A BRIEF INTRODUCTION INTO HYDRODYNAMICS

Variational point of view

Denys Dutykh
CNRS INSMI – LAMA UMR 5127
Université Savoie Mont Blanc

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To my parents.

ABSTRACT

In this manuscript we present some basic notions of Hydrodynamics in view of free surface flows modelling. The text is composed of 11 Chapters. We start the exposition with the brief description of classical Lagrangian and Hamiltonian mechanics. Then, we move to the continuous case where we present the governing equations of the incompressible fluid mechanics in the Eulerian description. The variational structure is highlighted for irrotational flows. Finally, we present the Lagrangian (fluid particle) description of incompressible flows along with some exact solutions. Throughout this manuscript we make a particular emphasis on the variational structures arising in classical and fluid mechanics. Finally, this manuscript contains an Appendix containing a brief description of the main mathematical tools needed in the main part of the text.

ACKNOWLEDGMENTS

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The hospitality of the Johannes Kepler Universität Linz (Austria) and of the Basque Center for Applied Mathematics (BCAM) is also acknowledged, where some parts of this manuscript were prepared. Of course, the main part of this work was done at my home laboratory — LAMA UMR #5127 at the University of Savoie Mont Blanc, France. The constant support of my friends and colleagues Dr. Marx Chhay, Dr. Marguerite Gisclon, Dr. Michel Raibaut and many others is also greatly acknowledged. I would like to thank also my collaborators, especially Prof. Didier Clamond (University of Nice Sophia Antipolis, France), Prof. Dimitrios Mitsotakis (Victoria University of Wellington, New Zealand) and Dr. Ashkan Rafiee (Carnegie Wave Energy, Australia), from whome I learnt a lot.

CONTENTS

PREFACE

In order to unveil the general philosophy of the book, the Author would like to quote some passages from the appendix "*Mathematical formalities and style*" to the book by E. JAYNES [Jay03], published posthumously as "*Probability Theory: The Logic of Science*":

> [...] Obviously, mathematical results cannot be communicated without some decent standards of precision in our statements. But a fanatical insistence on one particular form of precision and generality can be carried so far that it defeats its own purpose; XXth century mathematics often degenerates into an idle adversary game instead of a communication process.
>
> The fanatic is not trying to understand your substantive message at all, but only trying to find fault with your style of presentation. He will strive to read nonsense into what you are saying, if he can possibly find any way of doing so. In self-defense, writers are obliged to concentrate their attention on every tiny, irrelevant, nit-picking detail of how things are said rather than on what is said. The length grows; the content shrinks.
>
> Mathematical communication would be much more efficient and pleasant if we adopted a different attitude. For one who makes the courteous interpretation of what others write, the fact that x is introduced as a variable already implies that there is some set X of possible values. Why should it be necessary to repeat that incantation every time a variable is introduced, thus using up two symbols where one would do? (Indeed, the range of values is usually indicated more clearly at the point where it matters, by adding conditions such as $(0 < x < 1)$ after an equation.)
>
> For a courteous reader, the fact that a writer differentiates $f(x)$ twice already implies that he considers it twice differentiable; why should he be required to say everything twice? If he proves proposition A in enough generality to cover his application, why should he be obliged to use additional space for irrelevancies about the most general possible conditions under which A would be true?
>
> A scourge as annoying as the fanatic is his cousin, the compulsive mathematical nitpicker. We expect that an author will define his technical terms, and then use them in

> a way consistent with his definitions. But if any other author has ever used the term with a slightly different shade of meaning, the nitpicker will be right there accusing you of inconsistent terminology. The writer has been subjected to this many times; and colleagues report the same experience.
>
> Nineteenth century mathematicians were not being nonrigorous by their style; they merely, as a matter of course, extended simple civilized courtesy to others, and expected to receive it in return. This will lead one to try to read sense into what others write, if it can possibly be done in view of the whole context; not to pervert our reading of every mathematical work into a witch-hunt for deviations from the Official Style. [. . .]

The Author of this manuscript subscribes to every single word of this quotation.

INTRODUCTION

Physics is, hopefully, simple. Physicists are not.

— Edward Teller

Mechanics as a science was established mainly during the XVIIth century. Copernicus introduced the reference system in 1543 and Galileo Galilei stated the principle of inertia in 1638 in his "*Discorsi e dimostrzioni mathematiche, intorno a due nove scienze alla meccanica ed i movimenti locali*". This period was finished in 1687 when Isaac Newton published his book "*PhilosophiæNaturalis Principia Mathematica*".

For instance, since Galilei we know that a linear uniform motion is a state and not a process!

The beginning of the variational era is due to Pierre de Fermat, a judge in Toulouse, who proposed the principle least time in geometrical optics. It is interesting to note that Fermat is particularly known for his works in Mathematics. This discovery resulted from the criticism about René Descartes's chapter "*Dioptrique*", a part of his celebrated "*Discours de la Méthode*". During this quarrel Fermat criticized in 1637 the lack of rigor in Descartes's "*pseudo proof*". To resolve this cntroversy, Fermat in 1661 formulated his principle of *least time*. In his original work it was named the *principle of natural economy*.

The principle of *least action* in mechanics was stated for the first time by Pierre-Louis Moreau de Maupertuis in 1744, who was a popularizator of Newton's ideas in France. He understood that Newton's equations are equivalent to the minimality conditions of a quantity that he called the *action*. However, the mathematical proof of Maupertuis's principle was given later by Leonhard Euler in his tretise "*Methodus inveniendi lineas curvas maximi minimive proprietate gaudens*". The understanding that the laws of nature can be viewed as optimization principles had a great impact in the XVIIIth century.

The application of variational principles is not restricted exclusively to mechanics and optics. For example, the electric current is distributed in an electrical network such that the energy loss by Joule heating is minimal. Mathematically it is expressed as the celebrated Kirchhoff's laws.

Leonhard Euler, Joseph-Louis Lagrange and William R. Hamilton in their subsequent works set the foundations of modern theoretical physics with consequences which can be found today in general relativity, gauge theory of fundamental interactions and many other fields.

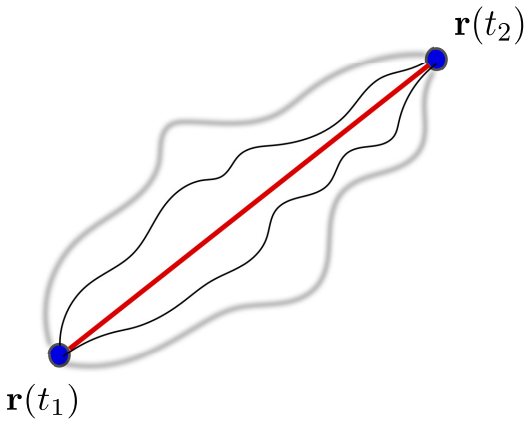


Figure 0.1.: *Optimal trajectory (in red) joining two points in space-time* $\mathbf{r}(t_1)$ *and* $\mathbf{r}(t_2)$.

The main mathematical tool developed by Euler is the calculus of variations. The major contributions of Lagrange in statics as well as in dynamics were published in 1788 in his "*Mécanique Analytique*". In this work Lagrange changed the paradigm of considering a mechanical problem. Instead of finding instantaneous values of the position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ of a particle, given its initial state $\{\mathbf{r}(t_0), \mathbf{v}(t_0)\}$, Lagrange proposed to seek the global trajectory of the particle starting at $\{\mathbf{r}(t_1), \mathbf{v}(t_1)\}$ and arriving at $\{\mathbf{r}(t_2), \mathbf{v}(t_2)\}$, in analogy to Fermat's reasoning in geometric optics.

Hamilton spent all his life in Dublin, Ireland where he served as a Professor of Astronomy and as an Astronomer in Dunsink Observatory. Hamilton noticed the similarity between the Maupertuis's principle in mechanics and Fermat's principle in optics. In 1830 he made a profound observation that Newtonian mechanics corresponds to the same limit as the geometrical optics derives from the wave optics. His remark did not attract much attention until 1890 when Felix Klein made it widely known. Thus, Hamilton's mechanics can be considered as a precursos of quantum mechanics.

Variational principles are universal. All physical laws can be recast in the variational form which leads the usual local form (*i.e.* Euler–Lagrange equations). However, the global form is richer and more powerful. In these lecture notes we shall show how these ideas found applications in Hydrodynamics.

These Lecture Notes are based on several sources. The first two Chapters were greatly inspired by the excellent book of J.-L. Basdevant [Bas07] and lectures of John C. Baez on Classical Mechanics [BWS05]. A few geometrical remarks were inspired by [Fer06]. The classical book of Landau & Lifshitz (volume 1) [LL76] is also highly recommended.

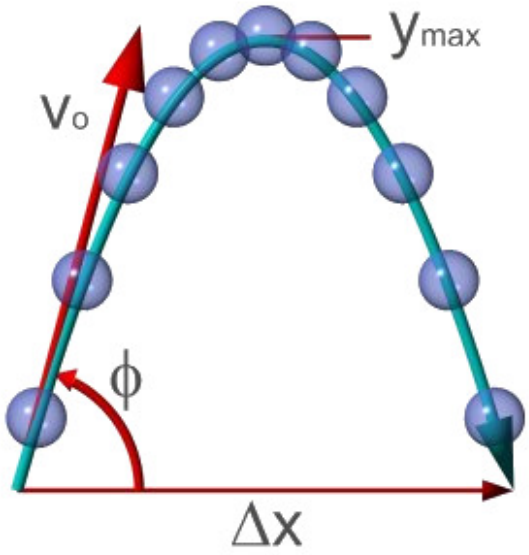
EXERCISES

1. Find in the neighbourhood of Dublin the bridge, under which Sir Hamilton wrote his formulas of quaternions multiplication.

Part I

CLASSICAL MECHANICS

Classical mechanics is a part of classical physics dealing with deterministic systems of point particles or rigid bodies. Sometimes the systems with infinitely many degrees of freedom are also included (*e.g.* infinite arrays of particles). The equations of motions are expressed as systems of Ordinary Differential Equations (ODEs). Nondissipative systems may be described geometrically using the formalism of *symplectic manifolds*, or more generally *Poisson manifolds*. Classical mechanics can be conventionally divided into three parts — *statics*, *kinematics* and *dynamics*. Statics studies the balance of forces in states of equilibria. Kinematics describes the relation between the position, velocity and acceleration, without reference to the causes of motion. Finally, Dynamics studies the forces as causes of motion and as the mean of mechanical interaction between bodies.



A parabolic shooting. ©Wikimedia Commons

1 LAGRANGIAN MECHANICS

In the beginning there was the Action.

— Johann Wolfgang Goethe

Nature always acts by the shortest paths.

— Pierre de Fermat

1.1 LEAST ACTION PRINCIPLE

The variational principle presented below was actually formulated by Hamilton in 1834 and we use, since it is simpler than the original one.

Any conservative mechanical system can be characterized by the Lagrangian density $\mathcal{L}$. For an idealized system consisting of a single particle the Lagrangian $\mathcal{L} = \mathcal{L}(x, \dot{x}, t)$, where $x(t)$ is the particle's position and $\dot{x}(t) := \frac{dx}{dt}$. The quantities $\left(x, \dot{x}\right)$ are the state variables of the particle. For this system the Lagrangian $\mathcal{L}$ takes the form

The choice of the Lagrangian function is non-unique and it will be discussed below.

$$\mathcal{L} := \frac{1}{2} m \dot{x}^2 - V(x, t),$$

where m is the particle's mass, $V(x, t)$ is the potential which determines the forces acting on the particle $f(x, t) := -\frac{\partial V}{\partial x}$.

For an arbitrary trajectory $x(t)$ we can define the *action* $\mathscr{L}$ as the following integral

$$\mathscr{L} := \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)\, dt.$$

The *least action principle* states that the trajectory realized in physics will be such that the action $\mathscr{L}$ has an extremum. In other words, the physical trajectories are stationary points of the Lagrangian functional.

1.2 EULER–LAGRANGE EQUATIONS

1.3 FORM OF THE LAGRANGIAN

1.3.1 Maupertuis's action

The action $\mathcal{A}$ of Maupertuis is the product of three terms: the mass, the velocity and the distance

$$\mathcal{A} := \int_{x_1}^{x_2} m v \, dl.$$

By assuming that the energy E is the integral of motion.

2 HAMILTONIAN MECHANICS

2.1 A GEOMETRICAL PARENTHESIS

Contrary to one's intuition, the letter $\mathcal{H}$ in the Hamiltonian functional comes from the name of a Dutch mathematician Christiaan Huygens.

The field of the modern geometry which studies the phase space of Hamiltonian systems is the symplectic geometry [Sou97]. A symplectic manifold $\mathcal{M}$ is a usual manifold equipped with a symplectic (*i.e.* closed, non-degenerate, 2–form) ω. Darboux's theorem states that in the neighbourhood of any point $x \in \mathcal{M}$ there exist local coordinates (q_i, p_i) such that the symplectic form ω takes the standard form [MS98]. Consequently, all symplectic manifolds, at least locally, have the same structure.

A different interpretation of Hamiltonian equations leads to the so-called Poisson Geometry. In analogy to the symplectic manifolds, a Poisson manifold $\mathcal{P}$ is a manifold equipped with a Poisson bracket $\{\cdot,\cdot\}$ (*i.e.* a Lie bracket on the algebra of smooth functions $C^\infty(\mathcal{P})$ which satisfies the Leibniz identity). Then, the following result can be proven [Wei83]

Theorem 2.1. *Let $\big(\mathcal{P}, \{\cdot,\cdot\}\big)$ be a Poisson manifold. For every $x \in \mathcal{P}$ there exist coordinates $(q_1,\dots,q_n,p_1,\dots,p_n,y_1,\dots,y_l)$ centered at x such that*

$$\{F,G\} = \sum_{i=1}^{n}\Big(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\Big) + \sum_{j,k=1}^{l}\pi_{jk}(y)\frac{\partial F}{\partial y_j}\frac{\partial G}{\partial y_k},$$

where $\pi_{jk}(y) = -\pi_{kj}(y)$ are certain functions of the $(y_1,\dots,y_l)$ alone which vanish at 0.

Therefore, contrary to the symplectic geometry, in Poisson geometry it is important to understand the local structure as well.

Part II

CONTINUUM MECHANICS: EULERIAN DESCRIPTION

Continuum mechanics is a branch of classical physics which studies the mechanical properties of continuous media such as fluids or deformable solid bodies. Continuum mechanics is related to the mechanics of point particles by various limiting processes. Equations of motions are expressed as systems of Partial Differential Equations (PDEs) describing the classical fields such as the velocities, displacements, deformations, strains, *etc.*

A vortex created by this agricultural plane' wing. ©Wikimedia Commons.

3 FLUID DYNAMICS EQUATIONS

We absolutely must leave room for doubt or there is no progress and no learning. There is no learning without posing a question. And a question requires doubt... Now the freedom of doubt, which is absolutely essential for the development of science, was born from a struggle with constituted authorities.

— Richard Feynmann (1964)

In this book we consider the Fluid Mechanics only. For the introduction into the Solid Mechanics theory the reader will have to consult the more specific literature, *e.g.* [LL86, MH94].

Fluid dynamics is a branch of continuum mechanics whics studies the motion of liquids and gases (or fluids in general). A fluid is in general a material which deforms constantly even when it is released from any action of the force. In contrast, an elastic solid recovers to its original state, once it is released from the force, while the fluid will keep deforming. Common examples of fluids include water and air.

The ultimate goal of this discipline is to study and predict the fluid motion or at least some of its main characteristics (for turbulent flows, for example). In this Chapter we present an elementary derivation of basic equations of the Fluid Dynamics. The reader can refer to almost any classical book on this topic if more details are needed. We can recommend the book of A. Chorin & J. Marsden [CM93] for its conciseness and mathematical clarity. The exposition below will follow the great lines of this reference.

3.1 DERIVATION OF THE GOVERNING EQUATIONS

For the sake of clarity all the derivations given below will be performed in Cartesian coordinates. A special care has to be used in other (curvilinear) coordinate systems. We would like to mention that it is possible to develop intrinsic coordinate free formulas valid in any system of coordinates. We refer to [AMR88] for more information on this approach.

3.1.1 Choice of the coordinate system

Consider the standard Euclidean space with a Cartesian coordinates system $Oxyz$ and let Ω be a domain (bounded or unbounded) in $\mathbb{R}^d$ ($d = 2, 3$) filled with a fluid. Let $\mathbf{x} = (x, y, z) \in \Omega$ be a point of the fluid domain. We assume that a *continuum assumption* holds. This means that physical properties such as the density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, *etc.* can be well-defined in every geometrical point $\mathbf{x} \in \Omega$. For most macroscopic phenomena this physical assumption is very accurate.

For instance, this assumption is not verified for the rarefied gases in the stratosphere, where the Boltzmann mesoscopic description is more appropriate.

Let $\rho(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ denote respectively the density and velocity fields in the fluid at the time instance $t \in \mathbb{R}^+$. From the physical sense the density function $\rho(\mathbf{x}, t) > 0$, $\forall t \geqslant 0$ has to be positive in the fluid domain. For each fixed time $t \geqslant 0$ the velocity $\mathbf{u}(\mathbf{x}, t)$ defines a vector field on Ω. Below we shall assume that the functions $\rho(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ are smooth enough in order to perform the standard calculus operations (*e.g.* integration, differentiation, *etc.*) on them.

The derivation of Fluid Mechanics equations is based on three basic principles:

- Mass is neither created nor destroyed
- The rate of change of momentum of the fluid parcel equals the sum of all forces applied to it
- Energy is neither created nor destroyed

3.1.2 Mass conservation

Let us take a (bounded) elementary volume $V \subseteq \Omega$, which does not change with time (for the illustration see Figure 3.1). The mass $m_V(t)$ of the fluid contained in V is then given by[1]

$$m_V(t) = \int_V \rho(\mathbf{x}, t)\, dV,$$

The mass conservation principle was formulated for the first time independently by M. Lomonosov (1748) in Russia and thirty years later by A. Lavoisier (1774) in France.

where dV is a volume element. The rate of change (in time) of the fluid mass contained in a region V is given by the time derivative

$$\frac{d\, m_V}{dt} = \frac{d}{dt} \int_V \rho(\mathbf{x}, t)\, dV = \int_V \frac{\partial\, \rho(\mathbf{x}, t)}{\partial t}\, dV,$$

where the time derivative could be put under the integral sign since the integration domain is time-independent. The change of mass inside V can be possible only due to the fluid transport through the boundary ∂V by the velocity field $\mathbf{u}(\mathbf{x}, t)$ (see Figure 3.2 for the illustration). Since the mass is neither being generated nor destroyed,

The rate of volume flow across ∂V per unit area is $\mathbf{u} \cdot \mathbf{n}$, while the mass flow rate per unit area is $\rho \mathbf{u} \cdot \mathbf{n}$.

1 Normally one has to write triple $\iiint$ and double $\iint$ integrals in this Section. However, we do not do it for the sake of brevity. It will become clear from the context and from the differentials dV and $d\sigma$ which integral is being considered.

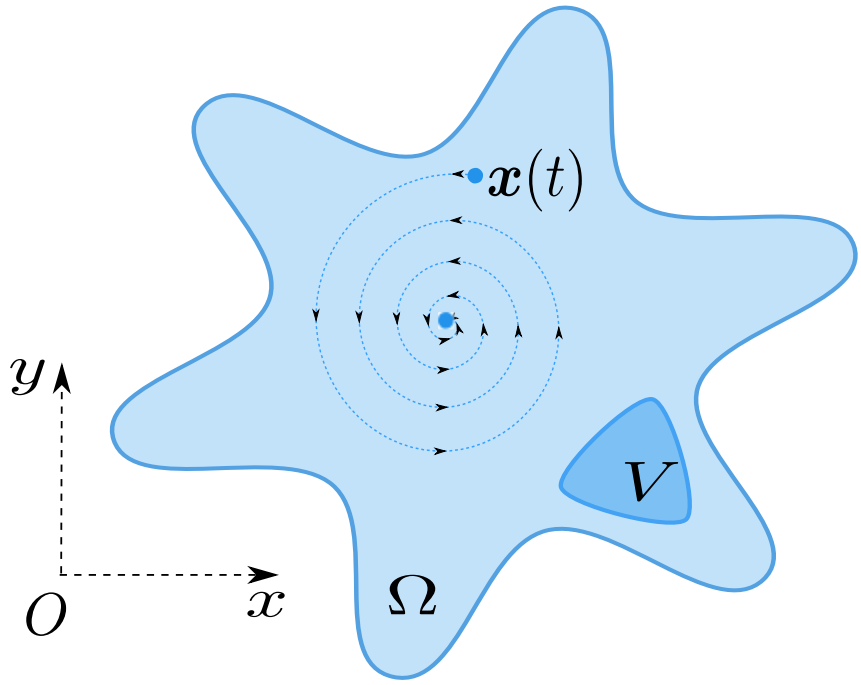


Figure 3.1.: *Sketch of the flow domain Ω along with a possible particle trajectory and a control volume* dV.

both rates of change have to be equal

$$\frac{d}{dt}\int_V \rho(\mathbf{x},t)\,dV = -\int_{\partial V} \rho\mathbf{u}\cdot\mathbf{n}\,d\sigma, \tag{3.1}$$

where $d\sigma$ is the element of surface. The sign $-$ comes from the convention that the positive normal $\mathbf{n}$ to ∂V points in the outward direction. Thus, the incoming mass flux is negative. By applying the Gauß–Ostrogradsky theorem (see Section A.3.7.3) to equation (3.1), one obtains

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right] dV = 0.$$

Since the control volume $V \subseteq \Omega$ is arbitrary, the last integral equation is equivalent to the following differential equation of continuity, which expresses the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{3.2}$$

Both equations (3.1) and (3.2) express the same physical fact. The former is the integral form, while the latter is the differential one. In cases where one of the functions $\rho(\mathbf{x},t)$ or $\mathbf{u}(\mathbf{x},t)$ has discontinuities (*i.e.* shock waves), the integral form (3.1) has to be used.

3.1.3 Momentum conservation

The balance of momentum states that the amount of momentum contained inside a given volume cannot change unless some forces

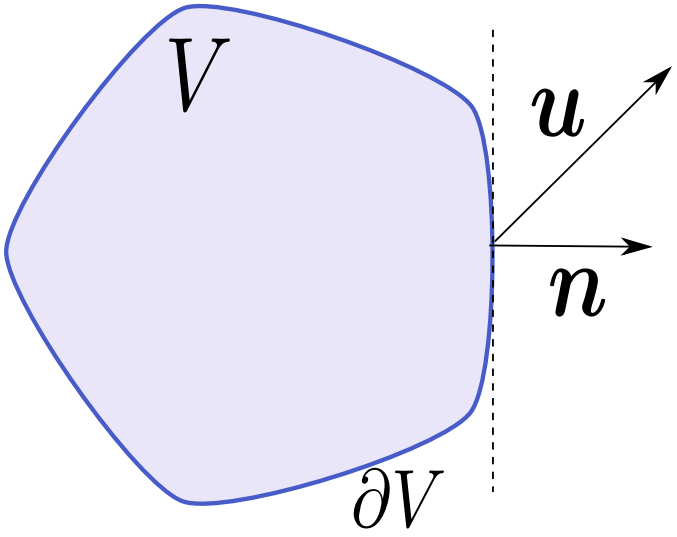


Figure 3.2.: *An elementary volume V with an outward normal* $\mathbf{n}$ *and fluid particle velocity* $\mathbf{u}$ *at this point on the boundary. The mass flux is positive since* $\mathbf{u} \cdot \mathbf{n} > 0$.

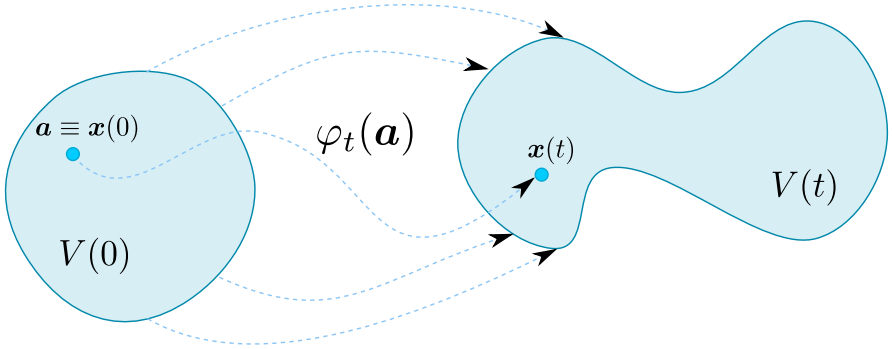


Figure 3.3.: *Action of the flow map on an elementary volume* $V(0)$.

cause it. In other words, the momentum rate of change inside a volume has to be in equilibrium with the forces applied to it.

Consider a fluid particle trajectory $\mathbf{x}(t) = \big(x(t), y(t), z(t)\big)$. We shall assume that there exists a so-called *flow map* $\mathbf{a} \mapsto \mathbf{x}(t) = \phi_t(\mathbf{a})$, which gives the particle position at time t provided that initially it was located at $\mathbf{a} = \mathbf{x}(0)$[2] (see Figure 3.3 for illustration).

The index t *in* $\phi_t(\mathbf{a})$ *should not be confused with a derivative operator in time. Here it means that the flow map depends on time as on the parameter.*

Remark 3.1. *The existence and regularity of the flow map can be shown under quite generic assumptions on the flow field* $\mathbf{u}(\mathbf{x}, t)$. *For example, it is sufficient to assume that* $\mathbf{u}$ *is Lipschitz continuous in* $\mathbf{x}$ *and just continuous*

2 Variable $\mathbf{a}$ is called in general a particle label. It is used extensively in the Lagrangian description of fluid flows. See Chapter 7 for more details.

in t. *Than, the application of Picard–Lindelöf–Cauchy–Lipschitz*[3] *theorem to the following system of ODEs*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(\mathbf{x}(t), t), \quad \mathbf{x}(0) = \mathbf{a}, \tag{3.3}$$

gives us the existence and uniqueness of the flow map locally in time. *A(n) (analytical or more often a numerical) solution to the ODE system* (3.3) *provides us with a practical way of computing the flow map* $\phi_t(\mathbf{a})$.

We can compute a particle velocity and acceleration using the flow map $\phi_t(\mathbf{a})$. For instance, the velocity follows directly from the system (3.3), since

$$\frac{d\,\phi_t(\mathbf{a})}{dt} \equiv \frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(\mathbf{x}(t), t) \equiv \mathbf{u}(\phi_t(\mathbf{a}), t). \tag{3.4}$$

The acceleration of fluid particles is given by the second derivative of the particle trajectory $\mathbf{x} = \mathbf{x}(t)$. Namely,

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \frac{d}{dt}\mathbf{u}(\phi_t(\mathbf{a}), t) = \partial_t\mathbf{u} + \frac{dx}{dt}\partial_x\mathbf{u} + \frac{dy}{dt}\partial_y\mathbf{u} + \frac{dz}{dt}\partial_z\mathbf{u} =$$
$$\partial_t\mathbf{u} + u\partial_x\mathbf{u} + v\partial_y\mathbf{u} + w\partial_z\mathbf{u} = \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} =: \frac{D\mathbf{u}}{Dt}.$$

In the last equality we introduced the *material derivative* operator:

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla). \tag{3.5}$$

Thus, we just showed that the acceleration of fluid particles is given by

Remark 3.2. *We would like to underline that the notation* $(\mathbf{u}\cdot\nabla)$ *means the scalar product of vectors* $\mathbf{u} = (u, v, w)$ *and* $\nabla = \left(\partial_x, \partial_y, \partial_z\right)$. *As a result we obtain an operator (not a function)*

$$\mathbf{u}\cdot\nabla = u\partial_x + v\partial_y + w\partial_z.$$

The notation $(\mathbf{u}\cdot\nabla)$ *is not to be confused with* $\nabla\cdot\mathbf{u}$ *which gives the divergence (see, for example, Lemma 3.1 below).*

The general integral form of balance of momentum states that the rate of change of momentum inside a flowing elementary volume $V(t)$ is equal to the sum of forces acting on this volume and on its surface $\partial V(t)$:

$$\frac{d}{d}\int_{V(t)} \rho\mathbf{u}\, dV = \int_{\partial V(t)} \boldsymbol{\sigma}\cdot\mathbf{n}\, d\sigma + \int_{V(t)} \rho\mathbf{g}\, dV, \tag{3.6}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor [MH94] and $\mathbf{g}$ is the vector representing the volume (body) force per unit mass. In most practical situations $\mathbf{g} = (0, 0, -g)$, where $g = 9.80665\ m/s^2$ is the standard gravity acceleration. The stress tensor $\boldsymbol{\sigma}$ can be easily computed by *ideal* (or perfect) fluids.

The value of the constant g *given in the text is prescribed by current standards. However, the local gravity acceleration varies with the position on the Earth.*

3 This theorem is named after Émile Picard (1856 – 1941), Ernst Lindelöf (1870 – 1946), Rudolf Lipschitz(1832 – 1903) and Augustin-Louis Cauchy (1789 – 1857).

Definition 3.1. *A fluid is called* ideal *if there exists a scalar function* $p(\mathbf{x}, t)$ *called the* pressure *such that the stress tensor* $\boldsymbol{\sigma}$ *inside the fluid is equal to*

$$\boldsymbol{\sigma} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

In this case the force per unit area acting on an elementary surface neighbourhood $\mathcal{S}$ *of a point* $\mathbf{x} \in \mathcal{S}$ *is given by*

$$\boldsymbol{\sigma} \cdot \mathbf{n} = -p(\mathbf{x}, t)\mathbf{n},$$

where $\mathbf{n}$ *is the exterior unit normal to the surface* $\mathcal{S}$ *in the point* $\mathbf{x}$*. The total force* $\mathcal{F}$ *applied to the surface of a closed compact volume* V *can be obtained by computing the following surface integral*

$$\mathcal{F}_{\partial V} = \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma = -\int_{\partial V} p\mathbf{n} \, d\sigma.$$

In order to differentiate under the first integral sign in (3.6), one has to change the variables to work on a fixed integration domain (which is time independent, in other words). For this purpose we shall use the flow map $\mathbf{x} = \phi_t(\mathbf{a})$ to come back to the initial positions of fluid particles[4] (see Figure 3.3 for illustration):

$$\frac{d}{dt} \int_{V(t)} \rho\mathbf{u} \, dV = \frac{d}{dt} \int_{V(0)} (\rho\mathbf{u})\big(\phi_t(\mathbf{a}), t\big) \, J(\mathbf{a}, t) \, dV,$$

where $J(\mathbf{a}, t)$ is the determinant of the Jacobian matrix of the map ϕ_t. Even if the expression under the integral on the right hand side became more complex, the integration domain is fixed, which allows to pull the derivative into the integral sign. By the product rule we have to differentiate the functions $(\rho\mathbf{u})\big(\phi_t(\mathbf{a})$ and $J(\mathbf{a}, t)$. The former can be differentiated by applying the material derivative operator (3.5):

$$\frac{d}{dt}(\rho\mathbf{u})\big(\phi_t(\mathbf{a}), t\big) = \frac{D(\rho\mathbf{u})}{Dt}\big(\phi_t(\mathbf{a}), t\big),$$

while the derivative of the latter is given by the following

Lemma 3.1.

$$\frac{d}{dt} J(\mathbf{a}, t) = \nabla \cdot \mathbf{u}\big(\phi_t(\mathbf{a}), t\big) \, J(\mathbf{a}, t).$$

Proof. In the proof we shall extensively use the well-known fact that a determinant is a multilinear function of its colums. By denoting the components of the map ϕ_t by $(\phi_t^1, \phi_t^2, \phi_t^3)$ and the components of $\mathbf{a}$ by (a, b, c), we have by definition of the determinant:

$$J(\mathbf{a}, t) = \frac{1}{3!} \sum_{\tau \in S_3} (-1)^{\deg(\tau)} \phi_{ta}^{\tau(1)}(\mathbf{a}, t) \phi_{tb}^{\tau(2)}(\mathbf{a}, t) \phi_{tc}^{\tau(3)}(\mathbf{a}, t),$$

4 It is always possible since the flow map is a one-to-one correspondence. Otherwise it would contradict the unicity result for the system (3.3). In other words, the particles trajectories cannot cross.

where S_3 is the symmetric group of permutations of 3 symbols $(1,2,3)$ and $\tau \in S_3$ is an arbitrary permutation. The subscripts a, b and c denote partial derivatives with respect to these independent variables. Now we can differentiate the determinant $J(\mathbf{a},t)$ with respect to time by using the usual product rule:

$$\frac{\partial J}{\partial t} = \frac{1}{3!}\sum_{\tau\in S_3}(-1)^{\deg(\tau)}\Big[\frac{\partial \phi_{ta}^{\tau(1)}(\mathbf{a},t)}{\partial t}\phi_{tb}^{\tau(2)}(\mathbf{a},t)\phi_{tc}^{\tau(3)}(\mathbf{a},t) + \phi_{ta}^{\tau(1)}(\mathbf{a},t)\frac{\partial \phi_{tb}^{\tau(2)}(\mathbf{a},t)}{\partial t}\phi_{tc}^{\tau(3)}(\mathbf{a},t) + \phi_{ta}^{\tau(1)}(\mathbf{a},t)\phi_{tb}^{\tau(2)}(\mathbf{a},t)\frac{\partial \phi_{tc}^{\tau(3)}(\mathbf{a},t)}{\partial t}\Big]. \quad (3.7)$$

By using the chain rule and equation (3.4), one can compute the partial derivatives with respect to time (for the sake of illustration we take the first element of the first column):

$$\frac{\partial \phi_{ta}^{1}(\mathbf{a},t)}{\partial t} = \frac{\partial}{\partial a}\Big(\underbrace{\frac{\partial \phi_{t}^{1}(\mathbf{a},t)}{\partial t}}_{=u(\phi(\mathbf{a},t),t) \text{ by } (3.4)}\Big) = \frac{\partial u(\phi(\mathbf{a},t),t)}{\partial a} = u_x\frac{\partial \phi_t^1}{\partial a} + u_y\frac{\partial \phi_t^2}{\partial a} + u_z\frac{\partial \phi_t^3}{\partial a}.$$

Substituting similar expressions into (3.7), many terms will vanish, since the determinant of a matrix with repeating columns iz zero. What is left is summarized below

$$\frac{\partial J}{\partial t} = u_x J + v_y J + w_z J = (\nabla\cdot\mathbf{u})J.$$

□

Now we can return to our integral and continue its differentiation with respect to time:

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)}\rho\mathbf{u}\,dV &= \\
&= \int_{V(0)}\Big[\frac{D(\rho\mathbf{u})}{Dt} + (\nabla\cdot\mathbf{u})(\rho\mathbf{u})\Big]J\,dV = \\
&= \int_{V(t)}\Big[\frac{D(\rho\mathbf{u})}{Dt} + \rho(\nabla\cdot\mathbf{u})\mathbf{u}\Big]dV = \\
&= \int_{V(t)}\Big[\frac{D\rho}{Dt}\mathbf{u} + \rho\frac{D\mathbf{u}}{Dt} + \rho(\nabla\cdot\mathbf{u})\mathbf{u}\Big]dV = \\
&= \int_{V(t)}\Big[\underbrace{\Big(\frac{D\rho}{Dt}\mathbf{u} + \rho(\nabla\cdot\mathbf{u})\Big)}_{=0 \text{ by } (3.2)}\mathbf{u} + \rho\frac{D\mathbf{u}}{Dt}\Big]dV = \\
&= \int_{V(t)}\rho\frac{D\mathbf{u}}{Dt}\,dV.
\end{aligned}$$

In these computations we actually showed the following important

Theorem 3.1 (Reynolds transport theorem)**.**

$$\frac{d}{dt}\int_{V(t)} \rho f \, dV = \int_{V(t)} \rho \frac{Df}{Dt} \, dV.$$

One can derive another form of the Reynolds transport theorem, which does not involve the mass density:

$$\frac{d}{dt}\int_{V(t)} f \, dV = \int_{V(t)} \left(\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{u}) \right) dV.$$

Remark 3.3. *For instance, the mass conservation equation* (3.2) *can be readily obtained from the last integral identity by setting* $f = \rho(\mathbf{x}, t)$.

However, the main goal of this section is to derive the momentum conservation equation. Indeed, we have assembled all necessary ingredients. After pulling the time derivative under the integral sign in (3.6) and taking into account that the volume $V(t)$ is arbitrary, we obtain the following balance of momentum:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}. \tag{3.8}$$

In situations where the regularity of solutions is low, it is interesting to have the integral form of momentum balance in a fixed volume V. We give it here without the proof (which can be found in [CM93]):

$$\frac{d}{dt}\int_V \rho \mathbf{u} \, dV = -\int_{\partial V} (p\mathbb{I} + \rho \mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{n} \, d\sigma + \int_V \rho \mathbf{g} \, dV,$$

where $\mathbb{I}$ is the identity matrix and $\otimes$ is the tensorial product, *i.e.*

$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{pmatrix}.$$

3.1.4 Energy conservation

A fluid moving in a domain Ω has a kinetic energy K given by

$$K = \frac{1}{2}\int_\Omega \rho |\mathbf{u}|^2 \, dV.$$

In order to obtain the total energy E, one has to add also the internal energy ϵ

$$E = K + \epsilon.$$

Please, note that ϵ does not have to be small ☺

The internal energy ϵ, as its name indicates it, comes from internal sources such as intermolecular potentials and molecular motion/vibrations. This energy cannot be measured (at the macroscopic level),

but it can be deduced from the laws of Thermodynamics (see Section 3.1.5.3) and measurable mascroscopic quantities (*e.g.* volume V, temperature T and pressure p).

The rate of change of the kinetic energy can be computed by applying the Reynolds transport theorem 3.1 and the momentum conservation equation (3.8):

$$\begin{aligned}\frac{dK}{dt} &= \frac{d}{dt}\frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2\,dV = \frac{1}{2}\int_\Omega \rho\frac{D|\mathbf{u}|^2}{Dt}\,dV \\ &= \int_\Omega \rho\frac{D\mathbf{u}}{Dt}\cdot\mathbf{u}\,dV \\ &= -\int_\Omega \nabla p\cdot\mathbf{u}\,dV + \int_\Omega \rho\mathbf{g}\cdot\mathbf{u}\,dV.\end{aligned}$$

The first term on the right hand side is the mechanical work done by pressure forces, while the second one is the work of body forces.

If additionally we assume the flow to be incompressible, then the integral involving the pressure can be integrated by parts:

$$\frac{dK}{dt} = -\int_\Omega p\mathbf{u}\cdot\mathbf{n}\,dV + \int_\Omega \rho\mathbf{g}\cdot\mathbf{u}\,dV,$$

where we used the Gauss–Ostrogradsky theorem and the following relation:

$$\nabla\cdot(p\mathbf{u}) = \nabla p\cdot\mathbf{u} + p\underbrace{\nabla\cdot\mathbf{u}}_{=0}.$$

3.1.5 Special flows

3.1.5.1 *Incompressible flows*

Definition 3.2. *A fluid flow is called* incompressible *if the measure of any elementary volume $V(t)$ is preserved by the flow,* i.e.

The mass is always conserved. The volume conservation is an additional condition for the incompressibility.

$$\int_{V(t)} dV \equiv \int_{V(0)} dV = \text{const}, \quad \forall t > 0.$$

The following conditions are just avatars of the flow incompressibility:

1. $\nabla\cdot\mathbf{u} = 0, \quad \forall\mathbf{x}\in\Omega(t), \quad \forall t > 0,$
2. $J(\mathbf{a},t) \equiv 1, \quad \forall\mathbf{a}\in\Omega(0), \quad \forall t > 0,$
3. $\frac{D\rho}{Dt} = 0, \quad \forall\mathbf{x}\in\Omega(t), \quad \forall t > 0.$

The equivalence of first two conditions (1), (2) follows from the following simple computation valid for an arbitrary fluid volume $V(t)$:

$$0 = \frac{d}{dt}\int_{V(t)} dV = \frac{d}{d}\int_{V(0)} J\,dV = \int_{V(0)}(\nabla\cdot\mathbf{u})\,J\,dV = \int_{V(t)}\nabla\cdot\mathbf{u}\,dV.$$

The last avatar (3) readily follows from the mass conservation and the divergence-free condition (1).

The following set of equations describes an incompressible flow of an ideal fluid:

$$\begin{aligned}\rho\frac{D\mathbf{u}}{Dt} &= -\nabla p + \rho\mathbf{g}, \quad (\mathbf{x},t) \in \Omega \times \mathbb{R}^+\\ \frac{D\rho}{Dt} &= 0, \quad (\mathbf{x},t) \in \Omega \times \mathbb{R}^+\\ \nabla\cdot\mathbf{u} &= 0, \quad (\mathbf{x},t) \in \Omega \times \mathbb{R}^+,\end{aligned}$$

along with appropriate boundary conditions. On a wall one can require that fluid particles do not penetrate the solid boundary, *i.e.* the normal component of the velocity vanishes

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega.$$

3.1.5.2 *Flows of homogenous fluids*

Definition 3.3. *A fluid is called* homogeneous *if its mass density* ρ *is constant in space.*

A homogeneous fluid is also incompressible if and only if the density ρ is also constant in time (not only in space as require by the definition 3.3). Note, that a fluid which is initially homogeneous remains so if and only if the flow is also incompressible. Otherwise, the density variations will appear due to compressible effects.

3.1.5.3 *Isentropic flow*

Definition 3.4. *A flow is called* isentropic *if it is not accompanied by any change in* entropy.

The notion of entropy was introduced for the first time by Rudolf Clausius

If a flow is taking place in mild periodic conditions (*e.g.* a very gradual compression followed by a gradual expansion), the flow variables may return to their original values. In Physics such processes are called *reversible*. According to the Second law of Thermodynamics, a reversible flow implies a constant value of the entropy. In other words, such flows are called *isentropic*. According to Gibbs relation

*In Greek "*iso*" means the same.*

$$Tds = d\epsilon + pdV,$$

where T is the absolute temperature, ϵ is the internal energy per unit mass and s is the entropy. Since, the flow is *isentropic*, $ds = 0$ and we have

$$d\epsilon = \frac{p}{\rho^2}d\rho.$$

Thus, by integrating the last relation from 0 to ρ one obtains the expression of the internal energy for isentropic flows:

$$\epsilon = \int_0^\rho \frac{p(\rho)}{\rho^2}\, d\rho.$$

3.2 NAVIER–STOKES EQUATIONS

So far we worked with ideal fluids only. In such fluids the forces acting across an internal surface are always normal to this surface. In this Section we shall consider more general fluids which involve some tangential (frictional) forces as well. These effects are important for the description of shear flows, for example.

The mass conservation is obviously not affected by tangential stresses. However, we have to review the momentum balance equation (3.6). Previously we assumed that the stress $\boldsymbol{\sigma}$ reduces only to normal components (*i.e.* the pressure p) in an ideal fluid. In viscid fluids we have to include also the tangential forces across an arbitrary material surface:

$$\boldsymbol{\sigma} = -p\mathbb{I} + \boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ is the *tangential stress tensor*. Consequently, the momentum balance equation (3.6) takes the following form:

$$\frac{\mathrm{d}}{\mathrm{d}} \int_{V(t)} \rho \mathbf{u} \, \mathrm{d}V = -\int_{\partial V(t)} p\mathbf{n} \, \mathrm{d}\sigma + \int_{\partial V(t)} \boldsymbol{\tau}\cdot\mathbf{n} \, \mathrm{d}\sigma + \int_{V(t)} \rho \mathbf{g} \, \mathrm{d}V.$$

Moreover, if we assume the fluid to be *Newtonian*, the symmetric tensor $\boldsymbol{\tau}$ has the following form

$$\boldsymbol{\tau} = \lambda(\nabla\cdot\mathbf{u})\mathbb{I} + 2\mu\mathbb{D}, \tag{3.9}$$

where λ, μ are viscosity coefficients[5] and $\mathbb{D} = \frac{1}{2}\left(\nabla\mathbf{u} + {}^{\top}(\nabla\mathbf{u})\right)$ is the *deformation tensor*. The coefficient λ is called the *bulk viscosity* and μ is the *dynamic viscosity*. By applying the Reynolds transport theorem 3.1 and taking into account that the volume V is arbitrary we obtain the following local form of the Cauchy momentum equation:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{g}.$$

The divergence of the tangential stress tensor can be easily computed taking into account (3.9) (it is easier to perform the computations in tensorial notations):

δ_{ij} *is the Kronecker symbol,* i.e. $\delta_{ii} = 1$ *and* $\delta_{ij} = 0$, *if* $i \neq j$.

$$\begin{aligned}
\nabla\cdot\boldsymbol{\tau} = \partial_j\tau_{ij} &= \\
\partial_j[\lambda(\partial_k u_k)\delta_{ij} + 2\mu\frac{1}{2}(\partial_i u_j + \partial_j u_i)] &= \\
\lambda\partial_i(\partial_k u_k) + \mu\left[\partial_i\partial_j u_j + \partial^2_{jj} u_i\right] &= \\
(\lambda+\mu)\partial_i(\partial_k u_k) + \mu\partial_{jj} u_i.&
\end{aligned}$$

As a result, the momentum balance in Navier–Stokes equations can be written as

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho\mathbf{g},$$

5 We can assume λ, μ to be constant in the first approximation. For more complex and especially compressible fluids the viscosity coefficients may depend on the local fluid density*i.e.* $\lambda = \lambda(\rho)$, $\mu = \mu(\rho)$.

the operator $\nabla^2 = \nabla \cdot \nabla$ is the celebrated Laplace operator. The last equation can be simplified for incompressible flows to give a more conventional form of the Navier–Stokes equations:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \frac{D\rho}{Dt} &= 0, \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\nabla p}{\rho} &= \nu \nabla^2 \mathbf{u} + \mathbf{g}, \end{aligned}$$

where $\nu := \frac{\mu}{\rho}$ is the kinematic viscosity.

3.2.1 Dimensionless Navier–Stokes equations

3.3 VORTICITY

Consider a velocity field $\mathbf{u}(\mathbf{x}, t)$ defined on an open domain $\Omega \subseteq \mathbb{R}^3$. Let us study how this field changes locally around any interior point $\mathbf{x} \in \Omega$. By applying the Taylor formula up to the 2^{nd} order in the vicinity of $\mathbf{x}$ we obtain (in this Section we shall consider the velocity field $\mathbf{u}(\mathbf{x})$ to be stationary to just simplify the notation):

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \nabla \mathbf{u}(\mathbf{x}) \cdot \Delta \mathbf{x} + \mathcal{O}(\|\Delta \mathbf{x}\|^2), \tag{3.10}$$

where $\Delta \mathbf{x} := \mathbf{y} - \mathbf{x} = (\Delta x_1, \Delta x_2, \Delta x_3)$. The Jacobian matrix $\nabla \mathbf{u}(\mathbf{x})$, as any other second order tensor, can be decomposed into the sum of its symmetric $\mathbb{D}$ and skew-symmetric $\mathbb{A}$ parts

$$\begin{aligned} \mathbb{D} &:= \frac{1}{2}\big(\nabla \mathbf{u} + {}^{\top}(\nabla \mathbf{u})\big), \\ \mathbb{A} &:= \frac{1}{2}\big(\nabla \mathbf{u} - {}^{\top}(\nabla \mathbf{u})\big). \end{aligned}$$

The tensor $\mathbb{D}$ was called above the deformation tensor. So, the Taylor expansion (3.10) becomes:

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \mathbb{D}(\mathbf{x}) \cdot \Delta \mathbf{x} + \mathbb{A}(\mathbf{x}) \cdot \Delta \mathbf{x} + \mathcal{O}(\|\Delta \mathbf{x}\|^2).$$

By performing some simple algebraic computations the anti-symmetric part can be transformed as follows

$$\begin{aligned}
\mathbb{A}(\mathbf{x})\cdot\Delta\mathbf{x} &= \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z} & \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} & 0 \end{pmatrix}\cdot\begin{pmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\Delta x_2 + \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\Delta x_3 \\ \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\Delta x_1 + \left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)\Delta x_3 \\ \left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)\Delta x_1 + \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\Delta x_2 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\Delta x_3 - \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\Delta x_2 \\ \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\Delta x_1 - \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\Delta x_3 \\ \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\Delta x_2 - \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\Delta x_1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \end{pmatrix}\times\begin{pmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{pmatrix} = \frac{1}{2}(\nabla\times\mathbf{u})\times\Delta\mathbf{x} = \frac{1}{2}\boldsymbol{\omega}\times\Delta\mathbf{x},
\end{aligned}$$

where we defined the so-called vorticity vector $\boldsymbol{\omega} := \nabla\times\mathbf{u}$. Consequently, the local change of the velocity field around a fixed location $\mathbf{x}$ can be represented as

$$\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}) = \mathbb{D}(\mathbf{x})\cdot\Delta\mathbf{x} + \boldsymbol{\omega}(\mathbf{x})\times\Delta\mathbf{x} + \mathcal{O}(\|\Delta\mathbf{x}\|^2).$$

The last formula has a simple physical interpretation: the local change of the velocity field $\mathbf{u}(\mathbf{x})$ is the superposition of deformation with rate $\mathbb{D}(\mathbf{x})$ and a rigid rotation with vector $\boldsymbol{\omega}(\mathbf{x})$ given by the rotor (or curl) operator applied to the velocity field at this point $\mathbf{x}$. Please notice that this transformation preserves the volume for incompressible fields.

3.3.1 Irrotational flows

Definition 3.5. *A flow is called* irrotational *if the velocity field* $\mathbf{u}(\mathbf{x},t)$ *satisfies* $\boldsymbol{\omega}(\mathbf{x},t) = \nabla\times\mathbf{u} = 0$, $\forall t \geqslant 0$.

From the geometrical sense of vorticity $\boldsymbol{\omega}$ it is clear now why such flows are called irrotational.

As it is known from the vector calculus, an irrotational vector field $\mathbf{u}$ can be expressed as the gradient of a scalar field $\phi(\mathbf{x},t)$ called the *velocity potential*:

$$\mathbf{u}(\mathbf{x},t) = \nabla\phi(\mathbf{x},t).$$

Conversely, when a velocity potential exists, *i.e.* $\mathbf{u} = \nabla\phi$, the flow is necessarily irrotational. This fact follows from the following well-known vector identity:

$$\boldsymbol{\omega}(\mathbf{x},t) = \nabla\times\mathbf{u}(\mathbf{x},t) = \nabla\times\nabla\phi(\mathbf{x},t) \equiv 0.$$

It is for this reason that irrotational flows are often referred to in the literature as the *potential flows*. Irrotational (or potential) flows are

particularly suitable for the description of waves on the surface of a fluid layer. The water wave problem will be discussed in more details in Chapter 4.

3.4 DIMENSIONAL ANALYSIS

The methods of dimensional analysis have been proved to be very useful in Physics in general and they can be also successfully applied to Fluid Mechanics problems. That is why we include this Section into the Chapter devoted to Fluid Dynamics equations. Moreover, it allows in some cases to answer on physical questions without solving the initial problem.

Theorem 3.2 (π-theorem)**.** *Let a physical problem involves n dimensional variables $v_1, v_2, \ldots, v_n$*

$$F(v_1, v_2, \ldots, v_n) = 0,$$

which depend only on r different independent dimensional units u_1, u_2, $\ldots u_r$. Then, the problem can be recast in the form

$$G(\pi_1, \pi_2, \ldots, \pi_{n-r}) = 0,$$

where $\{\pi_k\}_{k=1}^{n-r}$ are $n - r$ dimensionless variables:

$$\pi_k = v_1^{\mu_{k1}} \cdot v_2^{\mu_{k2}} \cdot \ldots \cdot v_n^{\mu_{kn}},$$

where $\mu_{kj} \in \mathbb{Q}$ are some rational numbers which make the product π_k dimensionless.

The π-theorem was first proved by a french mathematician Joseph Bertrand in 1878. However, in anglo-saxons literature it is often called Buckingham π theorem, named after Edgar Buckingham.

Proof. The proof of this theorem relies on some basic facts of the Linear Algebra. In particular, one has to construct the *dimension matrix* of the problem. So, any dimensional quantity v_i can be expressed as a product of fundamental units $u_1, u_2, \ldots u_r$:

$$v_k = u_1^{a_{1k}} \cdot u_2^{a_{2k}} \cdot \ldots \cdot u_r^{a_{rk}}, \quad k = 1, 2, \ldots, n. \tag{3.11}$$

The dimensional matrix $A = (a_{ik})$, $1 \leqslant i \leqslant r$, $1 \leqslant k \leqslant n$ is composed of the exponents a_{ik} written in columns. By assumptions of the π-theorem the rank of this matrix A is equal to r (the number of different independent fundamental units), *i.e.* $\operatorname{rank}(A) = r$. Now let us compose an arbitrary dimensionless combination π_k in the framework of our physical problem:

$$\pi_k = v_1^{\mu_{k1}} \cdot v_2^{\mu_{k2}} \cdot \ldots \cdot v_n^{\mu_{kn}}.$$

Substituting the representation (3.11) into the last formula and after equating all the exposants to zero (since the result has to be dimensionless), we obtain the following system of equations with respect to the unknown vector $\{\mu_{kj}\}_{j=1}^{n}$:

$$\sum_{j=1}^{n} a_{ij}\mu_{kj} = 0, \qquad i = 1, 2, \ldots, r.$$

Under the assumptions of our theorem the dimension of null space of this linear system is equal to $n - r$. □

Remark 3.4. *The π-theorem 3.2 does not guarantee the unicity of the set of dimensionless parameters. Basically, the non-unicity comes from the fact that the choice of the null space basis is not unique. However, in every problem the choice of the "most meaningful" dimensionless numbers is usually guided by the physical intuition and some other considerations.*

This theorem should be considered as a recipe to compute the sets dimensionless variables for a given physical problem in hand. Notice that dimensionless parameters can be computed even if the equations are unknown (one such example will be given in Section 3.4.3). One has just to know the physical variables involved in the problem.

Two physical problems which share the same set of dimensionless parameters are called *similar* and they form similarity classes with respect to this equivalence relation. In fact, such problems are completely equivalent from the mathematical point of view, while being possibly situated at very different scales in the real world (*e.g.* dynamite and nuclear blast explosion). The good side of it is that it is sufficient to solve one representative physical problem to know the solutions for the whole similarity class. Indeed, the solutions to *all* other *similar* problems can be obtained by a simple *rescaling* of variables.

This similarity transposes the notion of homothety transformation *of geometrical figures to physical problems.*

The following examples illustrate how to use the π theorem 3.2 in three physical problems coming from different physical horizons (*e.g.* Classical Mechanics, Fluid Mechanics and Detonation).

3.4.1 Harmonic oscillator

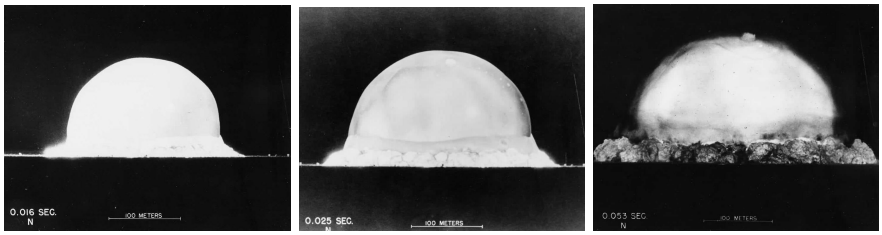
3.4.2 Viscous drag

3.4.3 Nuclear explosion

3.5 FLOW VISUALIZATION

In general there are four main types of characteristic curves, which can be used to visualize better a fluid flow. In this Section we describe briefly these curves, which appear regularly in CFD literature.

CFD stands for the Computational Fluid Dynamics. Sometimes to joke it is also called ***Coloured*** *Fluid Dynamics ☺*



(a) t = 0.016 s (b) t = 0.025 s (c) t = 0.053 s

Figure 3.4.: *Time laps of the Trinity test on July 16, 1945. © Atomic Heritage Foundation*

Suppose that a vector field $\mathbf{u}(\mathbf{x}, t) = \big(u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t)\big)$ is defined in a domain $\Omega \subseteq \mathbb{R}^3$. We assume that for any fixed time instance t this vector field is smooth enough to ensure the existence and unicity of integral curves. Then, a *stream line* $\mathbf{x}(s) = \big(x(s), y(s), z(s)\big)$ passing through the point $\mathbf{x}(0) = \mathbf{a}$ is a solution to the following system of ODEs:

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{a}. \tag{3.12}$$

In the last system the time variable t is regarded as a parameter, which prescribes a continuous family of vector fields $\mathbf{u}(\mathbf{x}, t)$. By specifying various initial conditions $\mathbf{x}(0)$ distributed more or less uniformely in the domain Ω, one obtains a family of curves at each instant t. Notice that a stream line corresponds to a particle's trajectory in an autonomous vector field $\mathbf{u}(\mathbf{x})$.

A *path line* (or *particle path*) is the trajectory of a massless particle in a time-dependent vector field. Path lines are represented by solutions to the following system of ODEs:

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{a}. \tag{3.13}$$

One can notice that the systems (3.12) and (3.13) are identical for autonomous ODEs. It implies that stream lines and path lines are identical for steady fields. By the way, this system realizes also the connection between the Eulerian and Lagrangian descriptions of fluid flows (see Chapter 7 for more details).

A *streak line* is the locus of fluid particles which pass through a fixed location over time. In other words, a streak line is not a trajectory, but rather a collection of particles wich passed by the same (specific) point. If we denote this point by $\mathbf{a}$, the streak-line will be represented parametrically as $\mathbf{x}(s) = \mathbf{x}(t - s; \mathbf{a})$, where $\mathbf{x}(t; \mathbf{a})$ solves the system (3.13). In experiments the streak lines can be visualized by releasing constantly the dye (or smoke in aerodynamics) into the flow in a constant point.

For steady flows, the stream, path and streak lines coincide and all give the same "portrait" of the flow. This is in contrast to *time lines*,

which differ from the other characteristic curves in both steady and unsteady vector fields. A *time line* is the collection of fluid particles advected over time starting from a fixed line arbitrarily chosen in the flow domain. So, a time line is not seeded at a point location, but along a line. The analogy in the real world experiment would consist in putting a perfectly flexible massless wire in the flow and watching its transport over the time. However, in contrast to the wire, a time line can change its length (become shorter or longer) depending on the flow configuration. For the computational recipes, but also to see some beautiful examples of how these concepts can be used in practice to visualize complex fluid flows, we refer to [WT12].

EXERCISES

1. Prove the following identities for the material derivative:

$$\begin{aligned}
\frac{D(f+g)}{Dt} &= \frac{Df}{Dt} + \frac{Dg}{Dt} && \text{(linearity)},\\
\frac{D(f\cdot g)}{Dt} &= f\frac{Dg}{Dt} + g\frac{Df}{Dt} && \text{(product rule)},\\
\frac{D(f\circ g)}{Dt} &= (f'\circ g)\frac{Dg}{Dt} && \text{(chain rule)}.
\end{aligned}$$

2. (1 000 000$ exercise ☺) Prove the *global existence* of 3D Navier–Stokes equations. To claim the reward, the reader has to publish his/her proof in a peer-reviewed journal and address it to the Clay Institute.

4 WATER WAVE PROBLEM

Sunset and the surf wave at Andaman Sea. ©Vitaly Sokol

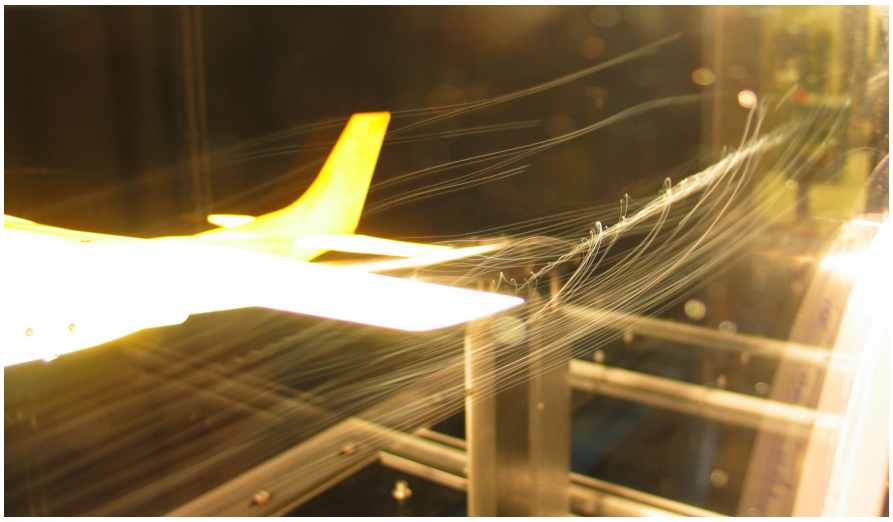
5 LAGRANGIAN VARIATIONAL PRINCIPLE

6 HAMILTONIAN FORMULATION

Part III

CONT. MECHANICS: LAGRANGIAN DESCRIPTION

There are at least two different descriptions of the flow field: Lagrangian and Eulerian (and all the mixed approaches). In the Lagrangian framework the observer follows individual fluid parcels. This can be visualized as sitting in a boat and drifting down a river. On the other hand, the Eulerian description focuses on specific locations in space as the fluid flows. It corresponds to sitting on the bank of a river and watching the water pass the fixed location. The Lagrangian formulation is analogue to Heisenberg's representation in Quantum Mechanics, while the Eulerian description is similar to Schrödinger's one. The names Eulerian/Lagrangian are completely conventional since both descriptions were proposed by Leonhard Euler (see a few historical remarks in Section 7.1 below).



A wind tunnel model of a Cessna 182 showing a wingtip vortex. Tested in the Rensselaer Polytechnic Institute Subsonic Wind Tunnel. ©Wikimedia Commons

7 LAGRANGIAN FLUID DYNAMICS

One owes to Euler the first general formulas for fluid motion... presented in the simple and luminous notation of partial differences... By this discovery, all fluid mechanics was reduced to a single point analysis, and if the equations involved were integrable, one could determine completely, in all cases the motion of a fluid moved by any forces.

— J.-L. Lagrange (*Mécanique analytique*, 1788)

The Lagrangian description of fluid flows seems to be more natural than the Eulerian one since it is related directly to the motion of fluid particles (see Section 7.2 for the definition of fluid particle). The topics for which the Lagrangian description is particularly appropriate include various *transport* and *mixing* phenomena. The main applications include (but not limited to) geophysical flows (*e.g.* pollutant transport), combustion, microfluidics, *etc.* The main reason is that the advection of passive scalars can be treated without any numerical diffusion inherent to Eulerian methods.

The governing equations derived in Lagrangian setting (especially for viscous flows) seem to be intractable (so far). Consequently, nowadays the Eulerian formulation is used to determine the velocity field. Then, in the second time, the kinematic equations are integrated to obtain the trajectory of any fluid particle.

The relation between the Eulerian and Lagrangian formulations is really subtle. It seems that the Lagrangian equations have a potential to generate a much richer dynamics. There are clear experimental evidences [KO92] wich show Lagrangian turbulence for simple Eulerian velocity fields at Reynolds numbers $\mathrm{Re} \propto 1$. In other words, when a velocity field is turbulent in the Eulerian setting, it is also turbulent in the Lagrangian one. However, one can have a purely kinematic Lagrangian turbulence, which has no Eulerian counterpart (since the velocity field is laminar). It is manifested in experiments by a very complex structure of the passive tracer field, while the corresponding velocity field can be quite simple. The reason for it is the following: the Eulerian turbulence is a dynamical phenomenon, while the Lagrangian one may be a purely kinematic one. We shall not discuss deeper the question of Lagrangian/Eulerian turbulence, since it is out of scope of the present manuscript. For this topic the interested reader can refer to [Tsi09].

Unfortunately, there is a very limited literature available on this topic. For instance, the classical monograph by H. Lamb [Lam32] (1932) contains a few sections on the Lagrangian formulation of Fluid Dynamics equations. The later monographs Landau & Lifscitz [LL87] (the 1 st edition in 1959) and by G. Batchelor [Bat67] (1967) do not even provide the governing equations in the Lagrangian form. We are aware of only two modern books entirely devoted to the Lagrangian fluid mechanics: by A. Bennett [Ben06] and by A. Abrashkin & E. Yakubovich [AY06] (in Russian).

7.1 HISTORICAL REMARKS

The names Eulerian/Lagrangian are completely conventional since both descriptions were proposed by Leonhard Euler about 30 years before Joseph-Louis Lagrange. Moreover, there was a correspondance between L. Euler and J.-L. Lagrange and Euler communicated these ideas to Lagrange. For example, in the monograph of Lamb [Lam32] the name Lagrangian is written in quotes: *"Lagrangian" form of the equations of motion and of the equation of continuity*. There is a more complete discussion of this misnomer by C. Truesdell in [Tru54] (see pages 30–32 partially quoted here):

> By the middle nineteenth century the history of fluid dynamics in the eighteenth century had apparently sunk into obscurity. Euler's papers were not often read, of his results which were not forgotten several were attributed to more recent authors who had appropriated them without acknowledgement or discovered them afresh, and indeed his supreme achievements in mathematics, mechanics, and mathematical physics were undervalued then, though not so much as now. The erroneous terminology still current was introduced in the posthumous memoir of Dirichlet [1860, 1, Introd.], edited by Dedekind, where [1757, 2] was quoted as the source of the "Eulerian" method, while it was stated that Lagrange in the *Méchanique Analytique* [1788, 1, Part II, Sect. II, ¶4-7] had introduced the "Lagrangian" method, but had immediately converted the resulting equations to "Eulerian" form.
>
> [...] The whole matter is easily clarified, however. In a letter, written to Lagrange under the date 27 October 1759, Euler after expressing his admiration for Lagrange's first memoir on the propagation of sound stated that one had reason to doubt that propagation in two or three dimensions would follow the same law as in the one dimensional case, since he has already found the fundamental equations to be of different form. The equations he gives are

the linearized equations of plane flow of a perfect fluid expressed in terms of the variables X, Y.

The beginning of the modern vortex dynamics can be associated with the pioneering paper of H. Helmholtz (1858) [Hel58]. The "'golden age" of the study of vortex dynamics using the Lagrangian description can be ascribed approximatively to 1870 – 1920 when important results were obtained starting from the important work of G. Kirchoff (1876) [Kir76]. The application of Lagrangian methods for the study of water waves in the presence of vorticity starts with the work of F. Gerstner (1809) [Ger09] and continues with important contributions by M.-L. Dubreil-Jacotin [DJ34] and R. Gouyon [Gou58]. In general, exact solutions obtained in Lagrangian description are much more precious than those obtained in the Eulerian one since a Lagrangian solution provides also the history of all particle trajectories in addition to the velocity field, given by the Eulerian solutions.

7.2 ON THE NOTION OF A FLUID PARTICLE

When a fluid flow is considered, it is assumed to be continuous. However, it is useful to introduce a discrete concept to develop a different point of view. This concept is usually referred to as the *fluid particle*. However, the term *fluid parcel* would be more accurate, but it is not generally accepted. So, a fluid particle in Fluid Mechanics has nothing to do with water H_2O or air O_2 molecules. A fluid particle is the fluid mass contained in a small volume dV whose size is small enough from the macroscopic point of view. On the other hand, this volume dV has to be large enough from the microscopic (*i.e.* intermolecular distance) viewpoint. The main contstraint is that the number of molecules has to be sufficient to give sense to the statistical description, since we would like to get rid of local fluctuations. For example, if intermolecular distances scale with 10^{-6} mm under normal conditions, the fluid particle "*diameter*" will scale with 10^{-3} mm or even larger. Again, under the same normal conditions (*e.g.* 0° C and 760 mm Hg) a cube with a side equal to 10^{-3} mm contains about 10^7 molecules. It is considered to be enough to have stable statistical averages over this ensemble. Of course, for rarefied gases all these estimations have to be readjusted.

7.3 DERIVATION OF EQUATIONS

The basic idea of Lagrangian description consists in following the motion of fluid particles marked with some labels $\mathbf{a} = (a, b, c) \in \mathbb{R}^3$. If in the Eulerian description the spatial coordinates $\mathbf{x} = (x, y, z)$ are considered as independent variables, in Lagrangian description they

become unknown functions of particle labels $\mathbf{a}$ and time $t \in \mathbb{R}^+$. To underline this fact we will use capital letters to denote this functional dependence, *i.e.*

$$\mathbf{x} = \mathbf{X}(\mathbf{a}, t), \qquad t \geqslant 0. \tag{7.1}$$

Let us describe how these labels appear in the fluid flow description. As before, the components of the velocity field are denoted with $\mathbf{u}(\mathbf{x}, t)$ and by definition we have

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t).$$

A general solution to this system of ODEs can be formally written as

$$\mathbf{x} = \mathbf{X}(t; \mathbf{a}),$$

where $\mathbf{X} = (X, Y, Z)$ are some functions of t and $\mathbf{a} = (a, b, c)$. The parameters (a, b, c) are three arbitrary integration constants, which have to be determined from the initial conditions:

$$\mathbf{X}(0; \mathbf{a}) = \mathbf{x}_0 \quad \Longrightarrow \quad \mathbf{a} = \mathbf{X}^{-1}(0; \mathbf{x}_0).$$

It can happen that integration constants coincide with the initial positions of fluid particles $\mathbf{x}(0) = \mathbf{a}$, but in general it does not have to be the case. So, these integration constants (a, b, c) can be viewed as a curvilinear coordinate system defined in the fluid domain. However, the "fluid" coordinate axes $\{(a, 0, 0) \,|\, a \in \mathbb{R}\}$, $\{(0, b, 0) \,|\, b \in \mathbb{R}\}$ and $\{(0, 0, c) \,|\, c \in \mathbb{R}\}$ can become quite stretched when represented in the original Euclidean space $(x, y, z) \in \mathbb{R}^3$ as the time (and flow) evolves. It is unnecessary to say that this coordinate system is in general *non-orthogonal* and *non-stationary*. In the rest of this Chapter we shall denote the arguments of particle trajectories on equal footing, *i.e.* $\mathbf{x} = \mathbf{X}(\mathbf{a}, t)$. In other words we forget what is the argument and what is the parameter. Particle labels $\mathbf{a}$ and time t will be both the arguments.

The relation between two ways of fluid description is given by the following equation

$$\frac{\partial \mathbf{X}(\mathbf{a}, t)}{\partial t} = \mathbf{u}\big(\mathbf{X}(\mathbf{a}, t), t\big). \tag{7.2}$$

In the last equation the Lagrangian velocity field $\mathbf{V}(\mathbf{a}, t) = \frac{\partial \mathbf{X}(\mathbf{a}, t)}{\partial t}$ is related to the Eulerian velocity $\mathbf{u}(\mathbf{x}, t)$ by the nonlinear relation (7.2). We can summarize now. In order to pass from Lagrangian to Eulerian coordinates, one has to do three steps:

- Differentiate the function $\mathbf{X}(\mathbf{a}, t)$ with respect to time in order to obtain the velocity field as a function of labels $\mathbf{V}(\mathbf{a}, t) = \frac{\partial \mathbf{X}(\mathbf{a}, t)}{\partial t}$
- Invert relations (7.1) in order to express particle labels $\mathbf{a}$ as functions of Eulerian coordinates, *i.e.* $\mathbf{a} = \mathbf{X}^{-1}(\mathbf{x}, t)$

- Substitute these expressions $\mathbf{a} = \mathbf{X}^{-1}(\mathbf{x}, t)$ into the Lagrangian velocity field $\mathbf{V}(\mathbf{a}, t)$ in order to obtain the Eulerian velocities $\mathbf{u}(\mathbf{x}, t) = \mathbf{V}\big(\mathbf{X}^{-1}(\mathbf{x}, t), t\big)$.

However, these transformations can be performed analytically only in some exceptional cases.

7.3.1 Mass conservation

7.3.2 Momentum conservation

8 VARIATIONAL STRUCTURE

9 LONG WAVES

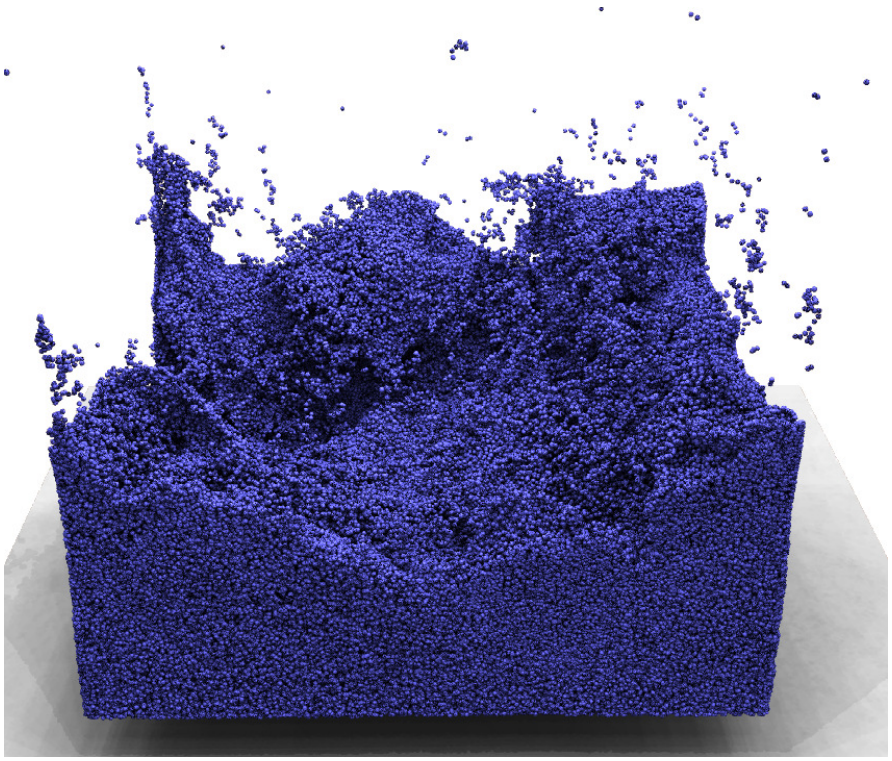
10 SOME EXACT SOLUTIONS

11 SMOOTHED PARTICLE HYDRODYNAMICS

In this Chapter we shall describe a numerical method which is situated in between the Eulerian and Lagrangian descriptions. Namely, the underlying formulation is Eulerian, strictly speaking. However, the idea to describe the flow in terms of fluid parcels (*i.e.* "*particles*") is borrowed from the Lagrangian description.

The Smoothed Particle Hydrodynamics (SPH) method was proposed by L. Lucy [Luc77] & R. Gingold & J. Monaghan [GM77] in Astrophysics to simulate star and galaxy formation. Much later it was applied to the simulation of free surface flows in a seminal paper of J. Monaghan [Mon94]. Some key ideas of this method (without the numerical implementation) can be also found in works of R. Salmon [Sal88], before it has become one of the standard methods in hydrodynamics.

As general comprehensive references on the SPH method we could probably suggest the review paper by J. Monaghan [Mon05]. An extension to MHD is explained in [Pri12a]. For Russian-speaking readers we could recommend the review paper by A. Aliev & G. Tarnavsky [AT07]. Below we shall follow the great lines of these excellent expositions.



SPH fluid simulation. ©Frank Zimmer

11.1 WHY PARTICLE METHODS?

There are well-established grid-based methods such as finite differences, finite volumes and finite elements. So, what is the motivation to develop particle-based (or meshless) methods? It turns out that the absence of a rigid topological structure of the mesh is beneficial in situations where topological changes may occur. An emblematic example is the wave breaking phenomenon. So, the main advantages of particle-based methods over mesh-based ones are summarized here:

- Treatment of large deformations is easy
- The advection is solved exactly
- Identifying the free surface or another interface is easy
- Atomization (*e.g.* splashing) and other topological changes (*e.g.* fragmentation, collision) are seamless.

From this description particle-based methods appear to be the ideal choice for most free surface flows simulations. However, some drawbacks of the SPH method will be discussed below in Section 11.3.

We would like to stress out one more time that the particles used in the SPH method are purely numerical concepts which are introduced in order to discretize the governing equations. These "particles" have nothing to do with the fluid particles discussed in Section 7.2.

11.2 PARTICLE INTERPOLATION

The smoothing kernels appeared already in works of Russian/Soviet mathematicians Vladimir A. Steklov and Sergei L. Sobolev.

11.3 A BLACK FLY IN YOUR CHARDONNAY

Several other "popular" critics of the SPH method (such as the tensile instability, particles pairing, *etc.*) were addressed in [Pri12b]. We invite the readers to have a look on this quite interesting paper.

The same expression in Russian-speaking cultures would read: "A spoon of tar in a barrel of honey."

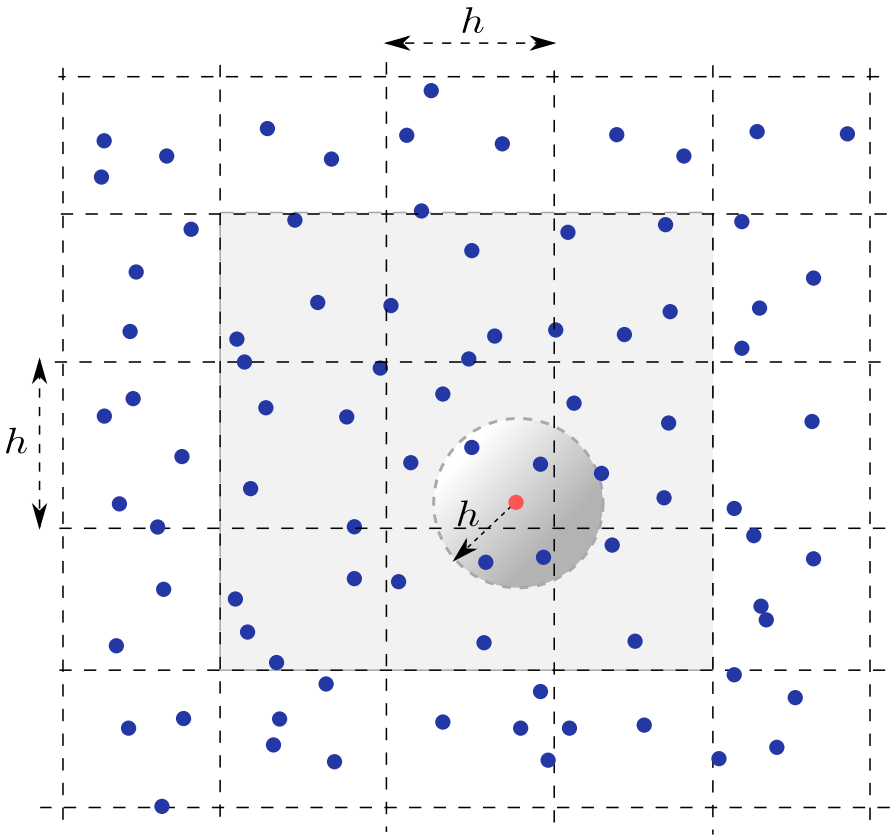


Figure 11.1.: *Interpolation in the SPH method. The active particle is shown in red. The underlying grid is used only to locate the candidates of neighbouring particles. The surrounding cells are shown in grey.*

Part IV

APPENDIX

A MATHEMATICAL TOOLS

Do not worry about your difficulties in Mathematics.
I can assure you mine are still greater.

— Albert Einstein

I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.

— Freeman Dyson (*Missed Opportunities*, 1972)

In this auxiliary Chapter we shall briefly introduce some mathematical tools which are used in the main text above. Some of these tools (such as the differential forms, for example) can make some objects more precise (*e.g.* the physical force, the mechanical work, *etc.*) and simplify greatly the work with them in the Continuum Mechanics.

A.1 FRÉCHET DERIVATIVE

Named after Maurice Fréchet, the Fréchet derivative generalizes the operation of differentiation on Banach spaces. It defines the functional derivative extensively used in the calculus of variations. The Fréchet derivative has numerous applications in physics and nonlinear functional analysis.

Definition A.1. *Let* U *and* V *be Banach spaces. A functional* $F : U \mapsto V$ *is said to be* Fréchet differentiable *at* $u \in U$ *if there exists a bounded linear operator* $dF(u) : U \mapsto V$ *such that*

$$\lim_{h \to 0} \frac{\|F(u+h) - F(u) - dF(u)[h]\|_V}{\|h\|_U} = 0.$$

If the limit exists $dF(u)$ *is called the Fréchet derivative of the functional* F *at* u.

A functional $F : U \mapsto V$ which has a Fréchet derivative in every point $u \in U$ is said to be of class C^1 if the following map is continuous

$$\begin{aligned} dF : U &\mapsto L(U, V), \\ u &\mapsto dF_u, \end{aligned}$$

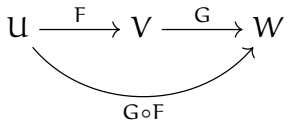
where $L(U, V)$ is a set of linear bounded functionals. Note that this notion is fundamentally different from the continuity of the derivative functional $dF(u)[h]$.

A.1.1 Properties

The Fréchet differentiation operation is linear, *i.e.* if both $F, G : U \mapsto V$ are differentiable at u, then $\alpha F + \beta G$ is also differentiable and

$$d\big(\alpha F + \beta G\big)(u) \;=\; \alpha dF(u) \;+\; \beta dG(u).$$

The usual chain rule is also valid for the Fréchet derivative. Let the functionals $F : U \mapsto V$ and $G : V \mapsto W$ are such that the following diagram commutes



We assume that F is differentiable at u and the functional G is differentiable at $v = F(u)$. Then, the chain rule reads

$$d\big(G \circ F\big)(u) \;=\; dG(v) \circ dF(u).$$

If U and V are finite dimensional spaces, the Fréchet derivative reduces to the usual derivation operation. In this case, the linear differentiation operator d can be conveniently represented with the Jacobian matrix J:

$$dF : \mathbb{R}^n \mapsto \mathbb{R}^m, \qquad J_u \;:=\; dF(u), \qquad J_u \in \mathrm{Mat}_{n\times m}(\mathbb{R}).$$

For the Fréchet differentiability in the finite dimensional settings it is sufficient that all partial derivatives of a function F exist and are continuous.

A.1.2 Higher derivatives

The derivative $dF : U \mapsto L(U, V)$ of a differentiable functional $F : U \mapsto V$ is also a function from U to another Banach space (of linear bounded operators operators from U to V). This function dF may also be differentiable. In this case, this derivative will be called the *second order derivative*

This particular Banach space is called the dual space.

$$d^2F : \; U \mapsto L(U, L(U, V)) \;\simeq\; L^2(U \times U, V),$$

where the dual space $L(U, L(U, V))$ is isomorphic to the space of bounded continuous bilinear operators $L^2(U \times U, V)$. This process can be continued by induction to higher order derivatives if they exist.

A.2 GÂTEAUX DERIVATIVE

The Gâteaux derivative, named after René GÂTEAUX, a young brilliant French mathematician who died during the World War I, is a generalization of the classical directional derivative to Banach spaces. Similarly to Fréchet derivative, the Gâteaux differential is used to formalize the functional derivatives commonly used in physics and calculus of variations.

Banach spaces can be replaced by more general locally convex topological vector spaces.

Definition A.2. *Let U and V be Banach (or other locally convex topological) spaces and $F : U \mapsto V$ is a map. If the following limit exists*

$$dF(u)[h] := \lim_{\varepsilon \to 0} \frac{F(u + \varepsilon h) - F(u)}{\varepsilon} = = \frac{d}{d\varepsilon} F(u + \varepsilon h)\Big|_{\varepsilon = 0}, \qquad (A.1)$$

it is called the Gâteaux differential $dF(u)[h]$ at $u \in U$ in the direction $h \in U$. If the limit (A.1) exists for all $\forall h \in U$ then F is said to be Gâteaux differentiable at u.

A.2.1 Properties

In every point $u \in U$ where the map F is differentiable, its Gâteaux differential defnes a homogeneous function in the second argument

$$dF(u)[\cdot] : U \mapsto V \quad \text{such that} \quad dF(u)[\alpha h] \equiv \alpha dF(u)[h].$$

However, this map is not necessarily additive and continuous in h, as it will be shown below.

The chain rule holds also for the Gâteaux derivative and reads

$$d(G \circ F)(u)[h] = dG\big(F(u)\big)\big[dF(u)[h]\big],$$

where the maps F and G satisfy the same conditions as in the previous section. However, the chain rule is not generally true if the derivatives are discontinuous.

A.2.1.1 *Relation with the Fréchet derivative*

If a map F is differentiable in the sense of Fréchet, then it is also differentiable in the sense of Gâteaux. The converse is not true since the Fréchet derivative is necessarily linear, while the Gâteaux derivative may be a nonlinear function. So, the Gâteaux differentiability does not necessarily imply the existence of a derivative in the sense of Fréchet.

EXAMPLE IN FINITE DIMENSION. In order to illustrate these different notions, let us consider the following finite-dimensional real-valued function $F : \mathbb{R}^2 \mapsto \mathbb{R}$:

$$F(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

The Gâteaux derivative of this function can be easily computed:

$$dF(0,0)[s,t] = \begin{cases} \dfrac{t^3}{s^2+t^2}, & \text{if } (s,t) \neq (0,0) \\ 0, & \text{if } (s,t) = (0,0). \end{cases}$$

Now it is obvious that the derivative $dF(0,0)$ is not linear in the directions (s,t).

A.2.2 Higher derivatives

Higher order Gâteaux derivatives in the direction h are defined as

$$d^n F(u)[h] := \frac{d^n}{d\varepsilon^n} F(u+\varepsilon h)\Big|_{\varepsilon=0}, \quad n > 0.$$

Note that function $d^n F(u)[h]$ is not a multilinear function in its second argument h, in contrast to the Fréchet derivative. However, it is still a homogeneous function of order n in the argument h.

Using higher order derivatives, one can write also a Taylor expansion if $F \in C^k$

$$F(u+h) = F(u) + dF(u)[h] + \frac{1}{2!} d^2F(u)[h] + \ldots \\ \ldots + \frac{1}{(k-1)!} d^{k-1}F(u)[h] + R_k(u)[h],$$

where R_k is the remainder term. If the segment $[u, u+h] \in U$, then one can write the remainder term R_k in the integral form given by

$$R_k(u)[h] = \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} d^k F(u+th)[h]\, dt.$$

A.2.3 Example in infinite dimensions

Finally, let us show how these tools can be applied to compute the variation of a real-valued *nonlinear* functional $\mathcal{F} : L^2(\Omega) \mapsto \mathbb{R}$, where $L^2(\Omega)$ is the Hilbert space of square integrable functions on a Lebesgue measurable set $\Omega \subseteq \mathbb{R}^n$. This functional $\mathcal{F}$ is given by the following integral

$$\mathcal{F}(u) = \int_\Omega F\big(u(x)\big)\, dx, \quad u : \Omega \mapsto \mathbb{R}, \quad F(u) \in L^2(\Omega), \tag{A.2}$$

where the kernel $F(\cdot)$ is assumed to be a differentiable function. Let us compute the Gâteaux derivative of (A.2) by following the Definition A.2:

$$\frac{\mathcal{F}(u+\varepsilon h)-\mathcal{F}(u)}{\varepsilon} = \frac{1}{\varepsilon}\Big[\int_\Omega F(u+\varepsilon h)\,dx - \int_\Omega F(u)\,dx\Big] =$$
$$= \frac{1}{\varepsilon}\int_\Omega\int_0^1 \frac{d}{d\tau}F(u+\varepsilon\tau h)\,ds\,dx = \int_\Omega\int_0^1 F'(u+\varepsilon\tau h)h\,ds\,dx.$$

After taking the limit $\varepsilon\to 0$ we obtain the required Gâteaux derivative

$$d\mathcal{F}(u)[h] = \int_\Omega F'\big(u(x)\big)\,h(x)\,dx.$$

The latter integral is nothing else but the inner scalar product $\langle F', h\rangle$ in the Hilbert space $L^2(\Omega)$.

A.3 DIFFERENTIAL FORMS

To Thales the primary question was not what do we know, but how do we know it.

— Aristotle

In this Section we shall introduce informally some notions of the exterior algebra and exterior differential calculus. Moreover, we shall illustrate their usage on several examples from the Calculus and Mechanics.

The. vectorial calculus turns out to be very useful in the three-dimensional space $\mathbb{R}^3$ due to its very special mathematical structure. For instance, the usual vectorial product $a\times b$ of two vectors a and b is, in fact, a mathematical trick which works only in $\mathbb{R}^3$. A mathematically sound generalization of the Vectorial Calculus to higher dimensional (and eventually non-flat) spaces is given by the Exterior Calculus, whose flavour we shall give below. The exposition will be conducted for simplicity in $\mathbb{R}^d$, $d = 2,3$. However, the generalization to any other Euclidean space $\mathbb{R}^n$, $n > 3$ is straightforward.

Without entering into details for the moment, we shall say that $n = 3$ turns out to be the only solution to the equation $C_2^n = n$, where C_k^n is a binomial coefficient. See below for the explanations.

The basics of the exterior calculus stem from the pioneering works of H. Poincaré and É. Cartan in the beginning of the XX^{th} century.

A.3.1 0-forms

By convention, any function $\omega = f(x,y,z)$ will be called a 0-form.

A.3.2 1-forms

The application of a 1-form on a vector is simply a real number. A basis in $\mathbb{R}^3$ is given by three independent (and orthonormal) vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, a 1-form can be defined simply by its action on the basis elements, since any other vector $\mathbf{u} \in \mathbb{R}^3$ can be decomposed into a linear combination

$$\mathbf{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3.$$

In a similar way we can introduce a basis in the space of 1-forms. For example, there exists a unique 1-form, that we will denote as dx, such as

$$dx(\boldsymbol{e}_1) = 1, \quad dx(\boldsymbol{e}_2) = 0, \quad dx(\boldsymbol{e}_3) = 0.$$

We would like to underline that dx *is not related to any infinitesimal quantity*. It is just a generally adopted notation to denote one particular differential form. In a similar way one can define 1-forms dy and dz. Now it is easy to compute the action of dx, dy, dz on the vector $\mathbf{u}$:

$$dx(\mathbf{u}) = u_1, \quad dy(\boldsymbol{e}_2) = u_2, \quad dz(\boldsymbol{e}_3) = u_3.$$

In the Classical Mechanics the forces correspond to 1-forms. The *mechanical work*, for example, is given by the application of this differential form on the displacement vector (it is assumed implicitly that the displacement is done under the action of this force).

We note also that the integral calculus of 1-differential forms can be developed as well. It results in a generalization of the usual curvlinear integral from the Calculus.

A.3.3 2-forms

The elementary 1-forms introduced in the previous Section serve as *atoms* to construct more complex objects. For this person we shall need an additional algebraic operation — the so-called *exterior* or *wedge* product[1], which allows to construct 2-forms from the "atoms", *e.g.* $\omega = dx \wedge dy$. The formal definition can be found in any textbook on the exterior algebra. Here we shall recall only the main properties

We can also recommend a very comprehensive book by M. Spivak [Spi71].

1 The proper definition of the wedge product of two polylinear forms $p(e_1, \ldots, e_m)$ and $q(e_1, \ldots, e_n)$ involves the alternator operation

$$p \wedge q := \frac{(m+n)!}{m!n!}\,\mathrm{Alt}[p \otimes q],$$

where the alternator $\mathrm{Alt}(\omega)$, $\deg\omega = n$, is defined as

$$\mathrm{Alt}[\omega](e_1, e_2, \ldots, e_n) := \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sign}\,\sigma \cdot \omega(e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}).$$

In this definition the summation is taken ovel all the elements of the symmetric group S_n (*i.e.* the permutations). Notice the similarities with the definition of the determinant in the Linear Algebra.

of this operation:

ASSOCIATIVITY: $(de_1 \wedge de_2) \wedge de_3 = de_1 \wedge (de_2 \wedge de_3)$

ANTICOMMUTATIVITY: (valid only for the forms of degree one)

$$de_1 \wedge de_2 = -de_2 \wedge de_1$$

DISTRIBUTIVITY ON THE LEFT:

$$de_1 \wedge (\alpha de_2 + \beta de_3) = \alpha de_1 \wedge de_2 + \beta de_1 \wedge de_3$$

DISTRIBUTIVITY ON THE RIGHT:

$$(\alpha de_1 + \beta de_2) \wedge de_3 = \alpha de_1 \wedge de_3 + \beta de_2 \wedge de_3$$

NILPOTENCY: (it is a direct corollary of the anticommutativity) $de \wedge de = 0$ (valid only for the forms of degree one)

In the Euclidean space $\mathbb{R}^n$ the basis of 2-forms consists of all the elements $de_i \wedge de_j$, where $\{e_i\}$, $i = 1, \ldots, n$ is an orthonormal basis in $\mathbb{R}^n$. The dimension of this space is equal to $C_2^n = \frac{n(n-1)}{2}$. It can be noticed that for $n = 3$ (and *only* for $n = 3$!) the dimension of this space is equal to n as well. As a consequence, it is custom to associate in $\mathbb{R}^3$ the wedge product $de_i \wedge de_j$ with the vector product and represent it with the vector normal to the plane spanned by e_i and e_j. However, we would like to bring the reader's attention to the fact that this representation is *impossible* in any other dimension $n \neq 3$.

A.3.4 k–forms

In a similar way, the 1-forms (or "atoms") can be combined together using the wedge product to produce k-forms, *e.g.* $\omega = de_{i_1} \wedge \ldots \wedge de_{i_k}$. The number k is also called the degree of a form ω and it is noted as $k = \deg \omega$. It is easy to see that the dimension of the space of all k-forms is equal to $C_k^n = \frac{n!}{k!(n-k)!}$. The wedge product for k-forms is not commutative in general, as a direct consequence of this formula:

$$p \wedge q = (-1)^{\deg p \cdot \deg q} q \wedge p.$$

As it was noticed earlier, the nilpotency property $\omega \wedge \omega = 0$ holds for the differential forms of degree one. However, the nilpotency does not hold in general, as it follows from the example below. Consider a 2-differential form $\omega = dt \wedge dx + dy \wedge dz$. Direct computations show that

$$\omega \wedge \omega = 2dt \wedge dx \wedge dy \wedge dz.$$

A.3.5 n-forms

The n-forms in $\mathbb{R}^n$ have a special geometrical meaning. Namely, it is the right type of product to compute an element of the volume

$$dV = de_1 \wedge de_2 \wedge \ldots \wedge de_n.$$

This form can be directly used to compute the signed volume of parallelepipeds and through the integration the volumes of more general domains.

Remark A.1. *For the sake of completeness we shall mention that the volume V of a domain $\Omega \subset \mathbb{R}^3$ bounded by a closed surface $\mathcal{S} = \partial\Omega$ can be computed using the following surface integral:*

$$V = \frac{1}{3} \iint_{\mathcal{S}} x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

A.3.6 External derivation

The external derivative operation transforms a k-form into a $(k+1)$-form. Again, instead of giving formal definitions we shall provide the reader with several important examples. The general principle can be understood from them:

- Let $\omega = f(x,y,z)$ be a 0-form. Then, trivially one can obtain

  $$d\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

  We can see that $d\omega$ represents the gradient of the function $f(x,y,z)$. From this example one can see that the gradient is not a vector, but a 1-form.

- Now let us take a 1-form $\omega = p(x,y,z)\ dx + q(x,y,z)\ dy + r(x,y,z)\ dz$. Its derivation can be performed in three steps. The first step will be given explicitly here:

  $$d\big(p(x,y,z)dx\big) = \Big(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz\Big) \wedge dx = \\ = -\frac{\partial p}{\partial y} dx \wedge dy - \frac{\partial p}{\partial z} dx \wedge dz$$

  After performing similar manipulations with other terms and gathering all the results, one obtains

  $$d\omega = \Big(-\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\Big) dx \wedge dy + \Big(-\frac{\partial q}{\partial z} + \frac{\partial r}{\partial y}\Big) dy \wedge dz + \\ + \Big(-\frac{\partial p}{\partial z} + \frac{\partial r}{\partial x}\Big) dx \wedge dz \quad (A.3)$$

  This time $d\omega$ coincides with the rotational operator of a vector field.

- As a final example, let us take a 2-form $\omega = r(x,y,z)\, dx \wedge dy + p(x,y,z)\, dy \wedge dz + q(x,y,z)\, dz \wedge dx$. Its derivative can be readily computed to give

$$d\omega = \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}\right) dx \wedge dy \wedge dz$$

  As a result, one can recognize the divergence operator of a vector field.

The rotation $\nabla \times \mathbf{F}$ and divergence $\nabla \cdot \mathbf{F}$ operators were both introduced by J. C. Maxwell. However, he considered the quantity $-\nabla \cdot \mathbf{F}$ that he referred to as the convergence.

From these examples one can see that all common operators of the vectorial calculus in 3D can be obtained as exterior derivatved of some differential forms. The main advantage of the language of differential forms is twofold:

- It is independent of the dimension,
- It is independent of the chosen coordinate system.

The general algorithm to compute the exterior derivative is the following. One takes an elementary differential form $\omega = f(x_1, \ldots x_n) de_{i_1} \wedge \ldots \wedge de_{i_k}$ and computes the usual differential of the coefficient $df = \frac{\partial f}{\partial x_1} de_1 + \ldots + \frac{\partial f}{\partial x_n} de_n$. Then, every obtained term in the direction de_i has to be multiplied externally by $de_{i_1} \wedge \ldots \wedge de_{i_k}$ to produce a $(k+1)$-form $de_i \wedge de_{i_1} \wedge \ldots \wedge de_{i_k}$. Finally, the resulting expression $d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} de_i \wedge de_{i_1} \wedge \ldots \wedge de_{i_k}$ has to be simplified using the properties of the wedge product.

The following result describes all higher order exterior derivatives:

Lemma A.1 (H. Poincaré)**.**

$$d^2\omega = d(d\omega) = 0.$$

This Lemma shows that the structure of the exterior derivation is very simple and all higher order derivatives are equal precisely to zero. Instead of making the complete proof, we will just provide here an explicit calculation for a particular 0-form $\omega = f(x,y)$, which illustrates the main idea of the proof:

$$d\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

$$d^2\omega = d(d\omega) = \left(-\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y}\right) dx \wedge dy \equiv 0.$$

A.3.6.1 *Derivative of the product*

The following result generalizes the usual formula for the derivative of the product of two functions to the exterior product of two differential forms:

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2. \tag{A.4}$$

In some cases it can facilitate the derivation of differential forms, which can be decomposed into the products of simpler ones. The integration by parts of differential forms is based on this result as well (see Section A.3.7.1 for more details).

A.3.6.2 *Application to conservation laws*

The conservation laws are omnipresent in Physics in general and, in particular, in Continuum Mechanics. Let $u(\mathbf{x}, t)$ denote a conservative quantity (*e.g.* the density, momentum, energy, *etc.*) and $\mathbf{F}(u)$ be its flux. Then, the conservation equation takes the following form:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) = 0. \tag{A.5}$$

In the language of differential forms this equation has a simple geometric interpretation. For simplicity, we consider the case with one spatial dimension first. Consider an arbitrary element in space-time with dimensions dx and dt along with the differential form $\omega = u dx - \mathbf{F} dt$. Then, the conservation law (A.5) has the following elegant form $d\omega = 0$. In the space of higher dimensions this elegant writing remains invariant. However, one has to redefine the differential form ω. In three spatial dimensions one has to consider the following differential form:

The differential forms having the property $d\omega = 0$ are called closed.

$$\omega = u\, dx \wedge dy \wedge dz - \left(F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy\right) \wedge dt.$$

The sign $-$ in front of the second term comes from the convention to consider the outgoing normal, which implies that the incoming flux is, thus, negative.

A.3.7 Integration of differential forms

The integration theory for differential forms on manifolds can be also developed [Spi71]. Here we provide only the central result of this theory:

Theorem A.1. *Let $\mathcal{S}$ be a compact oriented k-dimensional manifold with a border $\partial\mathcal{S}$, which has an implied orientation. Let ω be a $(k-1)$-form on $\mathcal{S}$. Then the following equality holds*

$$\int_{\mathcal{S}} d\omega = \int_{\partial\mathcal{S}} \omega,$$

Below we will consider the most important corrolaries of this fundamental theorem of the calculus on manifolds.

A.3.7.1 *Integration by parts*

We assume that all assumptions of the Stokes theorem A.1 hold. Using the Stokes theorem along with a product derivation formula (A.4), one readily obtains the following rule to integrate the differential forms by parts:

$$\int_{\mathcal{S}} d\omega_1 \wedge \omega_2 = \int_{\partial\mathcal{S}} \omega_1 \wedge \omega_2 - (-1)^{\deg \omega_1} \int_{\mathcal{S}} \omega_1 \wedge d\omega_2.$$

A.3.7.2 *Green theorem*

Theorem A.2. *Let $\mathcal{S} \subset \mathbb{R}^2$ be a compact two-dimensional manifold with a border. Additionally we assume that $\mathcal{S}$ has a standard orientation and $\partial\mathcal{S}$ has an implied orientation (also known as the counter-clockwise direction). Let $p, q : \mathcal{S} \mapsto \mathbb{R}$ are two differentiable functions. Then, the following equality holds:*

$$\int_{\partial\mathcal{S}} p\,dx + q\,dy = \iint_{\mathcal{S}} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx \wedge dy$$

Proof. It is sufficient to consider the differential form $\omega = p\,dx + q\,dy$ and to notice that $d\omega = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx \wedge dy$. □

A.3.7.3 *Gauß–Ostrogradsky theorem*

This theorem is also known in the literature as the divergence theorem.

Theorem A.3. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a compact three-dimensional manifold with a border, $\mathbf{n} = (n_1, n_2, n_3)$ be the unitary normal to $\partial\mathcal{S}$ pointing in the outward direction and $\mathbf{F} = (F^1, F^2, F^3)$ be a differentiable vector field on $\mathcal{S}$. Then the following equality holds*

$$\int_{\mathcal{S}} \nabla \cdot \mathbf{F}\, dV = \int_{\partial\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where dV, $d\sigma$ are volume and surface elements correspondingly.

Proof. In order to prove this Theorem it is enough to consider the 2-differential form $\omega = F^1\, dy \wedge dz + F^2\, dz \wedge dx + F^3\, dx \wedge dy$. Then, it is straightforward to show that $d\omega = \nabla \cdot \mathbf{F}\, dV$. Finally, by noticing that

$$\begin{aligned} n_1\, d\sigma &= dy \wedge dz, \\ n_2\, d\sigma &= dz \wedge dx, \\ n_3\, d\sigma &= dx \wedge dy, \end{aligned}$$

then on $\partial\mathcal{S}$ we have

$$\mathbf{F} \cdot \mathbf{n}\, d\sigma = F^1 n_1\, d\sigma + F^2 n_2\, d\sigma + \mathcal{F}^3 n_3\, d\sigma = \omega.$$

□

A.3.7.4 *Stokes theorem*

Theorem A.4. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a compact oriented manifold with a border, $\partial\mathcal{S}$ is supplied with the induced orientation and $\mathbf{n} = (n_1, n_2, n_3)$ be the unitary normal to $\partial\mathcal{S}$ pointing in the outward direction. Let $\mathbf{F} = (F^1, F^2, F^3)$ be a differentiable vector field on an open set containing the manifold $\mathcal{S}$. Then the following equality holds*

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_{\partial\mathcal{S}} \mathbf{F} \, ds, \tag{A.6}$$

where $d\sigma$, ds are surface and length elements correspondingly.

Remark A.2. *Equality (A.6) can be written also as*

$$\int_{\partial\mathcal{S}} F^1 \, dx + F^2 \, dy + F^3 \, dz = \iint_{\mathcal{S}} \Big[n_1 \Big(\frac{\partial F^3}{\partial y} - \frac{\partial F^2}{\partial z} \Big) + \\ + n_2 \Big(\frac{\partial F^1}{\partial z} - \frac{\partial F^3}{\partial x} \Big) + n_3 \Big(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y} \Big) \Big] \, d\sigma.$$

Proof. The proof of this theorem can be done by considering the 1-form $\omega = F^1 \, dx + F^2 \, dy + F^3 \, dz$ and by noticing that $d\omega = (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$ (see formula (A.3) and the proof of the previous Theorem for the details). □

A.3.8 The Hodge operator

This Section can be omitted for the first reading.

In mathematical physics the Laplace operator plays the fundamental rôle. However, from the Poincaré lemma we saw that a straightforward application of two successive differential operators annihilates any differential k-form. In other words, any differential form is a "harmonic function". Consequently, something else has to be done to define a meaningul analogue of the Laplace operator.

In the flat space $\mathbb{R}^n$ there is a particular differential form which corresponds to the volume element $\omega = de_1 \wedge \ldots \wedge de_n \equiv dV$. Once it is fixed, we can define the Hodge operator $\star$ which associates to a k-form ω the unique $(n-k)$-form $\star\omega$ such that

$$\omega \wedge (\star\omega) \equiv dV.$$

This operator has already been used above without giving it a name. For instance, in $\mathbb{R}^3$ we have the following examples:

$$\star dx = dydz,$$
$$\star dy = dzdx,$$
$$\star dz = dxdy.$$

A.3.8.1 *Laplace operator*

Now we can proceed to the meaningful definition of the Laplace operator. Let us take a 0-form $\omega = u(x, y, z)$. The following equalities can be readily obtained:

$$\begin{aligned} d\omega &= \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy + \frac{\partial u}{\partial z}\, dz, \\ \star d\omega &= \frac{\partial u}{\partial x}\, dy \wedge dz + \frac{\partial u}{\partial y}\, dz \wedge dx + \frac{\partial u}{\partial z}\, dx \wedge dy, \\ d(\star d\omega) &= \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big)\, dx \wedge dy \wedge dz. \end{aligned}$$

The last 3-form provides us with the usual Laplace operator. It is a good sign since we worked in a flat Euclidean space $\mathbb{R}^3$. However, the expression $d(\star d\omega)$ remains valid in any coordinate system.

A.3.8.2 *d'Alembert operator*

In the Minkowsky space-time $\mathbb{M}^3$ an element of the volume is given by the differential form $\omega = -dt \wedge dx \wedge dy$. As we proceeded above for the Laplace operator, we will define the differential 0-form $\omega = u(t, x, y)$. Then, we compute

$$\begin{aligned} d\omega &= -\frac{\partial u}{\partial t}\, dt + \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy, \\ \star d\omega &= \frac{\partial u}{\partial t}\, dx \wedge dy + \frac{\partial u}{\partial x}\, dt \wedge dy + \frac{\partial u}{\partial y}\, dt \wedge dx, \\ d(\star d\omega) &= \Big(-\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big)\, dt \wedge dx \wedge dy. \end{aligned}$$

The last operator is a 3-form representing the celebrated d'Alembert operator which appears in all kinds of wave equations arising in the mathematical physics.

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INDEX

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