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Series de Tiempo I

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1. Let $\{w_t; t = 0, 1, \ldots\}$ be a white noise process with variance σ_w^2 and let $|\phi| < 1$ be a constant. Consider the process $x_0 = w_0$, and
$$x_t = \phi x_{t-1} + w_t, \quad t = 1, 2, \ldots.$$
We might use this method to simulate an AR(1) process from simulated white noise.
   a) Show that $x_t = \sum_{j=0}^{t} \phi^j w_{t-j}$ for any $t = 0, 1, \ldots$.
   b) Find the $\mathbb{E}(x_t)$.
   c) Show that, for $t = 0, 1, \ldots$,
$$\mathbb{V}ar(x_t) = \frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)})$$
   d) Show that, for $h \geq 0$,
$$\mathbb{C}ov(x_{t+h}, x_t) = \phi^h \mathbb{V}ar(x_t)$$
   e) Is x_t stationary?
   f) Argue that, as $t \to \infty$, the process becomes stationary, so in a sense, x_t is "asymptotically stationary."
   g) Comment on how you could use these results to simulate n observations of a stationary Gaussian AR(1) model from simulated iid $N(0,1)$ values.
   h) Now suppose $x_0 = w_0/\sqrt{1-\phi^2}$. Is this process stationary?

2. Suppose we would like to predict a single stationary series x_t with zero mean and autocorrelation function $\gamma(h)$ at some time in the future, say, $t + l$, for $l > 0$.
   a) If we predict using only x_t and some scale multiplier A, show that the mean-square prediction error
$$MSE(A) = \mathbb{E}[(x_{t+l} - Ax_t)^2]$$
   is minimized by the value
$$A = \rho(l).$$
   b) Show that the minimum mean-square prediction error is
$$MSE(A) = \gamma(0)[1 - \rho^2(l)].$$
   c) Show that if $x_{t+l} = Ax_t$, then $\rho(l) = 1$ if $A > 0$, and $\rho(l) = -1$ if $A < 0$.

3. Una serie de 400 observaciones presentó los siguientes resultados:

h	1	2	3	4	5	6	7
ϕ_{hh}	0.8	-0.5	0.07	-0,02	-0,01	0.05	0.04

con $\bar{x}_t = 8$ y $\mu_0 = 9$.

a) Explique por qué podemos ajustar a la serie un modelo AR(2).

b) Obtenga las estimativas $\hat{\phi_1}$ y $\hat{\phi_2}$ del modelo AR(2) utilizando las ecuaciones de Yule-Walker.

c) Verifique que el modelo ajustado satisface las condiciones de estacionaridad.

d) Usando $\hat{\phi_1}$ y $\hat{\phi_2}$ como verdaderos, describa el comportamiento general de la ACF de ese proceso.

e) Obtenga los pronósticos para los próximos 4 periodos.

4. Suppose

$$y_t = \beta_0 + \beta_1 t + \ldots + \beta_q t^q + x_t, \qquad \beta_q \neq 0,$$

where x_t is stationary. First, show that $\nabla^k x_t$ is stationary for any $k = 1, 2, \ldots$, and then show that $\nabla^k y_t$ is not stationary for $k < q$, but is stationary for $k \geq q$.

5. For the ARIMA$(1, 1, 0)$ model with drift, $(1 - \phi B)(1 - B)x_t = \delta + w_t$, let $y_t = (1 - B)x_t = \nabla x_t$.

a) Noting that y_t is AR(1), show that, for $j \geq 1$,

$$y_{n+j}^n = \delta[1 + \phi + \ldots + \phi^{j-1}] + \phi^j y_n.$$

b) Use part (a) to show that, for $m = 1, 2, \ldots,$

$$x_{n+m}^n = x_n + \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right] + (x_n - x_{n-1})\frac{\phi(1-\phi^m)}{1-\phi}$$

Hint: From (a), $x_{n+j}^n - x_{n+j-1}^n = \delta\frac{1-\phi^j}{1-\phi} + \phi^j(x_n - x_{n-1})$. Now sum both sides over j from 1 to m.

Las bases de datos indicadas en los siguientes ejercicios, se encuentran en la librería *astsa* de R.

6. Crude oil prices in dollars per barrel are in *oil*. Fit an ARIMA(p, d, q) model to the growth rate performing all necessary diagnostics. Comment.

7. Fit an ARIMA(p, d, q) model to the global temperature data *globtemp* performing all of the necessary diagnostics. After deciding on an appropriate model, forecast (with limits) the next 10 years. Comment.

8. Fit an ARIMA(p, d, q) model to the sulfur dioxide series, *so2*, performing all of the necessary diagnostics. After deciding on an appropriate model, forecast the data into the future four time periods ahead (about one month) and calculate 95 % prediction intervals for each of the four forecasts. Comment.

9. Consider the AR(2) model, $x_t = 0{,}2 + 1{,}8x_{t-1} - 0{,}81x_{t-2} + w_t$, where the w_t is the Gaussian white noise with mean 0 and variance 4, and $x_{47} = 19$, $x_{48} = 22$, $x_{49} = 17$ and $x_{50} = 21$.

a) Find the mean and variance of the process.

b) Find forecast, $\hat{x}_{50}(l)$, for $l = 1, 2, 3$.

c) Find the 95 % forecast limits in part (c).

d) Find the eventual forecast function for the forecast made at $t = 50$ and its limiting value.

10. Consider the ARIMA model

$$x_t = w_t + \Theta w_{t-2}.$$

a. Identify the model using the notation ARIMA$(p, d, q) \times (P, D, Q)_s$.

b. Show that the series is invertible for $|\Theta| < 1$, and find the coefficients in the representation

$$w_t = \sum_{k=0}^{\infty} \pi_k x_{t-k}.$$

c. Develop equations for the $m-$step ahead forecast, and its variance.

11. Consider the following model:

$$(1 - B^{12})(1 - B)x_t = (1 - \theta_1 B)(1 - \Theta_1 B^{12})w_t$$

with $\theta_1 = 0{,}2$, $\Theta_1 = 0{,}8$, and $\sigma_w^2 = 1$.

a) Express the model in terms of the AR form. Compute and plot the π weights.

b) Compute and plot the ψ weights that are needed to evaluate the forecast variance.

c) Find the forecast and the their 95 % forecast limits for the next 12 periods.

12. Fit a seasonal ARIMA model of your choice to the chicken price data in *chicken*.Use the estimated model to forecast the next 12 months.

13. Fit a seasonal ARIMA model of your choice to the unemployment data in *unemp*. Use the estimated model to forecast the next 12 months.

14. Fit a seasonal ARIMA model of your choice to the U.S. Live Birth Series (*birth*). Use the estimated model to forecast the next 12 months.

15. Fit an appropriate seasonal ARIMA model to the log-transformed Johnson and Johnson earnings series (*jj*). Use the estimated model to forecast the next 4 quarters.