# 2072U Computational Science I
# Winter 2022

| Week | Topic |
|---|---|
| 1 | Introduction |
| 1–2 | Solving nonlinear equations in one variable |
| 3–4 | Solving systems of (non)linear equations |
| 5–6 | Computational complexity |
| 6–8 | Interpolation and least squares |
| 8–10 | Integration & differentiation |
| 10–12 | Additional Topics |

1. Error plotting
2. Recursion
3. Newton-Raphson iteration

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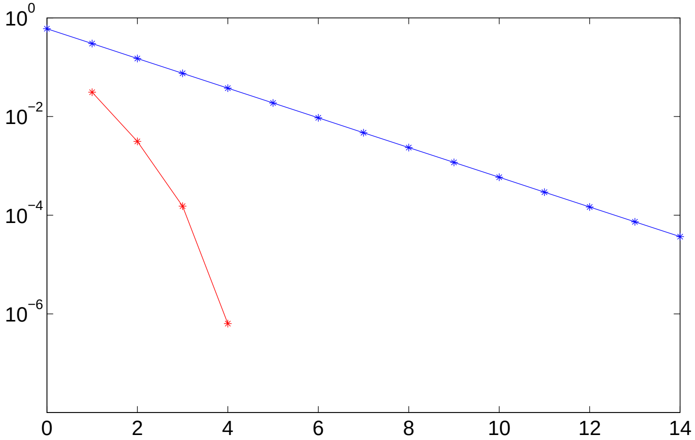
- Steeper than linear on semilog plot.



$10^{0}$
$10^{-2}$
$10^{-4}$
$10^{-6}$
0
2
4
6
8
10
12
14

Simple example of *recursive* programming:

```
def bisection(a, b, kMax, epsX, epsF):
    mid = (a+b)/2 ; x = mid
    err = abs(a-b)/2
    fm = func(mid)
    res = abs(fmid)
    if (kMax > 0):
        if err > epsX or res > epsF:
            fa = func(a)
            if (fm*fa > 0):
                a = mid
            else:
                b = mid
            x,err,res = bisection(a, b, kMax-1, epsX
                                  ,epsF)
    else:
        print("Warning: no convergence...")
    return x,err,res
```

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Cons:

- code harder to follow;
- uses more memory;
- small mistake can give bad crash!

Newton iteration can be generalized to $n$ equations with $n$ unknowns.
Alternative derivation in 1D:

$$f(x+\delta x) \approx f(x) + f'(x)\delta x = 0 \quad \Rightarrow \quad \delta x = -\frac{f(x)}{f'(x)}$$

Now in 2D. We want to find $x_1$ and $x_2$ such that

$$f_1(x_1, x_2) = 0$$
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Note that, in general, we need the same number of equations and unknowns to find (isolated) solutions...

$$f_1(x_1+\delta x_1, x_2+\delta x_2) \approx f_1(x_1,x_2) + \frac{\partial f_1}{\partial x_1}(x_1,x_2)\delta x_1 + \frac{\partial f_1}{\partial x_2}(x_1,x_2)\delta x_2$$
$$f_2(x_1+\delta x_1, x_2+\delta x_2) \approx f_2(x_1,x_2) + \frac{\partial f_2}{\partial x_1}(x_1,x_2)\delta x_1 + \frac{\partial f_2}{\partial x_2}(x_1,x_2)\delta x_2$$

which gives:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1,x_2) & \frac{\partial f_1}{\partial x_2}(x_1,x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1,x_2) & \frac{\partial f_2}{\partial x_2}(x_1,x_2) \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix} = -\begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix}$$