PROBLEMS ON PARITY AND PIGEONHOLE PRINCIPLE

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1. Parity

1.1. Part I: Direct problems.

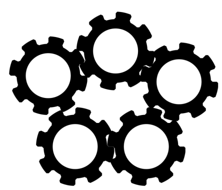
(1) Prove that the sum of 2 odd numbers or 2 even numbers is always even. Further, show that sum of an odd and an even number is always odd.
(2) Show that if the sum of two numbers is odd, their product is always even.
(3) Given a,b integers, is $(a.b.(a-b))$ always odd, always even or neither?

1.2. Part II: Word(y) problems.

(1) Peter has a notebook with 96 pages(excluding covers). He numbered them $1, 2, 3, ..., 192$ in order. Tony tore 25 pages from the notebook, and added the 50 numbers on them. Could tony possibly get 2020 as a sum of those page numbers?
(2) Suppose you have \$25. You want to change it using exactly 10 bills of \$1 , \$3 , and \$5. Can you possibly get the change?

1.3. Part III: Games and Parity.

(1) Can a 5x5 square checkerboard be covered with 1x2 dominos?
(2) Suppose, on a chess board, a knight is ploaced at $a1$. After several moves, the knight returns to position $a1$. Has the knight moved even number of times, or odd number of times?
(3) 5 gears are arranged in a cycle. Can all the gears move simultaneously?



2. Pigeonhole Principle

2.1. Part I: Basic Principles.

(1) A bag has balls of two colors: white and black. What is the smallest number of balls that you need to take out(without looking) so that we surely have two balls of same colour?
(2) Given 12 integers, show that two of them can be chosen such that their difference is divisible by 11.
(3) Show that an equilateral triangle cannot be covered by two smaller equilateral triangles.
(4) $1,900,000$ trees grow in a forest. it is known that each tree may have atmost $600,000$ leaves. Can we have two trees with same number of leaves? Can you say something more?

2.2. Part II: General Pigeons!

(1) Ten students solved a total of 35 problems in a math olympiad. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved exactly two problems, and at least one student who solved exactly three problems. Prove that there is also at least one student who solved at least 5 problems.

2.3. **Part III: Some divisiblity problems.**

(1) Show that there are two powers of two, such that their difference is divisible by 2019.
(2) Show that there is a number written entirely with the digit 1, such that the number is divisible by 2019.