ASTR 600: Problem Set 3

Isaque Dutra

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Problem 1: Reviewing the Background

Assume cosmology with $\Omega_{m,0} = 0.3$, $\Omega_\Lambda = 0.7$, and $h = 0.7$.

Density Parameters

At $z = 0.5$,

Starting from the scaling relation $\rho_m = \rho_{m,0} a^{-3}$ and $\rho_m = \rho_{\text{crit}}\Omega$, we find

$$\Omega_m = \Omega_{m,0} \frac{\rho_{\text{crit},0}}{\rho_{\text{crit}}} (1+z)^3$$

where $\rho_{\text{crit},0}/\rho_{\text{crit}}$ can be found from $\rho_{\text{crit}} = 3H^2/8\pi G$

$$\frac{\rho_{\text{crit},0}}{\rho_{\text{crit}}} = \frac{H_0^2}{H^2}$$

Therefore,

$$\begin{aligned}\Omega_m &= \Omega_{m,0} \frac{\rho_{\text{crit},0}}{\rho_{\text{crit}}} (1+z)^3 \\ &= 0.3 \cdot 0.584 \cdot (1+0.5)^3 \\ &= 0.591\end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_\Lambda &= \Omega_{\Lambda,0} \frac{\rho_{\text{crit},0}}{\rho_{\text{crit}}} (1+z)^3 \\ &= 0.7 \cdot 0.584 \cdot (1+0.5)^3 \\ &= 0.409 \\ &= 1 - \Omega_m\end{aligned}$$

Luminosity and Angular Diameter distances

As can be seen in the following plot, the angular diameter distance increases until a small redshift, then decreases again. It has a maximum at $z = 1.605$ ($z = 1.250$) for a universe with $\Omega_m = 0.3$ ($\Omega_m = 1.0$).

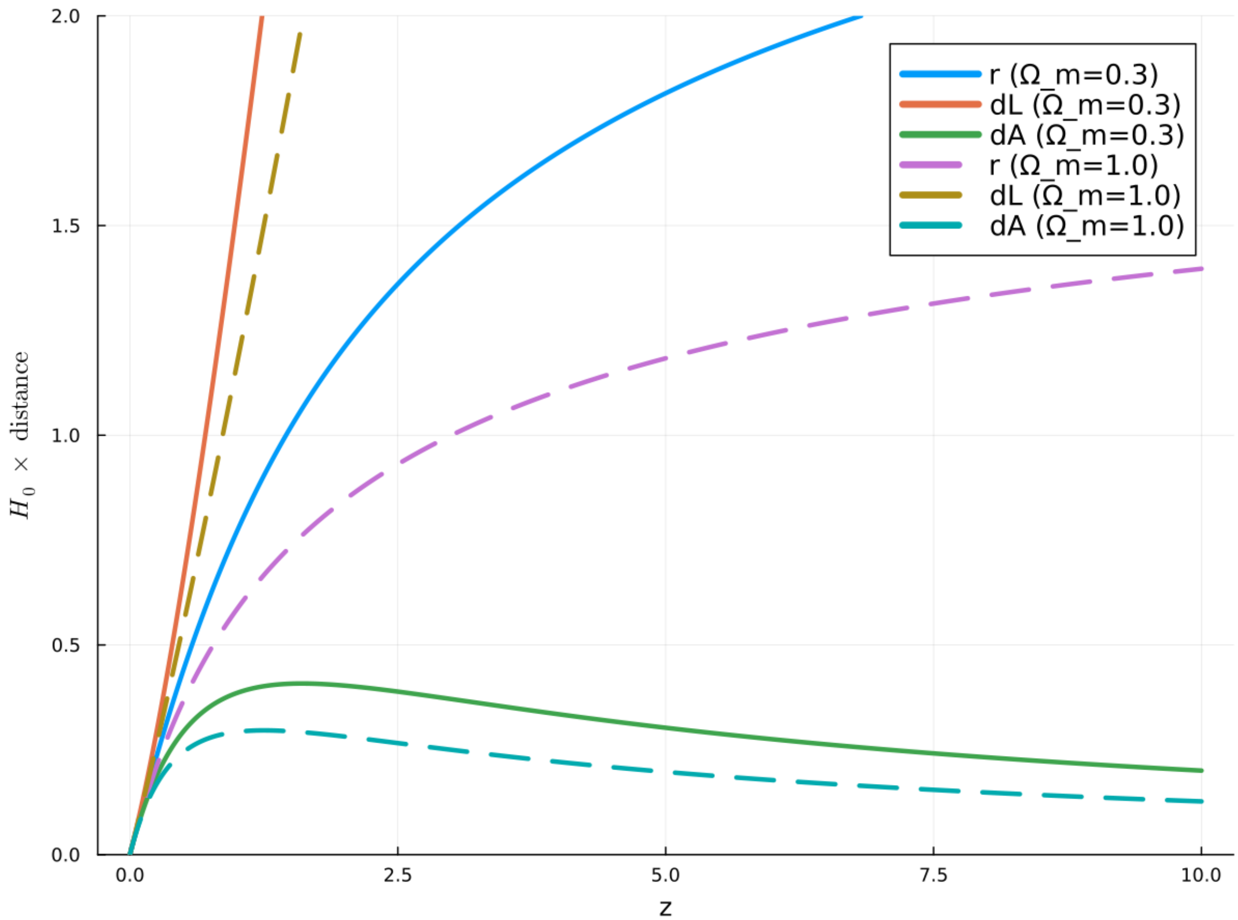

Figure 1: "Plot of proper, luminosity and angular diameter distance for $\Omega_m = 0.3$ (solid lines) and $\Omega_m = 1.0$ (dashed lines) universe"

```
using Integrals
using Plots
using LaTeXStrings
import PhysicalConstants.CODATA2018 as c
using Unitful, UnitfulAstro

E(z, Ω_m) = sqrt(Ω_m*(1+z)^3 + (1-Ω_m))

integral(z; Ω_m) = solve(IntegralProblem((z,p) -> 1/E(z, Ω_m), 0.0, z, Ω_m), QuadGKJL()).u

# define proper distance
r(z; Ω_m) = integral(z; Ω_m=Ω_m)

# define luminosity distance
dL(z; Ω_m) = r(z; Ω_m) * (1+z)

# define angular distance
dA(z; Ω_m) = r(z; Ω_m) / (1+z)

# plot distances for redshifts between 0 and 10
```

```
z = 0:0.001:10.0

plot(legendfontsize=12, dpi=150, size=(800,600),
      xlabel="z", ylabel=L"H_0 \; \times \; \mathrm{distance}",
      ylim=(0,2))
plot!(z, r.(z; Ω_m=0.3), linewidth=3, label="r (Ω_m=0.3)")
plot!(z, dL.(z; Ω_m=0.3), linewidth=3, label="dL (Ω_m=0.3)")
plot!(z, dA.(z; Ω_m=0.3), linewidth=3, label="dA (Ω_m=0.3)")
plot!(z, integral.(z; Ω_m=1.0), linewidth=3, line=:dash, label="r (Ω_m=1.0)")
plot!(z, dL.(z; Ω_m=1.0), linewidth=3, line=:dash, label="dL (Ω_m=1.0)")
plot!(z, dA.(z; Ω_m=1.0), linewidth=3, line=:dash, label="dA (Ω_m=1.0)")
```

Looking back

The redshift $z = 1.856$ corresponds to 10 billion years ago.

```
H0 = 100 * 0.7u"km/s / Mpc" |> u"yr^(-1)"
Ω_M = 0.3
Ω_Λ = 0.7
E(z) = (1.0+z) * sqrt(Ω_M*(1.0+z)^3 + Ω_Λ)

integral(x) = solve(IntegralProblem((z,p) -> 1.0/H0.val * 1.0/E(z), 0.0, x), QuadGKJL()).u

x = 0.0:0.001:5.0
i = abs.(abs.(integral.(x)) .- 10e9)
x[argmin(i)]
```

A Λ-dominated Universe

We can find $a(t)$ and the age of a universe with only a cosmological constant by starting with the Friedmann equation $H^2 = H_0^2\Omega_\Lambda$, then

$$\begin{aligned}\frac{\dot{a}}{a} &= H_0\sqrt{\Omega_\Lambda} \\ \frac{\mathrm{d}a}{\mathrm{d}t} &= aH_0\sqrt{\Omega_\Lambda} \\ a(t) &= a_0 e^{H_0\sqrt{\Omega_\Lambda}t}\end{aligned}$$

The age of such universe can be found by

$$\begin{aligned}\mathrm{d}t &= \frac{1}{H_0\sqrt{\Omega_\Lambda}}\int_0^1 \frac{\mathrm{d}a}{a} \\ t_0 &= \frac{1}{H_0\sqrt{\Omega_\Lambda}}\ln(a)|_0^1 \\ &= \infty\end{aligned}$$

A universe with only a cosmological constant is infinitely old.

Problem 2: Massive Neutrinos

Energy density at decoupling

Assuming neutrinos are in thermal equilibrium immediately prior to decoupling, their energy density is given by

$$\rho(T) = \frac{g}{(2\pi)^3} \int \left(\mathrm{d}^3 p f(p,T) \sqrt{m^2 + p^2} \right)$$

where g, the internal degree of freedom for neutrinos is $2 \times 2 \times 2 = 8$, and distribution function $f(p,T)$ is given by the Fermi-Dirac distribution for fermions:

$$f(p,T) = \frac{1}{e^{\frac{\sqrt{m^2+p^2}}{T}} + 1}$$

where we neglected the chemical potential μ, which is much smaller than the temperature, $\mu \ll T$ at early times.

Following Baumann's Section 3, we define the dimensionless variable $\xi \equiv p/T$, and rewrite the previous equation as

$$\rho_\nu = \frac{T_\nu^4}{\pi^2} \int \mathrm{d}\xi \frac{\xi^2 \sqrt{\xi^2 + \frac{m_\nu^2}{T_\nu^2}}}{e^\xi + 1}$$

Series expansion

For small $x \equiv m_\nu / T_\nu$, we can expand the integrand around $x = 0$ then integrate with respect to ξ:

$$\begin{aligned}
\sqrt{\xi^2 + x^2} &\approx \sqrt{\xi^2 + x^2}\Big|_{x=0} + \frac{x}{\sqrt{\xi^2 + x^2}}\Bigg|_{x=0} x + \frac{1}{2!} \frac{\xi^2}{(\xi^2 + x^2)^{\frac{3}{2}}}\Bigg|_{x=0} x^2 + \ldots \\
&\approx \xi + \frac{1}{2\xi} x^2 + \ldots \\
&\approx \xi + \frac{1}{2\xi} \left(\frac{m_\nu}{T_\nu} \right)^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_\nu &= \frac{T_\nu^4}{\pi^2} \int \mathrm{d}\xi \frac{\xi^2 \sqrt{\xi^2 + \frac{m_\nu^2}{T_\nu^2}}}{e^\xi + 1} \\
&\approx \frac{T_\nu^4}{\pi^2} \int \mathrm{d}\xi \frac{\xi^2 \left(\xi + \frac{1}{2\xi} \left(\frac{m_\nu}{T_\nu} \right)^2 \right)}{e^\xi + 1} \\
&\approx \frac{T_\nu^4}{\pi^2} \int \mathrm{d}\xi \frac{\xi^3 + \frac{\xi}{2} \left(\frac{m_\nu}{T_\nu} \right)^2}{e^\xi + 1}
\end{aligned}$$

However, for massless neutrinos ($m_\nu = 0$),

$$
\begin{aligned}
\rho_{\nu,0} &= \frac{T_\nu^4}{\pi^2}\int \mathrm{d}\xi \frac{\xi^3}{e^\xi+1} \\
&= \frac{7\pi^2}{120}T_\nu^4
\end{aligned}
$$

Thus, we can rewrite ρ_ν as

$$
\begin{aligned}
\rho_\nu &\approx \rho_{\nu,0} + \frac{T_\nu^4}{\pi^2}\int \mathrm{d}\xi \frac{\frac{\xi}{2}\left(\frac{m_\nu}{T_\nu}\right)^2}{e^\xi+1} \\
&= \rho_{\nu,0} + \frac{T_\nu^4}{\pi^2}\frac{\pi^2}{12}\frac{1}{2}\left(\frac{m_\nu}{T_\nu}\right)^2 \\
&= \rho_{\nu,0} + \frac{7\pi^2}{120}T_\nu^4 \cdot \frac{5}{7\pi^2}\left(\frac{m_\nu}{T_\nu}\right)^2 \\
&= \rho_{\nu,0}\left(1 + \frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right)
\end{aligned}
$$

Estimate smallest neutrino mass detectable in the CMB

If ρ_ν is "significantly" larger than $\rho_{\nu,0}$ at the epoch of the CMB ($z \sim 1000$), then the neutrinos can affect the CMB.

From previous section,

$$
\rho_\nu = \rho_{\nu,0}\left(1 + \frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right)
$$

Therefore, for the neutrinos to affect the CMB, let's estimate that $\rho_\nu \gg \rho_{\nu,0}$ is satisfied by $\rho_\nu \approx 2\rho_{\nu,0}$, or

$$
\frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2} \approx 1
$$

$$
\rightarrow m_\nu \approx \sqrt{\frac{7\pi^2}{5}}T_\nu
$$

Given the CMB temperature today $T_{\gamma,0} = 0.235$ meV, we can use the fact that $T \propto a^{-1}$ and after e^+e^- anihilation, the neutrino temperature is $T_\nu = (4/11)^{1/3}T_\gamma$, we find that the neutrino temperature at the epoch of CMB ($z \sim 1000$) was approximately

$$
\begin{aligned}
T_\nu &\approx (1+z)\left(\frac{4}{11}\right)^{1/3} 0.235 \text{ meV} \\
&= 0.168 \text{ eV}
\end{aligned}
$$

where we assumed that g_* does not change considerably since $g_* T^3 a^3 \approx$ constant.

Therefore, neutrino mass m_ν can affect the CMB is approximately

$$
\begin{aligned}
m_\nu &\approx \sqrt{\frac{7\pi^2}{5}} T_\nu \\
&= 0.625 \text{ eV}
\end{aligned}
$$

Redshift neutrinos become non-relativistic

Assuming that one of the neutrinos species has a mass m_ν, it becomes non-relativistic when $T_\nu \sim m_\nu$. Given that $T_\nu = T_{\nu,0}(1+z)$, we find the redshift neutrinos become non-relativistic is

$$
\begin{aligned}
z &\approx \frac{m_\nu}{T_{\nu,0}} \\
&\approx \frac{m_\nu}{(4/11)^{1/3} T_{\gamma,0}} \\
&\approx \frac{m_\nu}{(4/11)^{1/3} 0.235 \text{ meV}} \\
&\approx \frac{m_\nu}{0.168 \text{ meV}}
\end{aligned}
$$

Number density of neutrinos today

Similarly to the previous part, we start with the number density at thermal equilibrium given by

$$
n = \frac{g}{2\pi^2} T^3 \int \mathrm{d}\xi \frac{\xi^2}{e^{\sqrt{\xi^2 + \frac{m_\nu^2}{T^2}}} + 1}
$$

in the relativistic limit where $m_\nu \ll T_\nu$, we find that

$$
\begin{aligned}
n_\nu &= \frac{\zeta(3)}{\pi^2} g T_\nu^3 \cdot \frac{3}{4} \\
&= \frac{6\zeta(3)}{\pi^2} T_\nu^3
\end{aligned}
$$

Using the the fact that neutrinos decouple at approximated $T \approx 1$ MeV (from Baumann's book), and restoring the factors of c and $\hbar$,

$$
\begin{aligned}
n_\nu &= \frac{6\zeta(3)}{\pi^2} \frac{T_\nu^3}{(\hbar c)^3} \\
&= \frac{6\zeta(3)}{\pi^2} \left(\frac{T_\nu}{1 \text{ MeV}} \right)^3 \left(\left(6.5821 \times 10^{-22} * \text{MeV } s\right) \cdot \left(2.998 \times 10^{10} \text{ cm } s^{-1}\right) \right)^{-3} \\
&\approx 9.51 \times 10^{31} \text{cm}^{-3}
\end{aligned}
$$

In order to calculate the redshift which neutrinos decouple, we first calculate the neutrino temperature today. The neutrino temperature today $T_{\nu,0}$ can be calculated from the CMB temperature today

$$
\begin{aligned}
T_{\nu,0} &= \left(\frac{4}{11}\right)^{\frac{1}{3}} T_{\gamma,0} \\
&= \left(\frac{4}{11}\right)^{\frac{1}{3}} \cdot 0.235 \text{ meV} \\
&= 0.168 \text{ meV}
\end{aligned}
$$

We know that $g_{*S}a^3T^3 = \text{constant}$, and since the neutrino decoupling epoch g_{*S} has decreased from ≈ 11 to ≈ 4, therefore, the redshift which neutrinos decouple, z_D

$$
\begin{aligned}
z_D &\approx \frac{1 \text{ MeV}}{0.168 \text{ meV}} \left(\frac{11}{4}\right)^{\frac{1}{3}} \\
&\approx 8 \times 10^9
\end{aligned}
$$

The neutrino number density today is therefore

$$
\begin{aligned}
n_{\nu,0} &= \frac{n_\nu}{(1+z_D)^3} \\
&= 9.51 \times 10^{31} \text{cm}^{-3} \frac{1}{\left(8 \times 10^9\right)^3} \\
&\approx 163 \text{cm}^{-3}
\end{aligned}
$$

Energy density today

Given that the critical density today is

$$
\begin{aligned}
\rho_{c,0} &= \frac{3H_0^2}{8\pi G} \\
&= 1.878 \times 10^{-29} h^2 g \text{cm}^{-3} \\
&= 1.054 \times 10^4 h^2 \text{ eV cm}^{-3}
\end{aligned}
$$

and non relativistic particles have energy density $\rho_\nu \approx n_\nu m_\nu$, we find

$$
\begin{aligned}
\Omega_{\nu,0} h^2 &= \frac{\rho_{\nu,0}}{\rho_{c,0}} h^2 \\
&= \frac{n_{\nu,0} m_\nu}{\rho_{c,0}} h^2 \\
&= \frac{163 \text{cm}^{-3} \cdot m_\nu}{1.054 \times 10^4 h^2 \text{ eV cm}^{-3}} h^2 \\
&\approx \frac{m_\nu}{65 \text{ eV}}
\end{aligned}
$$

Problem 3: Measuring the Expansion History with Standard Candles

Let

$$\chi = \int \mathrm{d}z \frac{c}{H_0\left[\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} + \left(1-\Omega_{m,0}-\Omega_{\Lambda,0}\right)(1+z)^2\right]^{1/2}}$$

Series expansion to order z^3

Let

$$f(z) = \left(\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} + \left(1-\Omega_{m,0}-\Omega_{\Lambda,0}\right)(1+z)^2\right)^{-1/2}$$

Then

$$\begin{aligned}
\chi &= \int \mathrm{d}z \frac{c}{H_0} f(z) \\
&= \int \mathrm{d}z \frac{c}{H_0}\left[f(0) + f'(0)z + \frac{1}{2!}f''(0)z^2 + \cdots\right] \\
&= \frac{c}{H_0}\left[f(0)z + f'(0)\frac{z^2}{2} + \frac{1}{2!}f''(0)\frac{z^3}{3} + O(z^4)\right]
\end{aligned}$$

where

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= \frac{1}{2}(2\Omega_\Lambda - \Omega_m - 2) \\
f''(0) &= \frac{1}{4}\left(3\Omega_m^2 + 4\Omega_m - 12\Omega_m\Omega_\Lambda + 12\Omega_\Lambda^2 - 20\Omega_\Lambda + 8\right)
\end{aligned}$$

Therefore, to third order,

$$\chi = \frac{c}{H_0}\left[z + \frac{2\Omega_\Lambda - \Omega_m - 2}{4}z^2 + \frac{3\Omega_m^2 + 4\Omega_m - 12\Omega_m\Omega_\Lambda + 12\Omega_\Lambda^2 - 20\Omega_\Lambda + 8}{24}z^3\right]$$

Accurate to 10%?

Assuming $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ and $h = 0.7$ (values from Problem 1), we can numerically integrate χ and compare to our approximation χ_{approx}, and calculate the value of redshift z where

$$\left|\frac{\chi(z) - \chi_{\text{approx}}(z)}{\chi(z)}\right| \approx 10\%$$

we find a value $z \approx 1.204$.

```julia
using Integrals
import PhysicalConstants.CODATA2018 as c
using Unitful, UnitfulAstro

Ω_M = 0.3
Ω_Λ = 0.7
c_0 = c.c_0 |> u"km/s"
H0 = 100 * 0.7u"km/s / Mpc"
A = (c_0/H0 |> u"km").val

E(z) = (Ω_M * (1+z)^3 + Ω_Λ + (1 - Ω_M - Ω_Λ)*(1+z)^2)^(1/2)

integral(z) = solve(IntegralProblem((z_,p) ->  A/E(z_), 0.0, z), QuadGKJL()).u

approx(z) = A * (z + 1/2 * (2Ω_Λ - Ω_M - 2) * z^2/2 + 1/2 * 1/4 * (3Ω_M^2+4Ω_M-12Ω_M*Ω_Λ
+ 12Ω_Λ^2 - 20Ω_Λ+8)*z^3/3)

error(z) = abs((approx(z)-integral(z))/(integral(z)+0.0001))

z = 0.0:0.001:10.0
i = abs.(error.(z) .- .1)
z[argmin(i)]
```

Listing 1: Julia code to calculate redshift where third-order expansion is accurate to 10% assuming Problem 1′s values for Ω_m and Ω_Λ

Only parameter that can be measured

For small enough redshift, only the first term in our expansion contributes significantly to ξ, while the second order and higher terms are negligible and vanish.

Therefore, for small redshift,

$$\chi \approx \frac{c}{H_0} z$$

and we see that we can only hope to measure H_0 in very low redshift measurements.

Which combination of Ω_m and Ω_Λ can be measured?

As we increase the redshift reach but still at low redshift measurement, the second order term in our expansion of χ becomes non-negligible:

$$\chi \approx \frac{c}{H_0} \left[z + \frac{2\Omega_\Lambda - \Omega_m - 2}{4} z^2 \right]$$

Therefore, we may hope to measure the combination $(2\Omega_\Lambda - \Omega_m - 2)$

Forecast errors on H_0, Ω_m and Ω_Λ

As discussed previously, the error on H_0 will be mostly strongly affected by the first order factor

$$\chi^{(0)} \approx \frac{c}{H_0} z$$