Notes on Smearing

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Abstract

Here are the details of the smearing we decided to use for all new calculations from now on. The only difference with respect to what was done before is the "projection" from U(3) to SU(3). The smearing is done in two steps: the first consists in applying an APE smearing to the gauge links. The second implements a gaussian smearing to the quark fields. We can provide three independent tested codes, in C, C++ and Fortran 90.

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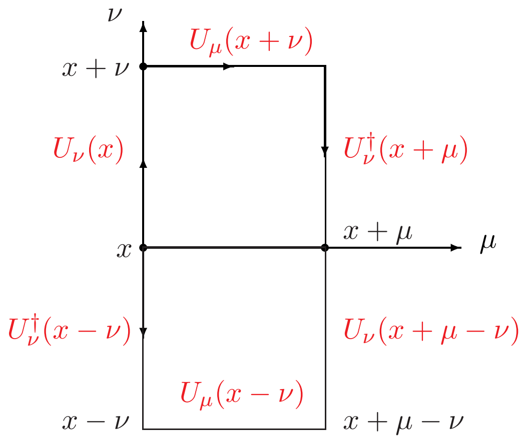
1 APE smearing

1.1 "Fuzzing"

$$\boxed{U_\mu(x) \to \hat{U}_\mu(x) = U_\mu(x) + \alpha V_\mu(x) \qquad \forall \mu \neq 0}$$

with

$$V_\mu(x) = \sum_{\nu=\pm1\pm2\pm3\neq\mu} U_\nu(x) U_\mu(x+\nu) U_\nu^\dagger(x+\mu)$$



1.2 SU(3) "projection"

Let us denote M the matrix we want to "project". If it is invertible, it can be uniquely decomposed as $M = U.H$, where U belongs to $U(3)$ and H is an hermitian positive definite matrix. We take as "projection of M over U(3)", the unitary part of this polar decomposition, i.e. U given by

$$U = M.(M^\dagger M)^{-1/2} \tag{1}$$

From $M = U.H$ one can see that H is given by

$$H^2 = M^\dagger M \tag{2}$$

so that $U = M.H^{-1}$. H^2 can be diagonalized and if $V = (V_1, V_2, V_3)$ is the orthonormal basis of eigenvectors and λ_i its eigenvalues:

$$H^{-1} = V^\dagger \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \frac{1}{\sqrt{\lambda_2}} & \\ & & \frac{1}{\sqrt{\lambda_3}} \end{pmatrix} V = V^\dagger D V \qquad\qquad V \in U(3)$$

Then

$$U = M.V^\dagger D V \tag{3}$$

It can be shown that this is equivalent to find the U(3) matrix X which minimises the Schur norm of the difference $M - X$, i.e. $Tr[(M-X)^\dagger(M-X)]$.

To restrict to $SU(3)$, we divide by:

$$\tilde{U} = \frac{U}{det^{1/3}U} \qquad\qquad \det(\tilde{U}) = 1 \tag{4}$$

We take the determination of the cubic root such that the argument is between $-\frac{\pi}{3}$ and $+\frac{\pi}{3}$.

Some comments:

- the choice of (4) does not minimize the Schur norm anymore; furthermore, the three possible determinations of the cubic root lead to three different norms.
- the main source of inaccuracy in this process is the diagonalization of H^2. Rémi, Vincent and I used three different diagonalization procedures, which are compatible at $\sim 10^{-12}$.
- the choice of the

2 Gaussian smearing

The local propagator reads:

$$G^{ab}_{\alpha\beta}(x,y) =< 0|q^a_\alpha(x)\bar{q}^b_\beta(y)|0 >$$

We define the smeared quark field by

$$Q^b_\beta(x) = \sum_{\vec{x'}\beta' b'} F^{bb'}_{\beta\beta'}(\vec{x};t,\vec{x'};t)\; q^{b'}_{\beta'}(\vec{x'},t)$$

with

$$F_{\beta\beta'}(\vec{y};t,\vec{y'};t) = \frac{\delta_{\beta\beta'}}{1+6\alpha}\left(\delta(\vec{y}-\vec{y'}) + \alpha\sum_{\mu=1}^{3}\left[U_\mu(\vec{y};t)\delta_{\vec{y}+\mu,\vec{y'}} + U^\dagger_\mu(\vec{y}-\mu;t)\delta_{\vec{y}-\mu,\vec{y'}}\right]\right) \tag{5}$$

where color indices are implicit.

2.1 Source smearing

The propagator smeared at the source is defined by

$$\tilde{G}^{ab}_{\alpha\beta}(x,y) =< 0|q^a_\alpha(x)\bar{Q}^b_\beta(y)|0 >$$

The conjugate of the smeared quark field is written as:

$$\bar{Q}^b_\beta(x) = [Q^\dagger\gamma_0]^b_\beta(x) = \sum_{\vec{x'}\beta' b'} \bar{q}^{b'}_{\beta'}(\vec{x'},t)\; \bar{F}^{b'b}_{\beta'\beta}(\vec{x'};t,\vec{x};t)$$

where

$$\bar{F}^{b'b}_{\beta'\beta}(\vec{x'};t,\vec{x};t) = \gamma^0_{\beta'\alpha}\; F^{*bb'}_{\alpha\alpha'}(\vec{x};t,\vec{x'};t)\; \gamma^0_{\alpha'\beta}$$

The source smeared propagator then reads

$$\tilde{G}^{ab}_{\alpha\beta}(x,y) \sum_{\vec{y'}\beta' b'} G^{ab'}_{\alpha\beta'}(x,y')\; \bar{F}^{b'b}_{\beta'\beta}(\vec{y'};t,\vec{y};t) \tag{6}$$

and is solution of Dirac equation with a source

$$\tilde{S}^{cb}_{\gamma\beta}(z,y) \quad = \quad \sum_{\vec{y'}\beta'b'} S^{cb'}_{\gamma\beta'}(z,y')\ \bar{F}^{b'b}_{\beta'\beta}(\vec{y'};t,\vec{y};t) \tag{7}$$

After n^{th} iterations:

$$S^{(n)cb}_{\gamma\beta}(z,y) \quad = \quad \sum_{\vec{y'}\beta'b'} S^{(n-1)cb'}_{\gamma\beta'}(z,y')\ \bar{F}^{b'b}_{\beta'\beta}(\vec{y'};t,\vec{y};t) \tag{8}$$

and in terms of matrix multiplication (spin-color-positions):

$$S^{(n)} \quad = \quad S^{(n-1)} \cdot \bar{F} = S^{(0)} \cdot \bar{F}^n \tag{9}$$

2.2 Sink smearing

The propagator smeared at the sink is defined by

$$\tilde{G}^{ab}_{\alpha\beta}(x,y) =< 0|Q^a_\alpha(x)\bar{q}^b_\beta(y)|0 >$$

i.e.

$$\tilde{G}^{ab}_{\alpha\beta}(x,y) = \sum_{\vec{x'}\beta'b'} F^{ab'}_{\alpha\beta'}(\vec{x};t,\vec{x'};t)\ G^{b'b}_{\beta'\beta}(x',y)$$

i.e. $\tilde{G} = F \cdot G$, and after n^{th} iterations: $\tilde{G}^{(n)} = F^n \cdot G$.

2.3 Codes

To summarize: using the fact that F is hermitian, the source smearing is given by $S^{(n)} = S^{(0)} \cdot F^n$ and the sink smearing by $\tilde{G}^{(n)} = F^n \cdot G$. We rewrite the smearing at the source developing the products. For two iterations:

$$S^{(2)cb}_{\gamma\beta}(z,y) \quad = \quad S^{(0)cb'}_{\gamma\beta'}(z,y')\ F^{b'b''}_{\beta'\beta''}(\vec{y'};t,\vec{y''};t)\ F^{b''b}_{\beta''\beta}(\vec{y''};t,\vec{y};t) \tag{10}$$

The source point in these notations is in y: it is fixed randomly. Before the first iteration, a matrix F^0 is initialized to $F^{0ab}_{\alpha\beta}(x,y) = \delta_{ab}\delta_{\alpha\beta}\delta_{xy}$. The first iteration calculates, for all (x,α,β,a,c):

$$\begin{aligned}
F^{(1)ac}_{\alpha\beta}(x,y) \quad &= \quad \sum_{\mu=1,3}\sum_b \left[U^{ab}_\mu(\vec{x};t)F^{(0)bc}_{\alpha\beta}(x+\mu,y) + U^{ab\dagger}_\mu(\vec{x}-\mu;t)F^{(0)bc}_{\alpha\beta}(x-\mu,y)\right] \\
&= \quad \sum_{\mu=1,3} \left[U^{ac}_\mu(\vec{x};t)\delta_{x+\mu,y} + U^{ac\dagger}_\mu(\vec{x}-\mu;t)\delta_{x-\mu,y}\right]\delta_{\alpha\beta} \\
&= \quad F^{ac}_{\alpha\beta}(x,y)
\end{aligned}$$

which is exactly the smearing function (5). Similarly, the second iteration gives:

$$\begin{aligned}
F^{(2)ac}_{\alpha\beta}(x,y) \quad &= \quad \sum_{\mu=1,3}\sum_b \left[U^{ab}_\mu(\vec{x};t)F^{(1)bc}_{\alpha\beta}(x+\mu,y) + U^{ab\dagger}_\mu(\vec{x}-\mu;t)F^{(1)bc}_{\alpha\beta}(x-\mu,y)\right] \\
&= \quad \sum_{\mu=1,3}\sum_{\vec{x'}}\sum_{\alpha'} \left[U^{ab}_\mu(\vec{x};t)\delta_{x+\mu,x'} + U^{ab\dagger}_\mu(\vec{x}-\mu;t)\delta_{x-\mu,x'}\right]\delta_{\alpha'\alpha}F^{(1)bc}_{\alpha'\beta}(x',y) \\
&= \quad \sum_{\vec{x'}} F^{ab}_{\alpha\alpha'}(x,x')F^{bc}_{\alpha'\beta}(x',y)
\end{aligned}$$

So at the end of n iteration, we get the term F^n, or more explicitly, refering to expression (10), $[F^{(n)}]^{b'b}_{\beta'\beta}(y', y)$, for all y'. So we still need to multiply by $S^{(0)cb'}_{\gamma\beta'}(z, y')$ to get the smeared source $S^{(n)cb}_{\gamma\beta}(z, y)$. If $S^{(0)cb'}_{\gamma\beta'}(z, y') = \delta_{cb'}\delta_{\gamma\beta'}\delta_{zy'}$, then

$$\begin{aligned} S^{(2)cb}_{\gamma\beta}(z, y) &= \delta_{cb'}\delta_{\gamma\beta'}\delta_{zy'}\ F^{b'b''}_{\beta'\beta''}(\vec{y'}; t, \vec{y''}; t)\ F^{b''b}_{\beta''\beta}(\vec{y''}; t, \vec{y}; t) \\ &= F^{cb''}_{\gamma\beta''}(\vec{z}; t, \vec{y''}; t)\ F^{b''b}_{\beta''\beta}(\vec{y''}; t, \vec{y}; t) \end{aligned}$$

So we do not need to explicitly multiply on the left by $S^{(0)}$, only if this latter is a local source. In case it is non-local (as it will be the case for generalized sources), the multiplication by $S^{(0)}$ must be done explicitly at the end of the loop over the smearing iterations (!!!).

At the sink, the initial matrix is set to the local propagator $F^{(0)ab}_{\alpha\beta}(x, y) = G^{ab}_{\alpha\beta}(x, y)$ and then multiplied on the left hand side by the smearing function, similarly to the source smearing.