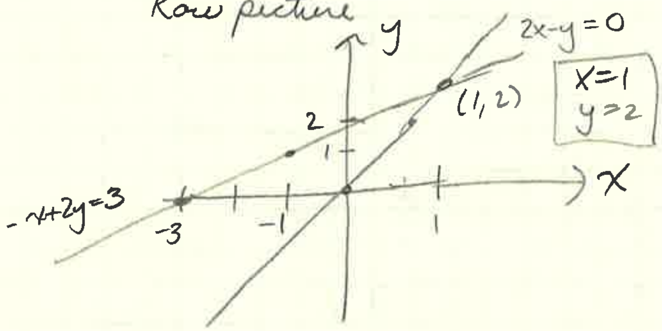
n linear equations, n unknowns

Row picture
* Column picture
Matrix form

$$2x - y = 0$$
$$-x + 2y = 3$$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{b}$$

Row picture

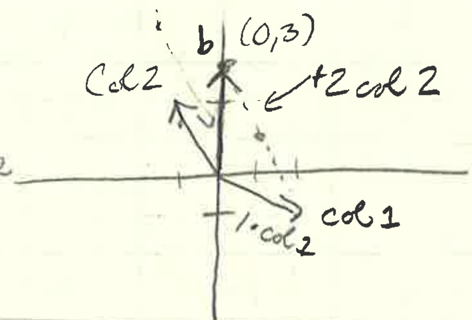


Column picture

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Find the linear combination of the columns to produce $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Take $x=1$ and $y=2$



Take all of the combinations

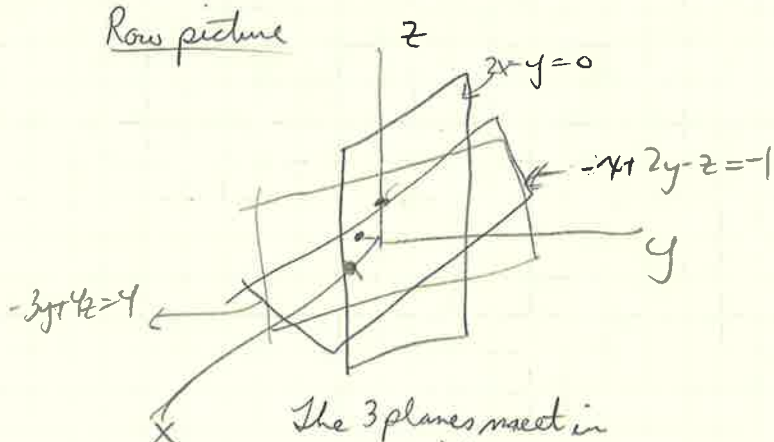
$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix}$; which fills the plane

---

$$2x - y \quad = 0$$
$$-x + 2y - z = -1$$
$$-3y + 4z = 4$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row picture

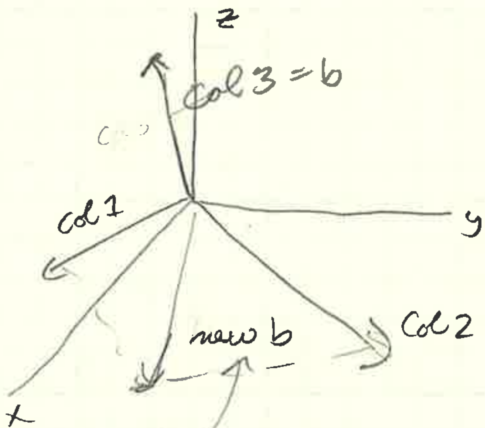


The 3 planes meet in one point

Column picture

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

Linear combination of 3-3D vectors



$x=0,\ y=0,\ z=1$

Suppose $b=\begin{bmatrix}1\\1\\-3\end{bmatrix}$

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}1\\1\\-3\end{bmatrix} \Rightarrow x=1, y=1, z=0$$

Can I solve $Ax=b$ for every b?

$\Leftrightarrow$ Do the linear combinations of the columns fill 3-D space?

For this A, the answer is yes.

$Ax=b$

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}$$

Ax is a combination of the columns of A

Lecture 2

Elimination < Success / failure

Back-Substitution
Elimination Matrices
Matrix Multiplication

$$\begin{aligned} x+2y+z&=2\\ 3x+8y+z&=12\\ 4y+z&=2 \end{aligned}$$

$$\begin{bmatrix}\boxed{1}&2&1\\3&8&1\\0&4&1\end{bmatrix}\xrightarrow[(2,1)\ (3,1)]{R_2-3R_1,\ R_3-0R_1}\begin{bmatrix}1&2&1\\0&\boxed{2}&-2\\0&4&1\end{bmatrix}\xrightarrow[(3,2)]{R_3-2R_2}\underbrace{\begin{bmatrix}1&2&1\\0&2&-2\\0&0&\boxed{5}\end{bmatrix}}_{U \text{ (upper triangular)}}$$

first pivot

Second pivot

third pivot

Elimination failure (not getting 3 pivots)

Back Substitution

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]}_{A \quad\quad b} \xrightarrow[R_3-0R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

$$\underbrace{\quad\quad}_{U} \quad \underbrace{\quad}_{c}$$

$Ux=c$:

$$x+2y+z=2 \quad , \quad x=2$$
$$2y-2z=6 \quad , \quad y=1$$
$$5z=-10 \quad , \quad z=-2$$

Elimination Matrices

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3\cdot \text{col } 1 + 4\cdot \text{col } 2 + 5 \cdot \text{col } 3$$

Matrix × column = column

$$\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}} = 1\cdot\text{row } 1 + 2\cdot\text{row } 2 + 7\cdot\text{row } 3$$

row × matrix = row

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 1. Subtract 3 row 1 from row 2 and gives new row 2

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Step 2: Subtract 2 row 2 from row 3 to get new row 3

$$E_{32}(E_{21}A) = U$$
$$\Rightarrow \text{associativity}$$
$$(E_{32}E_{21})A = U$$

Permutation matrix ⇒ Exchange rows

$$\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Multiply on left does row operations

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Exchange columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Multiply on right does column operations

$AB \neq BA$ in general (not commutative in general)

Inverses

$$\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

add 3 row 1 to row 2

Lecture 3

Matrix Multiplication (4 ways!)
Inverse of A, AB, A^T
Gauss-Jordan / find A^{-1}

$$\text{row 3}\underset{A_{m\times n}}{\begin{bmatrix} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \\ \end{bmatrix}}\underset{B_{n\times p}}{\begin{bmatrix} & b_{14} \text{ col 4} & \\ & b_{24} & \\ & \vdots & \\ & b_{n4} & \end{bmatrix}} = \underset{\substack{C=AB \\ m\times p}}{\begin{bmatrix} \\ & c_{34} & \\ \\ \end{bmatrix}}$$

$$c_{34} = (\text{row 3 of } A)\cdot(\text{col 4 of } B)$$
$$= a_{31}b_{14} + a_{32}b_{24} + \dots = \sum_{k=1}^{n} a_{3k}b_{k4}$$

$$\underset{A_{m\times n}}{\begin{bmatrix} \quad \end{bmatrix}}\underset{B_{n\times p}}{\begin{bmatrix} \quad \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} | & | & & | \\ A\cdot\text{col}_1 B & A\cdot\text{col}_2 B & \cdots & A\cdot\text{col}_p B \\ | & | & & | \end{bmatrix}}$$

Columns of C are combinations of columns of A

$$\underset{A_{m\times n}}{\begin{bmatrix} \\ \text{---} \\ \\ \end{bmatrix}}\underset{B_{n\times p}}{\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} \text{row}_1 A\cdot B \\ \\ \\ \end{bmatrix}}$$

← rows of C are combinations of rows of B

Column of A × row of B
(m×1) (1×p)

4th way

AB = Sum of (columns of A) × (rows of B)

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Block Multiplication

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$$

$A_1B_1 + A_2B_3$

$A_1B_2 + A_2B_4$

Inverses (square matrices)

$A^{-1}A = I = AA^{-1}$ called invertible, nonsingular

if this exists

Singular case: no inverse

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ can't form $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of columns of A

no inverse because we can find a vector $x \neq 0$ with $Ax = 0$

$$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$$

but $x \neq 0$

Back to invertible case

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$A \times$ column j A^{-1} = column j of I

Gauss-Jordan (solve 2 eqns at once)

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\quad \begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\underset{A\qquad I}{\left[\begin{array}{cc|cc}1&3&1&0\\2&7&0&1\end{array}\right]} \Rightarrow \left[\begin{array}{cc|cc}1&3&1&0\\0&1&-2&1\end{array}\right] \Rightarrow \underset{I\qquad A^{-1}}{\left[\begin{array}{cc|cc}1&0&7&-3\\0&1&-2&1\end{array}\right]}$$

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}7&-3\\-2&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$E\,[A\ I] = [EA\ EI] = [I\ A^{-1}]$ because $EA=I$ $\Leftrightarrow E=A^{-1}$

product of E's for the elimination steps

Lecture 4

Inverse of AB, A^T
Product of Elimination Matrices
$A=LU$ (no row exchanges)

$AA^{-1}=I=A^{-1}A$

If AB are both invertible, $ABB^{-1}A^{-1} = AIA^{-1}=I$

$\Rightarrow (AB)^{-1}=B^{-1}A^{-1}$

$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B=I$

$(AA^{-1})^T=(I)^T$

$(A^{-1})^TA^T = I$

This is the inverse of A^T $\Rightarrow (A^T)^{-1}=(A^{-1})^T$

$$\underset{E_{21}}{\begin{bmatrix}1&0\\-4&1\end{bmatrix}}\underset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}}=\underset{U}{\begin{bmatrix}2&1\\0&3\end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}} = \underset{L=E_{21}^{-1}}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\underset{U}{\begin{bmatrix}2&1\\0&3\end{bmatrix}}$$

$$= \underset{L}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\underset{D}{\begin{bmatrix}2&0\\0&3\end{bmatrix}}\underset{U}{\begin{bmatrix}1&1/2\\0&1\end{bmatrix}}$$

$E_{32}E_{31}E_{21}A_{3\times 3} = U$ (no row exchanges)

$$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L}\, U$$

$E_{31} = I$

$$\overset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}}\overset{E_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}} = \overset{E}{\begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}} \qquad EA = U$$

Inverses

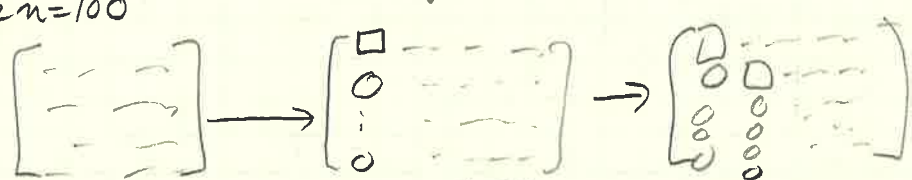
$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix} = L \qquad A = LU$$

$\underline{A = LU}$

If no row exchanges, the multipliers go directly into L.

How many operations on an $n \times n$ matrix A? (multiply + subtract)

Take $n = 100$



100×100 — like 99×99 (about 100^2) — like 98×98 (about 99^2 ops)

$$\#\text{ of ops} \sim n^2 + (n-1)^2 + \cdots + 3^2 + 2^2 + 1^2 \approx \tfrac{1}{3}n^3 \text{ on } A$$

Cost for every RHS is n^2

<u>Permutations</u> 3×3

$$I = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad P_{12} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \quad P_{13} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \quad P_{23} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},$$

switch rows 1 & 2

$$P_{123} = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \qquad P_{132} = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

cycle

Group of 6 matrices

$P_{12}^{-1} = P_{12}$ $\qquad P^{-1} = P^T$ for permutation matrices

$\#$ 4×4 permutation matrices $= 24$

Section 2.7 $PA = LU$

Section 3.1 Vector Spaces and Subspaces

Permutations P : execute row exchanges

$$A = LU = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ * & & 1 \end{bmatrix} \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix} \quad \text{(no row exchanges)}$$

becomes $PA = LU$ (with row exchanges)

for any invertible A

$P =$ identity matrix with reordered rows.

$n!$ permutation matrices of size n.

$P^{-1} = P^T \iff P^T P = I$

Transposes

$$R^T \rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}}_{R}$$

$R^T R$ is always symmetric

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

$$(A^T)_{ij} = A_{ji}$$

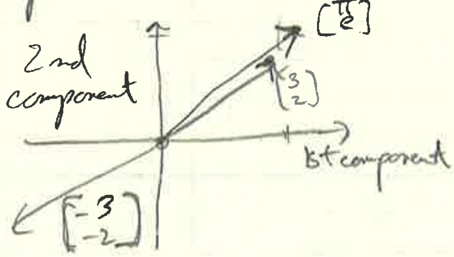
Symmetric Matrices : $A^T = A$

Ex. $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$$(R^T R)^T = R^T R^{T^T} = R^T R$$

Vector Spaces

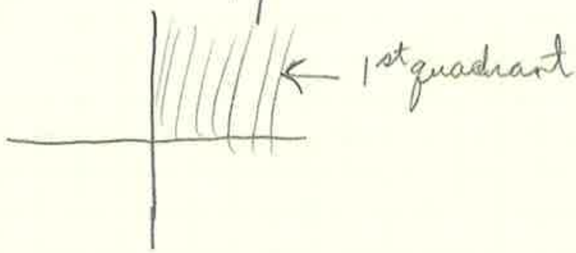
Examples : R^2 = all 2-dim. real vectors such as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \ldots$



R^2 = "x-y plane"

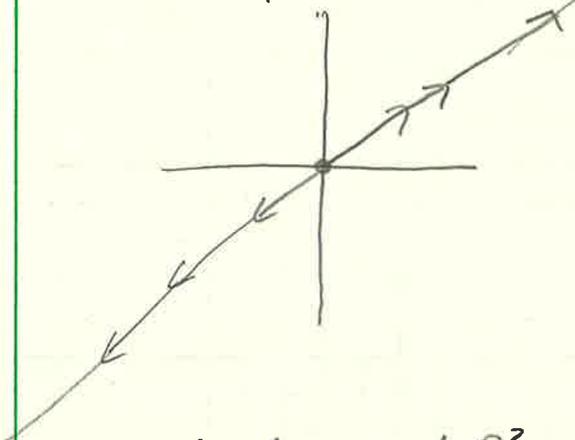
R^3 = all vectors with 3 real components $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

$\mathbb{R}^n$ = all $\underset{\text{column}}{\uparrow}$ vectors with n real components not a vector space



Can add and stay in the 1st quadrant, but scalar multiplication may bring us out of the space

A vector space inside $\mathbb{R}^2$ called a subspace of $\mathbb{R}^2$



a line in $\mathbb{R}^2$ through the zero vector

All subspaces of $\mathbb{R}^2$

1. All of $\mathbb{R}^2$
2. any line through $\begin{bmatrix}0\\0\end{bmatrix}$, L
3. zero vector only, Z

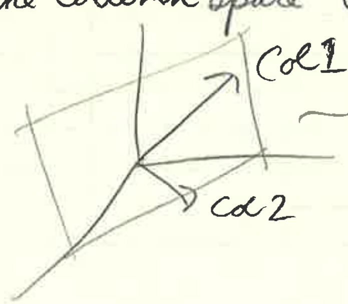
All subspaces of $\mathbb{R}^3$

1. All of $\mathbb{R}^3$
2. any line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$
3. any plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$
4. zero vector only

$$A = \begin{bmatrix}1 & 3\\2 & 3\\4 & 1\end{bmatrix}$$

<u>columns in $\mathbb{R}^3$</u>

all their linear combinations form a subspace called the column space $C(A)$

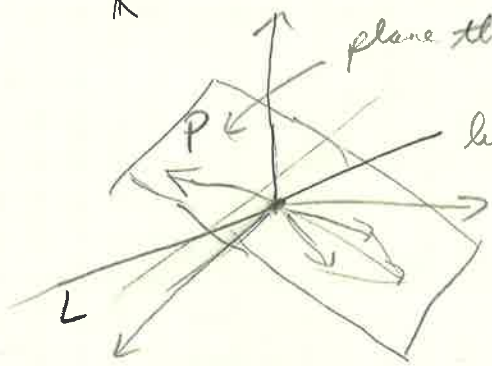


→ this space is a plane

Vector Spaces and Subspaces
Column space of A : Solving $Ax=b$
Null space of A

Vector Space requirements: $v+w$ and cw are in the space
all combinations $\boxed{cv+dw}$ are in the space

$\mathbb{R}^3$



plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of $\mathbb{R}^3$

line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of $\mathbb{R}^3$

2 subspaces: P and L
$P \cup L$ = all vectors in P or L or both
This $\cancel{(is)}$ $\boxed{(\text{is not})}$ a subspace

$P \cap L$ = all vectors in both P and L
This is a vector space

Subspaces S and T: $S \cap T$ is a subspace

C(A): Column Space of A (is a subspace of $\mathbb{R}^4$ here)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$C(A)$ = all linear combinations of the columns

Does $Ax=b$ have a solution for every b? No

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

For some RHS b this can be solved.

Which b allow this system to be solved?? $b=0$ always

$b = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $x = \begin{bmatrix}1\\0\\0\end{bmatrix}$ works $\qquad$ $b = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $x = \begin{bmatrix}0\\1\\0\end{bmatrix}$ works

Exactly when $b \in C(A)$

In this case $C(A)$ is a 2-D subspace of $\mathbb{R}^4$

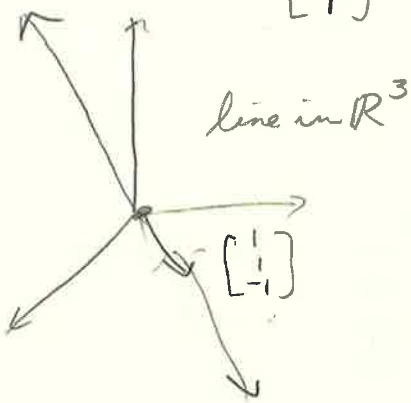
$N(A)$: <u>Nullspace of A</u> = all x's that solve $Ax=0$ (in $\mathbb{R}^3$ here)

$$Ax=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$N(A)$ contains $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}c\\c\\-c\end{bmatrix}$

$0 \in N(A)$ always

$N(A)$ contains $c\begin{bmatrix}1\\1\\-1\end{bmatrix}$ for any c



Check that solutions to $Ax=0$ always give a subspace

If $Av=0$ and $Aw=0$ then $A(v+w)=0$

$$A(v+w)=Av+Aw=0+0=0 \ \checkmark$$

If $Av=0$ then $A(cv)=0$

$$A(cv)=cAv=c0=0 \ \checkmark$$

$$Ax=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

Do the solutions form a subspace? <u>No</u>

No 0-vector to start

$x=\begin{bmatrix}1\\0\\0\end{bmatrix}$ is a solution

$x=\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is a solution

Lecture 7

Computing the nullspace ($Ax=0$)
Pivot variables - free variables
Special solutions - $\mathrm{rref}(A)=R$

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\to\begin{bmatrix}\boxed{1}&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}\to\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$

echelon form

2 pivot columns

The rank of A = number of pivots = 2 in this case

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$\quad\;\uparrow\quad\;\uparrow$
free columns

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0$

$x_3 = -2x_4 = 0$

$x_1 + 2 = 0$

$$c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$$

Choose $x_2 = 0,\ x_4 = 1 \Rightarrow$

$x_1 + 2x_3 + 2 = 0$

$2x_3 + 4 = 0$

$x_3 = -2$

$x_1 - 4 + 2 = 0 \Rightarrow x_1 = 2$

$$x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

$$d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

$$c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

Rank $r = 2$ = # of pivot variables

$n - r = 4 - 2 = 2$ free variables

R = reduced row echelon form has zeros above and below the pivots

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot rows and columns

$$\begin{aligned} x_1 + 2x_2 \qquad - 2x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned} \qquad Rx = 0$$

$I \rightarrow \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}\ \boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}} \leftarrow F$

pivot cols free cols

Signs are flipped for free variables

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ r pivot rows

r pivot cols $\quad n-r$ free cols

$Rx = 0$

nullspace matrix N

(columns are special solutions)

$RN = 0$

$N = \begin{bmatrix} -F \\ I \end{bmatrix}$

$Rx = 0$

$[I\ F]\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$

$\boxed{x_{pivot} = -F x_{free}}$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

Rank $r = 2$ again!

pivot cols free col

$n - r = 3 - 2 = 1$

set free var to 1

$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = -1$

$2x_2 + 2x_3 = 0 \Rightarrow x_2 = -1$

$c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = N(A^T) = \text{a line in } \mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \quad I \qquad F \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = R$$

$c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c\underbrace{\begin{bmatrix} -F \\ I \end{bmatrix}}_{N}$

Lecture 8

Complete Solution of $Ax = b$

Rank r

$r = m$: Solution exists $\qquad r = n$: Solution is unique

$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$
$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$
$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

Augmented matrix $[A\ b]$ $\quad$ pivot cols

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix} \qquad 0 = b_3 - b_2 - b_1$$

$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ $\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is OK

Solvability : Condition on b

$Ax = b$ is solvable iff $b \in C(A)$

If a combination of the rows of A gives a zero row, then the same combination of the entries of b must give 0.

To find the complete solution to $Ax = b$

① $x_{particular}$: Set all free variables to zero. Solve $Ax=b$ for pivot variables

$x_2 = 0, \; x_4 = 0$

$x_1 + 2x_3 = 1$

$2x_3 = 3 \quad \Rightarrow x_3 = 3/2$ and $x_1 = -2$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

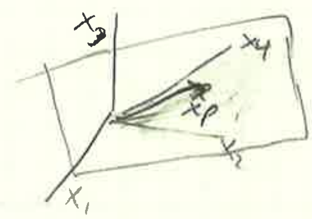
② $x_{nullspace}$ (x_n)

③ Complete solution $x = x_p + x_n$

$$\begin{aligned} Ax_p &= b \\ + \; Ax_n &= 0 \\ \hline A(x_p + x_n) &= b \end{aligned}$$

$$x = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Plot all solutions x in R^4



Solution is not a subspace

m by n matrix A of rank r (know $r \le m$, $r \le n$)

Full column rank means $\underline{r = n}$: no free variables

$N(A) = \{0\}$, Solution to $Ax = b$: $x = x_p$ unique if it exists

either 0 or 1 solutions.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{if } b = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix} \Rightarrow x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Full row rank means $\underline{r = m}$: m pivots

Can solve $Ax = b$ for every b (Exists)

Left with $n - r$ free variables $(n - m)$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \quad \text{rank } r = 2, \quad R = \begin{bmatrix} 1 & 0 & F \\ 0 & 1 & \end{bmatrix}$$

$r = m = n$ (Invertible)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \qquad R = I \qquad N(A) = \{0\}$$

b can be anything

$r = m = n$	$r = n < m$	$r = m < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = [\, I \; F \,]$ columns mixed
1 solution to $Ax = b$	0 or 1 solution to $Ax = b$	∞ solutions to $Ax = b$

$r < m$, $r < n$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ 0 or ∞ solutions to $Ax = b$

columns mixed

Lecture 9

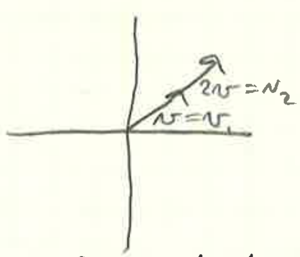
Linear independence
Spanning a space
Basis and dimension

Suppose A is m by n with $m < n$, then there are <u>nonzero</u> solutions to <u>$Ax = 0$</u>.
More unknowns than equations

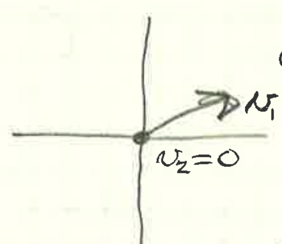
Reason: There will be at least one free variable!! ($n-m$ free variables)

<u>Independence</u> Vectors $x_1, x_2, \dots, x_n$ are linearly independent if no combination gives the zero vector except for the zero combination (all $c_i = 0$)

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n \neq 0 \quad \text{for all } c_i \neq 0$$

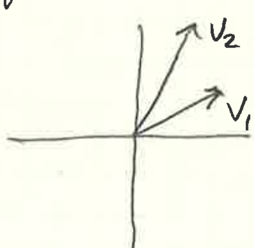


$2v_1 - v_2 = 0$
dependent

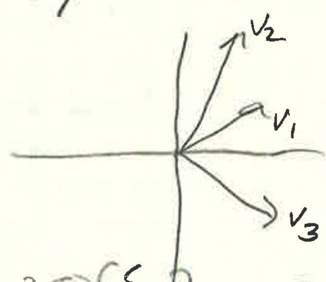


$0v_1 + cv_2 = 0$

If any of the v_i are 0, then they are dependent



independent



dependent

$$A = [v_1 \; v_2 \; v_3] = \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix}$$

dependence $\Leftrightarrow$ $N(A)$ is more than $\{0\}$

$$\begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Repeat when $v_1, \dots, v_n$ are columns of A. They are independent if $N(A)$ is $\{0\}$. They are dependent if $Ac = 0$ for some nonzero $c \in N(A)$.

rank $= n$ and $N(A) = \{0\}$ for independence
rank $< n$ and yes free variables $\Leftrightarrow$ dependence

Spanning a Space

Vectors $v_1, \dots, v_\ell$ span a space means that the space consists of all combinations of those vectors.

A Basis for a vector space is a sequence of vectors $v_1, v_2, \dots, v_d$ with 2 properties:

1. They are independent
2. They span the space

Example:
Space is R^3:

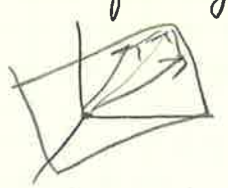
One basis is: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\quad c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$ iff $c_1 = c_2 = c_3 = 0$

$A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I$, $N(I) = \{0\}$

Another basis $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$, $A = \begin{bmatrix}1&2&3\\1&2&3\\2&5&8\end{bmatrix}$ matrix must be square and invertible

R^n: n vectors give a basis if the $n \times n$ matrix with those columns is invertible.

Are $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ a basis for any space? They are independent and span a plane so yes



Bases are not unique, but they all have the same number of vectors.
Given a space, every basis for the space has the same number of vectors.
The number is the dimension of the space.

Examples: $C(A)$

$A = \begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}$ $\quad \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} \in N(A) \Rightarrow$ the columns span but are not independent

$\uparrow\uparrow$ pivot cols $\quad \uparrow\uparrow$ free

rank $A = 2$

1) Correct error in Lecture 9
2) Four fundamental subspaces (for matrix A)

Example Space is $\mathbb{R}^3$ (from lecture 9)

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$ standard basis

Another basis

$$\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}, \quad A = \begin{bmatrix}1&2&3\\1&2&3\\2&5&8\end{bmatrix}$$ is Not invertible (two equal rows)

4 Fundamental Subspaces

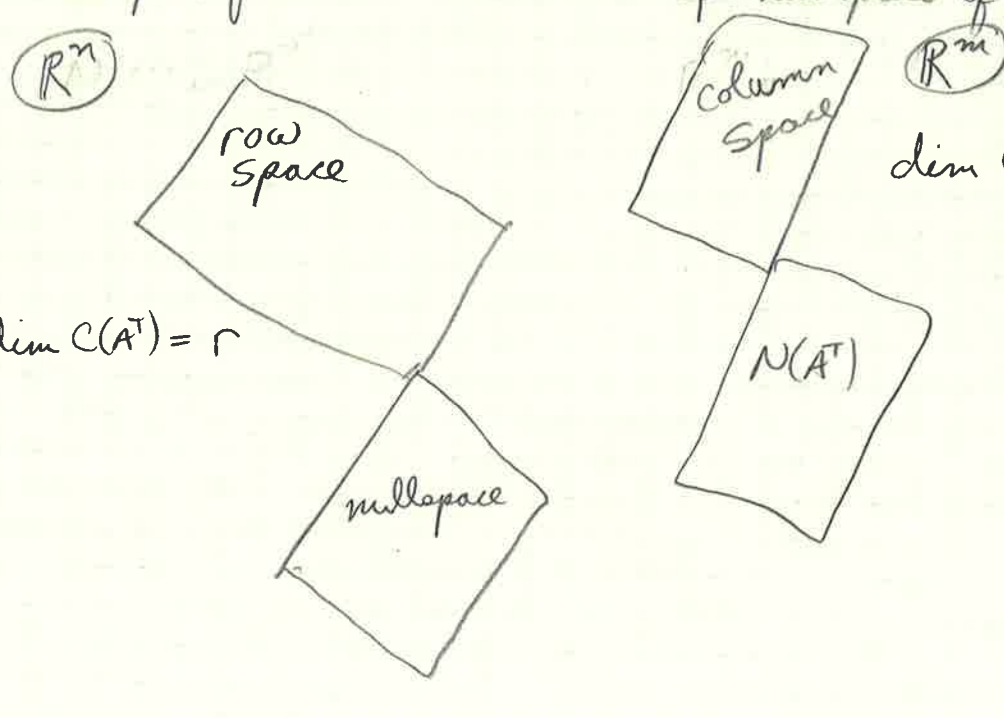
Column space $C(A) \in \mathbb{R}^m$ $\quad$ A is $m \times n$

Nullspace $N(A) \in \mathbb{R}^n$

Rowspace = all combinations of rows = all combinations of columns of A^T

$= C(A^T) \in \mathbb{R}^n$

Nullspace of A^T = $N(A^T)$ = left nullspace of $A \in \mathbb{R}^m$



$\dim C(A) = \text{rank} = r$

$\dim C(A^T) = r$

	$C(A)$	$C(A^T)$	$N(A)$	$N(A^T)$
basis	pivot cols	see below	special solutions for each free variable	See below
dimension	r	r	$n-r$	$m-r$

$n-r \leftarrow$ number of free variables in A

$m-r \leftarrow$ # of free variables in A^T

$$A=\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}\rightarrow\begin{bmatrix}1&2&3&1\\0&-1&-1&0\\0&0&0&0\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1&2&3&1\\0&1&1&0\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&\overbrace{1\;\;1}^{F}\\0&1&1\;\;0\\0&0&0\;\;0\end{bmatrix}=R$$

$C(R)\neq C(A)$ different column spaces

$\boxed{\text{Basis of row space for } A \text{ and } R \text{ is first } r \text{ rows of } R \text{ (not of } A)}$

$\boxed{\text{same row space}}$

$\underline{N(A^T)}$: $A^Ty=0 \Rightarrow y\in N(A^T)$

$$\begin{bmatrix}\;\end{bmatrix}\begin{bmatrix}\;\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$$

transpose $y^TA=0^T$

$$\begin{bmatrix}y^T\end{bmatrix}\begin{bmatrix}A\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$$

rref $\begin{bmatrix}A_{m\times n} & I_{m\times m}\end{bmatrix}\rightarrow\begin{bmatrix}R_{m\times n} & E_{m\times m}\end{bmatrix}$

$\Leftrightarrow E\begin{bmatrix}A_{m\times n} & I_{m\times m}\end{bmatrix}\rightarrow\begin{bmatrix}R_{m\times n} & E_{m\times m}\end{bmatrix}$

$\Rightarrow EA=R$

In Chapter 2, R was I and E was A^{-1}

$$\left[\begin{array}{cccc|ccc} 1&2&3&1&1&0&0\\ 1&1&2&1&0&1&0\\ 1&2&3&1&0&0&1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc} 1&2&3&1&1&0&0\\ 0&-1&-1&0&-1&1&0\\ 0&0&0&0&-1&0&1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|ccc} 1&2&3&1&1&0&0\\ 0&1&1&0&1&-1&0\\ 0&0&0&0&-1&0&1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1&0&1&1& \\ 0&1&1&0& \underbrace{\begin{matrix}-1&2&0\\1&-1&0\\-1&0&1\end{matrix}}_{E} \\ 0&0&0&0& \end{array}\right]$$

$$\begin{bmatrix} -1&2&0\\ 1&-1&0\\ -1&0&1 \end{bmatrix}\begin{bmatrix} 1&2&3&1\\ 1&1&2&1\\ 1&2&3&1 \end{bmatrix} = \begin{bmatrix} 1&0&1&1\\ 0&1&1&0\\ 0&0&0&0 \end{bmatrix}$$

Basis for $N(A^T)$

New vector space!

$\underline{M}$: All 3x3 matrices!! $A+B$, cA (not AB for now)

Subspaces of $\underline{M}$
- all upper triangular
- all symmetric matrices
- all diagonal matrices D: $\dim D = 3$

$$\left\{ \begin{bmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\ 0&3&0\\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&7 \end{bmatrix} \right\}$$ is a basis for D