ASTR 600: Problem Set 4

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Problem 1: Recombination

Derive the Saha equation

Following Baumann Section 3.1 closely, we start from the equation of density for particles in equilibrium (for $i = \{e, p, H\}$).

$$n_i^{\text{eq}} = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T} \right)$$

In order to remove the dependence on the chemical potentials, consider the following ratio

$$\left(\frac{n_H}{n_e n_p} \right)_{\text{eq}} = \frac{g_H}{g_e g_p} \left(\frac{m_H}{m_e m_p} \frac{2\pi}{T} \right)^{3/2} \exp\left(\frac{m_p + m_e - m_H}{T} \right)$$

We can simplify this equation by using the fact that $n_e = n_p$ and the number of internal degrees of freedom are $g_p = g_e = 2$, $g_H =$. Moreover, $m_H \approx m_p$ and knowing that the ionization energy of hydrogen is

$$E_I \equiv m_p + m_e - m_H = 13.6 \text{ eV}$$

Therefore,

$$\left(\frac{n_H}{n_e^2} \right)_{\text{eq}} = \left(\frac{2\pi}{m_e T} \right)^{3/2} e^{E_I/T}$$

Defining the free-electron fraction X_e as

$$X_e \equiv \frac{n_e}{n_p + n_H} = \frac{n_e}{n_e + n_H}$$

and noticing that the denominator of the previous equation is the baryon density n_b given by

$$n_b = \eta n_\gamma = \eta \times \frac{2\zeta(3)}{\pi^2} T^3$$

We can then rewrite $\frac{n_H}{n_e^2} n_b = \frac{1-X_e}{X_e^2}$, therefore we arrive at the Saha equation,

$$\left(\frac{1 - X_e}{X_e^2} \right)_{\text{eq}} = \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi T}{m_e} \right)^{3/2} e^{E_I/T}$$

Solve the Saha equation

We can solve the Saha equation by letting

$$f(T) \equiv \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T}{m_e}\right)^{3/2} e^{E_I/T}$$

then, we rewrite the Saha equation as

$fX_e^2 + X_e - 1 = 0$ with positive root solution

$$X_e(T) = \frac{-1+\sqrt{1+4f}}{2f(T)}$$

Since $T = T_0(1+z)$ where $T_0 = 2.73K$, we can rewrite $f(T)$ in terms of z as

$$f(z) \equiv \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T_0(1+z)}{m_e}\right)^{3/2} e^{E_I/T_0(1+z)}$$

We find that the redshift at which $X_e = 0.1$ is at $z = 1258$, while we find that the redshift at which $X_e = 0.5$ is at $z = 1376$, a difference of $\Delta z = 118$.

Plot

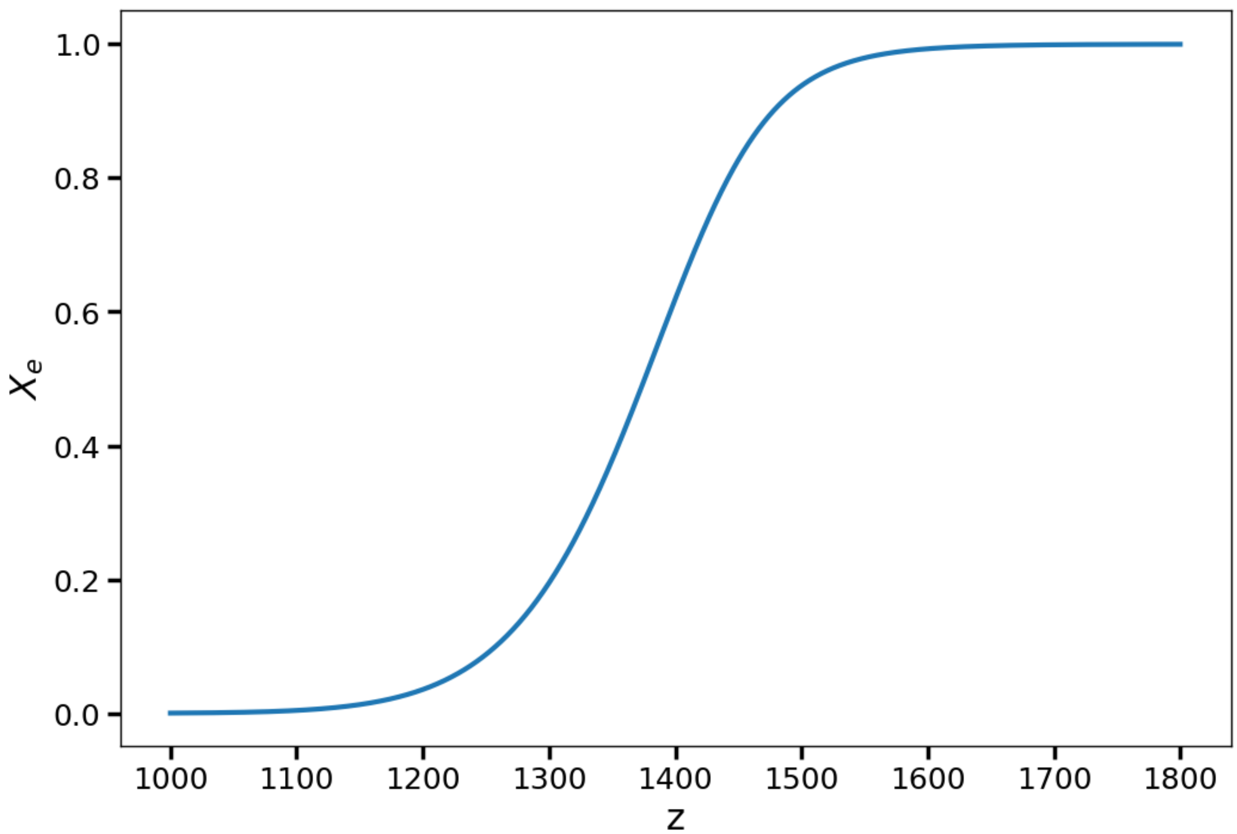

Figure 1: Plot of ionization fraction X_e vs redshift.

Age of universe

In order to find the age of the universe, we can integrate

$$t(a) = \int_0^a \frac{\mathrm{d}a}{H_0 a\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}}$$

At the redshift $z = 1258$ at which $X_e = 0.1$, we find that the universe age is approximately 380,000 years for an universe with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ and $H_0 = 70$ km/s / Mpc.

estimate the decoupling redshift at which the Thomson scattering rate equals the Hubble rate

$$\Gamma = n_b X_e \sigma_t \approx H_0\sqrt{\Omega_m (1+z)^3}$$

Using that $n_b = \eta n_\gamma = \eta \times \frac{2\zeta(3)}{\pi^2} T^3$ and $T = T_0(1+z)$,

We find that

$$(1+z)X_e(z)^{2/3} = \left[\frac{H_0\sqrt{\Omega_m}\pi^2}{\eta \times 2\zeta(3) T_0^3 \sigma_T}\right]^{2/3}$$

Solving it numerically (see attached notebook), we find $z = 1583$. The ionization fraction $X_e(z = 1583) = 0.99$ and the age of the universe is $\approx 270,000$ years.

"What-if" BBN

Following Baumann Section 3.2 closely, we again start from the equation of density for particles in equilibrium (for $i = \{n, p\}$).

$$n_i^{\mathrm{eq}} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)$$

Assuming that the chemical potentials of the electrons and neutrinos are negligibly small such as $\mu_n = \mu_p$, we find

$$\left(\frac{n_n}{n_p}\right)_{\mathrm{eq}} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(m_n - m_p)/T}$$

We can ignore the small difference between the proton and neutron masses in the coefficient, but not in the exponent

$$\left(\frac{n_n}{n_p}\right)_{\mathrm{eq}} = e^{-Q/T}$$

where $Q \equiv m_n - m_p = 1.30$ MeV

Defining the neutron fraction as

$$X_n \equiv \frac{n_n}{n_n + n_p}$$

we find

$$
\begin{aligned}
X_n &\equiv \frac{n_n}{n_n + n_p} \\
&= \left(\frac{n_n + n_p}{n_n}\right)^{-1} \\
&= \left(1 + \frac{n_p}{n_n}\right)^{-1} \\
&= \left(1 + e^{Q/T}\right)^{-1} \\
&= \frac{e^{-Q/T}}{1 + e^{-Q/T}}
\end{aligned}
$$

Therefore,

$$
X_n^{\text{eq}}(T) = \frac{e^{-Q/T}}{1 + e^{-Q/T}}
$$

Freezeout abundance of neutrons

Assuming the freezeout temperature of 0.8 MeV,

$$
\begin{aligned}
X_e^{\text{eq}}(T = 0.8\ \text{MeV}) &= \frac{e^{-1.3/0.8}}{1 + e^{-1.3/0.8}} \\
&= 0.165
\end{aligned}
$$

Mass fraction of helium

Since a helium atom is made out of two neutrons, we find that $n_{\text{He}} = 1/2 n_n$. Moreover, since $X_n \equiv n_n/(n_n + n_p)$, we have

$$
\frac{n_n}{n_p} = \frac{X_n}{1 - X_n}
$$

Therefore,

$$
\begin{aligned}
Y_P &= 4 n_{\text{He}}/n_H \\
&= 4 n_{\text{He}}/n_p \\
&= 4\frac{1}{2} n_n/n_p \\
&= 2 n_n/n_p \\
&= \frac{2X_n}{1 - X_n} \\
&= 0.395
\end{aligned}
$$

Impact on helium abundance

If the mass difference between neutrons and protons was actually 2.6 MeV, but the freeze out temperature remains the same, then

$$
\begin{aligned}
X_e^{\text{eq}}(T = 0.8 \text{ MeV}) &= \frac{e^{-2.6/0.8}}{1 + e^{-2.6/0.8}} \\
&= 0.037
\end{aligned}
$$

and

$$
\begin{aligned}
Y_P &= 4n_{\text{He}}/n_H \\
&= \frac{2X_n}{1 - X_n} \\
&= 0.078
\end{aligned}
$$

Therefore, the abundance of Helium would decrease significantly.

Problem 3: Freeze-in of particle species

For the freeze-in scenario, Boltzmann equation reads

$$
\frac{1}{a^3}\frac{\mathrm{d}(na^3)}{\mathrm{d}t} = 2\Gamma h(t) n_{\sigma,\text{eq}}(t)
$$

We want to rewrite the previous equations in terms of $x \equiv m_\sigma/T$ and $Y \equiv n/s$.

Since we only care on the overall dependence on x and can write the final result in terms of the undefined constant λ_1

Let's rewrite the left-side of the equation. First, the entropy density $s \propto a^{-3}$, thus $Y \equiv n/s \propto na^3$. Moreover, the temperature $T \propto a^{-1}$, thus $x \equiv m/T \propto a$.

Hence, $\frac{\mathrm{d}x}{\mathrm{d}t} \propto \dot{a} \propto \frac{\dot{a}}{a} x = Hx$. Moreover, in the radiation domination, $x \propto a \propto t^{1/2}$, thus $\mathrm{d}x \propto \mathrm{d}a \propto t^{-1/2} = a^{-1}\,\mathrm{d}t = x^{-1}\,\mathrm{d}t$. Thus, $\mathrm{d}t \propto x\,\mathrm{d}x$ and

$$
\begin{aligned}
\frac{1}{a^3}\frac{\mathrm{d}(na^3)}{\mathrm{d}t} &\propto \frac{1}{x^3}\frac{\mathrm{d}Y}{\mathrm{d}t} \\
&\propto x^{-4}\frac{\mathrm{d}Y}{\mathrm{d}x}
\end{aligned}
$$

On the right hand side of the original equation, we first notice that Γ is a constant. Then, we can take $h(t) \to h(x) \simeq x/(x+2)$ (according to Hunterer). Rewriting $n(t)$ in terms of Y, we find $n \propto Ys \propto a^{-3}Y \propto x^{-3}Y$. Thus,

$$
2\Gamma h(t) n_{\sigma,\text{eq}}(t) \propto h(x) x^{-3} Y_{\sigma,\text{eq}}
$$

Therefore, equating both sides, we find

$$
x^{-4}\frac{\mathrm{d}Y}{\mathrm{d}x} \propto h(x) x^{-3} Y_{\sigma,\text{eq}}
$$

and hence, the final result

$$
\frac{\mathrm{d}Y}{\mathrm{d}x} = \lambda_1 x h(x) Y_{\sigma,\text{eq}}
$$

Plots

We can plot $Y(x)$ and $Y_{\text{eq}}(x)$ by using the full form of n from Baumann Eq. 3.13,

$$n = \frac{g}{2\pi^2} T^3 I_{\pm}(x)$$

where

$$I_{\pm}(x) \equiv \int_0^{\infty} \mathrm{d}\eta \frac{\eta^2}{\exp\left[\sqrt{\eta^2 + x^2}\right] \pm 1}$$

Thus,

$$\begin{aligned} Y_{\text{eq}}(x) &\equiv \frac{n}{s} = \frac{n}{T^3} \\ &= \frac{g}{2\pi^2} I_{\pm}(x) \end{aligned}$$

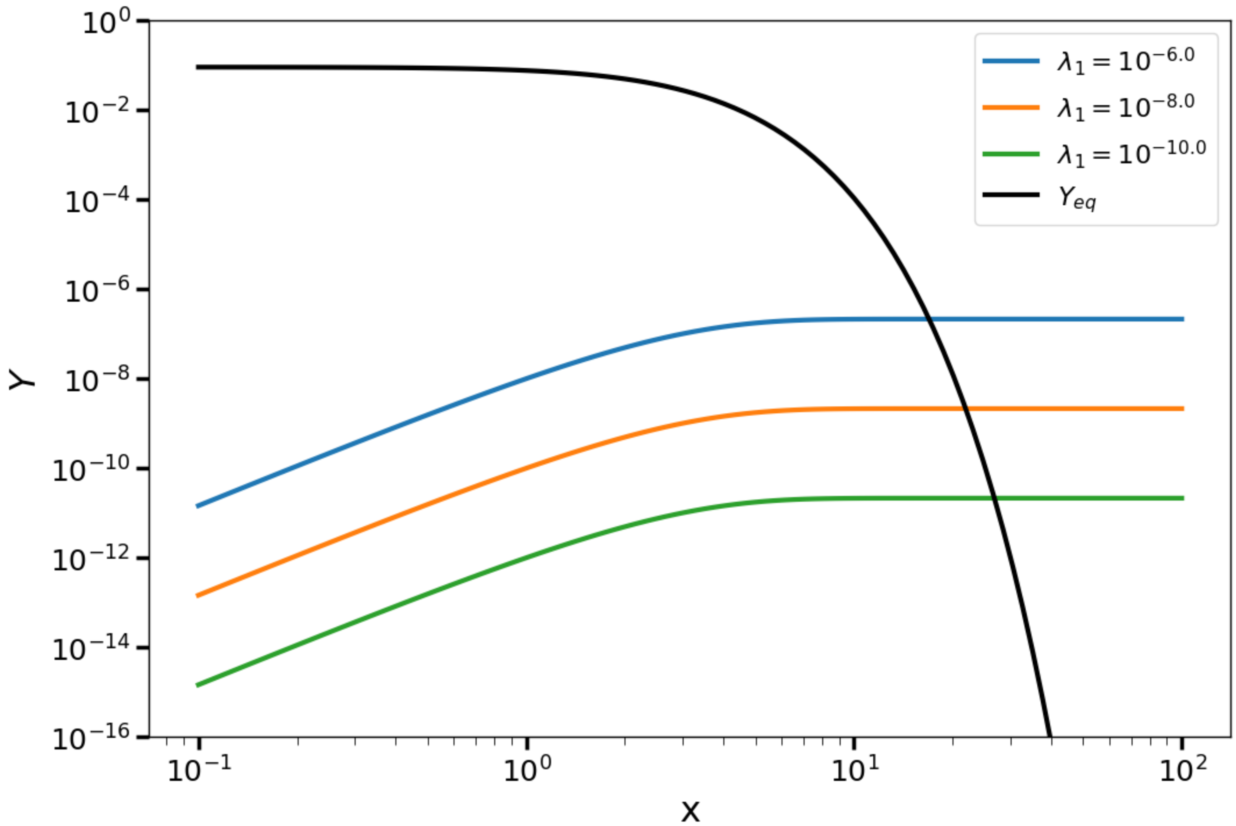

Figure 2: "Plot of Dark Matter freeze-in abundance."

Key difference

As seen in the freeze-in figure, as the anihilation/decay rate Γ – captured in the λ_1 parameter, increases, more Dark Matter is produced and remain after freeze-in. That is the opposite of the freeze-out scenario, where the larger anihilation/decay rate results in smaller dark matter density left out after freeze-out.

Moreover, in the freeze-out scenario, the universe starts with a large amount of Dark Matter that gets anihilated until freeze-out, while in the freeze-in scenario the universe starts with little Dark Matter, which is produced by the decay of an intermediate particle, until freeze-in.