UNIVERSIDAD DE ANTIOQUIA
Facultad de Ciencias Exactas y Naturales
Instituto de Matemáticas
Series de Tiempo II
Taller # 2

Profesor: Duván Cataño

1. A time series was generated by first drawing the white noise series wt from a normal distribution with mean zero and variance one. The observed series x_t was generated from

$$x_t = w_t - \theta w_{t-1}, \quad t = 0, \pm 1, \pm 2, \dots,$$

where θ is a parameter.

   a) Derive the theoretical mean value and autocovariance functions for the series x_t and w_t . Are the series x_t and w_t stationary? Give your reasons.

   b) Give a formula for the power spectrum of x_t , expressed in terms of θ and w.

2. In applications, we will often observe series containing a signal that has been delayed by some unknown time D, i.e.,

$$x_t = s_t + As_{t-D} + n_t,$$

where s_t and n_t are stationary and independent with zero means and spectral densities $f_s(w)$ and $f_n(w)$, respectively. The delayed signal is multiplied by some unknown constant A. Show that

$$f_x(w) = [1 + A^2 + 2Acos(2\pi wD)]f_s(w) + f_n(w).$$

3. Consider $y_t = h_t x_t$, for $t = 0, 1, \dots, T-1$, and using the modified DFT:

$$Y(v_k) = c^{-1/2} \sum_{t=0}^{T-1} x_t h_t e^{-2\pi i v_k t},$$

with $c = \sum_{t=0}^{T-1} h_t^2$.
Prove that

$$\mathbb{E}|Y(v_k)|^2 = \int_{-1/2}^{1/2} |H(v_k - v)|^2 f_x(v) dv,$$

with $H(v) = c^{-1/2} \sum_{t=0}^{T-1} h_t e^{-2\pi i v t}$.

4. Suppose x_t and y_t are stationary zero-mean time series with x_t independent of y_s for all s and t. Consider the product series

$$z_t = x_t y_t.$$

Prove the spectral density for z_t can be written as

$$f_z(w) = \int_{-1/2}^{1/2} f_x(w - v) f_y(v) dv.$$

5. With sample size to $n = 128$, generate and plot the time series as

$$x_{t1} = 2\cos(2\pi 0{,}06t) + 3\sin(2\pi 0{,}06t)$$
$$x_{t2} = 4\cos(2\pi 0{,}10t) + 5\sin(2\pi 0{,}10t)$$
$$x_{t3} = 6\cos(2\pi 0{,}40t) + 7\sin(2\pi 0{,}40t)$$
$$x_t = x_{t1} + x_{t2} + x_{t3}.$$

   a) Compute and plot the periodogram of the series, x_t , generated above and comment.

   b) Repeat the analyses adding noise to x_t ; that is

$$x_t = x_{t1} + x_{t2} + x_{t3} + w_t$$

   where $w_t \sim iid\ \mathcal{N}(0, 25)$. That is, you should simulate and plot the data, and then plot the periodogram of x_t and comment.

6. Figure 4.22 shows the biyearly smoothed (12-month moving average) number of sunspots from June 1749 to December 1978 with $n = 459$ points that were taken twice per year; the data are contained in *sunspotz*. With Example 4.13 as a guide, perform a periodogram analysis identifying the predominant periods and obtaining confidence intervals for the identified periods. Interpret your findings.

7. The levels of salt concentration known to have occurred over rows, corresponding to the average temperature levels for the soil science data considered in Figs. 1.18 and 1.19, are in *salt* and *saltemp*. Plot the series and then identify the dominant frequencies by performing separate spectral analyses on the two series. Include confidence intervals for the dominant frequencies and interpret your findings.

8. Let $\{w_t; t = 0, 1, \ldots\}$ be a white noise process with variance σ_w^2 and let $|\phi| < 1$ be a constant. Consider the process $x_0 = w_0$, and

$$x_t = \phi x_{t-1} + w_t, \quad t = 1, 2, \ldots.$$

   We might use this method to simulate an AR(1) process from simulated white noise.

   a) Show that $x_t = \sum_{j=0}^{t} \phi^j w_{t-j}$ for any $t = 0, 1, \ldots.$

   b) Find the $\mathbb{E}(x_t)$.

   c) Show that, for $t = 0, 1, \ldots,$

$$\mathbb{V}ar(x_t) = \frac{\sigma_w^2}{1-\phi^2}(1 - \phi^{2(t+1)})$$

   d) Show that, for $h \geq 0$,

$$\mathbb{C}ov(x_{t+h}, x_t) = \phi^h \mathbb{V}ar(x_t)$$

   e) Is x_t stationary?

   f) Argue that, as $t \to \infty$, the process becomes stationary, so in a sense, x_t is "asymptotically stationary."

   g) Comment on how you could use these results to simulate n observations of a stationary Gaussian AR(1) model from simulated iid $N(0, 1)$ values.

   h) Now suppose $x_0 = w_0/\sqrt{1-\phi^2}$. Is this process stationary? Hint: Show $\mathbb{V}ar(x_t)$ is constant.

9. Suppose we would like to predict a single stationary series x_t with zero mean and autocorrelation function $\gamma(h)$ at some time in the future, say, $t + l$, for $l > 0$.

   a) If we predict using only x_t and some scale multiplier A, show that the mean-square prediction error

$$MSE(A) = \mathbb{E}[(x_{t+l} - Ax_t)^2]$$

   is minimized by the value

$$A = \rho(l).$$

b) Show that the minimum mean-square prediction error is

$$MSE(A) = \gamma(0)[1 - \rho^2(l)].$$

c) Show that if $x_{t+l} = Ax_t$, then $\rho(l) = 1$ if $A > 0$, and $\rho(l) = -1$ if $A < 0$.

10. For the process $y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}$, with $\sum_{j=-\infty}^{\infty} |a_j| < \infty$, if x_t has spectrum $f_x(w)$, then the spectrum of the filtered output, y_t , say $f_y(w)$, is related to the spectrum of the input x_t by

$$f_y(w) = |A(w)|^2 f_x(w)$$

where the frequency response function $A(w) = \sum_{j=-\infty}^{\infty} a_j e^{-2\pi i w j}$.

11. Suponga que los residuos $\hat{a}_t$ del modelo $(1 - B)Z_t = (1 + 0,6B)a_t$, ajustado de una serie de 80 observaciones, proporcionan las siguientes autocorrelaciones:

h	1	2	3	4	5	6	7	8	9	10
$\rho_{\hat{a}}(h)$	0.39	0.20	0.09	0,04	0,09	-0.13	-0.05	0.06	0.11	-0.02

a) Analice la adecuación del modelo ajustado y si existe alguna indicación de falta de ajustamiento del modelo. Si esto ocurre, sugiera un modelo modificado y pruébelo.

b) Calcular la densidad espectral del modelo encontrado en el numeral anterior. Haga las suposiciones necesarias para garantizar la existencia del mismo.

12. Probar que

$$\gamma(h) = \begin{cases} 1, & h = 0 \\ \rho, & h = \pm 1 \\ 0, & o.c \end{cases}$$

es una función de autocovarianza si y sólo si $|\rho| < 1/2$.

13. Obtener el espectro del proceso cuya función de autocovarianza es dada por

$$\gamma(t) = Me^{-\alpha|\tau|} cos(\beta t),$$

donde $M > 0$, $\alpha > 0$, $\beta > 0$ y $\tau \in \mathbb{R}$.

14. Sea $Z_t = a_t + ca_{t-1} + ca_{t-2} + \ldots + ca_1$, para $t > 0$, donde $c \in \mathbb{R}$ y $a_t \sim RB(0, \sigma_a^2)$.

a) Calcular la media y autocovarianza de Z_t. ¿Ella es estacionaria? Justifique.

b) Calcular la media y autocovarianza de $(1 - B)Z_t$. ¿Ella es estacionaria? Justifique.

c) En caso de estacionaridad en alguno de los items anteriores, calcular el espectro.

15. Una serie de 400 observaciones presentó los siguientes resultados:

h	1	2	3	4	5	6	7
ϕ_{hh}	0.8	-0.5	0.07	-0,02	-0,01	0.05	0.04

con $\bar{Z}_t = 8$ y $\mu_o = 9$.

a) Explique por que podemos ajustar a la series un modelo AR(2).

b) Obtenga las estimativas $\hat{\phi_1}$ y $\hat{\phi_2}$ del modelo AR(2) utilizando las ecuaciones de Yule-Walker.

c) Verifique el modelo ajustado satisface las condiciones de estacionaridad.

d) Usando $\hat{\phi_1}$ y $\hat{\phi_2}$ como verdaderos, describa el comportamiento general de la ACF de ese proceso.