n linear equations, n unknowns

Row picture

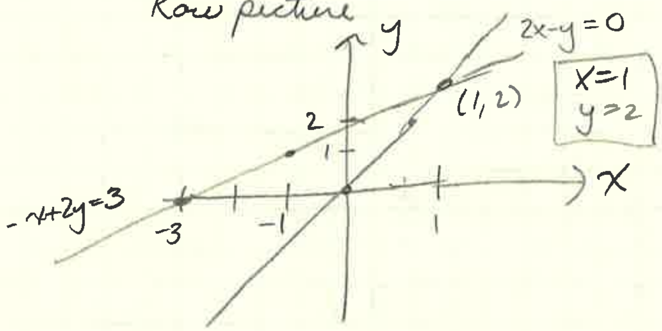
* Column picture

Matrix form

$$2x - y = 0$$
$$-x + 2y = 3$$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{b}$$

Row picture



Column picture

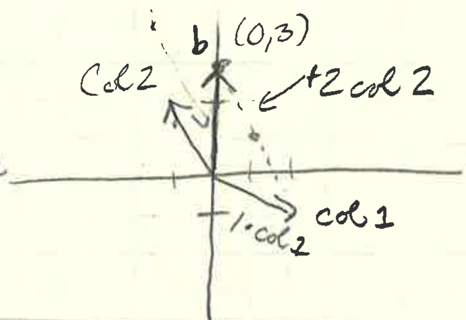
$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Find the linear combination of the columns to produce $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Take $x=1$ and $y=2$

Take all of the combinations

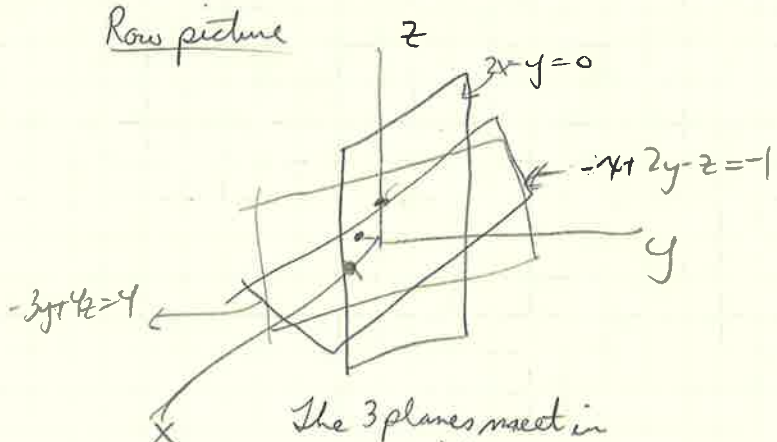
$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix}$; which fills the plane



$$2x - y = 0$$
$$-x + 2y - z = -1$$
$$-3y + 4z = 4$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row picture

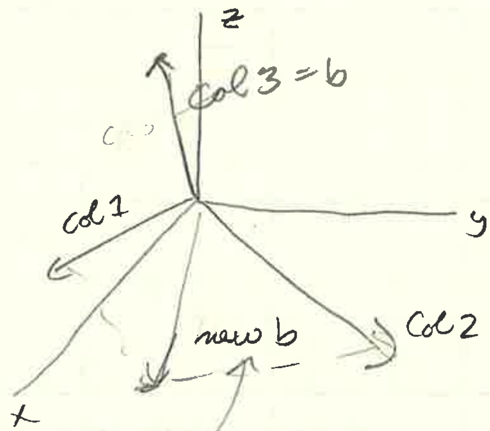


The 3 planes meet in one point

Column picture

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

Linear combination of 3-3D vectors



$x=0,\ y=0,\ z=1$

Suppose $b=\begin{bmatrix}1\\1\\-3\end{bmatrix}$

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}1\\1\\-3\end{bmatrix}\Rightarrow x=1, y=1, z=0$$

Can I solve $Ax=b$ for every b?

$\Leftrightarrow$ Do the linear combinations of the columns fill 3-D space?

For this A, the answer is yes.

$Ax=b$

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}$$

Ax is a combination of the columns of A

Lecture 2

Elimination < Success / failure

Back-Substitution
Elimination Matrices
Matrix Multiplication

$$\begin{aligned} x+2y+z&=2\\ 3x+8y+z&=12\\ 4y+z&=2\end{aligned}$$

$$\begin{bmatrix}\boxed{1}&2&1\\3&8&1\\0&4&1\end{bmatrix}\xrightarrow[(2,1)\ (3,1)]{R_2-3R_1,\ R_3-0R_1}\begin{bmatrix}1&2&1\\0&\boxed{2}&-2\\0&4&1\end{bmatrix}\xrightarrow[(3,2)]{R_3-2R_2}\underbrace{\begin{bmatrix}1&2&1\\0&2&-2\\0&0&\boxed{5}\end{bmatrix}}_{U\text{ (upper triangular)}}$$

first pivot

Second pivot

third pivot

Elimination failure (not getting 3 pivots)

Back Substitution

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} \left| \underbrace{\begin{matrix} 2 \\ 12 \\ 2 \end{matrix}}_{b} \right. \xrightarrow[R_3-0R_1]{R_2-3R_1} \left[\begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{matrix} \left| \begin{matrix} 2 \\ 6 \\ 2 \end{matrix}\right.\right] \xrightarrow{R_3-2R_2} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}_{U} \left| \underbrace{\begin{matrix} 2 \\ 6 \\ -10 \end{matrix}}_{C} \right.$$

$Ux = c$:

$$\begin{aligned} x + 2y + z &= 2, & x &= 2 \\ 2y - 2z &= 6, & y &= 1 \\ 5z &= -10, & z &= -2 \end{aligned}$$

Elimination Matrices

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3 \cdot \text{col } 1 + 4 \cdot \text{col } 2 + 5 \cdot \text{col } 3$$

Matrix × column = column

$$\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}} = 1 \cdot \text{row } 1 + 2 \cdot \text{row } 2 + 7 \cdot \text{row } 3$$

row × matrix = row

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 1.
Subtract 3 row 1 from row 2 and gives new row 2

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Step 2:
Subtract 2 row 2 from row 3 to get new row 3

$$E_{32}(E_{21}A) = U \Rightarrow \text{associativity}$$
$$(E_{32}E_{21})A = U$$

Permutation matrix ⇒ Exchange rows

$$\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Multiply on left does row operations

Exchange columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Multiply on right does column operations

$AB \neq BA$ in general (not commutative in general)

Inverses

$$\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

add 3 row 1 to row 2

Lecture 3

Matrix Multiplication (4 ways!)

Inverse of A, AB, A^T

Gauss-Jordan / find A^{-1}

$$\text{row } 3 \underset{A_{m\times n}}{\begin{bmatrix} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \\ \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} b_{14} \text{ col } 4 \\ b_{24} \\ \vdots \\ b_{n4} \end{bmatrix}} = \underset{\substack{C=AB \\ m\times p}}{\begin{bmatrix} & c_{34} \\ & \searrow \cdot \\ \\ \end{bmatrix}}$$

$$c_{34} = (\text{row } 3 \text{ of } A)\cdot(\text{col } 4 \text{ of } B)$$
$$= a_{31}b_{14} + a_{32}b_{24} + \cdots = \sum_{k=1}^{n} a_{3k}\, b_{k4}$$

$$\underset{A_{m\times n}}{\begin{bmatrix} \\ \\ \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} \\ \\ \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} | & | & & | \\ A\cdot\text{col}_1 B & A\,\text{col}_2 B & \cdots & A\cdot\text{col}_p B \\ | & | & & | \end{bmatrix}}$$

Columns of C are combinations of columns of A

$$\underset{A_{m\times n}}{\begin{bmatrix} \\ \text{---} \\ \\ \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} \text{row}_1 A\cdot B \\ \\ \\ \end{bmatrix}}$$

← rows of C are combinations of rows of B

Column of A $\times$ row of B
(m×1) (1×p)

4th way
$AB =$ Sum of (columns of A) $\times$ (rows of B)

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Block Multiplication

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$$

$A_1B_1 + A_2B_3$

$A_1B_2 + A_2B_4$

Inverses (square matrices)

$A^{-1}A = I = AA^{-1}$ called invertible, nonsingular

if this exists

Singular case: no inverse

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ can't form $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of columns of A

no inverse because we can find a vector $x \neq 0$ with $Ax = 0$

$$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$$

but $x \neq 0$

Back to invertible case

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$A \times$ column j of A^{-1} = column j of I

Gauss-Jordan (solve 2 eqns at once)

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\quad \begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\underset{A\quad\ \ I}{\left[\begin{array}{cc|cc}1&3&1&0\\2&7&0&1\end{array}\right]}\Rightarrow\left[\begin{array}{cc|cc}1&3&1&0\\0&1&-2&1\end{array}\right]\Rightarrow\underset{I\quad\ \ A^{-1}}{\left[\begin{array}{cc|cc}1&0&7&-3\\0&1&-2&1\end{array}\right]}$$

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}7&-3\\-2&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$E\,[A\ I] = [EA\ EI] = [I\ A^{-1}]$ because $EA=I$ $\Leftrightarrow E=A^{-1}$

↑ product of E's for the elimination steps

Lecture 4

Inverse of AB, A^T

Product of Elimination Matrices

$A=LU$ (no row exchanges)

$AA^{-1}=I=A^{-1}A$

If A, B are both invertible, $ABB^{-1}A^{-1} = AIA^{-1}=I$

$\Rightarrow (AB)^{-1}=B^{-1}A^{-1}$

$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B=I$

$(AA^{-1})^T=(I)^T$

$(A^{-1})^TA^T = I$

↑ This is the inverse of A^T $\Rightarrow (A^T)^{-1}=(A^{-1})^T$

$$\underset{E_{21}}{\begin{bmatrix}1&0\\-4&1\end{bmatrix}}\underset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}}=\underset{U}{\begin{bmatrix}2&1\\0&3\end{bmatrix}}\qquad \underset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}}=\underset{L=E_{21}^{-1}}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\underset{U}{\begin{bmatrix}2&1\\0&3\end{bmatrix}}$$

$$=\underset{L}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\underset{D}{\begin{bmatrix}2&0\\0&3\end{bmatrix}}\underset{U}{\begin{bmatrix}1&1/2\\0&1\end{bmatrix}}$$

$E_{32}E_{31}E_{21}A_{3\times 3} = U$ (no row exchanges)

$$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L} U$$

$E_{31} = I$

$$\begin{array}{ccc} E_{32} & E_{21} & & E \\ \begin{bmatrix} 1&0&0\\0&1&0\\0&-5&1 \end{bmatrix} & \begin{bmatrix} 1&0&0\\-2&1&0\\0&0&1 \end{bmatrix} & = & \begin{bmatrix} 1&0&0\\-2&1&0\\10&-5&1 \end{bmatrix} \end{array} \qquad EA = U$$

Inverses

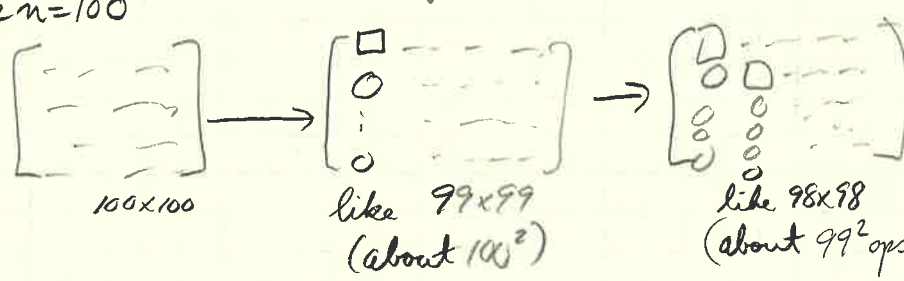
$$\begin{bmatrix} 1&0&0\\2&1&0\\0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0\\0&1&0\\0&5&1 \end{bmatrix} = \begin{bmatrix} 1&0&0\\2&1&0\\0&5&1 \end{bmatrix} = L \qquad A = LU$$

<u>$A = LU$</u>

If no row exchanges, the multipliers go directly into L.

!!How many operations on an $n \times n$ matrix A?
(multiply + subtract)

Take $n = 100$



$\#$ of ops $\sim n^2 + (n-1)^2 + \cdots + 3^2 + 2^2 + 1^2 \approx \frac{1}{3}n^3$ on A

Cost for every RHS is n^2

<u>Permutations</u> 3×3

$$I = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix},$$

switch rows 1&2

$$P_{123} = \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} \qquad P_{132} = \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}$$

cycle

Group of 6 matrices

$P_{12}^{-1} = P_{12}$ $\qquad P^{-1} = P^T$ for permutation matrices

$\#$ 4×4 permutation matrices $= 24$

Section 2.7 $PA = LU$

Section 3.1 Vector Spaces and Subspaces

Permutations P : execute row exchanges

$$A = LU = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ * & & 1 \end{bmatrix} \begin{bmatrix} & & * \\ & \diagdown & \\ 0 & & \end{bmatrix} \quad \text{(no row exchanges)}$$

becomes $PA = LU$ (with row exchanges)

for any invertible A

P = identity matrix with reordered rows.

$n!$ permutation matrices of size n.

$$P^{-1} = P^T \iff P^TP = I$$

<u>Transposes</u>

$$R^T \rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} \leftarrow R$$

R^TR is always symmetric

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

$$(A^T)_{ij} = A_{ji}$$

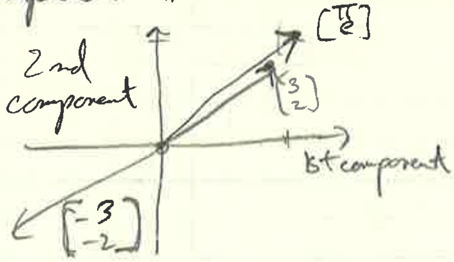
<u>Symmetric Matrices</u> : $A^T = A$

Ex. $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$$(R^TR)^T = R^T R^{TT} = R^TR$$

<u>Vector Spaces</u>

Examples: R^2 = all 2-dim. real vectors such as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \ldots$



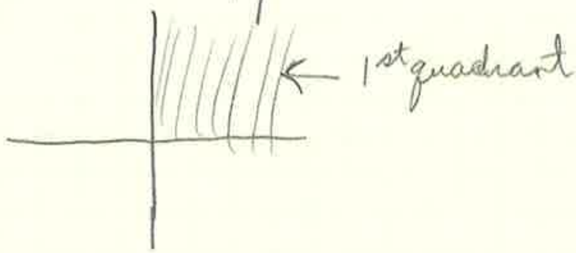
R^2 = "x-y plane"

R^3 = all vectors with 3 real components $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

R^n = all vectors with n real components

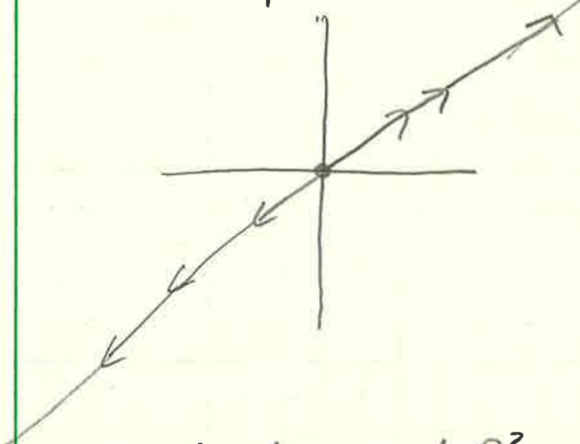
column

not a vector space



Can add and stay in the 1st quadrant, but scalar multiplication may bring us out of the space

A vector space inside $\mathbb{R}^2$ called a subspace of $\mathbb{R}^2$



a line in $\mathbb{R}^2$ through the zero vector

All subspaces of $\mathbb{R}^2$

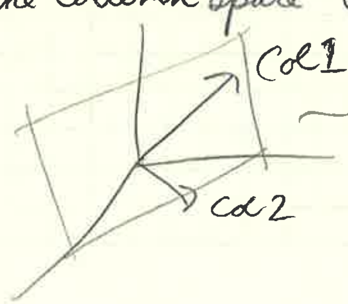
1. All of $\mathbb{R}^2$
2. any line through $\begin{bmatrix}0\\0\end{bmatrix}$, L
3. zero vector only, Z

All subspaces of $\mathbb{R}^3$

1. All of $\mathbb{R}^3$
2. any line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$
3. any plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$
4. zero vector only

$A = \begin{bmatrix}1 & 3\\2 & 3\\4 & 1\end{bmatrix}$ columns in $\mathbb{R}^3$

all their linear combinations form a subspace called the column space $C(A)$



→ this space is a plane

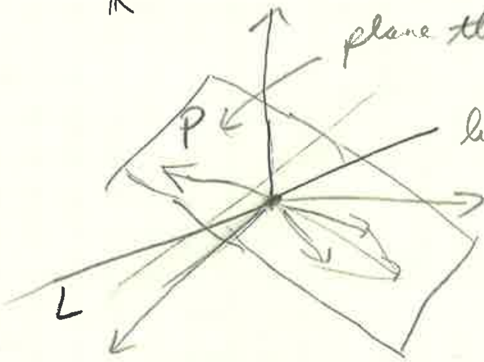
Vector Spaces and Subspaces
Column space of A : Solving $Ax=b$
Null space of A

Vector Space requirements: $v+w$ and cw are in the space
all combinations $\boxed{cv+dw}$ are in the space

$\mathbb{R}^3$

plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of $\mathbb{R}^3$

line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of $\mathbb{R}^3$



2 subspaces: P and L
$P \cup L$ = all vectors in P or L or both
This ~~(is)~~ $\boxed{\text{(is not)}}$ a subspace

$P \cap L$ = all vectors in both P and L
This is a vector space

Subspaces S and T: $S \cap T$ <u>is</u> a subspace

<u>$C(A)$: Column Space of A</u> (is a subspace of $\mathbb{R}^4$ here)

$$A=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}$$ $C(A)$ = all linear combinations of the columns

Does $Ax=b$ have a solution for every b? <u>No</u>

$$Ax=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

For some RHS b this can be solved.

Which b allow this system to be solved?? $b=0$ always

$b=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $x=\begin{bmatrix}1\\0\\0\end{bmatrix}$ works $\qquad b=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $x=\begin{bmatrix}0\\1\\0\end{bmatrix}$ works

Exactly when $b \in C(A)$

In this case $C(A)$ is a 2-D subspace of $\mathbb{R}^4$

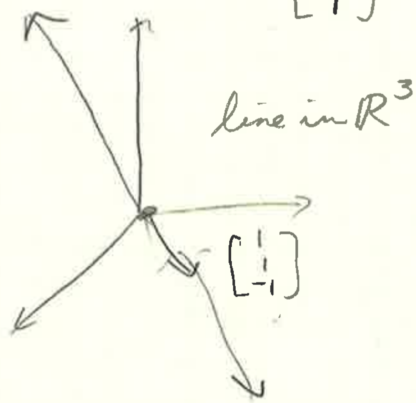
$N(A)$: Nullspace of A = all x's that solve $Ax=0$ (in $\mathbb{R}^3$ here)

$$Ax=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$N(A)$ contains $\begin{bmatrix}0\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}c\\c\\-c\end{bmatrix}$

$0 \in N(A)$ always

$N(A)$ contains $c\begin{bmatrix}1\\1\\-1\end{bmatrix}$ for any c



Check that solutions to $Ax=0$ always give a subspace

If $Av=0$ and $Aw=0$ then $A(v+w)=0$

$$A(v+w)=Av+Aw=0+0=0 \checkmark$$

If $Av=0$ then $A(cv)=0$

$$A(cv)=cAv=c0=0 \checkmark$$

$$Ax=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

Do the solutions form a subspace? No

No 0-vector to start

$x=\begin{bmatrix}1\\0\\0\end{bmatrix}$ is a solution

$x=\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is a solution

Lecture 7

Computing the nullspace ($Ax=0$)
Pivot variables - free variables
Special solutions - $\mathrm{rref}(A)=R$

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\to\begin{bmatrix}\boxed{1}&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}\to\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$

echelon form

2 pivot columns

The rank of A = number of pivots = 2 in this case

$$\begin{array}{c}\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & \uparrow & & \uparrow \end{array} \\ \text{free columns}\end{array} \qquad x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 0$$
$$x_3 = -2x_4 = 0$$
$$x_1 + 2 = 0$$

$$c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$$

Choose $x_2 = 0$, $x_4 = 1$ $\Rightarrow$

$$x_1 + 2x_3 + 2 = 0$$
$$2x_3 + 4 = 0$$
$$x_3 = -2$$
$$x_1 - 4 + 2 = 0 \Rightarrow x_1 = 2$$

$$x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

$$d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

$$c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

Rank $r = 2$ = # of pivot variables

$n - r = 4 - 2 = 2$ free variables

R = reduced row echelon form has zeros above and below the pivots

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot rows and columns

$$\begin{aligned} x_1 + 2x_2 - 2x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned} \qquad Rx = 0$$

$$I \rightarrow \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \; \boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}} \leftarrow F$$

pivot cols free cols

Signs are flipped for free variables

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \quad r \text{ pivot rows}$$

r pivot cols $\quad n - r$ free cols

$Rx = 0$

nullspace matrix N
(columns are special solutions)

$RN = 0$

$N = \begin{bmatrix} -F \\ I \end{bmatrix}$

$Rx = 0$

$[I\ F]\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$

$\boxed{x_{pivot} = -F x_{free}}$

$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$

Rank $r = 2$ again!

pivot cols free col

$n - r = 3 - 2 = 1$

set free var to 1

$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = -1$

$2x_2 + 2x_3 = 0 \Rightarrow x_2 = -1$

$c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = N(A^T) = \text{a line in } \mathbb{R}^3$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$ (columns 1–2: I, column 3: F)

$c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c\begin{bmatrix} -F \\ I \end{bmatrix}$ ← N

Lecture 8

Complete Solution of $Ax = b$

Rank r

$r = m$: Solution exists $r = n$: Solution is unique

$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$

$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$

$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$

Augmented matrix $[A\ b]$ pivot cols

$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$

$0 = b_3 - b_2 - b_1$

$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is OK

Solvability: Condition on b

$Ax=b$ is solvable iff $b \in C(A)$

If a combination of the rows of A gives a zero row, then the same combination of the entries of b must give 0.

To find the complete solution to $Ax=b$

① $x_{particular}$: Set all free variables to zero. Solve $Ax=b$ for pivot variables

$x_2=0,\ x_4=0$

$x_1+2x_3=1$

$2x_3=3 \Rightarrow x_3=3/2$ and $x_1=-2$

$$x_p=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}$$

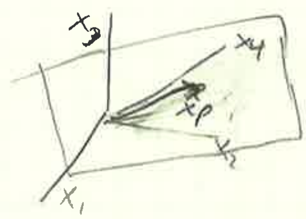
② $x_{nullspace}$ (x_n)

③ Complete solution $x=x_p+x_n$

$$\begin{aligned} Ax_p &= b\\ + \quad Ax_n &= 0\\ \hline A(x_p+x_n) &= b\end{aligned}$$

$$x=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}+c_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

Plot all solutions x in R^4



Solution is not a subspace

m by n matrix A of rank r (know $r \le m,\ r \le n$)

Full column rank means $\underline{r=n}$: no free variables

$N(A)=\{0\}$, Solution to $Ax=b$: $x=x_p$ unique if it exists

either 0 or 1 solutions.

$$A=\begin{bmatrix}1&3\\2&1\\6&1\\5&1\end{bmatrix} \rightarrow R=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix} \quad \text{if } b=\begin{bmatrix}4\\3\\7\\6\end{bmatrix} \Rightarrow x_p=\begin{bmatrix}1\\1\end{bmatrix}$$

Full row rank means $\underline{r=m}$: m pivots

Can solve $Ax=b$ for every b (Exists)

Left with $n-r$ free variables $(n-m)$

$$A=\begin{bmatrix}1&2&6&5\\3&1&1&1\end{bmatrix} \quad \text{rank } r=2, \quad R=\begin{bmatrix}1&0&-F-\\0&1&--\end{bmatrix}$$

$r = m = n$ (Invertible)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $\quad R = I \quad N(A) = \{0\}$

b can be anything

$r=m=n$	$r=n<m$	$r=m<n$
$R=I$	$R=\begin{bmatrix} I \\ 0 \end{bmatrix}$	$R=[\,I\ F\,]$ columns mixed
1 solution to $Ax=b$	0 or 1 solution to $Ax=b$	∞ solutions to $Ax=b$

$r<m,\ r<n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ columns mixed $\quad$ 0 or ∞ solutions to $Ax=b$

Lecture 9

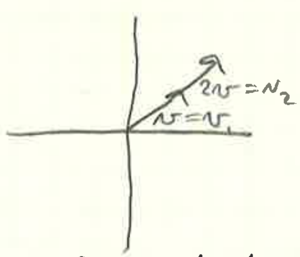
Linear independence
Spanning a space
Basis and dimension

Suppose A is m by n with $m<n$, then there are <u>nonzero</u> solutions to <u>$Ax=0$</u>.
More unknowns than equations

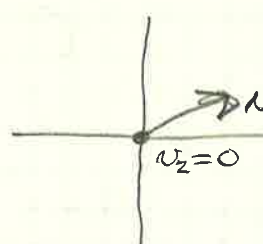
Reason: There will be at least one free variable!! ($n-m$ free variables)

<u>Independence</u> Vectors $x_1, x_2, \dots, x_n$ are linearly independent if no combination gives the zero vector except for the zero combination (all $c_i=0$)

$c_1x_1 + c_2x_2 + \dots + c_nx_n \neq 0$ for all $c_i \neq 0$

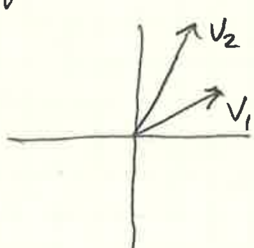


$2v_1 - v_2 = 0$
dependent

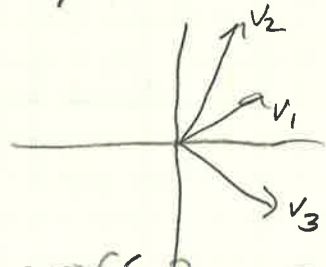


$0v_1 + cv_2 = 0$

If any of the v_i are 0, then they are dependent



independent



dependent

$A = [v_1\ v_2\ v_3] = \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix}$

dependence $\Leftrightarrow$ $N(A)$ is more than $\{0\}$ $\quad \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Repeat when $v_1, \dots, v_n$ are columns of A. They are independent if $N(A)$ is $\{0\}$. They are dependent if $Ac = 0$ for some nonzero $c \in N(A)$.
rank $= n$ and $N(A) = \{0\}$ for independence
rank $< n$ and yes free variables $\Leftrightarrow$ dependence

Spanning a Space

Vectors $v_1, \dots, v_\ell$ span a space means that the space consists of all combinations of those vectors.

A **Basis** for a vector space is a sequence of vectors $v_1, v_2, \dots, v_d$ with 2 properties:

1. They are independent
2. They span the space

Example:
Space is R^3:

One basis is: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\quad c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$ iff $c_1 = c_2 = c_3 = 0$

$A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I$, $N(I) = \{0\}$

Another basis $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$, $A = \begin{bmatrix}1&2&3\\1&2&3\\2&5&8\end{bmatrix}$ matrix must be square and invertible

R^n: n vectors give a basis if the $n \times n$ matrix with those columns is invertible.

Are $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ a basis for any space? They are independent and span a plane so yes

Bases are not unique, but they all have the same number of vectors.
Given a space, every basis for the space has the same number of vectors.
The number is the dimension of the space.

Examples: $C(A)$

$A = \begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}$ $\quad \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} \in N(A) \Rightarrow$ the columns span but are not independent

$\uparrow\uparrow$ pivot cols $\quad \uparrow\uparrow$ free

rank $A = 2$

1) Correct error in Lecture 9
2) Four fundamental subspaces (for matrix A)

Example Space is $\mathbb{R}^3$ (from lecture 9)

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ standard basis

Another basis

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$ is Not invertible (two equal rows)

4 Fundamental Subspaces

Column space $C(A) \in \mathbb{R}^m$ $\qquad$ A is $m \times n$

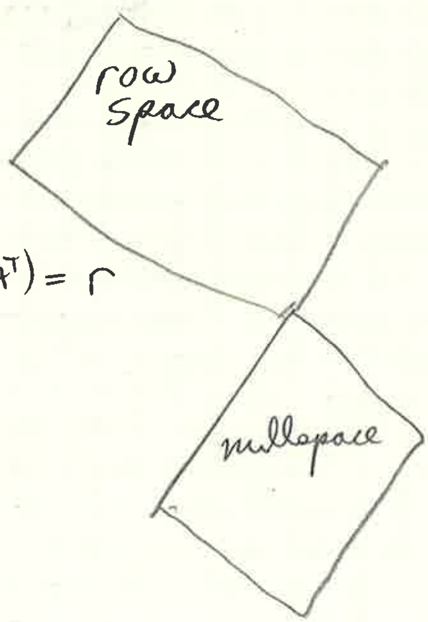
Nullspace $N(A) \in \mathbb{R}^n$

Rowspace = all combinations of rows = all combinations of columns of A^T

$= C(A^T) \in \mathbb{R}^n$

Nullspace of A^T = $N(A^T)$ = left nullspace of $A \in \mathbb{R}^m$

$\mathbb{R}^n$



dim $C(A^T) = r$

$\mathbb{R}^m$

dim $C(A)$ = rank = r

	$C(A)$	$C(A^T)$	$N(A)$	$N(A^T)$
basis	pivot cols	see below	special solutions for each free variable	See below
dimension	r	r	$n-r$	$m-r$

$n-r$ ← number of free variables in A

$m-r$ ← # of free variables in A^T

$$A=\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}\rightarrow\begin{bmatrix}1&2&3&1\\0&-1&-1&0\\0&0&0&0\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1&2&3&1\\0&1&1&0\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}=R$$

F

$C(R)\neq C(A)$ different column spaces

$\boxed{\text{Basis of row space for } A \text{ and } R \text{ is first } r \text{ rows of } R \text{ (not of } A)}$

$\boxed{\text{same row space}}$

$N(A^T)$: $A^Ty=0 \Rightarrow y\in N(A^T)$

$$\begin{bmatrix}\;\end{bmatrix}\begin{bmatrix}\;\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$$

transpose $y^TA=0^T$

$$\begin{bmatrix}y^T\end{bmatrix}\begin{bmatrix}A\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$$

rref $\begin{bmatrix}A_{m\times n} & I_{m\times m}\end{bmatrix}\rightarrow\begin{bmatrix}R_{m\times n} & E_{m\times m}\end{bmatrix}$

$\Leftrightarrow E\begin{bmatrix}A_{m\times n} & I_{m\times m}\end{bmatrix}\rightarrow\begin{bmatrix}R_{m\times n} & E_{m\times m}\end{bmatrix}$

$\Rightarrow EA=R$

In Chapter 2, R was I and E was A^{-1}

$$\left[\begin{array}{cccc|ccc} 1&2&3&1&1&0&0\\ 1&1&2&1&0&1&0\\ 1&2&3&1&0&0&1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc} 1&2&3&1&1&0&0\\ 0&-1&-1&0&-1&1&0\\ 0&0&0&0&-1&0&1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|ccc} 1&2&3&1&1&0&0\\ 0&1&1&0&1&-1&0\\ 0&0&0&0&-1&0&1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc} 1&0&1&1&-1&2&0\\ 0&1&1&0&1&-1&0\\ 0&0&0&0&-1&0&1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -1&2&0\\ 1&-1&0\\ -1&0&1 \end{bmatrix}}_{E}$$

$$\begin{bmatrix} -1&2&0\\ 1&-1&0\\ -1&0&1 \end{bmatrix}\begin{bmatrix} 1&2&3&1\\ 1&1&2&1\\ 1&2&3&1 \end{bmatrix} = \begin{bmatrix} 1&0&1&1\\ 0&1&1&0\\ 0&0&0&0 \end{bmatrix}$$

Basis for $N(A^T)$

New vector space!

M: All 3x3 matrices!! $A+B$, cA (not AB for now)

Subspaces of M
- all upper triangular
- all symmetric matrices
- all diagonal matrices D: $\dim D = 3$

$$\left\{ \begin{bmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\ 0&3&0\\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&7 \end{bmatrix} \right\}$$ is a basis for D

Lecture 11

- Bases of new vector spaces
- Rank one matrices
- Small world graphs

M = all 3x3 matrices dim $M = 9$

Subspaces $\begin{cases} \text{symmetric } 3\times3 = S, \ \dim S = 6 \\ \text{upper triangular } 3\times3 = U, \ \dim U = 6 \\ \text{diagonal } 3\times3 \end{cases}$ 3

Bases for M = all 3x3 matrices

$$\left\{ \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \dots, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix} \right\} \quad \dim M = 9$$

Basis for S is not in basis for M

$S \cap U$ = symmetric and upper triangular = diagonals, $\dim(S \cap U) = 3$

$S \cup U$ is not a subspace

$S + U$ = any element of S + any element of U = all 3x3's

$\dim(S+U) = 9$

$\dim S + \dim U = \dim(S \cap U) + \dim(S+U)$

$\frac{d^2y}{dx^2} + y = 0 \ni y = \cos x, \sin x$

complete solution: $y = c_1 \cos x + c_2 \sin x$

$\{\sin x, \cos x\}$ is a basis

dim(solution space) = 2

Rank one matrices

$A_{2\times3} = \begin{bmatrix} 1&4&5\\2&8&10 \end{bmatrix}$ Basis for $C(A^T) = [1\ 4\ 5]$

$\dim C(A) = \text{rank} = \dim C(A^T) = 1 = r$

$A = \begin{bmatrix} 1\\2 \end{bmatrix} [1\ 4\ 5]$ Rank 1 matrices : $A = uv^T$

u, v are column vectors

M = all 5×17 matrices

Subset of rank 4 matrices, is this a subspace? no

$\text{rank}(A+B) \le \text{rank}\,A + \text{rank}\,B$

In R^4, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, S = all $v \in R^4$ with $v_1+v_2+v_3+v_4=0$

S is a subspace since $c[v_1+v_2+v_3+v_4] = 0$ and

for $v, w \in S$ $\quad v_1+v_2+v_3+v_4+w_1+w_2+w_3+w_4=0$

$\dim S = 3$, S = null space of $A = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}}_{m\times n \,=\, 1\times 4}$ $\quad (Av=0)$

rank $A = 1 = r$

$\dim N(A) = n-r = 4-1 = 3$

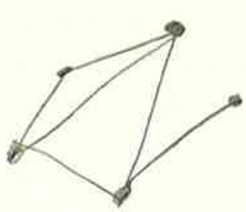
Basis for $S = N(A)$

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$C(A) = R^1$

$N(A^T) = \{0\}$ $\quad \dim N(A^T) = 0$

Graph = {nodes, edges}

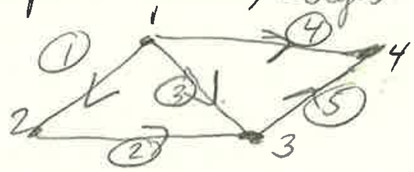


5 nodes
6 edges

Lecture 12

- Graphs & Networks
- Incidence Matrices
- Kirchhoff's Laws

Graph: Nodes, Edges



$n=4$ nodes
$m=5$ edges

Incidence Matrix

$$A = \begin{array}{c} \text{nodes } 1 \;\; 2 \;\; 3 \;\; 4 \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

edges 1, 2, 3: loop

$Ax=0$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2-x_1 \\ x_3-x_2 \\ x_3-x_1 \\ x_4-x_1 \\ x_4-x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x=(x_1,x_2,x_3,x_4)$
Potentials at nodes
$e=Ax \downarrow$
x_2-x_1, etc. Potential differences across edges
$y=Ce \downarrow$ (Ohm's law)
currents on edges
y_1, y_2, y_3, y_4, y_5
$\downarrow \; A^Ty=f$ (current source)
Kirchhoff's current law.

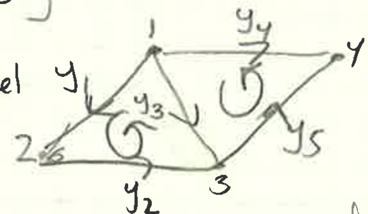
$$x=c\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = N(A) \qquad \dim N(A)=1$$

Set $x_4=0$ to ground a node. rank $=3$

$A^Ty=0$ to get $N(A^T)$, $\dim N(A^T) = 5-3=2$

$$A^Ty = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y_1-y_3-y_4=0$ current into node 1
$y_1-y_2=0$ current into node 2
$y_2+y_3-y_5=0$ node 3
$y_4+y_5=0$ node 4



Basis for $N(A^T) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

around big loop

$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ = sum of basis vectors

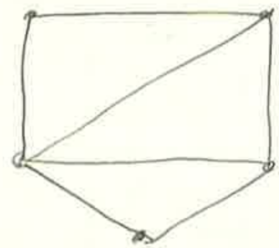
Tree : Graph with no loops

$\dim N(A^T) = m - r$ $\qquad r = n-1$

$\#\text{ loops} = \#\text{ edges} - (\#\text{ nodes} - 1)$

$\boxed{\#\text{ nodes} - \#\text{ edges} + \#\text{ loops} = 1}$ Euler's formula

$5 - 7 + 3 = 1$



$\boxed{A^T C A x = f}$ for equilibrium

Lecture 13 : review for exam 13.1

Q1) Let $U_{5\times3}$ be in rref with $r=3$

$U = \begin{bmatrix} | & | & | \end{bmatrix} \Rightarrow N(U) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$B = \begin{bmatrix} U \\ 2U \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} U \\ 0 \end{bmatrix}$ by block operations

$C_{10\times6} = \begin{bmatrix} U & U \\ U & 0 \end{bmatrix} \rightarrow \begin{bmatrix} U & U \\ 0 & -U \end{bmatrix} \rightarrow \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \rightarrow \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$

rank $C = 6$, $\dim N(C^T) = 10 - 6 = 4$

Q2) $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A is 3X3 , rank $r = 1$ because $\dim N(A) = 2$

$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ $\qquad Ax = b$ can be solved if b has the form $b = c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q) If $N(A) = 0$, then $N(A^T) = 0$

Q) The space of invertible matrices is NOT a subspace.

Q) If $B^2 = 0 \Rightarrow B = 0$? False

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Q) If $A_{n \times n}$ has independent columns, is $Ax = b$ always solvable?
Yes

Q)
$$\underset{3\times4}{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
don't multiply them

basis for $N(B) \subseteq \mathbb{R}^4$

If $B = CD$, and C is invertible $\Rightarrow N(CD) = N(D)$

basis for $N(B) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Complete solution to $Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

since 1st column of $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Q) If $m = n \Rightarrow$ then $C(A) = C(A^T)$? False

Q) A and $-A$ share the same 4 subspaces? True

Q) If A, B have the same 4 subspaces, then $A = cB$ for some $c \neq 0$?
False. For example, A & B are any invertible 6x6 matrices.

Q) If we exchange two rows of A, which subspaces stay the same?
$C(A^T)$, $N(A)$

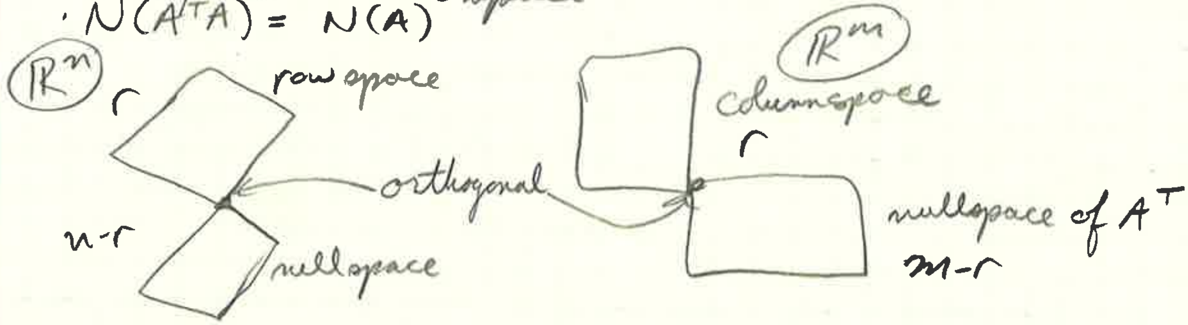
Q) $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can't be in nullspace and be a row of A. Why not?

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

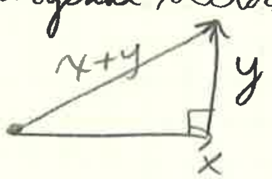
$$N(A) \cap C(A^T) = 0$$

Lecture 14 14.1

- Orthogonal vectors and subspaces
- nullspace $\perp$ row space
- $N(A^TA) = N(A)$



Orthogonal vectors



Pythagoras $\qquad x^Ty = 0$

$\|x\|^2 + \|y\|^2 = \|x+y\|^2$ (only for right triangles)

$$x^Tx + y^Ty = (x+y)^T(x+y)$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$\|x\|^2 = 14, \quad \|y\|^2 = 5, \quad \|x+y\|^2 = 19$$

$$\cancel{x^Tx} + \cancel{y^Ty} = \cancel{x^Tx} + \cancel{y^Ty} + \underbrace{x^Ty}_{0} + \underbrace{y^Tx}_{0}$$

Subspace S is orthogonal to subspace T means: every vector in S is orthogonal to every vector in T.

row space is orthogonal to nullspace
Why?

$$x \in N(A) \Rightarrow Ax = 0$$

$$\begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{matrix} c_1(\text{row } 1)^Tx = 0 \\ c_2(\text{row } 2)^Tx = 0 \\ \vdots \\ c_n(\text{row } n)^Tx = 0 \end{matrix}$$

$$c_1(\text{row}1)^Tx + c_2(\text{row}2)^Tx + \cdots + c_n(\text{row } n)^Tx = 0$$

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$, $\dim C(A^T) = 1$, $\dim N(A) = 2$

$n=3, r=1$ $N(A)$ is a plane $\perp$ to $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

Nullspace and rowspace are orthogonal complements in $\mathbb{R}^n$; i.e., null space contains all vectors $\perp$ rowspace.

Coming: "solve" $Ax=b$ when there is no solution
$m > n$
$\hookrightarrow A^T A \hat{x} = A^T b$

$A^T A$ is symmetric
$\underset{n \times m}{A^T} \underset{m \times n}{A}$
$\underline{n \times n}$ $(A^T A)^T = A^T A^{T^T} = A^T A$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$ rank $r = 2$
3×2

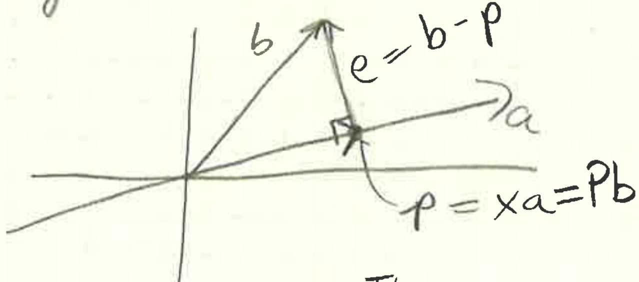
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$

$N(A^T A) = N(A)$
rank$(A^T A) =$ rank(A)

rank=1
$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$ is not invertible

$A^T A$ is invertible exactly if A has independent columns

Lecture 15 15.1
- Projections
- Least Squares
- Projection Matrix



$a^T(b - xa) = 0$
$x a^T a = a^T b$
$x = \dfrac{a^T b}{a^T a} \Rightarrow p = a \dfrac{a^T b}{a^T a}$

$p = a \dfrac{a^T b}{a^T a} = Pb \Rightarrow P = \dfrac{a a^T}{a^T a}$
projection matrix

$C(P) =$ line through a
rank$(P) = 1$

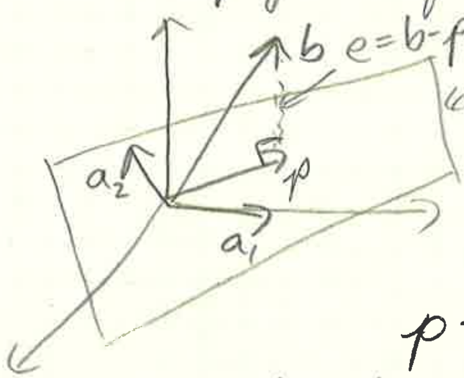
$\boxed{\begin{matrix} P^T = P \\ P^2 = P \end{matrix}}$ Two properties of a projection matrix

Why project?

Because $Ax=b$ may have no solution.

Solve $A\hat{x}=p$ instead

p ↘ projection of b onto column space



plane of a_1, a_2 ← basis $=$ column space of A

$$A=\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

e is perpendicular to plane

$$p=\hat{x}_1 a_1+\hat{x}_2 a_2 = A\hat{x}$$

$p=A\hat{x}$, find $\hat{x}$

Key: $b-A\hat{x} \perp$ plane

$a_1^T(b-A\hat{x})=0$ and $a_2^T(b-A\hat{x})=0$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b-A\hat{x})=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff A^T(\underbrace{b-A\hat{x}}_{e})=0$$

$$e\in N(A^T) \iff e\perp C(A)$$

$$\boxed{A^TA\hat{x}=A^Tb}$$

$$\hat{x}=(A^TA)^{-1}A^Tb$$

$$p=A\hat{x}=A(A^TA)^{-1}A^Tb = Pb$$

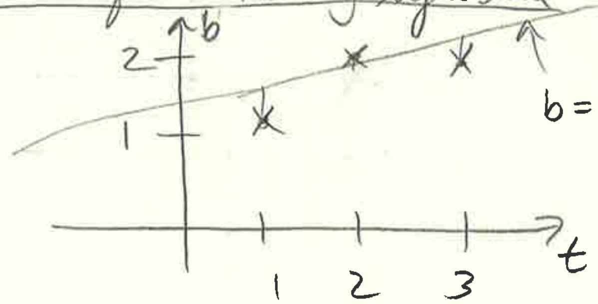
$$\boxed{P=A(A^TA)^{-1}A^T}$$ (A is not a square matrix)

If A is square, $C(A)=\mathbb{R}^n$ and $P=I$ and $b\in C(A)$

$P^T=P$

$$P^2=A\cancel{(A^TA)^{-1}A^TA}(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T=P$$

Least Squares Fitting by a line



$b=C+Dt$

$(1,1), (2,2), (3,2)$

$C+D=1$

$C+2D=2$

$C+3D=2$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x}=\underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

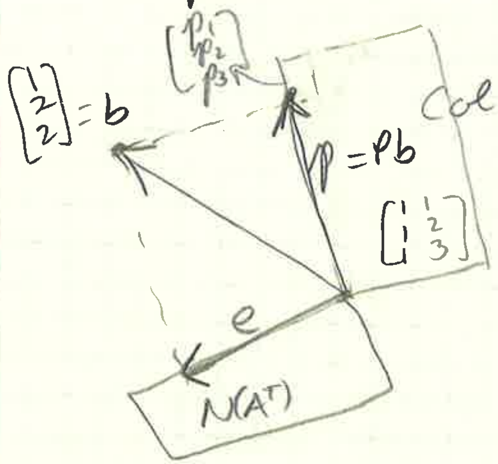
Lecture 16
- Projections
- Least Squares and best straight line

Projection matrix
$$P = A(A^TA)^{-1}A^T \qquad p = Pb$$

If b in column space, $Pb = b$ $\qquad b = Ax$

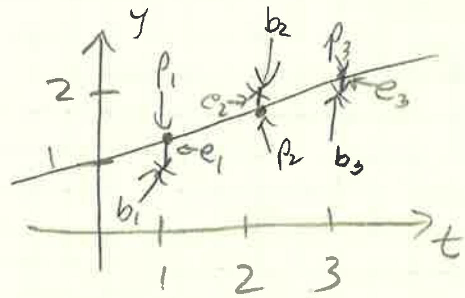
If $b \perp$ column space, $Pb = 0$ $\qquad (b \in N(A^T))$

$$p = A(A^TA)^{-1}A^TAx = Ax = b$$



$$b = p + e = (I-P)b$$
↑ Projection onto $\perp$ space

$$b = Pb + (I-P)b \Rightarrow (I-P)^T = I-P$$



$y = C + Dt$

$C + D = 1$

$C + 2D = 2$

$C + 3D = 2$

$Ax = b$
$$\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Want to minimize $\|Ax-b\|^2 = \|e\|^2$
$= e_1^2 + e_2^2 + e_3^2$

Find $\hat{x} = \begin{bmatrix}\hat{C}\\\hat{D}\end{bmatrix}$, p : $A^TA\hat{x} = A^Tb$ (Normal equations)

$$\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix} = \begin{bmatrix}3&6\\6&14\end{bmatrix} = A^TA \qquad A^Tb = \begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}5\\11\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3&6&5\\6&14&11\end{array}\right] \Leftrightarrow \begin{array}{l}3C+6D=5\\6C+14D=11\end{array}$$

$$e_1^2+e_2^2+e_3^2 = (C+D-1)^2+(C+2D-2)^2+(C+3D-2)^2 = \|e\|^2$$

$\frac{\partial\|e\|^2}{\partial D} = 0$ and $\frac{\partial\|e\|^2}{\partial C} = 0$ give the normal equations

$\left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & 14 & 11 \\ 0 & 2 & 1 \end{array}\right] \Rightarrow 2D=1 \Leftrightarrow D=\frac{1}{2}$

$3C+3=5 \Rightarrow C=\frac{2}{3}$

Best line is $y=\frac{2}{3}+\frac{1}{2}t$

$p_1 = \frac{2}{3}+\frac{1}{2} = \frac{7}{6} \Rightarrow e_1 = \frac{-1}{6}$

$p_2 = \frac{2}{3}+1 = \frac{5}{3} \Rightarrow e_2 = \frac{1}{3}$

$p_3 = \frac{2}{3}+\frac{3}{2} = \frac{13}{6} \Rightarrow e_3 = \frac{-1}{6}$ $\quad (p+e=b)$

$b=p+e$

$\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}7/6\\5/3\\13/6\end{bmatrix} + \begin{bmatrix}-1/6\\1/3\\-1/6\end{bmatrix}$

$p^Te=0$

$e \perp$ to column space

$e^T\begin{bmatrix}1\\1\\1\end{bmatrix}=0$, $e^T\begin{bmatrix}1\\2\\3\end{bmatrix}=0$

If A has independent columns, then A^TA is invertible

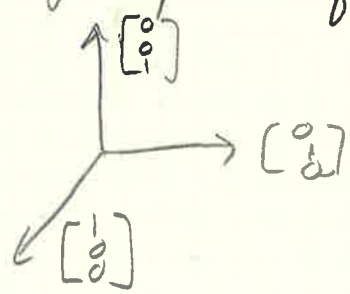
Proof: Suppose $A^TAx=0$, we want to show that x must be 0

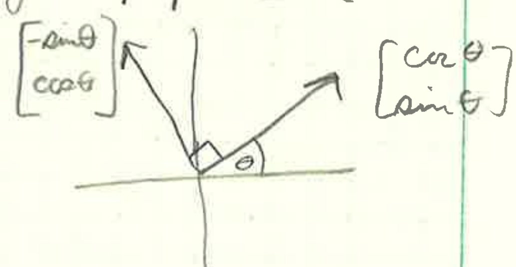
Idea: $x^TA^TAx=0 = (Ax)^TAx \overset{\text{square}}{\Rightarrow} Ax=0$

$\overset{\text{ind cols of } A}{\Rightarrow} x=0$

Columns are definitely independent if they are ~~perpendicular~~ unit vectors

orthonormal vectors





Lecture 17

- Orthogonal basis $q_1, \dots, q_n$
- Orthogonal matrix Q : square
- Gram - Schmidt $A \to Q$

Orthonormal vectors

$$q_i^T q_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\underset{m \times n}{Q} = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad Q^T Q = \begin{bmatrix} — q_1^T — \\ \vdots \\ — q_n^T — \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_1 \\ | & & | \end{bmatrix} = I$$

If Q is square, then $Q^TQ = I$ tells us that $Q^T = Q^{-1}$

Examples

Permutation matrix $\quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q^T} = I$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad Q = \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

Q has orthonormal columns, project onto its column space

$$P = Q(Q^TQ)^{-1}Q^T = QQ^T \quad (= I \text{ if } Q \text{ is square})$$

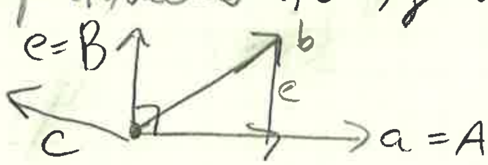
$$P^2 = Q\underbrace{Q^TQ}_{I}Q^T = QQ^T = P$$

$$A^TA\hat{x} = A^Tb$$

Now $A = Q \Rightarrow Q^TQ\hat{x} = Q^Tb \Leftrightarrow \hat{x} = Q^Tb \Rightarrow \hat{x}_i = q_i^T b$

Gram - Schmidt

(independent) Vectors $a, b \to$ get orthogonal vectors $A, B \to$ orthonormal vectors $q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|}$



$$B = b - \frac{A^Tb}{A^TA} A$$

Check $A \perp B$

$$A^TB = A^T\left(b - \frac{A^Tb}{A^TA} A\right) = 0$$

Independent vectors $a, b, c \rightarrow$ orthogonal vectors A, B, C
$\rightarrow$ orthogonal vectors $q_1 = \frac{A}{\|A\|}, q_2 = \frac{B}{\|B\|}, q_3 = \frac{C}{\|C\|}$

$$A = a$$
$$B = b - \frac{A^Tb}{A^TA}A$$
$$C = c - \underbrace{\frac{A^Tc}{A^TA}A}_{\text{comp in A dir}} - \underbrace{\frac{B^Tc}{B^TB}B}_{\text{comp in B dir}} \qquad C \perp A, \quad C \perp B$$

$$\underset{\substack{\| \\ A}}{a} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad b = \begin{bmatrix}1\\0\\2\end{bmatrix} \qquad B = \begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{3}\underset{A}{\begin{bmatrix}1\\1\\1\end{bmatrix}} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$Q = \begin{bmatrix}| & | \\ q_1 & q_2 \\ | & |\end{bmatrix} = \begin{bmatrix}1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2}\end{bmatrix}$$

$$A = \begin{bmatrix}1 & 1\\1 & 0\\1 & 2\end{bmatrix} \qquad C(A) = C(Q)$$

$$A = Q\underset{\text{upper triangular}}{R} \quad \text{(Gram-Schmidt)}$$

$$\begin{bmatrix}| & | \\ a & b \\ | & |\end{bmatrix} = \begin{bmatrix}| & | \\ q_1 & q_2 \\ | & |\end{bmatrix}\begin{bmatrix}a^Tq_1 & * \\ a^Tq_2 & *\end{bmatrix}$$

Lecture 18

- determinants $\det A = |A|$
- Properties 1, 2, 3, 4–10
- $\pm$ signs

Properties ① $\det I = 1$
② exchange rows: reverse the sign of det

$\det P = \pm 1$, even or odd exchanges of P - permutation matrix

① $\begin{vmatrix}1 & 0 \\ 0 & 1\end{vmatrix} = 1$, ② $\begin{vmatrix}0 & 1 \\ 1 & 0\end{vmatrix} = -1$

$$\begin{vmatrix}a & b \\ c & d\end{vmatrix} = ad - bc$$

Property (3a) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(3b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

$$\det(A+B) \neq \det A + \det B$$

det is linear for each row

(4) 2 equal rows $\Rightarrow \det = 0$
Exchange those rows $\rightarrow$ same matrix

(5) Subtract $l \times$ row i from row k $\rightarrow$ determinant doesn't change

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \overset{(3b)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

$$\overset{(3a)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{0 \; (4)}$$

(6) Row of zeros $\rightarrow \det A = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} \overset{(3b)}{=} \left. \begin{vmatrix} t\cdot a & t\cdot b \\ c & d \end{vmatrix} \right|_{t=0} = 0 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

(7) $U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ & & \ddots & * \\ 0 & & & d_n \end{bmatrix}$, $\det U = d_1 d_2 \cdots d_n$
(upper triangular) (product of pivots)

$\det U = d_n d_{n-1} \cdots d_2 d_1 \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ & & \ddots & & 0 \\ 0 & & & & 1 \end{vmatrix}$ after 5, 3a, 1, 2
elimination

(8) $\det A = 0$ exactly when A is singular $\rightarrow$ row of zero
$\Leftrightarrow \det A \neq 0$ when A is invertible $\rightarrow U \rightarrow \underbrace{D}_{\text{diagonal}} \rightarrow d_1 d_2 \cdots d_n$

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| \overset{(5)}{\rightarrow} \left| \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix} \right| = ad - \frac{c}{a} ab$$
$$= ad - bc$$

(9) $\det AB = (\det A)(\det B)$

$$A^{-1}A = I \Rightarrow \det A^{-1} = \frac{1}{\det A}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \quad \text{and } \det A \det A^{-1} = 6\left(\tfrac{1}{6}\right) = 1$$

Proved for diagonal matrices

$$\det A^2 = (\det A)^2$$

$$\det 2A = 2^n (\det A) \quad \text{(factor 2 from every row and use (3a))}$$

(10) $\det A^T = \det A$ $\Rightarrow$ properties about rows hold equally for columns

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Proof, using 1-9

$$|A^T| = |A|$$

$\downarrow$ elimination (in almost every case)

$$|U^T L^T| = |LU|$$

$$|U^T||L^T| = |L||U|$$

lower triangular with 1's on diagonal
upper " " " " "

$$\det L^T = 1, \quad \det L = 1 \quad \Rightarrow |U^T| = |U| \quad \square$$

Lecture 19 19.1

- Formula for det A (n! terms)
- Cofactor formula
- Tridiagonal Matrices

(1) $\det I = 1$
(2) Sign reverse with row exchange
(3) det is linear in each row separately

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{(3)}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

0 0
column of zeros

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad - \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} +$$

one row exchange — one row exchange

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{matrix}\text{determinants} \\ \text{with columns} \\ \text{of all zeros} \\ \| \\ 0\end{matrix}$$

2 exchanges — 2 exchanges — 1 exchange

$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The Big Formula

$$\det A = \sum_{n! \text{ terms}} \pm\, a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$$

($\pm$ = sign of permutation of columns)

$(\alpha, \beta, \gamma, \ldots, \omega)$ = permutation of $(1, 2, \ldots, n)$

<u>Example</u>

$$\det A = \begin{vmatrix} 0 & 0 & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & 0 \\ \underline{1} & \underline{1} & 0 & 0 \\ \underline{1} & 0 & 0 & \underline{1} \end{vmatrix}$$

column permutations
$\underline{1}$: $(4,3,2,1) \rightarrow +1$
$\underline{1}$: $(3,2,1,4) \rightarrow -1$

$\Rightarrow \det A = 0$

<u>Cofactors 3x3</u> (in parentheses)

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} \qquad \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Cofactor of $a_{ij} = \pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i, \text{ column } j \text{ erased} \end{pmatrix} = C_{ij}$

$+$ when $i+j$ is even
$-$ when $i+j$ is odd

Cofactor formula (along row 1)

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c)$$

Ex: Tridiagonal matrix

$$|A_4| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \qquad |A_1| = 1, \quad |A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$$|A_4| = 1 \cdot \underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}}_{= A_3} + 1\left(- \underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}}_{= 0} \right) = 1 \cdot |A_3| - 1 \cdot 1 \cdot |A_2| = -1$$

$$|A_n| = |A_{n-1}| - |A_{n-2}| \quad \Rightarrow |A_5| = -1 - (-1) = 0$$

$|A_6| = 0 - (-1) = 1$, $|A_7| = 1$ they have period 6 $\Rightarrow |A_{61}| = |A_1| = 1$

Lecture 20 20.1

1. Formula for A^{-1}
2. Cramer's Rule for $x = A^{-1}b$
3. $|\det A|$ = volume of box

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T$$

C ← matrix of cofactors (products of $n-1$ entries)

$\det A$ ← products of n entries

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}$$

Check $AA^{-1} = AC^T = (\det A) I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & & C_{n1} \\ C_{12} & & \vdots \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} = \det A & & 0 \\ & \det A & \\ 0 & & \sum_{i=1}^{n} a_{ni} C_{ni} = \det A \end{bmatrix} \checkmark$$

$$a_{i1} C_{j1} + a_{i2} C_{j2} + \cdots + a_{in} C_{jn} = 0, \quad i \neq j$$

Why?

When multiplying like this, it's like the matrix has 2 equal columns.

$Ax = b$

$$x = A^{-1} b = \frac{1}{\det A} C^T b$$

Cramer's Rule

$$x_1 = \frac{\det B_1}{\det A} \qquad B_1 = \begin{bmatrix} | & n-1 \\ b & \text{columns} \\ | & \text{of } A \end{bmatrix} = A \text{ with column 1 replaced by } b$$

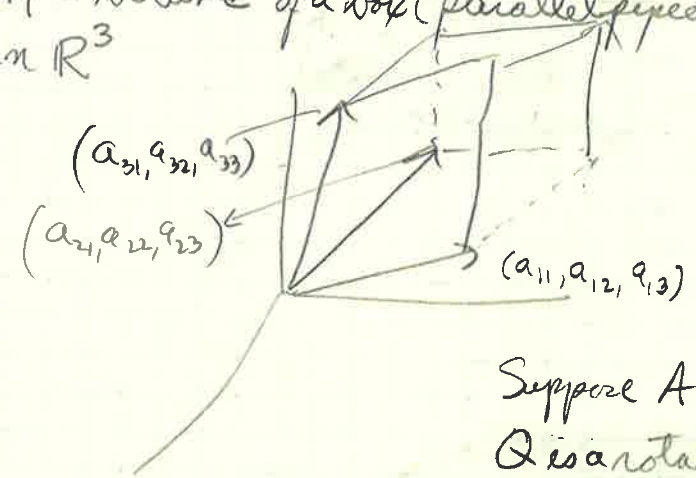
$$\vdots$$

$$x_j = \frac{\det B_j}{\det A} \qquad \det B_1 = b_1 C_{11} + b_2 C_{21} + \dots + b_n C_{n1}$$

$B_j =$ A with column j replaced by b

$|\det A|$ = volume of a box (parallelepiped)

In R^3



$A = I \Rightarrow$ the box is the unit cube and $\det I = 1$

Suppose $A = Q$ orthogonal matrix

Q is a rotated unit cube (volume $= 1$)

$\det(Q^TQ) = \det I = 1 \Rightarrow \det Q = \pm 1$

Since $\det Q^T = \det Q$

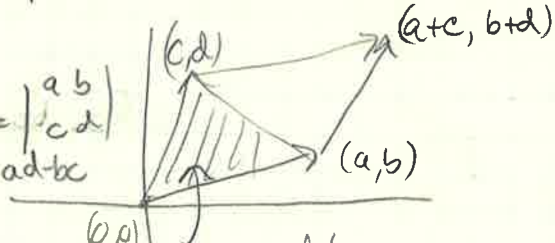
double row 1 ⇒ 2 identical boxes next to each other ⇒ volume doubles and the determinant also doubles

$|\det A|$ = volume of box has det properties 1, 2, 3a, 3b

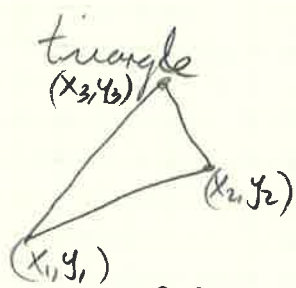
I, ±, t works too

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

area $= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $= ad - bc$



area $= \dfrac{ad-bc}{2}$



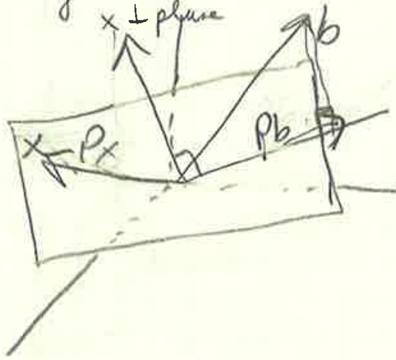
$$\text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Lecture 21
- Eigenvalues - Eigenvectors
- $\det[A-\lambda I]=0$
- Trace $= \lambda_1+\lambda_2+\dots+\lambda_n$

Ax parallel to x are the eigenvectors

$\boxed{Ax=\lambda x}$ $\quad\lambda$'s are eigenvalues

If A is singular, i.e. $Ax=0$ for some $x\neq 0$, then $\lambda=0$ is an eigenvalue.



What are the eigenvalues and eigenvectors for a projection matrix?

Any x in the plane is an eigenvector:

$$Px=x \Rightarrow \lambda=1$$

Any $x \perp$ to the plane is an eigenvector.

$$Px=0x \Rightarrow \lambda=0$$

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}\quad x=\begin{bmatrix}1\\1\end{bmatrix}\quad Ax=\begin{bmatrix}1\\1\end{bmatrix}=x \Rightarrow \lambda=1$$

The x's are $\perp$

$$x=\begin{bmatrix}-1\\1\end{bmatrix}\quad Ax=\begin{bmatrix}1\\-1\end{bmatrix}=-x \Rightarrow \lambda=-1$$

Fact: Sum of the eigenvalues equals $\operatorname{tr}(A)$.

How to solve $Ax=\lambda x$:

Rewrite: $(A-\lambda I)x=0 \Rightarrow A-\lambda I$ must be singular for $x\neq 0$

$\Rightarrow \det(A-\lambda I)=0$ (Characteristic eqn).

Find n λ's first:

$$A=\begin{bmatrix}3&1\\1&3\end{bmatrix}\qquad \det(A-\lambda I)=\begin{vmatrix}3-\lambda&1\\1&3-\lambda\end{vmatrix}=(3-\lambda)^2-1$$

$$(3-\lambda)^2-1=0 \Leftrightarrow \lambda^2-\underbrace{6}_{\operatorname{tr}A}\lambda+\underbrace{8}_{\det A}=0$$

$$(\lambda-4)(\lambda-2)=0 \Rightarrow \lambda_1=4,\ \lambda_2=2$$

$$(A-4I)x_1=\begin{bmatrix}-1&1\\1&-1\end{bmatrix}x_1=0 \Rightarrow x_1=\begin{bmatrix}1\\1\end{bmatrix},\ \lambda_1=4$$

$$(A-\lambda I)x_2=\begin{bmatrix}1&1\\1&1\end{bmatrix}x_2=0 \Rightarrow x_2=\begin{bmatrix}-1\\1\end{bmatrix},\ \lambda_2=2$$

How are $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ related?

Same eigenvectors, $\lambda_{21} = 3 + \lambda_{11}$, $\lambda_{22} = 3 + \lambda_{12}$

If we add kI to a matrix, its eigenvectors don't change but its eigenvalues increase by k.

If $Ax = \lambda x \Rightarrow (A + kI)x = \lambda x + kx = (\lambda + k)x$ (Great)

Not so great

If $Ax = \lambda x$, B has eigenvalues α. We have no reason to believe that x is an eigenvector of B.

$Ax = \lambda x$, $By = \alpha y$ $\Rightarrow$ $A+B$ or AB can't be combined to find eigenvectors given x & y.

Example
90° rotation

$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad$ $\operatorname{tr} Q = 0 = \lambda_1 + \lambda_2$
$\det Q = 1 \Rightarrow \lambda_1 \lambda_2$

$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \begin{matrix} \lambda_1 = i \\ \lambda_2 = -i \end{matrix}$

If λ is a complex eigenvalue $\Rightarrow$ λ and $\bar{\lambda}$ are a complex conjugate pair

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ eigenvalues of a $\triangle$ matrix are on the diagonal

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0 \Rightarrow \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 3 \end{matrix}$

$(A - \lambda I)x = 0 \Leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda_1 = 3$

repeated eigenvalues $\Rightarrow$ no 2nd independent eigenvector

Lecture 22

- Diagonalizing a matrix $S^{-1}AS=\Lambda$
- Powers of A / equation $u_{k+1}=Au_k$

$A-\lambda I$ singular $\Leftrightarrow Ax=\lambda x$

Suppose n independent eigenvectors of A. Put them in the columns of matrix S.

$$AS = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = S\Lambda$$

S eigenvector matrix, Λ eigenvalue matrix

$AS=S\Lambda$

n independent eigenvectors $\Rightarrow S^{-1}$ exists $\Rightarrow S^{-1}AS=\Lambda$

$\Leftrightarrow A = S\Lambda S^{-1}$

Ex: If $Ax=\lambda x$, $A^2x=\lambda Ax=\lambda^2 x$, squared eigenvalues, same e-vectors

$A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$. Then, since Λ is diagonal, the e-vals get squared.

$\Rightarrow A^k = \lambda^k x \Leftrightarrow A^k = S\Lambda^k S^{-1}$

Theorem

$A^k \to 0$ as $k\to\infty$ if all $|\lambda_i|<1$. (Needs n-indep eigenvectors)

A is sure to have n-independent eigenvectors (and be diagonalizable) if all λ's are distinct (no repeated eigenvalues).

Repeated eigenvalues $\Rightarrow$ may or may not have n-independent eigenvectors

$$A=\begin{bmatrix}2&1\\0&2\end{bmatrix} \Rightarrow \lambda_1=\lambda_2=2 \quad \text{(algebraic multiplicity = 2)}$$

$$A-2I=\begin{bmatrix}0&1\\0&0\end{bmatrix} \Rightarrow \begin{bmatrix}0&1\\0&0\end{bmatrix}x=0 \Rightarrow x_1=\begin{bmatrix}1\\0\end{bmatrix} \quad \text{(geometric multiplicity = 1)}$$

Equation Start with given u_0. solve $u_{k+1} = Au_k$

$$u_1 = Au_0 \,,\quad u_2 = Au_1 = A^2 u_0 \Rightarrow \boxed{u_k = A^k u_0}$$

To really solve, write $u_0 = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = Sc$

$$Au_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_n \lambda_n x_n$$

$$\Rightarrow A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \cdots + c_n \lambda_n^{100} x_n \quad (= u_{100})$$

$$= S \Lambda^{100} c$$

Fibonacci example: $\overset{F_0}{0}, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots, \boxed{F_{100} = ?}$

$$F_{k+2} = F_{k+1} + F_k \quad (2^{nd} \text{ order})$$

Trick $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \Rightarrow \begin{array}{l} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{array}$

$$\Rightarrow u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Au_k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = 0$$

$$\Leftrightarrow \underbrace{\lambda^2 - \lambda - 1 = 0}_{\substack{\text{same recursion} \\ \text{as in original} \\ F_{k+2} - F_{k+1} - F_k = 0}} \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow \begin{array}{l} \lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.618\ldots \\ \lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -.618\ldots \end{array}$$

$$F_{100} \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

Fibonacci numbers grow at an approximate rate of λ_1

$$F_{100} = c_1 \lambda_1^{100} x_1 + c_2 \underset{\uparrow \text{small}}{\lambda_2^{100}} x_2$$

$$(A - \lambda I)x = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \\ x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \end{array}$$

$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives c_1 and c_2

Lecture 23
- Differential Eqns $\frac{du}{dt} = Au$
- Exponential e^{At} of a matrix

Example

$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad \frac{du_2}{dt} = u_1 - 2u_2 \Rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}, \quad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda_1 = 0$ (singular matrix)
$\lambda_2 = -3$

$\lambda_1 = 0$: $A x_1 = 0 x_1 \Rightarrow x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = -3$ $(A + 3I)x_2 = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ $\quad (c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$ for difference eqn$)$

Check: $\frac{du}{dt} = Au$, plug in $e^{\lambda_1 t} x_1 \Rightarrow \lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$

$u(t) = c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}_{S} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \Rightarrow c_1 = 1/3, \; c_2 = 1/3$

$u(t) = \frac{1}{3}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

Steady state: $u(\infty) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

1. When do we get stability, $u(t) = 0$? Need $e^{\lambda t} \to 0$, $\operatorname{Re} \lambda < 0$
   $|e^{(-3+6i)t}| = e^{-3t}$, $|e^{6it}| = 1$
2. Steady state, $u(\infty) = c$: $\lambda_1 = 0$ and other $\operatorname{Re} \lambda < 0$
3. Blow up if $\operatorname{Re} \lambda_i > 0$ for any i

2X2 stability: $\operatorname{Re} \lambda_1 < 0$, $\operatorname{Re} \lambda_2 < 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\operatorname{tr} A = a + d = \lambda_1 + \lambda_2 < 0$
$\det A = \lambda_1 \lambda_2 > 0$

trace < 0
but still blow up
$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 1$, $\lambda_2 = -2$

$\frac{du}{dt} = Au$ Set $u = Sv$ ← eigenvector matrix

$S\frac{dv}{dt} = ASv \iff \frac{dv}{dt} = S^{-1}ASv = \Lambda v$

$$\frac{dv_1}{dt} = \lambda_1 v_1$$
$$\vdots$$
$$\frac{dv_n}{dt} = \lambda_n v_n$$

$\Rightarrow v(t) = e^{\Lambda t}v(0)$

$\boxed{u(t) = Se^{\Lambda t}S^{-1}u(0)} = e^{At}u_0$

$e^{At} = Se^{\Lambda t}S^{-1}$

Matrix exponential $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots + \frac{(At)^n}{n!} + \dots$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \dots$ for good approximation when t is small

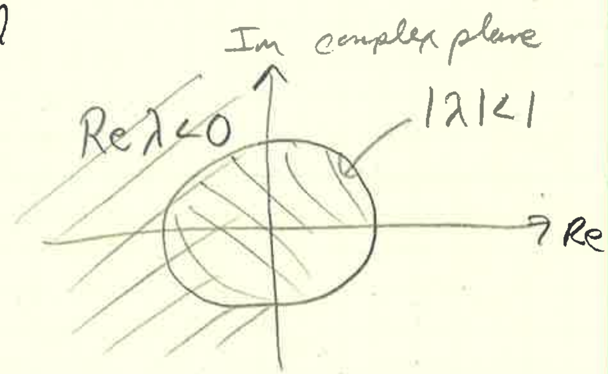
all $|\lambda_i| < 1$; eigenvalues of At

$e^{At} = SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2!} + \dots + \frac{S\Lambda^n S^{-1}t^n}{n!} + \dots = Se^{\Lambda t}S^{-1}$

only if A can be diagonalized

$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots\; e^{\lambda_n t} \end{bmatrix}$$



$y'' + by' + ky = 0$ 1 - 2nd order eqn → 2x2 1st order system

$u = \begin{bmatrix} y' \\ y \end{bmatrix} \Rightarrow u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y' \\ y \end{bmatrix}$

$$\begin{bmatrix} - & - & - & - & - \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$$ ⇒ 5th order to 5x5 1st order

Lecture 24

- Markov Matrices
  Steady State : $\lambda = 1$
- Fourier Series and Projections

Markov Matrices

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

① all entries ≥ 0
② all columns sum to 1

$$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n$$

as $k \to \infty$, $u(\infty) = c_1 x_1$ since $\lambda_1 = 1$ is the steady state

1. $\lambda = 1$ is an eigenvalue
2. All other $|\lambda_i| < 1$
3. all components of $x_1 \geq 0$

$$A - 1I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$$

This is singular : row 2 + row 3 + row 1 = 0
or all columns sum to 0.

$A - I$ is singular $\Leftrightarrow$ the rows are dependent $\Leftrightarrow$ $(1,1,1) \in N(A^T)$
$\Leftrightarrow$ $x_1 \in N(A)$

eigenvalues of A
eigenvalues of A^T } are the same

$$\det(A - \lambda I) = 0$$
$$\det(A - \lambda I) = \det(A - \lambda I)^T$$
$$= \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$$

$$\begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix} \begin{bmatrix} .6 \\ 33 \\ .7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$u_{k+1} = A u_k$, A is Markov

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{array}{c} \text{stay in Cal} \quad \text{Mass} \to \text{Cal} \\ \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \\ \text{cal} \to \text{mass} \quad \text{stay in Mass} \end{array} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k}$$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

$$\begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix} \Rightarrow \lambda_1 = 1, \; \lambda_2 = .7 \quad (\text{trace} = 1.7)$$

$$(A - I)x_1 = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Steady state:
$$u(\infty) = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A - .7I)x_2 = \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $c_1 = \frac{1000}{3}$, $c_2 = \frac{2000}{3}$

Projections with orthonormal basis

$q_1, \dots, q_n$ (orthonormal),

Any $v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n \rightarrow \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v \iff Qx = v$

$\iff x = Q^{-1} v$

$q_1^T v = x_1 \underbrace{q_1^T q_1}_{=1} + 0 + 0 + \dots + 0 \Rightarrow x_1 = q_1^T v$ $\iff x = Q^T v$

Fourier Series

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$ Fourier series

basis $= \{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$ (orthogonal functions)

for vectors: $v^T w = v_1 w_1 + \dots + v_n w_n$

for functions: $\langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx$ $f(x) = f(x + 2\pi)$ periodic

$\int_0^{2\pi} \sin x \cos x\, dx = \frac{1}{2} \sin^2 x \Big|_0^{2\pi} = 0$

a_1: $\int_0^{2\pi} f(x) \cos x\, dx = \int_0^{2\pi} a_0 \cos x\, dx + \underbrace{\int_0^{2\pi} a_1 \cos^2 x\, dx}_{\pi} + \int_0^{2\pi} b_1 \cos x \sin x\, dx + \dots$

$\iff a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x\, dx$

Lecture 24b

Review for Quiz 2

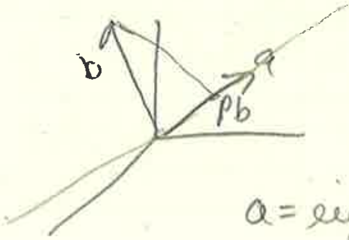
① $Q = [q_1 \cdots q_n]$
Projections - Least Squares
Gram-Schmidt
$Q^T Q = I$

② det A
properties 1-3
Big formula
($n!$ terms $\pm$)
Cofactors / A^{-1}

③ Eigenvalues
$Ax = \lambda x$
$\det(A - \lambda I) = 0$
Diagonalize $S^{-1} A S = \Lambda$
Powers A^k

Q1) $a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $P = A(A^T A)^{-1} A^T$

$= \frac{a a^T}{a^T a} = \frac{1}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$



rank $P = 1$ $\lambda = 0, 0, \frac{9}{9} = 1$

$C(P) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

a = eigenvector for $\lambda = 1 \iff Pa = a$

Solve $u_{k+1} = P u_k$, $u_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$

$$u_1 = P u_0 = a \frac{a^T u_0}{a^T a} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \frac{1}{9} \cdot 27 = 3a = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

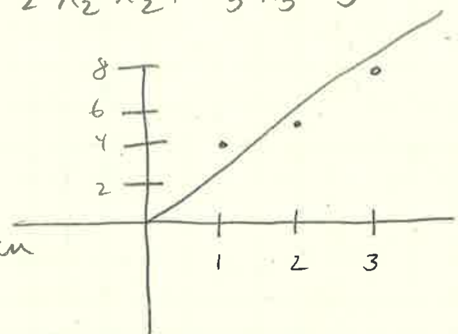
$\hat{x}$

$$u_k = P^k u_0 = P u_0 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$u_0 = c_1 x_1 + c_2 x_2 + c_3 x_3$$
$$A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + c_3 \lambda_3^k x_3$$

Q2) $t = 1$, $y = 4$
$t = 2$ $y = 5$
$t = 3$ $y = 8$

Fit line through origin



$y = Dt$
$1 \cdot D = 4$
$2 \cdot D = 5$
$3 \cdot D = 8$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$
$$A \hat{x} = b$$

Best D

$$A^T A \hat{D} = A^T b$$
$$14 \hat{D} = 38 \Rightarrow \hat{D} = 38/14$$

Projecting b onto C(A) (line)

Q3) $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\to$ plane = C(A)
Find 2 orthogonal vectors

$$B \perp a_1 : \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{a_1^T a_2}{a_1^T a_1} a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Q4) 4×4 matrix $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. What are conditions on λ_i for matrix to be invertible?

a) Invertible $\Leftrightarrow$ no zero eigenvalues

(b) $\det A^{-1} = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_4}\right)$

(c) $tr(A+I) = (\lambda_1+1) + (\lambda_2+1) + (\lambda_3+1) + (\lambda_4+1) = \lambda_1+\lambda_2+\lambda_3+\lambda_4+4$

Q5) $A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $D_n := \det A_n$

D_{n-1}

Use cofactors:

$$D_n = \frac{1}{}D_{n-1} + \frac{-1}{}D_{n-2} = D_{n-1} - D_{n-2}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ \boxed{1} & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

D_{n-2}

Solve $D_n = D_{n-1} - D_{n-2}$

$D_1 = 1$ $\qquad D_1 = [1]$

$D_2 = 0$ $\qquad D_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$

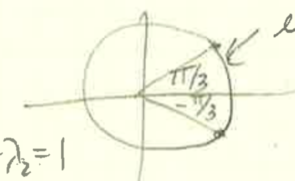
$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0 \qquad \lambda = e^{\pm i\pi/3}$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow \lambda_1 = \frac{1+i\sqrt{3}}{2}, \; \lambda_2 = \frac{1-i\sqrt{3}}{2}$$

$$|\lambda| = \frac{1}{4} + \frac{3}{4} = 1$$

unit circle $|\lambda| = 1$



Given $\lambda_1^6 = \lambda_2^6 = 1 \Rightarrow A^6$ has $\lambda_1 = \lambda_2 = 1$

$\Rightarrow A^6 = I$

Q6)

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} = A_4^T \qquad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \text{ (singular)}$$

$$P_3 = A_3(A_3^T A_3)^{-1} A_3^T$$

$$|A_3 - \lambda I| = \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{bmatrix} = -\lambda^3 + 5\lambda = 0 \Leftrightarrow \lambda(-\lambda^2 + 5) = 0$$

$$\lambda_1 = 0, \; \lambda_2 = \sqrt{5}, \; \lambda_3 = -\sqrt{5}$$

Find P_4: $\quad P_4 = I$ if $C(A_4) = \mathbb{R}^4$

$$\det A_4 = -1 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{vmatrix} = -1 \cdot 1 \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = 9 \Rightarrow P_4 = I$$

Probably A_n are $\begin{cases} \text{invertible}, & n \text{ even} \\ \text{singular}, & n \text{ odd} \end{cases}$

Lecture 25:

- Symmetric Matrices (real)
  Eigenvalues / Eigenvectors
- Start: Positive definite matrices

$A = A^T$ (real)

① The eigenvalues are real.

② The eigenvectors are perpendicular
can be chosen → leads to orthonormal eigenvectors
$\Rightarrow$ columns of Q

Usual case: $A = S\Lambda S^{-1}$

Symmetric case: $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ - spectral theorem

Why real eigenvalues?

$Ax = \lambda x \xRightarrow{\text{always}} \bar{A}\bar{x} = \bar{\lambda}\bar{x}$ complex conjugate

$= A\bar{x} = \bar{\lambda}\bar{x}$ for real matrix

$\Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda} \Leftrightarrow \bar{x}^T A = \bar{x}^T \bar{\lambda}$

$\bar{x}^T A x = \lambda \bar{x}^T x$ and $\bar{x}^T A x = \bar{x}^T \bar{\lambda} x$

$\Rightarrow \lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x \Leftrightarrow \lambda = \bar{\lambda}$ provided $\bar{x}^T x \neq 0$

$\Leftrightarrow \lambda$ is real

$$\bar{x}^T x \Rightarrow \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n$$

$= \|x\|^2$ = length squared

Good matrices: Real λ's and $\perp$ x's

$A = A^T$ if real

$A = \bar{A}^T$ if complex (conjugate transpose)

$A = Q\Lambda Q^T \quad (A = A^T)$

$$= \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} = \lambda_1 \underbrace{q_1 q_1^T}_{\text{projection matrix}} + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

- Every symmetric matrix is a combo of mutually $\perp$ projection matrices
- Signs of pivots (for $A=A^T$) are same as the signs of the λ's
  $\Leftrightarrow$ # positive pivots = # positive λ's
- product of pivots ($A=A^T$) = product of λ's.

Positive definite symmetric matrix

- All λ's are positive
- All pivots are positive

$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$ pivots are $5, \frac{11}{5}$ since det = 11

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 8 + 11 = 0 \Rightarrow \lambda = 4 \pm \sqrt{5}$$

- All subdeterminants are positive

Lecture 26 26.1

Complex vectors, matrices
inner products
Discrete Fourier (FAST) transform = FFT $O(n \log_2 n)$
Fourier matrix F_n

length

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n \qquad z^T z \text{ is no good} \qquad [1 \ i]\begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

$$\bar{z}^T z = [\bar{z}_1 \ \bar{z}_2 \ \cdots \ \bar{z}_n]\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \Rightarrow [1 \ -i]\begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$z^H = \bar{z}^T \quad \text{(Hermitian)}$$

Inner product is $z^H z = |z_1|^2 + \cdots + |z_n|^2$

Symmetric : $A^T = A$ no good if A is complex

Hermitian $A^H = A$ if A is complex

$$A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} = \bar{A}^T = A^H$$

Perpendicular

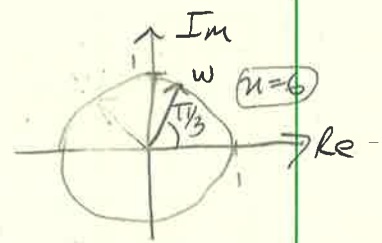
$q_1, q_2, \ldots, q_n \Rightarrow q_i^H q_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \qquad Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$

$Q^H Q = I$ unitary (orthogonal complex matrix)

$n \times n$ Fourier matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2n-1} & \cdots & w^{(n-1)^2} \end{bmatrix} \qquad (F_n)_{ij} = w^{ij} \qquad i,j = 0, \dots, n-1$$

$$w^n = 1$$
$$w = e^{i\,2\pi/n}$$



$\underline{n=4}$

$$w^4 = 1 \Leftrightarrow w = e^{i\,2\pi/4} = e^{i\pi/2} = i$$
$$w = i,\ w^2 = i^2 = -1,\ w^3 = -i,\ w^4 = 1$$

$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \\ \begin{array}{cccc} \quad & a & \quad & b \end{array} \end{array} \qquad a^H b = [1\ -i\ -1\ i]\begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} = 1 - 1 + 1 - 1 = 0$$

All columns are orthonormal

$$F_4^H F_4 = I$$

$$w_n = e^{i\,2\pi/n} \Rightarrow w_{64}^2 = w_{32} = e^{i\,2\pi/32}$$

$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix}\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}\begin{bmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{bmatrix} \leftarrow \text{Permutation matrix} = P$$

even

odd

$\hookrightarrow 64^2$ calcs $\qquad \hookrightarrow [2(32^2) + 32]$ calcs

$$D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \\ & & & & w^{31} \end{bmatrix}$$

$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix}\begin{bmatrix} \begin{matrix} I & D \\ I & -D \end{matrix} & 0 \\ 0 & \begin{matrix} I & D \\ I & -D \end{matrix} \end{bmatrix}\begin{bmatrix} F_{16} & & & \\ & F_{16} & & \\ & & F_{16} & \\ & & & F_{16} \end{bmatrix}\begin{bmatrix} P & \\ & P \end{bmatrix}\begin{bmatrix} P_{64} \end{bmatrix}$$

$$2[2(16)^2 + 16] + 32 \rightarrow 6 \times 32 \leftarrow \log_2 64 \cdot 64/2$$

$$\text{calcs} = \frac{1}{2} n \log_2 n$$

$$n = 1024 = 2^{10}, \quad n^2 > 10^6$$
$$\frac{1}{2} n \log_2 n = \frac{1}{2}(1024)(10) = 5 \cdot 1024$$

Lecture 27

- Positive Definite Matrix (Tests)
- Tests for Minimum $(x^TAx)>0$
- Ellipsoids in R^n

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

① $\lambda_1>0, \lambda_2>0$
② dets of principal minors >0
③ pivots $a>0$, $\frac{ac-b}{a}>0$
④ $x^TAx>0$ (new) $\forall x \neq 0$

} complete tests for positive defeteness

Examples

$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ positive semidefinite : $\lambda=0, 20$
eigenvalues ≥ 0

Pivots: 2

$[x_1 \; x_2]\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \; x_2]\begin{bmatrix} 2x_1+6x_2 \\ 6x_1+18x_2 \end{bmatrix} = 2x_1^2+12x_1x_2+18x_2^2$

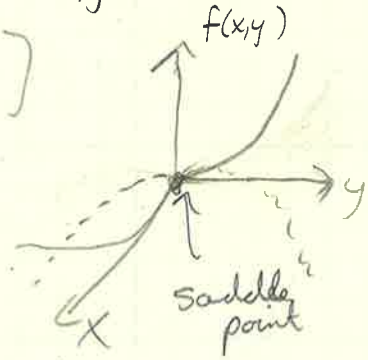
$ax^2+2bxy+cy^2 \overset{?}{>} 0$

quadratic form

Graph of $f(x,y)=\vec{x}^TA\vec{x} = ax^2+2bxy+cy^2$

$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

Indefinite



$2x^2+12xy+7y^2$

1st derivatives $=0$

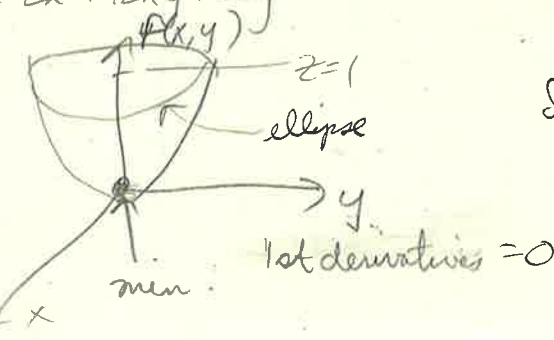
$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \Rightarrow 2x^2+12x_1x_2+20x_2^2$ (positive definite)

det $=4$, trace $=22$ $\Rightarrow \lambda_1, \lambda_2>0$

$x^TAx>0$ except at $x=0$

$f(x,y)=2x^2+12xy+20y^2$

paraboloid



Calculus: Min $\sim \frac{d^2u}{dx^2}>0$
$\sim \frac{du}{dx}=0$

In linear algebra:
MIN: $f(x_1,x_2,\ldots,x_n)$
matrix of 2nd derivatives is positive definite

$f(x,y) = 2x^2 + 12xy + 20y^2$

$= 2(x+3y)^2 + 2y^2 > 0 \quad \forall x,y \neq 0$ since it is all squares

pivot, multiplier, pivots

If we cut this at $z=1$, the curve is an ellipse

$$\overset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}} \longrightarrow \overset{U}{\begin{bmatrix} \boxed{2} & 6 \\ 0 & \boxed{2} \end{bmatrix}}, \quad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Matrix of 2nd derivatives (2x2)

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad f_{xy} = f_{yx}$$

3x3 example

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

dets are $2, 3, 4$

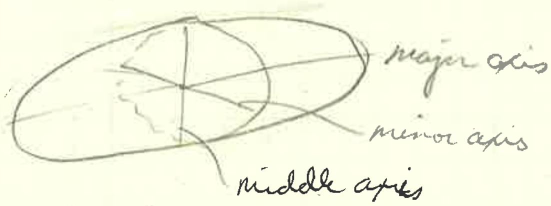
pivots $2, \frac{3}{2}, \frac{4}{3}$ product of pivots = det

eigenvalues are all > 0 $\quad 2-\sqrt{2}, 2, 2+\sqrt{2}$

$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

Cut through at $f(x_1, x_2, x_3) = 1$ to get an ellipsoid

$A = Q \Lambda Q^T$



length = eigenvalues
directions = eigenvectors

If two eigenvalues are equal, the minor and middle axes have equal length

<u>Lecture 28:</u> 28.1

- A^TA is positive definite!
- Similar Matrices A, B $\quad B = M^{-1}AM$
- Jordan Form

If A, B are positive definite, what about $A+B$? Yes

$x^T(A+B)x = x^TAx + x^TBx > 0$

Now, $A_{m \times n}$: A^TA (rank $A^TA = n$)

$x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2 > 0, \ \forall x \neq 0$

square, symmetric, pos def

A and B are similar (not necessarily symmetric), $n \times n$
means: for some M, $B = M^{-1}AM$

Example: A is similar to Λ

$S^{-1}AS = \Lambda$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{M} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \overbrace{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}^{B}$$

Similar matrices have the same eigenvalues and same # of eigenvectors
(but needs more to complete the picture)

$\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$ are in the family.

$Ax = \lambda x$ $(B = M^{-1}AM)$

$AMM^{-1}x = \lambda x \iff M^{-1}AMM^{-1}x = \lambda M^{-1}x$

$\iff BM^{-1}x = \lambda M^{-1}x \Rightarrow \lambda$ is an eigenvalue of B

eigenvector of B is M^{-1}(eigenvector of A)

BAD CASE $\lambda_1 = \lambda_2 = 4$.

One family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ which is not similar to $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$
(only member)

Big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ (all others)

$$M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}M = 4M^{-1}IM = 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ is (nondiagonalizable) called the Jordan Form

More members of family (same trace, same det)

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$, $\begin{bmatrix} a & \#\# \\ \# & 8-a \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; all $\lambda = 0$, rank $= 2$, dim Nullspace $= 2 =$ # of indep eigenvectors

$$\begin{bmatrix} 0&1&7&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \lambda=0,0,0,0, \text{ rank } 2, \ldots \text{ similar to } \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 0&1&0&0\\ 0&0&0&0\\ \hline 0&0&0&1\\ 0&0&0&0 \end{array}\right], \lambda=0,0,0,0, \text{ rank}=2, \text{ has 2 eigenvectors but not similar to } \left[\begin{array}{ccc|c} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ \hline 0&0&0&0 \end{array}\right]$$

(different Jordan blocks)

Jordan block (has 1 eigenvector only)

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Every square matrix A is similar to a Jordan matrix J

$$J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_2 \end{bmatrix}$$ # blocks = # of eigenvectors

Good Case: J is Λ

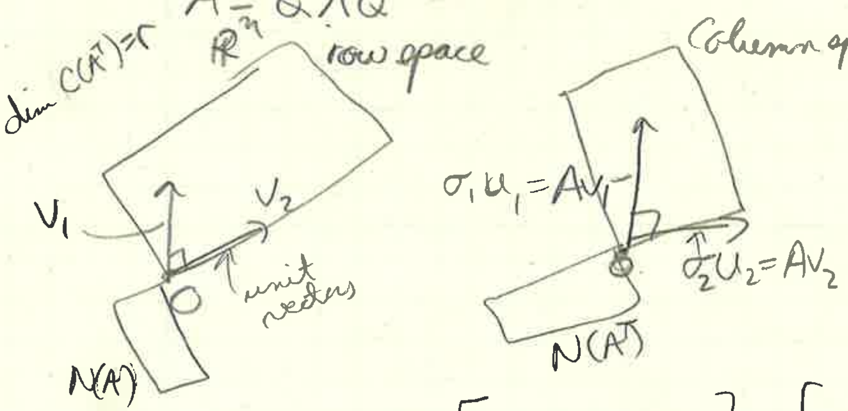
Lecture 29: 29.1

- Singular Value Decomposition (SVD) $A = U\Sigma V^T$

Σ - diagonal, U,V - orthogonal

Symmetric positive definite:

$A = Q\Lambda Q^T$



Orthogonal basis in row space → orthogonal bases in column space

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r & u_{r+1} & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ 0 & & & & \ddots \; 0 \end{bmatrix}$$

$AV = U\Sigma$

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

v_1, v_2 orthonormal in row space $= \mathbb{R}^2$
u_1, u_2 in col space $= \mathbb{R}^2$
$\sigma_1 > 0,\ \sigma > 0$

$Av_1 = \sigma_1 u_1$
$Av_2 = \sigma_2 u_2$

$$AV = U\Sigma \Rightarrow A = U\Sigma V^{-1} = U \Sigma V^T$$

$$A^TA = V\Sigma^T \underbrace{U^TU}_{I} \Sigma V^T$$

$$= V\Sigma^T\Sigma V^T = V\Sigma^2 V^T = V\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix}V^T$$

V are the eigenvectors of A^TA, Σ^2 are the eigenvalues

$$AA^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma^2 U^T$$

U are the eigenvectors of AA^T, Σ^2 are the eigenvalues

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\underbrace{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$ eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 32\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 18\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \overset{U}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\overset{\sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}}\overset{V^T}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}$$

To finish sign error

Find U

$$AA^T = U\Sigma \underbrace{V^TV}_{I}\Sigma^T U^T = U\Sigma^2 U^T$$

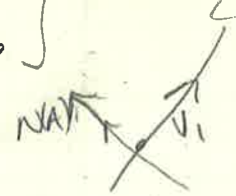
AA^T: Sym positive semidefinite

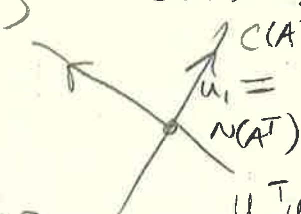
$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad C(A^T) = \left\{k\begin{bmatrix} 4 \\ 3 \end{bmatrix} \middle| k \in \mathbb{R}\right\} \qquad C(A) = \left\{k\begin{bmatrix} 4 \\ 8 \end{bmatrix} \middle| k \in \mathbb{R}\right\}$$





$u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \qquad v_2^Tv_1 = 0 \Rightarrow v_2 = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$$

$u_2^Tu_1 = 0$
$\Rightarrow u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\underset{A}{\begin{bmatrix}4&3\\8&6\end{bmatrix}} = \frac{1}{\sqrt{5}}\underset{U}{\begin{bmatrix}1&2\\2&-1\end{bmatrix}}\underset{\Sigma}{\begin{bmatrix}\sqrt{125}&0\\0&0\end{bmatrix}}\underset{V^T}{\begin{bmatrix}4/5&3/5\\3/5&-4/5\end{bmatrix}}$$

$$A^TA = \begin{bmatrix}4&8\\3&6\end{bmatrix}\begin{bmatrix}4&3\\8&6\end{bmatrix} = \begin{bmatrix}80&60\\60&45\end{bmatrix}$$

$\lambda_1 = 0, \ \lambda_2 = 125$

dim

r $\{v_1, \ldots v_r\}$ – orthonormal basis for the row space

r $\{u_1, \ldots, u_r\}$ – " " " " col "

$n-r$ $\{v_{r+1}, \ldots, v_n\}$ – " " " " null " of A

$m-r$ $\{u_{r+1}, \ldots, u_m\}$ – " " " " null " of A^T

and $Av_i = \sigma_i u_i$

Lecture 30: 30.1

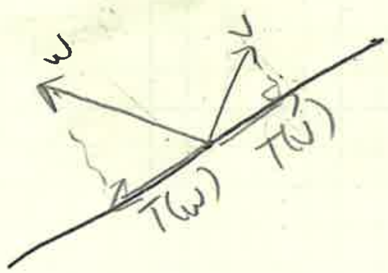
Linear Transformations T
Without coordinates: no matrix
With coordinates: matrix

$$\left.\begin{aligned} T(v+w) &= T(v) + T(w) \\ T(cv) &= cT(v)\end{aligned}\right\} \quad T(cv+dw) = cT(v) + dT(w)$$

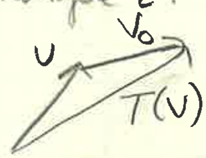
$T(0) = 0$

Example 1: Projection

$T: \mathbb{R}^2 \to \mathbb{R}^2$



(Non) Example 2: Shift whole plane by v_0

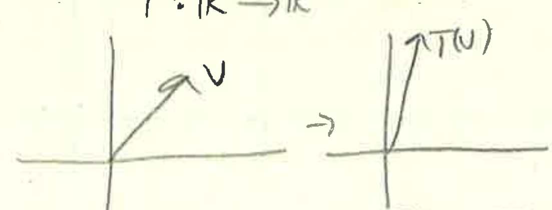


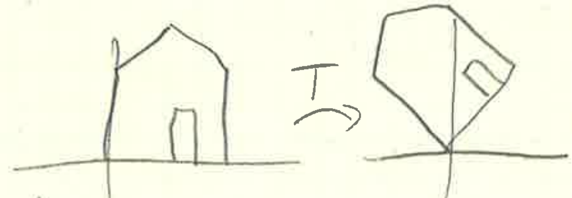
$T(v) = v + v_0$
This is not linear

(Non) Example 3: $T(v) = \|v\|$
$T: \mathbb{R}^3 \to \mathbb{R}^1$ This is not linear

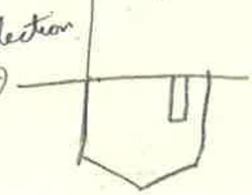
Example 4: Rotation by 45°

$T: \mathbb{R}^2 \to \mathbb{R}^2$





$A = \begin{bmatrix}1&0\\0&-1\end{bmatrix} \xrightarrow{\text{reflection}}$



Example 5: Matrix A

$T(v) = Av$

$A(v+w) = Av + Aw$, $A(cv) = cAv$ ✓ linear

Start: $T: \mathbb{R}^3 \to \mathbb{R}^2$

Example $\underbrace{T(v)}_{\text{in } \mathbb{R}^2} = A\underbrace{v}_{\text{in } \mathbb{R}^3}$, $A_{2\times3}$

Information needed to know $T(v)$ for all inputs

$T(v_1), T(v_2), \ldots, T(v_n)$ for any input basis $\{v_1, \ldots, v_n\}$

Every $v = c_1 v_1 + \cdots + c_n v_n \Rightarrow T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$

Coordinates come from a basis. The coordinates of v are the c's:

$T(v) = c_1 v_1 + \cdots + c_n v_n$

$$v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

coordinates for standard basis

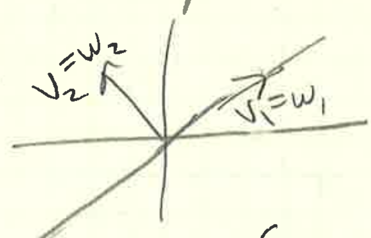
Construct matrix A that represents the linear transformation T

$T: \mathbb{R}^n \to \mathbb{R}^m$. Choose a basis $v_1, \ldots, v_n$ for the inputs in $\mathbb{R}^n$

Choose a basis $w_1, \ldots, w_m$ for the outputs in $\mathbb{R}^m$.

Want matrix A

Projection Example



$v = c_1 v_1 + c_2 v_2$

$T(v) = c_1 T(v_1) + c_2 T(v_2)$

$= c_1 v_1 + c_2 \cdot 0 = c_1 v_1$

$(c_1, c_2) \to (c_1, 0)$

$$Ac = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{input coords}} = \underbrace{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}_{\text{output coords}}$$

eigenvector basis leads to diagonal matrix Λ

Projecting onto 45° line, use standard basis $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$

matrix is $\dfrac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Rule to find A. Given bases $v_1, \dots, v_n$ + $w_1, \dots, w_m$

1st column of A : Write $T(v_1) = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$

2nd column of A : $T(v_2) = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m$

$$A\begin{pmatrix}\text{input}\\ \text{coords}\end{pmatrix} = \begin{pmatrix}\text{output}\\ \text{coords}\end{pmatrix}$$

$T = \frac{d}{dx}$ linear

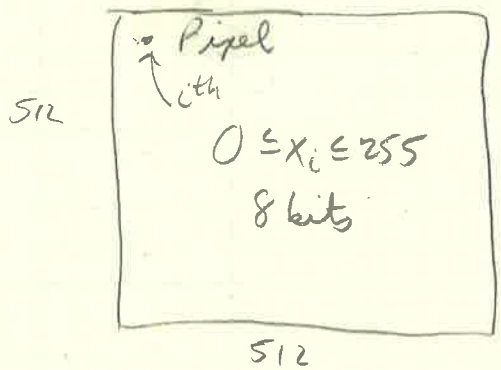
Input space : $c_1 + c_2 x + c_3 x^2$, basis $= \{1, x, x^2\}$

Output $= c_2 + 2c_3 x$, basis $= \{1, x\}$

$$A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Lecture 31 31.1

- Change of Basis
- Compression of Images
- Transformation $\Longleftrightarrow$ Matrix



$x \in \mathbb{R}^n$, $n = 512^2$

JPEG

$$x = \begin{bmatrix} \vdots \\ 121 \\ 124 \\ \vdots \end{bmatrix}$$

Standard basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Better Basis

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$$

Fourier Basis (JPEG u..) 8x8

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ w \\ w^2 \\ \vdots \\ w^{n-1} \end{bmatrix}$$

Signal p

(lossless) $\downarrow$ change basis

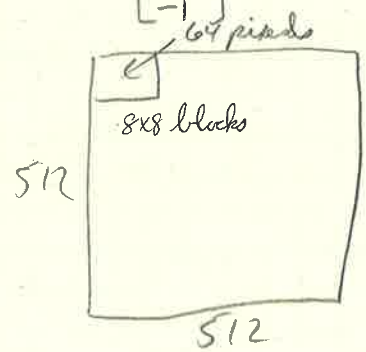
coeffs c

(lossy) $\downarrow$ Compression

$\hat{c}$ (many zeros)

$\downarrow$ reconstruct signal

$\hat{p} = \sum \hat{c}_i v_i$, $i \ll 64$



Video

Sequence of images - highly correlated

Wavelets in $\mathbb{R}^8$

$$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix}\begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}$$

$w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8$ ← orthogonal

standard basis $\begin{bmatrix}p_1\\p_2\\ \vdots\\p_8\end{bmatrix}$

$$p = c_1 w_1 + c_2 w_2 + \cdots + c_8 w_8$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_8 \\ | & | & & | \end{bmatrix}}_{W} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}$$

$$p = Wc$$

$$c = W^{-1}p$$

Good basis

① Fast in multiplication and inverse
FFT and FWT

② Few is enough

Change of Basis

Columns of W = new basis vectors

$$[x]_{\text{old basis}} \to [c]_{\text{new basis}} \qquad x = Wc$$

T with respect to $v_1, \dots, v_8$ it has a matrix A
with respect to $w_1, \dots, w_8$ it has a matrix B

What's the relation between A and B? A is similar to B

$$B = M^{-1} A M$$

What is A? Using basis $v_1, \dots, v_8$

Know T completely from $T(v_1), \dots, T(v_8)$ because every $x = c_1 v_1 + \cdots + c_8 v_8$

$\Rightarrow T(x) = c_1 T(v_1) + \cdots + c_8 T(v_8)$

Write $T(v_1) = a_{11} v_1 + a_{21} v_2 + \cdots a_{81} v_8$

$T(v_2) = a_{12} v_1 + a_{22} v_2 + \cdots + a_{82} v_8$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & a_{81} \\ \vdots & \vdots & \\ a_{18} & a_{28} & a_{88} \end{bmatrix}$$