2072U Computational Science I
Winter 2022

Week	Topic
1	Introduction
1–2	Solving nonlinear equations in one variable
3–4	Solving systems of (non)linear equations
5–6	Computational complexity
6–8	Interpolation and least squares
8–10	Integration & differentiation
10–12	Additional Topics

1. Central questions
2. LU decomposition with partial pivoting
3. Using LU decompositions to solve linear systems

Central questions:

- For any square $A \in \mathbb{R}^{n\times n}$, does a decomposition $A = LU$ exist?
- What is (partial) pivoting?
- What is a permutation matrix? How can it be represented?
- How is an LU decomposition $PA = LU$ computed?
- How do LU decompositions $A = LU$ or $PA = LU$ relate to solving linear systems of equations?
- How can LU decompositions be computed and used in PYTHON?

Existence of LU decomposition $A = LU$.

Proposition

For a given nonsingular matrix $A \in \mathbb{R}^{n \times n}$, the LU decomposition $A = LU$ exists and is unique iff all the leading principal submatrices of A are nonsingular.

Note: a *leading submatrix* is obtained from a matrix A by extracting its first k rows and columns: $A(1:k, 1:k)$.

- LU decomposition $A = LU$ has L unit lower triangular and U upper triangular
- Not always possible to find $A = LU$ for A nonsingular
- When A nonsingular, always possible to find permutation P such that $PA = LU$, i.e., so that PA has a Gauss (LU) factorisation

Permutation matrices:

Definition (Permutation matrix)

A permutation matrix is any matrix obtained from interchanging the rows or columns of the identity matrix.

- e.g., $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
- For any permutation matrix, $P^{-1} = P^T$
- PA permutes rows of matrix A
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Example: left multiplication by permutation matrix permutes rows

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$$

Example: right multiplication by transpose of permutation matrix permutes rows

$$AP^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 4 \\ 8 & 9 & 7 \end{bmatrix}$$

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- Instead, use permutation matrix to swap rows 1 & 2

$$PA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{if} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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$$PA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{if} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- PA is upper triangular, so LU decomposition is $\boxed{PA = LU}$ with $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

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- In finite precision arithmetic, $1 - 10^{20} \simeq -10^{20}$, so

$$\tilde{L} = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \text{ and } \tilde{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

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- Notice $\tilde{L} = L$ and $\tilde{U} \simeq U$ but $\tilde{L}\tilde{U} = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 0 \end{bmatrix} \neq A$

We want to find P, L, U, such that $PA = LU$.

$\rightarrow$ Use LU with partial pivoting.

Partial pivoting refers to swapping rows, not columns:

- Initialise $U = A$, $L = I$, and $P = I$
- Loop through pivot columns ($k = 1 : n - 1$):
  - Find largest entry in pivot column k, say $U_{\ell,k}, \ell \geq k$
  - Interchange rows $k \leftrightarrow \ell$ in P and U
  - Interchange rows $k \leftrightarrow \ell$ in L up to (not including) pivot column, i.e. just the first $k - 1$ columns
  - Eliminate rows below row k as usual (Gaussian elimination)
- End result is three matrices P, L, and U such that $PA = LU$

Pseudo-code for LU decomposition with partial pivoting

Input: $A \in \mathbb{R}^{n \times n}$

$U \leftarrow A$, $L \leftarrow I$, $P \leftarrow I$	(initialise matrices)
for $k = 1 : n-1$	(loop through pivot columns)
Select $\ell \geq k$ to maximise $\lvert U_{\ell,k} \rvert$	(choose pivot element)
$U_{k,k:n} \leftrightarrow U_{\ell,k:n}$	(swap rows of U)
$L_{k,1:k-1} \leftrightarrow L_{\ell,1:k-1}$	(swap rows of L up to pivot)
$P_{k,:} \leftrightarrow P_{\ell,:}$	(swap rows of P)
for $j = k+1 : n$	(loop through rows under pivot)
$L_{j,k} \leftarrow U_{j,k} / U_{k,k}$	(store multiplier in L matrix)
$U_{j,k:n} \leftarrow U_{j,k:n} - L_{j.k} U_{k,k:n}$	(update row j of U matrix)
end for	
end for	

Output: Matrices L, U and P

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix}$, construct the LU decomposition with partial pivoting, i.e. find $PA = LU$.

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Initialise matrices; pivot column $k = 1$

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$$U = \begin{bmatrix} 3 & 6 & 4 \\ & -1 & \frac{5}{3} \\ & -2 & -\frac{2}{3} \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - \tfrac{1}{3} R_1 \qquad L_{2,1} \leftarrow \tfrac{1}{3}$$
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Eliminate beneath pivot column $k = 1$ in U, store multipliers in L

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Swap rows 2 and 3 in U and P; swap up to pivot column $k = 2$ in L

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix}$, construct the LU decomposition with partial pivoting, i.e. find $PA = LU$.

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$$R_3 \leftarrow R_3 - \tfrac{1}{2} R_2 \qquad L_{3,2} \leftarrow \tfrac{1}{2}$$

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We arrive at the decomposition $PA = LU$.

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

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Initialise matrices; pivot column $k = 1$

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$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \qquad R_1 \leftrightarrow R_3$$

Swap rows 1 and 3 in U and P

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Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{4} & 0 & 1 & 0 \\ \frac{3}{4} & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - \tfrac{1}{2} R_1 \qquad L_{2,1} \leftarrow \tfrac{1}{2}$$
$$R_3 \leftarrow R_3 - \tfrac{1}{4} R_1 \qquad L_{3,1} \leftarrow \tfrac{1}{4}$$
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Eliminate beneath pivot column $k = 1$ in U, store multipliers in L

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Swap rows 2 and 4 in U and P; swap up to pivot column $k = 2$ in L

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$$R_3 \leftarrow R_3 + \tfrac{3}{7} R_2 \qquad L_{3,2} \leftarrow -\tfrac{3}{7}$$
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Eliminate beneath pivot column $k = 2$ in U, store multipliers in L

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$$R_4 \leftarrow R_4 - \tfrac{1}{3} R_3 \qquad\qquad L_{4,3} \leftarrow \tfrac{1}{3}$$

Eliminate beneath pivot column $k = 3$ in U, store multiplier in L

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We arrive at the decomposition $PA = LU$

Solving $A\mathbf{x} = \mathbf{b}$ from $PA = LU$

- Multiply by P:
$$PA\mathbf{x} = P\mathbf{b}$$
- Replace PA by LU:
$$LU\mathbf{x} = P\mathbf{b}$$
- Define $\mathbf{y} = U\mathbf{x}$ and solve
$$L\mathbf{y} = P\mathbf{b}$$
for $\mathbf{y}$ by forward substitution.
- Solve
$$U\mathbf{x} = \mathbf{y}$$
for $\mathbf{x}$ by backward substitution.

Using `scipy.linalg`:

```
import scipy

A = np.array([[0.2,1.4,-0.4,12.3],
              [-2.3,4.2,1.1,-0.9],
              [9.2,-2.3,-0.1,2.2],
              [3.4,3.3,-10.1,4.0]])
b = np.array([[0.2],[-1.2],[3.3],[0.2]])
Pt,L,U = scipy.linalg.lu(A)
Pb = scipy.dot(Pt.T,b)
y = scipy.linalg.solve_triangular(L,Pb,lower=True)
x = scipy.linalg.solve_triangular(U,y,lower=False)
```

- `scipy.linalg.solve` computes solution directly.
- Better to compute LU decomposition first if you have many systems to solve with different right hand sides.

Remarks

- Partial pivoting: interchanging rows to use small multipliers.
- Partial pivoting: multipliers $L_{k,\ell}$ satisfy $|L_{k,\ell}| \leq 1$.
- Pivoting stabilises differences in elimination.
- Complete pivoting: swapping rows and columns.
- $PA = LU$ decomposition is the default way to solve linear systems.

Representing permutation matrices by a vector:

- Unnecessary to store entire matrix P; single vector suffices
- Represent permutation matrix P by column vector $\mathbf{p}$
- Initialise $\mathbf{p} = (1:n)^T$
- Swap rows of $\mathbf{p}$ at each pivot step of Gaussian elimination
- End result: vector $\mathbf{p}$ with order of rows of identity matrix to give P

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \rightarrow \qquad \mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

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Initialise matrices; pivot column $k = 1$

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Swap rows 1 and 2 in U and $\mathbf{p}$

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$$U = \begin{bmatrix} 3 & 6 & 4 \\ & -1 & \frac{5}{3} \\ & -2 & -\frac{2}{3} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - \tfrac{1}{3} R_1 \qquad L_{2,1} \leftarrow \tfrac{1}{3}$$
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Eliminate beneath pivot column $k = 1$ in U, store multipliers in L

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$$U = \begin{bmatrix} 3 & 6 & 4 \\ & -2 & -\frac{2}{3} \\ & & {\color{red}2} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & {\color{red}\frac{1}{2}} & 1 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - \tfrac{1}{2} R_2 \qquad\qquad L_{3,2} \leftarrow \tfrac{1}{2}$$

Eliminate beneath pivot column $k = 2$ in U, store multiplier in L

Exercise: Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 3 & 6 & 4 \\ & -2 & -\frac{2}{3} \\ & & 2 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}$$

We arrive at the decomposition $PA = LU$.

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Initialise matrices; pivot column $k = 1$

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \qquad\qquad R_1 \leftrightarrow R_3$$

Swap rows 1 and 3 in U and $\mathbf{p}$

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{4} & 0 & 1 & 0 \\ \frac{3}{4} & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - \tfrac{1}{2} R_1 \qquad L_{2,1} \leftarrow \tfrac{1}{2}$$
$$R_3 \leftarrow R_3 - \tfrac{1}{4} R_1 \qquad L_{3,1} \leftarrow \tfrac{1}{4}$$
$$R_4 \leftarrow R_4 - \tfrac{3}{4} R_1 \qquad L_{4,1} \leftarrow \tfrac{3}{4}$$

Eliminate beneath pivot column $k = 1$ in U, store multipliers in L

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$ $\qquad$ $R_2 \leftrightarrow R_4$ $\qquad$ $R_2 \leftrightarrow R_4$

Swap rows 2 and 4 in U and **p**; swap up to pivot column $k = 2$ in L

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{2}{7} & \frac{4}{7} \\ & & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{3}{7} & 1 & 0 \\ \frac{1}{2} & -\frac{2}{7} & 0 & 1 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + \tfrac{3}{7} R_2 \qquad L_{3,2} \leftarrow -\tfrac{3}{7}$$
$$R_4 \leftarrow R_4 + \tfrac{2}{7} R_2 \qquad L_{4,2} \leftarrow -\tfrac{2}{7}$$

Eliminate beneath pivot column $k = 2$ in U, store multipliers in L

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{6}{7} & -\frac{2}{7} \\ & & -\frac{2}{7} & \frac{4}{7} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{2}{7} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{7} & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$ $\qquad$ $R_3 \leftrightarrow R_4$ $\qquad$ $R_3 \leftrightarrow R_4$

Swap rows 3 and 4 in U and $\mathbf{p}$; swap up to pivot column $k = 3$ in L

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{6}{7} & -\frac{2}{7} \\ & & & \frac{2}{3} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{2}{7} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{7} & \frac{1}{3} & 1 \end{bmatrix}$$

$$R_4 \leftarrow R_4 - \tfrac{1}{3} R_3 \qquad\qquad L_{4,3} \leftarrow \tfrac{1}{3}$$

Eliminate beneath pivot column $k = 3$ in U, store multiplier in L

Exercise:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$, construct the LU decomposition $PA = LU$.

$$U = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{6}{7} & -\frac{2}{7} \\ & & & \frac{2}{3} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{2}{7} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{7} & \frac{1}{3} & 1 \end{bmatrix}$$

We arrive at the decomposition $PA = LU$

Summary

- Diagonal & triangular systems are simple to solve
- Gaussian elimination: converting square matrix to triangular form
- Gauss factorisation: finding L, U such that $A = LU$
- Permutation matrices (representation as matrices and vectors)
- Gaussian elimination with partial pivoting always possible
- Pivoting reduces magnitude of multipliers, stabilising elimination
- Decomposition $PA = LU$ useful for repeated solves