DISCRET

R3.5. CuLTS

$H(z) = \frac{1}{z^2-2z+1}$, $u(k) = 1(k)$

D

Sol: $U(z) = \frac{z}{z-1} \Rightarrow Y(z) = H(z)\,U(z) =$

$= \frac{1}{(z-1)^2} \cdot \frac{z}{z-1} \Rightarrow \frac{Y(z)}{z} = \frac{1}{(z-1)^3} \leftarrow$ Met 2. F.s.

Met 1: Folosind formula reziduurilor

$y[k] = \text{Rez}\left[Y(z), 1\right] = \frac{1}{(3-1)!}\left[(z-1)^3 \cdot \frac{z \cdot z^{k-1}}{(z-1)^3}\right]''\Big|_{z=1}$

$= \frac{1}{2!}(z^k)''\Big|_{z=1} = \frac{1}{2}k(k-1)\cdot z^{k-2}\Big|_{z=1} =$

$= \frac{k(k-1)}{2} 1(k).$

R3.6. CuLTS

D

Calculati y_p, y_t al $H(z) = \frac{1}{z+0.2}$

la $u(k) = e^{j\omega k} 1(k)$.

Apoi la $v[k] = \sin \omega k \, 1(k)$.

Sol $Y(z) = H(z)U(z) = H(z)\frac{z}{z-e^{j\omega}} = \frac{z}{(z+0.2)(z-e^{j\omega})}$

$\Rightarrow \frac{Y(z)}{z} = \frac{1}{(z+0,2)(z-e^{j\omega})} = \frac{A}{(z+0,2)} + \frac{B}{z-e^{j\omega}}$

$= \frac{Az - Ae^{j\omega} + Bz + 0,2B}{(z+0,2)(z-e^{j\omega})} = \frac{z(A+B) + 0,2B - Ae^{j\omega}}{}$

$\Leftrightarrow \begin{cases} A+B=0 \Rightarrow B=-A \\ 0,2B - Ae^{j\omega} = 1 \end{cases} \Bigg| \Rightarrow A = \frac{-1}{0,2+e^{j\omega}} = -B$

$\Rightarrow \frac{Y(z)}{z} = \frac{1}{0,2+e^{j\omega}}\left(\frac{1}{z-e^{j\omega}} - \frac{1}{z+0,2}\right)$

$$= \underbrace{\frac{1}{0{,}2+e^{j\omega}}}_{H(e^{j\omega})} (\underbrace{e^{j\omega k}}_{y_p} - \underbrace{(-0{,}2)^k}_{y_t})1(k)$$

OBS. Știam din capitolul de sist. de convoluție că $y_\sim(k) = |H(e^{j\omega})| e^{(\omega k + \arg H(e^{j\omega}))}$

Pentru $v[k] = \sin\omega k \, 1(k)$

Met 1 $V(s) = \mathcal{Z}\{\sin\omega k\} = \frac{z \sin\omega}{z^2+2z\cos\omega+1}$

și apoi desfacem în fracții simple sau reziduuri.

Met 2: Obs. că $\sin\omega k = \frac{e^{j\omega k} - e^{-j\omega k}}{2j}$

$$\Rightarrow \tilde{y}(k) = \frac{y_\sim(k) - \overline{y_\sim(k)}}{2j}$$

$$= \frac{1}{2j}\left(\frac{e^{j\omega k}-(-0{,}2)^k}{0{,}2+e^{j\omega}} - \frac{e^{-j\omega k}-(-0{,}2)^k}{0{,}2+e^{-j\omega}}\right)\cdot 1(k)$$

$$= \frac{1}{2j}\left(\frac{\cos\omega k-(-0{,}2)^k+j\sin\omega k}{0{,}2+\cos\omega+j\sin\omega} - \frac{\cos\omega k-(-0{,}2)^k-j\sin\omega k}{0{,}2+\cos\omega-j\sin\omega}\right)1(k)$$

$$= \frac{1}{2j}\cdot\frac{(0{,}2+\cos\omega-j\sin\omega)[(\cos\omega k-(-0{,}2)^k)+j\sin\omega k] - (0{,}2+\cos\omega+j\sin\omega)[(\cos\omega k-(-0{,}2)^k)-j\sin\omega k]}{0{,}04+0{,}4\cos\omega+\underbrace{\cos^2+\sin^2}}$$

num $= 1{,}04+0{,}4\cos\omega$

Pentru ușurarea calculului, facem următoarele notații:

$a = 0{,}2+\cos\omega$, $c = \cos\omega k-(-0{,}2)^k$

$b = \sin\omega$, $d = \sin\omega k$

R3.7 Cool TS   Cât este $y_1(k)$ pt...

D $F(z) = \frac{z+0,1}{(z-0,1)(z+0,5)^2}$

Sol: Folosim calculul rezidual

$$y_1(k) = f(k) = \sum_{i=1}^{2} \text{Rez}\left[z^{k-1} F(z); p_i\right]$$

Poli: $p_1 = 0,1$; $p_{2,3} = -0,5$

p_1: $\text{Rez}\left[z^{k-1} F(z), 0,1\right] = \frac{1}{0!}\left[(z-0,1)z^k F(z)\right]\Big|_{z=0,1}$

$$= \frac{1}{0!}\left[\frac{z+0,1}{(z+0,5)^2} \cdot z^k\right]\Bigg|_{0,1} = \left[0,1^k \cdot \frac{0,2}{0,36}\right] \cdot 1(k-1)$$

p_2: $\text{Rez}\left[z^{k-1} F(z); -0,5\right] = \frac{1}{(2-1)!}\frac{d}{dz}\left[\frac{z+0,1}{z-0,1} z^{k-1}\right]\Bigg|_{-0,5}$

$$= \frac{d}{dz}\left[\frac{z^k + 0,1 z^{k-1}}{(z-0,1)}\right]\Bigg|_{0,5} =$$

$$= \frac{\left[k z^{k-1} + (k-1)\cdot 0,1\, z^{k-2}\right](z-0,1) - (z^k + 0,1 z^{k-1})}{(z-0,1)^2}\Bigg|_{0,5}$$

$$= \left[\frac{2}{3}(k-1)\left(-\frac{1}{2}\right)^{k-2} - \frac{5}{9}\left(-\frac{1}{2}\right)^{k-1}\right] 1(k-1)$$

R3.8 Cool TS   Răsp pt al $H(z) = \frac{1}{(2z-1)(4z+1)}$

D   cu $u(k) = 1(k)$

Sol: $Y(z) = H(z)U(z) = H(z) \cdot \frac{z}{z-1} =$

$$= \frac{z}{(2z-1)(4z+1)(z-1)} \Rightarrow \frac{A}{2z-1} + \frac{B}{4z+1} + \frac{C}{z-1} = \frac{Y(z)}{z}$$

$$\Leftrightarrow \frac{Y(z)}{z} = \frac{A(4z+1)(z-1) + B(2z-1)(z-1) + C(2z-1)(4z+1)}{(2z-1)(4z+1)(z-1)}$$

Cel mai simplu: Calculul rezidual

$$A = \left[(2z-1)\frac{Y(z)}{z}\right]\Bigg|_{z=\frac{1}{2}} = \left[\frac{1}{(4z+1)(z-1)}\right]\Bigg|_{\frac{1}{2}}$$

$$= \frac{1}{(2+1)(-\frac{1}{2})} = -\frac{2}{3}$$

$$B = \left[(4z+1)\frac{Y(z)}{z}\right]\Bigg|_{z=-\frac{1}{4}} = \left[\frac{1}{(2z-1)(z-1)}\right]\Bigg|_{-\frac{1}{4}}$$

$$= \frac{1}{(-\frac{1}{2}-1)(-\frac{1}{4}-1)} = \frac{1}{-\frac{3}{2}\cdot(-\frac{5}{4})} = \frac{8}{15}$$

OBS: Deoarece $H(z)$ este stabil $\Rightarrow$

$$\Rightarrow C = H(1) = \frac{1}{(4+1)(2-1)} = \frac{1}{5}$$

$\Downarrow$

$$Y(z) = \frac{2}{3}\cdot\frac{z}{z-\frac{1}{2}}\cdot\frac{1}{2} + \frac{8}{15}\cdot\frac{z}{z+\frac{1}{4}}\cdot\frac{1}{4}$$

$$+ \frac{1}{5}\frac{z}{z-1}$$

OBS Recall

$$\mathcal{Z}(a^k\,1(k))(z) = \frac{a^{-1}z}{a^{-1}z-1} = \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{z-a}!$$

$$\Rightarrow y(z) = \underbrace{\frac{1}{5}1(k)}_{y_p(k)} + \underbrace{\frac{1}{3}\left(\frac{1}{2}\right)^k 1(k) + \frac{2}{15}\left(-\frac{1}{4}\right)^k 1(k)}_{y_t(k)}$$

P3.25 CoolTS

$h(k) = ?$ $H(z) = \frac{3z}{z-3}$

D

$y_5(k) = ?$

Sol: Din prop tr $\mathcal{Z}$ $\Rightarrow h(k) = 3\cdot(3)^k 1(k)$

$= 3^{k+1}\cdot 1(k)$ → instabil!

Dc $u(k) = 1(k) \Rightarrow Y(z) = \frac{3z}{z-3}\cdot\frac{z}{z-1}$

$$\Rightarrow \frac{Y(z)}{z} = \frac{3z}{(z-3)(z-1)} = \frac{A}{z-3} + \frac{B}{z-1}$$

$$A = \left[(z-3)\frac{Y(z)}{z}\right]_{z=3} = \left[\frac{3z}{(z-1)}\right]\Bigg|_{z=3} = \frac{9}{2}$$

$$B = \left[(z-1)\frac{Y(z)}{z}\right]\Bigg|_{z=1} = \left(\frac{3z}{z-3}\right)\Bigg|_{z=1} = -\frac{3}{2}$$

$$\Rightarrow Y(z) = -\frac{3}{2}\cdot\frac{z}{z-1} + \frac{9}{2}\frac{z}{z-3}$$

$$\Rightarrow y(k) = -\frac{3}{2}1(k) + \frac{9}{2}3^k 1(k)$$

P3.26 CoolTS

$H(z) = \frac{1}{z+2}$ la $u(k) = e^{j\omega k}1(k)$

D

Sol: Poli sist. $p = -2 \Rightarrow |p| > 1$

$\Rightarrow$ sist instabil $\Rightarrow \cancel{(\exists)}$ regim perm și tranz.

P3.27 CoolTS

$H(z) = \frac{6z^2+3z}{(2z-1)(3z+1)}$, $u(k) = 1(k)$

D

Sol: Poli: $p_1 = \frac{1}{2}$, $p_2 = -\frac{1}{3}$ $\in$ disc unit.

$\Rightarrow$ sist este stabil $\Rightarrow (\exists)$ regim p și t

$$\frac{Y(z)}{z} = \frac{6z^2+3z}{(2z-1)(3z+1)}\cdot\frac{1}{z-1}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{H(1)}{z-1} + \frac{A}{(2z-1)} + \frac{B}{3z+1}$$

$$A = \left[(2z-1)\frac{Y(z)}{z}\right]_{z=\frac{1}{2}} = \left[\frac{6z^2+z}{(3z+1)(z-1)}\right]_{z=\frac{1}{2}}$$

$$= \left[\frac{\frac{3}{2}+\frac{1}{2}}{\left(\frac{3}{2}+1\right)\left(-\frac{1}{2}\right)}\right] = \frac{2}{-\frac{5}{4}} = -\frac{8}{5}$$

$$B = \left[(3z+1)\frac{Y(z)}{z}\right]_{z=-\frac{1}{3}} = \left[\frac{6z^2+z}{(2z-1)(z-1)}\right]_{z=-\frac{1}{3}}$$

$$= \frac{\frac{6}{9} - {}^{3/}\frac{1}{3}}{\left(-\frac{2}{3}-1\right)\left(-\frac{1}{3}-1\right)} = \frac{\frac{1}{3}}{-\frac{5}{3}\cdot\left(-\frac{4}{3}\right)} = \frac{1}{20}$$

$H(1) = \frac{9}{4}$, Prin urmare,

$$Y(z) = \frac{9}{4}\frac{z}{z-1} - \frac{8}{5}\cdot\frac{1}{2}\frac{z}{z-\frac{1}{2}} + \frac{1}{20}\cdot\frac{1}{3}\frac{z}{z+\frac{1}{3}}$$

$$\Rightarrow y(k) = \Big[\underbrace{\frac{9}{4}}_{y_p} \underbrace{- \frac{4}{5}\left(\frac{1}{2}\right)^k + \frac{1}{20}\left(-\frac{1}{3}\right)^k}_{y_t}\Big]1(k)$$

A 3.34 cont TS $\quad H(z) = \frac{z+a}{z^2+4}$; a) Este stabil $a \in \mathbb{R}$?

D $\quad$ b) $y_S = ?$

Sol: Soluție: Poli $z_{12} = \pm 2j \Rightarrow |z| = 2 > 1$

$\Rightarrow$ sist. este instabil $(\forall)\ a \in \mathbb{R}$.

b) $Y(z) = H(z)U(z) = \frac{z+a}{z^2+4}\cdot\frac{z}{z-1}$,

Desc în fr. simple complex pe $\frac{Y(z)}{z} = \frac{z+a}{(z^2+4)(z-1)}$

$$\Rightarrow \frac{Y(z)}{z} = \frac{A}{z+2j} + \frac{\bar{A}}{z-2j} + \frac{B}{z-1}$$

Folosind calculul rezidual

$$A = \left[(z+2j)\frac{Y(z)}{z} \right]\Big|_{z=-2j} =$$

$$= \frac{z+a}{(z-1)(z-2j)}\Big|_{z=-2j} = \frac{-2j+a}{(1-2j)\cdot(-4j)}$$

$$= \left(-\frac{a}{10}+\frac{4}{10}\right)+2\left(\frac{a}{10}-\frac{1}{10}\right)j =$$

$$= \frac{-a+4}{10}+\frac{1}{5}(a-1)j$$

$$\bar{A} = \frac{4-a}{10}-\frac{1}{5}(a-1)j$$

$$B = \frac{a+1}{5}$$

Cum recuperăm semnalele din $z\frac{A(\bar{A})}{z\pm 2j}$?

OBS. $$\mathcal{Z}^{-1}\left\{\frac{z}{z\pm 2j}\right\}(k) = (\mp 2j)^k$$

$$= (-2^k)\, e^{\pm j\pi/2 k}$$

$$= (-2^k)\left(\cos\frac{k\pi}{2} \pm j\sin\frac{k\pi}{2}\right)$$

Deci $$Y(z) = A\frac{z}{z+2j}+\bar{A}\frac{z}{z-2j}+B\cdot\frac{z}{z-1}$$

$$\Rightarrow y(k) = A(2j)^k+\bar{A}(-2j)^k+B\,1(k)$$

$$= 2^k A\left(\cos\frac{k\pi}{2}+j\sin\frac{k\pi}{2}\right)+(-2)^k\bar{A}\cdot$$

$$\cdot\left(\cos\frac{k\pi}{2}-j\sin\frac{k\pi}{2}\right)+B\cdot 1(k)$$

etc!