ASTR 600: Problem Set 5

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I: Growth of Matter Perturbations

Given the growth of matter perturbation equation

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G \rho_m \delta_m$$

We start by showing that the left hand side can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{\delta}_m\right) &= 2a\dot{a}\dot{\delta}_m + a^2\ddot{\delta}_m \\
&= 2a^2H\dot{\delta}_m + a^2\ddot{\delta}_m \\
&= a^2\left(2H\dot{\delta}_m + \ddot{\delta}_m\right)
\end{aligned}$$

Therefore, we can rewrite the first equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{\delta}_m\right) = a^2 4\pi G \rho_m \delta_m$$

Moreover,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} &= \frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}a} \\
&= a\frac{\dot{a}}{a}\frac{\mathrm{d}}{\mathrm{d}a} \\
&= aH\frac{\mathrm{d}}{\mathrm{d}a}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{\delta}_m\right) &= a^2 4\pi G \rho_m \delta_m \\
aH\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3H\frac{\mathrm{d}\delta}{\mathrm{d}a}\right) &= a^2 4\pi G \rho_m \delta_m \\
\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3H\frac{\mathrm{d}\delta}{\mathrm{d}a}\right) &= \frac{a}{H} 4\pi G \rho_m \delta_m \\
\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3H\frac{\mathrm{d}\delta}{\mathrm{d}a}\right) &= 4\pi G \rho_{m,0}\frac{\delta_m}{a^2H}
\end{aligned}$$

Defining $y \equiv a/a_{\mathrm{eq}}$, and using the fact that at equality $\rho_{r,\,\mathrm{eq}} = \rho_{m,\,\mathrm{eq}}$, we write the Hubble parameter

$$
\begin{aligned}
H^2 &= \frac{8\pi G}{3}(\rho_m a^{-3} + \rho_r a^{-4}) \\
&= \frac{8\pi G}{3}\rho_{m,\text{ eq}}\left(\frac{a^{-3}}{a_{\text{eq}}^{-3}} + \frac{a^{-4}}{a_{\text{eq}}^{-4}}\right) \\
&= \frac{8\pi G}{3}\rho_{m,\text{ eq}}(y^{-3} + y^{-4}) \\
&= \frac{8\pi G}{3}\rho_{m,\text{ eq}}\frac{1}{y^4}(1+y)
\end{aligned}
$$

Therefore,

$$
H = \sqrt{\frac{8\pi G}{3}\rho_{m,\text{ eq}}}\frac{1}{y^2}\sqrt{1+y}
$$

II: Spherical Collapse

Starting with the parametric solution for a spherical overdensity of mass M and energy E:

$$
\begin{aligned}
r(\theta) &= A(1-\cos\theta) \\
t(\theta) &= B(\theta - \sin\theta)
\end{aligned}
$$

with $A = GM/2|E|$ and $A^3 = GMB^2$,

we can check they are solutions to the equation

$$
\frac{1}{2}\dot{r} - \frac{GM}{r} = E
$$

Start by taking the time derivative by using the chain rule

$$
\begin{aligned}
\dot{r} &= A\sin\theta\dot{\theta} \\
\dot{t} &= 1 = \dot{\theta}B(1-\cos\theta)
\end{aligned}
$$

therefore, `dot` $\theta = 1/B(1-\cos\theta)$ and

$$
\begin{aligned}
\dot{r}^2 &= \left(\frac{A}{B}\right)^2\left(\frac{\sin\theta}{1-\cos\theta}\right)^2 \\
&= 2|E|\left(\frac{\sin\theta}{1-\cos\theta}\right)^2
\end{aligned}
$$

Substituting into the previous equation,

$$
\begin{aligned}
|E|\left(\frac{\sin\theta}{1-\cos\theta}\right)^2 - \frac{GM}{A(1-\cos\theta)} &\stackrel{?}{=} E \\
\frac{1}{(1-\cos\theta)^2}\left[|E|\sin^2\theta - \frac{GM}{A}(1-\cos\theta)\right] &\stackrel{?}{=} E \\
\frac{|E|}{(1-\cos\theta)^2}\left[1-\cos\theta^2 - 2(1-\cos\theta)\right] &\stackrel{?}{=} E \\
-\frac{|E|}{(1-\cos\theta)^2}\left[1-2\cos\theta+\cos\theta^2\right] &\stackrel{?}{=} E \\
-\frac{|E|}{(1-\cos\theta)^2}(1-\cos\theta)^2 &\stackrel{?}{=} E \\
-|E| &= E
\end{aligned}
$$

Assuming $E < 0$, the solution is verified.

III: Equality Scale

Using the fact that at equality, $\Omega_R = a_{\text{eq}}\Omega_M$, we have

$$
\begin{aligned}
k_{\text{eq}} &= a_{\text{eq}} H_{\text{eq}} \\
&= a_{\text{eq}} H_0 \sqrt{\Omega_R a_{\text{eq}}^{-4} + \Omega_M a_{\text{eq}}^{-3}} \\
&= a_{\text{eq}} H_0 \sqrt{2\Omega_M a_{\text{eq}}^{-3}} \\
&= H_0 \sqrt{\frac{2\Omega_M}{a_{\text{eq}}}}
\end{aligned}
$$

For a cosmology roughly the one we live in, $\Omega_R = 9\times10^{-5}$, $\Omega_M = 0.31$, $H_0 = 67.7$ km $s^{-1}\text{Mpc}^{-1}$, $a_{\text{eq}} = \Omega_R/\Omega_M = 2.9\times10^{-4}$, we find

$$
\begin{aligned}
k_{\text{eq}} &= 67.7\frac{\text{km}/s}{\text{Mpc}}\sqrt{\frac{2\cdot 0.31}{2.9\times10^{-4}}} \\
&\approx 3130\frac{\text{km}/s}{\text{Mpc}} \\
&\approx \frac{3130}{c}\text{Mpc}^{-1} \\
&\approx 0.0104\text{Mpc}^{-1}
\end{aligned}
$$

IV: A Study in Simulations

Orienting Yourself: The Linear Power Spectrum

From the linear power spectrum, we calculate $n_s = 0.962$. By looking at where P_k reaches its maximum, we estimate $k_{\text{eq}} \approx 0.17\text{Mpc}^{-1}$.

We integrate the power spectrum to find

$$\sigma_8 = \int_0^\infty \Delta^2(k)\left(\frac{3j_1(kR)}{kR}\right)^2 \mathrm{d}\ln k \approx 1.35$$

which is not close to the value 0.834.

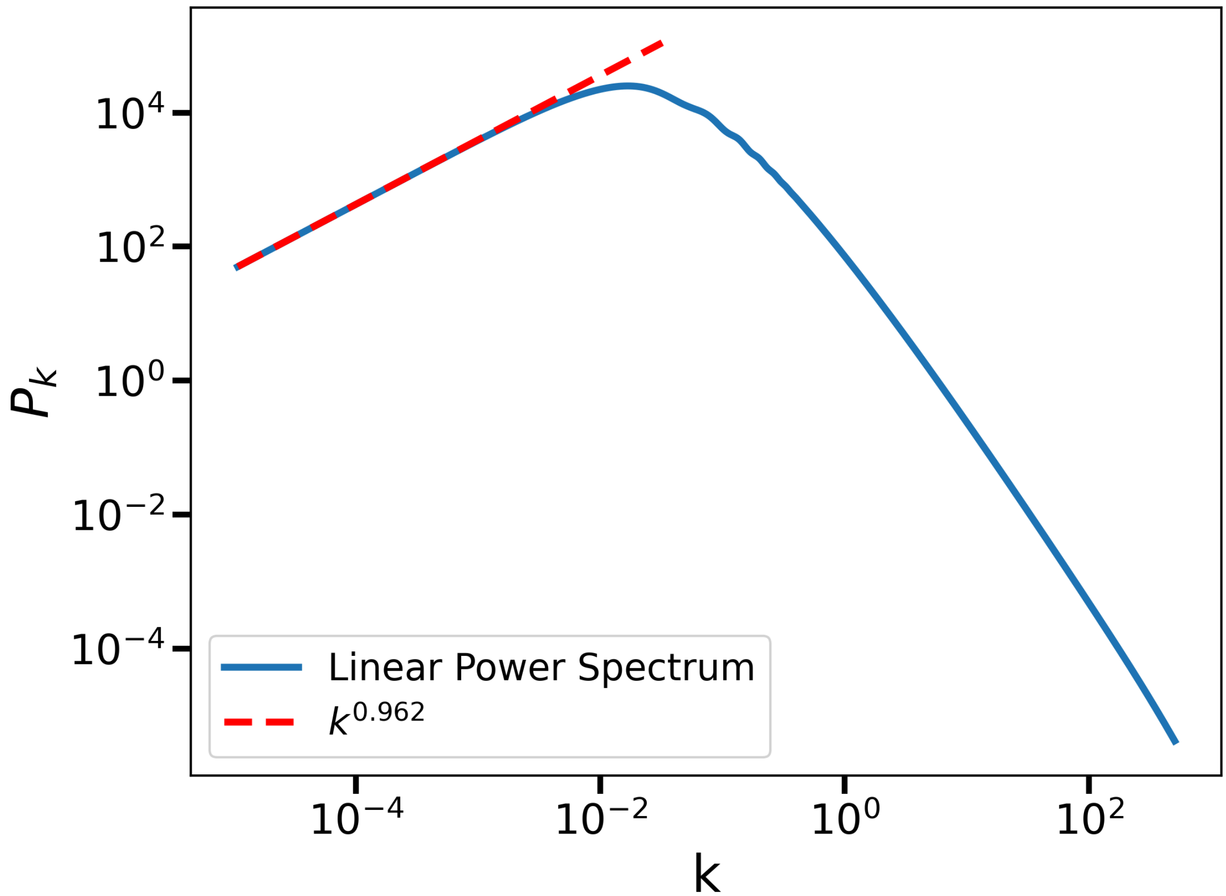

Figure 1: "Plot of Linear Power Spectrum."

We estimate the Transfer Function by using the BBKS transfer function as shown in Hunterer:

$$T(q) \equiv \frac{\log(1+2.34q)}{2.34q}\left(1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4\right)^{-1/4}$$

where $q \equiv \Gamma^{-1}k$ and $\Gamma \equiv \Omega_M e^{-\Omega_b - 1.3*\Omega_b/\Omega_M}$.

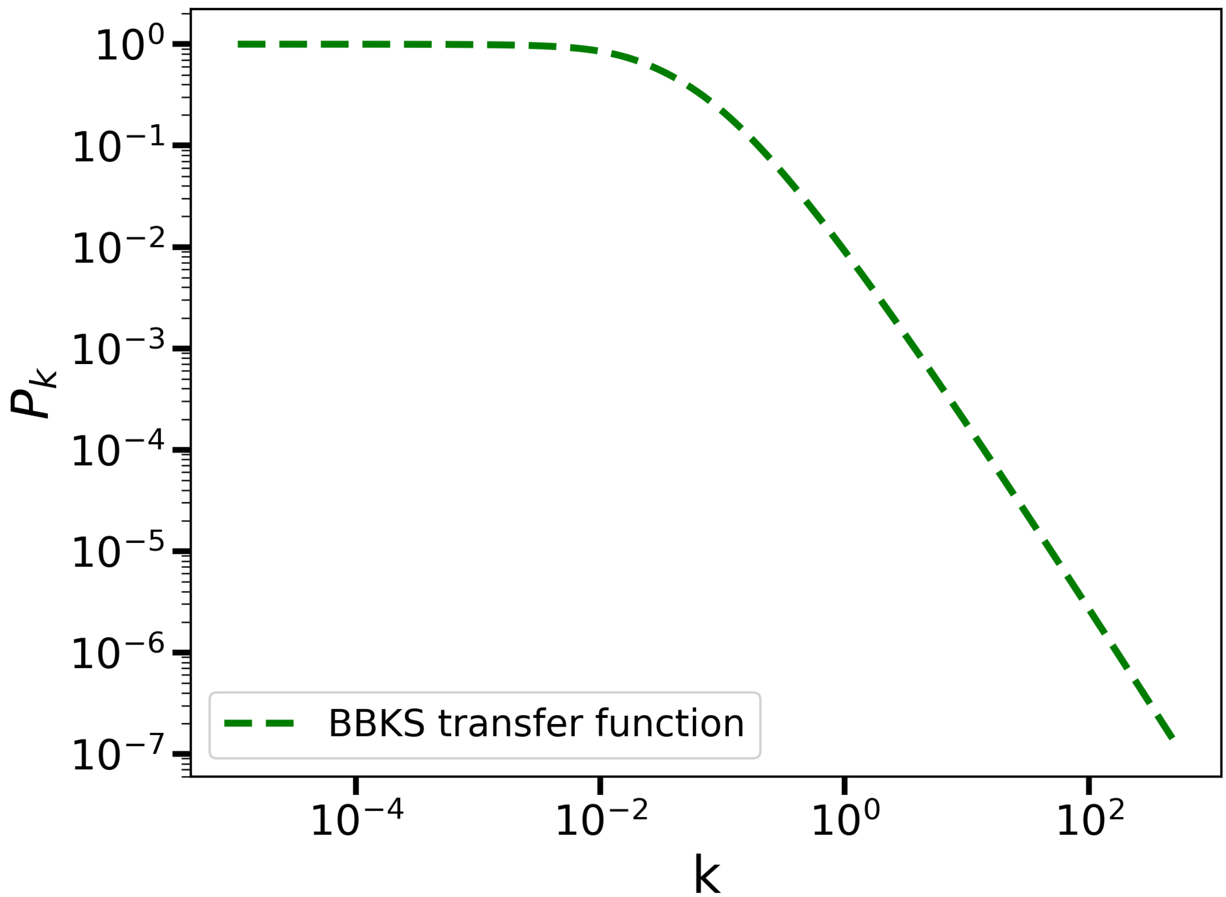

Figure 2: "Plot of Linear Power Spectrum with Transfer Function."