ASTR 600: Problem Set 2

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Problem 1: Friedmann Equation II

Friedmann Equation I

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k^2}{a^2} + \frac{\Lambda}{3}$$

Continuity Equation

$$\dot{\rho} + 3H(\rho + P) = 0$$

Differentiating FEI with respect to time and replacing $\frac{\dot{\rho}}{H}$ from the CE,

$$\begin{aligned}
\dot{a}^2 &= \frac{8\pi G}{3}\rho a^2 - k^2 + \frac{\Lambda}{3}a^2 \\
2\dot{a}\ddot{a} &= \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\dot{a}) + \frac{2\Lambda}{3}a\dot{a} \\
\frac{\dot{a}}{a}\frac{\ddot{a}}{a} &= \frac{4\pi G}{3}\left(\dot{\rho} + 2\rho\frac{\dot{a}}{a}\right) + \frac{\Lambda}{3}\frac{\dot{a}}{a} \\
\frac{\ddot{a}}{a} &= \frac{4\pi G}{3}\left(\frac{\dot{\rho}}{H} + 2\rho\right) + \frac{\Lambda}{3} \\
&= \frac{4\pi G}{3}(-3(\rho + P) + 2P) + \frac{\Lambda}{3} \\
&= -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}
\end{aligned}$$

Thus, we recover Friedmann Equation II

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

Problem 2: Cosmological Dimming

Surface brightness of an object as seen by an observer is defined as the total flux received by the observed divided by the angular area of the object as seen by the observer.

Since the total flux is proportional to $1/D_L^2$ while the angular area is proportional to D_A^2, and $D_A = \frac{D_L}{(1+z)^2}$, we find that

$$I \propto \frac{D_A^2}{D_L^2}$$

$$I \propto \frac{1}{(1+z)^4}$$

$$I_0 = I_e (1+z)^{-4}$$

Problem 3: Magnitudes and K-corrections

Show that $m = M + \mathrm{DM}(z)$

Define the apparent magnitude of an observed flux f as

$$m = -2.5 \log_{10}\left(\frac{f}{f_0}\right)$$

where the observed flux of object with luminosity L at flux distance D_l is defined as $f = \frac{L}{4\pi D_l^2}$. Thus

$$\begin{aligned}
m &= -2.5 \log\left(\frac{L}{4\pi D_L^2}\frac{1}{f_0}\right) \\
&= -2.5 \log\left(\frac{L}{4\pi 10\text{ pc}}\frac{10\text{ pc}}{D_L^2}\frac{1}{f_0}\right) \\
&= -2.5\left[\log\left(\frac{L}{4\pi 10\text{ pc}}\frac{1}{f_0}\right) + \log\left(\frac{10\text{ pc}}{D_L^2}\right)\right] \\
&= -2.5\left[\log\left(\frac{L}{4\pi 10\text{ pc}}\frac{1}{f_0}\right) - 2\log\left(\frac{D_L}{10\text{ pc}}\right)\right] \\
&= -2.5 \log\left(\frac{L}{4\pi 10\text{ pc}}\frac{1}{f_0}\right) + 5\log\left(\frac{D_L}{10\text{ pc}}\right) \\
&= M + \mathrm{DM}(z)
\end{aligned}$$

K modification

Object originally emitted luminosity $L_{\nu(1+z)}$ at frequency $\nu(1+z)$, different than we observe today (L_ν) due to expansion of universe.

The differential flux density S_ν observed today at frequency ν is given by the differential luminosity when emitted at frequency $\nu(1+z)$ divided by the proper surface area of a sphere $4\pi D_L^2$.

The photos observed today at frequency ν were emitted at frequency $\nu(1+z)$, and since energy of a photon $E_\gamma \propto \nu$, the energy flux per unit bandwidth is rescaled by $(1+z)$.

Therefore, the differential flux S_ν is

$$
\begin{aligned}
S_\nu &= (1+z)\frac{L_{\nu(1+z)}}{4\pi D_L^2} \\
&= (1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi D_L^2}
\end{aligned}
$$

And the observed apparent magnitude formula gets modified to

$$
\begin{aligned}
m &= -2.5\log\left[(1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi D_L^2}\frac{1}{f_0}\right] \\
&= -2.5\log\left[(1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi 10\text{pc}}\frac{1}{f_0}\frac{10\text{ pc}}{D_L^2}\right] \\
&= -2.5\log\left(\frac{L_\nu}{4\pi 10\text{pc}}\frac{1}{f_0}\right) + 5\log\left(\frac{D_L}{10\text{ pc}}\right) - 2.5\log\left((1+z)\frac{L_{\nu(1+z)}}{L_\nu}\right) \\
&= M + \text{DM} + K
\end{aligned}
$$

Problem 4: A Static Universe

Second Friedmann equation for this model

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}
$$

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}
$$

Let $\rho \to \rho' = \rho + \frac{\Lambda}{8\pi G}$, then the first Friedmann Equation becomes

$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \\
&= \frac{8\pi G}{3}\left(\rho + \frac{\Lambda}{8\pi G}\right) - \frac{k}{a^2} \\
&= \frac{8\pi G}{3}\rho' - \frac{k}{a^2}
\end{aligned}
$$

and the second Friedmann Equation

$$
\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \\
&= -\frac{4\pi G}{3}\left(\rho' - \frac{\Lambda}{8\pi G} + 3P\right) + \frac{\Lambda}{3} \\
&= -\frac{4\pi G}{3}\left(\rho' - \frac{\Lambda}{8\pi G} - \frac{\Lambda}{4\pi G} + 3P\right) \\
&= -\frac{4\pi G}{3}\left(\rho' - 3\frac{\Lambda}{8\pi G} + 3P\right) \\
&= -\frac{4\pi G}{3}(\rho' + 3P')
\end{aligned}
$$

where we defined $P \to P' = P - \frac{\Lambda}{8\pi G}$

Value of Λ and k such as $\dot{a} = 0$ and $\ddot{a} = 0$.

Let $\ddot{a} = 0$ in second Friedmann Equation,

$$
\begin{aligned}
\rho' + 3P' &= 0 \\
\rho + \frac{\Lambda}{8\pi G} - \frac{3\Lambda}{8\pi G} &= 0 \\
\rho - \frac{\Lambda}{4\pi G} &= 0
\end{aligned}
$$

Thus,

$$\ddot{a} = 0 \to \rho = \frac{\Lambda}{4\pi G}$$

Let $\dot{a} = 0$ in first Friedmann Equation,

$$
\begin{aligned}
\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k^2}{a^2} &= 0 \\
\frac{2\Lambda}{3} + \frac{\Lambda}{3} - \frac{k^2}{a^2} &= 0 \\
\Lambda - \frac{k^2}{a^2} &= 0
\end{aligned}
$$

Thus,

$$\dot{a} = 0 \to k = \sqrt{a^2\Lambda}$$

implying an open (positive curvature) universe.

Pertubations

Imagine pertubing the scale factor by

$$a(t) = 1 + \delta a(t)$$

and

$$\rho_{m(t)} = \rho_{m[1-3\delta a(t)]}$$

Substituting this into the second Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m + 3P) + \frac{\Lambda}{3}$$

$$\frac{\ddot{\delta a}}{1+\delta a} = -\frac{4\pi G}{3}(\rho_m[1-3\delta a]) + \frac{\Lambda}{3}$$

Substituting $\rho_m = \frac{\Lambda}{4\pi G}$ found previously,

$$\begin{aligned}\frac{\ddot{\delta a}}{1+\delta a} &= -\frac{4\pi G}{3}\frac{\Lambda}{4\pi G}[1-3\delta a] + \frac{\Lambda}{3} \\ &= -\frac{\Lambda}{3}[1-3\delta a] + \frac{\Lambda}{3} \\ &= \Lambda\delta a\end{aligned}$$

Thus,

$$\ddot{\delta a} = \Lambda\delta a(1+\delta a)$$

Dropping second order terms,

$$\ddot{\delta a} - \Lambda\delta a = 0$$

Assuming a positive cosmological constant, the solutions are

$$\delta a \propto e^{-\sqrt{\Lambda}t} + e^{+\sqrt{\Lambda}t}$$

Assuming initial conditions $\delta a(0) = \delta a_0$ and $\dot{\delta a}(0) = 0$, we find

$$\delta a(t) = \frac{\delta a_0}{2}\left(e^{-\sqrt{\Lambda}t} + e^{+\sqrt{\Lambda}t}\right)$$

This solution is clearly unstable since the positive exponential term goes to infinity as $t \to \infty$.

Problem 5: Redshift Drift

$$1 + z = \frac{a(t_0)}{a(t_1)}$$

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t_0} &= \frac{1}{a(t_1)}\frac{\mathrm{d}a(t_0)}{\mathrm{d}t_0}\frac{\mathrm{d}t_0}{\mathrm{d}t_0} - \frac{a(t_0)}{a(t_1)^2}\frac{\mathrm{d}a(t_1)}{\mathrm{d}t_1}\frac{\mathrm{d}t_1}{\mathrm{d}t_0} \\ &= \frac{1}{a(t_1)}\frac{\mathrm{d}a(t_0)}{\mathrm{d}t_0} - \frac{a(t_0)}{a(t_1)^2}\frac{\mathrm{d}a(t_1)}{\mathrm{d}t_1}\frac{\mathrm{d}t_1}{\mathrm{d}t_0}\end{aligned}$$

What is $\frac{\mathrm{d}t_1}{\mathrm{d}t_0}$? For light, $\mathrm{d}t = a(t)d\chi$, and for light emitted at t_1 and observed today at t_0, the coordinate distance $\mathrm{d}\chi$ between two peaks remains unchanged. Therefore, $\mathrm{d}\chi_1 = \mathrm{d}\chi_0$, and

$$\frac{\mathrm{d}t_1}{a(t_1)} = \frac{\mathrm{d}t_0}{a(t_0)}$$

$$\frac{\mathrm{d}t_1}{\mathrm{d}t_0} = \frac{a(t_1)}{a(t_0)}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t_0} &= \frac{1}{a(t_1)}\frac{\mathrm{d}a(t_0)}{\mathrm{d}t_0} - \frac{a(t_0)}{a(t_1)^2}\frac{\mathrm{d}a(t_1)}{\mathrm{d}t_1}\frac{a(t_1)}{a(t_0)} \\ &= (1+z)H_0 - H(t_1)\end{aligned}$$

where we used the fact that $\frac{1}{a(t_1)} = 1 + z$, $a(t_0) = 1$ and

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

$$H_0 = \frac{\mathrm{d}a(t)}{\mathrm{d}t}\frac{1}{a(t)}\bigg|_{t_0}$$

$$H_0 = \frac{\mathrm{d}a(t_0)}{\mathrm{d}t_0}$$

$$H_1 = \frac{1}{a(t)}\frac{\mathrm{d}a(t)}{\mathrm{d}t}\bigg|_{t_1}$$

Estimating drift for object at $z = 1$

For a matter dominated flat Universe,

$$H(z) = H_0(1+z)^{\frac{3}{2}}$$

and assuming $H_0 = 100$ h km/s/Mpc and 1 km $= 3.25 \times 10^{-20}$ Mpc

$$\begin{aligned}\Delta z &= H_0\Big[(1+z) - (1+z)^{\frac{3}{2}}\Big]\Delta t \\ &= 100h\frac{\mathrm{km}}{\mathrm{Mpc}}\Big[(1+z) - (1+z)^{\frac{3}{2}}\Big]\frac{\Delta t}{s} \\ &= 100h \cdot 3.25 \times 10^{-20}\Big[(1+z) - (1+z)^{\frac{3}{2}}\Big] \cdot \frac{\Delta t}{3.17 \times 10^{-8}\ \text{year}} \\ &\approx 10^{-10}h\Big[(1+z) - (1+z)^{\frac{3}{2}}\Big]\left(\frac{\Delta t}{1\ \text{year}}\right)\end{aligned}$$

For an object at $z = 1$,

$$\Delta z \approx 10^{-10}h\left(\frac{\Delta t}{1\ \text{year}}\right)$$