can we be sure there are enough eigenvectors to diagonalize? We give a precise account in the remainder of this section.

Let me present a small argument to ease you into the idea of the proof below. Suppose A has eigenvectors v_1, v_2 with eigenvalues $\lambda_1 \neq \lambda_2$. Suppose $c_1v_1 + c_2v_2 = 0$. Multiply by $(A - \lambda_1 I)$ to obtain:

$$c_1(A - \lambda_1 I)v_1 + c_2(A - \lambda_1 I)v_2 = c_1(\lambda_2 - \lambda_1)v_2 = 0$$

hence $c_1 = 0$ as $\lambda_2 - \lambda_1 \neq 0$ and $v_2 \neq 0$. Hence $c_2v_2 = 0$ which yields $c_2 = 0$. Thus $\{v_1, v_2\}$ is LI.

Theorem 8.1.27. *Linear independence of eigenvectors with distinct eigenvalues:*

> If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of a $A \in \mathbb{F}^{n\times n}$ then the corresponding eigenvectors $v_1, v_2, \dots, v_k$ form a LI set. Likewise, eigenvectors with distinct eigenvalues for $T : V \to V$ form a LI set.

Proof: I begin with a direct proof. Suppose $v_1, v_2, \dots, v_k$ are e-vectors with e-values $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{F}$ such that $\lambda_i \neq \lambda_j$ for all $i \neq j$. Suppose $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$. Multiply by $\Pi_{i=1}^{k-1}(A - \lambda_i I)$,

$$c_1 \prod_{i=1}^{k-1}(A - \lambda_i I)v_1 + \cdots + c_{k-1}\prod_{i=1}^{k-1}(A - \lambda_i I)v_{k-1} + c_k\prod_{i=1}^{k-1}(A - \lambda_i I)v_k = 0 \ \star$$

Consider that the terms in the product commute as:

$$(A - \lambda_i I)(A - \lambda_j I) = A^2 - (\lambda_i - \lambda_j)A + \lambda_i\lambda_j I = (A - \lambda_j I)(A - \lambda_i I).$$

It follows that we can bring $(A - \lambda_j I)$ to the right of the product multiplying the j-th summand:

$$c_1 \prod_{i\neq 1}^{k-1}(A - \lambda_i I)(A - \lambda_1 I)v_1 + \cdots + c_{k-1}\prod_{i\neq k-1}^{k-1}(A - \lambda_i I)(A - \lambda_{k-1} I)v_{k-1} + c_k\prod_{i=1}^{k-1}(A - \lambda_i I)v_k = 0 \ \star^2$$

Notice, for $i \neq j$, $(A - \lambda_j I)v_i = \lambda_i v_i - \lambda_j v_i = (\lambda_i - \lambda_j)v_i \neq 0$ as $\lambda_i \neq \lambda_j$ and $v_i \neq 0$. On the other hand, if $i = j$ then $(A - \lambda_i I)v_i = \lambda_i v_i - \lambda_i v_i = 0$. Therefore, in $\star$ we find that terms with coefficients $c_1, c_2, \dots, c_{k-1}$ all vanish. All that remains is:

$$c_k\prod_{i=1}^{k-1}(A - \lambda_i I)v_k = 0 \ \ \star^3$$

We calculate,

$$\begin{aligned} c_k\prod_{i=1}^{k-1}(A - \lambda_i I)v_k = \prod_{i=1}^{k-2}(A - \lambda_i I)(A - \lambda_{k-1} I)v_k &= c_k(\lambda_k - \lambda_{k-1})\prod_{i=1}^{k-2}(A - \lambda_i I)v_k \\ &= c_k(\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k-2})\prod_{i=1}^{k-3}(A - \lambda_i I)v_k \\ &= c_k(\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k-2})\cdots(\lambda_k - \lambda_1)v_k. \end{aligned}$$

However, as $v_k \neq 0$ and $\lambda_k \neq \lambda_i$ for $i = 1, \dots k-1$ it follows from the identity above that $\star^3$ implies $c_k = 0$. Next, we repeat the argument, except only multiply $\star$ by $\prod_{i=1}^{k-2}(A - \lambda_i)$ which yields $c_{k-1} = 0$. We continue in this fashion until we have shown $c_1 = c_2 = \cdots = c_k = 0$. Hence $\{v_1, \dots, v_k\}$ is linearly independent as claimed. $\square$

I am fond of the argument which was just offered. Technically, it could be improved by including explicit induction arguments in place of $\cdots$. The next argument is standard.

Proof: Let e-vectors $v_1, v_2, \dots, v_k$ have e-values $\lambda_1, \lambda_2, \dots, \lambda_k$. Let us prove the claim by induction on k. Note $k = 1$ and $k = 2$ we have already shown in previous work. Suppose inductively the claim is true for $k-1$. Consider, towards a contradiction, that there is some vector v_j which is a nontrivial linear combination of the other vectors:

$$v_j = c_1 v_1 + c_2 v_2 + \cdots + \widehat{c_j v_j} + \cdots + c_k v_k$$

Multiply by A,

$$Av_j = c_1 Av_1 + c_2 Av_2 + \cdots + \widehat{c_j Av_j} + \cdots + c_k Av_k$$

Which yields,

$$\lambda_j v_j = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + \widehat{c_j \lambda_j v_j} + \cdots + c_k \lambda_k v_k$$

But, we can replace v_j on the l.h.s with the linear combination of the other vectors. Hence

$$\lambda_j \left[c_1 v_1 + c_2 v_2 + \cdots + \widehat{c_j v_j} + \cdots + c_k v_k\right] = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + \widehat{c_j \lambda_j v_j} + \cdots + c_k \lambda_k v_k$$

Consequently,

$$c_1(\lambda_j - \lambda_1)v_1 + c_2(\lambda_j - \lambda_2)v_2 + \cdots + c_j(\widehat{\lambda_j - \lambda_j})v_j + \cdots + c_k(\lambda_j - \lambda_k)v_k = 0$$

However, this is a set of $k-1$ e-vectors with distinct e-values linearly combined to give zero. It follows from the induction claim that each coefficient is trivial. As $\lambda_j \neq \lambda_i$ for $i \neq j$ it is thus necessary that $c_1 = c_2 = \cdots = c_k = 0$. But, this implies $v_j = 0$ which contradicts $v_j \neq 0$ as is known since v_j was assumed an e-vector. Hence $\{v_1, \dots, v_k\}$ is LI as claimed and by induction on $k \in \mathbb{N}$ we find the proposition is true. $\square$

Finally, notice T has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ implies by Proposition 8.1.11 that $[T]_{\beta,\beta}$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ and thus by either proof already offered there are LI eigenvectors $[v_1]_\beta, \dots, [v_k]_\beta \in \mathbb{F}^n$ for $[T]_{\beta,\beta}$. We claim $\{v_1, \dots, v_k\}$ are LI. Note:

$$c_1 v_1 + \cdots + c_k v_k = 0 \quad \Rightarrow \quad c_1[v_1]_\beta + \cdots + c_k[v_k]_\beta = 0 \quad \Rightarrow \quad c_1 = 0, \dots, c_k = 0. \ \square$$

Eigenspaces with distinct eigenvalues are independent subspaces: In fact, we have diagonalizability of T if and only if T is split with each geometric multiplicity matching the algebraic multiplicity.

Theorem 8.1.28.

If $T : V \to V$ is a linear transformation on an n-dimensional vector space V over $\mathbb{F}$ then the following are equivalent:

(1.) T is diagonalizable

(2.) V is a direct sum of eigenspaces of distinct eigenvalues; that is for $\lambda_1, \dots, \lambda_s$ distinct we have $V = \mathcal{E}_{\lambda_1} \oplus \mathcal{E}_{\lambda_2} \oplus \cdots \oplus \mathcal{E}_{\lambda_s}$

(3.) the characteristic polynomial for T is split over $\mathbb{F}$ and for each distinct eigenvalue $\lambda_1, \dots, \lambda_s$ we have equality of the algebraic and geometric multiplicity; that is

$$\det(T - x) = (-1)^n (x - \lambda_1)^{m_1} (x - \lambda_2)^{m_2} \cdots (x - \lambda_s)^{m_s}$$

where $m_i = \dim(\mathcal{E}_{\lambda_i})$ for $i = 1, 2, \dots, s$.

Proof: Assume (1.) is true. If T is diagonalizable then V has an eigenbasis β for T. Arrange $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_s$ where β_j contains eigenvectors of T with common eigenvalue λ_j and we assume $\lambda_i \neq \lambda_j$ for $i \neq j$. Notice $\mathcal{E}_{\lambda_j} = \text{span}(\beta_j)$ and since β is a basis for V we thus find

$$V = \mathcal{E}_{\lambda_1} + \mathcal{E}_{\lambda_2} + \cdots + \mathcal{E}_{\lambda_s}.$$

Suppose $x_j \in \mathcal{E}_{\lambda_j}$ for $j = 1, \dots, s$ and $x_1 + x_2 + \cdots + x_s = 0$. Towards a contradiction, collect the nonzero elements $x_{i_1}, \dots, x_{i_r}$ in this linear combination. Hence $x_{i_1}, \dots, x_{i_r}$ are eigenvectors with distinct eigenvalues $\lambda_{i_1}, \dots, \lambda_{i_r}$ and $x_{i_1} + \cdots + x_{i_r} = 0$ which contradicts the LI given by Theorem 8.1.27. Therefore, all the vectors $x_1, \dots, x_s$ are zero and we have established (5.) of Theorem 7.7.4 from which we conclude $V = \mathcal{E}_{\lambda_1} \oplus \mathcal{E}_{\lambda_2} \oplus \cdots \oplus \mathcal{E}_{\lambda_s}$. In short, (1.) implies (2.).

Assume there exist $\lambda_1, \dots, \lambda_s$ distinct for which $V = \mathcal{E}_{\lambda_1} \oplus \mathcal{E}_{\lambda_2} \oplus \cdots \oplus \mathcal{E}_{\lambda_s}$. Then by Theorem 7.7.4 we may form a basis for V be combining the bases $\beta_1, \dots, \beta_s$ for $\mathcal{E}_{\lambda_1}, \dots \mathcal{E}_{\lambda_s}$ respective. Hence,

$$\dim(\mathcal{E}_{\lambda_1}) + \dim(\mathcal{E}_{\lambda_2}) + \cdots + \dim(\mathcal{E}_{\lambda_s}) = \dim(V) = n$$

Let m_j be the algebraic multiplicity of λ_j for $j = 1, \dots, s$ and note $m_1 + m_2 + \cdots m_s \leq n$ since $\deg(\det(T - x)) = n$. Recall Theorem 8.1.23 provides $1 \leq \dim(\mathcal{E}_{\lambda_j}) \leq m_j$. If there existed j for which $m_j > \dim(\mathcal{E}_{\lambda_j})$ then we would find the degree of the characteristic polynomial was larger than $\dim(V)$. Therefore, $\dim(\mathcal{E}_{\lambda_j}) = m_j$ for $j = 1, 2, \dots, s$. Consequently, (2.) implies (3.).

If we assume (3.) then there exist distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s \in \mathbb{F}$ for which $det(T - x) = (-1)^n (x - \lambda_1)^{m_1} \cdots (x - \lambda_s)^{m_s}$ and each eigenspace $\mathcal{E}_{\lambda_j} = \text{span}(\beta_j)$ where $\#\beta_j = m_j$ for $j = 1, \dots, s$. Thus $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_s$ is an eigenbasis and we find T is diagonalizable.

Thus, (1.) $\Rightarrow$ (2.) $\Rightarrow$ (3.) $\Rightarrow$ (1.) and the equivalence of (1.), (2.) and (3.) follows. $\square$

Corollary 8.1.29.

If $T : V \to V$ is a linear transformation on an n-dimensional vector space V over $\mathbb{F}$ then if T has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$ then

$$\mathcal{E}_{\lambda_1} + \mathcal{E}_{\lambda_2} + \cdots + \mathcal{E}_{\lambda_s} = \mathcal{E}_{\lambda_1} \oplus \mathcal{E}_{\lambda_2} \oplus \cdots \oplus \mathcal{E}_{\lambda_s}$$

Proof: the proof of the independence of distinct eigenspaces is already inside the proof of the preceding theorem. The difference here is that I make no assumption that T is diagonalizable, the eigenspaces need not cover all of V, yet, they are still interesting and useful. They are just part of the larger story we tell next:

8.2 Jordan form

We have seen linear transformations may fail to diagonalize for a variety of reasons. In this Section we describe one generalization. We study $T : V \to V$ where $\dim(V) = n$ or $A \in \mathbb{F}^{n\times n}$ such that the characteristic polynomial splits over $\mathbb{F}$. In other words, we study transformations or matrices which have a full set of eigenvalues. In this context, we describe how to find a basis β for V for which $[T]_{\beta,\beta}$ is in Jordan form. Likewise, we explain how to make a similarity transformation from A to its Jordan form. Diagonal matrices are examples of Jordan forms, however, not every Jordan form matrix is diagonal. Usually there are blocks. Let me begin with the definition of a Jordan form then I'll explain their connection to **generalized eigenvectors** and **generalized eigenspaces**.

Definition 8.2.1.

A $k \times k$ **Jordan block** for eigenvalue λ has the form $J_k(\lambda) = \lambda I_k + N$ where $N = E_{12} + E_{23} + \cdots + E_{k-1,k}$. Explictly,

$$J_1(\lambda) = [\lambda], \quad J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad J_4(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

A matrix M is in **Jordan form** if it is formed by the direct sum of Jordan blocks:

$$M = J_{k_1}(\lambda_1) \oplus J_{k_2}(\lambda_2) \oplus \cdots \oplus J_{k_s}(\lambda_s)$$

The terminology *eigenvalue* is completely reasonable since the eigenvalue of $J_k(\lambda)$ is precisely λ. Moreover, it is simple to verify that $\det(J_k(\lambda) - xI) = (\lambda - x)^k$.

Example 8.2.2. *The matrices below are all in Jordan form.*

$$A_1 = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} = J_3(7) \oplus J_1(7) \quad A_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3+2i & 1 \\ 0 & 0 & 0 & 3+2i \end{bmatrix} = J_2(2) \oplus J_2(3+2i)$$

$$A_3 = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix} = J_3(6) \quad A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = J_2(0) \oplus J_1(2), \quad A_5 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = J_1(1) \oplus J_1(0).$$

Eigenvectors are to diagonal matrices as generalized eigenvectors are to matrices in Jordan form. The next page or two unravels this statement.

Definition 8.2.3.

Let $T : V \to V$ be a linear transformation on a vector space V over $\mathbb{F}$ and suppose $\lambda \in \mathbb{F}$. A nonzero vector $x \in V$ is a **generalized eigenvector** of T corresponding to λ if $(T-\lambda)^p(x) = 0$ for some $p \in \mathbb{N}$. Likewise, a generalized eigenvector with eigenvalue λ for $A \in \mathbb{F}^{n\times n}$ is a nonzero vector $x \in \mathbb{F}^n$ for which $(A-\lambda I)^p x = 0$ for some $p \in \mathbb{N}$.

In particular, each eigenvector with eigenvalue λ for T is a generalized eigenvector with eigenvalue λ for T. If a matrix A is in Jordan form then the standard basis is a basis of generalized eigenvectors for L_A.

Example 8.2.4. *Consider* $A = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$ *observe* $A - 7I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *thus* $(A-7I)e_1 = 0$. *Furthermore,* $(A-7I)e_2 = e_1$ *thus* $(A-7I)^2e_2 = (A-7I)e_1 = 0$. *Also,* $(A-7I)e_3 = e_2$ *hence* $(A-7I)^3e_3 = (A-7I)^2e_2 = (A-7I)e_1 = 0$. *Thus* e_1, e_2, e_3 *are all solutions of* $(A-7I)^3 = 0$ *which means they are generalized eigenvectors corresponding to eigenvalue* $\lambda = 7$.

It is helpful to distinguish between the eigenvectors and generalized eigenvectors in a precise fashion. Thus introduce the **order** of a generalized eigenvector with eigenvalue λ:

Definition 8.2.5.

If $x \in V$ has $(T-\lambda)^p(x) = 0$ yet $(T-\lambda)^{p-1}(x) \neq 0$ then x is a **generalized eigenvector of order** p with eigenvalue λ for T. Likewise, $x \in \mathbb{F}^n$ is a generalized eigenvector of order p for $A \in \mathbb{F}^{n\times n}$ if x is a generalized eigenvector of order p for L_A.

To each Jordan block we have a *chain* of generalized eigenvectors.

Example 8.2.6. *Examine the results of Example 8.2.4,*

$$\begin{aligned}
&(A-7I)^3e_3 = 0, \quad (A-7I)^2e_3 \neq 0 \;\Rightarrow\; e_3 \textit{ is generalized e-vector of order } 3 \textit{ for } \lambda = 7 \\
&(A-7I)^2e_2 = 0, \quad (A-7I)e_2 \neq 0 \;\Rightarrow\; e_2 \textit{ is generalized e-vector of order } 2 \textit{ for } \lambda = 7 \\
&(A-7I)e_1 = 0, \;\Rightarrow\; e_1 \textit{ is generalized e-vector of order } 1 \textit{ for } \lambda = 7:
\end{aligned}$$

Notice, if we begin with e_3 *then we can* **generate** e_2 *and* e_1 *simply by multiplying by* $(A-3I)$. *We say* e_1, e_2, e_3 *is a* **chain of generalized eigenvectors** *of length* 3 *generated by* e_3 *for* $\lambda = 7$

$$e_3 \xrightarrow{\textit{multiply by } A-7I} e_2 \xrightarrow{\textit{multiply by } A-7I} e_1$$

Example 8.2.7. $A = \begin{bmatrix} 8 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ *has generalized eigenvectors* e_1, e_2, e_3 *in a chain and* e_4 *a separate single element chain. You can check:* $A - 8I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *thus it is easy to calculate that:*

$$(A-8I)e_1 = 0 \qquad (A-8I)e_2 = e_1 \qquad (A-8I)e_3 = e_2 \qquad (A-8I)e_4 = 0$$

Thus e_1, e_4 *are eigenvectors with* $\lambda = 8$ *and* e_3 *generates the* $\lambda = 8$ *chain of length* 3:

$$e_3 \xrightarrow{\textit{multiply by } A-8I} e_2 \xrightarrow{\textit{multiply by } A-8I} e_1$$

Example 8.2.8. *Let $T(y) = (D-6)[y]$ where $D = d/dx$ and $y \in span\{v_1, v_2\}$ where $v_1 = e^{6x}$ and $v_2 = xe^{6x}$. Calculate:*

$$T[v_1] = D(e^{6x}) - 6e^{6x} = 0, \qquad T[v_2] = D(xe^{6x}) - 6xe^{6x} = e^{6x} + 6xe^{6x} - 6xe^{6x} = v_1$$

Thus $\beta = \{v_1, v_2\}$ gives matrix $[T]_{\beta,\beta} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = J_2(0)$.

The structure of the examples above is not accidental. Anytime we can find a generalized eigenvector of order p we can always build a chain of length p from the highest order eigenvector.

Lemma 8.2.9. *on chains and blocks via generalized eigenvectors.*

If $T : V \to V$ is a linear transformation with $x \neq 0$ a generalized eigenvector of T of order p with eigenvalue λ then $(T-\lambda)(x), (T-\lambda)^2(x), \dots, (T-\lambda)^{p-1}(x)$ are generalized eigenvectors of order $p-1, p-2, \dots, 1$ respective. Moreover, if we denote $x_j = (T-\lambda)^{p-j}(x)$ for $j = 1, \dots, p$ where $x_p = x$ then $\beta = \{x_1, \dots, x_p\}$ is linearly independent and $W = \text{span}(\beta)$ is a T-invariant subspace for which $T_W : W \to W$ has matrix $[T_W]_{\beta,\beta} = J_p(\lambda)$.

Proof: suppose there exists $\lambda \in \mathbb{F}$ and $x \neq 0$ for which $(T-\lambda)^p(x) = 0$ yet $(T-\lambda)^{p-1}(x) \neq 0$. Observe that for $j = 1, 2, \dots, p-1$ we may calculate

$$(T-\lambda)^p(x) = (T-\lambda)^{p-j}((T-\lambda)^j(x)) = 0$$

Thus $(T-\lambda)^j(x) \in \text{Ker}(T-\lambda)^{p-j}$ for each $j = 1, 2, \dots, p-1$. Notice:

$$(T-\lambda)^{p-j-1}((T-\lambda)^j(x)) = (T-\lambda)^{p-1}(x) \neq 0$$

thus $(T-\lambda)^j(x)$ is a generalized eigenvector of order $p-j$ for $j = 1, 2, \dots, p-1$. Define,

$$x_1 = (T-\lambda)^{p-1}(x), \quad x_2 = (T-\lambda)^{p-2}(x), \ \dots \ , \ x_{p-1} = (T-\lambda)(x), \quad x_p = x.$$

Suppose $c_1x_1 + c_2x_2 + \cdots + c_px_p = 0$. Notice $(T-\lambda)^{p-1}(x_j) = 0$ for $j = 1, 2, \dots, p-1$ hence

$$(T-\lambda)^{p-1}(c_1x_1 + c_2x_2 + \cdots + c_px_p) = c_p(T-\lambda)^{p-1}(x_p) = c_px_{p-1} = 0$$

thus $c_p = 0$. Next, operate by $(T-\lambda)^{p-1}$ on $c_1x_1 + c_2x_2 + \cdots + c_{p-1}x_{p-1} = 0$ to obtain $c_{p-1}x_{p-2} = 0$ hence $c_{p-1} = 0$. Continue in this fashion to obtain $c_1 = 0, c_2 = 0, \dots, c_p = 0$ thus β is linearly independent. Let $W = \text{span}(\beta)$. If $w \in W$ then $w = c_1x_1 + \cdots + c_px_p$. Let us pause to appreciate how the vectors in β are related:

$$(T-\lambda)(x_p) = x_{p-1} \quad \Rightarrow \quad T(x_p) = \lambda x_p + x_{p-1}$$

in fact the story remains the same for $j = 2, 3, \dots, p$

$$(T-\lambda)(x_j) = x_{j-1} \quad \Rightarrow \quad T(x_j) = \lambda x_j + x_{j-1}$$

only when we reach the eigenvector x_1 does the pattern alter to $T(x_1) = \lambda x_1$. In view of these calculations return now to show the T-invariance of W:

$$\begin{aligned} T(w) &= T(c_1x_1 + c_2x_2 + \cdots + c_px_p) \\ &= c_1\lambda x_1 + c_2(\lambda x_2 + x_3) + \cdots + c_p(\lambda x_p + x_{p-1}) \in \text{span}(\beta) = W. \end{aligned}$$

Thus W is T-invariant and so $T_W : W \to W$ is well-defined and we can calculate:

$$\begin{aligned}
[T_W]_{\beta,\beta} &= [[T(x_1)]_\beta|[T(x_2)]_\beta|\cdots|[T(x_p)]_\beta] \\
&= [[\lambda x_1]_\beta|[\lambda x_2 + x_1]_\beta|\cdots|[\lambda x_p + x_{p-1}]_\beta] \\
&= \begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \\
&= J_p(\lambda). \quad \square
\end{aligned}$$

The Lemma above gives us some insight into how we can calculate the Jordan form for a given transformation. We should search for generalized eigenvectors for the given set of eigenvalues. Let me sketch an algorithm for finding all the chains for a given transformation: given $T : V \to V$ a linear transformation where $p(x) = \det(T - x)$ is split with distinct zeros $\lambda_1, \lambda_2, \dots, \lambda_s$ with algebraic multiplicities $m_1, m_2, \dots, m_s$. That is,

$$p(x) = (-1)^n (x - \lambda_1)^{m_1} (x - \lambda_2)^{m_2} \cdots (x - \lambda_s)^{m_s}$$

procede as follows:

(1.) Find chains of generalized eigenvectors for λ_1. Be careful to construct the chains to be linearly independent. When you have a set of chains containing a total of m_1 vectors you may go onto the next step.

(2.) Find chains of generalized eigenvectors for λ_2. Again, be careful the chains are not tangled. When you have a set of chains containing a total of m_2 vectors you may go on, ...

(3.) Finally, find chains of λ_s. When you find m_s total generalized eigenvectors with eigenvalue λ_s then this step is complete.

(4.) Construct a basis β by collecting all the chains. The matrix $[T]_{\beta,\beta}$ will be in Jordan form where the order of the blocks reflects the order in which you adjoined the chains.

Finding linearly independendent chains happens naturally if you work top-down. In other words, first find the largest order possible generalized eigenvector and use it to generate a chain. Then, find the next largest possible eigenvector which is not in the span of the chain just created and use it to create another chain. Continue in this fashion always looking for the largest order eigenvector for λ possible which is not alread covered in a previous chain. That said, it is often possible to work by finding all the eigenvectors then extending to generalized eigenvectors as needed. Just beware, there are obstacles if your eigenvectors happen to be in two chains which are tangled. Sorry if this is vague, see Chapter 7 of Insel Spence and Friedberg for a calculational scheme to sort through the subtlety here. I think I have a pdf from Bill which illustrates some examples you don't want to calculate by hand, much less type into these notes. I don't foresee the subtlety being an issue in this class. In my experience, it is usually permissible to begin with finding all the eigenvectors then going from there. But, I should caution.

Definition 8.2.10.

If $T : V \to V$ is a linear transformation and λ is an eigenvalue of T then K_λ denotes the **generalized eigenspace** of T for eigenvalue λ. In particular,

$$K_\lambda = \{x \in V \mid (T - \lambda)^p(x) = 0, \quad \text{for some } p \in \mathbb{N}\}$$

This terminology allows us to summarize the structure of Jordan forms as it relates to our previous results captured by Theorem 8.1.28

Theorem 8.2.11.

Let $T : V \to V$ be a linear transformation on an n-dimensional vector space V over $\mathbb{F}$ and suppose the characteristic polynomial for T is split over $\mathbb{F}$ with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$ and algebraic multiplicities $m_1, m_2, \dots, m_s$ for which

$$\det(T - x) = (-1)^n(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_s)^{m_s}.$$

Then:

(1.) V is a direct sum of generalized eigenspaces of distinct eigenvalues; that is for $\lambda_1, \dots, \lambda_s$ distinct we have $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_s}$ where $\dim(K_{\lambda_j}) = m_j$ for $j = 1, 2, \dots, s$.

(2.) Each generalized eigenspace K_{λ_j} has a basis β_j formed by chains of generalized eigenvectors with eigenvalue λ_j. Moreover, $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_s$ is a basis of generalized eigenvectors for T and $[T]_{\beta,\beta}$ is a matrix in Jordan form.

(3.) T is diagonalizable if and only if $K_{\lambda_j} = \mathcal{E}_{\lambda_j}$ for each $j = 1, 2, \dots, s$.

Proof: is found in §7.1 and §7.2 of Insel Spence and Friedberg. Of course, this is a central result of modern linear algebra and the proof can be found in dozens of texts at this level. We could prove it, but, I allocate our time elsewhere. $\square$

There is also an analog of eigenspace Corollary 8.1.29:

Corollary 8.2.12.

If $T : V \to V$ is a linear transformation on an n-dimensional vector space V over $\mathbb{F}$ then if T has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$ then

$$K_{\lambda_1} + K_{\lambda_2} + \cdots + K_{\lambda_s} = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_s}$$

This result assures us that chains with different eigenvalues are linearly independent. Finally, one last theoretical tidbit before we do a few examples.

Theorem 8.2.13.

Let $A \in \mathbb{F}^{n\times n}$ have a split characteristic polynomial then A is similar to a J in Jordan form. Moreover, B is similar to A if and only if B and A are similar to the Jordan form J.

Proof: if A has a split characteristic polynomial then L_A also has a split characteristic polynomial hence by part (2.) of Theorem 8.2.11 there exists a basis for which $[L_A]_{\beta,\beta} = J$ where J is a Jordan form. Thus, $[\beta]A[\beta]^{-1} = J$ which shows A is similar to a Jordan form.

Suppose B is similar to A then $B = PAP^{-1}$ for some P hence $B = P([\beta]^{-1}J[\beta])P^{-1} = QJQ^{-1}$ for $Q = P[\beta]^{-1}$ has $Q^{-1} = [\beta]P^{-1}$. Thus B is similar to J. Conversely, if B is similar to J then $B = RJR^{-1}$ and as $[\beta]A[\beta]^{-1} = J$ we find $B = R([\beta]A[\beta]^{-1})R^{-1} = (R[\beta])A(R[\beta])^{-1}$ which shows A is similar to B. $\square$

This gives us a simple method to check if two matrices are similar (well, over $\mathbb{C}$ for now):

> To see if $A, B \in \mathbb{C}^{n\times n}$ are similar just check if they have the same Jordan form (up to reordering of the blocks)

Example 8.2.14. *Let* $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ *are similar matrices since they have the same Jordan form up to reordering of the blocks. We observe* $A = J_2(2) \oplus J_2(3)$ *and* $B = J_2(3) \oplus J_2(2)$*. In fact, it's not too hard to see for* $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = P^{-1}$ *and* $B = PAP^{-1}$*. It's easiest to calculate if you look at the blocks decompositions;*

$$P = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} J_2(2) & 0 \\ 0 & J_2(3) \end{bmatrix}, \quad B = \begin{bmatrix} J_2(3) & 0 \\ 0 & J_2(2) \end{bmatrix}.$$

Of course, this example, like most of my examples in this Chapter, are very easy. I have avoided placing computationally heavy examples in our discussions thus far, but, in general finding the Jordan form is a lengthy calculation. I use Lemma 8.2.9 to guide the calculations in the examples which follow:

Example 8.2.15. *Let* $T(f(x)) = f''(x)$ *for each* $f(x) \in P_4(\mathbb{R})$*. Thus, for* $\beta = \{x^4, x^3, x^2, x, 1\}$*,*

$$T(ax^4 + bx^3 + cx^2 + dx + e) = 12ax^2 + 6bx + 2c \quad \Rightarrow \quad [T]_{\beta,\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

hence $\lambda = 0$ *with multiplicity* 5*. Notice both* 1 *and* x *are in* $\mathcal{E}_0 = Ker(T)$*. Furthermore, as* $T^2(ax^4+bx^3+cx^2+dx+e) = 24a$ *and* $T^3(f(x)) = 0$*. for all* $f(x) \in P_4(\mathbb{R})$ *we see* $f(x) = x^4$ *serves as a generalized eigenvector of order* 3 *with eigenvalue* $\lambda = 0$*. We find the chain* $x^4, 12x^2, 24$*. Notice the chain of* x^4 *picks up our eigenvector* 1*, but, not the* x *eigenvector. The* $Ker(T^2) = span\{1, x, x^2, x^3\}$ *and we know* $1, x$ *are eigenvectors, and* x^2 *is in the* 3*-chain generated by* x^4 *hence the interesting vector in* $Ker(T^2)$ *is* x^3*. Notice,* x^3 *has* $T(x^3) = 6x$*. In summary, construct the Jordan basis*

$\gamma = \{24, 12x^2, x^4, 6x, x^3\}$ *and calculate:*

$$\begin{aligned}
[T]_{\gamma,\gamma} &= [[T(24)]_\gamma | [T(12x^2)]_\gamma | [T(x^4)]_\gamma | [T(6x)]_\gamma | [T(x^3)]_\gamma] \\
&= [0]_\gamma | [24]_\gamma | [12x^2]_\gamma | [0]_\gamma | [6x]_\gamma] \\
&= \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

Example 8.2.16. *Let*[4] $A = \begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{bmatrix}$. *You can calculate* $det(A - xI) = (x-2)^2(x-4)^2$ *hence* $\lambda_1 = 2$ *and* $\lambda_2 = 4$ *both with algebraic multiplicity* 2*. There are three eigenvectors for* A *you can easily verify that*

$$\begin{aligned}
(A - 2I)(2, 1, 0, 2) &= 0, \\
(A - 2I)(0, 1, 2, 0) &= 0, \\
(A - 4I)(0, 1, 1, 1) &= 0
\end{aligned}$$

Moreover,

$$(A - 4I)(1, -1, -1, 0) = (0, 1, 1, 1)$$

hence for $\beta = \{(2, 1, 0, 2), (0, 1, 2, 0), (0, 1, 1, 1), (1, -1, -1, 0)\}$ *we find*

$$[\beta]^{-1} A[\beta] = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right].$$

Example 8.2.17. *Let*[5] $T(f(x)) = -f(x) - f'(x)$ *for* $f(x) \in P_2(\mathbb{R})$*. Calculate,*

$$T(a + bx + cx^2) = -a - bx - cx^2 - b - 2cx = -a - b + (-b - 2c)x - cx^2$$

Thus, for $\beta = \{1, x, x^2\}$ *we find* $[T]_{\beta,\beta} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$ *and identify* $\lambda = -1$ *with algebraic multiplicity* 3*. Observe:*

$$\Rightarrow \ [T]_{\beta,\beta} + I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \ \Rightarrow \ ([T]_{\beta,\beta} + I)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next, $([T]_{\beta,\beta} + I)^3 = 0$ *hence* $e_3 = (0, 0, 1)$ *serves as a generalized eigenvector of order* 3 *with eigenvalue* $\lambda = -1$*. Note,* $([T]_{\beta,\beta} + I)e_3 = -2e_2$ *and* $([T]_{\beta,\beta} + I)(-2e_2) = -2(-e_1) = 2e_1$ *and*

[4] this is Example 3 from section §7.2 of Insel Spence and Friedberg

[5] this is Example 3 from section §7.1 of Insel Spence and Friedberg

$([T]_{\beta,\beta} + I)(2e_1) = 0$. *Thus* $v_1 = (2, 0, 0), v_2 = (0, -2, 0)$ *and* $v_3 = (0, 0, 1)$ *suggest how to define a Jordan basis for* T, *set* $\gamma = \{2, -2x, x^2\}$ *and calculate*

$$\begin{aligned} [T]_\gamma &= [[T(2)]_\gamma|[T(-2x)]_\gamma|[T(x^2)]_\gamma] \\ &= [[-2]_\gamma|[-2-2x]_\gamma|[-x^2-2x]_\gamma] \\ &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

8.3 complexification and the real Jordan form

In this section we study how a given real vector space is naturally extended to a vector space over $\mathbb{C}$. Then we study how a real linear transformation can be complexified. It is interesting to note the construction of the complexification of V as a particular structure on $V \times V$ is the same in essence as Gauss' construction of the complex numbers from $\mathbb{R}^2$.

8.3.1 concerning matrices and vectors with complex entries

To begin, we denote the complex numbers by $\mathbb{C}$. As a two-dimensional real vector space we can decompose the complex numbers into the direct sum of the real and pure-imaginary numbers; $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$. In other words, any complex number $z \in \mathbb{R}$ can be written as $z = a + ib$ for **unique** $a, b \in \mathbb{R}$. It is convenient to define

$$\boxed{\text{If } \lambda = \alpha + i\beta \in \mathbb{C} \quad \text{for } \alpha, \beta \in \mathbb{R} \quad \text{then} \quad Re(\lambda) = \alpha, \quad Im(\lambda) = \beta}$$

The projections onto the real or imaginary part of a complex number are actually linear transformations from $\mathbb{C}$ to $\mathbb{R}$; $Re : \mathbb{C} \to \mathbb{R}$ and $Im : \mathbb{C} \to \mathbb{R}$. Next, complex vectors are simply n-tuples of complex numbers:

$$\boxed{\mathbb{C}^{\,n} = \{(z_1, z_2, \dots, z_n) \mid z_j \in \mathbb{C}\ \}}.$$

Definitions of scalar multiplication and vector addition follow the obvious rules: if $z, w \in \mathbb{C}^{\,n}$ and $c \in \mathbb{C}$ then

$$(z + w)_j = z_j + w_j \qquad (cz)_j = cz_j$$

for each $j = 1, 2, \dots, n$. The complex n-space is again naturally decomposed into the direct sum of two n-dimensional real spaces; $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. In particular, any complex n-vector can be written uniquely as the sum of real vectors are known as the real and imaginary vector components:

$$\boxed{\text{If } v = a + ib \in \mathbb{C}^{\,n} \text{ for } a, b \in \mathbb{R}^n \text{ then} \quad Re(v) = a, \quad Im(v) = b.}$$

Recall $z = x + iy \in \mathbb{C}$ has complex conjugate[6] $\bar{z} = x - iy$. Let $v \in \mathbb{C}^{\,n}$ we define the complex conjugate of the vector v to be $\bar{v}$ which is the vector of complex conjugates;

$$(\bar{v})_j = \overline{v_j}$$

for each $j = 1, 2, \dots, n$. For example, $[1 + i, 2, 3 - i]^* = [1 - i, 2, 3 + i]$. It is easy to verify the following properties for complex conjugation of numbers and vectors:

$$\overline{v + w} = \bar{v} + \bar{w}, \qquad \overline{cv} = \bar{c} \cdot \bar{v}, \qquad \bar{\bar{v}} = v.$$

[6] I'm not using z^* as is sometimes popular in various applications, we reserve $*$ for the **adjoint** discussed in an upcoming chapter

Notice $\mathbb{C}^{m\times n} = \mathbb{R}^{m\times n} \oplus i\mathbb{R}^{m\times n}$ via the real and imaginary part maps $(Re(Z))_{ij} = Re(Z_{ij})$ and $(Im(Z))_{ij} = Im(Z_{ij})$ for all i, j. There is an obvious isomorphism $\mathbb{C}^{\ m\times n} \approxeq \mathbb{R}^{\ 2m\times 2n}$ which follows from stringing out all the real and imaginary parts. Again, complex conjugation is also defined component-wise: $(\overline{X+iY})_{ij} = X_{ij} - iY_{ij}$. It's easy to verify that

$$\overline{Z+W} = \overline{Z} + \overline{W}, \qquad \overline{cZ} = \overline{c}\cdot\overline{Z}, \qquad \overline{ZW} = \overline{ZW}$$

for appropriately sized complex matrices Z, W and $c \in \mathbb{C}$. Conjugation gives us a natural operation to characterize the *reality* of a variable. Let $c \in \mathbb{C}$ then c is **real** iff $\overline{c} = c$. Likewise, if $v \in \mathbb{C}^{\ n}$ then we say that v is **real** iff $\overline{v} = v$. If $Z \in \mathbb{C}^{m\times n}$ then we say that Z is **real** iff $\overline{Z} = Z$. In short, an object is real if all its imaginary components are zero.

8.3.2 the complexification

Suppose V is a vector space over $\mathbb{R}$, we seek to construct a new vector space $V_{\mathbb{C}}$ which is a natural extension of V. In particular, define:

$$V_{\mathbb{C}} = \{(x,y) \mid x, y \in V\}$$

Suppose $(x,y), (v,w) \in V_{\mathbb{C}}$ and $a+ib \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Define:

$$(x,y) + (v,w) = (x+v, y+w) \qquad \& \qquad (a+ib)\cdot(x,y) = (ax - by, ay + bx).$$

I invite the reader to verify that $V_{\mathbb{C}}$ given the addition and scalar multiplication above forms a vector space over $\mathbb{C}$. In particular we may argue $(0,0)$ is the zero in $V_{\mathbb{C}}$ and $1\cdot(x,y) = (x,y)$. Moreover, as $x, y \in V$ and $a, b \in \mathbb{R}$ the fact that V is a real vector space yields $ax - by, ay + bx \in V$. The other axioms all follow from transferring the axioms over $\mathbb{R}$ for V to $V_{\mathbb{C}}$. Our current notation for $V_{\mathbb{C}}$ is a bit tiresome. Note $(1+0i)\cdot(x,y) = (x,y)$ and $(0+i)\cdot(x,y) = (-y,x)$. Thus, if we use the notation $(x,y) = x+iy$ then the identities above translate to

$$1\cdot(x+iy) = x+iy \qquad \& \qquad i\cdot(x+iy) = ix - y.$$

and generally, $(a+ib)\cdot(x+iy) = a\cdot x - b\cdot y + i(a\cdot x + b\cdot y)$ for all $a, b \in \mathbb{R}$ and $x, y \in V$. Since $\mathbb{R} \subset \mathbb{C}$ the fact that $V_{\mathbb{C}}$ is a complex vector space automatically makes $V_{\mathbb{C}}$ a real vector space. Moreover, with respect to the real vector space structure of $V_{\mathbb{C}}$, there are two natural subspaces of $V_{\mathbb{C}}$ which are isomorphic to V.

$$V = \{x + i(0) \mid x \in V\} \qquad \& \qquad iV = \{0 + i(x) \mid x \in V\}$$

Note $V + iV = V_{\mathbb{C}}$ and $V \cap (iV) = \{(0,0)\}$ hence $V_{\mathbb{C}} = V \oplus iV$. Here $\oplus$ could be denoted $\oplus_{\mathbb{R}}$ to emphasize it is a direct sum with respect to the real vector space structure of $V_{\mathbb{C}}$. This is perhaps the simplest way to think of the complexification:

> To find the complexification of $V(\mathbb{R})$ we simply consider $V(\mathbb{C})$. In other words, replace the real scalars with complex scalars.

This slogan is just a short-hand for the explicit construction outlined thus far in this section[7]. For future convenience we should agree to work with the notation given below:

[7] Another notation for this is $V_{\mathbb{C}} = \mathbb{C}\otimes V$ where $\otimes$ is the **tensor** product and $\mathbb{C}\otimes V = \{(a+ib)\otimes x \mid a+ib \in \mathbb{C},\ x \in V\}$ and the identification $1\otimes x = x$ allows us to see V as a subset of $\mathbb{C}\otimes V$. I'm not sure this way of thinking adds much at this point. We hopefully discuss $\otimes$ in Abstract Algebra II, see Chapter 10 of Dummit and Foote for the careful contruction of the formal algebraic tensor product $\otimes$

Definition 8.3.1.

If V is a vector space over $\mathbb{R}$ then define $V_{\mathbb{C}}$ to be the **complexification** of V which is a complex vector space $V_{\mathbb{C}} = \{x + iy \mid x, y \in V\}$ where we define:

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= x_1 + x_2 + i(y_1 + y_2) \\ (a + ib) \cdot (x + iy) &= ax - by + i(ay + bx)\end{aligned}$$

and $\text{Re}(x + iy) = x$ and $\text{Im}(x + iy) = y$ for all $x, y, x_1, x_2, y_1, y_2 \in V$ and $a, b \in \mathbb{R}$.

Concretely, $V_{\mathbb{C}} = V \times V$ as described before the Definition and $(x, y) = x + iy$ is the identification which is made. Notice this provides that $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Example 8.3.2. *If $V = \mathbb{R}$ then $V_{\mathbb{C}} = \mathbb{R} \oplus i\mathbb{R} = \mathbb{C}$.*

Example 8.3.3. *If $V = \mathbb{R}^n$ then $V_{\mathbb{C}} = \mathbb{R}^n \oplus i\mathbb{R}^n = \mathbb{C}^n$.*

Example 8.3.4. *If $V = \mathbb{R}^{m \times n}$ then $V_{\mathbb{C}} = \mathbb{R}^{m \times n} \oplus i\mathbb{R}^{m \times n} = \mathbb{C}^{m \times n}$.*

We might notice a simple result about the basis of $V_{\mathbb{C}}$ which is easy to verify in the examples given thus far: if $\text{span}_{\mathbb{R}}(\beta) = V$ then $\text{span}_{\mathbb{C}}(\beta) = V_{\mathbb{C}}$.

Proposition 8.3.5.

Suppose V over $\mathbb{R}$ has basis $\beta = \{v_1, \dots, v_n\}$ then β is also a basis for $V_{\mathbb{C}}$

Proof: let $\beta = \{v_1, \dots, v_n\}$ be a basis for V over $\mathbb{R}$. Notice, $\beta \subset V_{\mathbb{C}}$ under the usual identification of $V \leq V_{\mathbb{C}}$ as described in this section. Let $z \in V_{\mathbb{C}}$ then there exist $x, y \in V$ for which $z = x + iy$. Moreover, as $x, y \in \text{span}_{\mathbb{R}}(\beta)$ there exits $x_j, y_j \in \mathbb{R}$ for which $x = \sum_{j=1}^n x_j v_j$ and $y = \sum_{j=1}^n y_j v_j$. Thus,

$$z = x + iy = \sum_{j=1}^n x_j v_j + i \sum_{j=1}^n y_j v_j = \sum_{j=1}^n (x_j + iy_j) v_j \in \text{span}_{\mathbb{C}}(\beta)$$

Therefore, β is a generating set for $V_{\mathbb{C}}$. To prove linear independence of β over $\mathbb{C}$ suppose $c_j = a_j + ib_j$ are complex constants with real parts $a_j \in \mathbb{R}$ and imaginary coefficients $b_j \in \mathbb{R}$. Consider,

$$\sum_{j=1}^n c_j v_j = 0 \quad \Rightarrow \quad \sum_{j=1}^n (a_j + ib_j) v_j = \sum_{j=1}^n a_j v_j + i \left(\sum_{j=1}^n b_j v_j \right) = 0$$

Therefore, both $\sum_{j=1}^n a_j v_j = 0$ and $\sum_{j=1}^n b_j v_j = 0$. By the real LI of β we find $a_j = 0$ and $b_j = 0$ for all $j \in \mathbb{N}_n$ hence $c_j = a_j + ib_j = 0$ for all $j \in \mathbb{N}_n$ and we conclude β is linearly independent in $V_{\mathbb{C}}$ and thus β is a basis for the complex vector space $V_{\mathbb{C}}$. $\square$

If we view $V_{\mathbb{C}}$ as real vector space and if β is a basis for V then $\gamma = \beta \cup i\beta$ is a natural basis for $V_{\mathbb{C}}$. In particular, if $z = x + iy \in V_{\mathbb{C}}$ then

$$[z]_\gamma = [x]_\gamma + [iy]_\gamma = ([x]_\beta, 0) + (0, [y]_\beta) = ([x]_\beta, [y]_\beta).$$

Although, it is often useful to order the real basis for $V_{\mathbb{C}}$ as follows: given $\beta = \{v_1, v_2, \dots, v_n\}$ construct $\beta_{\mathbb{C}}$ as $\beta_{\mathbb{C}} = \{v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n\}$. These are the common methods to induce a real basis for $V_{\mathbb{C}}$ from a given real basis for V. In Problem 46 of Spring 2017 we explored these two options in a particular example.

Example 8.3.6. *If $V = \mathbb{R}[t]$ then $V_{\mathbb{C}} = \mathbb{R}[t] \oplus i\mathbb{R}[t] = \mathbb{C}[t]$. Likewise for polynomials of limited degree. For example $W = P_2$ is given by $span_{\mathbb{R}}\{1, t, t^2\}$ whereas $W_{\mathbb{C}} = span_{\mathbb{R}}\{1, i, t, it, t^2, it^2\}$.*

From a purely complex perspective viewing an n-complex-dimensional space as a $2n$-dimensional real vector space is strange. However, in the application we are most interested, the complex vector space viewed as a real vector space yields data of interest to our study. We are primarily interested in solving real problems, but a complexification of the problem at times yields a simpler problem which is easily solved. Once the complexification has served its purpose of solvablility then we have to drop back to the context of real vector spaces. This is the game plan, and the reason we are spending some effort to discuss the complexification technique.

Example 8.3.7. *If $V = \mathcal{L}(U, W)$ then $V_{\mathbb{C}} = \mathcal{L}(U, W) \oplus i\mathcal{L}(U, W)$. If $T \in V_{\mathbb{C}}$ then $T = L_1 + iL_2$ for some $L_1, L_2 \in V$. However, if β is a basis for U then β is a complex basis for $U_{\mathbb{C}}$ thus T extends uniquely to a complex linear map $T_{\mathbb{C}} : U_{\mathbb{C}} \to W_{\mathbb{C}}$. Therefore, we find $V_{\mathbb{C}} = \mathcal{L}_{\mathbb{C}}(U_{\mathbb{C}}, W_{\mathbb{C}})$*

Example 8.3.8. *An application of the last example: if $V = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ then $V_{\mathbb{C}} = \mathcal{L}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m)$.*

The last example brings us to the main-point of this discussion. If we consider $T : \mathbb{R}^n \to \mathbb{R}^m$ and we extend to $T_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^m$ then this simply amounts to allowing the matrix of T be complex.

Definition 8.3.9.

If $T : V \to V$ is a linear transformation over $\mathbb{R}$ then the **complexification of** T is $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ where $V_{\mathbb{C}} = V \oplus iV$ defined for all $x, y \in V$ by:

$$T_{\mathbb{C}}(x + iy) = T(x) + iT(y).$$

This idea is at times tacitly used without any explicit mention of the complexification. In view of our discussion in this section that omission is not too dangerous. Indeed, that is why in other courses I at times just *allow* the variable to be complex. This amounts to the complexification procedure defined in this section. Let me illustrate the process with a particular example from the introductory differential equations course:

Example 8.3.10. complexification for 2nd order constant coefficient problem: *To solve $ay'' + by' + cy = 0$ we try to use the* **real** *solution $y = e^{\lambda t}$. Since $y' = \lambda e^{\lambda t}$ and $y'' = \lambda^2 e^{\lambda t}$*

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0$$

then as $e^{\lambda t} \neq 0$ we can divide by it to reveal the[8] **auxillary equation** *of $a\lambda^2 + b\lambda + c = 0$. If the solution to the auxillary equation is real then we get at least one solution. For example, $y'' + 3y' + 2y = 0$ gives $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ this $\lambda_1 = -1$ and $\lambda_2 = -2$ and the general solution is simply $y = c_1 e^{-t} + c_2 e^{-2t}$.*

On the other hand, if we try the same real approach to solve $y'' + y = 0$ then we face $\lambda^2 + 1 = 0$ which has no real solutions. Therefore, we complexify the problem and study $z'' + z = 0$ where $z = x + iy$ and x, y are real-valued functions of t. Conveniently, if $\lambda = \alpha + i\beta$ is complex then $z = e^{\lambda t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$ and we can derive $z' = \lambda e^{\lambda t}$ and $z'' = \lambda^2 e^{\lambda t}$ hence, by the same argument as in the real case, we look for (possibly complex) solutions of $a\lambda^2 + b\lambda + c = 0$. Returning to our $y'' + y = 0$ example, we now solve $\lambda^2 + 1 = 0$ to obtain $\lambda = \pm i$. It follow that $z_1 = e^{it}$ and

[8] I usually call it the characteristic equation, but, I'd rather not at the moment

$z_2 = e^{-it}$ are complex solutions for $z'' + z = 0$. Notice, $z = x + iy$ has $z' = x' + iy'$ and $z'' = x'' + iy''$ thus $z'' + z = 0$ implies $x'' + iy'' + x + iy = 0$ thus $x'' + x + i(y'' + y) = 0$ (call this $\star$). Notice the real and imaginary components of $\star$ must both be zero hence $x'' + x = 0$ and $y'' + y = 0$. Notice, for $z = e^{it} = \cos t + i \sin t$ we have $x = \cos t$ and $y = \sin t$. It follows that $\cos t$ and $\sin t$ are **real** *solutions to $y'' + y = 0$. Indeed, the general solution to $y'' + y = 0$ is $y = c_1 \cos t + c_2 \sin t$. To summarize: we take the given real problem, extend to a corresponding complex problem, solve the complex problem using the added algebraic flexibility of $\mathbb{C}$, then extract a pair of real solutions from the complex solution. You might notice, we didn't need to use e^{-it} since $e^{-it} = \cos t - i \sin t$ only has data we already found in e^{it}.*

The case $y'' + 2y' + y = 0$ is more troubling. Here $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$ hence $\lambda = -1$ twice. We only get $y = e^{-t}$ as a solution. The other solution $y = te^{-t}$ arises from a generalized eigenvector for a system of differential equations which corresponds to $y'' + 2y' + y = 0$. The reason for the t is subtle. We will discuss this further when we study systems of differential equations.

By now it should be clear that as we consider problems of real vector spaces the general results, especially those algebraic in nature, invariably involve some complex case. However, technically it usually happens that the construction from which the complex algebra arose is no longer valid if the algebra requires complex solutions. The technique to capture data in the complex cases of the real problems is to **complexify** the problem. What this means is we replace the given vector spaces with their complexifications and we extend the linear transformations of interest in the same fashion. It turns out that solutions to the complexification of the problem reveal both the real solutions of the original problem as well as complex solutions which, while not real solutions, still yield useful data for unwrapping the general real problem. If this all seems a little vague, don't worry, we will get into all the messy details for the eigenvector problem.

Definition 8.3.11.

If $T : V \to V$ is a linear transformation over $\mathbb{R}$ then if $v \in V_{\mathbb{C}}$ is a nonzero vector and $\lambda \in \mathbb{C}$ for which $T_{\mathbb{C}}(v) = \lambda v$ then we say v is a **complex eigenvector with eigenvalue λ for T**. Furthermore, if $A \in \mathbb{R}^{n \times n}$ and $v \in \mathbb{C}^n$ is nonzero with $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ then we say v is a **complex eigenvector** with **complex eigenvalue** λ of the real matrix A.

Technically, a complex eigenvector for real linear transformation is not an eigenvector (because we only allow $\lambda \in \mathbb{R}$ for $T : V \to V$ where V is a real vector space). On the other hand, a complex eigenvector for a real linear transformation **is** an eigenvector for $T_{\mathbb{C}}$. Of course, $T \neq T_{\mathbb{C}}$ so logically this ought not be terribly troubling.

Example 8.3.12. *Consider $T = D$ where $D = d/dx$. If $\lambda = \alpha + i\beta$ then $e^{\lambda x} = e^{\alpha x}(\cos(\beta x) + i \sin(\beta x)$ by definition of the complex exponential. It is first semester calculus to show $D_{\mathbb{C}}(e^{\lambda x}) = \lambda e^{\lambda x}$. Thus $e^{\lambda x}$ is a complex e-vector of $T_{\mathbb{C}}$ with complex e-value λ. In other words, $e^{\lambda x}$ for complex λ are complex eigenfunctions of the differentiation operator.*

Suppose $\beta = \{f_1, \dots, f_n\}$ is a basis for V; $\text{span}_{\mathbb{R}}(\beta) = V$. Then, Proposition 8.3.5 showed us that β also serves as a complex basis for $V_{\mathbb{C}}$, $\text{span}_{\mathbb{C}}(\beta) = V_{\mathbb{C}}$. It follows that the matrix of $T_{\mathbb{C}}$ with respect to β over $\mathbb{C}$ is the same as the matrix of T with respect to β over $\mathbb{R}$. In particular:

$$[T_{\mathbb{C}}(f_i)]_\beta = [T(f_i)]_\beta.$$

Suppose v is a complex e-vector with e-value λ then note $T_{\mathbb{C}}(v) = \lambda v$ implies $[T_{\mathbb{C}}]_{\beta,\beta}[v]_\beta = \lambda [v]_\beta$ where $[v]_\beta \in \mathbb{C}^n$. However, $[T_{\mathbb{C}}]_{\beta,\beta} = [T]_{\beta,\beta}$. Conversely, if $[T]_{\beta,\beta}$ viewed as a matrix in $\mathbb{C}^{n\times n}$ has complex e-vector w with e-value λ then $v = \Phi_\beta^{-1}(w)$ is a complex e-vector for $T_{\mathbb{C}}$ with e-value λ. My point is simply this: we can exchange the problem of complex e-vectors of T for the associated problem of finding complex e-vectors of $[T]_{\beta,\beta}$. Just as we found in the case of real e-vectors it suffices to study the matrix problem. That is:

Proposition 8.3.13.

Let V be a finite dimensional real vector space and suppose $T : V \to V$ is a linear transformation. Then, T has complex eigenvalue λ if and only if $[T]_{\beta,\beta}$ has complex eigenvalue for any basis β of V.

The complex case is different than the real case for one main reason: the complex numbers are an algebraically closed field. In particular we have the Fundamental Theorem of Algebra:[9] if $f(x)$ is an n-th order polynomial complex coefficients then the equation $f(x) = 0$ has n-solutions where some of the solutions may be repeated. Moreover, if $f(x)$ is an n-th order polynomial with real coefficients then complex solutions to $f(x) = 0$ come in conjugate pairs. It follows that any polynomial with real coefficients can be factored into a unique product of repeated real and irreducible quadratic factors. The proof of the Fundamental Theorem of Algebra is often given in the complex analysis course. We should examine one aspect of the theorem, if $f(x) = c_0 + c_1x + \cdots + c_nx^n$ is a polynomial with real coefficients and $\lambda = \alpha + i\beta$ has $f(\lambda) = 0$ then as $\overline{c_j} = c_j$ for $j = 0, \dots, n$,

$$c_0 + c_1\lambda + \cdots + c_n\lambda^n = 0 \quad \Rightarrow \quad c_0 + c_1\overline{\lambda} + \cdots + c_n\overline{\lambda}^n = 0$$

by properties of complex conjugation. Thus, $f(\overline{\lambda}) = 0$. Essentially the same argument forces complex eigenvalues of a complexified real operator to come in conjugate pairs:

Proposition 8.3.14.

Let $T : V \to V$ be a real linear transformation. Suppose λ is an eigenvalue for $T_{\mathbb{C}}$ with eigenvector $z \in V_{\mathbb{C}}$ then $\overline{\lambda}$ is an eigenvalue for $T_{\mathbb{C}}$ with eigenvector $\overline{z}$. Likewise, if $\lambda \in \mathbb{C}$ is a complex eigenvalue of $A \in \mathbb{R}^{n\times n}$ with eigenvector $v \in \mathbb{C}^n$ then $\overline{\lambda}$ is a complex eigenvalue of A with eigenvector $\overline{v}$.

Proof: We define $T_{\mathbb{C}}(x + iy) = T(x) + iT(y)$ for all $x, y \in V$ and hence for each $x + iy \in V_{\mathbb{C}}$. Suppose $\lambda = \alpha + i\beta$ is an eigenvalue for $T_{\mathbb{C}}$ with eigenvector $z = x + iy$ then

$$T_{\mathbb{C}}(z) = \lambda z$$

thus,

$$T(x + iy) = (\alpha + i\beta)(x + iy)$$

and algebra provides

$$T(x) + iT(y) = \alpha x - \beta y + i(\alpha y + \beta x).$$

[9] sometimes this is stated as "there exists at least one complex solution to an n-th order complex polynomial equation" then the factor theorem repeated applied leads to the theorem I quote here.

thus $T(x) = \alpha x - \beta y$ and $T(y) = \beta x + \alpha y$. Therefore,

$$\begin{aligned} T_{\mathbb{C}}(\overline{z}) &= T(x - iy) \\ &= T(x) - iT(y) \\ &= \alpha x - \beta y - i(\beta x + \alpha y) \\ &= (\alpha - i\beta)(x - iy) \\ &= \overline{\lambda}\,\overline{z}. \end{aligned}$$

Since $z \neq 0$ implies $\overline{z} \neq 0$ we find $\overline{z}$ is an eigenvector with eigenvalue $\overline{\lambda} \in \mathbb{C}$ for $T_{\mathbb{C}}$. Likewise, for $A \in \mathbb{R}^{n\times n}$, if there exists $0 \neq v \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ for which $Av = \lambda v$ then complex conjugation $\overline{Av} = \overline{\lambda v}$ yields $A\overline{v} = \overline{\lambda}\overline{v}$. $\square$

This is a useful proposition. It means that once we calculate one complex e-vectors we almost automatically get a second e-vector merely by taking the complex conjugate.

8.3.3 real Jordan form

Theorem 8.3.15.

Let $T : V \to V$ be a linear map on a real vector space V and suppose $T_{\mathbb{C}}$ has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\mathrm{Im}(\lambda) = \beta \neq 0$ with eigenvector $z = a + ib$ for $a, b \in V$ then

(1.) $\{a, b\}$ is a (real) linearly independent subset of V,

(2.) $\mathrm{span}\{a, b\}$ is a T-invariant subspace of V

(3.) if $W = \mathrm{span}\{a, b\}$ is given basis $\gamma = \{a, b\}$ then $[T_W]_{\gamma,\gamma} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$.

Proof: (1.) Suppose $z = a + ib$ is eigenvector of $T_{\mathbb{C}}$ with eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$ then by Proposition 8.3.14 we have $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of $T_{\mathbb{C}}$ with eigenvector $a - ib$. Moreover, as $\beta \neq 0$ we have $\overline{\alpha + i\beta} = \alpha - i\beta \neq \alpha + i\beta$ thus $\{z, \overline{z}\}$ form a linearly independent subset of $V_{\mathbb{C}}$ by Theorem 8.1.27. Notice $z + \overline{z} = 2a$ whereas $z - \overline{z} = 2ib$ hence

$$a = \frac{1}{2}(z + \overline{z}) \qquad \& \qquad b = \frac{1}{2i}(z - \overline{z})$$

Suppose $c_1, c_2 \in \mathbb{R}$ and $c_1 a + c_2 b = 0$ hence

$$\frac{c_1}{2}(z + \overline{z}) + \frac{c_2}{2i}(z - \overline{z}) = 0 \quad \Rightarrow \quad (c_1 - ic_2)z + (c_1 + ic_2)\overline{z} = 0$$

Thus, by LI of $\{z, \overline{z}\}$, we find $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$ thus $c_1 = 0$ and $c_2 = 0$ and we find $\{a, b\}$ is a linearly independent subset of V.

Proof: (2. and (3.) continue with the notation given in (1.),

$$T_{\mathbb{C}}(z) = \lambda z \quad \Rightarrow \quad T(a) = \alpha a - \beta b \;\; \& \;\; T(b) = \beta a + \alpha b.$$

thus $T(\mathrm{span}\{a, b\}) \subseteq \mathrm{span}\{a, b\}$. Moreover, setting $\gamma = \{a, b\}$ and $W = \mathrm{span}(\gamma)$ we find

$$[T(a)]_\gamma = [\alpha a - \beta b]_\gamma = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \;\; \& \;\; [T(b)]_\gamma = [\beta a + \alpha b]_\gamma = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Consequently, $[T_W]_{\gamma,\gamma} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. $\square$

Theorem 8.3.16. *form of complex diagonalizable real matrix*

If $A \in \mathbb{R}^{\,n\times n}$ and $\lambda_j = \alpha_j + i\beta_j \in \mathbb{C}$ with $\alpha_j, \beta_j \in \mathbb{R}$ and $\beta_j \neq 0$ is an e-value with e-vector $v_j = a_j + ib_j \in \mathbb{C}^n$ and $a_j, b_j \in \mathbb{R}^n$ for $j = 1, 2 \dots, k$ and $\{v_1, \dots, v_k\}$ are linearly independent in $\mathbb{C}^n$ then $\{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ is a linearly independent set of real vectors. Moreover, if $M(\lambda_j) = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$ and $\gamma = \{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ then

$$[\gamma]A[\gamma]^{-1} = M(\lambda_1) \oplus M(\lambda_2) \oplus \cdots \oplus M(\lambda_k).$$

In particular, for a 2×2 matrix with complex eigenvector $a + ib$ with nonreal eigenvalue $\alpha + i\beta$ we have

$$[a|b]A[a|b]^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Proof: exercise for the reader. $\square$

It might be instructive to note the complexification has a different complex matrix than the real matrix we just exhibited. The key equations are $T_{\mathbb{C}}(v) = \lambda v$ and $T_{\mathbb{C}}(\overline{v}) = \overline{\lambda}\overline{v}$ thus if $\delta = \{v, \overline{v}\}$ is a basis for $V_{\mathbb{C}} = V \oplus iV$ then the complexification $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ has matrix:

$$[T_{\mathbb{C}}]_{\delta,\delta} = \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix}.$$

The matrix above is complex, but it clearly contains information about the linear transformation T of the real vector space V. Next, we study a repeated complex eigenvalue where the complexification is not complex diagonalizable.

Theorem 8.3.17.

If V is an 4-dimensional real vector space and $T : V \to V$ is a linear transformation with repeated complex eigenvalue $\lambda = \alpha + i\beta$ where $\beta \neq 0$ with complex eigenvector $v_1 = a_1 + ib_1 \in V_{\mathbb{C}}$ and generalized complex eigenvector $v_2 = a_2 + ib_2$ where $(T_{\mathbb{C}} - \lambda)(v_2) = v_1$ then the matrix of T with respect to $\gamma = \{a_1, b_1, a_2, b_2\}$ is $[T]_{\gamma,\gamma} = \begin{bmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix}$

Proof: we are given $T_{\mathbb{C}}(v_1) = \lambda v_1$ and $T_{\mathbb{C}}(v_2) = \lambda v_2 + v_1$. We simply need to extract real equations from this data: note $v_1 = a_1 + ib_1$ and $v_2 = a_2 + ib_2$ where $a_1, a_2, b_1, b_2 \in V$ and $\lambda = \alpha + i\beta$. Set $\gamma = \{a_1, b_1, a_2, b_2\}$. Calculate,

$$T_{\mathbb{C}}(a_1 + ib_1) = (\alpha + i\beta)(a_1 + ib_1) = \alpha a_1 - \beta b_1 + i(\beta a_1 + \alpha b_1)$$

$$T_{\mathbb{C}}(a_2 + ib_2) = (\alpha + i\beta)(a_2 + ib_2) + (a_1 + ib_1) = \alpha a_2 - \beta b_2 + a_1 i(\beta a_2 + \alpha b_2 + b_1).$$

Note $T_{\mathbb{C}}(a_1 + ib_1) = T(a_1) + iT(b_1)$ and $T_{\mathbb{C}}(a_2 + ib_2) = T(a_2) + iT(b_2)$. Thus

$$T(a_1) = \alpha a_1 - \beta b_1 \quad \Rightarrow \quad [T(a_1)]_\gamma = (\alpha, -\beta, 0, 0)$$

$$T(b_1) = \beta a_1 + \alpha b_1 \quad \Rightarrow \quad [T(b_1)]_\gamma = (\beta, \alpha, 0, 0)$$

$$T(a_2) = a_1 + \alpha a_2 - \beta b_2 \quad \Rightarrow \quad [T(a_2)]_\gamma = (1, 0, \alpha, -\beta)$$
$$T(b_2) = b_1 + \beta a_2 + \alpha b_2 \quad \Rightarrow \quad [T(b_2)]_\gamma = (0, 1, \beta, \alpha)$$

The theorem follows. $\square$

Once more, I write the matrix of the complexification of T for the linear transformation considered above. Let $\delta = \{v_1, v_2, \overline{v_1}, \overline{v_2}\}$ then

$$[T]_{\delta,\delta} = \left[\begin{array}{cc|cc} \alpha+i\beta & 1 & 0 & 0 \\ 0 & \alpha+i\beta & 0 & 0 \\ \hline 0 & 0 & \alpha-i\beta & 1 \\ 0 & 0 & 0 & \alpha-i\beta \end{array}\right]$$

The next case would be a complex eigenvalue repeated three times. If $\delta = \{v_1, v_2, v_3, \overline{v_1}, \overline{v_2}, \overline{v_3}\}$ where $(T_{\mathbb{C}} - \lambda)(v_3) = v_2$, $(T_{\mathbb{C}} - \lambda)(v_2) = v_1$ and $(T_{\mathbb{C}} - \lambda)(v_1) = 0$. The matrix of the complexification with respect to δ would be:

$$[T_{\mathbb{C}}]_{\delta,\delta} = \left[\begin{array}{ccc|ccc} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \overline{\lambda} & 1 & 0 \\ 0 & 0 & 0 & 0 & \overline{\lambda} & 1 \\ 0 & 0 & 0 & 0 & 0 & \overline{\lambda} \end{array}\right].$$

In this case, if we use the real and imaginary components of v_1, v_2, v_3 as the basis $\gamma = \{a_1, b_1, a_2, b_2, a_3, b_3\}$ then the matrix of $T : V \to V$ will be formed as follows:

$$[T]_{\gamma,\gamma} = \left[\begin{array}{cccccc} \alpha & \beta & 1 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 1 & 0 \\ 0 & 0 & -\beta & \alpha & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & -\beta & \alpha \end{array}\right]. \tag{8.1}$$

The proof is essentially the same as we already offered for the repeated complex eigenvalue case.

The **Kronecker product** or **tensor** product of matrices is a new way of combining matrices. Rather than define it, I'll illustrate by example:

$$\left[\begin{array}{cccc} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \otimes \left[\begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array}\right] + \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

then we can express the matrix in Equation 8.1 more concisely as:

$$[T]_{\gamma,\gamma} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \otimes \left[\begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array}\right] + \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

If T has a 4-chain of generalized complex e-vectors then we expect the pattern continues to:

$$[T]_{\gamma,\gamma} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \otimes \left[\begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array}\right] + \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

The term built from tensoring with the superdiagonal matrix will be **nilpotent**. This $\otimes$ product of matrices is interesting, I'll try to find a few simple exercises for your homework.

Remark 8.3.18.

I'll abstain from writing the general real Jordan form of a matrix. Sufficient to say, it is block diagonal where each block is either formed as discussed thus far in this section or it is a Jordan block. Any real matrix A is similar to a unique matrix in real Jordan form up to the ordering of the blocks.

Example 8.3.19. *To begin let's try an experiment using the e-vector and complex e-vectors for found in Example 8.1.26. We'll perform a similarity transformation based on this complex basis:* $\beta = \{(i,1,0),(-i,1,0),(0,0,1)\}$. *Notice that*

$$[\beta] = \begin{bmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad [\beta]^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then, we can calculate that

$$[\beta]^{-1}A[\beta] = \frac{1}{2}\begin{bmatrix} -i & 1 & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note that A is complex-diagonalizable in this case. Furthermore, A is already in real Jordan form.

We should take a moment to appreciate the significance of the 2×2 complex blocks in the real Jordan form of a matrix. It turns out there is a simple interpretation:

Example 8.3.20. *Suppose $b \neq 0$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We can calculate that $det(A - \lambda I) = (a-\lambda)^2 + b^2 = 0$ hence we have complex eigenvalues $\lambda = a \pm ib$. Denoting $r = \sqrt{a^2+b^2}$ (the modulus of $a+ib$). We can work out that*

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}$$

Therefore, a 2×2 matrix with complex eigenvalue will factor into a dilation by the modulus of the e-value $|\lambda|$ times a rotation by the argument of the eigenvalue. If we write $\lambda = rexp(i\beta)$ then we can identify that $r > 0$ is the modulus and β is an arugment (there is degeneracy here because angle is multiply defined).

Transforming a given matrix by a similarity transformation into real Jordan form is a generally difficult calculation. On the other hand, reading the eigenvalues as well as geometric and algebraic multiplicities is a simple matter given an explicit matrix in real Jordan form.

Example 8.3.21. *Suppose $A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. I can read $\lambda_1 = 2+3i$ with geometric and algebraic multiplicity one and $\lambda_2 = 5$ with geometric multiplicity one and algebraic multiplicity two. Of course, $\lambda = 2-3i$ is also an e-value as complex e-values come in conjugate pairs.*

Example 8.3.22. *Suppose* $A = \left[\begin{array}{cccccc} 0 & 3 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$. *I read* $\lambda_1 = 3i$ *with geometric multiplicity one and algebraic multiplicity two. Also* $\lambda_2 = 5$ *with geometric multiplicity and algebraic multiplicity two.*

Example 8.3.23. *Let* $A = \left[\begin{array}{cccc} 2 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{array}\right]$ *you can calculate* $\lambda = 2 \pm 3i$. *Note:*

$$A = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \otimes \left[\begin{array}{cc} 2 & 3 \\ -3 & 2 \end{array}\right] + \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

I can read $v_1 = e_1 + ie_2$ *has* $Av_1 = (2+3i)v_1$ *and* $v_2 = e_3 + ie_4$ *has* $(A - (2+3i)I)v_2 = v_1$ *from the given matrix.*

Theorem 8.3.24.

If V is an n-dimensional real vector space and $T : V \to V$ is a linear transformation then T has n-complex e-values. If $\zeta_1, \dots, \zeta_s$ are distinct real eigenvalues of T then we may form a real Jordan basis which spans

$$K_{\zeta_1} \oplus K_{\zeta_2} \oplus \cdots \oplus K_{\zeta_s}$$

if the direct sum above is all of V then all the complex e-values of T are actually real eigenvalues of T. However, generallly, we also have distinct nonreal eigenvalues $\alpha_1 + i\beta_1, \alpha_1 - i\beta_1, \dots, \alpha_r + i\beta_r, \alpha_r - i\beta_r$. Suppose $\alpha + i\beta$ is an eigenvalue of $T_{\mathbb{C}}$ and $\{v_1, v_2, \dots, v_k\}$ is a k-chain for λ generated by v_k then if $v_j = a_j + ib_j$ for real vectors a_j, b_j for $j = 1, 2, \dots, k$ then $\gamma = \{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ is a **real Jordan basis** for T and $W = \text{span}(\gamma)$ is a T-invariant subspace of V for which

$$[T_W]_{\gamma,\gamma} = I_k \otimes \left[\begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array}\right] + (E_{12} + E_{23} + \cdots + E_{k-1,k}) \otimes I_2$$

The real Jordan form of T is given by concatenating the real Jordan basis together with the γ-type chains. This form is unique up to a reordering of the blocks. Moreover, if $A \in \mathbb{R}^{n \times n}$ then A is similar to a real Jordan form and two matrices are similar if and only if they are similar to a common real Jordan form.

I have proven many parts of the Theorem above in this Section. Please forgive me if I stop here.

8.4 polynomials and operators

Our goal in this Section is to survey some well-known theorems on polynomials of operators. I omit proofs of the more difficult theorems, but, I provide references. I should mention, the approach to analyzing linear transformations is sometimes centered around the ideas in this Section. For example, see Curtis' *Linear Algebra* where the primary decomposition theorem is derived from polynomial algebra. Our approach in these notes centers around the Jordan form. The theory we begin to uncover in this Section is more general as it is the foundation of the **rational canonical form** which makes no assumptions about roots to polynomials.

Polynomials over the field $\mathbb{F}$ are commonly denoted $\mathbb{F}[x]$. In particular, $f(x) \in \mathbb{F}[x]$ then there exist $c_0, c_1, \dots, c_n \in \mathbb{F}$ for which

$$f(x) = c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0.$$

if $c_n \neq 0$ we say $\deg(f(x)) = n$. Since the sum, composite and scalar multiple of an endomorphism is once more an endomorphism it follows the polynomial of a linear transformation on V is once more a linear transformation on V:

Definition 8.4.1.

Let $f(x) = c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0 \in \mathbb{F}[x]$ and suppose $T : V \to V$ is a linear transformation then define $P(T) : V \to V$ by

$$(P(T))(x) = c_n T^n(x) + \cdots + c_2 T^2(x) + c_1 T(x) + c_0 x$$

for each $x \in V$ where T^n denotes the n-fold composite of T.

Since the identity holds for all $x \in V$ we can also express $P(T)$ as above by

$$P(T) = c_n T^n + \cdots + c_2 T^2 + c_1 T + c_0$$

here $c_0 = c_0 \cdot Id_V$ is defined by $c_0(x) = c_0 \cdot x$ for each $x \in V$. In particular, $c_0 \in \mathrm{End}(V)$ is the transformation of scalar multiplication by c_0. Polynomials in a given operator T behave just like polynomials in a real or complex variable.

Consider, for $a, b \in \mathbb{F}$ and $x \in V$, if $T \in \mathrm{End}(V)$ then

$$\begin{aligned}(T+a)(T+b)(x) &= (T+a)(T(x)+bx) \\ &= T(T(x)) + bT(x) + aT(x) + abx \\ &= (T^2 + (a+b)T + ab)(x)\end{aligned}$$

as $x \in V$ is arbitrary we've shown $(T+a)(T+b) = T^2 + (a+b)T + ab$. Hence $(T+a)(T+b) = (T+b)(T+a)$. Generalizing:

Proposition 8.4.2.

If $f(x), g(x) \in \mathbb{F}[x]$ and $h(x) = f(x)+g(x)$ and $k(x) = f(x)g(x)$ as defined by the usual rules of polynomial addition and multiplication then $h(T) = f(T) + g(T)$ and $k(T) = f(T)g(T)$. Moreover, polynomials in T commute; $f(T)g(T) = g(T)f(T)$.

Proof: If $f(x), g(x) \in \mathbb{F}[x]$ and $n = \max(\deg(f(x)), \deg(g(x)))$ then we can write $f(x) = a_nx^n + \cdots + a_1x + a_0$ and $g(x) = b_nx^n + \cdots + b_1x + b_0$. Thus,

$$h(x) = f(x) + g(x) = (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + a_0 + b_0$$

Hence, for $v \in V$,

$$\begin{aligned}(h(T))(v) &= ((a_n + b_n)T^n + \cdots + (a_1 + b_1)T + a_0 + b_0)\,(v) \\ &= a_nT^n(v) + \cdots + a_1T(v) + a_0v + b_nT^n(v) + \cdots + b_1T(v) + b_0v \\ &= (f(T) + g(T))(v).\end{aligned}$$

thus $h(T) = f(T) + g(T)$ as the identity above holds for all $v \in V$. Consider the monomial cx^k then multiplying cx^k by $f(x)$ gives:

$$cx^kf(x) = ca_nx^{k+n} + \cdots + ca_1x^{k+1} + ca_0x^k$$

consider for $v \in V$,

$$\begin{aligned}(cT^kf(T))(v) &= (cT^k)((f(T))(v)) \\ &= (cT^k)(a_nT^n(v) + \cdots + a_1T(v) + a_0v) \\ &= ca_nT^{k+n}(v) + \cdots + ca_1T^{k+1}(v) + ca_0v \\ &= (ca_nT^{k+n} + \cdots + ca_1T^{k+1} + ca_0T^k)(v)\end{aligned}$$

Hence, $cT^kf(T) = ca_nT^{k+n} + \cdots + ca_1T^{k+1} + ca_0T^k$. I leave the remainder of the proof that $k(T) = f(T)g(T)$ to the reader. I hope the details thus far shown give a good indication of how to procede. The proof that $f(T)g(T) = g(T)f(T)$ follows immediately from the fact that $f(x)g(x) = g(x)f(x)$. $\square$

An important application of this algebra is found in the study of ordinary differential equations:

Example 8.4.3. *Notice $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$ for any $\lambda \in \mathbb{R}$. Let $D = d/dx$ and observe the differential equation $y'' - 3y' + 2y = 0$ can be expressed in operator notation as $(D^2 - 3D + 2)[y] = 0$. Note,*

$$(D^2 - 3D + 2)[y] = (D - 1)(D - 2)[y] = (D - 2)(D - 1)[y]$$

it follows that $y = e^x$ and $y = e^{2x}$ are solutions since $(D - 2)[e^{2x}] = 0$ and $(D - 1)[e^x] = 0$. In fact, $y = c_1e^x + c_2e^{2x}$ forms the general solution to the given differential equation.

I spend several classes in Differential Equations explaining how polynomial factoring allows us to solve the general n-th order constant coefficient differential equation. The foundation of all those arguments is Proposition 8.4.2.

Proposition 8.4.4.

Let V be an n-dimensional vector space over $\mathbb{F}$ and suppose $T : V \to V$ is a linear transformation then there exists a polynomial $f(x)$ for which $f(T) = 0$.

Proof: the dimension of $\mathrm{End}(V)$ as a vector space over $\mathbb{F}$ is n^2. Consider, $\{1, T, T^2, \dots, T^{n^2}\}$ is a set of $n^2 + 1$ vectors hence there is some linear dependence among the vectors

$$c_o + c_1T + c_2T^2 + \cdots + c_{n^2}T^{n^2} = 0$$

where not all c_i are zero. Let $f(x) = c_o + c_1x + c_2x^2 + \cdots + c_{n^2}x^{n^2}$ and the Proposition follows. $\square$

This result is easy to obtain, and, it suggests the following Definition is reasonable; there must exist a smallest degree polynomial $m(x)$ for which $m(T) = 0$:

Definition 8.4.5.

Suppose V is finite dimensional and $T : V \to V$ is a linear transformation. The monic polynomial $m(x)$ of least degree for which $m(T) = 0$ is the **minimal polynomial** of T. The minimal polynomial for $A \in \mathbb{F}^{n\times n}$ is the minimal polynomial of the induced map L_A.

A polynomial is **monic** if it has a leading coefficient of 1. For example, x^2+3x+1 is monic whereas $2x^3 + 1$ is not monic. The minimal polynomial for T contains some useful information about T:

Proposition 8.4.6.

The characteristic polynomial and minimal polynomial share the same zeros.

Proof: see §7.3 of Insel Spence and Friedberg's *Linear Algebra*. Let $p(x)$ be the characteristic polynomial and $m(x)$ the minimal polynomial of $T \in \mathrm{End}(V)$. The argument is roughly this: first note $m(x)$ divides $p(x)$ hence each zero of $m(x)$ is also shared by $p(x)$. Conversely, if λ is an eigenvalue with eigenvector $v \neq 0$ then $m(T) = 0$ hence $m(T)(v) = m(\lambda)v = 0$ thus $m(\lambda) = 0$ hence eigenvalue is a zero of $m(x)$. Therefore, each zero of $p(x)$ is also a zero of $m(x)$. $\square$

Notice there is no requirement that $p(x)$ and $m(x)$ share the same multiplicity for each eigenvalue.

Example 8.4.7. *If T has $p(x) = (x-2)^2(x-3)^2$ then $m(x)$ could be $(x-2)(x-3)$ or $(x-2)^2(x-3)$ or $(x-2)(x-3)^2$ or $(x-2)^2(x-3)^2$. For example, A_1, A_2, A_3, A_4 given below all have $p(x) = det(A - xI) = (x-2)^2(x-3)^2$ whereas $m_1(x) = (x-2)(x-3)$, $m_2(x) = (x-2)^2(x-3)$, $m_3(x) = (x-2)(x-3)^2$ and $m_4(x) = (x-2)^2(x-3)^2$ are the minimal polynomials of A_1, A_2, A_3 and A_4 respective,*

$$A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

You can verify the alleged minimal polynomials by direct calculation.

Proposition 8.4.8.

Let V be an n-dimensional vector space over $\mathbb{F}$ and suppose $T : V \to V$ is a linear transformation and $f(x) \in \mathbb{F}[x]$ then $\mathrm{Ker}(f(T))$ is an invariant subspace of V.

Proof: let $v \in \mathrm{Ker}(f(T))$ and notice

$$(f(T))(T(v)) = (f(T)T)(v) = (Tf(T))(v) = T(f(T)(v)) = T(0) = 0$$

thus $T(v) \in \mathrm{Ker}(f(T))$ and we find $\mathrm{Ker}(f(T))$ is a T-invariant subspace of V. $\square$

Notice the above Proposition has many applications. For example, it provides the T-invariance of $\mathrm{Ker}(T)$ and eigenspaces $\mathrm{Ker}(T - \lambda)$ as well as generalized eigenspaces $\mathrm{Ker}(T - \lambda)^p$. This T-invariance stands behind the direct sum decompositions we have studied in this Chapter. On the other hand, we also have T-invariance of Range(T) for an endomorphism of V. Observe, if $T(x) \in (Range)(T)$ then $T(T(x)) \in \text{Range}(T)$ thus Range(T) is T-invariant. Another way to build a T-invariant subspace of T from a span is as follows:

Definition 8.4.9.

Suppose $T : V \to V$ is a linear transformation and $x \in V$ is a nonzero vector then the T-cyclic subspace generated by x is

$$\langle x \rangle = \text{span}\{x, T(x), T^2(x), T^3(x), \dots\}$$

Notice, for finite dimensional vector space we have $\langle x \rangle \leq V$ hence

$$\langle x \rangle = \text{span}\{x, T(x), \cdots, T^{k-1}(x)\}$$

where $T^k(x)$ has some nontrivial linear dependence:

$$c_0 x + c_1 T(x) + \cdots + c_{k-1} T^{k-1}(x) + T^k(x) = 0$$

Clearly $T(\langle x \rangle) \subset \langle x \rangle$ hence setting $W = \langle x \rangle$ we may consider $T_W : W \to W$ with respect to the **T-cyclic basis** generated by x:

$$\beta_x = \{x, T(x), \dots, T^{k-1}(x)\}$$

Notice, for $j = 1, 2, \dots, k-1$ we have $[T^j(x)]_{\beta_x} = e_{j+1}$ then for $j = k$,

$$[T^k(x)]_{\beta_x} = [-c_0 x - c_1 T(x) - \cdots - c_{k-1} T^{k-1}(x)]_{\beta_x} = (-c_0, -c_1, \dots, -c_{k-1}).$$

We've shown the following:

Theorem 8.4.10.

Let $T : V \to V$ be a linear transformation and suppose $W = \langle x \rangle$ has basis $\beta_x = \{x, T(x), \dots, T^{k-1}(x)\}$ where $c_0 x + c_1 T(x) + \cdots + c_{k-1} T^{k-1}(x) + c_k T^k(x) = 0$ then

$$[T_W]_{\beta_x, \beta_x} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{k-1} \end{bmatrix}$$

The matrix in the Proposition above is known as the **companion matrix** for the polynomial $c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} + c_k x^k = 0$. You can easily show that the characteristic of polynomial of a companion matrix is (up to $\pm$) simply the polynomial from which the matrix is formed

Example 8.4.11. *The companion matrix to $x^3 - 2x^2 + 3x + 1$ is given by* $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$

As is often the case, there is an interesting application of companion matrices to differential equations. Well, almost:

Example 8.4.12. *Consider $y''' - 2y'' + 3y' + y = 0$.* **Reduction of order** *allows us to convert this third order differential equations to three first order differential equations by making the substitutions:*

$$x_1 = y, \quad , x_2 = y', \quad x_3 = y''$$

hence $x_3' = y''' = 2y'' - 3y' - y = -x_1 - 3x_2 + 2x_3$ and

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1 - 3x_2 + 2x_3 \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Notice the matrix of this system of differential equations is formed by the transpose of the companion matrix to $x^3 - 2x^2 + 3x + 1$.

The example above illustrates once again how the study of differential equations naturally intertwines with linear algebra. Personally, I have gained great insight into linear algebra from techniques of differential equations, but, I suspect the usual pattern is the reverse.

Theorem 8.4.13. Cayley Hamilton:

Let V be an n-dimensional vector space over $\mathbb{F}$ and suppose $T : V \to V$ is a linear transformation with characteristic polynomial $p(x) = \det(T - x)$ then $p(T) = 0$.

Proof: see §5.4 in Insel, Spence and Friedberg's *Linear Algebra*. □

This theorem is very nice to know for certain examples. For instance, if $p(x) = \det(A - xI) = x^3 + 3x^2 + x - 1$ then $p(A) = A^3 + 3A^2 + A - I = 0$ hence $A^3 + 3A^2 + A = I$ or $A(A^2 + 3A + I) = I$ which provides $A^{-1} = A^2 + 3A + I$. Furthermore, it can be helpful in calculations where arbitrary matrix powers are of interest:

Example 8.4.14. *I'll explain that $e^X = I + X + \frac{1}{2}X^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ converges for any matrix X in a future chapter. It is easy to argue that $\frac{d}{dt} e^{tA} = Ae^{tA}$. Hence e^{tA} solves the differential equation $X' = AX$. If we have A for which $p(x) = x^2 - 1$ then $A^2 = I$ and we can calculate the matrix exponential directly: note $A^{2k} = (A^2)^k = I$ and $A^{2k+1} = A$ thus:*

$$\begin{aligned} e^{tA} &= I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \frac{1}{4!}t^4A^4 + \cdots \\ &= I(1 + t^2/2 + t^4/4! + \cdots) + A(t + t^3/3! + \cdots) \\ &= I\cosh(t) + A\sinh(t). \end{aligned}$$

I'll discuss systems of differential equations more systematically in a future chapter, our purpose in studying them here is simply to better appreciate the applicability of the theory of linear algebra. There is much more to say about polynomials and operators, if you wish to know more then I can recommend some reading.

8.4.1 a calculation forged in the fires of polynomial algebra

Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V over $\mathbb{F}$. Consider a factorization of the minimal polynomial $m(x)$ into prime power irreducible polynomials; $m(x) = p_1(x)^{n_1} p_2(x)^{n_2} \cdots p_s(x)^{n_s}$. If $\mathbb{F} = \mathbb{C}$ then the prime polynomials are merely linear polynomials whereas $\mathbb{F} = \mathbb{R}$ allows for both linear and irreducible quadratics as prime polynomials. In any event, the fascinating thing is that polynomial algebra in great analogy to integer arithmetic gives us that for each pair of relatively prime polynomials $f(x), g(x)$ there exist polynomials $a(x), b(x)$ for which $a(x)f(x) + b(x)g(x) = 1$. This extends to more items. Consider,

$$q_j(x) = m(x)/p_j(x)^{n_j} = p_1(x)^{n_1} \cdots p_{j-1}(x)^{n_{j-1}} p_{j+1}(x)^{n_{j+1}} \cdots p_s(x)^{n_s}$$

By construction $q_1(x), q_2(x), \dots, q_s(x)$ have no common factor except a constant. That is,

$$\gcd(q_1(x), q_2(x), \dots, q_s(x)) = 1$$

Hence, there exist polynomials $a_1(x), a_2(x), \dots, a_s(x) \in \mathbb{F}[x]$ for which

$$a_1(x)q_1(x) + a_2(x)q_2(x) + \cdots + a_s(x)q_s(x) = 1$$

Therefore,

$$a_1(T)q_1(T) + a_2(T)q_2(T) + \cdots + a_s(T)q_s(T) = Id_V$$

which is a very interesting decomposition of the identity into operators closely tied to the minimal polynomial. Ok, so just what are these $q_j(T)$ operators ?

By construction $m(T) = 0$ and no polynomial of smaller degree annihilates T. Hence, $q_j(T)$ is a nonzero endomorphism. Suppose T has a full set of eigenvalues in the sense that $\det(T - x)$ is split over $\mathbb{F}$. This implies $m(x)$ is split and each $p_j(x) = x - \lambda_j$. It follows that $q_j(T)$ sends every generalized eigenspace K_i to zero except $i = j$. But, $V = K_1 \oplus K_2 \oplus \cdots \oplus K_s$ hence $\text{Range}(q_j(T)) \subset K_j$.

Example 8.4.15. $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ *has characteristic polynomial* $p(x) = (x-3)^2(x-2)^2$ *and minimal polynomial* $m(x) = (x-3)^2(x-2)$ *thus if we set* $\lambda_1 = 3$ *and* $\lambda_2 = 2$ *we find* $q_1(x) = x - 2$ *and* $q_2(x) = (x-3)^2$. *Thus,*

$$q_2(A) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \& \quad q_1(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice $Col(q_1(A)) = K_1$ *and* $Col(q_2(A)) = K_2$. *Also, to find* $a_1(x), a_2(x)$ *for which* $a_1(x)q_1(x) + a_2(x)q_2(x) = 1$ *we could do long division, but, I'm a bit lazy so I'll use this nice website: Bill's Toy where I learn to choose* $a_1(x) = 4 - x$ *and* $a_2(x) = 1$. *Hence*

$$a_1(A)q_1(A) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so it is clear that $a_1(A)q_1(A) + a_2(A)q_2(A) = a_1(A)q_1(A) + q_2(A) = I_4$. *Moreover,* $a_j(A)q_j(A)$ *serves as a projection onto* K_j *for each* j.

This is more exciting when the matrix is further from Jordan form.

Example 8.4.16. $A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$ *has eigenvalues* $\lambda_1 = -5$, $\lambda_3 = 3$, $\lambda_3 = 6$ *hence* $m(x) = (x+5)(x-3)(x-6)$ *and*

$$q_1(x) = (x-3)(x-6), \quad q_2(x) = (x+5)(x-6), \quad q_3(x) = (x+5)(x-3)$$

Hence calculate

$$q_1(A) = (A-3I)(A-6I) = 2\begin{bmatrix} 28 & 22 & -10 \\ 14 & 11 & -5 \\ -14 & -11 & 5 \end{bmatrix}$$

$$q_2(A) = (A+5I)(A-6I) = 2\begin{bmatrix} -4 & 6 & -2 \\ 6 & -9 & 3 \\ 2 & -3 & 1 \end{bmatrix}$$

$$q_3(A) = (A+5I)(A-3I) = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 0 & 12 \\ 16 & 0 & 32 \end{bmatrix}$$

After some calculation, you can find $a_1 = 1/88, a_2 = -1/24, a_3 = 1/33$ *give*

$$a_1(x)q_1(x) + a_2(x)q_2(x) + a_3(x)q_3(x) = 1$$

and you can check

$$\frac{2}{88}\begin{bmatrix} 28 & 22 & -10 \\ 14 & 11 & -5 \\ -14 & -11 & 5 \end{bmatrix} - \frac{2}{24}\begin{bmatrix} -4 & 6 & -2 \\ 6 & -9 & 3 \\ 2 & -3 & 1 \end{bmatrix} + \frac{1}{33}\begin{bmatrix} 1 & 0 & 2 \\ 6 & 0 & 12 \\ 16 & 0 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculations reveal that the eigenspaces for $\lambda_1 = 5$, $\lambda_2 = 3$ *and* $\lambda_3 = 6$ *are respectively:*

$$\mathcal{E}_1 = span\{(-2,-1,1)\}, \quad \mathcal{E}_2 = span\{(-2,3,1)\}, \quad \mathcal{E}_3 = span\{(1,6,16)\},$$

The matrices $\frac{2}{88}\begin{bmatrix} 28 & 22 & -10 \\ 14 & 11 & -5 \\ -14 & -11 & 5 \end{bmatrix}, -\frac{2}{24}\begin{bmatrix} -4 & 6 & -2 \\ 6 & -9 & 3 \\ 2 & -3 & 1 \end{bmatrix}$ *and* $\frac{1}{33}\begin{bmatrix} 1 & 0 & 2 \\ 6 & 0 & 12 \\ 16 & 0 & 32 \end{bmatrix}$ *are projection matrices onto* $\mathcal{E}_1, \mathcal{E}_2$ *and* $\mathcal{E}_3$ *respective. We say* E *is a projection matrix for* $W \leq \mathbb{F}^n$ *if* $E^2 = E$ *and* $Col(E) = W$

This calculation was inspired from my reading in Curtis' *Linear Algebra* in the Spring 2016 linear algebra course. There are other results in Curtis' text which we have not discussed, but, I think placing the emphasis on Jordan forms and their applications is a better match for our course.

8.4.2 a word on quantum mechanics and representation theory

Physically measureable quantities are described by operators and states in quantum mechanics[10]. The operators are linear operators and the states are usually taken to be the eigenvectors with respect to a physical quantity of interest. For example:

$$\hat{p}|p>= p|p> \qquad \hat{J}^2|j>= j(j+1)|j> \qquad \hat{H}|E>= E|E>$$

[10]you can skip this if you're not a physics major, but maybe you're interested despite the lack of direct relevance to your major. Maybe your interested in an education not a degree. I believe this is possible so I write these words

In the above the eigenvalues were $p, j(j+1)$ and E. Physically, p is the momentum, $j(j+1)$ is the value of the square of the magnitude of the total angular momentum and E is the energy. The exact mathematical formulation of the eigenstates of momentum, energy and angular momentum is in general a difficult problem and well-beyond the scope of the mathematics we cover this semester. You have to study Hilbert space which is an infinite-dimensional vector space with rather special properties. In any event, if the physical system has nice boundary conditions then the quantum mechanics gives mathematics which is within the reach of undergraduate linear algebra. For example, one of the very interesting aspects of quantum mechanics is that we can only measure a certain pairs of operators simultaneously. Such operators have eigenstates which are simultaneously eigenstates of both operators at once. The careful study of how to label states with respect to the energy operator (called the Hamiltonian) and some other commuting symmetry operator (like isospin or angular momentum etc...) gives rise to what we call Chemistry. In other words, Chemistry is largely the tabulation of the specific interworkings of eigenstates as the correlate to the energy, momentum and spin operators (this is a small part of what is known as *representation theory* in modern mathematics).

The interplay between physics and symmetry is largely implemented by representation theory. For example, our standard model of particle physics currently is based on viewing particles as having wavefunctions which represent $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$. This encodes the way the particle behaves with respect to the strong, weak and electromagnetic forces. All of these are **gauge theories** at a classical level. I'd love to explain gauge theory to you if you are interested in an independent study after completing Advanced Calculus under my supervision.

Chapter 9

systems of differential equations

Systems of differential equations are found at the base of many nontrivial questions in physics, math, biology, chemistry, nuclear engineering, economics, etc... Consider this, anytime a problem is described by several quantities which depend on time and each other it is likely that a simple conservation of mass, charge, population, particle number,... force linear relations between the time-rates of change of the quantities involved. This means, we get a system of differential equations. To be specific, Newton's Second Law is a system of differential equations. Maxwell's Equations are a system of differential equations. Now, generally, the methods we discover in this chapter will not allow solutions to problems I allude to above. Many of those problems are **nonlinear**. There are researchers who spend a good part of their career just unraveling the structure of a particular partial differential equation. That said, once simplifying assumptions are made and the problem is linearlized one often faces the problem we solve in this chapter. We show how to solve **any** system of first order differential equations with constant coefficients. This is accomplished by the application of Jordan basis for the matrix of the system to the **matrix exponential**. I'm not sure the exact history of the method I show in this chapter. In my opinion, the manner in which the chains of generalized eigenvectors tame the matrix exponential are reason enough to study them.

I should mention, the results of this Chapter allow generalization. We could develop theorems for calculus of an $\mathbb{F}$-valued function of a real variable. But, we content ourselves to focus on $\mathbb{R}$ and $\mathbb{C}$ as is convenient to the applications of interest.

9.1 calculus of matrices

A more apt title would be "calculus of matrix-valued functions of a real variable".

Definition 9.1.1.

A matrix-valued function of a real variable is a function from $I \subseteq \mathbb{R}$ to $\mathbb{R}^{m\times n}$. Suppose $A : I \subseteq \mathbb{R} \to \mathbb{R}^{m\times n}$ is such that $A_{ij} : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable for each i, j then we define

$$\frac{dA}{dt} = \left[\frac{dA_{ij}}{dt}\right]$$

which can also be denoted $(A')_{ij} = A'_{ij}$. We likewise define $\int A dt = [\int A_{ij} dt]$ for A with integrable components. Definite integrals and higher derivatives are also defined componentwise.

Example 9.1.2. *Suppose* $A(t) = \begin{bmatrix} 2t & 3t^2 \\ 4t^3 & 5t^4 \end{bmatrix}$. *I'll calculate a few items just to illustrate the definition above. calculate; to differentiate a matrix we differentiate each component one at a time:*

$$A'(t) = \begin{bmatrix} 2 & 6t \\ 12t^2 & 20t^3 \end{bmatrix} \qquad A''(t) = \begin{bmatrix} 0 & 6 \\ 24t & 60t^2 \end{bmatrix} \qquad A'(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Integrate by integrating each component:

$$\int A(t)dt = \begin{bmatrix} t^2 + c_1 & t^3 + c_2 \\ t^4 + c_3 & t^5 + c_4 \end{bmatrix} \qquad \int_0^2 A(t)dt = \begin{bmatrix} t^2\big|_0^2 & t^3\big|_0^2 \\ & \\ t^4\big|_0^2 & t^5\big|_0^2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 16 & 32 \end{bmatrix}$$

Proposition 9.1.3.

Suppose A, B are matrix-valued functions of a real variable, f is a function of a real variable, c is a constant, and C is a constant matrix then

1. $(AB)' = A'B + AB'$ (product rule for matrices)
2. $(AC)' = A'C$
3. $(CA)' = CA'$
4. $(fA)' = f'A + fA'$
5. $(cA)' = cA'$
6. $(A + B)' = A' + B'$

where each of the functions is evaluated at the same time t and I assume that the functions and matrices are differentiable at that value of t and of course the matrices A, B, C are such that the multiplications are well-defined.

Proof: Suppose $A(t) \in \mathbb{R}^{m\times n}$ and $B(t) \in \mathbb{R}^{n\times p}$ consider,

$$\begin{aligned}
{(AB)'}_{ij} &= \tfrac{d}{dt}((AB)_{ij}) && \text{defn. derivative of matrix} \\
&= \tfrac{d}{dt}(\textstyle\sum_k A_{ik}B_{kj}) && \text{defn. of matrix multiplication} \\
&= \textstyle\sum_k \tfrac{d}{dt}(A_{ik}B_{kj}) && \text{linearity of derivative} \\
&= \textstyle\sum_k \big[\tfrac{dA_{ik}}{dt}B_{kj} + A_{ik}\tfrac{dB_{kj}}{dt}\big] && \text{ordinary product rules} \\
&= \textstyle\sum_k \tfrac{dA_{ik}}{dt}B_{kj} + \sum_k A_{ik}\tfrac{dB_{kj}}{dt} && \text{algebra} \\
&= (A'B)_{ij} + (AB')_{ij} && \text{defn. of matrix multiplication} \\
&= (A'B + AB')_{ij} && \text{defn. matrix addition}
\end{aligned}$$

this proves (1.) as i, j were arbitrary in the calculation above. The proof of (2.) and (3.) follow quickly from (1.) since C constant means $C' = 0$. Proof of (4.) is similar to (1.):

$$\begin{aligned}
{(fA)'}_{ij} &= \tfrac{d}{dt}((fA)_{ij}) && \text{defn. derivative of matrix} \\
&= \tfrac{d}{dt}(fA_{ij}) && \text{defn. of scalar multiplication} \\
&= \tfrac{df}{dt}A_{ij} + f\tfrac{dA_{ij}}{dt} && \text{ordinary product rule} \\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. matrix addition} \\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. scalar multiplication.}
\end{aligned}$$

The proof of (5.) follows from taking $f(t) = c$ which has $f' = 0$. I leave the proof of (6.) as an exercise for the reader. $\square$.

To summarize: the calculus of matrices is the same as the calculus of functions with the small qualifier that we must respect the rules of matrix algebra. The noncommutativity of matrix multiplication is the main distinguishing feature.

Since we're discussing this type of differentiation perhaps it would be worthwhile for me to insert a comment about complex functions here. Differentiation of functions from $\mathbb{R}$ to $\mathbb{C}$ is defined by splitting a given function into its real and imaginary parts then we just differentiate with respect to the real variable one component at a time. For example:

$$\begin{aligned} \frac{d}{dt}(e^{2t}\cos(t) + ie^{2t}\sin(t)) &= \frac{d}{dt}(e^{2t}\cos(t)) + i\frac{d}{dt}(e^{2t}\sin(t)) \\ &= (2e^{2t}\cos(t) - e^{2t}\sin(t)) + i(2e^{2t}\sin(t) + e^{2t}\cos(t)) \\ &= e^{2t}(2+i)(\cos(t) + i\sin(t)) = (2+i)e^{(2+i)t}. \end{aligned}$$

where I have made use of the identity[1] $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. We just saw that $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$ which seems obvious enough until you appreciate that we just proved it for $\lambda = 2+i$. We make use of this calculation in the next section in the case we have complex e-values.

9.2 introduction to systems of linear differential equations

A differential equation (DEqn) is simply an equation that is stated in terms of derivatives. The highest order derivative that appears in the DEqn is called the *order* of the DEqn. In calculus we learned to integrate. Recall that $\int f(x)dx = y$ iff $\frac{dy}{dx} = f(x)$. Everytime you do an integral you are solving a first order DEqn. In fact, it's an *ordinary* DEnq (ODE) since there is only one indpendent variable (it was x). A system of ODEs is a set of differential equations with a common independent variable. It turns out that any linear differential equation can be written as a system of ODEs in *normal form*. I'll define *normal form* then illustrate with a few examples.

Definition 9.2.1.

Let t be a real variable and suppose $x_1, x_2, \dots, x_n$ are functions of t. If A_{ij}, f_i are functions of t for all $1 \leq i \leq m$ and $1 \leq j \leq n$ then the following set of differential equations is defined to be a system of linear differential equations in **normal form**:

$$\begin{aligned} \tfrac{dx_1}{dt} &= A_{11}x_1 + A_{12}x_2 + \cdots A_{1n}x_n + f_1 \\ \tfrac{dx_2}{dt} &= A_{21}x_1 + A_{22}x_2 + \cdots A_{2n}x_n + f_2 \\ \vdots &= \vdots \qquad \vdots \quad \cdots \quad \vdots \\ \tfrac{dx_m}{dt} &= A_{m1}x_1 + A_{m2}x_2 + \cdots A_{mn}x_n + f_m \end{aligned}$$

In matrix notation, $\frac{dx}{dt} = Ax + f$. The system is called homogeneous if $f = 0$ whereas the system is called nonhomogeneous if $f \neq 0$. The system is called **constant coefficient** if $\frac{d}{dt}(A_{ij}) = 0$ for all i, j. If $m = n$ and a set of intial conditions $x_1(t_0) = y_1, x_2(t_0) = y_2, \dots, x_n(t_0) = y_n$ are given then this is called an **initial value problem** (IVP).

[1] or definition, depending on how you choose to set-up the complex exponential, I take this as the definition in calculus II

Example 9.2.2. *If x is the number of tigers and y is the number of rabbits then*

$$\tfrac{dx}{dt} = x + y \qquad\qquad \tfrac{dy}{dt} = -100x + 20y$$

is a model for the population growth of tigers and bunnies in some closed environment. My logic for my made-up example is as follows: the coefficient 1 is the growth rate for tigers which don't breed to quickly. Whereas the growth rate for bunnies is 20 since bunnies reproduce like, well bunnies. Then the y in the $\frac{dx}{dt}$ equation goes to account for the fact that more bunnies means more tiger food and hence the tiger reproduction should speed up (this is probably a bogus term, but this is my made up example so deal). Then the $-100x$ term accounts for the fact that more tigers means more tigers eating bunnies so naturally this should be negative. In matrix form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -100 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

How do we solve such a system? This is the question we seek to answer.

The preceding example is a *predator-prey* model. There are many other terms that can be added to make the model more realistic. Ultimately all population growth models are only useful if they can account for all significant effects. History has shown population growth models are of only limited use for humans.

Example 9.2.3. Reduction of Order *in calculus II you may have studied how to solve $y'' + by' + cy = 0$ for any choice of constants b, c. This is a second order ODE. We can reduce it to a system of first order ODEs by introducing new variables: $x_1 = y$ and $x_2 = y'$ then we have*

$$x_1' = y' = x_2$$

and,

$$x_2' = y'' = -by' - cy = -bx_2 - cx_1$$

As a matrix DEqn,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Similarly if $y'''' + 2y''' + 3y'' + 4y' + 5y = 0$ we can introduce variables to reduce the order: $x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$ then you can show:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is equivalent to $y'''' + 2y''' + 3y'' + 4y' + 5y = 0$. We call the matrix above the **companion matrix** *of the n-th order constant coefficient ODE. There is a beautiful interplay between solutions to n-th order ODEs and the linear algebra of the compansion matrix.*

Example 9.2.4. *Suppose $y'' + 4y' + 5y = 0$ and $x'' + x = 0$. The is a system of linear second order ODEs. It can be recast as a system of 4 first order ODEs by introducing new variables: $x_1 = y, x_2 = y'$ and $x_3 = x, x_4 = x'$. In matrix form the given system in normal form is:*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The companion matrix above will be found to have eigenvalues $\lambda = -2 \pm i$ and $\lambda = \pm i$. I know this without further calculation purely on the basis of what I know from DEqns and the interplay I alluded to in the last example.

Example 9.2.5. *If $y'''' + 2y'' + y = 0$ we can introduce variables to reduce the order: $x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$ then you can show:*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is equivalent to $y'''' + 2y'' + y = 0$. If we solve the matrix system then we solve the equation in y and vice-versa. I happen to know the solution to the y equation is $y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$. From this I can deduce that the companion matrix has a repeated e-value of $\lambda = \pm i$ and just one complex e-vector and its conjugate. This matrix would answer the bonus point question I posed a few sections back. I invite the reader to verify my claims.

Remark 9.2.6.

For those of you who will or have taken math 334 my guesswork above is predicated on two observations:

1. the "auxillarly" or "characteristic" equation in the study of the constant coefficient ODEs is identical to the characteristic equation of the companion matrix.

2. ultimately eigenvectors will give us exponentials and sines and cosines in the solution to the matrix ODE whereas solutions which have multiplications by t stem from generalized e-vectors. Conversely, if the DEqn has a t or t^2 multiplying cosine, sine or exponential functions then the companion matrix must in turn have generalized e-vectors to account for the t or t^2 etc...

I will not explain (1.) in this course, however we will hopefully make sense of (2.) by the end of this Chapter.

9.3 eigenvector solutions and diagonalization

Any system of linear differential equations with constant coefficients[2] can be reformulated into a single system of linear differential equations in **normal form** $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A \in \mathbb{R}^{\,n\times n}$ and $\vec{f} : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function of a real variable which is usually called the **inhomogeneous term**. To begin suppose $\vec{f} = 0$ so the problem becomes the homogeneous system $\frac{d\vec{x}}{dt} = A\vec{x}$. We wish to find a vector-valued function $\vec{x}(t) = (x_1(t), x_2(t), \dots, x_n(t)$ such that when we differentiate it we obtain the same result as if we multiplied it by A. This is what it means to solve the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$. Essentially, solving this DEqn is like performing n-integrations at once. For each integration we get a constant, these constants are fixed by initial conditions if we have n of them. In any event, the general solution has the form:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

[2]there are many other linear differential equations which are far more subtle than the ones we consider here, however, this case is of central importance to a myriad of applications

where $\{\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)\}$ is a LI set of solutions to $\frac{d\vec{x}}{dt} = A\vec{x}$ meaning $\frac{d\vec{x}_j}{dt} = A\vec{x}_j$ for each $j = 1, 2, \dots, n$. Therefore, if we can find these n-LI solutions then we've solved the problem. It turns out that the solutions are particularly simple if the matrix is diagonalizable: suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is an eigenbasis with e-values $\lambda_1, \lambda_2, \dots, \lambda_n$. Let $\vec{x}_j = e^{\lambda_j t}\vec{u}_j$ and observe that

$$\frac{d\vec{x}_j}{dt} = \frac{d}{dt}\big[e^{\lambda_j t}\vec{u}_j\big] = \frac{d}{dt}\big[e^{\lambda_j t}\big]\vec{u}_j = e^{\lambda_j t}\lambda_j\vec{u}_j = e^{\lambda_j t}A\vec{u}_j = Ae^{\lambda_j t}\vec{u}_j = A\vec{x}_j.$$

We find that each eigenvector $\vec{u}_j$ yields a solution $\vec{x}_j = e^{\lambda_j t}\vec{u}_j$. If there are n-LI e-vectors then we obtain n-LI solutions.

Example 9.3.1. *Consider for example, the system*

$$x' = x + y, \qquad y' = 3x - y$$

We can write this as the matrix problem

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{d\vec{x}/dt} = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}}$$

It is easily calculated that A has eigenvalue $\lambda_1 = -2$ with e-vector $\vec{u}_1 = (-1, 3)$ and $\lambda_2 = 2$ with e-vectors $\vec{u}_2 = (1, 1)$. The general solution of $d\vec{x}/dt = A\vec{x}$ is thus

$$\vec{x}(t) = c_1e^{-2t}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1e^{-2t} + c_2e^{2t} \\ 3c_1e^{-2t} + c_2e^{2t} \end{bmatrix}$$

So, the **scalar solutions** *are simply* $\boxed{x(t) = -c_1e^{-2t} + c_2e^{2t}}$ *and* $\boxed{y(t) = 3c_1e^{-2t} + c_2e^{2t}}$.

Thus far I have simply told you how to solve the system $d\vec{x}/dt = A\vec{x}$ with e-vectors, it is interesting to see what this means geometrically. For the sake of simplicity we'll continue to think about the preceding example. In it's given form the DEqn is **coupled** which means the equations for the derivatives of the dependent variables x, y cannot be solved one at a time. We have to solve both at once. In the next example I solve the same problem we just solved but this time using a change of variables approach.

Example 9.3.2. *Suppose we change variables using the diagonalization idea: introduce new variables $\bar{x}, \bar{y}$ by $P(\bar{x}, \bar{y}) = (x, y)$ where $P = [\vec{u}_1|\vec{u}_2]$. Note $(\bar{x}, \bar{y}) = P^{-1}(x, y)$. We can diagonalize A by the similarity transformation by P; $D = P^{-1}AP$ where $Diag(D) = (-2, 2)$. Note that $A = PDP^{-1}$ hence $d\vec{x}/dt = A\vec{x} = PDP^{-1}\vec{x}$. Multiply both sides by P^{-1}:*

$$P^{-1}\frac{d\vec{x}}{dt} = P^{-1}PDP^{-1}\vec{x} \quad \Rightarrow \quad \frac{d(P^{-1}\vec{x})}{dt} = D(P^{-1}\vec{x}).$$

You might not recognize it but the equation above is decoupled. In particular, using the notation $(\bar{x}, \bar{y}) = P^{-1}(x, y)$ we read from the matrix equation above that

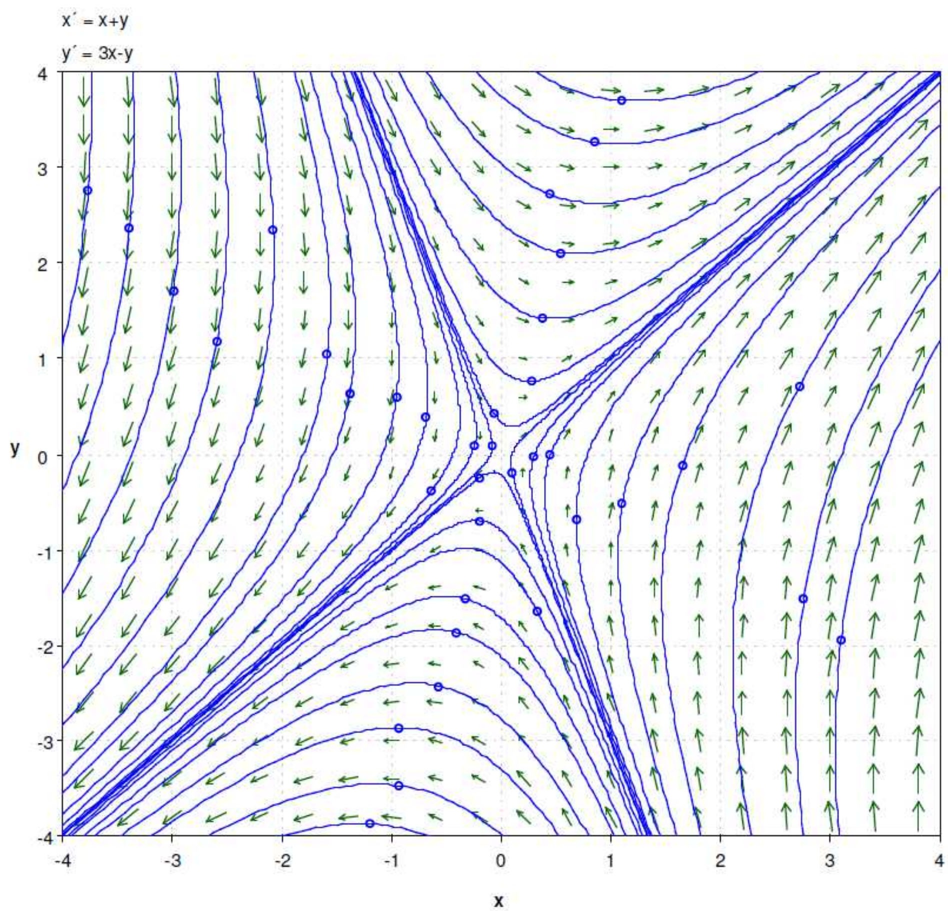
$$\frac{d\bar{x}}{dt} = -2\bar{x}, \qquad \frac{d\bar{y}}{dt} = 2\bar{y}.$$

Separation of variables and a little algebra yields that $\bar{x}(t) = c_1e^{-2t}$ and $\bar{y}(t) = c_2e^{2t}$. Finally, to find the solution back in the original coordinate system we multiply $P^{-1}\vec{x} = (c_1e^{-2t}, c_2e^{2t})$ by P to isolate $\vec{x}$,

$$\vec{x}(t) = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{-2t} \\ c_2e^{2t} \end{bmatrix} = \begin{bmatrix} -c_1e^{-2t} + c_2e^{2t} \\ 3c_1e^{-2t} + c_2e^{2t} \end{bmatrix}.$$

This is the same solution we found in the last example. Usually linear algebra texts present this solution because it shows more interesting linear algebra, however, from a pragmatic viewpoint the first method is clearly faster.

Finally, we can better appreciate the solutions we found if we plot the direction field $(x', y') = (x+y, 3x-y)$ via the "pplane" tool in Matlab. I have clicked on the plot to show a few representative trajectories (solutions):



9.4 the matrix exponential

Perhaps the most important first order ODE is $\frac{dy}{dt} = ay$. This DEqn says that the rate of change in y is simply proportional to the amount of y at time t. Geometrically, this DEqn states the solutions value is proportional to its slope at every point in its domain. The solution[3] is the exponential function $y(t) = e^{at}$.

We face a new differential equation; $\frac{dx}{dt} = Ax$ where x is a vector-valued function of t and $A \in \mathbb{R}^{\ n\times n}$. Given our success with the exponential function for the scalar case is it not natural to suppose that $x = e^{tA}$ is the solution to the matrix DEqn? The answer is yes. However, we need to define a few items before we can understand the true structure of the claim.

[3] ok, technically separation of variables yields the general solution $y = ce^{at}$ but I'm trying to focus on the exponential function for the moment.

Definition 9.4.1.

Let $A\mathbb{R}^{\ n\times n}$ define $e^A \in \mathbb{R}^{\ n\times n}$ by the following formula

$$e^A = \sum_{n=0}^{\infty} \tfrac{1}{n!}A^n = I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \cdots .$$

We also denote $e^A = exp(A)$ when convenient.

This definition is the natural extension of the Taylor series formula for the exponential function we derived in calculus II. Of course, you should be skeptical of this definition. How do I even know the series converges for an arbitrary matrix A? And, what do I even mean by "converge" for a series of matrices? (skip the next subsection if you don't care)

9.4.1 analysis for matrices

Remark 9.4.2.

The purpose of this section is to alert the reader to the gap in the development here. We will use the matrix exponential despite our inability to fully grasp the underlying analysis. Much in the same way we calculate series in calculus without proving every last theorem. I will attempt to at least sketch the analytical underpinnings of the matrix exponential. The reader will be happy to learn this is not part of the required material.

We use the Frobenius norm for $A \in \mathbb{R}^{\ n\times n}$, $||A|| = \sqrt{\sum_{i,j}(A_{ij})^2}$. We already proved this was a norm in a previous chapter. A sequence of square matrices is a function from $\mathbb{N}$ to $\mathbb{R}^{\ n\times n}$. We say the sequence $\{A_n\}_{n=1}^{\infty}$ converges to $L \in \mathbb{R}^{\ n\times n}$ iff for each $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $||A_n - L|| < \epsilon$ for all $n > M$. This is the same definition we used in calculus, just now the norm is the Frobenius norm and the functions are replaced by matrices. The definition of a series is also analogus to the definition you learned in calculus II.

Example 9.4.3. $A = \begin{bmatrix} 5 & 3 \\ 1 & -1 \end{bmatrix}$ *has* $||A|| = \sqrt{5^2 + 3^2 + 1^2 + (-1)^2} = 6.$

Definition 9.4.4.

Let $A_k \in \mathbb{R}^{m\times m}$ for all k, the sequence of partial sums of $\sum_{k=0}^{\infty} A_k$ is given by $S_n = \sum_{k=1}^{n} A_k$. We say the series $\sum_{k=0}^{\infty} A_k$ converges to $L \in \mathbb{R}^{m\times m}$ iff the sequence of partial sums converges to L. In other words,

$$\sum_{k=1}^{\infty} A_k = \lim_{n\to\infty} \sum_{k=1}^{n} A_k.$$

Many of the same theorems hold for matrices:

Proposition 9.4.5.

Let $t \to S_A(t) = \sum A_k(t)$ and $t \to S_B(t) = \sum_k B_k(t)$ be matrix valued functions of a real variable t where the series are uniformly convergent and $c \in \mathbb{R}$ then

1. $\sum_k cA_k = c\sum_k A_k$
2. $\sum_k(A_k + B_k) = \sum_k A_k + \sum_k B_k$
3. $\frac{d}{dt}\left[\sum_k A_k\right] = \sum_k \frac{d}{dt}\left[A_k\right]$
4. $\int\left[\sum_k A_k\right]dt = C + \sum_k \int A_k dt$ where C is a constant matrix.

The summations can go to infinity and the starting index can be any integer.

Uniform convergence means the series converge without regard to the value of t. Let me just refer you to the analysis course, we should discuss uniform convergence in that course, the concept equally well applies here. It is the crucial fact which one needs to interchange the limits which are implicit within $\sum_k$ and $\frac{d}{dt}$. There are counterexamples in the case the series is not uniformly convergent. Fortunately,

Proposition 9.4.6.

Let A be a square matrix then $exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$ is a uniformly convergent series of matrices.

Basically, the argument is as follows: The set of square matrices with the Frobenius norm is isometric to $\mathbb{R}^{n^2}$ which is a complete space. A complete space is one in which every Cauchy sequence converges. We can show that the sequence of partial sums for $exp(A)$ is a Cauchy sequence in $\mathbb{R}^{\,n\times n}$ hence it converges. Obviously I'm leaving some details out here. You can look at the excellent *Calculus* text by Apostle to see more gory details. Also, if you don't like my approach to the matrix exponential then he has several other ways to look it.

9.4.2 formulas for the matrix exponential

Now for the fun part.

Proposition 9.4.7.

Let A be a square matrix then $\frac{d}{dt}\left[exp(tA)\right] = Aexp(tA)$

Proof: consider the calculation below:

$$\begin{aligned}
\frac{d}{dt}(exp(tA)) &= \frac{d}{dt}(I + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{3!}t^3A^3 + \cdots) \\
&= \frac{d}{dt}(I) + \frac{d}{dt}(tA) + \tfrac{1}{2}\frac{d}{dt}(t^2A^2) + \tfrac{1}{3\cdot 2}\frac{d}{dt}(t^3A^3) + \cdots \\
&= A + tA^2 + \tfrac{1}{2}t^2A^3 + \cdots \\
&= A(I + tA + \tfrac{1}{2}t^2A^2 + \cdots) \\
&= Aexp(tA).
\end{aligned}$$

$\square$

Alternatively, we could provide proof using the summation notation:

$$\begin{aligned}
\frac{d}{dt}\left[exp(tA)\right] &= \frac{d}{dt}\left[\sum_{k=0}^{\infty}\frac{1}{k!}t^kA^k\right] && \text{defn. of matrix exponential} \\
&= \sum_{k=0}^{\infty}\frac{d}{dt}\left[\frac{1}{k!}t^kA^k\right] && \text{since matrix exp. uniformly conv.} \\
&= \sum_{k=0}^{\infty}\frac{k}{k!}t^{k-1}A^k && A^k \text{ constant and } \frac{d}{dt}(t^k) = kt^{k-1} \\
&= A\sum_{k=1}^{\infty}\frac{1}{(k-1)!}t^{k-1}A^{k-1} && \text{since } k! = k(k-1)! \text{ and } A^k = AA^{k-1}. \\
&= Aexp(tA) && \text{defn. of matrix exponential.}
\end{aligned}$$

Notice that we have all we need to see that $exp(tA)$ is a matrix of solutions to the differential equation $x' = Ax$. The following prop. follows from the preceding prop. and Prop. 2.3.14.

Proposition 9.4.8.

If $X = exp(tA)$ then $X' = Aexp(tA) = AX$. This means that each column in X is a solution to $x' = Ax$.

Let us illustrate this proposition with a particularly simple example.

Example 9.4.9. *Suppose $x' = x, y' = 2y, z' = 3z$ then in matrix form we have:*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The coefficient matrix is diagonal which makes the k-th power particularly easy to calculate,

$$A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

$$\Rightarrow \quad exp(tA) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty}\frac{t^k}{k!}1^k & 0 & 0 \\ 0 & \sum_{k=0}^{\infty}\frac{t^k}{k!}2^k & 0 \\ 0 & 0 & \sum_{k=0}^{\infty}\frac{t^k}{k!}3^k \end{bmatrix}$$

$$\Rightarrow \quad exp(tA) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

Thus we find three solutions to $x' = Ax$,

$$x_1(t) = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} \qquad x_2(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} \qquad x_3(t) = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$$

In turn these vector solutions amount to the solutions $x = e^t, y = 0, z = 0$ or $x = 0, y = e^{2t}, z = 0$ or $x = 0, y = 0, z = e^{3t}$. It is easy to check these solutions.

Usually we cannot calculate the matrix exponential explicitly by such a straightforward calculation. We need e-vectors and sometimes generalized e-vectors to reliably calculate the solutions of interest.

Proposition 9.4.10.

If A, B are square matrices such that $AB = BA$ then $e^{A+B} = e^A e^B$

Proof: I'll show how this works for terms up to quadratic order,

$$e^A e^B = (1 + A + \tfrac{1}{2}A^2 + \cdots)(1 + B + \tfrac{1}{2}B^2 + \cdots) = 1 + (A + B) + \tfrac{1}{2}A^2 + AB + \tfrac{1}{2}B^2 + \cdots .$$

However, since $AB = BA$ and

$$(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2.$$

Thus,

$$e^A e^B = 1 + (A + B) + \tfrac{1}{2}(A + B)^2 + \cdots = e^{A+B} \qquad \square$$

You might wonder what happens if $AB \neq BA$. In this case we can account for the departure from commutativity by the **commutator** of A and B.

Definition 9.4.11.

Let $A, B \in \mathbb{R}^{\ n\times n}$ then the **commutator** of A and B is $[A, B] = AB - BA$.

Proposition 9.4.12.

Let $A, B, C \in \mathbb{R}^{\ n\times n}$ then

1. $[A, B] = -[B, A]$
2. $[A + B, C] = [A, C] + [B, C]$
3. $[AB, C] = A[B, C] + [A, C]B$
4. $[A, BC] = B[A, C] + [A, B]C$
5. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

The proofs of the properties above are not difficult. In contrast, the following formula known as the Baker-Campbell-Hausdorff (BCH) relation takes considerably more calculation:

$$\boxed{e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]+\frac{1}{12}[[B,A],A]+\cdots} \qquad \text{BCH-formula}}$$

The higher order terms can also be written in terms of nested commutators. What this means is that if we know the values of the commutators of two matrices then we can calculate the product of their exponentials with a little patience. This connection between multiplication of exponentials and commutators of matrices is at the heart of Lie theory. Actually, mathematicians have greatly abstracted the idea of Lie algebras and Lie groups way past matrices but the concrete example of matrix Lie groups and algebras is perhaps the most satisfying. If you'd like to know more just ask. It would make an excellent topic for an independent study that extended this course.

Remark 9.4.13.

In fact the BCH holds in the abstract as well. For example, it holds for the Lie algebra of derivations on smooth functions. A *derivation* is a linear differential operator which satisfies the product rule. The derivative operator is a derivation since $D[fg] = D[f]g + fD[g]$. The commutator of derivations is defined by $[X, Y][f] = X(Y(f)) - Y(X(f))$. It can be shown that $[D, D] = 0$ thus the BCH formula yields

$$e^{aD}e^{bD} = e^{(a+b)D}.$$

If the coefficient of D is thought of as position then multiplication by e^{bD} generates a translation in the position. By the way, we can state Taylor's Theorem rather compactly in this operator notation: $f(x+h) = exp(hD)f(x) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$.

Proposition 9.4.14.

Let $A, P \in \mathbb{R}^{\ n\times n}$ and assume P is invertible then

$$exp(P^{-1}AP) = P^{-1}exp(A)P$$

Proof: this identity follows from the following observation:

$$(P^{-1}AP)^k = P^{-1}APP^{-1}APP^{-1}AP\cdots P^{-1}AP = P^{-1}A^kP.$$

Thus $exp(P^{-1}AP) = \sum_{k=0}^{\infty}\frac{1}{k!}(P^{-1}AP)^k = P^{-1}(\sum_{k=0}^{\infty}\frac{1}{k!}A^k)P = P^{-1}exp(A)P$. $\square$

Proposition 9.4.15.

Let A be a square matrix, $det(exp(A)) = exp(trace(A))$.

Proof I: If the matrix A is diagonalizable then the proof is simple. Diagonalizability means there exists invertible$P = [v_1|v_2|\cdots|v_n]$ such that $P^{-1}AP = D = [\lambda_1 v_1|\lambda_2 v_2|\cdots|\lambda_n v_n]$ where v_i is an e-vector with e-value λ_i for all i. Use the preceding proposition to calculate

$$\det(exp(D)) = \det(exp(P^{-1}AP) = \det(P^{-1}exp(A)P) = \det(P^{-1}P)\det(exp(A)) = det(exp(A))$$

On the other hand, the trace is cyclic $trace(ABC) = trace(BCA)$

$$trace(D) = trace(P^{-1}AP) = trace(PP^{-1}A) = trace(A)$$

But, we also know D is diagonal with eigenvalues on the diagonal hence $exp(D)$ is diagonal with e^{λ_i} on the corresponding diagonals

$$\det(exp(D)) = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} \ \text{ and } \ trace(D) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Finally, use the laws of exponents to complete the proof,

$$e^{trace(A)} = e^{trace(D)} = e^{\lambda_1+\lambda_2+\cdots+\lambda_n} = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} = det(exp(D)) = det(exp(A)).$$

I've seen this proof in texts presented as if it were the general proof. But, not all matrices are diagonalizable so this is a curious proof. Of course, the correction is just to use the Jordan form.

We know there exists diagonal matrix D with e-values of A on the diagonal and nilpotent matrix N for which $J = D + N$ is similar to A; $J = P^{-1}AP$. Then, $A = PJP^{-1}$ and we calculate

$$e^A = e^{PJP^{-1}} = Pe^J P^{-1} = Pe^{D+N}P^{-1} = Pe^D e^N P^{-1}$$

as $DN = ND$ is known for the Jordan decomposition of A. Thus,

$$\det(e^A) = \det(e^D)\det(e^N)$$

Now, since $N^k = 0$ we can calculate e^N explicitly as the finite sum:

$$e^N = I + N + \frac{1}{2}N^2 + \cdots + \frac{1}{(k-1)!}N^{k-1} =$$

Moreover, $N, N^2, \dots, N^{k-1}$ are all strictly upper triangular hence e^N is upper triangular with 1 on each diagonal entry. Therefore, $\det(e^N) = 1$. Then, our proof for the diagonal case completes the general proof[4]. $\square$

alternate calculus-based proof: The preceding proof shows it may be hopeful to suppose that $det(exp(tA)) = exp(t\,trace(A))$ for $t \in \mathbb{R}$. Notice that $y = e^{kt}$ satisfies the differential equation $\frac{dy}{dt} = ky$. Conversely, if $\frac{dy}{dt} = ky$ for some constant k then the general solution is given by $y = c_o e^{kt}$ for some $c_o \in \mathbb{R}$. Let $f(t) = det(exp(tA))$. If we can show that $f'(t) = trace(A)f(t)$ then we can conclude $f(t) = c_0 e^{t\,trace(A)}$. Consider:

$$\begin{aligned}
f'(t) &= \tfrac{d}{dh}\left(f(t+h) \right|_{h=0} \\
&= \tfrac{d}{dh}\left(det(exp[(t+h)A]) \right|_{h=0} \\
&= \tfrac{d}{dh}\left(det(exp[tA + hA]) \right|_{h=0} \\
&= \tfrac{d}{dh}\left(det(exp[tA]exp[hA]) \right|_{h=0} \\
&= det(exp[tA])\tfrac{d}{dh}\left(det(exp[hA]) \right|_{h=0} \\
&= f(t)\tfrac{d}{dh}\left(det(I + hA + \tfrac{1}{2}h^2A^2 + \tfrac{1}{3!}h^3A^3 + \cdots) \right|_{h=0} \\
&= f(t)\tfrac{d}{dh}\left(det(I + hA)) \right|_{h=0}
\end{aligned}$$

[4] I gave a more linear presentation of this proof in the Lecture on March 25 of 2016

Let us discuss the $\frac{d}{dh}(det(I + hA))$ term seperately for a moment:[5]

$$\begin{aligned}
\tfrac{d}{dh}(det(I + hA)) &= \tfrac{d}{dh}[\sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n}(I + hA)_{i_1 1}(I + hA)_{i_2 2}\cdots(I + hA)_{i_n n}]_{h=0} \\
&= \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n}\tfrac{d}{dh}[(I + hA)_{1 i_1}(I + hA)_{1 i_2}\cdots(I + hA)_{n i_n}]_{h=0} \\
&= \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n}(A_{1i_1}I_{1i_2}\cdots I_{ni_n} + I_{1i_1}A_{2i_2}\cdots I_{ni_n} + \cdots + I_{1i_1}I_{2i_2}\cdots A_{ni_n}) \\
&= \sum_{i_1}\epsilon_{i_1 2\dots n}A_{1i_1} + \sum_{i_2}\epsilon_{1 i_2\dots n}A_{2i_2} + \cdots + \sum_{i_n}\epsilon_{12\dots I_n}A_{ni_n} \\
&= A_{11} + A_{22} + \cdots + A_{nn} \\
&= trace(A)
\end{aligned}$$

Therefore, $f'(t) = trace(A)f(t)$ consequently, $f(t) = c_o e^{t\,trace(A)} = det(exp(tA))$. However, we can resolve c_o by calculating $f(0) = det(exp(0)) = det(I) = 1 = c_o$ hence

$$e^{t\,trace(A)} = det(exp(tA))$$

Take $t = 1$ to obtain the desired result. $\square$

Remark 9.4.16.

The formula $det(exp(A)) = exp(trace(A))$ is very important to the theory of matrix Lie groups and Lie algebras. Generically, if G is the Lie group and $\mathfrak{g}$ is the Lie algebra then they are connected via the matrix exponential: $exp : \mathfrak{g} \to G_o$ where I mean G_o to denoted the connected component of the identity. For example, the set of all nonsingular matrices $GL(n)$ forms a Lie group which is disconnected. Half of $GL(n)$ has positive determinant whereas the other half has negative determinant. The set of all $n \times n$ matrices is denoted $gl(n)$ and it can be shown that $exp(gl(n))$ maps onto the part of $GL(n)$ which has positive determinant. One can even define a matrix logarithm map which serves as a local inverse for the matrix exponential near the identity. Generally the matrix exponential is not injective thus some technical considerations must be discussed before we could put the matrix log on a solid footing. This would take us outside the scope of this course. However, this would be a nice topic to do a follow-up independent study. The theory of matrix Lie groups and their representations is ubiqitious in modern quantum mechanical physics.

Finally, we come to the formula that is most important to our study of systems of DEqns. Let's call this the magic formula.

Proposition 9.4.17.

Let $\lambda \in \mathbb{C}$ and suppose $A \in \mathbb{R}^{\,n\times n}$ then

$$\boxed{exp(tA) = e^{\lambda t}(I + t(A - \lambda I) + \tfrac{t^2}{2}(A - \lambda I)^2 + \tfrac{t^3}{3!}(A - \lambda I)^3 + \cdots).}$$

[5] I use the definition of the identity matrix $I_{ij} = \delta_{ij}$ in eliminating all but the last summation in the fourth line. Then the levi-civita symbols serve the same purpose in going to the fifth line as $\epsilon_{i_1 2\dots n} = \delta_{1i_1}$,$\epsilon_{1 i_2\dots n} = \delta_{2i_2}$ etc...

Proof: Notice that $tA = t(A - \lambda I) + t\lambda I$ and $t\lambda I$ commutes with all matrices thus,

$$\begin{aligned} exp(tA) &= exp(t(A - \lambda I) + t\lambda I) \\ &= exp(t(A - \lambda I))exp(t\lambda I) \\ &= e^{\lambda t} exp(t(A - \lambda I)) \\ &= e^{\lambda t}\big(I + t(A - \lambda I) + \tfrac{t^2}{2}(A - \lambda I)^2 + \tfrac{t^3}{3!}(A - \lambda I)^3 + \cdots\big) \end{aligned}$$

In the third line I used the identity proved below,

$$exp(t\lambda I) = I + t\lambda I + \tfrac{1}{2}(t\lambda)^2 I^2 + \cdots = I(1 + t\lambda + \tfrac{(t\lambda)^2}{2} + \cdots) = Ie^{t\lambda}. \qquad \square$$

While the proofs leading up to the magic formula only dealt with real matrices it is not hard to see the proofs are easily modified to allow for complex matrices.

9.5 solutions for systems of DEqns with real eigenvalues

Let us return to the problem of solving $\vec{x}\,' = A\vec{x}$ for a constant square matrix A where $\vec{x} = [x_1, x_2, \dots, x_n]$ is a vector of functions of t. I'm adding the vector notation to help distinguish the scalar function x_1 from the vector function $\vec{x}_1$ in this section. Let me state one theorem from the theory of differential equations. The existence of solutions theorem which is the heart of of this theorem is fairly involved to prove. It requires a solid understanding of real analysis.

Theorem 9.5.1.

If $\vec{x}\,' = A\vec{x}$ and A is a constant matrix then any solution to the system has the form

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

where $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ is a linearly independent set of solutions defined on $\mathbb{R}$ (this is called the **fundamental solution set**). Moreover, these fundamental solutions can be concatenated into a single invertible solution matrix called the **fundamental matrix** $X = [\vec{x}_1|\vec{x}_2|\cdots|\vec{x}_n]$ and the general solution can be expressed as $\vec{x}(t) = X(t)\vec{c}$ where $\vec{c}$ is an arbitrary vector of real constants. If an initial condtion $\vec{x}(t_o) = \vec{x}_o$ is given then the solution to the IVP is $\vec{x}(t) = X^{-1}(t_o)X(t)\vec{x}_o$.

We proved in the previous section that the matrix exponential $exp(tA)$ is a solution matrix and the inverse is easy enought to guess: $exp(tA)^{-1} = exp(-tA)$. This proves the columns of $exp(tA)$ are solutions to $\vec{x}\,' = A\vec{x}$ which are linearly independent and as such form a fundamental solution set.

Problem: we cannot directly calculate $exp(tA)$ for most matrices A. We have a solution we can't calculate. What good is that?

When can we explicitly calculate $exp(tA)$ without much thought? Two cases come to mind: (1.) if A is diagonal then it's easy, saw this in Example 9.4.9, (2.) if A is a **nilpotent** matrix then there is some finite power of the matrix which is zero; $A^k = 0$. In the nilpotent case the infinite series defining the matrix exponential truncates at order k:

$$exp(tA) = I + tA + \tfrac{t^2}{2}A^2 + \cdots + \tfrac{t^{k-1}}{(k-1)!}A^{k-1}$$

Example 9.5.2. *Let* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *we calculate* $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *thus*

$$exp(tA) = I + tA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Incidentally, the solution to $\vec{x}\,' = A\vec{x}$ *is generally* $\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} t \\ 1 \end{bmatrix}$*. In other words,* $x_1(t) = c_2 + c_2 t$ *whereas* $x_2(t) = c_2$*. These solutions are easily seen to solve the system* $x_1' = x_2$ *and* $x_2' = 0$*.*

Unfortunately, the calculation we just did in the last example almost never works. For example, try to calculate an arbitrary power of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, let me know how it works out. We would like for all examples to truncate. The **magic formula** gives us a way around this dilemma:

Proposition 9.5.3.

Let $A \in \mathbb{R}^{\,n\times n}$. Suppose v is an e-vector with e-value λ then $exp(tA)v = e^{\lambda t}v$.

Proof: we are given that $(A-\lambda I)v = 0$ and it follows that $(A-\lambda I)^k v = 0$ for all $k \geq 1$. Use the magic formula,

$$exp(tA)v = e^{\lambda t}(I + t(A-\lambda I) + \cdots)v = e^{\lambda t}(Iv + t(A-\lambda I)v + \cdots = e^{\lambda t}v$$

noting all the higher order terms vanish since $(A-\lambda I)^k v = 0$. $\square$

We can't hope for the matrix exponential itself to truncate, but when we multiply $exp(tA)$ on an e-vector something special happens. Since $e^{\lambda t} \neq 0$ the set of vector functions $\{e^{\lambda_1 t}v_1, e^{\lambda_2 t}v_2, \dots, e^{\lambda_k t}v_k\}$ will be linearly independent if the e-vectors v_i are linearly independent. If the matrix A is diagonalizable then we'll be able to find enough e-vectors to construct a fundamental solution set using e-vectors alone. However, if A is not diagonalizable, and has only real e-values, then we can still find a Jordan basis $\{v_1, v_2, \dots, v_n\}$ which consists of generalized e-vectors and it follows that $\{e^{tA}v_1, e^{tA}v_2, \dots, e^{tA}v_n\}$ forms a fundamental solution set. Moreover, this is not just of theoretical use. We can actually calculate this solution set.

Proposition 9.5.4.

Let $A \in \mathbb{R}^{\,n\times n}$. Suppose A has a chain $\{v_1, v_2, \dots, v_k\}$ is of generalized e-vectors with e-value λ, meaning $(A-\lambda)v_1 = 0$ and $(A-\lambda)v_{k-1} = v_k$ for $k \geq 2$, then

1. $e^{tA}v_1 = e^{\lambda t}v_1$,
2. $e^{tA}v_2 = e^{\lambda t}(v_2 + tv_1)$,
3. $e^{tA}v_3 = e^{\lambda t}\big(v_3 + tv_2 + \frac{t^2}{2}v_1\big)$,
4. $e^{tA}v_k = e^{\lambda t}\big(v_k + tv_{k-1} + \cdots + \frac{t^{k-1}}{(k-1)!}v_1\big)$.

Proof: Study the chain condition,

$$(A-\lambda I)v_2 = v_1 \quad \Rightarrow \quad (A-\lambda)^2 v_2 = (A-\lambda I)v_1 = 0$$

$$(A-\lambda I)v_3 = v_2 \quad \Rightarrow \quad (A-\lambda I)^2 v_3 = (A-\lambda I)v_2 = v_1$$

Continuing with such calculations[6] we find $(A-\lambda I)^j v_i = v_{i-j}$ for all $i > j$ and $(A-\lambda I)^i v_i = 0$. The magic formula completes the proof:

$$e^{tA}v_2 = e^{\lambda t}\big(v_2 + t(A-\lambda I)v_2 + \tfrac{t^2}{2}(A-\lambda I)^2 v_2 \cdots\big) = e^{\lambda t}\big(v_2 + tv_1\big)$$

likewise,

$$\begin{aligned} e^{tA}v_3 &= e^{\lambda t}\big(v_3 + t(A-\lambda I)v_3 + \tfrac{t^2}{2}(A-\lambda I)^2 v_3 + \tfrac{t^3}{3!}(A-\lambda I)^3 v_3 + \cdots\big) \\ &= e^{\lambda t}\big(v_3 + tv_2 + \tfrac{t^2}{2}(A-\lambda I)v_2\big) \\ &= e^{\lambda t}\big(v_3 + tv_2 + \tfrac{t^2}{2}v_1\big). \end{aligned}$$

We already proved the e-vector case in the preceding proposition and the general case follows from essentially the same calculation. $\square$

We have all the theory we need to solve systems of homogeneous constant coefficient ODEs.

Example 9.5.5. *Recall Example 8.1.17 we found* $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ *had e-values* $\lambda_1 = 0$ *and* $\lambda_2 = 4$ *and corresponding e-vectors*

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus we find the general solution to $\vec{x}\,' = A\vec{x}$ *is simply,*

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

just to illustrate the terms: we have fundmamental solution set and matrix:

$$\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} \right\} \qquad X = \begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t} \end{bmatrix}$$

Notice that a different choice of e-vector scaling would just end up adjusting the values of c_1, c_2 in the event an initial condition was given. This is why different choices of e-vectors still gives us the same general solution. It is the flexibility to change c_1, c_2 that allows us to fit any initial condition.

Example 9.5.6. *We can modify Example 9.2.2 and propose a different model for a tiger/bunny system. Suppose x is the number of tigers and y is the number of rabbits then*

$$\tfrac{dx}{dt} = x - 4y \qquad \tfrac{dy}{dt} = -10x + 19y$$

[6] keep in mind these conditions hold because of our current labling scheme, if we used a different indexing system then you'd have to think about how the chain conditions work out, to test your skill perhaps try to find the general solution for the system with the matrix from Example **??**

is a model for the population growth of tigers and bunnies in some closed environment. Suppose that there is initially 2 *tigers and* 100 *bunnies.* **Find the populations of tigers and bunnies at time $t > 0$:**

Solution: *notice that we must solve $\vec{x}\,' = A\vec{x}$ where $A = \begin{bmatrix} 1 & -4 \\ -10 & 19 \end{bmatrix}$ and $\vec{x}(0) = [2, 100]^T$. We can calculate the eigenvalues and corresponding eigenvectors:*

$$det(A - \lambda I) = 0 \ \Rightarrow \ \lambda_1 = -1, \ \lambda_2 = 21 \ \Rightarrow \ u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Therefore, using Proposition 9.5.4, the general solution has the form:

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{21t} \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

However, we also know that $\vec{x}(0) = [2, 100]^T$ hence

$$\begin{bmatrix} 2 \\ 100 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} \ \Rightarrow \ \begin{bmatrix} 2 \\ 100 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 5 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 100 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 110 \\ 198 \end{bmatrix}$$

Finally, we find the vector-form of the solution to the given initial value problem:

$$\vec{x}(t) = 10e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \tfrac{198}{11} e^{21t} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Which means that $x(t) = 20e^{-t} - \frac{198}{11}e^{21t}$ and $y(t) = 1020e^{-t} + 90e^{21t}$ are the number of tigers and bunnies respective at time t.

Notice that a different choice of e-vectors would have just made for a different choice of c_1, c_2 in the preceding example. Also, notice that when an initial condition is given there ought not be any undetermined coefficients in the final answer[7].

Example 9.5.7. *We found that in Example 8.1.20 the matrix $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ has e-values $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = 2$ with corresponding e-vectors*

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \ \vec{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Hence, using Proposition 9.5.4 and Theorem 9.5.1 the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is simply:

$$\vec{x}(t) = c_1 e^{4t}\vec{u}_1 + c_2 e^{4t}\vec{u}_2 + c_3 e^{2t}\vec{u}_3 = c_1 e^{4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

[7]Assuming of course that there are enough initial conditions given to pick a unique solution from the family of solutions which we call the "general solution".

Example 9.5.8. *Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that:*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We analyzed this matrix in Example **??**. *We found a pair of chains of generalized e-vectors all with eigenvalue $\lambda = 1$ which satisfied the following conditions:*

$$(A-I)\vec{u}_3 = \vec{u}_1, \ (A-I)\vec{u}_1 = 0 \qquad (A-I)\vec{u}_4 = \vec{u}_2, \ (A-I)\vec{u}_2 = 0$$

In particular, $\vec{u}_j = e_j$ for $j = 1,2,3,4$. We can use the magic formula to extract 4 solutions from the matrix exponential, by Proposition 9.5.4 we find:

$$\begin{aligned} \vec{x}_1 &= e^{At}\vec{u}_1 = e^t\vec{u}_1 = e^te_1 \\ \vec{x}_2 &= e^{At}\vec{u}_2 = e^t(e_2 + te_1) \\ \vec{x}_3 &= e^{At}\vec{u}_3 = e^te_3 \\ \vec{x}_4 &= e^{At}\vec{u}_4 = e^t(e_4 + te_3) \end{aligned} \tag{9.1}$$

Let's write the general solution in vector and scalar form, by Theorem 9.5.1,

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + c_4\vec{x}_4 = c_1e^te_1 + c_2e^t(e_2+te_1) + c_3e^te_3 + c_4e^t(e_4+te_3) = \begin{bmatrix} c_1e^t + tc_2e^t \\ c_2e^t \\ c_3e^t + tc_4e^t \\ c_4e^t \end{bmatrix}$$

In other words, $x_1(t) = c_1e^t + tc_2e^t$, $x_2(t) = c_2e^t$, $x_3(t) = c_3e^t + tc_4e^t$ and $x_4(t) = c_4e^t$ form the general solution to the given system of differential equations.

Example 9.5.9. *Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given (generalized)eigenvectors $\vec{u}_i$, $i = 1,2,3,4,5,6,7,8,9$ such that:*

$$(A-I)\vec{u}_1 = 0, \quad A\vec{u}_2 = \vec{u}_2, \quad A\vec{u}_3 = 7\vec{u}_3, \quad (A-I)\vec{u}_4 = \vec{u}_1$$

$$(A+5I)\vec{u}_5 = 0, \ (A-3I)\vec{u}_6 = \vec{u}_7 \ \ A\vec{u}_7 = 3\vec{u}_7, \ A\vec{u}_8 = 0, \quad (A-3I)\vec{u}_9 = \vec{u}_6$$

We can use the magic formula to extract 9 solutions from the matrix exponential, by Proposition 9.5.4 we find:

$$\begin{aligned} \vec{x}_1 &= e^{At}\vec{u}_1 = e^t\vec{u}_1 = e^t\vec{u}_1 \\ \vec{x}_2 &= e^{At}\vec{u}_2 = e^t\vec{u}_2 \\ \vec{x}_3 &= e^{At}\vec{u}_3 = e^{7t}\vec{u}_3 \\ \vec{x}_4 &= e^{At}\vec{u}_4 = e^t(\vec{u}_4 + t\vec{u}_1) \qquad \text{can you see why?} \\ \vec{x}_5 &= e^{At}\vec{u}_5 = e^{-5t}\vec{u}_5 \\ \vec{x}_6 &= e^{At}\vec{u}_6 = e^{3t}(\vec{u}_6 + t\vec{u}_7) \qquad \text{can you see why?} \\ \vec{x}_7 &= e^{At}\vec{u}_7 = e^{3t}\vec{u}_7 \\ \vec{x}_8 &= e^{At}\vec{u}_8 = \vec{u}_8 \\ \vec{x}_9 &= e^{At}\vec{u}_9 = e^{3t}(\vec{u}_9 + t\vec{u}_6 + \tfrac{1}{2}t^2\vec{u}_7) \qquad \text{can you see why?} \end{aligned} \tag{9.2}$$

Let's write the general solution in vector and scalar form, by Theorem 9.5.1,

$$\vec{x}(t) = \sum_{i=1}^{9} c_i \vec{x}_i$$

where the formulas for each solution $\vec{x}_i$ was given above. If I was to give an explicit matrix A with the eigenvectors given above it would be a 9×9 matrix.

Challenge: *find the matrix exponential e^{At} in terms of the given (generalized)eigenvectors.*

Hopefully the examples have helped the theory settle in by now. We have one last question to settle for systems of DEqns.

Theorem 9.5.10.

The nonhomogeneous case $\vec{x}\,' = A\vec{x} + \vec{f}$ the general solution is $\vec{x}(t) = X(t)c + \vec{x}_p(t)$ where X is a fundamental matrix for the corresponding homogeneous system and $\vec{x}_p$ is a particular solution to the nonhomogeneous system. We can calculate $\vec{x}_p(t) = X(t)\int X^{-1}\vec{f}dt$.

Proof: suppose that $\vec{x}_p = X\vec{v}$ for X a fundamental matrix of $\vec{x}\,' = A\vec{x}$ and some vector of unknown functions $\vec{v}$. We seek conditions on $\vec{v}$ which make $\vec{x}_p$ satisfy $\vec{x}_p\,' = A\vec{x}_p + \vec{f}$. Consider,

$$(\vec{x}_p)' = (X\vec{v})' = X'\vec{v} + X\vec{v}' = AX\vec{v} + X\vec{v}'$$

But, $\vec{x}_p\,' = A\vec{X}_p + \vec{f} = AX\vec{v} + \vec{f}$ hence

$$X\tfrac{d\vec{v}}{dt} = \vec{f} \quad \Rightarrow \quad \tfrac{d\vec{v}}{dt} = X^{-1}\vec{f}$$

Integrate to find $\vec{v} = \int X^{-1}\vec{f}dt$ therefore $x_p(t) = X(t)\int X^{-1}\vec{f}dt$. $\square$

If you ever work through variation of parameters for higher order ODEqns then you should appreciate the calculation above. In fact, we can derive n-th order variation of parameters from converting the n-th order ODE by reduction of order to a system of n first order linear ODEs. You can show that the so-called Wronskian of the fundamental solution set is precisely the determinant of the fundamental matrix for the system $\vec{x}\,' = A\vec{x}$ where A is the companion matrix. I have this worked out in an old test from a DEqns course I taught at NCSU[8]

Example 9.5.11. *Suppose that $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$, find the general solution of the nonhomogenous DEqn $\vec{x}\,' = A\vec{x} + \vec{f}$. Recall that in Example 9.5.5 we found $\vec{x}\,' = A\vec{x}$ has fundamental matrix $X = \begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t} \end{bmatrix}$. Use variation of parameters for systems of ODEs to constuct $\vec{x}_p$. First calculate the inverse of the fundamental matrix, for a 2×2 we know a formula:*

$$X^{-1}(t) = \tfrac{1}{e^{4t}-(-3)e^{4t}}\begin{bmatrix} e^{4t} & -e^{4t} \\ 3 & 1 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 1 & -1 \\ 3e^{-4t} & e^{-4t} \end{bmatrix}$$

[8] see solution of Problem 6 in *www.supermath.info/ma341f07test2_sol.pdf* for the $n = 2$ case of this comment, also §6.4 of Nagel Saff and Snider covers n-th order variation of parameters if you want to see details

Thus,

$$\begin{aligned}
x_p(t) = X(t)\int \tfrac{1}{4}\begin{bmatrix} 1 & -1 \\ 3e^{-4t} & e^{-4t}\end{bmatrix}\begin{bmatrix} e^t \\ e^{-t}\end{bmatrix} dt &= \tfrac{1}{4}X(t)\int \begin{bmatrix} e^t - e^{-t} \\ 3e^{-3t} + e^{-5t}\end{bmatrix} dt \\
&= \tfrac{1}{4}\begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t}\end{bmatrix}\begin{bmatrix} e^t + e^{-t} \\ -e^{-3t} - \frac{1}{5}e^{-5t}\end{bmatrix} \\
&= \tfrac{1}{4}\begin{bmatrix} 1(e^t + e^{-t}) + e^{4t}(-e^{-3t} - \frac{1}{5}e^{-5t}) \\ -3(e^t + e^{-t}) + e^{4t}(-e^{-3t} - \frac{1}{5}e^{-5t})\end{bmatrix} \\
&= \tfrac{1}{4}\begin{bmatrix} e^t + e^{-t} - e^t - \frac{1}{5}e^{-t} \\ -3e^t - 3e^{-t} - e^t - \frac{1}{5}e^{-t}\end{bmatrix} \\
&= \tfrac{1}{4}\begin{bmatrix} \frac{4}{5}e^{-t} \\ -4e^t - \frac{16}{5}e^{-t}\end{bmatrix}
\end{aligned}$$

Therefore, the general solution is

$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ -3\end{bmatrix} + c_2e^{4t}\begin{bmatrix} 1 \\ 1\end{bmatrix} + \tfrac{1}{5}\begin{bmatrix} e^{-t} \\ -e^t - 4e^{-t}\end{bmatrix}.$$

The general scalar solutions implicit within the general vector solution $\vec{x}(t) = [x(t), y(t)]^T$ *are*

$$x(t) = c_1 + c_2e^{4t} + \tfrac{1}{5}e^{-t} \qquad y(t) = -3c_1 + c_2e^{4t} - \tfrac{1}{5}e^t - \tfrac{4}{5}e^{-t}.$$

I'll probably ask you to solve a 3×3 system in the homework. The calculation is nearly the same as the preceding example with the small inconvenience that finding the inverse of a 3×3 requires some calculation.

Remark 9.5.12.

You might wonder how would you solve a system of ODEs $x' = Ax$ such that the coefficients A_{ij} are not constant. We will not cover such problems in this course. We do cover how to solve an $n - th$ order ODE with nonconstant coefficients via series techniques in Math 334. It's probably possible to extend some of those techniques to systems. Laplace Transforms also extend to systems of ODEs. It's just a matter of algebra. Nontrivial algebra.

9.6 solutions for systems of DEqns with complex eigenvalues

The calculations in the preceding section still make sense for a complex e-value and complex e-vector. However, we usually need to find real solutions. How to fix this? The same way as always. We extract real solutions from the complex solutions. Fortunately, our previous work on linear independence of complex e-vectors insures that the resulting solution set will be linearly independent.

Proposition 9.6.1.

Let $A \in \mathbb{R}^{\,n\times n}$. Suppose A has a chain $\{v_1, v_2, \dots, v_k\}$ is of generalized complex e-vectors with e-value $\lambda = \alpha + i\beta$, meaning $(A-\lambda)v_1 = 0$ and $(A-\lambda)v_{k-1} = v_k$ for $k \geq 2$ and $v_j = a_j + ib_j$ for $a_j, b_j \in \mathbb{R}^n$ for each j, then

1. $e^{tA}v_1 = e^{\lambda t}v_1$,
2. $e^{tA}v_2 = e^{\lambda t}(v_2 + tv_1)$,
3. $e^{tA}v_3 = e^{\lambda t}\big(v_3 + tv_2 + \frac{t^2}{2}v_1\big)$,
4. $e^{tA}v_k = e^{\lambda t}\big(v_k + tv_{k-1} + \cdots + \frac{t^{k-1}}{(k-1)!}v_1\big)$.

Furthermore, the following are the $2k$ linearly independent real solutions that are implicit within the complex solutions above,

1. $x_1 = Re(e^{tA}v_1) = e^{\alpha t}\big[(\cos\beta t)a_1 - (\sin\beta t)b_1\big]$,
2. $x_2 = Im(e^{tA}v_1) = e^{\alpha t}\big[(\sin\beta t)a_1 + (\cos\beta t)b_1\big])$,
3. $x_3 = Re(e^{tA}v_2) = e^{\alpha t}\big[(\cos\beta t)(a_2 + ta_1) - (\sin\beta t)(b_2 + tb_1)\big]$,
4. $x_4 = Im(e^{tA}v_2) = e^{\alpha t}\big[(\sin\beta t)(a_2 + ta_1) + (\cos\beta t)(b_2 + tb_1)\big]$,
5. $x_5 = Re(e^{tA}v_3) = e^{\alpha t}\big[(\cos\beta t)(a_3 + ta_2 + \frac{t^2}{2}a_1) - (\sin\beta t)(b_3 + tb_2 + \frac{t^2}{2}b_1)\big]$,
6. $x_6 = Im(e^{tA}v_3) = e^{\alpha t}\big[(\cos\beta t)(a_3 + ta_2 + \frac{t^2}{2}a_1) - (\sin\beta t)(b_3 + tb_2 + \frac{t^2}{2}b_1)\big]$.

Proof: the magic formula calculations of the last section just as well apply to the complex case. Furthermore, we proved that

$$Re\big[e^{\alpha t + i\beta t}(v + iw)\big] = e^{\alpha t}\big[(\cos\beta t)v - (\sin\beta t)w\big]$$

and

$$Im\big[e^{\alpha t + i\beta t}(v + iw)\big] = e^{\alpha t}\big[(\sin\beta t v + (\cos\beta t)w\big],$$

the proposition follows. $\square$

Example 9.6.2. *This example uses the results derived in Example 8.1.24. Let* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and find the e-values and e-vectors of the matrix. Observe that* $det(A - \lambda I) = \lambda^2 + 1$ *hence the eigevalues are* $\lambda = \pm i$*. We find* $u_1 = [1, i]^T$*. Notice that*

$$u_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This means that $\vec{x}\,' = A\vec{x}$ *has general solution:*

$$\vec{x}(t) = c_1\left(\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2\left(\sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

The solution above is the "vector-form of the solution". We can add the terms together to find the scalar solutions: denoting $\vec{x}(t) = [x(t), y(t)]^T$,

$$x(t) = c_1\cos(t) + c_2\sin(t) \qquad y(t) = -c_1\sin(t) + c_2\cos(t)$$

These are the parametric equations of a circle with radius $R = \sqrt{c_1^2 + c_2^2}$.

Example 9.6.3. *We solved the e-vector problem for* $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *in Example 8.1.26. We found one real e-value* $\lambda_1 = 3$ *and a pair of complex e-values* $\lambda_2 = 1 \pm i$. *The corresponding e-vectors were:*

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We identify that $Re(\vec{u}_2) = e_2$ *and* $Im(\vec{u}_2) = e_1$. *The general solution of* $\vec{x}\,' = A\vec{x}$ *should have the form:*

$$\vec{x}(t) = c_1 e^{At}\vec{u}_1 + c_2 Re(e^{At}\vec{u}_2) + c_3 Im(e^{At}\vec{u}_2)$$

The vectors above are e-vectors so these solution simplify nicely:

$$\vec{x}(t) = c_1 e^{3t} e_3 + c_2 e^t(\cos(t)e_2 - \sin(t)e_1) + c_3 e^t(\sin(t)e_2 + \cos(t)e_1)$$

For fun let's look at the scalar form of the solution. Denoting $\vec{x}(t) = [x(t), y(t), z(t)]^T$,

$$x(t) = -c_2 e^t \sin(t) + c_3 e^t \cos(t), \qquad y(t) = c_2 e^t \cos(t) + c_3 e^t \sin(t), \qquad z(t) = c_1 e^{3t}$$

Believe it or not this is a spiral helix which has an exponentially growing height and radius.

Example 9.6.4. *Let's suppose we have a chain of* 2 *complex eigenvectors* $\vec{u}_1, \vec{u}_2$ *with eigenvalue* $\lambda = 2 + i3$. *I'm assuming that*

$$(A - (2+i)I)\vec{u}_2 = \vec{u}_1, \qquad (A - (2+i)I)\vec{u}_1 = 0.$$

We get a pair of complex-vector solutions (using the magic formula which truncates since these are e-vectors):

$$\vec{z}_1(t) = e^{At}\vec{u}_1 = e^{(2+i)t}\vec{u}_1, \qquad \vec{z}_2(t) = e^{At}\vec{u}_2 = e^{(2+i)t}(\vec{u}_2 + t\vec{u}_1),$$

The real and imaginary parts of these solutions give us 4 *real solutions which form the general solution:*

$$\begin{aligned}
\vec{x}(t) = c_1 e^{2t}&\big[\cos(3t)Re(\vec{u}_1) - \sin(3t)Im(\vec{u}_1)\big] \\
&+ c_2 e^{2t}\big[\sin(3t)Re(\vec{u}_1) + \cos(3t)Im(\vec{u}_1)\big] \\
&+ c_3 e^{2t}\big[\cos(3t)[Re(\vec{u}_2) + tRe(\vec{u}_1)] - \sin(3t)[Im(\vec{u}_2) + tIm(\vec{u}_1)]\big] \\
&+ c_4 e^{2t}\big[\sin(3t)[Re(\vec{u}_2) + tRe(\vec{u}_1)] + \cos(3t)[Im(\vec{u}_2) + tIm(\vec{u}_1)]\big].
\end{aligned}$$

Chapter 10

euclidean geometry

The concept of a geometry is very old. Philosophers in the nineteenth century failed miserably in their analysis of geometry and the physical world. They became mired in the popular misconception that mathematics must be physical. They argued that because 3 dimensional Eulcidean geometry was the only geometry familar to everyday experience it must surely follow that a geometry which differs from Euclidean geometry must be nonsensical. However, why should physical intuition factor into the argument? We understand now that geometry is a mathematical construct, not a physical one. There are many possible geometries. On the other hand, it would seem the geometry of space and time probably takes just one form. We are tempted by this misconception every time we ask "but what is this math really". That question is usually wrong-headed. A better question is "is this math logically consistent" and if so what physical systems is it known to model.

The modern view of geometry is stated in the langauge of manifolds, fiber bundles,algebraic geometry and perhaps even more fantastic structures. There is currently great debate as to how we should model the true intrinsic geometry of the universe. Branes, strings, quivers, noncommutative geometry, twistors, ... this list is endless. However, at the base of all these things we must begin by understanding what the geometry of a flat space entails.

Vector spaces are flat manifolds. They possess a global coordinate system once a basis is chosen. Up to this point we have only cared about algebraic conditions of linear independence and spanning. There is more structure we can assume. We can ask what is the length of a vector? Or, given two vectors we might want to know what is the angle bewtween those vectors? Or when are two vectors orthogonal?

If we desire we can also insist that the basis consist of vectors which are *orthogonal* which means "perpendicular" in a generalized sense. A geometry is a vector space plus an idea of orthogonality and length. The concepts of orthogonality and length are encoded by an inner-product. Inner-products are symmetric, positive definite, bilinear forms, they're like a dot-product. Once we have a particular geometry in mind then we often restrict the choice of bases to only those bases which preserve the length of vectors.

The mathematics of orthogonality is exhibited by the dot-products and vectors in calculus III. However, it turns out the concept of an *inner-product* allows us to extend the idea or perpendicular to abstract vectors such as functions. This means we can even ask interesting questions such as "how close is one function to another" or "what is the closest function to a set of functions".

Least-squares curve fitting is based on this geometry.

This chapter begins by studying inner product spaces and the induced norm. Then we discuss orthogonality, the Gram Schmidt algorithm, orthogonal complements and finally the application to the problem of least square analysis. We also look at how Fourier analysis naturally flows from our finite dimensional discussions of orthogonality. [1]. Then we turn to study linear transformations with the help of inner product-based geometry. The adjoint is introduced and we see how normal transformations and self-adjoint transformations play a special role. In particular, we discuss the spectral theorems for complex and real inner product spaces. Applications of the spectral theorems are vast, we consider several in the next Chapter.

Let me digress from linear algebra for a little while. In physics it is customary to only allow coordinates which fit the physics. In classical mechanics one often works with intertial frames which are related by a rigid motion. Certain quantities are the same in all intertial frames, notably force. This means Newtons laws have the same form in all intertial frames. The geometry of special relativity is 4 dimensional. In special relativity, one considers coordinates which preserve Einstein's three axioms. Allowed coordinates are related to other coordinates by Lorentz transformations. These Lorentz transformations include rotations and velocity boosts. These transformations are designed to make the speed of a light ray invariant in all frames. For a linear algebraist the vector space is the starting point and then coordinates are something we add on later. Physics, in contrast, tends to start with coordinates and if the author is kind he might warn you which transformations are allowed.

What coordinate transformations are allowed actually tells you what kind of physics you are dealing with. This is an interesting and nearly universal feature of modern physics. The allowed transformations form what is known to physicsists as a "group" (however, strictly speaking these groups do not always have the strict structure that mathematicians insist upon for a group). In special relativity the group of interest is the Poincaire group. In quantum mechanics you use unitary groups because unitary transformations preserve probabilities. In supersymmetric physics you use the super Poincaire group because it is the group of transformations on superspace which preserves supersymmetry. In general relativity you allow general coordinate transformations which are locally lorentzian because all coordinate systems are physical provided they respect special relativity in a certain approximation. In solid state physics there is something called the renormilzation group which plays a central role in physical predictions of field-theoretic models. My point? Transformations of coordinates are important if you care about physics. We study the basic case of vector spaces in this course. If you are interested in the more sophisticated topics just ask, I can show you where to start reading.

[1]we ignore analytical issues of convergence since we have only in mind a Fourier approximation, not the infinite series

10.1 inner product spaces

Definition 10.1.1.

Let V be a vector space over $\mathbb{F}$. An **inner product** on V is a function which maps pairs $(x, y) \in V \times V$ to a scalar denoted $\langle x, y\rangle \in \mathbb{F}$ such that for all $x, y, z \in V$ and $c \in \mathbb{F}$,

(a.) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,

(b.) $\langle cx, y\rangle = c\langle x, y\rangle$,

(c.) $\overline{\langle x, y\rangle} = \langle y, x\rangle$,

(d.) $\langle x, x\rangle > 0$ if $x \neq 0$.

In the case $\mathbb{F} = \mathbb{R}$ condition (c.) reduces to symmetry of the inner product; $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V(\mathbb{R})$.

Example 10.1.2. *Observe that $(x, y) \mapsto x \bullet y$ defines an inner product on $\mathbb{R}^n$. We say $\mathbb{R}^n$ paired with the dot-product forms* **Euclidean n-space***. We'll see how the dot-product allows us to define the usual concepts in geometry such as vector length and the angle between nonzero vectors.*

The following notation is helpful for the complex examples.

Definition 10.1.3.

Let $A \in \mathbb{C}^{n\times n}$ then define the **conjugate transpose** or **adjoint** to be $A^* = \overline{A}^T$

In physics literature this is often called the **Hermitian adjoint** and the notation $A^\dagger = A^*$ is used. In fact, it is quite common for A^* to denote the complex conjugate of A. Beware of notation.

Example 10.1.4. *The standard inner product on complex n-vectors is given by [2]:*

$$\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n = z^T\overline{w} = w^* z.$$

for $z, w \in \mathbb{C}^n$. Notice, $\overline{\langle z, w\rangle} = \overline{z^T\overline{w}} = \overline{z}^T w = z^* w = \langle w, z\rangle$. Furthermore,

$$\langle z, z\rangle = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \geq 0$$

and we see $\langle z, z\rangle > 0$ if $z \neq 0$. The reader can easily verify the other axioms to affirm $\langle , \rangle$ forms an inner product on $\mathbb{C}^n$. I may refer to it as the **complex inner product** *on $\mathbb{C}^n$ to distinguish from the next example. For $\mathbb{C}^2$, $z = (1, 1)$ and $w = (-i, 1)$ provide $\langle z, w\rangle = 1(i) + 1(1) = i + 1$.*

Example 10.1.5. *Let V be a complex vector space with inner product $\langle , \rangle : V \times V \to \mathbb{C}$. Then V can also be thought of as a real vector space. Moreover, V can be given a real inner product $\langle , \rangle_{\mathbb{R}} : V \times V \to \mathbb{R}$ which is induced from the given complex inner product:*

$$\langle x, y\rangle_{\mathbb{R}} = Re(\langle x, y\rangle)$$

In invite the reader to confirm this satifies all the axioms of a real inner product. Recall the formula for the real part of $z \in \mathbb{C}$ is given by $Re(z) = \frac{1}{2}(z + \overline{z})$ hence:

$$\langle x, y\rangle_{\mathbb{R}} = \frac{1}{2}\left[\ \langle x, y\rangle + \overline{\langle x, y\rangle}\ \right] = \frac{1}{2}[\langle x, y\rangle + \langle y, x\rangle].$$

The formula above provides $\langle x, y\rangle_{\mathbb{R}} = \langle y, x\rangle_{\mathbb{R}}$ for all $x, y \in V$. As a parting thought for $V = \mathbb{C}^2$, $z = (1, 1)$ and $w = (-i, 1)$ provide $\langle z, w\rangle_{\mathbb{R}} = Re(i + 1) = 1$.

[2] to see the second formula you can transpose the first since the transpose of a number is just the number once more and $(z^T\overline{w})^T = \overline{w}^T(z^T)^T = w^* z$

Example 10.1.6. *Let $A, B \in \mathbb{R}^{n\times n}$ then a natural dot-product is given by stringing out the entries of A and B as n^2-dimensional vectors and taking the dot-product:*

$$\langle A, B\rangle = \sum_{i,j=1}^{n} A_{ij}B_{ij} = A_{11}B_{11} + A_{12}B_{12} + \cdots + A_{nn}B_{nn}$$

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *then* $\langle A, B\rangle = 1(0) + 2(1) + 3(1) + 4(0) = 5$. *For many questions, the following reformulation of this inner product is helpful:*

$$\langle A, B\rangle = \sum_{i,j=1}^{n} A_{ij}B_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B^T_{ji} = \sum_{i=1}^{n}(AB^T)_{ii} = trace(AB^T).$$

Example 10.1.7. *If $Z, W \in \mathbb{C}^{n\times n}$ then define $\langle Z, W\rangle = trace(ZW^*)$. In terms of components,*

$$\langle Z, W\rangle = \sum_{i,j=1}^{n} Z_{ij}\overline{W}_{ij} \qquad \& \qquad \langle Z, Z\rangle = \sum_{i,j=1}^{n} Z_{ij}\overline{Z}_{ij} = \sum_{i,j=1}^{n} |Z_{ij}|^2.$$

We can also consider $\langle Z, W\rangle_{\mathbb{R}} = Re(\langle Z, W\rangle)$ as introduced in Example 10.1.5.

Example 10.1.8. *Let $C^0[0, 1]$ denote the real vector space of continuous real-valued functions with domain $[0, 1]$. Define for $f, g \in C^0[0, 1]$,*

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

Observe,

$$\begin{aligned} \langle cf_1 + f_2, g\rangle &= \int_0^1 (cf_1 + f_2)(x)g(x)dx \\ &= c\int_0^1 f_1(x)g(x)dx + \int_0^1 f_2(x)g(x)dx \\ &= c\langle f_1, g\rangle + \langle f_2, g\rangle. \end{aligned}$$

Furthermore, it is easy to see $\langle f, g\rangle = \langle g, f\rangle$ and $\langle f, f\rangle = \int_0^1 f(x)^2dx > 0$ if $f(x) \neq 0$. Note, for a continuous function the values of $f(x)$ cannot be nonzero unless they are nonzero in some open set. Thus continuity provides that $\langle f, f\rangle = 0$ if and only if $f(x) = 0$ for all $x \in [0, 1]$. In summary, $\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx$ defines an inner product for $C^0[0, 1]$.

It's subtle, but, for $C^0(\mathbb{R})$ the map $(f, g) \mapsto \int_0^1 f(x)g(x)dx$ does **not** define an inner product. You can construct a nonzero function f which is zero on $[0, 1]$ hence $\langle f, f\rangle = 0$ and $f \neq 0$. Such a construction is not possible with polynomials.

Example 10.1.9. *Let $\mathbb{R}[x]$ denote the vector space of real polynomials. Define $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$. In invite the reader to confirm this defines an inner product on $\mathbb{R}[x]$.*

In quantum mechanics one studies the wave function $\Psi : \mathbb{R} \to \mathbb{C}$ for which $\int_{-\infty}^{\infty} \overline{\Psi}(x)\Psi(x)dx = 1$. The idea is that Ψ is a **probability density**. Some of the terminology in this part of math is reflective of the motivating influence of quantum mechanics on the development of the math of inner product spaces. In particular, the study of infinite dimensional Hilbert spaces and distributions (so-called *Rigged Hilbert Space*) was largely motivated by the physical relevance of such spaces to quantum mechanics. If you'd like to know more, I can recommend some reading.

Example 10.1.10. *Let V be the space of complex-valued functions on $\mathbb{R}$ with $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$. If $f, g \in V$ then we define a complex inner product by:*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

In particular, a wave function has $\langle \Psi, \Psi \rangle = 1$.

This list of examples should suffice to illustrate the generality of the concept of inner product space. We wish to abstract the structure of a dot-product for real vector spaces. Likewise, the modulus in $\mathbb{C}$ is generalized by the complex inner product.

Proposition 10.1.11. *algebraic properties of the inner product:*

Let V be an inner product space over $\mathbb{F}$ with inner product $\langle , \rangle$. If $x, y, z \in V$ and $c \in \mathbb{F}$ then

(a.) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$,

(b.) $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$,

(c.) $\langle x, 0 \rangle = 0 = \langle 0, x \rangle$,

(d.) $\langle x, x \rangle = 0$ if and only if $x = 0$,

(e.) if $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$ then $y = z$

Proof: we did some of these in Lecture. Also see Problem 112 of Spring 2017 for (b). All of these make nice entry level test questions on inner product spaces. $\square$

Definition 10.1.12.

Let $(V, \langle , \rangle)$ be an inner product space. We define the **norm** of $x \in V$ by $\|x\| = \sqrt{\langle x, x \rangle}$

We also say $\langle x \rangle$ is the **length** of x. Furthermore, you can think about a normed linear space independent of the existence of the inner product. Not all norms are induced from an inner product. For example, $\|(x, y)\| = |x| + |y|$ defines a norm, yet, there is no inner product on $\mathbb{R}^2$ which induces this norm.

Example 10.1.13. *Let $v = (1, 2, 2)$ then $v \bullet v = 1 + 4 + 4 = 9$ thus $\|v\| = \sqrt{v \bullet v} = \sqrt{9} = 3$.*

Example 10.1.14. *Let $A = \begin{bmatrix} 2+i & -3 & i \\ 0 & 0 & 0 \\ 3+i & 0 & 0 \end{bmatrix}$ then*

$$\langle A, A \rangle = \sum_{i,j=1}^{3} |A_{ij}|^2 = |2+i|^2 + |-3|^2 + |i|^2 + |3+i|^2 = 5 + 9 + 1 + 10 = 25$$

Thus $\|A\| = \sqrt{25} = 5$. The length of A with respect to the Frobenius norm is 5.

Example 10.1.15. *Consider $f(x) = x + ix^2$. Find the length of $f(x)$ with respect to the inner product $\langle f(x), g(x) \rangle = \int_0^1 f(x)\overline{g(x)}dx$. Calculate,*

$$\langle x + ix^2, x + ix^2 \rangle = \int_0^1 (x + ix^2)(x - ix^2)dx = \int_0^1 (x^2 + x^4)dx = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

Hence $\|x + ix^2\| = \sqrt{8/15}$.

Proposition 10.1.16. *properties of the induced norm:*

Let V be an inner product space over $\mathbb{F}$ with inner product $\langle , \rangle$. If $x, y \in V$ and $c \in \mathbb{F}$ then the following properties hold for $\|x\| = \sqrt{\langle x, x\rangle}$:

(a.) $\|cx\| = |c|\,\|x\|$

(b.) $\|x\| = 0$ if and only if $x = 0$,

(c.) $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ (the Cauchy Schwarz inequality),

(d.) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Proof: we discussed the proof of (a.) and (c.) in Lecture of Spring 2017. The proofs for (c.) and (d.) below are not as general as we desire. In particular, the proofs I share here are for the dot-product in $\mathbb{R}^n$. One of your homework problems is to adapt these proofs to the more general context of an inner product space (possibly complex). I don't anticipate this adaptation to be particularly difficult and I think it will be a good way for you to better understand why these inequalities are true.

Specialized Proof of (c.): let $x, y \in \mathbb{R}^n$. If either $x = 0$ or $y = 0$ then the inequality is clearly true. Suppose then that both x and y are nonzero vectors. It follows that $||x||, ||y|| \neq 0$ and we can define vectors of unit-length; $\hat{x} = \frac{x}{||x||}$ and $\hat{y} = \frac{y}{||y||}$. Notice that $\hat{x} \bullet \hat{x} = \frac{x}{||x||} \bullet \frac{x}{||x||} = \frac{1}{||x||^2}\hat{x} \bullet x = \frac{x \bullet x}{x \bullet x} = 1$ and likewise $\hat{y} \bullet \hat{y} = 1$. Consider,

$$\begin{aligned} 0 \leq ||\hat{x} \pm \hat{y}||^2 &= (\hat{x} \pm \hat{y}) \bullet (\hat{x} \pm \hat{y}) \\ &= \hat{x} \bullet \hat{x} \pm 2(\hat{x} \bullet \hat{y}) + \hat{y} \bullet \hat{y} \\ &= 2 \pm 2(\hat{x} \bullet \hat{y}) \\ &\Rightarrow \quad -2 \leq \pm 2(\hat{x} \bullet \hat{y}) \\ &\Rightarrow \quad \pm \hat{x} \bullet \hat{y} \leq 1 \\ &\Rightarrow \quad |\hat{x} \bullet \hat{y}| \leq 1 \end{aligned}$$

Therefore, noting that $x = ||x||\hat{x}$ and $y = ||y||\hat{y}$,

$$|x \bullet y| = |\, ||x||\hat{x} \bullet ||y||\hat{y}\,| = ||x||\,||y||\,|\hat{x} \bullet \hat{y}| \leq ||x||\,||y||.$$

The use of unit vectors is what distinguishes this proof from the others I've found.

Specialized Proof of (d.): (triangle inequality for $\mathbb{R}^n$) Let $x, y \in \mathbb{R}^n$,

$$\begin{aligned} ||x + y||^2 &= |(x + y) \bullet (x + y)| && \text{defn. of norm} \\ &= |x \bullet (x + y) + y \bullet (x + y)| && \text{prop. of dot-product} \\ &= |x \bullet x + x \bullet y + y \bullet x + y \bullet y| && \text{prop. of dot-product} \\ &= |\; ||x||^2 + 2x \bullet y + ||y||^2 \;| && \text{prop. of dot-product} \\ &\leq ||x||^2 + 2|x \bullet y| + ||y||^2 && \text{triangle ineq. for } \mathbb{R} \\ &\leq ||x||^2 + 2||x||\,||y|| + ||y||^2 && \text{Cauchy-Schwarz ineq.} \\ &\leq (||x|| + ||y||)^2 && \text{algebra} \end{aligned}$$

Notice that both $||x+y||, ||x||+||y|| \geq 0$ hence the inequality above yields $||x+y|| \leq ||x||+||y||$. $\square$

There are two natural concepts of angle we can consider. For real vector space the concept of angle is the familar one where $0 \leq \theta \leq \pi$ for θ between a given pair of nonzero vectors. In constrast, for a complex vector space[3] we may study the **complex angle** which is found in $0 \leq \tilde{\theta} \leq \pi/2$. To give an explicit example to contrast; the angle between $z = 1$ and $z = i$ in terms of the real geometry of the complex plane is just $\theta = \pi/2$. However, the complex angle between $z = 1$ and $z = i$ is zero. This is not unreasonable as the idea of a complex vector space includes complex scalar multiplication. The numbers $z = 1$ and $z = i$ are in the same *complex direction* as they are related by scalar mulitplication over $\mathbb{C}$.

Definition 10.1.17.

Let $(V, \langle , \rangle)$ be an inner product space over $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$ then the **angle** between nonzero vectors $x, y \in V$ is given by $\cos\theta = \frac{\langle x,y\rangle}{\|x\|\|y\|}$ for $0 \leq \theta \leq \pi$. If $\mathbb{F} = \mathbb{C}$ then the **complex angle** between nonzero vectors $x, y \in V$ is given by $\cos\tilde{\theta} = \frac{|\langle x,y\rangle|}{\|x\|\|y\|}$ for $0 \leq \tilde{\theta} \leq \pi/2$.

There is special significance when $\langle x, y\rangle = 0$ for nonzero x, y. We see in the next section that this condition provides $\{x, y\}$ is linearly independent.

10.2 orthogonality of vectors

Definition 10.2.1.

Let $(V, \langle , \rangle)$ be an inner product space. We say x, y are **orthogonal** if $\langle x, y\rangle = 0$. A subset S of V is orthogonal if each pair $x, y \in S$ is orthogonal. If S is an orthogonal subset of V in which $\|x\| = 1$ for each $x \in S$ then S is an **orthonormal** set.

The following is one important aspect of orthogonality:

Proposition 10.2.2.

An orthogonal set of nonzero vectors is linearly independent. An orthonormal set of vectors is linearly independent.

Proof: since an orthonormal set is a nonzero orthogonal set it suffices to consider S orthogonal. Suppose $v_1, \dots, v_k \in S$ and there exist $c_1, \dots, c_k \in \mathbb{F}$ with

$$\sum_{i=1}^{k} c_i v_i = 0 \quad \Rightarrow \quad \left\langle \sum_{i=1}^{k} c_i v_i, v_j \right\rangle = \langle 0, v_j\rangle \quad \Rightarrow \quad \sum_{i=1}^{k} c_i\langle v_i, v_j\rangle = 0 \quad \Rightarrow \quad c_j\langle v_j, v_j\rangle = 0$$

Since $v_j \neq 0$ we have $\langle v_j, v_j\rangle > 0$ hence $c_j = 0$. Thus $c_1 = 0, \dots, c_k = 0$ and we deduce that S is linearly independent. $\square$

Notice that orthogonality depends on the choice of field and type of geometry considered. For example, $\{1, i\}$ is an orthonormal set with respect to the real geometry of the complex plane. In constrast, $\{1, i\}$ is not orthogonal with respect to the complex geometry of $\mathbb{C}$.

[3] every complex vector space can also be viewed as a real vector space and Example 10.1.5 details how the complex inner product may be related to a corresponding real inner product

Proposition 10.2.3.

If $S \subseteq V$ is orthogonal and V has induced norm $\|x\| = \sqrt{\langle x, x\rangle}$ and $x_1, \dots, x_n \in S$ then

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2.$$

Proof: I'll prove $n = 2$ and leave the general case as an induction exercise for the reader. Suppose $\{x, y\}$ is an orthogonal subset of an inner product space V with induced norm $\|x\| = \sqrt{\langle x, x\rangle}$. We assume $\langle x, y\rangle = 0$ and it follows that $\langle y, x\rangle = 0$ since $\overline{\langle y, x\rangle} = \langle x, y\rangle = 0$ implies $\langle y, x\rangle = 0$. Consider, by properties of the inner product,

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2. \quad \square$$

It is a simple matter to take an orthogonal set of nonzero vectors and create an orthonormal set which generates the same span:

Example 10.2.4. *It is simple to verify $\{(1,1),(1,-1)\}$ is an orthogonal set with respect to the dot-product on $\mathbb{R}^2$. Since $\|(1,1)\| = \sqrt{2}$ and $\|(1,-1)\| = \sqrt{2}$ it follows that*

$$\beta = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

is an orthonormal set. Indeed, β forms an orthonormal basis for $\mathbb{R}^2$.

Example 10.2.5. *A somewhat silly example: let $S = \{E_{11}, 2E_{12}, 3E_{21}, 4E_{22}\}$ then we can show S is orthogonal with respect the Frobenius norm on 2×2 real matrices. Note that $\|E_{ij}\| = 1$ for any i, j. Hence,*

$$S'' = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

is an orthonormal set and it's not hard to see $span(S) = span(S'')$.

Definition 10.2.6.

Let $(V, \langle , \rangle)$ be an inner product space. If β is a basis for V and β is **orthonormal** then β is an **orthonormal basis**.

If $\beta = \{v_1, v_2, \dots, v_n\}$ is orthonormal then $\langle v_i, v_j\rangle$ is either zero or $\langle v_i, v_i\rangle = \|v_i\|^2 = 1$. Thus, we capture orthonormality of a basis for a finite dimensional inner product space by the simple formula $\langle v_i, v_j\rangle = \delta_{ij}$

Example 10.2.7. *Recall $span\{e_1, \dots, e_n\} = \mathbb{R}^n$ and $e_i \bullet e_j = \delta_{ij}$ thus the standard basis is an orthonormal basis. Indeed, Cartesian coordinates are the quintessential example of an orthogonal coordinate system.*

In fact, whenever we find an orthonormal basis β for V the coordinate system paired to β behaves as Cartesian coordinates. This claim is established in theorems we soon uncover.

Example 10.2.8. *The standard matrix basis E_{ij} is orthonormal in $\mathbb{R}^{n \times n}$ since $E_{ij}E_{kl} = \delta_{jk}E_{il}$ and*

$$\langle E_{ij}, E_{lk}\rangle = trace(E_{ij}E_{lk}^T) = trace(E_{ij}E_{kl}) = trace(\delta_{jk}E_{il}) = \delta_{jk}trace(E_{il}) = \delta_{ij}\delta_{il}$$

The formula above indicates that $\|E_{ij}\| = 1$ for all i, j and $\langle E_{ij}, E_{kl}\rangle = 0$ whenever E_{ij} and E_{kl} are distinct.

Example 10.2.9. *Consider $\mathbb{C}^n$ as a complex vector space. Observe $\langle e_i, e_j\rangle = e_i^T\overline{e_j} = e_i^T e_j = \delta_{ij}$.*

Example 10.2.10. *Let $W = span\{\sin(nx), \cos(nx), 1/\sqrt{\pi} \mid n \in \mathbb{N}\}$ and define*

$$\langle f(x), g(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Calculate the length of $f(x) = \sin(nx)$:

$$\begin{aligned}\langle \sin(nx), \sin(nx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(nx)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\big(1-\cos(2nx)\big)dx \\ &= \frac{1}{\pi}\frac{2\pi}{2} \\ &= 1.\end{aligned}$$

Likewise, $\langle \cos(nx), \cos(nx)\rangle = 1$ and $\langle 1/\sqrt{\pi}, 1/\sqrt{\pi}\rangle = 1$. Hence $|\sin(nx)\| = 1$ and $\|\cos(nx)\| = 1$ and $\|1/\sqrt{\pi}\| = 1$. You can also verify that $\langle f(x), g(x)\rangle = 0$ for $f(x) \neq g(x)$ hence

$$\{\sin(nx), \cos(nx), 1/\sqrt{\pi} \mid n \in \mathbb{N}\}$$

forms an **orthonormal** *basis for W.*

The observation below allows us to create many new inner product spaces from those we have already discussed:

Proposition 10.2.11. *inner product on subspaces*

If V is an inner product space over $\mathbb{F}$ with inner product $\langle , \rangle : V \times V \to \mathbb{F}$ and $W \leq V$ is a nonzero subspace then $\langle\ ,\ \rangle_W = \langle\ ,\ \rangle|_{W\times W} : W \times W \to \mathbb{F}$ is an inner product for W. Customarily, we still use $\langle , \rangle$ for this restricted inner product.

Proof: it is simple to verify that $\langle\ ,\ \rangle_W$ satifies all the axioms of an inner product since $\langle\ ,\ \rangle$ satisfies the axioms for V and $W \subseteq V$. I leave the explicit details to the reader. $\square$

This Proposition fails when we generalize inner products to **scalar products**. For example, in Lorentzian geometry the Minkowski product vanishes on lines in the light cone.

Example 10.2.12. *In Example 10.2.10 we found an orthonormal basis for W formed by the span of sinusoids and the constant function. If we studied $W_1 = span\{1, \sin(x), \cos(2x)\}$ then clearly $W_1 \leq W$ and hence the inner product for W naturally restricts to W_1. We can see $\{1/\sqrt{\pi}, \sin(x), \cos(2x)\}$ forms an orthonormal basis for W_1.*

Orthonormal coordinate systems are extremely simple. Notice how (b.) ties in with what you know from multivariate calculus[4]; if $\vec{A} = a\hat{x} + b\hat{y} + c\hat{z}$ then $a = \vec{A}\bullet\hat{x}$, $b = \vec{A}\bullet\hat{y}$ and $c = \vec{A}\bullet\hat{z}$ and $\|\vec{A}\| = \sqrt{a^2+b^2+c^2}$.

[4]my apologies once more to the reader who has been cheated out of a course in multivariate calculus, for the curious, my lectures on vectors are on my You Tube lecture and we get through vectors in about the week and a half.

Proposition 10.2.13. *coordinates with respect to orthogonal or orthonormal basis:*

Let $\beta = \{w_1, \dots, w_k\}$ serve as a basis for $W = \text{span}(\beta) \leq V$ where V is an inner product space with inner product $\langle , \rangle$. Then:

(a.) if β is orthogonal and $y \in W$ then $y = \displaystyle\sum_{j=1}^{k} \frac{\langle y, w_j \rangle}{\|w_j\|^2} w_j$

(b.) if β is orthonormal and $y \in W$ then $y = \displaystyle\sum_{j=1}^{k} \langle y, w_j \rangle w_j$. Moreover, the length of $y = y_1w_1 + y_2w_2 + \cdots + y_kw_k$ is given by $\|y\| = \sqrt{y_1^2 + y_2^2 + \cdots + y_k^2}$.

Proof: Let $\beta = \{w_1, \dots, w_k\}$ be a basis for $W \leq V$ where β is orthogonal with respect to the inner product $\langle , \rangle$. Let $y \in W$ then $y = \sum_{i=1}^{k} c_i w_i$ observe $\langle w_i, w_j \rangle = \delta_{ij} \langle w_j, w_j \rangle = \delta_{ij} \|w_j\|^2$ hence

$$\langle y, w_j \rangle = \left\langle \sum_{i=1}^{k} c_i w_i, w_j \right\rangle = \sum_{i=1}^{k} c_i \langle w_i, w_j \rangle = \sum_{i=1}^{k} c_i \delta_{ij} \langle w_j, w_j \rangle = c_j \langle w_j, w_j \rangle \quad \Rightarrow \quad c_j = \frac{\langle y, w_j \rangle}{\|w_j\|^2}$$

Thus (a.) is true and (b.) follows immediately as orthonormality provides $\|w_j\| = 1$ for all j. $\square$

Example 10.2.14. *Consider $A \in \mathbb{R}^{2\times 2}$ with the Frobenius norm. If $A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}$ then*

$$\|A\| = \sqrt{1^2 + 3^2 + 5^2 + 1^2} = \sqrt{36} = 6.$$

The following Corollary for Euclidean space is very important for applications.

Corollary 10.2.15. *coordinates with respect to orthogonal or orthonormal basis in Euclidean space:*

Let $\beta = \{w_1, \dots, w_k\}$ serve as an orthonormal basis for $W = \text{span}(\beta) \leq \mathbb{R}^n$ with respect to the dot-product. Then for $y \in W$ we find the coordinate vector with respect to β is given by $[y]_\beta = [\beta]^T y$. If $k = n$ then we also find $[\beta]^T = \beta^{-1}$.

Proof: let $\beta = \{w_1, \dots, w_k\} \subset \mathbb{R}^n$ be an orthonormal basis for W. If $y \in W$ then by Proposition 10.2.13 part (b.) we have $y = (y \bullet w_1)w_1 + (y \bullet w_2)w_2 + \cdots + (y \bullet w_k)w_k$ hence

$$[y]_\beta = (y \bullet w_1, y \bullet w_2, \cdots, y \bullet w_k).$$

Calculate, for $[\beta] = [w_1|w_2|\cdots|w_k]$,

$$[\beta]^T y = \underbrace{\begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_k^T \end{bmatrix}}_{k\times n} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} w_1^T y \\ w_2^T y \\ \vdots \\ w_k^T y \end{bmatrix}}_{n\times 1} = \begin{bmatrix} w_1 \bullet y \\ w_2 \bullet y \\ \vdots \\ w_k \bullet y \end{bmatrix}.$$

Hence $[\beta]^T y = [y]_\beta$. When we studied coordinate change earlier in our study we found $[y]_\beta = [\beta]^{-1} y$ for y a column vector. Hence $[\beta]^T y = [y]_\beta = [\beta]^{-1} y$ for all $y \in \mathbb{R}^n$ and we deduce $[\beta]^T = [\beta]^{-1}$. $\square$

Notice Problem 115 of Spring 2017 provides an example of the above result.

Definition 10.2.16.

If $R \in \mathbb{R}^{n \times n}$ then we say R is an **orthogonal matrix**. The set of all such matrices is:

$$O(n, \mathbb{R}) = \{R \in \mathbb{R}^{n \times n} \mid R^T R = I\}.$$

A short calculation reveals that a matrix R for which $R^T = R^{-1}$ is an *orthogonal matrix*. Thus, the proof of Proposition 10.2.15 shows that we may glue together orthonormal vectors to form an orthogonal matrix. Somewhat annoyingly, the concatenation of orthogonal vectors does not give an orthogonal matrix.

Theorem 10.2.17. *Gram Schmidt Algorithm (GSA)*

Let V be an inner product space and $S = \{w_1, w_2, \dots, w_k\}$ a linearly independent subset of V. Then $S' = \{w_1', w_2', \dots, w_k'\}$ where

$$w_j' = w_j - \sum_{i=1}^{j-1} \frac{\langle w_j, w_i' \rangle}{\|w_i'\|^2} w_i'$$

is an orthogonal set for which $\text{span}(S) = \text{span}(S')$. Furthermore,

$$S'' = \left\{ \frac{1}{\|w_i'\|} w_i' \mid i = 1, 2, \dots, k \right\}$$

is an orthonormal set with $\text{span}(S'') = \text{span}(S')$.

Proof: see Theorem 6.4 on page 244 of in Insel Spence and Friedberg. If you see the older versions of my Linear Algebra notes you'll find a derivation in the case of Euclidean space which is built from successive applications of projection operators. I thought the approach in Insel Spence and Friedberg was more efficient so I removed that story arc from these notes. $\square$

It might be helpful to write this out the GSA in Euclidean space.

Proposition 10.2.18. The Gram-Schmidt Process

If $S = \{v_1, v_2, \dots, v_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ then $S' = \{v_1', v_2', \dots, v_k'\}$ is an orthogonal set of vectors in $\mathbb{R}^n$ if we define v_i' as follows:

$$\begin{aligned}
v_1' &= v_1 \\
v_2' &= v_2 - \frac{v_2 \bullet v_1'}{v_1' \bullet v_1'} v_1' \\
v_3' &= v_3 - \frac{v_3 \bullet v_2'}{v_2' \bullet v_2'} v_2' - \frac{v_3 \bullet v_1'}{v_1' \bullet v_1'} v_1' \\
v_k' &= v_k - \frac{v_k \bullet v_{k-1}'}{v_{k-1}' \bullet v_{k-1}'} v_{k-1}' - \frac{v_k \bullet v_{k-2}'}{v_{k-2}' \bullet v_{k-2}'} v_{k-2}' - \cdots - \frac{v_k \bullet v_1'}{v_1' \bullet v_1'} v_1'.
\end{aligned}$$

Furthermore, $S'' = \{v_1'', v_2'', \dots, v_k''\}$ is an orthonormal set if we define $v_i'' = \widehat{v_i'} = \frac{1}{||v_i'||} v_i'$ for each $i = 1, 2, \dots, k$.

Proposition 10.2.19.

Let V be a finite dimensional inner product space then there exists an orthonormal basis for V.

Proof: apply the Gram Schmidt Algorithm to any basis for V. $\square$.

Example 10.2.20. *Suppose $v_1 = (1,0,0,0), v_2 = (3,1,0,0), v_3 = (3,2,0,3)$. Let's use the Gram-Schmidt Process to orthogonalize these vectors: let $v_1' = v_1 = (1,0,0,0)$ and calculate:*

$$v_2' = v_2 - \frac{v_2 \bullet v_1}{v_1 \bullet v_1} v_1 = (3,1,0,0) - 3(1,0,0,0) = (0,1,0,0).$$

Next,

$$v_3' = v_3 - \frac{v_3 \bullet v_2'}{v_2' \bullet v_2'} v_2' - \frac{v_3 \bullet v_1'}{v_1' \bullet v_1'} v_1' = (3,2,0,3) - 2(0,1,0,0) - 3(1,0,0,0) = (0,0,0,3).$$

We find the orthogonal set of vectors $\{e_1, e_2, e_4\}$. It just so happens this is also an orthonormal set of vectors.

Example 10.2.21. *Suppose $v_1 = (1,1,1), v_2 = (1,2,3), v_3 = (0,0,3)$ find an orthonormal set of vectors that spans span$\{v_1, v_2, v_3\}$. We can use Gram-Schmidt followed by a normalization, let $v_1' = (1,1,1)$ then calculate*

$$v_2' = (1,2,3) - \left(\frac{1+2+3}{3}\right)(1,1,1) = (1,2,3) - (2,2,2) = (-1,0,1).$$

as a quick check on my arthimetic note $v_1' \bullet v_2' = 0$ (good). Next,

$$v_3' = (0,0,3) - \left(\frac{0(-1)+0(0)+3(1)}{2}\right)(-1,0,1) - \left(\frac{0(1)+0(1)+3(1)}{3}\right)(1,1,1)$$

$$\Rightarrow v_3' = (0,0,3) + (\tfrac{3}{2}, 0, -\tfrac{3}{2}) - (1,1,1) = (\tfrac{1}{2}, -1, \tfrac{1}{2})$$

again it's good to check that $v_2' \bullet v_3' = 0$ and $v_1' \bullet v_3' = 0$ as we desire. Finally, note that $||v_1'|| = \sqrt{3}$, $||v_2'|| = \sqrt{2}$ and $||v_3'|| = \sqrt{3/2}$ hence

$$\boxed{v_1'' = \tfrac{1}{\sqrt{3}}(1,1,1), \qquad v_2'' = \tfrac{1}{\sqrt{2}}(-1,0,1), \qquad v_3'' = \sqrt{\tfrac{2}{3}}(\tfrac{1}{2}, -1, \tfrac{1}{2})}$$

are orthonormal vectors.

Example 10.2.22. *Let $W = span\{(1,0,0,0), (3,1,0,0), (3,2,0,3)\}$. Find an orthonormal basis for $W \leq \mathbb{R}^4$. Recall from Example 10.2.20 we applied Gram-Schmidt and found the orthonormal set of vectors $\{e_1, e_2, e_4\}$. That is an orthonormal basis for W.*

Example 10.2.23. *In Example 10.2.21 we found $\{v_1'', v_2'', v_3''\}$ is an orthonormal set of vectors. Since orthogonality implies linear independence it follows that this set is in fact a basis for $\mathbb{R}^{3\times 1}$. It is an* **orthonormal basis**. *Of course there are other bases which are orthogonal. For example, the standard basis is orthonormal.*

Example 10.2.24. *Let us define $S = \{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^4$ as follows:*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \ v_4 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

It is easy to verify that S defined below is a linearly independent set vectors basis for span$(S) \leq \mathbb{R}^{4\times 1}$. Let's see how to find an orthonormal basis for span(S). The procedure is simple: apply the Gram-Schmidt algorithm then normalize the vectors.

$$v_1' = v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2' = v_2 - \left(\frac{v_2 \bullet v_1'}{v_1' \bullet v_1'}\right) v_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3' = v_3 - \left(\frac{v_3 \bullet v_2'}{v_2' \bullet v_2'}\right) v_2' - \left(\frac{v_3 \bullet v_1'}{v_1' \bullet v_1'}\right) v_1' = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{0}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} v_4' &= v_4 - \left(\tfrac{v_4 \bullet v_3'}{v_3' \bullet v_3'}\right) v_3' - \left(\tfrac{v_3 \bullet v_2'}{v_2' \bullet v_2'}\right) v_2' - \left(\tfrac{v_3 \bullet v_1'}{v_1' \bullet v_1'}\right) v_1' \\ &= \begin{bmatrix} 3 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \tfrac{1}{14}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \tfrac{1}{14}\begin{bmatrix} 9 \\ 0 \\ -27 \\ 18 \end{bmatrix} \end{aligned}$$

Then normalize to obtain the orthonormal basis for Span(S) below:

$$\beta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{42}}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \tfrac{1}{9\sqrt{14}}\begin{bmatrix} 9 \\ 0 \\ -27 \\ 18 \end{bmatrix} \Bigg\}$$

Example 10.2.25. *Let $v = [1, 2, 3, 4]$. Find the coordinates of v with respect to the orthonormal basis β found in Example 10.2.24.*

$$\beta = \{f_1, f_2, f_3, f_4\} = \left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{42}}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \tfrac{1}{9\sqrt{14}}\begin{bmatrix} 9 \\ 0 \\ -27 \\ 18 \end{bmatrix} \right\}$$

Let us denote the coordinates vector $[v]_\beta = [w_1, w_2, w_3, w_4]$ we know we can calculate these by taking the dot-products with the vectors in the orthonormal basis β:

$$w_1 = v \bullet f_1 = \frac{1}{\sqrt{3}}[1, 2, 3, 4][1, 0, 1, 1]^T = \frac{8}{\sqrt{3}}$$

$$w_2 = v \bullet f_2 = [1,2,3,4][0,1,0,0]^T = 2$$
$$w_3 = v \bullet f_3 = \frac{1}{\sqrt{42}}[1,2,3,4][-5,0,1,4]^T = \frac{14}{\sqrt{42}}$$
$$w_4 = v \bullet f_4 = \tfrac{1}{9\sqrt{14}}[1,2,3,4][9,0,-27,18]^T = \tfrac{0}{9\sqrt{14}} = 0$$

Therefore, $[v]_\beta = [\frac{8}{\sqrt{3}}, 2, \frac{14}{\sqrt{42}}, 0]$. *Now, let's check our answer. What should this mean if it is correct? We should be able verify* $v = w_1f_1 + w_2f_2 + w_3f_3 + w_4f_4$*:*

$$\begin{aligned} w_1f_1 + w_2f_2 + w_3f_3 + w_4f_4 &= \frac{8}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{14}{\sqrt{42}}\frac{1}{\sqrt{42}}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix} \\ &= \frac{8}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 8/3 - 5/3 \\ 2 \\ 8/3 + 1/3 \\ 8/3 + 4/3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \end{aligned}$$

Well, that's a relief.

Example 10.2.26. *Consider the inner-product* $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$ *for* $f, g \in C[-1,1]$*. Let's calculate the length squared of the standard basis:*

$$\langle 1, 1\rangle = \int_{-1}^{1} 1 \cdot 1 dx = 2, \qquad \langle x, x\rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\bigg|_{-1}^{1} = \frac{2}{3}$$
$$\langle x^2, x^2\rangle = \int_{-1}^{1} x^4 dx = \frac{x^5}{5}\bigg|_{-1}^{1} = \frac{2}{5}$$

Notice that the standard basis of P_2 *are not all* $\langle\ ,\ \rangle$*-orthogonal:*

$$\langle 1, x\rangle = \int_{-1}^{1} x dx = 0 \qquad \langle 1, x^2\rangle = \langle x, x\rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3} \qquad \langle x, x^2\rangle = \int_{-1}^{1} x^3 dx = 0$$

We can use the Gram-Schmidt process on $\{1, x, x^2\}$ *to find an orthonormal basis for* P_2 *on* $[-1,1]$*. Let,* $u_1(x) = 1$ *and*

$$u_2(x) = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} = x$$
$$u_3(x) = x^2 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle}x - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} = x^2 - \frac{1}{3}$$

We have an orthogonal set of functions $\{u_1, u_2, u_3\}$ we already calculated the length of u_1 and u_2 so we can immediately normalize those by dividing by their lengths; $v_1(x) = \frac{1}{\sqrt{2}}$ and $v_2(x) = \sqrt{\frac{3}{2}}x$. We need to calculate the length of u_3 so we can normalize it as well:

$$\langle u_3, u_3\rangle = \int_{-1}^{1} (x^2 - \tfrac{1}{3})^2 dx = \int_{-1}^{1} (x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9}) dx = \tfrac{2}{5} - \tfrac{4}{9} + \tfrac{2}{9} = \tfrac{8}{45}$$

Thus $v_3(x) = \sqrt{\frac{8}{45}}(x^2 - \frac{1}{3})$ has length one. Therefore, $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{8}{45}}(x^2 - \frac{1}{3}) \right\}$ is an orthonormal basis for P_2 restricted to $[-1, 1]$. Other intervals would not have the same basis. This construction depends both on our choice of inner-product and the interval considered. Incidentally, these are the first three **Legendre Polynomials**. *These arise naturally as solutions to certain differential equations. The theory of* **orthogonal polynomials** *is full of such calculations. Orthogonal polynomials are quite useful as approximating functions. If we offered a second course in differential equations we could see the full function of such objects.*

10.3 orthogonal complements and projections

We can extend the concept of orthgonality to subspaces. Throughout this section V denotes an inner product space.

Definition 10.3.1.

Suppose W_1 and W_2 are subspaces of an inner product space V then we say W_1 is **orthogonal** to W_2 and write $W_1 \perp W_2$ iff $\langle w_1, w_2\rangle = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$.

Example 10.3.2. *Let $W_1 = span\{e_1, e_2\}$ and $W_2 = span\{e_3\}$ then $W_1, W_2 \leq \mathbb{R}^n$. Let $w_1 = ae_1 + be_2 \in W_1$ and $w_2 = ce_3 \in W_2$ calculate,*

$$w_1 \bullet w_2 = (ae_1 + be_2) \bullet (ce_3) = ace_1 \bullet e_3 + bce_2 \bullet e_3 = 0$$

Hence $W_1 \perp W_2$. Geometrically, we have shown the xy-plane is orthogonal to the z-axis.

We notice that orthogonality relative to the basis will naturally extend to the span of the basis via the properties of the inner product:

Proposition 10.3.3.

Suppose W_1, W_2 are subspaces of an inner product space V and W_1 has basis $\{w_i\}_{i=1}^r$ and W_2 has basis $\{v_j\}_{j=1}^s$. Then W_1 is orthogonal to W_2 iff $w_i \bullet v_j = 0$ for all $1 \leq i \leq r, 1 \leq j \leq s$.

Proof: Suppose $\{w_i\}_{i=1}^r$ is a basis for $W_1 \leq V$ and $\{v_j\}_{j=1}^s$ for $W_2 \leq V$. If $W_1 \perp W_2$ then clearly $\{w_i\}_{i=1}^r$ is orthogonal to $\{v_j\}_{j=1}^s$. Conversely, suppose $\{w_i\}_{i=1}^r$ is orthogonal to $\{v_j\}_{j=1}^s$ then let $x \in W_1$ and $y \in W_2$:

$$\langle x, y\rangle = \left\langle \sum_{i=1}^{r} x_i w_i \ , \ \sum_{i=1}^{s} y_j w_j \right\rangle = \sum_{i=1}^{r}\sum_{j=1}^{s} x_i \overline{y_j}\langle w_i, v_j\rangle = 0. \qquad \square$$

Given a subspace W which lives in $\mathbb{R}^n$ we might wonder what is the largest subspace which is orthogonal to W? In $\mathbb{R}^{3\times 1}$ it is clear that the xy-plane is the largest subspace which is orthogonal to the z-axis, however, if the xy-plane was viewed as a subset of $\mathbb{R}^{4\times 1}$ we could actually find a volume which was orthogonal to the z-axis (in particular $span\{e_1, e_2, e_4\} \perp span\{e_3\}$).

Definition 10.3.4.

Let $S \subseteq V$ then $S^\perp$ is defined as follows:

$$S^\perp = \{x \in V | \langle x, s\rangle = 0 \text{ for all } s \in S\}$$

It is clear that $S^\perp$ is the largest subset in V which is orthogonal to span(S). Better than just that, it's the largest subspace orthogonal to span(S).

Proposition 10.3.5.

Let $S \subseteq V$ then $S^\perp \leq V$.

Proof: Let $x, y \in S^\perp$ and let $c \in \mathbb{F}$. Furthermore, suppose $s \in S$ and note

$$\langle cx + y, s\rangle = c\langle x, s\rangle + \langle y, s\rangle = c(0) + 0 = 0 \quad \Rightarrow \quad cx + y \in S^\perp$$

since $\langle 0, s\rangle = 0$ we know $0 \in S^\perp$ hence $S^\perp$ is a nonempty subset of V which is closed under vector addition and scalar multiplication. Thus, $S^\perp \leq V$. $\square$

Example 10.3.6. *$\{0\}^\perp = V$ since $\langle 0, x\rangle = 0$ for each $x \in V$. Likewise, $V^\perp = \{0\}$ as $\langle x, s\rangle = 0$ for all $s \in V$ implies $x = 0$. Notice, $(\{0\}^\perp)^\perp) = \{0\}$ and $(V^\perp)^\perp = V$.*

Example 10.3.7. *Find the orthogonal complement to $W = span\{v_1 = (1, 1, 0, 0), v_2 = (0, 1, 0, 2)\}$. Let's treat this as a matrix problem. We wish to describe a typical vector in $W^\perp$. Towards that goal, let $r = (x, y, z, w) \in W^\perp$ then the conditions that r must satisfy are $v_1 \bullet r = v_1^T r = 0$ and $v_2 \bullet r = v_2^T r = 0$. But this is equivalent to the single matrix equation below:*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad r = \begin{bmatrix} 2w \\ -2w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $W^\perp = span\{(0, 0, 1, 0), (2, -2, 0, 1)\}$.

If you study the preceding example it becomes clear that finding the orthogonal complement of a set of column vectors is equivalent to calculating the null space of a particular matrix. We have considerable experience in such calculations so this is a welcome observation.

Proposition 10.3.8.

If $S = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$ and $A = [v_1|v_2|\cdots|v_k]$ then $S^\perp = Null(A^T)$

Proof: Denote $A = [v_1|v_2|\cdots|v_k] \in \mathbb{R}^{\ n\times k}$ and $x = [x_1, x_2, \dots, x_k]^T$. Observe that:

$$\begin{aligned} x \in Null(A^T) &\Leftrightarrow A^T x = 0 \\ &\Leftrightarrow [row_1(A^T)x, row_2(A^T)x, \cdots, row_k(A^T)x] = 0 \\ &\Leftrightarrow [(col_1(A))^T x, (col_2(A))^T x, \cdots, (col_k(A))^T x] = 0 \\ &\Leftrightarrow [v_1 \bullet x, v_2 \bullet x, \cdots, v_k \bullet x] = 0 \\ &\Leftrightarrow v_j \bullet x = 0 \text{ for } j = 1, 2, \dots, k \\ &\Leftrightarrow x \in S^\perp \end{aligned}$$

Therefore, $Null(A^T) = S^{\perp}$. $\square$

Given the correspondence above we should be interested in statements which can be made about the row and column space of a matrix. That seems like a good homework problem. See Problem 126 of Spring 2017.

Proposition 10.3.9.

Let $W_1, W_2 \leq V$, if $W_1 \perp W_2$ then $W_1 \cap W_2 = \{0\}$

Proof: let $z \in W_1 \cap W_2$ then $z \in W_1$ and $z \in W_2$ and since $W_1 \perp W_2$ it follows $\langle z, z\rangle = 0$ hence $z = 0$ and $W_1 \cap W_2 \subseteq \{0\}$. The reverse inclusion $\{0\} \subseteq W_1 \cap W_2$ is clearly true since 0 is in every subspace. Therefore, $W_1 \cap W_2 = \{0\}$ $\square$

We defined the direct sum of two subspaces in the Section 7.7. The fact that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$ was sufficient to prove $V \approxeq W_1 \times W_2$ so, by our definition, we can write $V = W_1 \oplus W_2$. The theorem below is at the heart of many geometric arguments in multivariate calculus. Intuitively I think of it like this: if we show $x \notin W$ then by process of elimination it must be in $W^{\perp}$. Intuition fails unless $W^{\perp}$ is a **complementary subspace**. We say W_1 and W_2 are complementary subspaces of V iff $V = W_1 \oplus W_2$.

Theorem 10.3.10.

Let $W \leq V$ where V is an inner product space over $\mathbb{F}$ with $\dim(V) = n$ then

(a.) for each $x \in V$ there exist unique $x_o \in W$ and $x_1 \in W^{\perp}$ for which $x = x_o + x_1$

(b.) if $\beta = \{w_1, \dots, w_k\}$ is an orthonormal basis for W and if $x = x_o + x_1$ for$x_o \in W$ and $x_1 \in W^{\perp}$ then x_o is given by the **projection** operator:

$$\text{Proj}_W(x) = \sum_{i=1}^{k} \langle x, w_i\rangle w_i = x_o$$

and x_1 is given by the **orthogonal projection**

$$\text{Orth}_W(x) = x - \text{Proj}_W(x)$$

Thus $x = \text{Proj}_W(x) + \text{Orth}_W(x)$ for each $x \in V$.

(c.) $V = W \oplus W^{\perp}$.

(d.) $dim(W) + dim(W^{\perp}) = n$,

(e.) $(W^{\perp})^{\perp} = W$,

Proof: let $\beta = \{w_1, \dots, w_k\}$ be an orthonormal basis for $W \leq V$ and suppose $x \in V$. Construct

$x_o = \sum_{i=1}^{k} \langle x, w_i \rangle w_i$ and observe $x_o \in W$. Let $x_1 = x - x_o$ hence $x = x_o + x_1$. Consider,

$$\begin{aligned}
\langle x_1, w_j \rangle &= \langle x - x_o, w_j \rangle \\
&= \langle x, w_j \rangle - \langle x_o, w_j \rangle \\
&= \langle x, w_j \rangle - \left\langle \sum_{i=1}^{k} \langle x, w_i \rangle w_i, w_j \right\rangle \\
&= \langle x, w_j \rangle - \sum_{i=1}^{k} \langle x, w_i \rangle \langle w_i, w_j \rangle \\
&= \langle x, w_j \rangle - \sum_{i=1}^{k} \langle x, w_i \rangle \delta_{ij} \\
&= \langle x, w_j \rangle - \langle x, w_j \rangle \\
&= 0.
\end{aligned}$$

Thus, by Problem 124 of Spring 2017, $\langle x_1, w \rangle = 0$ for each $w \in W$ hence $x_1 \in W^\perp$. To establish uniqueness for (a.) suppose $x_o' \in W$ and $x_1' \in W^\perp$ also provide $x = x_o' + x_1'$ hence $x_o + x_1 = x_o' + x_1'$ which gives $x_o - x_o' = x_1 - x_1' \in W \cap W^\perp$. Then, by Proposition 10.3.9 we find $x_o - x_o' = 0 = x_1 - x_1'$ hence $x_o = x_o'$ and $x_1 = x_1'$. Note parts (b.), (c.) and (d.) follow from the proof we give here and our previous work on direct sum decompositions. See Theorem 7.7.4.

To prove part (e.) let $U = W^\perp$ and note by (c.) we have $U \oplus U^\perp = V$. Thus $V = W \oplus W^\perp$ and $V = W^\perp \oplus (W^\perp)^\perp$. In invite the reader[5] to prove this data implies $W = (W^\perp)^\perp$. $\square$

The proof above allows us to logically offer the definition below for the projection and orthogonal projection with respect to a subspace W of an inner product space of V:

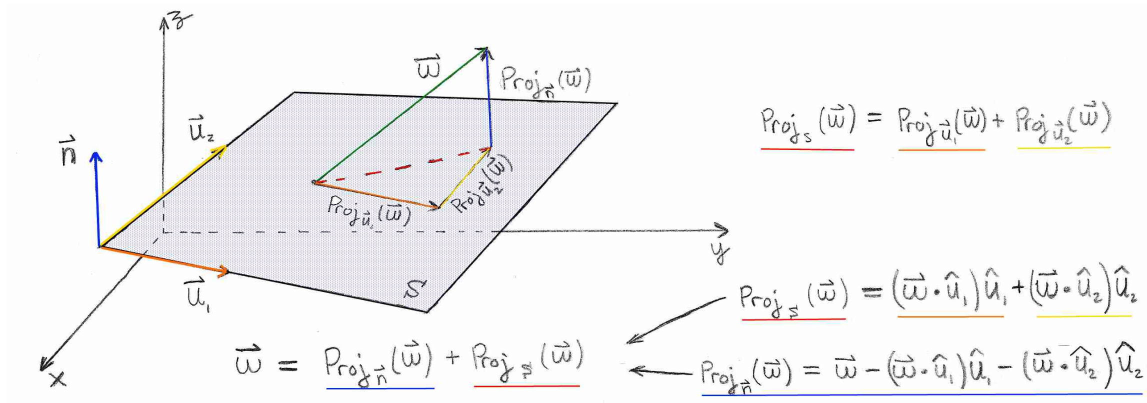
Definition 10.3.11.

Let $W \leq V$ if $z \in V$ and $z = u + w$ for some $u \in W$ and $w \in W^\perp$ then we define $u = Proj_W(z)$ and $w = Orth_W(z)$. Equivalently, choose an orthonormal basis $\beta = \{v_1, v_2, \dots v_k\}$ for W then if $z \in V$ we define

$$Proj_W(z) = \sum_{i=1}^{k} \langle z, v_i \rangle v_i \qquad \text{and} \qquad Orth_W(z) = z - Proj_W(z).$$

Perhaps the following picture helps: here I show projections onto a plane with basis $\{\vec{u}_1, \vec{u}_2\}$ and its normal $\vec{n}$.

[5] see Problem 125



Example 10.3.12. *Let $W = span\{e_1+e_2, e_3\}$ and $x = (1,2,3)$ calculate $Proj_W(x)$. To begin I note that the given spanning set is orthogonal and hence linear indpendent. We need only orthonormalize to obtain an orthonormal basis β for W*

$$\beta = \{v_1, v_2\} \qquad with \qquad v_1 = \tfrac{1}{\sqrt{2}}(1,1,0),\; v_2 = (0,0,1)$$

Calculate, $v_1 \bullet x = \frac{3}{\sqrt{2}}$ and $v_2 \bullet x = 3$. Thus,

$$Proj_W((1,2,3)) = (v_1 \bullet x)v_1 + (v_2 \bullet x)v_2 = \tfrac{3}{\sqrt{2}}v_1 + 3v_2 = (\tfrac{3}{2}, \tfrac{3}{2}, 3)$$

Then it's easy to calculate the orthogonal part,

$$Orth_W((1,2,3)) = (1,2,3) - (\tfrac{3}{2}, \tfrac{3}{2}, 3) = (-\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

As a check on the calculation note that $Proj_W(x) + Orth_W(x) = x$ and $Proj_W(x) \bullet Orth_W(x) = 0$.

Example 10.3.13. *Let $W = span\{u_1, u_2, u_3\} \leq \mathbb{R}^4$ where*

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

calculate $Proj_W([0,6,0,6]^T)$ [6]. Notice that the given spanning set appears to be linearly independent but it is not orthogonal. Apply Gram-Schmidt to fix it:

$$\begin{aligned}
v_1 &= u_1 = (2,1,2,0) \\
v_2 &= u_2 - \tfrac{u_2 \bullet v_1}{v_1 \bullet v_1}v_1 = u_2 = (0,-2,1,1) \\
v_3 &= u_3 - \tfrac{u_3 \bullet v_1}{v_1 \bullet v_1}v_1 - \tfrac{u_3 \bullet v_2}{v_2 \bullet v_2}v_2 = u_3 + \tfrac{5}{6}v_2 = (-1,2,0,-1) + (0, -\tfrac{10}{6}, \tfrac{5}{6}, \tfrac{5}{6})
\end{aligned}$$

We calculate,

$$v_3 = (-1,\; 2 - \tfrac{5}{3},\; \tfrac{5}{6},\; -1 + \tfrac{5}{6}\;) = (-1,\; \tfrac{1}{3},\; \tfrac{5}{6},\; -\tfrac{1}{6}\;) = \tfrac{1}{6}(-6,2,5,-1)$$

The normalized basis follows easily,

$$v_1' = \tfrac{1}{3}(2,1,2,0) \qquad v_2' = \tfrac{1}{\sqrt{6}}(0,-2,1,1) \qquad v_3' = \tfrac{1}{\sqrt{66}}(-6,2,5,-1)$$

[6]this problem is inspired from Anton & Rorres' §6.4 homework problem 3 part d.

Calculate dot-products in preparation for the projection calculation,

$$v_1' \bullet x = \tfrac{1}{3}[2,1,2,0](0,6,0,6) = 2$$

$$v_2' \bullet x = \tfrac{1}{\sqrt{6}}[0,-2,1,1](0,6,0,6) = \tfrac{1}{\sqrt{6}}(-12+6) = -\sqrt{6}$$
$$v_3' \bullet x = \tfrac{1}{\sqrt{66}}[-6,2,5,-1](0,6,0,6) = \tfrac{1}{\sqrt{66}}(12-6) = \tfrac{6}{\sqrt{66}}$$

Now we calculate the projection of $x = (0,6,0,6)$ onto W with ease:

$$\begin{aligned} Proj_W(x) &= (x \bullet v_1')v_1' + (x \bullet v_2')v_2' + (x \bullet v_3')v_3' \\ &= (2)\tfrac{1}{3}(2,1,2,0) - (\sqrt{6})\tfrac{1}{\sqrt{6}}(0,-2,1,1) + (\tfrac{6}{\sqrt{66}})\tfrac{1}{\sqrt{66}}(-6,2,5,-1) \\ &= (\tfrac{4}{3}, \tfrac{2}{3}, \tfrac{4}{3}, 0) + (0,2,-1,-1) + (\tfrac{-6}{11}, \tfrac{2}{11}, \tfrac{5}{11}, \tfrac{-1}{11}) \\ &= (\tfrac{26}{33}, \tfrac{94}{33}, \tfrac{26}{33}, \tfrac{-36}{33}) \end{aligned}$$

and,

$$Orth_W(x) = (\tfrac{-26}{33}, \tfrac{104}{33}, \tfrac{-26}{33}, \tfrac{234}{33})$$

10.4 approximation by projection

In this section we consider results which ultimately show how to find the best approximation to problems which have no exact solution. In other words, we consider how to almost solve inconsistent systems in the best way possible. Once again we assume V is an inner product space throughout this section.

10.4.1 the closest vector problem

Suppose we are given a subspace and a vector not in the subspace, which vector in the subspace is closest to the external vector ? Naturally the projection answers this question. The projection of the external vector onto the subspace will be closest. Let me be a bit more precise:

Proposition 10.4.1. *Closest vector inequality.*

If $W \leq V$ and $b \in V$ such that $b \notin W$ then for all $u \in W$ with $u \neq Proj_W(b)$,

$$||b - Proj_W(b)|| < ||b - u||.$$

This means $Proj_W(b)$ is the closest vector to b in W.

Proof: Noice that $b - u = b - Proj_W(b) + Proj_W(b) - u$. Furthermore note that $b - Proj_W(b) = Orth_W(b) \in W^\perp$ whereas $Proj_W(b) - u \in W$ hence these are orthogonal vectors and we can apply the Pythagorean Theorem,

$$||b-u||^2 = ||b - Proj_W(b)||^2 + ||Proj_W(b) - u||^2$$

Notice that $u \neq Proj_W(b)$ implies $Proj_W(b) - u \neq 0$ hence $||Proj_W(b) - u||^2 > 0$. It follows that $||b - Proj_W(b)||^2 < ||b-u||^2$. And as the $||\cdot||$ is nonnegative[7] we can take the squareroot to obtain $||b - Proj_W(b)|| < ||b-u||$. $\square$

[7] notice $a^2 < b^2$ need not imply $a < b$ in general. For example, $(5)^2 < (-7)^2$ yet $5 \not< -7$. Generally, $a^2 < b^2$ together with the added condition $a, b > 0$ implies $a < b$.

Remark 10.4.2.

In calculus III I show at least three distinct methods to find the point off a plane which is closest to the plane. We can minimize the distance function via the 2nd derivative test for two variables, or use Lagrange Multipliers or use the geometric solution which invokes the projection operator. It's nice that we have an explicit proof that the geometric solution is valid. We had argued on the basis of geometric intuition that $Orth_S(b)$ is the shortest vector from the plane S to the point b off the plane[8] Now we have proof. Better yet, our proof equally well applies to subspaces of $\mathbb{R}^n$. In fact, this discussion extends to the context of inner product spaces.

Example 10.4.3. *Consider $\mathbb{R}^2$ let $W = span\{(1,1)\}$. Find the point on the line W closest to the point $(4,0)$.*

$$Proj_W((4,0)) = \tfrac{1}{2}((1,1)\bullet(4,0))(1,1) = (2,2)$$

Thus, $(2,2) \in W$ is the closest point to $(4,0)$. Geometrically, this is something you should have been able to derive for a few years now. The points $(2,2)$ and $(4,0)$ are on the perpendicular bisector of $y = x$ (the set W is nothing more than the line $y = x$ making the usual identification of points and vectors)

Example 10.4.4. *In Example 10.3.13 we found that $W = span\{u_1, u_2, u_3\} \leq \mathbb{R}^4$ where*

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

has $Proj_W((0,6,0,6)) = (\frac{26}{33}, \frac{94}{33}, \frac{26}{33}, \frac{-36}{33})$. We can calculate that

$$rref\left[\begin{array}{rrr|r} 2 & 0 & -1 & 0 \\ 1 & -2 & 2 & 6 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

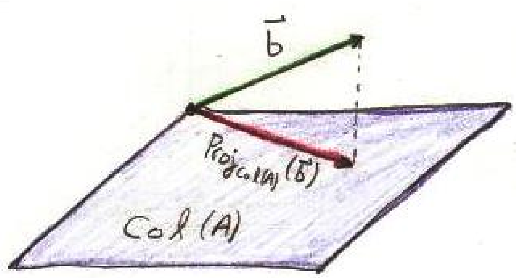
This means that $(0,6,0,6) \notin W$. However, we learned in Proposition 10.4.1 that $Proj_W((0,6,0,6))$ is the vector in W which is closest to $(0,6,0,6)$. Notice that we can deduce that the orthogonal basis from Example 10.3.13 unioned with $Orth_W((0,6,0,6))$ will form an orthogonal basis for $\mathbb{R}^4$. To modify it to an orthonormal basis we could simply normalize each vector to length one.

Example 10.4.5. *Example 10.3.12 shows that $W = span\{e_1 + e_2, e_3\}$ and $x = (1,2,3)$ yields $Proj_W(x) = (\frac{3}{2}, \frac{3}{2}, 3)$. Again we can argue that $x \notin Col[e_1 + e_2|e_3] = W$ but $Proj_W(x)$ is in fact in W. Moreover, $Proj_W(x)$ is the closest vector to x which is in W. The geometric interpretation here is that $Orth_W(x) = (-\frac{1}{2}, \frac{1}{2}, 0)$ is precisely the normal vector to the plane W.*

The examples above are somewhat special in that the subspaces considered have only one dimension less than the total vector space. This means that the orthogonal projection of any vector outside the subspace will return the same vector modulo a nonzero constant. In other words, the orthogonal complement is selecting the normal vector to our subspace. In general if we had a subspace which was two or more dimensions smaller than the total vector space then there would be more variety in the output of the orthogonal projection with respect to the subspace. For example, if we consider a plane inside $\mathbb{R}^4$ then there is more than just one direction which is orthogonal to the plane, the orthogonal projection would itself fill out a plane in $\mathbb{R}^4$.

10.4.2 the least squares approximation

We've spent considerable time solving systems of equations which were *consistent*. What if a system of equations $Ax = b$ is inconsistent? What if anything can we say? Let $A \in \mathbb{R}^{m \times n}$ then we found in Proposition 6.8.3 $\boxed{Ax = b \text{ is consistent iff } b \in Col(A).}$ In other words, the system has a solution iff there is some linear combination of the columns of A such that we obtain b. Here the columns of A and b are both m-dimensional vectors. If $rank(A) = dim(Col(A)) = m$ then the system is consistent no matter which choice for b is made. However, if $rank(A) < m$ then there are some vectors in $\mathbb{R}^m$ which are not in the column space of A and if $b \notin Col(A)$ then there will be no $x \in \mathbb{R}^n$ such that $Ax = b$. We can picture it as follows: the $Col(A)$ is a subspace of $\mathbb{R}^m$ and $b \in \mathbb{R}^m$ is a vector pointing out of the subspace. The shadow of b onto the subspace $Col(A)$ is given by $Proj_{Col(A)}(b)$.



Notice that $Proj_{Col(A)}(b) \in Col(A)$ thus the system $Ax = Proj_{Col(A)}(b)$ has a solution for any $b \in \mathbb{R}^m$. In fact, we can argue that x which solves $Ax = Proj_{Col(A)}(b)$ is the solution which comes closest to solving $Ax = b$. Closest in the sense that $||Ax - b||^2$ is minimized. We call such x the least squares solution to $Ax = b$ (*which is kind-of funny terminology since x is not actually a solution, perhaps we should really call it the "least squares approximation"*).

Theorem 10.4.6. *Least Squares Solution:*

> If $Ax = b$ is inconsistent then the solution of $Au = Proj_{col(A)}(b)$ minimizes $||Ax - b||^2$.

Proof: We can break-up the vector b into a vector $Proj_{Col(A)}(b) \in Col(A)$ and $Orth_{col(A)}(b) \in Col(A)^{\perp}$ where

$$b = Proj_{Col(A)}(b) + Orth_{Col(A)}(b).$$

Since $Ax = b$ is inconsistent it follows that $b \notin Col(A)$ thus $Orth_{Col(A)}(b) \neq 0$. Observe that:

$$\begin{aligned} ||Ax - b||^2 &= ||Ax - Proj_{Col(A)}(b) - Orth_{Col(A)}(b)||^2 \\ &= ||Ax - Proj_{Col(A)}(b)||^2 + ||Orth_{Col(A)}(b)||^2 \end{aligned}$$

Therefore, the solution of $Ax = Proj_{Col(A)}(b)$ minimizes $||Ax - b||^2$ since any other vector will make $||Ax - Proj_{Col(A)}(b)||^2 > 0$. $\square$

Admittably, there could be more than one solution of $Ax = Proj_{Col(A)}(b)$, however it is usually the case that this system has a unique solution. Especially for expermentally determined data sets.

We already have a technique to calculate projections and of course we can solve systems but it is exceedingly tedious to use the proposition above from scratch. Fortunately there is no need:

Proposition 10.4.7.

If $Ax = b$ is inconsistent then the solution(s) of $Au = Proj_{Col(A)}(b)$ are solutions of the so-called **normal equations** $A^TAu = A^Tb$.

Proof: Observe that,

$$\begin{aligned} Au = Proj_{Col(A)}(b) \quad &\Leftrightarrow \quad b - Au = b - Proj_{Col(A)}(b) = Orth_{Col(A)}(b) \\ &\Leftrightarrow \quad b - Au \in Col(A)^{\perp} \\ &\Leftrightarrow \quad b - Au \in Null(A^T) \\ &\Leftrightarrow \quad A^T(b - Au) = 0 \\ &\Leftrightarrow \quad A^TAu = A^Tb, \end{aligned}$$

where we used Problem 126 of Spring 2017 in the third step. $\square$

The proposition below follows immediately from the preceding proposition.

Proposition 10.4.8.

If $det(A^TA) \neq 0$ then there is a unique solution of $Au = Proj_{Col(A)}(b)$.

The proposition above is the calculational core of the least squares method.

In experimental studies we often have some model with coefficients which appear linearly. We perform an experiment, collect data, then our goal is to find coefficients which make the model fit the collected data. Usually the data will be inconsistent with the model, however we'll be able to use the idea of the last section to find the so-called *best-fit* curve. I'll begin with a simple linear model. This linear example contains all the essential features of the least-squares analysis.

linear least squares problem

Problem: find values of c_1, c_2 such that $y = c_1x + c_2$ most closely models a given data set: $\{(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)\}$

Solution: Plug the data into the model and see what equations result:

$$y_1 = c_1x_1 + c_2, \quad y_2 = c_1x_2 + c_2, \quad \dots \quad y_k = c_1x_k + c_2$$

arrange these as a matrix equation,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_k & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Rightarrow \vec{y} = M\vec{v}$$

where $\vec{y} = (y_1, y_2, \dots, y_k)$ and $v = (c_1, c_2)$ and M is defined in the obvious way. The system $\vec{y} = M\vec{v}$ will be inconsistent due to the fact that error in the data collection will[9] make the results bounce above and below the true solution. We can solve the normal equations $M^T\vec{y} = M^TM\vec{v}$ to find c_1, c_2 which give the best-fit curve[10].

[9] almost always

[10] notice that if x_i are not all the same then it is possible to show $det(M^TM) \neq 0$ and then the solution to the normal equations is unique

Example 10.4.9. *Find the best fit line through the points* $(0, 2), (1, 1), (2, 4), (3, 3)$*. Our model is* $y = c_1 + c_2 x$*. Assemble* M *and* $\vec{y}$ *as in the discussion preceding this example:*

$$\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad M^T M = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}$$

$$\textit{and we calculate:} \qquad M^T y = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$$

The normal equations[11] *are* $M^T M \vec{v} = M^T \vec{y}$*. Note that* $(M^T M)^{-1} = \frac{1}{20} \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$ *thus the solution of the normal equations is simply,*

$$\vec{v} = (M^T M)^{-1} M^T \vec{y} = \frac{1}{20} \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} 18 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, $\boxed{y = 0.6x + 1.6}$ *is the best-fit line. This solution minimizes the vertical distances squared between the data and the model.*

It's really nice that the order of the normal equations is only as large as the number of coefficients in the model. If the order depended on the size of the data set this could be much less fun for real-world examples. Let me set-up the linear least squares problem for 3-coefficients and data from $\mathbb{R}^3$, the set-up for more coefficients and higher-dimensional data is similar. We already proved this in general in the last section, the proposition simply applies mathematics we already derived. I state it for your convenience.

Proposition 10.4.10.

Given data $\{\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n\} \subset \mathbb{R}^3$, with $\vec{r}_k = [x_k, y_k, z_k]^T$, the best-fit of the linear model $z = c_1 x + c_2 y + c_3$ is obtained by solving the normal equations $M^T M \vec{v} = M^T \vec{z}$ where

$$\vec{z} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad M = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} \qquad \vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

Example 10.4.11. *Find the plane which is closest to the points*

$$(0, 0, 0), (1, 2, 3), (4, 0, 1), (0, 3, 0), (1, 1, 1).$$

[11] notice my choice to solve this system of 2 equations and 2 unknowns is just a choice, You can solve it a dozen different ways, you do it the way which works best for you.

An arbitrary[12] *plane has the form* $z = c_1x + c_2y + c_3$. *Work on the normal equations,*

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad M^TM = \begin{bmatrix} 0 & 1 & 4 & 0 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 3 & 6 \\ 3 & 14 & 6 \\ 6 & 6 & 5 \end{bmatrix}$$

$$also, \quad M^T\vec{z} = \begin{bmatrix} 0 & 1 & 4 & 0 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

We solve $M^TM\vec{v} = M^T\vec{z}$ *by row operations, after some calculation we find:*

$$rref[M^TM|M^T\vec{z}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 89/279 \\ 0 & 1 & 1 & 32/93 \\ 0 & 0 & 1 & 19/93 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} c_1 = 89/279 \\ c_2 = 32/93 \\ c_3 = 19/93 \end{array}$$

Therefore, $z = \frac{89}{293}x + \frac{32}{93}y + \frac{19}{93}$ *is the plane which is "closest" to the given points. Technically, I'm not certain that is is the absolute closest. We used the vertical distance squared as a measure of distance from the point. Distance from a point to the plane is measured along the normal direction, so there is no garauntee this is really the absolute "best" fit. For the purposes of this course we will ignore this subtle and annoying point. When I say "best-fit" I mean the least squares fit of the model.*

nonlinear least squares

Problem: find values of c_1, c_2 such that $y = c_1f_1(x)x + c_2f_2(x) + \cdots + c_nf_n(x)$ most closely models a given data set: $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$. We assume the coefficients c_1, c_2 appear linearly on (possibly nonlinear) functions $f_1, f_2, \ldots f_n$.

Solution: Plug the data into the model and see what equations result:

$$\begin{aligned} y_1 &= c_1f_1(x_1) + c_2f_2(x_1) + \cdots + c_nf_n(x_1), \\ y_2 &= c_1f_1(x_2) + c_2f_2(x_2) + \cdots + c_nf_n(x_2), \\ \vdots \quad &\quad \vdots \qquad\qquad \vdots \\ y_k &= c_1f_1(x_k) + c_2f_2(x_k) + \cdots + c_nf_n(x_k) \end{aligned}$$

arrange these as a matrix equation,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_k) & f_2(x_k) & \cdots & f_n(x_k) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \Rightarrow \vec{y} = M\vec{v}$$

[12]technically, the general form for a plane is $ax + by + cz = d$, if $c = 0$ for the best solution then our model misses it. In such a case we could let x or y play the role that z plays in our set-up.

where $\vec{y} = [y_1, y_2, \dots, y_k]^T$, $v = [c_1, c_2, \dots, c_n]^T$ and M is defined in the obvious way. The system $\vec{y} = M\vec{v}$ will be inconsistent due to the fact that error in the data collection will[13] make the results bounce above and below the true solution. We can solve the normal equations $M^T\vec{y} = M^T M\vec{v}$ to find $c_1, c_2, \dots, c_n$ which give the best-fit curve[14].

Remark 10.4.12.

Nonlinear least squares includes the linear case as a subcase, take $f_1(x) = x$ and $f_2(x) = 1$ and we return to the linear least squares examples. We will use data sets from $\mathbb{R}^2$ in this subsection. These techniques do extend to data sets with more variables as I demonstrated in the simple case of a plane.

Example 10.4.13. *Find the best-fit parabola through the data $(0,0), (1,3), (4,4), (3,6), (2,2)$. Our model has the form $y = c_1x^2 + c_2x + c_3$. Identify that $f_1(x) = x^2$, $f_2(x) = x$ and $f_3(x) = 1$ thus we should study the normal equations: $M^T M\vec{v} = M^T\vec{y}$ where:*

$$M = \begin{bmatrix} f_1(0) & f_2(0) & f_3(0) \\ f_1(1) & f_2(1) & f_3(1) \\ f_1(4) & f_2(4) & f_3(4) \\ f_1(3) & f_2(3) & f_3(3) \\ f_1(2) & f_2(2) & f_3(2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 16 & 4 & 1 \\ 9 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \vec{y} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 6 \\ 2 \end{bmatrix}.$$

Hence, calculate

$$M^T M = \begin{bmatrix} 0 & 1 & 16 & 9 & 4 \\ 0 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 16 & 4 & 1 \\ 9 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 354 & 100 & 30 \\ 100 & 30 & 10 \\ 30 & 10 & 5 \end{bmatrix}$$

and,

$$M^T\vec{y} = \begin{bmatrix} 0 & 1 & 16 & 9 & 4 \\ 0 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 129 \\ 41 \\ 15 \end{bmatrix}$$

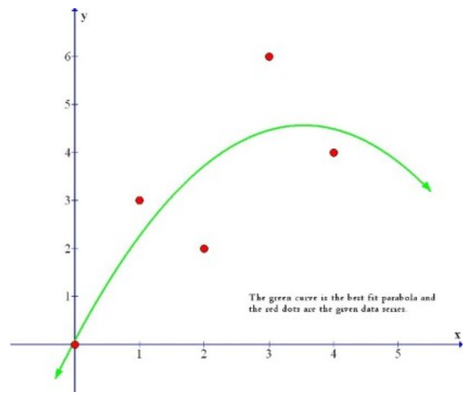
After a few row operations we can deduce,

$$rref[M^T M | M^T\vec{y}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & -5/14 \\ 0 & 1 & 1 & 177/70 \\ 0 & 0 & 1 & 3/35 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} c_1 = -5/14 \approxeq -0.357 \\ c_2 = 177/70 \approxeq 2.529 \\ c_3 = 3/35 = 0.086 \end{array}$$

We find the best-fit parabola is $\boxed{y = -0.357x^2 + 2.529x + 0.086}$

[13] almost always

[14] notice that if $f_j(x_i)$ are not all the same then it is possible to show $det(M^T M) \neq 0$ and then the solution to the normal equations is unique



Yes..., but what's this for?

Example 10.4.14. *Suppose you land on a mysterious planet. You find that if you throw a ball it's height above the ground y at time t is measured at times $t = 0, 1, 2, 3, 4$ seconds to be $y = 0, 2, 3, 6, 4$ meters respective. Assume that Newton's Law of gravity holds and determine the gravitational acceleration from the data. We already did the math in the last example. Newton's law approximated for heights near the surface of the planet simply says $y'' = -g$ which integrates twice to yield $y(t) = -gt^2/2 + v_o t + y_0$ where v_o is the initial velocity in the vertical direction. We find the best-fit parabola through the data set $\{(0,0),(1,3),(4,4),(3,6),(2,2)\}$ by the math in the last example,*

$$y(t) = -0.357t^2 + 2.529 + 0.086$$

we deduce that $g = 2(0.357)m/s^2 = 0.714m/s^2$. Apparently the planet is smaller than Earth's moon (which has $g_{moon} \approx \frac{1}{6}9.8m/s^2 = 1.63m/s^2$.

Remark 10.4.15.

> If I know for certain that the ball is at $y = 0$ at $t = 0$ would it be equally reasonable to assume y_o in our model? If we do it simplifies the math. The normal equations would only be order 2 in that case.

Example 10.4.16. *Find the best-fit parabola that passes through the origin and the points* $(1,3),(4,4),(3,6),(2,2)$. *To begin we should state our model: since the parabola goes through the origin we know the y-intercept is zero hence $y = c_1x^2 + c_2x$. Identify $f_1(x) = x^2$ and $f_2(x) = x$. As usual set-up the M and $\vec{y}$,*

$$M = \begin{bmatrix} f_1(1) & f_2(1) \\ f_1(4) & f_2(4) \\ f_1(3) & f_2(3) \\ f_1(2) & f_2(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 16 & 4 \\ 9 & 3 \\ 4 & 2 \end{bmatrix} \qquad \textit{and} \qquad \vec{y} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix}.$$

Calculate,

$$M^TM = \begin{bmatrix} 1 & 16 & 9 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 16 & 4 \\ 9 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 354 & 100 \\ 100 & 30 \end{bmatrix} \Rightarrow (M^TM)^{-1} = \frac{1}{620}\begin{bmatrix} 30 & -100 \\ -100 & 354 \end{bmatrix}$$

and,

$$M^T\vec{y} = \begin{bmatrix} 1 & 16 & 9 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 129 \\ 41 \end{bmatrix}$$

We solve $M^TM\vec{v} = M^T\vec{y}$ *by multiplying both sides by* $(M^TM)^{-1}$ *which yeilds,*

$$\vec{v} = (M^TM)^{-1}M^T\vec{y} = \frac{1}{620}\begin{bmatrix} 30 & -100 \\ -100 & 354 \end{bmatrix}\begin{bmatrix} 129 \\ 41 \end{bmatrix} = \begin{bmatrix} -23/62 \\ 807/310 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = -23/62 \approxeq -0.371 \\ c_2 = 807/310 \approxeq 2.603 \end{array}$$

Thus the best-fit parabola through the origin is $\boxed{y = -0.371x^2 + 2.603x}$

Sometimes an application may not allow for direct implementation of the least squares method, however a rewrite of the equations makes the unknown coefficients appear linearly in the model.

Example 10.4.17. *Newton's Law of Cooling states that an object changes temperature* T *at a rate proportional to the difference between* T *and the room-temperature. Suppose room temperature is known to be* 70^o *then* $dT/dt = -k(T-70) = -kT + 70k$. *Calculus reveals solutions have the form* $T(t) = c_0e^{-kt} + 70$. *Notice this is very intuitive since* $T(t) \to 70$ *for* $t >> 0$. *Suppose we measure the temperature at successive times and we wish to find the best model for the temperature at time* t. *In particular we measure:* $T(0) = 100$, $T(1) = 90$, $T(2) = 85$, $T(3) = 83$, $T(4) = 82$. *One unknown coefficient is* k *and the other is* c_1. *Clearly* k *does not appear linearly. We can remedy this by working out the model for the natural log of* $T - 70$. *Properties of logarithms will give us a model with linearly appearing unknowns:*

$$\ln(T(t) - 70) = \ln(c_0e^{-kt}) = \ln(c_0) + \ln(e^{-kt}) = \ln(c_0) - kt$$

Let $c_1 = \ln(c_0), c_2 = -k$ *then identify* $f_1(t) = 1$ *while* $f_2(t) = t$ *and* $y = \ln(T(t) - 70)$. *Our model is* $y = c_1f_1(t) + c_2f_2(t)$ *and the data can be generated from the given data for* $T(t)$*:*

$$\begin{aligned} t_1 = 0 : y_1 &= \ln(T(0) - 70) = \ln(100 - 70) = \ln(30) \\ t_2 = 1 : y_2 &= \ln(T(1) - 90) = \ln(90 - 70) = \ln(20) \\ t_3 = 2 : y_3 &= \ln(T(2) - 85) = \ln(85 - 70) = \ln(15) \\ t_4 = 3 : y_4 &= \ln(T(2) - 83) = \ln(83 - 70) = \ln(13) \\ t_5 = 4 : y_5 &= \ln(T(2) - 82) = \ln(82 - 70) = \ln(12) \end{aligned}$$

Our data for (t, y) *is* $(0, \ln 30), (1, \ln 20), (2, \ln 15), (3, \ln 13), (4, \ln 12)$. *We should solve normal equations* $M^TM\vec{v} = M^T\vec{y}$ *where*

$$M = \begin{bmatrix} f_1(0) & f_2(0) \\ f_1(1) & f_2(1) \\ f_1(2) & f_2(2) \\ f_1(3) & f_2(3) \\ f_1(4) & f_2(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} \ln 30 \\ \ln 20 \\ \ln 15 \\ \ln 13 \\ \ln 12 \end{bmatrix}.$$

We can calculate $M^TM = \begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix}$ *and* $M^T\vec{y} \approxeq \begin{bmatrix} 14.15 \\ 26.05 \end{bmatrix}$. *Solve* $M^TM\vec{v} = M^T\vec{y}$ *by multiplication by inverse of* M^TM*:*

$$\vec{y} = (M^TM)^{-1}M^T\vec{y} = \begin{bmatrix} 3.284 \\ -0.2263 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 \approxeq 3.284 \\ c_2 \approxeq -0.2263 \end{array}.$$

Therefore, $y(t) = \ln(T(t) - 70) = 3.284 - 0.2263$ *we identify that* $k = 0.2263$ *and* $ln(c_0) = 3.284$ *which yields* $c_0 = e^{3.284} = 26.68$. *We find the best-fit temperature function is*

$$\boxed{T(t) = 26.68e^{-0.2263t} + 70.}$$

Now we could give good estimates for the temperature $T(t)$ *for other times. If Newton's Law of cooling is an accurate model and our data was collected carefully then we ought to be able to make accurate predictions with our model.*

Remark 10.4.18.

The accurate analysis of data is more involved than my silly examples reveal here. Each experimental fact comes with an error which must be accounted for. A real experimentalist never gives just a number as the answer. Rather, one must give a number and an uncertainty or error. There are ways of accounting for the error of various data. Our approach here takes all data as equally valid. There are weighted best-fits which minimize a weighted least squares. Technically, this takes us into the realm of math of inner-product spaces. Finite dimensional inner-product spaces also allows for least-norm analysis. The same philosophy guides the analysis: the square of the norm measures the sum of the squares of the errors in the data. The collected data usually does not precisely fit the model, thus the equations are inconsistent. However, we project the data onto the plane representative of model solutions and this gives us the best model for our data. Generally we would like to minimize χ^2, this is the notation for the sum of the squares of the error often used in applications. In statistics finding the best-fit line is called doing "linear regression".

10.4.3 approximation and Fourier analysis

Example 10.4.19. *Clearly* $f(x) = e^x \notin P_2$. *What is a good approximation of* f? *Use the projection onto* P_2: $Proj_{\ P_2}(f) = \langle f, v_1\rangle v_1 + \langle f, v_2\rangle v_2 + \langle f, v_3\rangle v_3$. *We calculate,*

$$\langle f, v_1\rangle = \int_{-1}^{1} \tfrac{1}{\sqrt{2}} e^x dx = \tfrac{1}{\sqrt{2}}(e^1 - e^{-1}) \approxeq 1.661$$

$$\langle f, v_2\rangle = \int_{-1}^{1} \sqrt{\tfrac{3}{2}} x e^x dx = \sqrt{\tfrac{3}{2}}(xe^x - e^x)|_{-1}^{1} = \sqrt{\tfrac{3}{2}}[-(-e^{-1} - e^{-1})] = \sqrt{6}e^{-1} \approxeq 0.901$$

$$\langle f, v_3\rangle = \int_{-1}^{1} \sqrt{\tfrac{8}{45}}(x^2 - \tfrac{1}{3}) e^x dx = \tfrac{2e}{3} - \tfrac{14e^{-1}}{3} \approxeq 0.0402$$

Thus,

$$\begin{aligned} Proj_{\ P_2}(f)(x) &= 1.661 v_1(x) + 0.901 v_2(x) + 0.0402 v_3(x) \\ &= 1.03 + 1.103x + 0.017x^2 \end{aligned}$$

This is closest a quadratic can come to approximating the exponential function on the interval $[-1, 1]$. *What's the giant theoretical leap we made in this example? We wouldn't face the same leap if we tried to approximate* $f(x) = x^4$ *with* P_2. *What's the difference? Where does* e^x *live?*

Example 10.4.20. *Consider $C[-\pi,\pi]$ with inner product $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. The set of sine and cosine functions $\{1,\cos(x),\sin(x),\cos(2x),\sin(2x),\dots,\cos(kx),\sin(kx)\}$ is an orthogonal set of functions.*

$$\langle \cos(mx),\cos(nx)\rangle = \int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx = \pi\delta_{mn}$$

$$\langle \sin(mx),\sin(nx)\rangle = \int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx = \pi\delta_{mn}$$

$$\langle \sin(mx),\cos(nx)\rangle = \int_{-\pi}^{\pi}\sin(mx)\cos(nx)dx = 0$$

Thus we find the following is a set of <u>*orthonormal*</u> *functions*

$$\boxed{\beta_{trig} = \{\tfrac{1}{\sqrt{2\pi}},\ \tfrac{1}{\sqrt{\pi}}\cos(x),\ \tfrac{1}{\sqrt{\pi}}\sin(x),\ \tfrac{1}{\sqrt{\pi}}\cos(2x),\ \tfrac{1}{\sqrt{\pi}}\sin(2x),\ \dots,\ \tfrac{1}{\sqrt{\pi}}\cos(kx),\ \tfrac{1}{\sqrt{\pi}}\sin(kx)\}}$$

The idea of Fourier analysis is based on the least-squares approximation and the example above. We wish to represent a function with a sum of sines and cosines, this is called a **Fourier sum**. Much like a power series, the more terms we use to approximate the function the closer the approximating sum of functions gets to the real function. In the limit the approximation can become exact, the Fourier sum goes to a Fourier series. I do not wish to confront the analytical issues pertaining to the convergence of Fourier series. As a practical matter, it's difficult to calculate infinitely many terms so in practice we just keep the first say 10 or 20 terms and it will come very close to the real function. The advantage of a Fourier sum over a polynomial is that sums of trigonometric functions have natural periodicities. If we approximate the function over the interval $[-\pi,\pi]$ we will also find our approximation repeats itself outside the interval. This is desireable if one wishes to model a wave-form of some sort.

Example 10.4.21. *Suppose* $f(t) = \begin{cases} 1 & 0<t<\pi \\ -1 & -\pi<t<0 \end{cases}$ *and $f(t+2n\pi) = f(t)$ for all $n\in\mathbb{Z}$. This is called a* **square wave** *for the obvious reason (draw its graph). Find the first few terms in a Fourier sum to represent the function. We'll want to use the projection: it's convenient to bring the normalizing constants out so we can focus on the integrals without too much clutter.* [15]

$$\begin{aligned} Proj_W(f)(t) = \tfrac{1}{2\pi}\langle f,1\rangle + \tfrac{1}{\pi}\langle f,\cos t\rangle\cos t + \tfrac{1}{\pi}\langle f,\sin t\rangle \sin t + \\ + \tfrac{1}{\pi}\langle f,\cos 2t\rangle\cos 2t + \tfrac{1}{\pi}\langle f,\sin 2t\rangle\sin 2t + \cdots \end{aligned}$$

Where $W = span(\beta_{trig})$. The square wave is constant on $(0,\pi]$ and $[-\pi,0)$ and the value at zero is not defined (you can give it a particular value but that will not change the integrals that calculate the Fourier coefficients). Calculate,

$$\langle f,1\rangle = \int_{-\pi}^{\pi} f(t)dt = 0$$

$$\langle f,\cos t\rangle = \int_{-\pi}^{\pi}\cos(t)f(t)dt = 0$$

[15] In fact, various texts put these little normalization factors in different places so when you look up results on Fourier series beware conventional discrepancies

Notice that $f(t)$ and $\cos(t)f(t)$ are odd functions so we can conclude the integrals above are zero without further calculation. On the other hand, $\sin(-t)f(-t) = (-\sin t)(-f(t)) = \sin t f(t)$ thus $\sin(t)f(t)$ is an even function, thus:

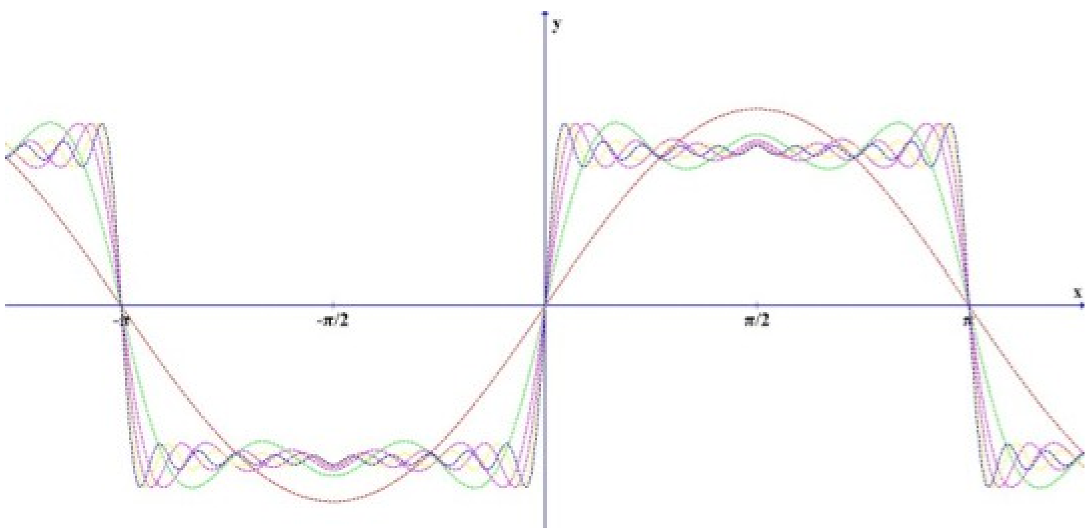
$$\langle f, \sin t\rangle = \int_{-\pi}^{\pi} \sin(t)f(t)dt = 2\int_0^{\pi} \sin(t)f(t)dt = 2\int_0^{\pi} \sin(t)dt = 4$$

Notice that $f(t)\cos(kt)$ is odd for all $k \in \mathbb{N}$ thus $\langle f, \cos(kt)\rangle = 0$. Whereas, $f(t)\sin(kt)$ is even for all $k \in \mathbb{N}$ thus

$$\begin{aligned}\langle f, \sin kt\rangle &= \int_{-\pi}^{\pi} \sin(kt)f(t)dt = 2\int_0^{\pi} \sin(kt)f(t)dt \\ &= 2\int_0^{\pi} \sin(kt)dt = \frac{2}{k}\big[1-\cos(k\pi)\big] = \begin{cases} 0, & k \text{ even} \\ \frac{4}{k}, & k \text{ odd}\end{cases}\end{aligned}$$

Putting it all together we find (the $\sim$ indicates the functions are nearly the same except for a finite subset of points),

$$f(t) \sim \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + +\frac{1}{5}\sin 5t + \cdots\right) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin(2n-1)t$$



I have graphed the Fourier sums up the sum with 11 terms.

Remark 10.4.22.

The treatment of Fourier sums and series is by no means complete in these notes. There is much more to say and do. Our goal here is simply to connect Fourier analysis with the more general story of orthogonality. In the math 334 course we use Fourier series to construct solutions to partial differential equations. Those calculations are foundational to describe interesting physical examples such as the electric and magnetic fields in a waveguide, the vibrations of a drum, the flow of heat through some solid, even the vibrations of a string instrument.

10.5 introduction to geometry

To study geometry is to study shapes and the transformations which preserve such shapes. In particular, when we have a vector space and an inner product then the orthogonal, or unitary, transformations preserve the inner product and hence the length and angles between vectors. It follows that these linear isometries are the shape preserving transformations. Formally, we can say a vector space together with an inner product defines a **Euclidean geometry**. Other geometries can be defined by pairing a vector space with different type of product. We may study such **non-Euclidean** geometries in a later chapter.

If we begin with an orthogonal subset of $\mathbb{R}^n$ and we preform a linear transformation then will the image of the set still be orthogonal? We would like to characterize linear transformations which maintain orthogonality. These transformations should take an orthogonal basis to a new basis which is still orthogonal.

Definition 10.5.1.

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $T(x) \bullet T(y) = x \bullet y$ for all $x, y \in \mathbb{R}^n$ then we say that T is an **orthogonal transformation**

Example 10.5.2. *Let $\{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be a rotation of the coordinates by angle θ in the clockwise direction,*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

As a check on my sign conventions, consider rotating $(1,0)$ by $R(\pi/2)$, we obtain $(x', y') = (0,1)$. Intuitively, a rotation should not change the length of a vector, let's check the math: let $v, w \in \mathbb{R}^2$,

$$R(\theta)v \bullet R(\theta)w = [R(\theta)v]^T R(\theta)w = v^T R(\theta)^T R(\theta)w$$

Now calculate $R(\theta)^T R(\theta)$,

$$R(\theta)^T R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = I$$

Therefore, $R(\theta)v \bullet R(\theta) = v^T I w = v^T w = v \bullet w$ for all $v, w \in \mathbb{R}^2$ and we find $L_{R(\theta)}$ is an orthogonal transformation.

This shows the matrix of a rotation L_R satisfies $R^T R = I$. Is this always true or was this just a special formula for rotations? Or is this just a two-dimensional thing? What if we look at orthhogonal transformations on $\mathbb{R}^n$ what general condition is there on the matrix of the transformation?

Definition 10.5.3.

Let $A \in \mathbb{R}^{\,n\times n}$ then we say that A is an **orthogonal matrix** iff $A^T A = I$. Moreover, we say A is a **reflection matrix** if A is orthogonal and $det(A) = -1$ whereas we say A is a **rotation matrix** if A is orthogonal with $det(A) = 1$. The set of all orthogonal $n \times n$ matrices is denoted $O(n, \mathbb{R})$ and the set of all $n \times n$ rotation matrices is denoted $SO(n, \mathbb{R})$.

Proposition 10.5.4. matrix of an orthogonal transformation is orthogonal

If A is the matrix of an orthogonal transformation on $\mathbb{R}^n$ then $A^TA = I$ and either A is a rotation matrix or A is a reflection matrix.

Proof: Suppose $L(x) = Ax$ and L is an orthogonal transformation on $\mathbb{R}^n$. Notice that

$$L(e_i) \bullet L(e_j) = [Ae_i]^T Ae_j = e_i^T [A^TA] e_j$$

and $e_i \bullet e_j = e_i^T e_j = e_i^T I e_j$ hence $e_i^T [A^TA - I] e_j = 0$ for all i, j thus $A^TA - I = 0$ by Example 2.3.26 and we find $A^TA = I$. Consequently,

$$det(A^TA) = det(I) \quad \Leftrightarrow \quad det(A)det(A) = 1 \quad \Leftrightarrow det(A) = \pm 1$$

Thus $A \in SO(n)$ or A is a reflection matrix. $\square$

The proposition below is immediate from the definitions of length, angle and linear transformation.

Proposition 10.5.5. orthogonal transformations preserve lengths and angles

If $v, w \in \mathbb{R}^n$ and L is an orthogonal transformation such that $v' = L(v)$ and $w' = L(w)$ then the angle between v' and w' is the same as the angle between v and w, in addition the length of v' is the same as v.

Remark 10.5.6.

Reflections, unlike rotations, will spoil the "handedness" of a coordinate system. If we take a right-handed coordinate system and perform a reflection we will obtain a new coordinate system which is left-handed. If you'd like to know more just ask me sometime.

If orthogonal transformations preserve the geometry of $\mathbb{R}^n$ you might wonder if there are other non-linear transformations which also preserve distance and angle. The answer is yes, but we need to be careful to distinguish between the length of a vector and the distance bewtween points. It turns out that the translation defined below will preserve the distance, but not the norm or length of a vector.

Definition 10.5.7.

Fix $b \in \mathbb{R}^n$ then a translation by b is the mapping $T_b(x) = x + b$ for all $x \in \mathbb{R}^n$.

This is known as an **affine transformation**, it is not linear since $T(0) = b \neq 0$ in general. (if $b = 0$ then the translation is both affine and linear). Anyhow, affine transformations should be familar to you: $y = mx + b$ is an affine transformation on $\mathbb{R}$.

Proposition 10.5.8. translations preserve geometry

Suppose $T_b : \mathbb{R}^n \to \mathbb{R}^n$ is a translation then

(1.) If $\angle(xyz)$ denotes the angle formed by line segments $\overline{xy}, \overline{yz}$ which have endpoints x, y and y, z respectively then $\angle(T_b(x)T_b(y)T_b(z)) = \angle(xyz)$

(2.) The distance from x to y is the equal to the distance from $T_b(x)$ to $T_b(y)$.

Proof: I'll begin with (2.) since it's easy:

$$d(T_b(x), T_b(y)) = ||T_b(y) - T_b(x)|| = ||y + b - (x + b)|| = ||y - x|| = d(x, y).$$

Next, the angle $\angle(xyz)$ is the angle between $x - y$ and $z - y$. Likewise the angle $\angle T_b(x)T_b(y)T_b(z)$ is the angle between $T_b(x) - T_b(y)$ and $T_b(z) - T_b(y)$. But, these are the same vectors since $T_b(x) - T_b(y) = x + b - (y + b) = x - y$ and $T_b(z) - T_b(y) = z + b - (y + b) = z - y$. $\square$

Definition 10.5.9.

Suppose $T(x) = Ax + b$ where $A \in SO(n)$ and $b \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$ then we say T is a **rigid motion**.

In high-school geometry you studied the concept of *congruence*. To objects were congruent if they had the same size and shape. From the viewpoint of analytic geometry we can say two objects are congruent iff one is the image of the other with respect to some rigid motion. We leave further discussion of such matters to the modern geometry course where you study these concepts in depth.

Remark 10.5.10.

In Chapter 6 of my *Mathematical Models in Physics* notes I describe how Euclidean geometry is implicit and foundational in classical Newtonian Mechanics. The concept of a rigid motion is used to define what is meant by an *intertial frame*. Chapter 7 of the same notes describes how Special Relativity has hyperbolic geometry as its core. The dot-product is replaced with a Minkowski-product which yields all manner of curious results like time-dilation, length contraction, and the constant speed of light. If your interested in hearing a lecture or two on the geometry of Special Relativity please ask and I'll try to find a time and a place, I mean, we'll make it an event.

This concludes our short tour of Euclidean geometry on $\mathbb{R}^n$. Incidentally, you might look at Barret Oneil's *Elementary Diffferential Geometry* if you'd like to see a more detailed study of isometries of $\mathbb{R}^3$. Some notes are posted on my website from the Math 497, Spring 2014 course. Now let's turn our attention to abstract, possibly complex, vector spaces.

Definition 10.5.11.

If $T : V \to V$ is a linear transformation on an inner product space V over $\mathbb{F}$ such that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$ then if

(i.) $\mathbb{F} = \mathbb{R}$ we say T is an **orthogonal transformation**

(ii.) $\mathbb{F} = \mathbb{C}$ we say T is an **unitary transformation**

I invite the reader to verify that the matrix of an orthogonal transformation is an orthogonal matrix. The theory for real inner product spaces is essentially the same as we saw already for $\mathbb{R}^n$. The structure in the complex case is a bit different: I'll study the abstract case in the next section. For now consider $V = \mathbb{C}^n$ and suppose $T : V \to V$ is a unitary transformation defined by $T(v) = Uv$ for some $U \in \mathbb{C}^{n\times n}$. Notice:

$$\langle T(e_i), T(e_j)\rangle = \langle e_i, e_j\rangle \quad \Rightarrow \quad (Ue_j)^*Ue_i = \delta_{ij} \quad \Rightarrow \quad e_j^T U^*Ue_i = \delta_{ji}$$

That is, $(U^*U)_{ji} = I_{ji}$ for all j, i hence $U * U = I$ where $U^* = \overline{U}^T$. It turns out the same is true for any unitary transformation on a complex inner product space; the matrix of a unitary transformation with respect to an orthonormal basis is a unitary matrix:

Definition 10.5.12.

Let $U \in \mathbb{C}^{n\times n}$ then we say that A is an **unitary matrix** iff $U^*U = I$. The set of all unitary $n \times n$ matrices is denoted $U(n)$ and the set of all **special unitary matrices** is defined by

$$SU(n) = \{U \in U(n) \mid \det(U) = 1\}.$$

The special unitary matrices are used in the formulation of the standard model of particle physics. Allow me a brief digression, In 1929 Weyl showed how $U(1)$ could be used to derive electromagnetism as a gauge theory. This laid the foundation for later work by Yang, Mills and Utiyama in the 1950's. Yang-Mill's theory generalizes electromagnetics $U(1)$ symmetry to other gauge groups like $SU(2)$. If you're curious, I'm happy to arrange a longer discussion about the theory of fiber bundles and how we use them to understand physical *laws*.

Example 10.5.13. *Let $T : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ be a linear transformation defined by $T(A) = BA$ for B a fixed matrix in $SU(2)$. We can show T is a unitary transformation as follows:*

$$\langle T(X), T(Y)\rangle = \langle BX, BY\rangle = trace[(BX)(BY)^*] = trace[BXY^*B^*]$$

The trace satisfies the identity $trace(LM) = trace(ML)$ hence identifying $M = B^*$ and $L = BXY^*$ we find:

$$\langle T(X), T(Y)\rangle = trace[B^*BXY^*] = trace[XY^*] = \langle X, Y\rangle.$$

We used that $B^*B = I$ for a special unitary matrix. If $B \in U(2)$ then we also find $T(A) = BA$ is a unitary transformation.

Example 1 in §6.5 of Insel Spence and Friedberg inspired the Example above.

10.6 orthonormal diagonalization

In this section we set the ground work for stating the spectral theorems. We aim to understand how the concepts of orthogonality and diagonalizability are merged. The answer is given by the spectral theorems.

The approach which follows is coordinate-based. A coordinate free, but implicit, approach can be found in the §6.3 of the 4^{th} edition of Insel Spence and Friedberg's *Linear Algebra*. Their approach also has some infinite dimensional application.

Suppose V is a finite dimensional inner product space with orthonormal basis $\beta = \{v_1, \dots, v_n\}$.

Proposition 10.6.1. selecting components via inner products

Let $y \in V$ and suppose $T : V \to V$ is a linear transformation then:

(a.) $y = \sum_{i=1}^{n} \langle y, v_i\rangle v_i$,

(b.) $([T]_{\beta,\beta})_{ij} = \langle T(v_j), v_i\rangle$

Proof: see Proposition 10.2.13 for part (a.). For part (b.) recall that the j-th column of $[T]_{\beta,\beta}$ is given by $[T(v_j)]_\beta$. Hence, set $y = T(v_j)$ and note the i-th component of $[y]_\beta$ is given by $\langle T(v_j), v_i\rangle$

using part (a.). Hence, $([T]_{\beta,\beta})_{ij} = \langle T(v_j), v_i\rangle$. $\square$

We can easily show the matrix of an orthogonal(unitary) transformation is an orthogonal(unitary) matrix:

Proposition 10.6.2.

Let $T : V \to V$ be a linear transformation and β an orthonormal basis. Then,

(1.) T is orthogonal iff $[T]_{\beta,\beta}$ is a orthogonal matrix,

(2.) T is unitary iff $[T]_{\beta,\beta}$ is a unitary matrix.

Proof: Suppose V is a real vector space and T is a linear transformation, notice $T(v_i) = \sum_{k=1}^{n} A_{kj} v_k$ where $A = [T]_{\beta,\beta}$ hence

$$\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle \quad \Leftrightarrow \quad \left\langle \sum_{k=1}^{n} A_{ki} v_k, \sum_{k=1}^{n} A_{lj} v_l \right\rangle = \delta_{ij}$$

Thus,

$$\sum_{l,k=1}^{n} A_{ki} A_{lj} \delta_{kl} = \delta_{ij} \quad \Leftrightarrow \quad \sum_{k=1}^{n} A_{ki} A_{kj} = \delta_{ij} \quad \Leftrightarrow \quad \sum_{k=1}^{n} (A^T)_{ik} A_{kj} = \delta_{ij} \quad \Leftrightarrow \quad A^T A = I.$$

Thus the matrix $[T]_{\beta,\beta} = A$ is orthogonal if and only if T is orthogonal. If V is a complex vector space then nearly the same calculation shows that $[T]_{\beta,\beta} = U$ satisfies $U^*U = I$ if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$. $\square$

There is a small gap in the proof above. Can you show that if T has $\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle$ for all $v_i, v_j \in \beta$ a basis for V if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$? This is a good exercise to practice.

Definition 10.6.3.

Let $T : V \to V$ be a linear transformation on a finite dimensional inner product space with orthonormal basis $\beta = \{v_1, \dots, v_n\}$ then we define $T^* : V \to V$ by:

$$T^*(y) = \sum_{i=1}^{n} \langle y, T(v_i)\rangle v_i$$

Let me address the apparent coordinate dependence in this definition. Suppose $\gamma = \{w_1, \dots, w_n\}$ is another orthonormal basis for V. We can relate β and γ by some invertible matrix P for which $w_i = \sum_{k=1}^{n} P_{ik} v_k$. Since both β and γ are orthonormal we find

$$\delta_{ij} = \langle w_i, w_j\rangle = \left\langle \sum_k P_{ik} v_k, \sum_l P_{jl} v_l \right\rangle = \sum_{l,k} P_{ik} \overline{P_{jl}} \langle v_k, v_l\rangle = \sum_k P_{ik} \overline{P_{jk}} = \sum_k P_{ik} (\overline{P^T})_{kj}$$

hence $I = PP^*$ and thus $I = P^*P$ which provides $\delta_{ij} = \sum_k (P^*)_{ik} P_{kj} = \sum_k \overline{P_{ki}} P_{kj} = \sum_k P_{kj} \overline{P_{ki}}$. Let $y \in V$ and consider:

$$\begin{aligned}
\sum_k \langle y, T(w_k) \rangle w_k &= \sum_k \left\langle y, T\left(\sum_j P_{kj} v_j\right) \right\rangle \sum_i P_{ki} v_i \\
&= \sum_{i,k} P_{ki} \left\langle y, \sum_j P_{kj} T(v_j) \right\rangle v_i \\
&= \sum_{i,j,k} P_{ki} \overline{P_{kj}} \langle y, T(v_j) \rangle v_i \\
&= \sum_{i,j} \delta_{ij} \langle y, T(v_j) \rangle v_i \\
&= \sum_j \langle y, T(v_j) \rangle v_j.
\end{aligned}$$

Thus $T^* : V \to V$ defined by $T^*(y) = \sum_i \langle y, T(v_i) \rangle v_i$ does not depend on the choice of the basis β

Theorem 10.6.4.

Let $T : V \to V$ be a linear transformation and β an orthonormal basis. Then,

(1.) $T^* : V \to V$ is a linear transformation,

(2.) $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$

Proof: let $y_1, y_2 \in V$ and $c \in \mathbb{F}$,

$$\begin{aligned}
T^*(cy_1 + y_2) &= \sum_i \langle cy_1 + y_2, T(v_i) \rangle v_i \\
&= \sum_i \langle cy_1 + y_2, T(v_i) \rangle v_i \\
&= c \sum_i \langle y_1, T(v_i) \rangle v_i + \sum_i \langle y_2, T(v_i) \rangle v_i \\
&= cT^*(y_1) + T^*(y_2)
\end{aligned}$$

Thus $T*$ is a linear transformation on V. Let $y \in V$ and $v_j \in \beta$. Calculate,

$$\begin{aligned}
\langle v_j, T^*(y) \rangle &= \langle v_j, \sum_i \langle y, T(v_i) \rangle v_i \rangle \\
&= \sum_i \overline{\langle y, T(v_i) \rangle} \langle v_j, v_i \rangle \\
&= \sum_i \langle T(v_i), y \rangle \delta_{ji} \\
&= \langle T(v_j), y \rangle
\end{aligned}$$

A short calculation shows $\langle x, T^*(y) \rangle = \langle T(x), y \rangle$ for all $x, y \in V$. $\square$

We used the notation $A^* = \overline{A}^T$ to denote the **adjoint** or **Hermitian adjoint** of a matrix. The next result explains why we use the same notation:

Proposition 10.6.5.

If T has adjoint T^* and β is an orthonormal basis then $([T]_{\beta,\beta})^* = [T^*]_{\beta,\beta}$.

Proof: apply part (b.) of Proposition 10.6.1 to observe

$$([T]_{\beta,\beta})_{ij} = \langle T(v_j), v_i\rangle = \langle v_j, T^*(v_i)\rangle = \overline{\langle T^*(v_i), v_j\rangle} = \overline{[T^*]_{ji}} \quad \Rightarrow \quad [T]_{\beta,\beta} = \overline{([T^*]_{\beta,\beta})^T}.$$

Thus conjugating and transposing we find $\overline{[T]_{\beta,\beta}}^T = [T^*]_{\beta,\beta}$ hence $([T]_{\beta,\beta})^* = [T^*]_{\beta,\beta}$. $\square$

Many properties for the adjoint of a matrix naturally follow from properties of the transpose and conjugate. For example, since $\overline{AB} = \overline{A}\overline{B}$ and $(AB)^T = B^TA^T$ we naturally find $(AB)^* = B^*A^*$. Similarly, $(A+B)^* = A^* + B^*$, $(cA)^* = \overline{c}A^*$ and $I^* = I$. Given the correspondence between the matrix and transformation adjoints the following proposition is not surprising:

Proposition 10.6.6.

Suppose T, U are linear transformations on V and $c \in \mathbb{F}$ then

(a.) $(T+U)^* = T^* + U^*$,

(b.) $(cT)^* = \overline{c}T^*$,

(c.) $(TU)^* = U^*T^*$,

(d.) $T^{**} = T$,

(e.) $Id^* = Id$.

Proof: we could prove these from the definition. I'll prove (d.) via Proposition 10.6.5. Let $A = [T]_{\beta,\beta}$ then $A^* = [T^*]_{\beta,\beta}$ and as $A = A^{**}$ we find $A = ([T^*]_{\beta,\beta})^* = [T^{**}]_{\beta,\beta}$. Thus $[T]_{\beta,\beta} = [T^{**}]_{\beta,\beta}$ and it follows that $T^{**} = T$. In constrast, I'll prove (e.) from the definition of adjoint,

$$\langle Id^*(v_j), v_i\rangle = \langle v_j, Id(v_i)\rangle = \langle v_j, v_i\rangle = \delta_{ji}$$

hence $Id^*(v_j) = \sum_i \delta_{ji}v_i = v_j$ for each v_j which shows $Id^* = Id$. $\square$

The next result is interesting.

Theorem 10.6.7.

If T has eigenvector v with eigenvalue λ then T^* has an eigenvector with eigenvalue $\overline{\lambda}$.

Proof: let $x \in V$ and note $(T - \lambda Id)(v) = 0$ by assumption and hence

$$0 = \langle 0, x\rangle = \langle (T - \lambda Id)(v), x\rangle = \langle v, (T - \lambda Id)^*(x)\rangle = \langle v, (T^* - \overline{\lambda} Id)(x)\rangle$$

notice $(T^* - \overline{\lambda}Id)(x) \in \text{Range}(T^* - \overline{\lambda}Id)$ and hence $v \in (T^* - \overline{\lambda}Id)^\perp$. Thus $\text{Range}(T^* - \overline{\lambda}Id) \neq V$ which shows $T^* - \overline{\lambda}Id \in \text{End}(V)$ is not surjective hence not injective and we find $\text{Ker}(T^* - \overline{\lambda}Id) \neq 0$. Therefore, there exists $w \neq 0$ for which $(T^* - \overline{\lambda}Id)(w) = 0$. That is, T^* has an eigenvector w with eigenvalue $\overline{\lambda}$. $\square$

Notice I use Theorem 7.6.4 to connect surjective and injective for the linear map on V.

Theorem 10.6.8. Schur's Theorem

Let V be a finite dimensional inner product space over $\mathbb{F}$ and suppose $T : V \to V$ is a linear transformation for which $p(x) = \det(T - xId)$ factors into linear factors over $\mathbb{F}$. Then there exists an orthonormal basis β for which $[T]_{\beta,\beta}$ is upper triangular.

Proof: we use induction on the dimension. For $n = 1$ the result is clear. Suppose there exists an orthonormal basis to upper-triangularize and linear transformation on an $(n-1)$-dimensional space with split characterisitic polynomial. Consider $T : V \to V$ where V is n-dimensional where T has a split characteristic polynomial. Thus, by Theorem 10.6.7 we may assume there exists an eigenvector x with eigenvalue λ for T^*. Let $W = \text{span}\{x\}$ and suppose $y \in W^\perp$ consider for $w = cx \in W$,

$$\langle T(y), w\rangle = \langle y, T^*(cx)\rangle = \langle y, cT^*(x)\rangle = \overline{c}\langle y, \lambda x\rangle = \overline{c\lambda}\langle y, x\rangle = 0$$

Hence $W^\perp$ is a T-invariant subspace of dimension $n-1$. Moreover, $T|_W^\perp$ has a characterisitic polynomial which divides the characteristic polynomial of T. Thus $T_W^\perp$ is a linear transformation on an $n-1$ dimensional space whose characteristic polynomial is split. Thus there exists an orthonormal basis β' for which the matrix of $T_W^\perp$ is upper triangular. Let $\beta = \{\frac{1}{\|x\|}\} \cup \beta'$ and note $[T]_{\beta,\beta}$ is upper triangular. Here we use Theorem 7.7.14 to understand why the characteristic polynomial of the restriction must divide that of T as well as the block-decomposition which stems from the T-invariance of both W and $W^\perp$. $\square$

I found this induction argument Insel Spence and Friedberg's proof of their Theorem 6.14.

Definition 10.6.9.

Let V be an inner product space and suppose $T : V \to V$ is a linear transformation then T is **normal** iff $TT^* = T^*T$. Likewise, an $n \times n$ matrix A is normal iff $A^*A = AA^*$.

Example 10.6.10. *If $A^T = \pm A$ then $A^TA = \pm A^2 = AA^T$ hence any real symmetric or antisymmetric matrix is a normal matrix.*

Example 10.6.11. *Orthgonal matrices satisfy $R^TR = I = RR^T$ hence orthogonal matrices are normal. Similarly, unitary matrices are normal since $U^*U = I = UU^*$. In summary, linear isometries of inner product space are normal transformations.*

Normal operators have very nice properties:

Proposition 10.6.12.

Suppose T is a normal linear transformations on V then

(a.) $\|T(x)\| = \|T^*(x)\|$ for each $x \in V$,

(b.) $T - cId$ is normal for each $c \in \mathbb{F}$,

(c.) if $T(x) = \lambda x$ then $T^*(x) = \overline{\lambda}x$,

(d.) for $\lambda_1 \neq \lambda_2$ with corresponding eigenvectors x_1, x_2 we have $\{x_1, x_2\}$ orthogonal.

Proof: see Insel Spence and Friedberg page 371. Also, I intend to assign a homework problem which is related. $\square$

Theorem 10.6.13. normal operators are orthonormaly diagonalizable over $\mathbb{C}$:

Let T be a linear transformation on a finite dimensional complex inner product space V then T is normal if and only if there exists an orthonormal eigenbasis for T.

Proof: let T be a normal transformation on a finite dimensional inner product space over $\mathbb{C}$. The fundamental theorem of algebra provides that the characteristic polynomial of T is split. Apply Schur's Theorem to obtain a basis $\beta = \{v_1, v_2, \dots, v_n\}$ in which $[T]_{\beta,\beta}$ is upper triangular. Let $[T]_{\beta,\beta} = A$ to reduce clutter in calculation below. Observe v_1 is an eigenvector by the upper triangularity of A. Suppose inductively that $v_1, \dots, v_{k-1}$ are eigenvectors of T with eigenvalues $\lambda_1, \dots, \lambda_{k-1}$. Consider v_k. By upper triangularity of the matrix,

$$T(v_k) = A_{1k}v_1 + A_{2k}v_2 + \cdots + A_{k-1,k}v_{k-1} + A_{kk}v_k$$

However, by part (b.) of Proposition 10.6.1 and Theorem 10.6.7 we find:

$$A_{jk} = \langle T(v_k), v_j\rangle = \langle v_k, T^*(v_j)\rangle = \langle v_k, \overline{\lambda_j}v_j\rangle = \lambda_j\langle v_k, v_j\rangle = \lambda_j\delta_{kj}$$

hence $T(v_k) = A_{kk}v_k$ as all the other terms vanish by the othonormality of β. Hence the induction step is verified and we conclude that β is an orthonormal eigenbasis for T. Conversely, if β is an orthonormal eigenbasis for T then $[T]_{\beta,\beta} = D$ and as $DD^* = D^*D$ it follows that $TT^* = T^*T$. $\square$.

Insel Spence and Friedberg give Example 3 on page 372-373 to demonstrate the theorem above fails in the infinite dimensional context. We will focus on finite dimension here.

Next we turn to real inner product spaces. Recall that rotations in the plane are not usually diagonalizable, yet, we argued in Example 10.6.11 that orthogonal matrices and hence rotations are normal. Normality fails to gaurantee the existence of an orthonormal eigenbasis over $\mathbb{R}$.

Definition 10.6.14.

Let V be an inner product space and suppose $T : V \to V$ is a linear transformation then T is **self-adjoint** iff $T = T^*$. Likewise, an $n \times n$ matrix A is self-adjoint iff $A^* = A$.

We also call a self-adjoint transformation or matrix a **Hermitian** transformation or **Hermitian** matrix. The following results show the impact of the condition $T = T^*$:

Proposition 10.6.15.

Suppose T is self-adjoint on V then

(a.) each eigenvalue of T is real

(b.) if V is a real inner product space then the characteristic polynomial for T splits

Proof: suppose $T = T^*$ and suppose $v \neq 0$ with $T(v) = \lambda v$ hence $\lambda v = T(v) = T^*(v) = \overline{\lambda}v$ by Theorem 10.6.7. Hence $\lambda = \overline{\lambda}$ which proves (a.). Suppose V is a real inner product space. Consider the complexification of $T_{\mathbb{C}}$ on $V_{\mathbb{C}}$ then recall the matrix of T and $T_{\mathbb{C}}$ is the same for a given basis β for V. By the fundamental theorem of algebra the characteristic polynomial splits over $\mathbb{C}$ for $T_{\mathbb{C}}$. However, the same polynomial serves as the characteristic polynomial for T. It follows the eigenvalues of T are found in conjugate pairs. But, since all the eigenvalues are real we deduce the characteristic polynomial for T is split over $\mathbb{R}$. $\square$

If you forgot about the complexification concept then you might want to review § 8.3.2.

Theorem 10.6.16. self-adjoint operators are orthonormaly diagonalizable over $\mathbb{R}$:

Let T be a linear transformation on a finite dimensional real inner product space V then T is self-adjoint if and only if there exists an orthonormal eigenbasis for T.

Proof: let V be a finite dimensional real inner product space and T a linear transformation on V such that $T^* = T$. By Proposition 10.6.15 we find the characteristic polynomial splits over $\mathbb{R}$ hence by Schur's Theorem we find an orthonormal basis β for which $[T]_{\beta,\beta}$ is upper triangular. Yet, $T = T^*$ hence $[T^*]_{\beta,\beta}$ is also upper triangular. We know $[T^*]_{\beta,\beta} = ([T]_{\beta,\beta})^* = ([T]_{\beta,\beta})^T$. Thus, $[T]_{\beta,\beta} = ([T]_{\beta,\beta})^T$ and $[T]_{\beta,\beta}$ is upper triangular. Thus $[T]_{\beta,\beta}$ is diagonal and we find β is an eigenbasis for T. The converse direction is much easier. If β is an orthonormal eigenbasis then $[T]_{\beta,\beta} = D$ where D is diagonal. Thus $D^T = D$ and $([T]_{\beta,\beta})^T = [T]_{\beta,\beta}$. Hence $[T^*]_{\beta,\beta} = [T]_{\beta,\beta}$ as $A^T = A^*$ for A a real matrix. Consequently $T^* = T$. $\square$

You'll forgive me if I deprive you of the proof of the assertions below. Details can be found in Chapter 6 of Insel Spence and Friedberg.

Theorem 10.6.17.

There exists $P \in O(n, \mathbb{R})$ for which P^TAP is diagonal if and only if $A^T = A$.

In other words, a real matrix admits an orthonormal eigenbasis if and only if A is symmetric. Notice $A^T = A$ thus implies all the eigenvalues of A exist in $\mathbb{R}$. This observation is crucial to our study of quadratic forms in the next chapter.

Theorem 10.6.18.

There exists $P \in U(n)$ for which P^*AP is diagonal if and only if $A^*A = AA^*$.

That is, a complex matrix A admits an orthonormal eigenbasis for $\mathbb{C}^n$ if and only if $A^*A = AA^*$. Finally, we introduce a precise characterization of *orthogonal projection* which we need for the statement of the spectral theorems. The operator $\text{Proj}_W : V \to W$ is a good example to motivate the following:

Definition 10.6.19.

Let V be an inner product space and $S : V \to V$ a linear transformation for which $\text{Range}(S)^\perp = \text{Ker}(S)$ and $\text{Ker}(S)^\perp = \text{Range}(S)$ then S is an **orthogonal projection**.

Example 10.6.20. *Let $A = 3uu^T + 7vv^T$ be a 2×2 real matrix built from $\{u, v\}$ an orthnormal basis for $\mathbb{R}^2$. Then $Au = 3u$ and $Av = 7v$ hence the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 7$. Observe $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_1(x) = uu^Tx$ and $T_2(x) = vv^Tx$ define linear transformations for which $T_1T_2 = 0$ and $T_1T_1 = T_1$ and $T_2T_2 = T_2$. Moreover, $T_1 + T_2 = Id$ as can be seen from the fact $T_1 + T_2$ fixes the basis $\{u, v\}$:*

$$(T_1 + T_2)(u) = uu^Tu + vv^Tu = u \qquad \& \qquad (T_1 + T_2)(v) = uu^Tv + vv^Tv = v$$

At the level of matrices, $uu^T + vv^T = I$.

I used eigenvalues $3, 7$ to illustrate the idea, but, you might be able to see we could just as well replace them with any other real values. Once we have an orthonormal basis for $\mathbb{R}^n$ we can always play this game to construct matrices with eigenvalues of our choosing. The fascinating thing is this

observation can be reversed for normal matrices over $\mathbb{C}$ or symmetric matrices over $\mathbb{R}$. This is the celebrated *Spectral Theorem*. The term **spectrum** is parallel to the usage in *spectroscopy*. In the time this Theorem was popularized it's application to Quantum Mechanics had the interpretation that eigenvalues were **energy**. Mathematically, the **spectrum** is an ordered list of eigenvalues (in the real case). Physically, since the observables are described by Hermitian operators the eigenvalues are real. For example, the eigenvalue of the Hamiltonian operator is energy.
Finally, we arrive at the central result:

Theorem 10.6.21. Spectral Theorem:

Let V be a finite dimensional inner product space over $\mathbb{F}$ and suppose T is a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. If $\mathbb{F} = \mathbb{C}$ suppose T is normal. If $\mathbb{F} = \mathbb{R}$ then suppose T is self-adjoint. Denote $\mathcal{E}_i = \text{Ker}(T - \lambda_i Id)$ for each $i = 1, 2, \ldots, k$. There exist orthogonal projections T_i of V onto $\mathcal{E}_i$ for each $i = 1, 2, \ldots, k$. Moreover,

(a.) $V = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_k$

(b.) Let $\widehat{\mathcal{E}_j}$ denote the direct sum of all the eigenspaces except $\mathcal{E}_j$ then $\widehat{\mathcal{E}_j}^{\perp} = \mathcal{E}_j$

(c.) $T_i T_j = \delta_{ij} T_i$ for $1 \leq i, j \leq k$

(d.) $Id = T_1 + T_2 + \cdots + T_k$

(e.) $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$

We make good use of this Theorem in the next Chapter.

Chapter 11

quadratic forms

Quadradic forms arise in a variety of interesting applications. From geometry to physics these particular formulas arise. When there are no cross-terms it is fairly easy to analyze the behaviour of a given form. However, the appearance of cross-terms masks the true nature of a given form. Fortunately quadratic forms permit a matrix formulation and even more fantastically the matrix is necessarily symmetric and real. It follows the matrix is orthonormally diagonalizable and the spectrum (set of eigenvalues) completely describes the given form. We study this application of eigenvectors and hopefully learn a few new things about geometry and physics in the process.

11.1 conic sections and quadric surfaces

Some of you have taken calculus III others have not, but most of you still have much to learn about level curves and surfaces. Let me give two examples to get us started:

$$x^2 + y^2 = 4 \qquad \text{level curve; generally has form } f(x, y) = k$$

$$x^2 + 4y^2 + z^2 = 1 \qquad \text{level surface; generally has form } F(x, y, z) = k$$

Alternatively, some special surfaces can be written as a graph. The top half of the ellipsoid $F(x, y, z) = x^2 + 4y^2 + z^2 = 1$ is the $graph(f)$ where $f(x, y) = \sqrt{1 - x^2 - 4y^2}$ and $graph(f) = \{x, y, f(x, y) \ |(x, y) \in dom(f)\}$. Of course there is a great variety of examples to offer here and I only intend to touch on a few standard examples in this section. Our goal is to see what linear algebra has to say about conic sections and quadric surfaces.

11.2 quadratic forms and their matrix

Definition 11.2.1.

Generally, a **quadratic form** Q is a function $Q : \mathbb{R}^n \to \mathbb{R}$ whose formula can be written $Q(\vec{x}) = \vec{x}^T A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ where $A \in \mathbb{R}^{\ n\times n}$ such that $A^T = A$.

In particular, if $\vec{x} = (x, y)$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = ax^2 + bxy + byx + cy^2 = ax^2 + 2bxy + y^2.$$

The $n = 3$ case is similar,denote $A = [A_{ij}]$ and $\vec{x} = (x, y, z)$ so that

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = A_{11}x^2 + 2A_{12}xy + 2A_{13}xz + A_{22}y^2 + 2A_{23}yz + A_{33}z^2.$$

Generally, if $[A_{ij}] \in \mathbb{R}^{\ n\times n}$ and $\vec{x} = [x_i]^T$ then the associated quadratic form is

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \sum_{i,j} A_{ij}x_ix_j = \sum_{i=1}^{n} A_{ii}x_i^2 + \sum_{i<j} 2A_{ij}x_ix_j.$$

In case you wondering, yes you could write a given quadratic form with a different matrix which is not symmetric, but we will find it convenient to insist that our matrix is symmetric since that choice is always possible for a given quadratic form.

Also, you may recall (from the future) I said a **bilinear form** was a mapping from $V \times V \to \mathbb{R}$ which is linear in each slot. For example, an inner-product as defined in Definition 10.1.1 is a symmetric, positive definite bilinear form. When we discussed $\langle x, y\rangle$ we allowed $x \neq y$, in contrast a quadratic form is more like $\langle x, x\rangle$. Of course the dot-product is also an inner product and we can write a given quadratic form in terms of a dot-product:

$$\vec{x}^T A\vec{x} = \vec{x} \cdot (A\vec{x}) = (A\vec{x}) \cdot \vec{x} = \vec{x}^T A^T \vec{x}$$

Some texts actually use the middle equality above to define a symmetric matrix.

Example 11.2.2.

$$2x^2 + 2xy + 2y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 11.2.3.

$$2x^2 + 2xy + 3xz - 2y^2 - z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & 3/2 \\ 1 & -2 & 0 \\ 3/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition 11.2.4.

The values of a quadratic form on $\mathbb{R}^n - \{0\}$ is completely determined by it's values on the $(n-1)$-sphere $S_{n-1} = \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x}|| = 1\}$. In particular, $Q(\vec{x}) = ||\vec{x}||^2 Q(\hat{x})$ where $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$.

Proof: Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$. Notice that we can write any nonzero vector as the product of its magnitude $||x||$ and its direction $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$,

$$Q(\vec{x}) = Q(||\vec{x}||\hat{x}) = (||\vec{x}||\hat{x})^T A||\vec{x}||\hat{x} = ||\vec{x}||^2\hat{x}^T A\hat{x} = ||x||^2 Q(\hat{x}).$$

Therefore $Q(\vec{x})$ is simply proportional to $Q(\hat{x})$ with proportionality constant $||\vec{x}||^2$. $\square$

The proposition above is very interesting. It says that if we know how Q works on unit-vectors then we can extrapolate its action on the remainder of $\mathbb{R}^n$. If $f : S \to \mathbb{R}$ then we could say $f(S) > 0$ iff $f(s) > 0$ for all $s \in S$. Likewise, $f(S) < 0$ iff $f(s) < 0$ for all $s \in S$. The proposition below follows from the proposition above since $||\vec{x}||^2$ ranges over all nonzero positive real numbers in the equations above.

Proposition 11.2.5.

If Q is a quadratic form on $\mathbb{R}^n$ and we denote $\mathbb{R}^n_* = \mathbb{R}^n - \{0\}$

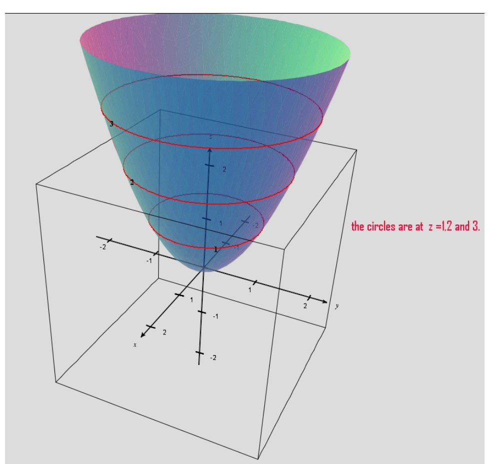
1.(negative definite) $Q(\mathbb{R}^n_*) < 0$ iff $Q(S_{n-1}) < 0$

2.(positive definite) $Q(\mathbb{R}^n_*) > 0$ iff $Q(S_{n-1}) > 0$

3.(non-definite) $Q(\mathbb{R}^n_*) = \mathbb{R} - \{0\}$ iff $Q(S_{n-1})$ has both positive and negative values.

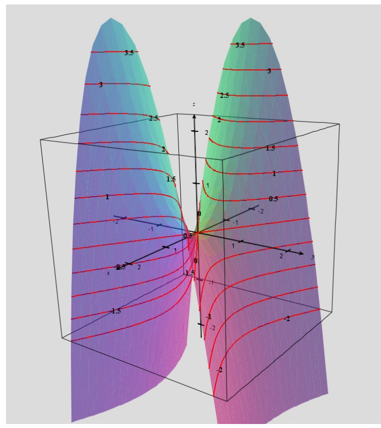
Before I get too carried away with the theory let's look at a couple examples.

Example 11.2.6. *Consider the quadric form* $Q(x, y) = x^2 + y^2$. *You can check for yourself that* $z = Q(x, y)$ *is a cone and* Q *has positive outputs for all inputs except* $(0, 0)$. *Notice that* $Q(v) = ||v||^2$ *so it is clear that* $Q(S_1) = 1$. *We find agreement with the preceding proposition. Next, think about the application of* $Q(x, y)$ *to level curves;* $x^2 + y^2 = k$ *is simply a circle of radius* $\sqrt{k}$ *or just the origin. Here's a graph of* $z = Q(x, y)$*:*



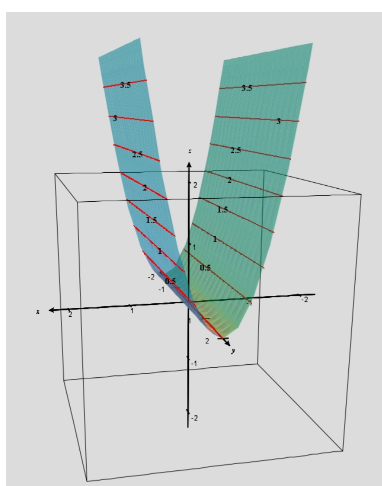
Notice that $Q(0, 0) = 0$ *is the absolute minimum for* Q. *Finally, let's take a moment to write* $Q(x, y) = [x, y]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are* $\lambda_1 = \lambda_2 = 1$.

Example 11.2.7. *Consider the quadric form $Q(x,y) = x^2 - 2y^2$. You can check for yourself that $z = Q(x,y)$ is a hyperboloid and Q has non-definite outputs since sometimes the x^2 term dominates whereas other points have $-2y^2$ as the dominent term. Notice that $Q(1,0) = 1$ whereas $Q(0,1) = -2$ hence we find $Q(S_1)$ contains both positive and negative values and consequently we find agreement with the preceding proposition. Next, think about the application of $Q(x,y)$ to level curves; $x^2 - 2y^2 = k$ yields either hyperbolas which open vertically ($k > 0$) or horizontally ($k < 0$) or a pair of lines $y = \pm\frac{x}{2}$ in the $k = 0$ case. Here's a graph of $z = Q(x,y)$:*



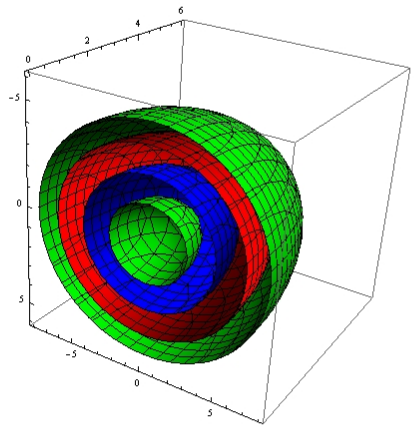
The origin is a **saddle point**. *Finally, let's take a moment to write $Q(x,y) = [x,y]\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = -2$.*

Example 11.2.8. *Consider the quadric form $Q(x,y) = 3x^2$. You can check for yourself that $z = Q(x,y)$ is parabola-shaped trough along the y-axis. In this case Q has positive outputs for all inputs except $(0,y)$, we would call this form* **positive semi-definite**. *A short calculation reveals that $Q(S_1) = [0,3]$ thus we again find agreement with the preceding proposition (case 3). Next, think about the application of $Q(x,y)$ to level curves; $3x^2 = k$ is a pair of vertical lines: $x = \pm\sqrt{k/3}$ or just the y-axis. Here's a graph of $z = Q(x,y)$:*



Finally, let's take a moment to write $Q(x,y) = [x,y]\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 3$ and $\lambda_2 = 0$.

Example 11.2.9. *Consider the quadric form $Q(x,y,z) = x^2 + 2y^2 + 3z^2$. Think about the application of $Q(x,y,z)$ to level surfaces; $x^2 + 2y^2 + 3z^2 = k$ is an ellipsoid. I can't graph a function of three variables, however, we can look at level surfaces of the function. I use Mathematica to plot several below:*



Finally, let's take a moment to write $Q(x,y,z) = [x,y,z]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = 2$ and $\lambda_3 = 3$.*

The examples given thus far are the simplest cases. We don't really need linear algebra to understand them. In contrast, e-vectors and e-values will prove a useful tool to unravel the later examples.

Proposition 11.2.10.

If Q is a quadratic form on $\mathbb{R}^n$ with matrix A and e-values $\lambda_1, \lambda_2, \dots, \lambda_n$ with orthonormal e-vectors $v_1, v_2, \dots, v_n$ then

$$Q(v_i) = {\lambda_i}^2$$

for $i = 1, 2, \dots, n$. Moreover, if $P = [v_1|v_2|\cdots|v_n]$ then

$$Q(\vec{x}) = (P^T\vec{x})^T P^T A P P^T \vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where we defined $\vec{y} = P^T\vec{x}$.

Let me restate the proposition above in simple terms: we can transform a given quadratic form to a diagonal form by finding orthonormalized e-vectors and performing the appropriate coordinate transformation. Since P is formed from orthonormal e-vectors we know that P will be either a rotation or reflection. This proposition says we can remove "cross-terms" by transforming the quadratic forms with an appropriate rotation.

Example 11.2.11. *Consider the quadric form $Q(x,y) = 2x^2 + 2xy + 2y^2$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice* $Q(x,y) = [x,y]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. *Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$$

Therefore, the e-values are $\lambda_1 = 1$ *and* $\lambda_2 = 3$.

$$(A - I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ *to give* $v = -u$ *choose* $u = 1$ *then normalize to get the vector above. Next,*

$$(A - 3I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

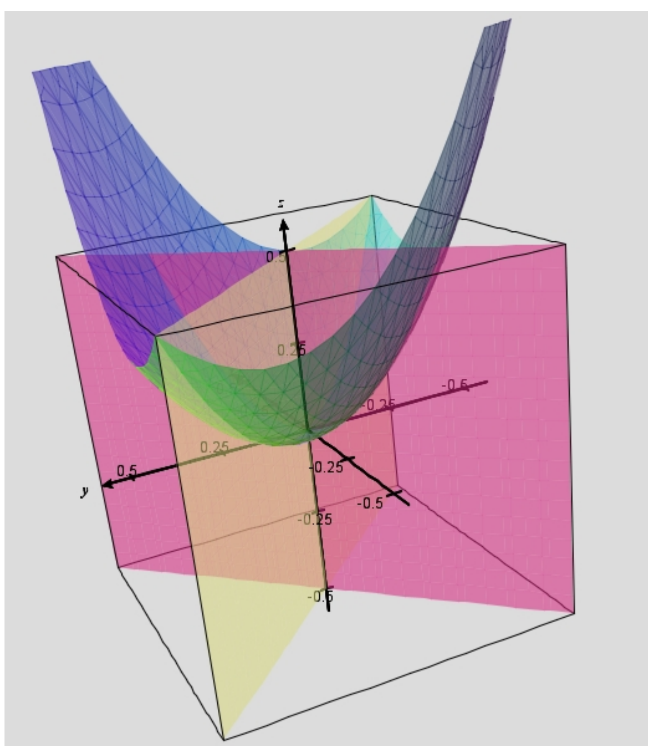
I just solved $u - v = 0$ *to give* $v = u$ *choose* $u = 1$ *then normalize to get the vector above. Let* $P = [\vec{u}_1|\vec{u}_2]$ *and introduce new coordinates* $\vec{y} = [\bar{x}, \bar{y}]^T$ *defined by* $\vec{y} = P^T\vec{x}$. *Note these can be inverted by multiplication by* P *to give* $\vec{x} = P\vec{y}$. *Observe that*

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x &= \frac{1}{\sqrt{2}}(\bar{x} + \bar{y}) \\ y &= \frac{1}{\sqrt{2}}(-\bar{x} + \bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} &= \frac{1}{\sqrt{2}}(x - y) \\ \bar{y} &= \frac{1}{\sqrt{2}}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q *yield*[1]*:*

$$\tilde{Q}(\bar{x}, \bar{y}) = \bar{x}^2 + 3\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x, y) = k$ *is an ellipse. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of* $Q(x, y) = 2x^2 + 2xy + 2y^2 = k$ *is an ellipse rotated by 45 degrees. Or, if you like, we can plot* $z = Q(x, y)$*:*



[1] technically $\tilde{Q}(\bar{x}, \bar{y})$ is $Q(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))$

Example 11.2.12. *Consider the quadric form $Q(x,y) = x^2+2xy+y^2$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2) = 0$$

Therefore, the e-values are $\lambda_1 = 0$ and $\lambda_2 = 2$.

$$(A - 0)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

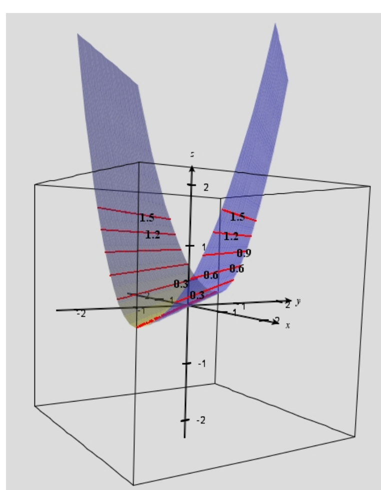
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x},\bar{y}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x & = \frac{1}{\sqrt{2}}(\bar{x}+\bar{y}) \\ y & = \frac{1}{\sqrt{2}}(-\bar{x}+\bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} & = \frac{1}{\sqrt{2}}(x-y) \\ \bar{y} & = \frac{1}{\sqrt{2}}(x+y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y}) = 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is a pair of paralell lines. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = x^2+2xy+y^2 = k$ is a line with slope -1. Indeed, with a little algebraic insight we could have anticipated this result since $Q(x,y) = (x+y)^2$ so $Q(x,y) = k$ implies $x + y = \sqrt{k}$ thus $y = \sqrt{k} - x$. Here's a plot which again verifies what we've already found:



Example 11.2.13. *Consider the quadric form $Q(x,y) = 4xy$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A-\lambda I) = det\begin{bmatrix} -\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda+2)(\lambda-2) = 0$$

Therefore, the e-values are $\lambda_1 = -2$ and $\lambda_2 = 2$.

$$(A+2I)\vec{u}_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u+v=0$ to give $v=-u$ choose $u=1$ then normalize to get the vector above. Next,

$$(A-2I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

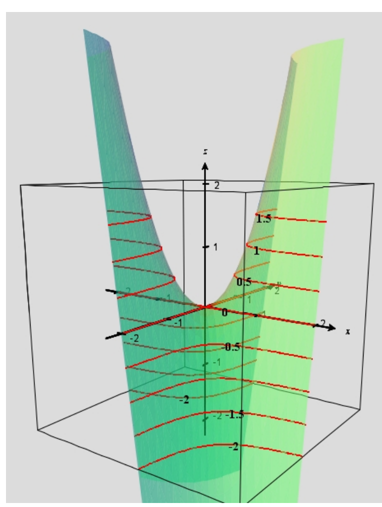
I just solved $u-v=0$ to give $v=u$ choose $u=1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x},\bar{y}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x & = \frac{1}{\sqrt{2}}(\bar{x}+\bar{y}) \\ y & = \frac{1}{\sqrt{2}}(-\bar{x}+\bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} & = \frac{1}{\sqrt{2}}(x-y) \\ \bar{y} & = \frac{1}{\sqrt{2}}(x+y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y}) = -2\bar{x}^2 + 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is a hyperbola. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = 4xy = k$ is a hyperbola rotated by 45 degrees. The graph $z = 4xy$ is thus a hyperbolic paraboloid:



The fascinating thing about the mathematics here is that if you don't want to graph $z = Q(x,y)$, but you do want to know the general shape then you can determine which type of quadraic surface you're dealing with by simply calculating the eigenvalues of the form.

Remark 11.2.14.

I made the preceding triple of examples all involved the same rotation. This is purely for my lecturing convenience. In practice the rotation could be by all sorts of angles. In addition, you might notice that a different ordering of the e-values would result in a redefinition of the barred coordinates. [2]

We ought to do at least one 3-dimensional example.

Example 11.2.15. *Consider the quadric form Q defined below:*

$$Q(x,y,z) = [x,y,z]\begin{bmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Denote the matrix of the form by A and calculate the e-values/vectors:

$$\begin{aligned} det(A-\lambda I) &= det\begin{bmatrix} 6-\lambda & -2 & 0 \\ -2 & 6-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \\ &= [(\lambda-6)^2-4](5-\lambda) \\ &= (5-\lambda)[\lambda^2-12\lambda+32](5-\lambda) \\ &= (\lambda-4)(\lambda-8)(5-\lambda) \end{aligned}$$

Therefore, the e-values are $\lambda_1 = 4,\ \ \lambda_2 = 8$ and $\lambda_3 = 5$. After some calculation we find the following orthonormal e-vectors for A:

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

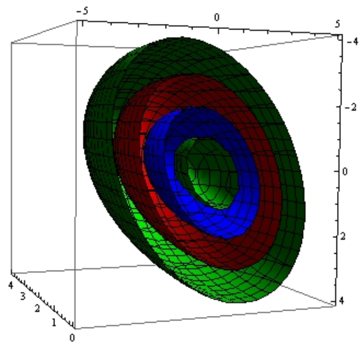
Let $P = [\vec{u}_1|\vec{u}_2|\vec{u}_3]$ and introduce new coordinates $\vec{y} = [\bar{x},\bar{y},\bar{z}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{\sqrt{2}}(\bar{x}+\bar{y}) \\ y &= \tfrac{1}{\sqrt{2}}(-\bar{x}+\bar{y}) \\ z &= \bar{z} \end{aligned} \quad or \quad \begin{aligned} \bar{x} &= \tfrac{1}{\sqrt{2}}(x-y) \\ \bar{y} &= \tfrac{1}{\sqrt{2}}(x+y) \\ \bar{z} &= z \end{aligned}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y},\bar{z}) = 4\bar{x}^2 + 8\bar{y}^2 + 5\bar{z}^2$$

It is clear that in the barred coordinate system the level surface $Q(x,y,z) = k$ is an ellipsoid. If we draw the barred coordinate system superposed over the xyz-coordinate system then you'll see that the graph of $Q(x,y,z) = k$ is an ellipsoid rotated by 45 degrees around the $z-axis$. Plotted below are a few representative ellipsoids:



Remark 11.2.16.

> If you would like to read more about conic sections or quadric surfaces and their connection to e-values/vectors I reccommend sections 9.6 and 9.7 of Anton's text. I have yet to add examples on how to include translations in the analysis. It's not much more trouble but I decided it would just be an unecessary complication this semester. Also, section 7.1,7.2 and 7.3 in Lay's text show a bit more about how to use this math to solve concrete applied problems. You might also take a look in Strang's text, his discussion of tests for positive-definite matrices is much more complete than I will give here.

11.2.1 summary of quadratic form analysis

There is a connection between the shape of level curves $Q(x_1, x_2, \dots, x_n) = k$ and the graph $x_{n+1} = f(x_1, x_2, \dots, x_n)$ of f. I'll discuss $n = 2$ but these comments equally well apply to $w = f(x, y, z)$ or higher dimensional examples. Consider a critical point (a, b) for $f(x, y)$ then the Taylor expansion about (a, b) has the form

$$f(a + h, b + k) = f(a, b) + Q(h, k)$$

where $Q(h, k) = \frac{1}{2}h^2 f_{xx}(a, b) + hk f_{xy}(a, b) + \frac{1}{2}h^2 f_{yy}(a, b) = [h, k][Q](h, k)$. Since $[Q]^T = [Q]$ we can find orthonormal e-vectors $\vec{u}_1, \vec{u}_2$ for $[Q]$ with e-values λ_1 and λ_2 respective. Using $U = [\vec{u}_1|\vec{u}_2]$ we can introduce rotated coordinates $(\bar{h}, \bar{k}) = U(h, k)$. These will give

$$Q(\bar{h}, \bar{k}) = \lambda_1 \bar{h}^2 + \lambda_2 \bar{k}^2$$

Clearly if $\lambda_1 > 0$ and $\lambda_2 > 0$ then $f(a, b)$ yields the local minimum whereas if $\lambda_1 < 0$ and $\lambda_2 < 0$ then $f(a, b)$ yields the local maximum. Edwards discusses these matters on pgs. 148-153. In short, supposing $f \approx f(p) + Q$, if all the e-values of Q are positive then f has a local minimum of $f(p)$ at p whereas if all the e-values of Q are negative then f reaches a local maximum of $f(p)$ at p. Otherwise Q has both positive and negative e-values and we say Q is non-definite and the function has a saddle point. If all the e-values of Q are positive then Q is said to be **positive-definite** whereas if all the e-values of Q are negative then Q is said to be **negative-definite**. Edwards gives a few nice tests for ascertaining if a matrix is positive definite without explicit computation of e-values. Finally, if one of the e-values is zero then the graph will be like a trough.

Remark 11.2.17. *summary of the summary.*

> In short, the behaviour of a quadratic form $Q(x) = x^T Ax$ is governed by it's spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Moreover, the form can be written as $Q(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_k y_k^2$ by choosing the coordinate system which is built from the orthonormal eigenbasis of $col(A)$. In this coordinate system questions of optimization become trivial (see section 7.3 of Lay for applied problems)

11.3 Taylor series for functions of two or more variables

It turns out that linear algebra and e-vectors can give us great insight into locating local extrema for a function of several variables. To summarize, we can calculate the multivariate Taylor series and we'll find that the quadratic terms correspond to a *quadratic form*. In fact, each quadratic form has a symmetric matrix representative. We know that symmetric matrices are diagonalizable hence the e-values of a symmetric matrix will be real. Moreover, the eigenvalues tell you what the min/max value of the function is at a critical point (usually). This is the n-dimensional generalization of the 2nd-derivative test from calculus. If you'd like to see further detail on these please consider taking Advanced Calculus (Math 332).

Our goal here is to find an analog for Taylor's Theorem for function from $\mathbb{R}^n$ to $\mathbb{R}$. Recall that if $g : U \subseteq \mathbb{R} \to \mathbb{R}$ is *smooth* at $a \in \mathbb{R}$ then we can compute as many derivatives as we wish, moreover we can generate the Taylor's series for g centered at a:

$$\boxed{g(a+h) = g(a) + g'(a)h + \frac{1}{2}g''(a)h^2 + \frac{1}{3!}g''(a)h^3 + \cdots = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}h^n}$$

The equation above assumes that g is analytic at a. In other words, the function actually matches it's Taylor series near a. This concept can be made rigorous by discussing the remainder. If one can show the remainder goes to zero then that proves the function is analytic. You might read pages 117-127 of Edwards *Advanced Calculus* for more on these concepts, I sometimes cover parts of that material in Advanced Calculus, Theorem 6.3 is particularly interesting.

11.3.1 deriving the two-dimensional Taylor formula

The idea is fairly simple: create a function on $\mathbb{R}$ with which we can apply the ordinary Taylor series result. Much like our discussion of directional derivatives we compose a function of two variables with linear path in the domain. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be smooth with smooth partial derivatives of all orders. Furthermore, let $(a, b) \in U$ and construct a line through (a, b) with direction vector (h_1, h_2) as usual:

$$\phi(t) = (a, b) + t(h_1, h_2) = (a + th_1, b + th_2)$$

for $t \in \mathbb{R}$. Note $\phi(0) = (a, b)$ and $\phi'(t) = (h_1, h_2) = \phi'(0)$. Construct $g = f \circ \phi : \mathbb{R} \to \mathbb{R}$ and differentiate, note we use the chain rule for functions of several variables in what follows:

$$\begin{aligned} g'(t) = (f \circ \phi)'(t) &= f'(\phi(t))\phi'(t) \\ &= \nabla f(\phi(t)) \cdot (h_1, h_2) \\ &= h_1 f_x(a + th_1, b + th_2) + h_2 f_y(a + th_1, b + th_2) \end{aligned}$$

Note $g'(0) = h_1 f_x(a, b) + h_2 f_y(a, b)$. Differentiate again (I omit $(\phi(t))$ dependence in the last steps),

$$\begin{aligned} g''(t) &= h_1 f_x'(a + th_1, b + th_2) + h_2 f_y'(a + th_1, b + th_2) \\ &= h_1 \nabla f_x(\phi(t)) \cdot (h_1, h_2) + h_2 \nabla f_y(\phi(t)) \cdot (h_1, h_2) \\ &= h_1^2 f_{xx} + h_1 h_2 f_{yx} + h_2 h_1 f_{xy} + h_2^2 f_{yy} \\ &= h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy} \end{aligned}$$

Thus, making explicit the point dependence, $g''(0) = h_1^2 f_{xx}(a,b) + 2h_1h_2 f_{xy}(a,b) + h_2^2 f_{yy}(a,b)$. We may construct the Taylor series for g up to quadratic terms:

$$\begin{aligned} g(0+t) &= g(0) + tg'(0) + \frac{1}{2}g''(0) + \cdots \\ &= f(a,b) + t[h_1 f_x(a,b) + h_2 f_y(a,b)] + \frac{t^2}{2}\left[h_1^2 f_{xx}(a,b) + 2h_1h_2 f_{xy}(a,b) + h_2^2 f_{yy}(a,b)\right] + \cdots \end{aligned}$$

Note that $g(t) = f(a + th_1, b + th_2)$ hence $g(1) = f(a + h_1, b + h_2)$ and consequently,

$$\begin{aligned} f(a+h_1, b+h_2) = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) + \\ + \frac{1}{2}\Big[h_1^2 f_{xx}(a,b) + 2h_1h_2 f_{xy}(a,b) + h_2^2 f_{yy}(a,b)\Big] + \cdots \end{aligned}$$

Omitting point dependence on the 2^{nd} derivatives,

$$\boxed{f(a+h_1, b+h_2) = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) + \tfrac{1}{2}\left[h_1^2 f_{xx} + 2h_1h_2 f_{xy} + h_2^2 f_{yy}\right] + \cdots}$$

Sometimes we'd rather have an expansion about (x,y). To obtain that formula simply substitute $x - a = h_1$ and $y - b = h_2$. Note that the point (a,b) is fixed in this discussion so the derivatives are not modified in this substitution,

$$\begin{aligned} f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \\ + \frac{1}{2}\Big[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big] + \cdots \end{aligned}$$

At this point we ought to recognize the first three terms give the tangent plane to $z = f(z,y)$ at $(a,b,f(a,b))$. The higher order terms are nonlinear corrections to the linearization, these quadratic terms form a *quadratic form.* If we computed third, fourth or higher order terms we'd find that, using $a = a_1$ and $b = a_2$ as well as $x = x_1$ and $y = x_2$,

$$\boxed{f(x,y) = \sum_{n=0}^{\infty}\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\cdots\sum_{i_n=0}^{n}\frac{1}{n!}\frac{\partial^{(n)} f(a_1,a_2)}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}}(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2})\cdots(x_{i_n} - a_{i_n})}$$

The multivariate Taylor formula for a function of j-variables for $j > 2$ is very similar. Rather than even state the formula I will show a few examples in the subsection that follows.

11.3.2 examples

Example 11.3.1. *Suppose $f(x,y) = exp(-x^2 - y^2 + 2y - 1)$ expand f about the point $(0,1)$:*

$$f(x,y) = exp(-x^2)exp(-y^2 + 2y - 1) = exp(-x^2)exp(-(y-1)^2)$$

expanding,

$$f(x,y) = (1 - x^2 + \cdots)(1 - (y-1)^2 + \cdots) = 1 - x^2 - (y-1)^2 + \cdots$$

Recenter about the point $(0,1)$ by setting $x = h$ and $y = 1 + k$ so

$$f(h, 1+k) = 1 - h^2 - k^2 + \cdots$$

If (h,k) is near $(0,0)$ then the dominant terms are simply those we've written above hence the graph is like that of a quadraic surface with a pair of negative e-values. It follows that $f(0,1)$ is a local maximum. In fact, it happens to be a global maximum for this function.

Example 11.3.2. *Suppose $f(x,y) = 4-(x-1)^2+(y-2)^2+Aexp(-(x-1)^2-(y-2)^2)+2B(x-1)(y-2)$ for some constants A, B. Analyze what values for A, B will make $(1,2)$ a local maximum, minimum or neither. Expanding about $(1,2)$ we set $x = 1+h$ and $y = 2+k$ in order to see clearly the local behaviour of f at $(1,2)$,*

$$\begin{aligned} f(1+h,2+k) &= 4-h^2-k^2+Aexp(-h^2-k^2)+2Bhk \\ &= 4-h^2-k^2+A(1-h^2-k^2)+2Bhk\cdots \\ &= 4+A-(A+1)h^2+2Bhk-(A+1)k^2+\cdots \end{aligned}$$

There is no nonzero linear term in the expansion at $(1,2)$ which indicates that $f(1,2) = 4+A$ may be a local extremum. In this case the quadratic terms are nontrivial which means the graph of this function is well-approximated by a quadraic surface near $(1,2)$. The quadratic form $Q(h,k) = -(A+1)h^2+2Bhk-(A+1)k^2$ has matrix

$$[Q] = \begin{bmatrix} -(A+1) & B \\ B & -(A+1)^2 \end{bmatrix}.$$

The characteristic equation for Q is

$$det([Q]-\lambda I) = det\begin{bmatrix} -(A+1)-\lambda & B \\ B & -(A+1)^2-\lambda \end{bmatrix} = (\lambda+A+1)^2-B^2 = 0$$

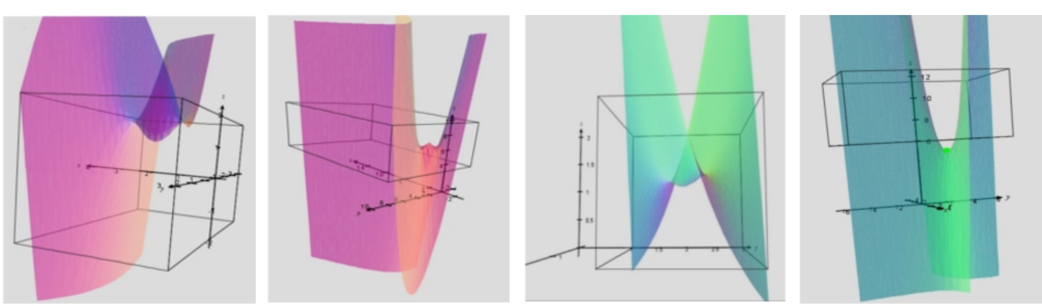
We find solutions $\lambda_1 = -A-1+B$ and $\lambda_2 = -A-1-B$. The possibilities break down as follows:

1. *if $\lambda_1, \lambda_2 > 0$ then $f(1,2)$ is local minimum.*
2. *if $\lambda_1, \lambda_2 < 0$ then $f(1,2)$ is local maximum.*
3. *if just one of λ_1, λ_2 is zero then f is constant along one direction and min/max along another so technically it is a local extremum.*
4. *if $\lambda_1\lambda_2 < 0$ then $f(1,2)$ is not a local etremum, however it is a saddle point.*

In particular, the following choices for A, B will match the choices above

1. *Let $A = -3$ and $B = 1$ so $\lambda_1 = 3$ and $\lambda_2 = 1$;*
2. *Let $A = 3$ and $B = 1$ so $\lambda_1 = -3$ and $\lambda_2 = -5$*
3. *Let $A = -3$ and $B = -2$ so $\lambda_1 = 0$ and $\lambda_2 = 4$*
4. *Let $A = 1$ and $B = 3$ so $\lambda_1 = 1$ and $\lambda_2 = -5$*

Here are the graphs of the cases above, note the analysis for case 3 is more subtle for Taylor approximations as opposed to simple quadraic surfaces. In this example, case 3 was also a local minimum. In contrast, in Example 11.2.12 the graph was like a trough. The behaviour of f away from the critical point includes higher order terms whose influence turns the trough into a local minimum.



Example 11.3.3. *Suppose $f(x,y) = \sin(x)\cos(y)$ to find the Taylor series centered at $(0,0)$ we can simply multiply the one-dimensional result $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$ and $\cos(y) = 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \cdots$ as follows:*

$$\begin{aligned} f(x,y) &= (x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots)(1 - \tfrac{1}{2!}y^2 + \tfrac{1}{4!}y^4 + \cdots) \\ &= x - \tfrac{1}{2}xy^2 + \tfrac{1}{24}xy^4 - \tfrac{1}{6}x^3 - \tfrac{1}{12}x^3y^2 + \cdots \\ &= x + \cdots \end{aligned}$$

The origin $(0,0)$ is a critical point since $f_x(0,0) = 0$ and $f_y(0,0) = 0$, however, this particular critical point escapes the analysis via the quadratic form term since $Q = 0$ in the Taylor series for this function at $(0,0)$. This is analogous to the inconclusive case of the 2nd derivative test in calculus III.

Example 11.3.4. *Suppose $f(x,y,z) = xyz$. Calculate the multivariate Taylor expansion about the point $(1,2,3)$. I'll actually calculate this one via differentiation, I have used tricks and/or calculus II results to shortcut any differentiation in the previous examples. Calculate first derivatives*

$$f_x = yz \qquad f_y = xz \qquad f_z = xy,$$

and second derivatives,

$$\begin{array}{ccc} f_{xx} = 0 & f_{xy} = z & f_{xz} = y \\ f_{yx} = z & f_{yy} = 0 & f_{yz} = x \\ f_{zx} = y & f_{zy} = x & f_{zz} = 0, \end{array}$$

and the nonzero third derivatives,

$$f_{xyz} = f_{yzx} = f_{zxy} = f_{zyx} = f_{yxz} = f_{xzy} = 1.$$

It follows,

$$\begin{aligned} &f(a+h, b+k, c+l) = \\ &\quad = f(a,b,c) \;+\; f_x(a,b,c)h \;+\; f_y(a,b,c)k \;+\; f_z(a,b,c)l \;+ \\ &\qquad \tfrac{1}{2}(\; f_{xx}hh + f_{xy}hk + f_{xz}hl + f_{yx}kh + f_{yy}kk + f_{yz}kl + f_{zx}lh + f_{zy}lk + f_{zz}ll \;) + \cdots \end{aligned}$$

Of course certain terms can be combined since $f_{xy} = f_{yx}$ etc... for smooth functions (we assume smooth in this section, moreover the given function here is clearly smooth). In total,

$$f(1+h, 2+k, 3+l) = 6 + 6h + 3k + 2l + \frac{1}{2}\big(3hk + 2hl + 3kh + kl + 2lh + lk\big) + \frac{1}{3!}(6)hkl$$

Of course, we could also obtain this from simple algebra:

$$f(1+h, 2+k, 3+l) = (1+h)(2+k)(3+l) = 6 + 6h + 3k + l + 3hk + 2hl + kl + hkl.$$

Remark 11.3.5.

One very interesting application of the orthogonal complement theorem is to the method of Lagrange multipliers. The problem is to maximize an objective function $f(x_1, x_2, \dots, x_n)$ with respect to a set of constraint functions $g_1(x_1, x_2, \dots, x_n) = 0$, $g_2(x_1, x_2, \dots, x_n) = 0$ and $g_k(x_1, x_2, \dots, x_n) = 0$. One can argue that extreme values for f must satisfy

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_k \nabla g_k$$

for a particular set of Lagrange multipliers $\lambda_1, \lambda_2, \dots, \lambda_k$. The crucial step in the analysis relies on the orthogonal decomposition theorem. It is the fact that forces the gradient of the objective function to reside in the span of the gradients of the constraints. See my Advanced Calculus notes, or consult many advanced calculus texts.

11.4 intertia tensor, an application of quadratic forms

We can use quadratic forms to elegantly state a number of interesting quantities in classical mechanics. For example, the translational kinetic energy of a mass m with velocity v is

$$T_{trans}(v) = \frac{m}{2}v^T v = [v_1, v_2, v_3]\begin{bmatrix} m/2 & 0 & 0 \\ 0 & m/2 & 0 \\ 0 & 0 & m/2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

On the other hand, the rotational kinetic energy of an object with moment of intertia I and angular velocity ω with respect to a particular axis of rotation is

$$T_{rot}(v) = \frac{I}{2}\omega^T \omega.$$

In addition you might recall that the force F applied at radial arm r gave rise to a torque of $\tau = r \times F$ which made the angular momentum $L = I\omega$ have the time-rate of change $\tau = \frac{dL}{dt}$. In the first semester of physics this is primarily all we discuss. We are usually careful to limit the discussion to rotations which happen to occur with respect to a particular axis. But, what about other rotations? What about rotations with respect to less natural axes of rotation? How should we describe the rotational physics of a rigid body which spins around some axis which doesn't happen to line up with one of the nice examples you find in an introductory physics text?

The answer is found in extending the idea of the moment of intertia to what is called the inertia tensor I_{ij} (in this section I is not the identity). To begin I'll provide a calculation which motivates the definition for the inertia tensor.

Consider a rigid mass with density $\rho = dm/dV$ which is a function of position $r = (x_1, x_2, x_3)$. Suppose the body rotates with angular velocity ω about some axis through the origin, however it is otherwise not in motion. This means all of the energy is rotational. Suppose that dm is at r then we define $v = (\dot{x_1}, \dot{x_2}, \dot{x_3}) = dr/dt$. In this context, the velocity v of dm is also given by the cross-product with the angular velocity; $v = \omega \times r$. Using the einstein repeated summation notation the k-th component of the cross-product is nicely expressed via the Levi-Civita symbol; $(\omega \times r)_k = \epsilon_{klm}\omega_l x_m$. Therefore, $v_k = \epsilon_{klm}\omega_l x_m$. The infinitesimal kinetic energy due to this little bit of rotating mass dm is hence

$$\begin{aligned}
dT &= \frac{dm}{2}v_k v_k \\
&= \frac{dm}{2}(\epsilon_{klm}\omega_l x_m)(\epsilon_{kij}\omega_i x_j) \\
&= \frac{dm}{2}\epsilon_{klm}\epsilon_{kij}\omega_l\omega_i x_m x_j \\
&= \frac{dm}{2}(\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\omega_l\omega_i x_m x_j \\
&= \frac{dm}{2}(\delta_{li}\delta_{mj}\omega_l\omega_i x_m x_j - \delta_{lj}\delta_{mi}\omega_l\omega_i x_m x_j) \\
&= \omega_l \frac{dm}{2}(\delta_{li}\delta_{mj} x_m x_j - \delta_{lj}\delta_{mi} x_m x_j)\omega_i \\
&= \omega_l \left[\frac{dm}{2}(\delta_{li}||r||^2 - x_l x_i)\right]\omega_i.
\end{aligned}$$

Integrating over the mass, if we add up all the little bits of kinetic energy we obtain the total kinetic energy for this rotating body: we replace dm with $\rho(r)dV$ and the integration is over the volume of the body,

$$T = \int \omega_l \left[\frac{1}{2}(\delta_{li}||r||^2 - x_l x_i)\right]\omega_i \rho(r)dV$$

However, the body is rigid so the angular velocity is the same for each dm and we can pull the components of the angular velocity out of the integration[3] to give:

$$T = \frac{1}{2}\omega_j \underbrace{\left[\int (\delta_{jk}||r||^2 - x_j x_k)\rho(r)dV\right]}_{I_{jk}} \omega_k$$

This integral defines the intertia tensor I_{jk} for the rotating body. Given the inertia tensor I_{lk} the kinetic energy is simply the value of the quadratic form below:

$$T(\omega) = \frac{1}{2}\omega^T \omega = [\omega_1, \omega_2, \omega_3] \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

The matrix above is not generally diagonal, however you can prove it is symmetric (easy). Therefore, we can find an orthonormal eigenbasis $\beta = \{u_1, u_2, u_3\}$ and if $P = [\beta]$ then it follows by orthonormality of the basis that $[I]_{\beta,\beta} = P^T[I]P$ is diagonal. The eigenvalues of the inertia tensor (the matrix $[I_{jk}]$) are called the **principle moments of inertia** and the eigenbasis $\beta = \{u_1, u_2, u_3\}$ define the **principle axes** of the body.

The study of the rotational dynamics flows from analyzing the equations:

$$L_i = I_{ij}\omega_j \qquad \text{and} \qquad \tau_i = \frac{dL_i}{dt}$$

If the initial angular velocity is in the direction of a principle axis u_1 then the motion is basically described in the same way as in the introductory physics course provided that the torque is also in the direction of u_1. The moment of intertia is simply the first principle moment of inertia and $L = \lambda_1 \omega$. However, if the torque is not in the direction of a princple axis or the initial angular velocity is not along a principle axis then the motion is more complicated since the rotational motion is connected to more than one axis of rotation. Think about a spinning top which is spinning in place. There is wobbling and other more complicated motions that are covered by the mathematics described here.

Example 11.4.1. *The intertia tensor for a cube with one corner at the origin is found to be*

$$I = \frac{2}{3}Ms^2 \begin{bmatrix} 1 & -3/8 & -3/8 \\ -3/8 & 1 & -3/8 \\ -3/8 & -3/8 & 1 \end{bmatrix}$$

Introduce $m = M/8$ to remove the fractions,

$$I = \frac{2}{3}Ms^2 \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

[3] I also relabled the indices to have nicer final formula, nothing profound here

You can calculate that the e-values are $\lambda_1 = 2$ *and* $\lambda_2 = 11 = \lambda_3$ *with principle axis in the directions*

$$u_1 = \frac{1}{\sqrt{3}}(1,1,1), \;\; u_2 = \frac{1}{\sqrt{2}}(-1,1,0), \;\; u_3 = \frac{1}{\sqrt{2}}(-1,0,1).$$

The choice of u_2, u_3 *is not unique. We could just as well choose any other orthonormal basis for* $span\{u_2, u_3\} = W_{11}$*.*

Finally, a word of warning, for a particular body there may be so much symmetry that no particular eigenbasis is specified. There may be many choices of an orthonormal eigenbasis for the system. Consider a sphere. Any orthonormal basis will give a set of principle axes. Or, for a right circular cylinder the axis of the cylinder is clearly a principle axis however the other two directions are arbitrarily chosen from the plane which is the orthogonal complement of the axis. I think it's fair to say that if a body has a unique (up to ordering) set of principle axes then the shape has to be somewhat ugly. Symmetry is beauty but it implies ambiguity for the choice of certain princple axes.

Chapter 12

Abstract Linear Algebra

In this chapter we study a number of abstract constructions in linear algebra which are essential to understanding applications of linear algbera to abstract mathematics[1]

We begin by studying the addition of sets. We find that if we add vectors to a subspace $W \leq V$ then cosets $x + W$ have a natural vector space structure. In particular,

$$(x + W) + (y + W) = (x + y) + W \quad \& \quad c(x + W) = cx + W$$

which means $0 + W = W$ is the additive identity for coset addition. The **quotient space** of V by W is denoted $V/W = \{x + W \mid x \in V\}$. We prove that V/W forms a vector space. The vectors in V/W are actually sets of vectors in V; a coset of V is a vector in V/W. You can think of each vector in the quotient space as a glued-coset.

Quotient space and cosets find application to the problem of constructing a bijection from a given linear transformation. This is part of a larger story. Let me briefly explain. It is a fact of set theory that any function can be made into a bijection by suitably adjusting the domain and codomain. Consider $f : A \to B$. For the codomain B we simply exchange the codomain for the image $f(A)$. For the domain, a selection can be made for each fiber of the function[2] in particular we choose $S \subseteq \text{dom}(f)$ for which $S \cap f^{-1}(p) = \{q\}$ for each $p \in \text{image}(f)$ and $f(S) = f(A)$. Notice $f|_S : S \to f(A)$ is a bijection since it is surjective and if $f(x) = f(y) = p$ for $x, y \in S$ then $x, y \in f^{-1}(p)$ thus $x = y$. The part of this construction which is most troublesome is the selection of S. For a linear transformation, the fibers have a uniformity which makes them particularly easy to understand; in particular, the fibers of T are cosets of $\ker(T)$. We could form a cross-section for $T : V \to W$ simply as $S = \{0\} \cup (V - \ker(T))$. Another way of thinking of this cross-section is the quotient of V by $\ker(T)$. In fact, we will prove the *First Isomorphism Theorem* which states there is a natural isomorphism from $V/\ker(T)$ to $T(V)$ for any linear transformation $T : V \to W$. We also study a number of elementary examples and applications of the first isomorphism theorem to linear algebra. We'll see it gives us a new and often easy method to construct isomorphisms.

If V is a vector space over $\mathbb{F}$ then $V^* = L(V, \mathbb{F})$ is the **dual space** of V. Furthermore, the **double dual** of V is denoted V^{**}. For a finite dimensional vector space we have that the vector space, its dual and also its double dual are all isomorphic. However, the explicit construction of an isomorphism of V and V^* requires the choice of a basis for V. In contrast, there is a natural, basis free,

[1]Jedi applied math student forcefully suggests **these are not the applications you're looking for...** unfortunately, your Jedi mind tricks do not work on me.

[2]a fiber is the inverse image of a singleton; $f^{-1}\{p\} = \{x \in \text{dom}(f) \mid f(x) = p\}$ is a fiber

isomorphism of V and V^{**}. In the context of an inner product space there is also a natural isomorphism of V and V^* which is known as the **musical morphism** in the larger study of tensors on an inner product space. We study the elementary theory of duals and we discover a few interesting applications to the theory of orthogonal complements. There is an important reformulation of $W^{\perp}$ in terms of **annihilators** in the dual.

There are about four different maps we can naturally associate with a given vector space with a given basis. In particular, we can study $L(V,V), L(V^*,V^*)$ or we can study bilinear maps on $V\times V$ or $V^*\times V^*$. Each type of map has a different coordinate change rule for its matrix. We have studied how the matrix of $T : V \to V$ transforms according to a similarity transformation. The matrix for $S \in L(V^*,V^*)$ also undergoes a similarity transformation if we change the basis for V^*. The **dual basis** β^* is naturally connected to a given basis β for V. In constrast, the matrix of a bilinear map does not transform via a similarity transformation. We'll discuss how to define the matrix of a bilinear map. We also discuss a number of elementary theorems for blinear forms. In addition, the concept of a quadratic form is introduced over an arbitrary vector space[3]. Finally, I state without proof a famous theorem by Sylvester which makes psuedo-inner products more boring than you might first expect. Basically, we learn that any inner product can be written as a dot-product with the right choice of basis. Or, a psuedo-inner product can be written as a Minkowski-product[4]

[3]I defined a quadratic form on $\mathbb{R}^n$ by its matrix formula as to avoid uncessary discussion about bilinear forms earlier in this course

[4]for example, $\eta(v,w) = -v_0w_0 + v_1w_1 + v_2w_2 + v_3w_3$ describes the spacetime geometry of special relativity on $\mathbb{R}^4$.

12.1 quotient space

Let us begin with a discussion of how to add sets of vectors. If $S, T \subseteq V$ a vector space over $\mathbb{F}$ then we define $S+T$ as follows:

$$S + T = \{s + t \mid s \in S,\ t \in T\}$$

In the particular case $S = \{x\}$ it is customary to write

$$x + T = \{x + t \mid t \in T\}$$

we drop the $\{\}$ around x in this special case.

Definition 12.1.1. *Coset*

Let V be a vector space with $x \in V$ and $W \leq V$ then $x + W$ is a **coset** of W.

Example 12.1.2. *If $W = span\{(1,0)\}$ is the x-axis in $\mathbb{R}^2$ then $(a,b) + W$ is the the horizontal line given by equation $y = b$. You can easily see $(0,b) + W = (a,b) + W$. Each coset is obtained by translating the x-axis to a parallel horizontal line.*

Example 12.1.3. *If $W = \{A \in \mathbb{R}^{n\times n} \mid A^T = A\}$ then $X + W$ is a coset in the square matrices. Geometrically, it is a linear manifold of the same dimension as W. I can't picture this one directly.*

Quotient space of V by W is the set of all such **cosets** of W. We now work towards motivating the definition of Quotient space. In particular, we need to show how it has a natural vector space structure induced from V.

Proposition 12.1.4.

Let V be vector space over $\mathbb{F}$ and $W \leq V$. Then $x + W = y + W$ iff $x - y \in W$.

Proof: Suppose $x + W = y + W$. If $p \in x + W$ then it follows there exists $w_1 \in W$ for which $p = x + w_1$. However, as $x + W \subseteq y + W$ we find $x + w_1 \in y + W$ and thus there exists $w_2 \in W$ for which $x + w_1 = y + w_2$. Therefore, $y - x = w_1 - w_2 \in W$ as W is a subspace of V.

Conversely, suppose $x, y \in V$ and $x - y \in W$. Thus, there exists $w \in W$ for which $x - y = w$ and so for future reference $x = y + w$ or $y = x - w$. Let $p \in x + W$ hence there exists $w_1 \in W$ for which $p = x + w_1$. Furthermore, as W is a subspace we know $w, w_1 \in W$ implies $w + w_1 \in W$. Consider then, $p = x + w_1 = y + w + w_1 \in y + W$. Therefore, $x + W \subseteq y + W$. A similar argument shows $y + W \subseteq x + W$ hence $x + W = y + W$. $\square$

Proposition 12.1.5.

Let V be vector space over $\mathbb{F}$ and $W \leq V$. Then $x + W = W$ iff $x \in W$.

Proof: if $x + W = W$ then $x + w \in W$ for some w hence $x + w = w_1$. But, it follows $x = w_1 - w$ which makes clear that $x \in W$ as $W \leq V$.

Conversely, if $x \in W$ then consider $p = x + w_1 \in x + W$ and note $x + w_1 \in W$ hence $p \in W$ and we find $x + W \subseteq W$. Likewise, if $w \in W$ then note $w = x + w - x$ and $w - x \in W$ thus $w \in x + W$

and we find $W \subseteq x + W$. Therefore, $x + W = W$. $\square$

Observe that Proposition 12.1.4 can be reformulated to say $x+W$ is the same as $y+W$ if $y = x+w$ for some $w \in W$. We say that x and y are coset representatives of the same coset iff $x+W = y+W$. Suppose $x_1 + W = x_2 + W$ and $y_1 + W = y_2 + W$; that is, suppose x_1, x_2 are representatives of the same coset and suppose y_1, y_2 are representatives of the same coset.

Proposition 12.1.6.

Let V be vector space over $\mathbb{F}$ and $W \leq V$. If $x_1 + W = x_2 + W$ and $y_1 + W = y_2 + W$ and $c \in \mathbb{F}$ then $x_1 + y_1 + W = x_2 + y_2 + W$ and $cx_1 + W = cx_2 + W$.

Proof: Suppose $x_1 + W = x_2 + W$ and $y_1 + W = y_2 + W$ then by Proposition 12.1.4 we find $x_2 - x_1 = w_x$ and $y_2 - y_2 = w_y$ for some $w_x, w_y \in W$. Consider

$$(x_2 + y_2) - (x_1 + y_1) = x_2 - x_1 + y_2 - y_1 = w_x + w_y.$$

However, $w_x, w_y \in W$ implies $w_x + w_y \in W$ hence by Proposition 12.1.4 we find $x_1 + y_1 + W = x_2 + y_2 + W$. I leave proof that $cx_1 + W = cx_2 + W$ as an exercise to the reader. $\square$

The preceding triple of propositions serves to show that the definitions given below are independent of the choice of coset representative. That is, while a particular coset represetative is used to make the definition, the choice is immaterial to the outcome.

Definition 12.1.7.

We define V/W to be the **quotient space of V by W**. In particular, we define:

$$V/W = \{x + W \mid x \in V\}$$

and for all $x + W, y + W \in V/W$ and $c \in \mathbb{F}$ we define:

$$(x + W) + (y + W) = x + y + W \qquad \& \qquad c(x + W) = cx + W.$$

Note, we have argued thus far that addition and scalar multiplication defined on V/W are well-defined functions. Let us complete the thought:

Theorem 12.1.8.

If $W \leq V$ a vector space over $\mathbb{F}$ then V/W is a vector space over $\mathbb{F}$.

Proof: if $x + W, y + W \in V/W$ note $(x + W) + (y + W)$ and $c(x + W)$ are single elements of V/W thus Axioms 9 and 10 of Definition 6.1.1 are true. Axiom 1: by commutativity of addition in V we obtain commutativity in V/W:

$$(x + W) + (y + W) = x + y + W = y + x + W = (y + W) + (x + W).$$

Axiom 2: associativity of addition follows from associativity of V,

$$\begin{aligned}
(x+W)+[(y+W)+(z+W)] &= x+W+[(y+z)+W] && \text{defn. of + in } V/W \\
&= x+(y+z)+W && \text{defn. of + in } V/W \\
&= (x+y)+z+W && \text{associativity of + in } V \\
&= [(x+y)+W]+(z+W) && \text{defn. of + in } V/W \\
&= [(x+W)+(y+W)]+(z+W) && \text{defn. of + in } V/W.
\end{aligned}$$

Axiom 3: note that $0+W=W$ and it follows that W serves as the additive identity in the quotient:

$$(x+W)+(0+W)=x+0+W=x+W.$$

Axiom 4: the additive inverse of $x+W$ is simply $-x+W$ as $(x+W)+(-x+W)=W$.
Axiom 5: observe that

$$1(x+W)=1\cdot x+W=x+W.$$

I leave verification of Axioms 6,7 and 8 for V/W to the reader. I hope you can see these will easily transfer of the Axioms 6,7 and 8 for V itself. $\square$

The notation $x+W$ is at times tiresome. An alternative notation is given below:

$$[x]=x+W$$

then the vector space operations on V/W are

$$[x]+[y]=[x+y] \qquad \& \qquad c[x]=[cx].$$

Naturally, the disadvantage of this notation is that it hides the particular subspace by which the quotient is formed. For a given vector space V many different subspaces are typically available and hence a wide variety of quotients may be constructed.

Example 12.1.9. *Suppose $V=\mathbb{R}^3$ and $W=span\{(0,0,1)\}$. Let $[(a,b,c)]\in V/W$ note*

$$[(a,b,c)]=\{(a,b,z)\ |\ z\in\mathbb{R}\}$$

thus a point in V/W is actually a line in V. The parameters a,b fix the choice of line so we expect V/W is a two dimensional vector space with basis $\{[(1,0,0)],[(0,1,0)]\}$.

Example 12.1.10. *Suppose $V=\mathbb{R}^3$ and $W=span\{(1,0,0),(0,1,0)\}$. Let $[(a,b,c)]\in V/W$ note*

$$[(a,b,c)]=\{(x,y,c)\ |\ x,y\in\mathbb{R}\}$$

thus a point in V/W is actually a plane in V. In this case, each plane is labeled by a single parameter c so we expect V/W is a one-dimensional vector space with basis $\{[(0,0,1)]\}$.

Example 12.1.11. *Let $V=\mathbb{R}[x]$ and let $W=\mathbb{R}$ the set of constant polynomials.*

$$[a_0+a_1x+\cdots+a_nx^n]=\{c+a_1x+\cdots+a_nx^n\ |\ c\in\mathbb{R}\}$$

Perhaps, more to the point,

$$[a_0+a_1x+\cdots+a_nx^n]=[a_1x+\cdots+a_nx^n]$$

In this quotient space, we identify polynomials which differ by a constant.

We could also form quotients of $\mathcal{F}(\mathbb{R})$ or P_n or $C^\infty(\mathbb{R})$ by $\mathbb{R}$ and it would have the same meaning; if we quotient by constant functions then $[f] = [f + c]$.

The quotient space construction allows us to modify a given transformation such that its reformulation is injective. For example, consider the problem of inverting the derivative operator $D = d/dx$.

$$D(f) = f' \qquad \& \qquad D(f + c) = f'$$

thus D is not injective. However, if we instead look at the derivative operator on[5] a quotient space of differentiable functions of a connected domain where $[f] = [f + c]$ then defining $D([f]) = f'$ proves to be injective. Suppose $D([f]) = D([g])$ hence $f' = g'$ so $f - g = c$ and $[f] = [g]$. We generalize this example in the next subsection.

12.1.1 the first isomorphism theorem

I'll begin by defining an standard linear map attached the quotient construction:

Definition 12.1.12. *Let V be a vector space with $W \leq V$.*

> The **quotient map** $\pi : V \to V/W$ is defined by $\pi(x) = x + W$ for each $x \in V$.

We observe π is a linear transformation.

Proposition 12.1.13.

> The quotient map $\pi : V \to V/W$ is a linear transformation.

Proof: suppose $x, y \in V$ and $c \in \mathbb{F}$. Consider

$$\pi(cx + y) = (cx + y) + W = (cx + W) + (y + W) = c(x + W) + (y + W) = c\pi(x) + \pi(y). \quad \square$$

When W is formed as the kernel of a linear transformation the mapping π takes on a special significance. The π map allows us to create isomorphisms as described in the theorem below:

Theorem 12.1.14. *First Isomorphism Theorem of Linear Algebra*

> If $T : V \to U$ is a linear transformation and $W = \text{Ker}(T)$ has quotient map π then the mapping $\overline{T} : V/W \to U$ defined implicity by $\overline{T} \circ \pi = T$ is an injection. In particular, $\overline{T}(x + \ker(T)) = T(x)$ for each $x + \ker(T) \in V/W$. Moreover, $V/\ker(T) \approx T(V)$.

Proof: to begin we show $\overline{T}$ is well-defined. Of course, $T(x) \in U$ for each $x \in V$ hence $\overline{T}$ is **into** U. Is $\overline{T}$ single-valued? Suppose $x + \text{Ker}(T) = y + \text{Ker}(T)$ then $y - x \in \text{Ker}(T)$ hence $T(y - x) = 0$ which gives $T(x) = T(y)$. Thus, $\overline{T}(x + \text{Ker}(T)) = T(x) = T(y) = \overline{T}(y + \text{Ker}(T))$. Therefore, $\overline{T}$ is single-valued.

Next, we show $\overline{T}$ is a linear transformation. Let $x, y \in V$ and $c \in \mathbb{F}$. Consider,

$$\overline{T}(c(x + W) + (y + W)) = \overline{T}(cx + y + W) = T(cx + y) = cT(x) + T(y) = c\overline{T}(x + W) + \overline{T}(y + W).$$

[5] to be careful, I only modify the domain of the derivative operator here, note the output of D is not an equivalence class. Furthermore, perhaps a different symbol like $\overline{D}$ should be used to write $\overline{D}([f]) = f'$ as $D \neq \overline{D}$

We find linearity of $\overline{T}$ follows naturally from the definition of V/W as a vector space and the linearity of T.

We now turn to the question of injectivity of $\overline{T}$. Let $x+W, y+W \in V/W$ where $W = \ker(T)$ and suppose $\overline{T}(x+W) = \overline{T}(y+W)$. It follows that $T(x) = T(y)$ thus $T(x-y) = 0$ and we find $x - y \in W = \ker(T)$ which proves $x + \ker(T) = y + \ker(T)$. We have shown $\overline{T}$ is injective.

The isomorphism of $V/\ker(T)$ and $T(V)$ is given by $T' : V/\ker(T) \to T(V)$ where $T'(x+\ker(T)) = \overline{T}(x+\ker(T)) = T(x)$. If $y = T(x) \in T(V)$ then clearly $T'(x+\ker(T)) = y$ hence T' is a surjection and hence an isomorphism as we have injectivity from our work on $\overline{T}$. $\square$

The last paragraph simply says that the injective map $\overline{T}$ can be made into a surjection by reducing its codomain to its range. This is not surprising. What may be surprising is how this theorem can be used to see isomorphisms in a terribly efficient manner:

Example 12.1.15. *Consider $V \times W/(\{0\} \times W)$ and V. To show these are isomorphic we consider $T(v,w) = v$. It is simple to verify that $T : V \times W \to V$ is a linear surjection. Moreover, Ker$(T) = \{(0,w) \mid w \in W\} = \{0\} \times W$. The first isomorphism theorem reveals $V \times W/(\{0\} \times W) \approx V$.*

Example 12.1.16. *Consider $S : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $S(A) = A + A^T$. Notice that the range of $S(A)$ is simply symmetric matrices as $(S(A))^T = (A + A^T)^T = A^T + (A^T)^T = A + A^T = S(A)$. Moreover, if $A^T = A$ the clearly $S(A/2) = A$ hence S is onto the symmetric matrices. What is the kernel of S? Suppose $S(A) = 0$ and note:*

$$A + A^T = 0 \qquad \Rightarrow \qquad A^T = -A.$$

Thus Ker(S) is the set of antisymmetric matrices. Therefore,

$$S'([A]) = A + A^T$$

is an isomorphism from $\mathbb{R}^{n\times n}/$Ker(S) to the set of symmetric $n \times n$ matrices.

Example 12.1.17. *Consider $D : P \to P$ defined by $D(f(x)) = df/dx$. Here I denote $P = \mathbb{R}[x]$,the set of all polynomials with real coefficients. Notice*

$$Ker(D) = \{f(x) \in P \mid df/dx = 0\} = \{f(x) \in P \mid f(x) = c\}.$$

In this case D is already a surjection since we work with all polynomials hence:

$$\overline{D}([f(x)]) = f'(x)$$

is an isomorphism. Just to reiterate in this case:

$$\overline{D}([f(x)]) = \overline{D}([g(x)]) \quad \Rightarrow \quad f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + c \quad \Rightarrow \quad [f(x)] = [g(x)].$$

Essentially, $\overline{D}$ is just d/dx on equivalence classes of polynomials. Notice that $\overline{D}^{-1} : P \to P/Ker(D)$ is a mapping you have already studied for several months! In particular,

$$\overline{D}^{-1}(f(x)) = \{F(x) \mid dF/dx = f(x)\}$$

Just to be safe, let's check that my formula for the inverse is correct:

$$\overline{D}^{-1}(\overline{D}([f(x)]) = \overline{D}^{-1}(df/dx) = \{F(x) \mid dF/dx = df/dx\} = \{f(x) + c \mid c \in \mathbb{R}\} = [f(x)].$$

Conversely, for $f(x) \in P$,

$$\overline{D}(\overline{D}^{-1}(f(x)) = \overline{D}(\{F(x) \mid dF/dx = f(x)\}) = f(x).$$

Perhaps if I use a different notation to discuss the preceding example then you will see what is happening: we usually call $\overline{D}^{-1}(f(x)) = \int f(x)dx$ and $\overline{D} = d/dx$ then

$$\frac{d}{dx}\int f\,dx = f \qquad \& \qquad \int \frac{d}{dx}(f + c_1)\,dx = f + c_2$$

In fact, if your calculus instructor was careful, then he should have told you that when we calculate the indefinite integral of a function the answer is not a function. Rather, $\int f(x)\,dx = \{g(x) \mid g'(x) = f(x)\}$. However, nobody wants to write a set of functions every time they integrate so we instead make the custom to write $g(x)+c$ to indicate the non-uniqueness of the answer. Antidifferentiation of f is finding a specific function F for which $F'(x) = f(x)$. Indefinite integration of f is the process of finding the set of all functions $\int f dx$ for which $\frac{d}{dx}\int f dx = f$. In any event, I hope you see that we can claim that differentiation and integration are inverse operations, however, this is in the understanding that we work on a quotient space of functions where two functions which differ by a constant are considered the same function. In that context, $f + c_1 = f + c_2$.

Example 12.1.18. *Consider $D : P_2 \to P_2$ defined by*

$$D(ax^2 + bx + c) = 2ax + b$$

Observe $\overline{D}([ax^2+bx+c]) = 2ax+b$ defines a natural isomorphism from $P_2/\mathbb{R}$ to P_1 where I denote Ker$(D) = \mathbb{R}$. In other words, when I write the quotient by $\mathbb{R}$ I am identifying the set of constant polynomials with the set of real numbers.

Example 12.1.19. *Consider $\mathcal{F}(\mathbb{R})$ the set of all functions on $\mathbb{R}$. Observe, any function can be written as a sum of an even and odd function:*

$$f(x) = \frac{1}{2}\Big(f(x) + f(-x)\Big) + \frac{1}{2}\Big(f(x) - f(-x)\Big)$$

Furthermore, if we denote the subspaces of even and odd functions as $\mathcal{F}_{even} \leq \mathcal{F}(\mathbb{R})$ and $\mathcal{F}_{odd} \leq \mathcal{F}(\mathbb{R})$ and note $\mathcal{F}_{even} \cap \mathcal{F}_{odd} = \{0\}$ hence $\mathcal{F}(\mathbb{R}) = \mathcal{F}_{even} \oplus \mathcal{F}_{odd}$. Consider the projection $T : \mathcal{F}(\mathbb{R}) \to \mathcal{F}_{even}$ clearly Null$(T) = \mathcal{F}_{odd}$ hence by the first isomorphism theorem, $\mathcal{F}(\mathbb{R})/\mathcal{F}_{odd} \approx \mathcal{F}_{even}$.

Example 12.1.20. *This example will be most meaningful for students of differential equations, however, there is something here for everyone to learn. An n-th order linear differential equation can be written as $L[y] = g$. Here y and g are functions on a connected interval $I \subseteq \mathbb{R}$. There is an existence theorem for such problems which says that* **any** *solution can be written as*

$$y = y_h + y_p$$

where $L[y_h] = 0$ and $L[y_p] = g$. The so-called **homogeneous solution** *y_h is generally formed from a linear combination of n-LI fundamental solutions $y_1, y_2, \dots, y_n$ as*

$$y_h = c_1y_1 + c_2y_2 + \cdots + c_ny_n.$$

Here $L[y_i] = 0$ for $i = 1, 2, \dots, n$. It follows that Null(L) is n-dimensional and the fundamental solution set forms a basis for this null-space. On the other hand the particular solution y_p can be formed through a technique known as **variation of parameters**. *Without getting into the technical details, the point is there is an explicit, although tedious, method to calculate y_p once we know the fundamental solution set and g. Techniques for finding the fundamental solution set vary from problem to problem. For the constant coefficient case or Cauchy Euler problems it is as simple as* *factoring the characteristic polynomial and writing down the homogeneous solutions. Enough about that, let's think about this problem in view of quotient spaces.*

The differential equation $L[y] = g$ can be instead thought of as a function which takes g as an input and produces y as an output. Of course, given the infinity of possible homogeneous solutions this would not really be a function, it's not single-valued. If we instead associate with the differential equation a function $H : V \to V/Null(L)$ where $H(g) = y + Null(L)$ then the formula can be compactly written as $H(g) = [y_p]$. For convenience, suppose $V = C^0(\mathbb{R})$ then $dom(H) = V$ as variation of parameters only requires integration of the forcing function g. Thus $H^{-1} : V/Null(L) \to V$ is an isomorphism. In short, the mathematics I outline here shows us there is a one-one correspondance between forcing functions and solutions modulo homogeneous terms. Linear differential equations have this beautiful feature; the net-response of a system L to inputs $g_1, \dots, g_k$ is nothing more than the sum of the responses to each forcing term. This is the principal of superposition which makes linear differential equations comparitively easy to understand.

Proposition 12.1.21.

If $V = A \oplus B$ then $V/A \approx B$.

Proof: Since $V \approx A \times B$ under $\eta : A \times B \to V$ with $\eta(a, b) = a + b$ it follows for each $v \in V$ there exists a unique pair (a, b) such that $v = a + b$. Given this decomposition of each vector in V we can define a projection onto B as follows: define $\pi_B : V \to B$ by $\pi_B(a + b) = b$. It is clear π_B is linear and $\mathrm{Ker}(\pi_B) = A$ thus the first isomorphism theorem gives $V/A \approx B$. $\square$

Proposition 12.1.22.

If $W \leq V$ and V is finite-dimensional then $\dim(V/W) = \dim(V) - \dim(W)$.

Proof: Suppose $\dim(V) = n$ and $\dim(W) = k$. Let $\beta = \{w_1, \dots, w_k\}$ be a basis for W. Extend β to $\gamma = \{w_1, \dots, w_k, v_1, \dots, v_{n-k}\}$ a basis for V. Observe that $w_j + W = W$ for $j = 1, \dots, k$ as $w_j \in W$ for each j. Since $O_{V/W} = W$ we certainly cannot form the basis for V/W with β. However, we can show $\{v_i + W\}_{i=1}^{n-k}$ serves as a basis for V/W. Suppose

$$c_1(v_1 + W) + c_2(v_2 + W) + \cdots + c_{n-k}(v_{n-k} + W) = O_{V/W} = 0 + W$$

thus, by the definitions of coset addition and scalar multiplication,

$$(c_1 v_1 + c_2 v_2 + \cdots + c_{n-k} v_{n-k}) + W = W$$

it follows $c_1 v_1 + c_2 v_2 + \cdots + c_{n-k} v_{n-k} \in W$. But, this must be the zero vector since by construction the vectors $v_1, \dots, v_{n-k}$ are outside $\mathrm{span}(\beta)$. Thus $c_1 v_1 + c_2 v_2 + \cdots + c_{n-k} v_{n-k} = 0$ and hence by linear independence of γ we find $c_1 = c_2 = \cdots = c_{n-k} = 0$. Suppose $x + W \in V/W$ then there exist $x_j, y_i \in \mathbb{F}$ for which $x = \sum_{j=1}^{k} x_j w_j + \sum_{i=1}^{n-k} y_i v_i$ thus

$$x + W = \sum_{j=1}^{k} x_j w_j + \sum_{i=1}^{n-k} y_i v_i + W = \sum_{i=1}^{n-k} y_i v_i + W = \sum_{i=1}^{n-k} y_i (v_i + W).$$

Thus, $\mathrm{span}\{v_1 + W, \dots, v_{n-k} + W\} = V/W$. It follows $\{v_1 + W, \dots, v_{n-k} + W\}$ is a basis for V/W and we count $\dim(V/W) = n - k = \dim(V) - \dim(W)$. $\square$

It is interesting to study how the matrix of T and the matrix of $\overline{T}$ are related. This is part of a larger story which I tell now[6].

Recall[7] , if V permits a direct sum decomposition in terms of invariant subspaces $W_1, \dots, W_k$ then there exists a basis β for V formed by concatenating the bases $\beta_1, \dots, \beta_k$ for $W_1, \dots, W_k$ respective. Moreover $[T]_{\beta,\beta}$ is in block-diagonal form where each block is simply the matrix of the restriction of $T|_{W_i} : W_i \to W_i$ with respect to β_i. What follows below is a bit different since we only assume that U is a T-invariant subspace.

Proposition 12.1.23.

Let V be finite a finite dimensional vector space over $\mathbb{F}$. If $T : V \to V$ is a linear transformation and $U \leq V$ for which $T(U) \leq U$ and if β_U is a basis of U and $\beta_U \cup \beta_2$ is a basis for V then

$$[T]_{\beta,\beta} = \left[\begin{array}{cc} A & B \\ 0 & C \end{array} \right]$$

where $A = [T|_U]_{\beta_U,\beta_U}$ and $C = [T_{V/U}]_{\beta_{V/U},\beta_{V/U}}$ where $\beta_{V/U} = \{v_l + U \mid v_l \in \beta_2\}$.

Proof: let T and U be as in the statement of the proposition. The fact that $A = [T|_U]_{\beta_U,\beta_U}$ follows from $T(U) \leq U$. Denote $\beta_U = \{u_1, \dots, u_k\}$ and $\beta_2 = \{v_1, \dots, v_{n-k}\}$. Notice, $T(v_j) = \sum_{i=1}^{k} B_{ij} u_i + \sum_{l=1}^{n-k} C_{lj} v_l$. We define $T_{V/U} : V/U \to V/U$ by $T_{V/U}(x + U) = T(x) + U$ (this is well-defined since we assumed $T(U) \leq U$). Notice,

$$\overline{T}(v_j + U) = T(v_j) = \sum_{i=1}^{k} B_{ij} u_i + \sum_{l=1}^{n-k} C_{lj} v_l + U = \sum_{l=1}^{n-k} C_{lj} v_l + U. \qquad \square$$

It is interesting to use the result above paired with Proposition 12.1.21. If $V = V_1 \oplus V_2$ and $T : V \to V$ is a linear transformation for which $T(V_1) \leq V_1$ and $T(V_2) \leq V_2$. We know from Theorem 7.7.14 that for the V_1, V_2 concatenated basis $\beta = \beta_1 \cup \beta_2$ we have $[T]_{\beta,\beta} = \left[\begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right]$. It follows, omitting explicit basis dependence, from Proposition 12.1.23 that

$$A_1 = [T_{V/V_2}] = [T|_{V_1}] \qquad \& \qquad A_2 = [T_{V/V_1}] = [T|_{V_2}].$$

In other words, given a block decomposition we can either view the blocks being attached to the restriction of the map to particular subspaces, or, we can see the blocks in terms of induced maps on quotients. Similar comments can be made for direct sums of more than two subspaces.

12.2 dual space

Definition 12.2.1.

Let V be a vector space over a field $\mathbb{F}$ then the **dual space** $V^* = L(V, \mathbb{F})$.

[6] here I follow pages 231-233 of Charles Curtis' text
[7] see Definition 7.7.13 and Theorem .

In the case that $\dim(V) = \infty$ this **algebraic dual space** is quite large and it is common to replace it with the set of bounded linear functionals. That said, our focus will be on the case $\dim(V) < \infty$. Roman's *Advanced Linear Algebra* is a good place to read more about the infiinte dimensional case, or, most functional analysis texts. Let it be understood that $\dim(V) = n$ in the remainder of this section.

We should recall $L(V, \mathbb{F})$ is a vector space over $\mathbb{F}$ hence V^* is also a vector space over $\mathbb{F}$. The definition which follows s a natural next step:

Definition 12.2.2.

Let V be a vector space over a field $\mathbb{F}$ then the **double dual space** $V^{**} = L(V^*, \mathbb{F})$.

We can exchange V, V^* and V^{**} in a given application of linear algebra. In the finite dimensional case these are all isomorphic it is often possible to exchange one of these for the other.

Theorem 12.2.3.

Let V be a finite dimensional vector space over a field $\mathbb{F}$ then $V \approx V^*$ and $V \approx V^{**}$

Proof: observe $\dim(L(V, \mathbb{F}) = \dim(\mathbb{F}^{1\times n}) = n = \dim(V)$ thus $V^* \approx V$. Next, since $V^{**} = (V^*)^*$ by construction we have $V^* \approx V^{**}$. Transitivity of isomorphism yields $V \approx V^{**}$. $\square$

The proof above is somewhat useless. I did not give it in lecture of 2016 because I wanted to illustrate the actual maps which make manifest the isomorphisms. To begin, the evaluation map is the key to connect the double dual and V.

Definition 12.2.4. *an* **evaluation** *map:*

For $x \in V$, let $\text{eval}_x : V^* \to \mathbb{F}$ be defined by $\text{eval}_x(\alpha) = \alpha(x)$ for each $\alpha \in V^*$.

I invite the reader to confirm that $\text{eval}_x \in L(V^*, \mathbb{F})$ hence $\text{eval}_x \in V^{**}$. Furthermore, the assignment $x \mapsto \text{eval}_x$ defines an explicit isomorphism of V and V^{**}. In other words, $\Psi : V \to V^{**}$ defined by

$$(\Psi(x))(\alpha) = \text{eval}_x(\alpha) = \alpha(x)$$

gives a bijective linear transformation from V to V^{**}. In appropriate contexts, we sometimes just write $x = \text{eval}_x$. If I make this abuse, I'll warn you[8].

The isomorphism $x \mapsto \text{eval}_x$ is *natural* in the sense that we could describe it without reference to a choice of basis. This is also possible for V^* if a metric[9] is given with V. Naturality aside, we can find an explicit isomorphism via the use a particular basis for V. In fact, the construction below is central to much of what follows in this chapter!

Definition 12.2.5.

Let $\beta = \{v_1, v_2, \dots, v_n\}$ form a basis for V over $\mathbb{F}$. For each $i = 1, 2, \dots, n$, define $v^i : V \to \mathbb{F}$ by linearly extending the formula $v^i(v_j) = \delta_{ij}$ for all $j = 1, 2, \dots, n$. We say $\beta^* = \{v^1, v^2, \dots, v^n\}$ is the **dual basis** to β.

[8] in these notes that is...

[9] slight generalization of our inner product concept, has to wait for next section's technology

The position of the indices up[10] or down[11] indicates how the given quantity transforms when we change coordinates. This notation is fairly popular in certain sectors of abstract math. We write for $x \in V$ that the coordinates with respect to $\beta = \{v_1, \dots, v_n\}$ are $x^1, \dots, x^n$ in the sense that:

$$x = \sum_{i=1}^{n} x^i v_i = x^1 v_1 + \cdots + x^n v_n.$$

I know this notation is a bit weird the first time you see it. Just keep in mind the upper-indices are not powers. We ought to confirm that the dual basis is not wrongly labeled. You know, just because I call something a basis it doesn't make it so.

Proposition 12.2.6.

If $\beta^* = \{v^1, \dots, v^n\}$ is dual to basis $\beta = \{v_1, \dots, v_n\}$ for V then:

(i.) if $x = \sum_{i=1}^{n} x^i v_i$ then $e^i(x) = x^i$.

(ii.) β^* is a linearly independent subset of V^*

(iii.) $\text{span}(\beta^*) = V^*$ and if $\alpha = \sum_{i=1}^{n} \alpha_i v^i$ then $\alpha(v_i) = \alpha_i$ for each $\alpha \in V^*$

Proof: most of these claims follow from the defining formula $v^i(v_j) = \delta_{ij}$. Suppose $x = \sum_{j=1}^{n} x^j v_j$ and calculate: by linearity of $v^i : V \to \mathbb{F}$ we have:

$$v^i(x) = v^i\left(\sum_{j=1}^{n} x^j v_j\right) = \sum_{j=1}^{n} x^j v^i(v_j) = \sum_{j=1}^{n} x^j \delta_{ij} = x^i.$$

To prove (ii.) suppose $\sum_{i=1}^{n} c_i v^i = 0$. Evaluate on v_j,

$$\left(\sum_{i=1}^{n} c_i v^i\right)(v_j) = 0(v_j) \quad \Rightarrow \quad \sum_{i=1}^{n} c_i v^i(v_j) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} c_i \delta_{ij} = 0 \quad \Rightarrow \quad c_j = 0.$$

Hence β^* is a LI subset of V^*. By construction, it is clear that $\beta^* \subseteq V^*$ hence $\text{span}(\beta^*) \subseteq V^*$. Conversely, suppose $\alpha \in V^*$. Calculate: for $x = \sum_{j=1}^{n} x^j v_j$, by linearity of α and (i.)

$$\alpha(x) = \alpha\left(\sum_{j=1}^{n} x^j v_j\right) = \sum_{j=1}^{n} x^j \alpha(v_j) = \sum_{j=1}^{n} \alpha(v_j) v^j(x) \quad \Rightarrow \quad \alpha = \sum_{j=1}^{n} \alpha(v_j) v^j$$

notice we have the desired formula as claimed in (iii.). We've shown $V^* \subseteq \text{span}(\beta^*)$ thus $V^* = \text{span}(\beta^*)$. $\square$

With the Proposition above in hand it is easy to provide an isomorphism of V and V^*. Simply define $\Psi : V \to V^*$ by linearly extending the formula $\Psi(v_i) = v^i$ for $i = 1, \dots, n$. This makes Ψ an isomorphism. Let me pause to include a few simple examples:

[10] up indices are called contravariant indices in physics

[11] down indices are called covariant indices in physics

Example 12.2.7. *Let $\alpha : \mathbb{F}^{n\times n} \to \mathbb{F}$ be defined by $\alpha(A) = trace(A)$. Since the trace is a linear map we have $\alpha \in (\mathbb{F}^{n\times n})^*$.*

Example 12.2.8. *Let $V = \mathcal{F}(\mathbb{R})$ and $x_o \in \mathbb{R}$. Define $\alpha(f) = f(x_o)$. Notice, for $f, g \in V$ and $c \in \mathbb{R}$,*

$$\alpha(cf + g) = (cf + g)(x_o) = cf(x_o) + g(x_o) = c\alpha(f) + \alpha(g)$$

thus $\alpha \in V^*$.

Example 12.2.9. *Let $v \in \mathbb{R}^n$ and define $\alpha(x) = x \bullet v$. It is easy to see α is linear hence $\alpha \in (\mathbb{R}^n)^*$.*

Example 12.2.10. *If $V = C^0[0,1]$ then define $\alpha(f) = \int_0^1 xf(x)\ dx$. Observe $\alpha(cf + g) = \int_0^1 x(cf(x) + g(x))\ dx = c\int_0^1 xf(x)\ dx + \int_0^1 xg(x))\ dx = c\alpha(f) + \alpha(g)$. Thus $\alpha \in V^*$.*

Definition 12.2.11.

If $W \leq V$ then define the **annihilator** of W in V^* by $\text{ann}(W) = \{\alpha \in V^* \mid \alpha(W) = 0\}$.

The condition $\alpha(W) = 0$ means $\alpha(w) = 0$ for each $w \in W$. Some texts denote $\text{ann}(W) = W^0$. There are many interesting theorems we can state for annihilators. We explore some such theorems in the homework.

Example 12.2.12. *Consider $V = \mathbb{R}^4$. Let $W = span\{(1,1,1,1),(1,0,0,0)\}$ then $ann(W) = \{\alpha \in (\mathbb{R}^4)^* \mid \alpha(1,1,1,1) = 0\ \&\ \alpha(1,0,0,0) = 0\}$. Thus, $\alpha \in ann(W)$ has*

$$\alpha_1 = 0 \quad \& \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \quad \Rightarrow \quad \alpha_2 = -\alpha_3 - \alpha_4$$

we deduce $ann(W) = span\{e^3 - e^2, e^4 - e^2\}$

Example 12.2.13. *Consider $V = \mathbb{R}^{2\times 2}$ and let W be the symmetric 2×2 matrices. If $\alpha \in ann(W)$ then for $A = A^T$ we have $\alpha(A) = 0$. In fact, the basis for W is given by $\{E_{11}, E_{12} + E_{21}, E_{22}\}$. We can extend this to a basis for V by adjoining $E_{12} - E_{21}$. The dual basis to the standard matrix basis is given by $E^{ij}(E_{kl}) = \delta_{ik}\delta_{jl}$. Notice, $\alpha = E^{12} - E^{21}$ is in the annihilator since:*

$$\alpha(A) = (E^{12} - E^{21})(A) = (E^{12} - E^{21})(A_{11}E_{11} + A_{12}E_{12} + A_{21}E_{21} + A_{22}E_{22}) = A_{12} - A_{21} = 0.$$

In fact, $span\{E^{12} - E^{21}\} = ann(W)$.

One of the homework problems ask us to show $\dim(W) + \dim(\text{ann}(W)) = \dim(V)$. To prove this, we take a basis for W and extend it to a basis for V. When we study the dual basis to the extended basis then the dual vectors which correspond to the vectors outside of W provide the basis for the annihilator of W. You can see an example of that story in the example above.

This concludes our brief look at dual spaces. There is more to learn in the homework. This topic is an excellent laboratory to hone our proof skill in abstract linear algebra.

12.3 bilinear forms

A bilinear form is simply a mapping which is linear in two arguments. For example, an inner product is a bilinear form. In this section we detail some of the basic structural results for bilinear forms then we study the generalization of inner products to what we[12] shall call a *metric*. All vector spaces in this section are assumed finite dimensional.

[12]to be precise, the inner product requires the positive definite condition whereas a metric merely is nondegenerate. Other authors need not agree!

Definition 12.3.1.

Let U, V, W be vector spaces over $\mathbb{F}$. If $b : V \times W \to U$ is a function for which

$$b(cv_1 + v_2, w) = cb(v_1, w) + b(v_2, w) \ \ \& \ \ b(v, cw_1 + w_2) = cb(v, w_1) + b(v, w_2)$$

for all $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$ and $c \in \mathbb{F}$ then we say b is a **bilinear map** from $V \times W$ to U and we write $b \in \mathcal{B}(V \times W, U)$.

The examples below show how we can use matrix multiplication to construct bilinear maps.

Example 12.3.2. *Let $b : \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}$ be defined by $b(x, y) = x^T M y$ for $M \in \mathbb{F}^{m\times n}$. Observe,*

$$b(cx + y, z) = [cx + y]^T Mz = cx^T Mz + y^T Mz = cb(x, z) + b(y, z).$$

for $x, y \in \mathbb{F}^m$ and $z \in \mathbb{F}^n$ and $c \in \mathbb{F}$. Likewise, for $c \in \mathbb{F}$ and $x \in \mathbb{F}^m$ and $y, z \in \mathbb{F}^n$

$$b(x, cy + z) = x^T M(cy + z) = cx^T My + x^T Mz = cb(x, y) + b(x, z).$$

Example 12.3.3. *Let $b : \mathbb{R}^{\,n\times n} \times \mathbb{R}^{\,n\times n} \to \mathbb{R}^{\,n\times n}$ be defined by $b(X, Y) = XMY + YX$ then*

$$\begin{aligned} b(cX_1 + X_2, Y) &= (cX_1 + X_2)MY + Y(cX_1 + X_2) \\ &= c(X_1MY + YX_1) + X_2MY + YX_2 \\ &= cb(X_1, Y) + b(X_2, Y) \end{aligned}$$

and

$$\begin{aligned} b(X, cY_1 + Y_2) &= XM(cY_1 + Y_2) + (cY_1 + Y_2)X \\ &= c(XMY_1 + Y_1X) + XMY_2 + Y_2X \\ &= cb(X, Y_1) + b(X, Y_2). \end{aligned}$$

Notice $b = b_M + b_2$ where $b_M(X, Y) = XMY$ and $b_2(X, Y) = YX$. We can see b_M and b_2 are bilinear from the calculation above.

I'll skip the proof of the proposition below, but, I think it's not difficult:

Proposition 12.3.4.

If $\mathcal{B}(V \times W, U)$ denotes the set of bilinear mappings on vector spaces V, W, U over $\mathbb{F}$ then $\mathcal{B}(V \times W, U)$ forms a vector space over $\mathbb{F}$ with respect to addition and scalar multiplication of functions. In particular, if $b_1, b_2 \in \mathcal{B}(V \times W, U)$ and $c \in \mathbb{F}$ then $cb_1 + b_2 \in \mathcal{B}(V \times W, U)$.

Given a choice of bases for V and W we can relate each bilinear map to a matrix (or matrices in the case that $U \neq \mathbb{F}$).

Proposition 12.3.5.

Let U, V, W be vector spaces over $\mathbb{F}$ with bases $\Upsilon = \{u_1, \dots, u_p\}, \beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ respective. Let $b : V \times W \to U$ be a blinear transformation then if $x = \sum_{i=1}^n x^i v_i$ and $y = \sum_{j=1}^m y^j w_j$ then there exist constants ${b_{ij}}^k \in \mathbb{F}$ for which

$$b(x, y) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^p x^i y^j {b_{ij}}^k u_k$$

Proof: with the notation as in the proposition consider, by bilinearity we obtain:

$$b(x,y) = b\left(\sum_{i=1}^{n} x^i v_i, \sum_{j=1}^{m} y^j w_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} x^i y^j b(v_i, w_j)$$

Recall, the basis Υ and dual basis Υ^* allow us to write $z \in U$ as $z = \sum_{k=1}^{p} z^k u_k = \sum_{k=1}^{p} u^k(z) u_k$. Applying this observation to $b(v_i, w_j) \in U$ yields:

$$b(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} x^i y^j u^k\left(b(v_i, w_j)\right) u_k$$

hence define ${b_{ij}}^k = u^k\left(b(v_i, w_j)\right)$ and the proposition follows. $\square$

Observe, any bilinear form $b : V \times W \to U$ can be written as $b = \sum_{k=1}^{p} b^k u_k$ where $b^k : V \times W \to \mathbb{F}$ is bilinear. Thus, the theory of $\mathbb{F}$-valued linear forms is still informs the theory of U-valued bilinear forms. An $\mathbb{F}$-valued bilinear form has no need of the k-index in the proposition above. In fact, we can simply write the formula for the bilinear map in terms of a matrix multiplication[13]. For the remainder of this section we only study this less cluttered case.

Proposition 12.3.6.

Let V, W be vector spaces over $\mathbb{F}$ with bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ respective. Let $b : V \times W \to \mathbb{F}$ be a bilinear transformation then if $x = \sum_{i=1}^{n} x^i v_i$ and $y = \sum_{j=1}^{m} y^j w_j$ then there exist constants $b_{ij} \in \mathbb{F}$ for which

$$b(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} x^i y^j b_{ij}$$

moreover, $b(x,y) = [x]_\beta^T M [y]_\gamma$ where $M_{ij} = b(v_i, w_j)$ is the i,j-th component of $M \in \mathbb{F}^{m\times n}$.

Proof: with the notation as in the proposition consider, by bilinearity we obtain:

$$b(x,y) = b\left(\sum_{i=1}^{n} x^i v_i, \sum_{j=1}^{m} y^j w_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} x^i y^j b(v_i, w_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} [x]^i M_{ij} [y]^j = [x]^T M [y].$$

where I used $[x] = [x]_\beta$ and $[y] = [y]_\gamma$ for brevity in the formula above. $\square$

The converse of the above proposition is also true, if there exists a matrix M for which $b(x,y) = [x]^T M [y]$ then b is bilinear. In short, there is a one-one correspondence between $\mathbb{F}$-valued bilinear forms on $V \times W$ and $m \times n$ matrices over $\mathbb{F}$. This assignment is different than that of matrices for $L(V, W)$. We can see this most clearly as we examine how the matrix of a bilinear form changes when we change the basis on V and W respective. Consider, $\beta, \bar{\beta}$ bases for V and basis $\gamma, \bar{\gamma}$ for W. Let us use the notations

$$\beta = \{v_1, \dots, v_n\} \quad \& \quad \bar{\beta} = \{\bar{v}_1, \dots, \bar{v}_n\}, \quad \& \quad \gamma = \{w_1, \dots, w_m\} \quad \& \quad \bar{\gamma} = \{\bar{w}_1, \dots, \bar{w}_m\}$$

[13] we could think of ${b_{ij}}^k$ as the components of a tensor which is a multilinear map from $V \times W \times U^*$ to $\mathbb{F}$. The tensor concept allows for more indices than just two. A rough intuition, two indices implies we can write the formula in terms of some matrix product, more than two implies a tensor product is required. That said, you can turn anything multilinear into a matrix product with suitable reformulation so this distinction is less clear than you might hope. Sorry about this comment, it only makes sense in view of the definition in the next section.

Since $\Phi_\beta(v_i) = e_i = \Phi_{\bar\beta}(\bar v_i)$ we have $\left(\Phi_\beta^{-1} \circ \Phi_{\bar\beta}\right)(\bar v_i) = \Phi_\beta^{-1}(e_i) = v_i$. Consider, for $b : V \times W \to \mathbb{F}$ bilinear, we may induce a bilinear form on $\mathbb{F}^n \times \mathbb{F}^m$ by the coordinate maps:

$$\tilde b = b \circ (\Phi_\beta^{-1}, \Phi_\gamma^{-1}) : \mathbb{F}^n \times \mathbb{F}^m \to \mathbb{F} \qquad \text{where} \qquad \tilde b([x]_\beta, [y]_\gamma) = [x]_\beta^T M [y]_\gamma = b(x, y). \ \star$$

likewise, in the other coordinates, we induce

$$\bar b = b \circ (\Phi_{\bar\beta}^{-1}, \Phi_{\bar\gamma}^{-1}) : \mathbb{F}^n \times \mathbb{F}^m \to \mathbb{F} \qquad \text{where} \qquad \bar b([x]_{\bar\beta}, [y]_{\bar\gamma}) = [x]_{\bar\beta}^T \overline{M} [y]_{\bar\gamma} = b(x, y). \ \star\star$$

Our goal is to relate M and $\bar M$. Solve $\tilde b$ and $\bar b$ for b to obtain:

$$\bar b \circ (\Phi_{\bar\beta}, \Phi_{\bar\gamma}) = \tilde b \circ (\Phi_\beta, \Phi_\gamma) \quad \star\star\star$$

Evaluate $\star\star\star$ evaluate at $(x, y) \in V \times W$ and

$$\tilde b([x]_\beta, [y]_\gamma) = \bar b([x]_{\bar\beta}, [y]_{\bar\gamma}),$$

now use $\star$ and $\star\star$

$$[x]_{\bar\beta}^T \overline{M} [y]_{\bar\gamma} = [x]_\beta^T M [y]_\gamma.$$

If $P[x]_\beta = [x]_{\bar\beta}$ and $Q[y]_\gamma = [y]_{\bar\gamma}$ describe the change of coordinates for V and W respective then we can use P and Q to relate M and $\overline{M}$;

$$(P[x]_\beta)^T \overline{M} Q[y]_\gamma = [x]_\beta^T M [y]_\gamma \quad \Rightarrow \quad [x]_\beta)^T P^T \bar M Q[y]_\gamma = [x]_\beta^T M [y]_\gamma \quad \Rightarrow \quad P^T \overline{M} Q = M.$$

Alternatively, we have $\overline{M} = (P^T)^{-1} M Q^{-1}$. It is instructive to compare this result to that of Proposition 7.5.5 where we studied $T : V \to W$. We used notation $P_{\beta,\bar\beta} = [\Phi_{\bar\beta} \circ \Phi_\beta^{-1}]$ and found:

$$[T]_{\bar\beta,\bar\gamma} = P_{\gamma,\bar\gamma} [T]_{\beta,\gamma} (P_{\beta,\bar\beta})^{-1}.$$

Note $P_{\beta,\bar\beta}[x]_\beta = [\Phi_{\bar\beta} \circ \Phi_\beta^{-1}] \Phi_\beta(x) = \Phi_{\bar\beta}(x) = [x]_{\bar\beta}$. In short, $P_{\beta,\bar\beta} = P$ of our preceding discussion in this section. Likewise, $Q = P_{\gamma,\bar\gamma}$ hence we have the contrast:

$$[T]_{\bar\beta,\bar\gamma} = Q[T]_{\beta,\gamma} P^{-1} \qquad \text{vs.} \qquad \overline{M} = (P^T)^{-1} M Q^{-1}.$$

The matrix of a linear transformation and the matrix of a bilinear form do not transform in the same way under a change of coordinates. I should mention, the tensor product of the next section gives us a more universal method to understand the coordinate change properties for linear maps of all shapes and sizes. We discussed quadratic forms over $\mathbb{R}$ in Chapter 11, we can generalize to V over $\mathbb{F}$ as follows:

Definition 12.3.7.

Let V be vector spaces over $\mathbb{F}$ and $b : V \times V \to \mathbb{F}$ a bilinear form then $Q : V \to \mathbb{F}$ defined by $Q(x) = b(x, x)$ is a **quadratic form on** V. The matrix $[Q]_\beta$ of Q with respect to $\beta = \{v_1, \dots, v_n\}$ is given by $[Q]_\beta = [\frac{1}{2}(b(v_i, v_j) + b(v_j, v_i))]$

Following our discussion of coordinate change, if $[Q]_\beta$ and $[Q]_{\bar\beta}$ are the matrices of Q with respect to β and $\bar\beta$ bases for V then we may relate these by:

$$[Q]_{\bar\beta} = (P^T)^{-1} [Q]_\beta P^{-1}.$$

In the column vector context, $P = [\beta]^{-1}$ hence the formula above reads

$$[Q]_{\bar{\beta}} = [\beta]^T [Q]_\beta [\beta].$$

We discussed in lecture that if we wrote $Q(v) = v^T Av$ for $v \in \mathbb{R}^n$ and $A^T = A$ then we could find an **orthonormal eigenbasis** β for which $Av_i = \lambda_i v_i$ with $\lambda_i \in \mathbb{R}$ for each $i = 1, \dots, n$. This was the content of the **Real Spectral Theorem**. Notice, this structure also is found for a real vector space V as $[Q]_\beta^T = [Q]_\beta$ implies the existence of an orthonormal eigenbasis for $[Q]_\beta$ and hence vectors $v \in V$ for which $Q(v) = [v]^T[Q]_\beta[v] = [v]^T(\lambda[v]) = \lambda[v]^T[v] = \lambda$. If $x = x^1 v_1 + \cdots + x^n v_n$ where $v_1, \dots, v_n$ is a basis of V which maps to an orthonormal basis of $[Q]_\beta$ then $Q(x) = \lambda_1(x^1)^2 + \cdots + \lambda_n(x^n)^2$. We could study quadratic forms on vector spaces over $\mathbb{C}$ or $\mathbb{F}$. However, we stop here.

When $b : V \times V \to \mathbb{F}$ is a bilinear form then we say b is a **bilinear form on** V and we can write $b \in \mathcal{B}(V)$. Since both arguments of a bilinear form on V are from V this allows us to make a few additional definitions[14]:

Definition 12.3.8.

Let V be vector spaces over $\mathbb{F}$ and $b : V \times V \to \mathbb{F}$ a bilinear form then we say:

(i.) b is **symmetric** if $b(x, y) = b(y, x)$ for all $x, y \in V$,

(ii.) b is **antisymmetric** if $b(x, y) = -b(y, x)$ for all $x, y \in V$

(iii.) b is **nongenenerate** if $b(x, y) = 0$ for all $y \in V$ implies $x = 0$.

Notice, the matrix of $b : V \times V \to \mathbb{F}$ is a square matrix.

Proposition 12.3.9.

Let V be vector spaces over $\mathbb{F}$ and $b : V \times V \to \mathbb{F}$ a bilinear form and M the matrix of V with respect to a basis $\beta = \{v_1, \dots, v_n\}$ for V; $M_{ij} = b(v_i, v_j)$. Then

(i.) b is **symmetric** iff $M^T = M$

(ii.) b is **antisymmetric** iff $M^T = -M$

(iii.) b is **nongenenerate** iff $\det(M) \neq 0$

Proof: let b be a bilinear form on V with matrix M with respect to basis $\beta = \{v_1, \dots, v_n\}$. If b is symmetric then $b(v_i, v_j) = b(v_j, v_i)$ hence $M_{ij} = M_{ji}$ which gives $M^T = M$. Conversely, if $M = M^T$ then $b_{ij} = b(v_i, v_j) = b(v_j, v_i) = b_{ji}$ and following Proposition 12.3.6 we calculate:

$$b(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} x^i y^j b_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m} x^i y^j b_{ji} = b(y, x).$$

the proof of (ii.) is exceedingly similar. Next, suppose b is nondenerate. Suppose $b(x, y) = 0$ for all $y \in V$ (we know $x = 0$ by nondegeneracy) this gives:

$$[x]^T M[y] = 0$$

[14] I suppose we could define nondegeneracy for a bilinear form on $V \times W$, but, I won't in these notes

for all $[y] \in \mathbb{F}^n$ implies $x = 0$ hence $[x] = 0$. Consider $y = v_i$ hence $[y] = e_i$ and $[x]^T M[y] = [x]^T M e_i = \text{col}_i([x]^T M)$. Therefore,

$$[x]^T M[y] = 0$$

for all $y \in V$ implies $[x]^T M = 0$. More to the point, we have $[x]^T M = 0$ implies $x = 0$. But, transposing yields $M^T[x] = 0$ implies $x = 0$ which shows $\text{Null}(M^T) = \{0\}$. Thus M^T is invertible and we conclude $\det(M^T) = \det(M) \neq 0$. I leave the converse direction of (iii.) to the reader. $\square$

12.3.1 geometry of metrics and musical morphisms

An important generalization of real inner product space is now given[15].

Definition 12.3.10.

Let V be a real vector space. If $g : V \times V \to \mathbb{R}$ is a symmetric, nongenerate bilinear form then we say g is a **metric** or **scalar product** on V. The pair (V, g) is known as a **geometry**. If $L : V \to L$ is a linear transformation such that $g(L(x), L(y)) = g(x, y)$ for all $x, y \in V$ then L is a **g-orthogonal mapping**. If V has basis $\beta = \{v_1, \dots, v_n\}$ then the matrix of the metric is denoted $G_{ij} = g(v_i, v_j)$.

Sorry for the uber definition, but, I'm trying to be concise. Incidentally, symplectic geometry involves a nongenerate bilinear form which is not symmetric, so, our definition of geometry is by no means universal.

Proposition 12.3.11.

Let (V, g) form a real geometry. If G is the matrix of g in the sense that there exists a basis $\beta = \{v_1, \dots, v_n\}$ of V and $G_{ij} = g(v_i, v_j)$ then $G^T = G$ and $\det(G) \neq 0$. Furthermore, if A is the matrix of a g-orthogonal mapping then $A^T G A = G$.

Proof: symmetry and nondegeneracy of G follow immediately from Proposition 12.3.9. I leave the proof that $A^T G A = G$ to the reader. $\square$

Example 12.3.12. *If $g(x, y) = x \bullet y$ for all $x, y \in \mathbb{R}^n$ then we can write $g(x, y) = x^T I y$ and identify $G = I$ for this* **Euclidean metric**. *Notice, the condition $A^T I A = I$ simply reads $A^T A = I$ which we recognize as the defining condition for an orthogonal matrix; $A \in O(n)$.*

Generalizing the example above a bit, we previously worked with **inner products**. An inner product $h(x, y) = \langle x, y \rangle$ is a symmetric, bilinear map which is **positive definite**. The positive definite condition reads:

$$h(x, x) \geq 0 \ \ \& \ \ h(x, x) = 0 \ \text{ iff } \ x = 0$$

you can show that the positive definite condition implies the matrix of h is invertible. This means, *every real inner product is a metric.* Furthermore, if (V, h) is a geometry with h an inner product then we say the geometry is **Euclidean**.

Example 12.3.13. *Let* $\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *then $g(x, y) = x^T \eta y$ defines the* **Minkowski metric** *on $\mathbb{R}^4$. In particular $g(x, y) = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3$. This is certainly not an inner product*

[15] see *Matrix Groups for Undergraduates* by Christopher Tapp for a rather nice extension to complex or even quaternionic valued products

since there are **null-vectors**. *In particular, if $n = (n^0, n^1, n^2, n^3) \neq 0$ and $g(n,n) = 0$ we have a null-vector. If n has $(n^0)^2 = (n^1)^2 + (n^2)^2 + (n^3)^2$ then $g(n,n) = 0$. Physically, $x^0 = ct$ where t is time and c is the speed of light. If you analyze the points for which $g(n,n) = 0$ you can see these are connected by paths to the origin which travel at the speed of light. For example, $x^0 = ct$ and $x^1 = ct$ and $x^2 = x^3 = 0$ gives $n = (ct, ct, 0, 0)$ with $\triangle x/\triangle t = c$. There are three possibilities for events in the time-space of $\mathbb{R}^4$:*

1. *if $g(x,x) > 0$ then x is a* **spacelike** *event.*
2. *if $g(x,x) = 0$ then x is a* **lightlike** *event.*
3. *if $g(x,x) < 0$ then x is a* **timelike** *event.*

If you analyze paths from the origin to timelike events then you will find they can be connected by **worldlines**[16] *which correspond to subluminal*[17] *travel. For example, the worldline with parameterization $x^0 = ct, x^1 = ct/2, x^2 = 0, x^3 = 0$ has $(ct, ct/2, 0, 0)$ and thus*

$$g((ct, ct/2, 0, 0), (ct, ct/2, 0, 0)) = -c^2t^2 + c^2t^2/4 = -3c^2t^2/4 < 0$$

for this worldline with speed $c/2$. Events which are connected by timelike vectors are **causally connected**. *In contrast, events which are connected by spacelike vectors can be acausal. For example, you could kill your own grandma is you could achieve travel via spacelike vectors. I'm not sure why you'd do such a thing, there must be better ways to illustrate aburdity in impossible physics, but, as explained in the Lectures by Professor Shankar of Yale it is a popular construct. Anyway, I don't expect you to become proficient in Special Relativity for this course, I merely mention a few terms for your amusement. If you'd like to understand it fairly deeply with not too much effort, I highly recommend Professor Shankar's lectures.*

If Λ is the matrix of g-orthogonal transformation on $\mathbb{R}^4$ then we have the condition $\Lambda^T \eta \Lambda = \eta$. Such matrices are known as **Lorentz matrices** *and the corresponding transformations are* **Lorentz transformations**. *Physically, these transformations are constructed from a combination of spatial rotations and time-space velocity transformations known as* **boosts**. *A nice notation which reflects our choice of η is that $\Lambda \in O(1,3)$.*

Mathematically, metrics are rather different than the inner products we studied in Chapter 10. Consider $W = span\{(1,1,0,0)\}$. If we define the g-orthogonal complement of W as usual:

$$W^\perp = \{(a,b,c,d) \mid g((1,1,0,0),(a,b,c,d)) = 0\}$$

the condition on (a,b,c,d) is just $-a + b = 0$ hence $a = b$. Thus $(a,b,c,d) \in W^\perp$ has the form $(a,a,c,d) = a(1,1,0,0) + c(0,0,1,0) + d(0,0,0,1)$. We find:

$$W^\perp = span\{(1,1,0,0),(0,0,1,0),(0,0,0,1)\}.$$

Well, this is a bit disturbing. We observe $W \leq W^\perp$ and $W + W^\perp \neq \mathbb{R}^4$. This in stark contrast to Theorem 10.3.10 and the corresponding theorem for real inner product spaces[18].

[16] just a path in Minkowski space, a trajectory

[17] slower than the speed of light

[18] I proved the real inner product space in the April 8 Lecture of Spring 2016 see this

The theory of orthogonal complements for metrics involves subtleties we have not yet exposed. You can read Serge Lang's *Linear Algebra* or, for more details and a rather systematic exposition, study Steven Roman's Advanced Linear Algebra.

The theorem below essentially says Euclidean space and variations on Minkowski space are all that are allowed for our definition of metric geometry.

Theorem 12.3.14. *(special case of Sylvester's Law of Inertia)*

Let (V, g) form a real geometry. There exists a basis β for V in which $G = \text{diag}(-1, \dots, -1, 1, \dots, 1)$ and there are p-copies of -1 and q copies of 1.

Proof Sketch: let γ be a basis for V and $\bar{G}$ be the matrix of g with respect to the γ basis. Apply the real spectral theorem to find an orthonormal basis for $\bar{G}$ where the eigenvectors $x_1, x_2, \dots, x_n$ are arranged into p-negative values and q positive values. We know there is no zero eigenvalue since $\det(G) \neq 0$. Next, define $y_1, \dots, y_p$ as $y_i = \frac{1}{\sqrt{\lambda_i}} x_i$ and $y_{p+1}, \dots, y_n$ as $y_i = \frac{1}{\sqrt{-\lambda_i}} x_i$ for $i = p+1, \dots, n$. Notice $\lambda_i < 0$ hence $\sqrt{-\lambda_i} \in \mathbb{R}$ for $i = p+1, \dots, p+q = n$. If $\beta = [y_1|\cdots|y_n]$ then we can show $[\beta]^T \, \overline{G} \, [\beta] = diag(-1, \dots, -1, 1, \dots, 1)$. $\square$

The invertibility of G allows us to construct the musical morphism.

Definition 12.3.15. *inverse of metric and the musical morphisms:*

If (V, g) is a real geometry and G is the matrix of its metric g with respect to $\beta = \{v_1, \dots, v_n\}$ then we define $g^{ij} = (G^{-1})_{ij}$ hence $\sum_{k=1}^n g^{ik} g_{kj} = \delta_{ij}$. Also we define **musical morphisms** as follows: for each $x = \sum_{i=1}^n x^i v_i \in V$ and $\alpha = \sum_{i=1}^n \alpha_i v^i \in V^*$ we define:

$$\flat x^i = x_i = \sum_{j=1}^n x^i g_{ij} \qquad \& \qquad \sharp \alpha_i = \alpha^i = \sum_{j=1}^n \alpha_i g^{ij}$$

We say $\flat$ **lowers the index** whereas $\sharp$ **raises the index**.

Theorem 12.3.16. *almost natural isomorphism of V and V^**

Let (V, g) form a real geometry and suppose V has basis $\beta = \{v_1, \dots, v_n\}$ with dual basis $\beta^* = \{v^1, \dots, v^n\}$ then $\Psi : V \to V^*$ defined by

$$\Psi(x) = \sum_{i=1}^n (\flat x^i) v^i$$

is an isomorphism.

Proof: in fact, the isomorphism Ψ is not dependent on the choice of basis despite appearances. Consider, by the definition of $\flat$,

$$\Psi(x) = \sum_{i=1}^n \sum_{j=1}^n g_{ij} x^j v^i$$

However, as $\Psi(x) \in V^*$ we are free to evaluate at $y \in V$ hence:

$$(\Psi(x))(y) = \left(\sum_{i=1}^n \sum_{j=1}^n g_{ij} x^j v^i \right)(y) = \sum_{i=1}^n \sum_{j=1}^n g_{ij} x^j y^i = g(y, x)$$

where I used $v^i(y) = y^i$ as well as Proposition 12.3.6. The formula above clearly demonstrates the linearity of Ψ is induced from that of g. Moreover, injectivity of Ψ follows from nondegeneracy of g. Let me be explicit,

$$\begin{aligned} \ker(\Psi) = \{x \in V \mid \Psi(x) = 0\} &= \{x \in V \mid (\Psi(x))(y) = 0 \text{ for all } y \in V\} \\ &= \{x \in V \mid g(x, y) = 0 \text{ for all } y \in V\} \\ &= \{0\}. \end{aligned}$$

Furthermore, if $\alpha \in V^*$ we can use $\sharp$ to prove surjectivity; note $x = \sum_{i=1}^n (\sharp\alpha_i)v_i$ has $x^i = \sharp\alpha_i$ hence

$$(\Psi(x))(v_k) = g(x, v_k) = g\left(\sum_{i=1}^n (\sharp\alpha_i)v_i, v_k\right) = (\sharp\alpha_i)g(v_i, v_k) = \sum_{i=1}^n \sum_{j=1}^n \alpha_j g^{ji} g_{ik} = \sum_{j=1}^n \alpha_j \delta_{jk} = \alpha_k.$$

This shows $\Psi(x)$ and α have the same values on the basis element v_k thus $\Psi(x) = \alpha$. This demonstrates that Ψ is a surjection. Therefore, $\Psi : V \to V^*$ is an isomorphism. $\square$

If I didn't wish to discuss the musical morphisms $\flat$ and $\sharp$ we could simply define $\Psi(x)$ by $(\Psi(x))(y) = g(x, y)$. We could argue injectivity and surjectivity directly from rank nullity arguments for the matrix of G.

Finally, we can extend the meaning of $\sharp$ and $\flat$ beyond the explicit formulas thus far given. In particular, if $x \in V$ then

$$\flat x = \sum_{i=1}^n (\flat x^i) v^i = \sum_{i,j=1}^n g_{ij} x^j v^i \qquad \& \qquad \sharp\alpha = \sum_{i=1}^n (\sharp\alpha_i) v_i = \sum_{i,j=1}^n g^{ij} \alpha_j v_i$$

We should recognize $\sharp\alpha \in V$ is precisely the **Riesz'** vector for α. Explicitly, notice:

$$g(\sharp\alpha, x) = g\left(\sum_{i,j=1}^n g^{ij}\alpha_j v_i, \sum_{k=1}^n x^k v_k\right) = \sum_{i,j,k=1}^n g^{ij}\alpha_j x^k g(v_i, v_k) = \sum_{j,k=1}^n \alpha_j x^k \sum_{i=1}^n g^{ji} g_{ik}$$

however, $\sum_{i=1}^n g^{ji} g_{ik} = \delta_{jk}$ hence

$$g(\sharp\alpha, x) = \sum_{j,k=1}^n \alpha_j x^k \delta_{jk} = \sum_{j=1}^n \alpha_j x^j = \sum_{j=1}^n \alpha_j v^j(x) = \left(\sum_{j=1}^n \alpha_j v^j\right)(x) = \alpha(x).$$

In words, the vector $\sharp\alpha$ allows us to express the operation of α via the operation of the metric where the one of the entries is held fixed at $\sharp\alpha$. Finally, I pause to note if (V, g) is a geometry then (V^*, g^*) is likewise a geometry where we define $g^*(\alpha, \beta) = g(\sharp\alpha, \sharp\beta)$. I leave it to the reader to check that g^* is indeed, symmetric, bilinear, and nondegenerate.