

The background of the slide features a large, faint watermark of the University of Pavia seal. The seal is circular and contains a central shield with a crown on top. Below the shield are several books and a quill pen. The words "ALMA MATER" and "UNIVERSITAS" are visible on the left and right sides of the seal respectively.

# GARCH MODELS

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# Stylized Facts

GARCH models have been developed to account for empirical regularities in financial data. Many financial time series have a number of characteristics in common.

- Asset prices are generally non stationary. Returns are usually stationary. Some financial time series are fractionally integrated.
- Return series usually show no or little autocorrelation.
- Serial independence between the squared values of the series is often rejected pointing towards the existence of non-linear relationships between subsequent observations.

# Stylized Facts

- Volatility of the return series appears to be clustered.
- Normality has to be rejected in favor of some thick-tailed distribution.
- Some series exhibit so-called *leverage effect*, that is changes in stock prices tend to be negatively correlated with changes in volatility. A firm with debt and equity outstanding typically becomes more highly leveraged when the value of the firm falls. This raises equity returns volatility if returns are constant. Black, however, argued that the response of stock volatility to the direction of returns is too large to be explained by leverage alone.
- Volatilities of different securities very often move together.

# The ARCH Model

## ARCH process

(Bollerslev, Engle and Nelson, 1994) The process  $\{\varepsilon_t(\theta_0)\}$  follows an ARCH (AutoRegressive Conditional Heteroskedasticity) model if

$$E_{t-1}[\varepsilon_t(\theta_0)] = 0 \quad t = 1, 2, \dots$$

and the conditional variance

$$\sigma_t^2(\theta_0) \equiv \text{Var}_{t-1}[\varepsilon_t(\theta_0)] = E_{t-1}[\varepsilon_t^2(\theta_0)] \quad t = 1, 2, \dots$$

depends non trivially on the  $\sigma$ -field generated by the past observations:  $\{\varepsilon_{t-1}(\theta_0), \varepsilon_{t-2}(\theta_0), \dots\}$ .

# The ARCH Model

Let  $\{y_t(\theta_0)\}$  denote the stochastic process of interest with conditional mean

$$\mu_t(\theta_0) \equiv E_{t-1}(y_t) \quad t = 1, 2, \dots$$

$\mu_t(\theta_0)$  and  $\sigma_t^2(\theta_0)$  are measurable with respect to the time  $t - 1$  information set. Define the  $\{\varepsilon_t(\theta_0)\}$  process by

$$\varepsilon_t(\theta_0) \equiv y_t - \mu_t(\theta_0).$$

# The ARCH Model

It follows from eq.(5) and (5), that the standardized process

$$z_t(\theta_0) \equiv \varepsilon_t(\theta_0) \sigma_t^2(\theta_0)^{-1/2} \quad t = 1, 2, \dots$$

with

$$\begin{aligned} E_{t-1}[z_t(\theta_0)] &= 0 \\ \text{Var}_{t-1}[z_t(\theta_0)] &= 1 \end{aligned}$$

will have conditional mean zero ( $E_{t-1}[z_t(\theta_0)] = 0$ ) and a time invariant conditional variance of unity.

# The ARCH Model

We can think of  $\varepsilon_t(\theta_0)$  as generated by

$$\varepsilon_t(\theta_0) = z_t(\theta_0) \sigma_t^2(\theta_0)^{1/2}$$

where  $\varepsilon_t^2(\theta_0)$  is unbiased estimator of  $\sigma_t^2(\theta_0)$ .

Let's suppose  $z_t(\theta_0) \sim NID(0, 1)$  and independent of  $\sigma_t^2(\theta_0)$

$$E_{t-1}[\varepsilon_t^2] = E_{t-1}[\sigma_t^2] \quad E_{t-1}[z_t^2] = E_{t-1}[\sigma_t^2] = \sigma_t^2$$

because  $z_t^2 | \Phi_{t-1} \sim \chi_{(1)}^2$ .



# The ARCH Model

If the conditional distribution of  $z_t$  is time invariant with a finite fourth moment, the fourth moment of  $\varepsilon_t$  is

$$E[\varepsilon_t^4] = E[z_t^4] E[\sigma_t^4] \geq E[z_t^4] E[\sigma_t^2]^2 = E[z_t^4] E[\varepsilon_t^2]^2$$

where the last equality follows from

$$E[\sigma_t^2] = E[E_{t-1}(\varepsilon_t^2)] = E[\varepsilon_t^2]$$

$$E[\varepsilon_t^4] \geq E[z_t^4] E[\varepsilon_t^2]^2$$

by Jensen's inequality. The equality holds true for a constant conditional variance only.

If  $z_t \sim NID(0, 1)$ , then  $E[z_t^4] = 3$ , the unconditional distribution for  $\varepsilon_t$  is therefore leptokurtic

$$\begin{aligned} E[\varepsilon_t^4] &\geq 3E[\varepsilon_t^2]^2 \\ E[\varepsilon_t^4] / E[\varepsilon_t^2]^2 &\geq 3 \end{aligned}$$

The kurtosis can be expressed as a function of the variability of the conditional variance.

# The ARCH Model

In fact, if  $\varepsilon_t | \Phi_{t-1} \sim N(0, \sigma_t^2)$

$$E_{t-1} [\varepsilon_t^4] = 3E_{t-1} [\varepsilon_t^2]^2$$

$$E [\varepsilon_t^4] = 3E [E_{t-1} (\varepsilon_t^2)^2] \geq 3 \{E [E_{t-1} (\varepsilon_t^2)]\}^2 = 3 [E (\varepsilon_t^2)]^2$$

$$E [\varepsilon_t^4] - 3 [E (\varepsilon_t^2)]^2 = 3E \{E_{t-1} [\varepsilon_t^2]^2\} - 3 \{E [E_{t-1} (\varepsilon_t^2)]\}^2$$

$$E [\varepsilon_t^4] = 3 [E (\varepsilon_t^2)]^2 + 3E \{E_{t-1} [\varepsilon_t^2]^2\} - 3 \{E [E_{t-1} (\varepsilon_t^2)]\}^2$$

$$k = \frac{E [\varepsilon_t^4]}{[E (\varepsilon_t^2)]^2} = 3 + 3 \frac{E \{E_{t-1} [\varepsilon_t^2]^2\} - \{E [E_{t-1} (\varepsilon_t^2)]\}^2}{[E (\varepsilon_t^2)]^2}$$

$$= 3 + 3 \frac{\text{Var} \{E_{t-1} [\varepsilon_t^2]\}}{[E (\varepsilon_t^2)]^2} = 3 + 3 \frac{\text{Var} \{\sigma_t^2\}}{[E (\varepsilon_t^2)]^2}.$$

# The ARCH Model

Another important property of the *ARCH* process is that the process is conditionally serially uncorrelated. Given that

$$E_{t-1} [\varepsilon_t] = 0$$

we have that with the Law of Iterated Expectations:

$$E_{t-h} [\varepsilon_t] = E_{t-h} [E_{t-1} (\varepsilon_t)] = E_{t-h} [0] = 0.$$

This orthogonality property implies that the  $\{\varepsilon_t\}$  process is conditionally uncorrelated:

$$\begin{aligned} \text{Cov}_{t-h} [\varepsilon_t, \varepsilon_{t+k}] &= E_{t-h} [\varepsilon_t \varepsilon_{t+k}] - E_{t-h} [\varepsilon_t] E_{t-h} [\varepsilon_{t+k}] = \\ &= E_{t-h} [\varepsilon_t \varepsilon_{t+k}] = E_{t-h} [E_{t+k-1} (\varepsilon_t \varepsilon_{t+k})] = \\ &= E_{t-h} [\varepsilon_t E_{t+k-1} [\varepsilon_{t+k}]] = 0 \end{aligned}$$

The ARCH model has showed to be particularly useful in modeling the temporal dependencies in asset returns.

# The ARCH( $q$ ) Model

The  $ARCH(q)$  model introduced by Engle (1982) is a linear function of past squared disturbances:

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2$$

In this model to assure a positive conditional variance the parameters have to satisfy the following constraints:  $\omega > 0$  e  $\alpha_1 \geq 0, \alpha_2 \geq 0, \dots, \alpha_q \geq 0$ . Defining

$$\sigma_t^2 \equiv \varepsilon_t^2 - v_t$$

where  $E_{t-1}(v_t) = 0$  we can write (13) as an  $AR(q)$  in  $\varepsilon_t^2$ :

$$\varepsilon_t^2 = \omega + \alpha(L) \varepsilon_t^2 + v_t$$

where  $\alpha(L) = \alpha_1 L + \alpha_2 L^2 + \dots + \alpha_q L^q$ .

# The ARCH(q) Model

$$(1 - \alpha(L))\varepsilon_t^2 = \omega + v_t$$

The process is weakly stationary if and only if  $\sum_{i=1}^q \alpha_i < 1$ ; in this case the unconditional variance is given by

$$E(\varepsilon_t^2) = \omega / (1 - \alpha_1 - \dots - \alpha_q).$$

# The ARCH(q) Model

The process is characterized by leptokurtosis in excess with respect to the normal distribution. In the case, for example, of  $ARCH(1)$  with  $\varepsilon_t | \Phi_{t-1} \sim N(0, \sigma_t^2)$ , the kurtosis is equal to:

$$E(\varepsilon_t^4) / E(\varepsilon_t^2)^2 = 3(1 - \alpha_1^2) / (1 - 3\alpha_1^2)$$

with  $3\alpha_1^2 < 1$ , when  $3\alpha_1^2 = 1$  we have

$$E(\varepsilon_t^4) / E(\varepsilon_t^2)^2 = \infty.$$

In both cases we obtain a kurtosis coefficient greater than 3, characteristic of the normal distribution.

# The ARCH Regression Model

We have an ARCH regression model when the disturbances in a linear regression model follow an ARCH process:

$$y_t = x_t' b + \varepsilon_t$$

$$E_{t-1}(\varepsilon_t) = 0$$

$$\varepsilon_t | \Phi_{t-1} \sim N(0, \sigma_t^2)$$

$$E_{t-1}(\varepsilon_t^2) \equiv \sigma_t^2 = \omega + \alpha(L) \varepsilon_t^2$$



# The GARCH(p,q) model

In order to model in a parsimonious way the conditional heteroskedasticity, Bollerslev (1986) proposed the Generalised *ARCH* model, i.e *GARCH*(p,q):

$$\sigma_t^2 = \omega + \alpha(L) \varepsilon_t^2 + \beta(L) \sigma_t^2.$$

where  $\alpha(L) = \alpha_1 L + \dots + \alpha_q L^q$ ,  $\beta(L) = \beta_1 L + \dots + \beta_p L^p$ .

The *GARCH*(1,1) is the most popular model in the empirical literature:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

# The GARCH(p,q) model

To ensure that the conditional variance is well defined in a  $GARCH(p,q)$  model all the coefficients in the corresponding linear  $ARCH(\infty)$  should be positive:

$$\begin{aligned}\sigma_t^2 &= \left(1 - \sum_{i=1}^p \beta_i L_i\right)^{-1} \left[\omega + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2\right] \\ &= \omega^* + \sum_{k=0}^{\infty} \phi_k \varepsilon_{t-k-1}^2\end{aligned}$$

$\sigma_t^2 \geq 0$  if  $\omega^* \geq 0$  and all  $\phi_k \geq 0$ . The non-negativity of  $\omega^*$  and  $\phi_k$  is also a necessary condition for the non negativity of  $\sigma_t^2$ .

# The GARCH(p,q) model

In order to make  $\omega^*$  e  $\{\phi_k\}_{k=0}^{\infty}$  well defined, assume that :

- i. the roots of the polynomial  $\beta(x) = 1$  lie outside the unit circle, and that  $\omega \geq 0$ , this is a condition for  $\omega^*$  to be finite and positive.
- ii.  $\alpha(x)$  e  $1 - \beta(x)$  have no common roots.

These conditions are establishing nor that  $\sigma_t^2 \leq \infty$  neither that  $\{\sigma_t^2\}_{t=-\infty}^{\infty}$  is strictly stationary. For the simple  $GARCH(1,1)$  almost sure positivity of  $\sigma_t^2$  requires, with the conditions (i) and (ii), that (Nelson and Cao, 1992),

$$\omega \geq 0$$

$$\beta_1 \geq 0$$

$$\alpha_1 \geq 0$$

# The GARCH(p,q) model

For the  $GARCH(1,q)$  and  $GARCH(2,q)$  models these constraints can be relaxed, e.g. in the  $GARCH(1,2)$  model the necessary and sufficient conditions become:

$$\begin{aligned}\omega &\geq 0 \\ 0 &\leq \beta_1 < 1 \\ \beta_1\alpha_1 + \alpha_2 &\geq 0 \\ \alpha_1 &\geq 0\end{aligned}$$

For the  $GARCH(2,1)$  model the conditions are:

$$\begin{aligned}\omega &\geq 0 \\ \alpha_1 &\geq 0 \\ \beta_1 &\geq 0 \\ \beta_1 + \beta_2 &< 1 \\ \beta_1^2 + 4\beta_2 &\geq 0\end{aligned}$$

# The GARCH(p,q) model

These constraints are less stringent than those proposed by Bollerslev (1986):

$$\begin{aligned}\omega &\geq 0 \\ \beta_i &\geq 0 \quad i = 1, \dots, p \\ \alpha_j &\geq 0 \quad j = 1, \dots, q\end{aligned}$$

These results cannot be adopted in the multivariate case, where the requirement of positivity for  $\{\sigma_t^2\}$  means the positive definiteness for the conditional variance-covariance matrix.

# Stationarity

The process  $\{\varepsilon_t\}$  which follows a GARCH(p,q) model is a martingale difference sequence. In order to study second-order stationarity it's sufficient to consider that:

$$\text{Var} [\varepsilon_t] = \text{Var} [E_{t-1} (\varepsilon_t)] + E [\text{Var}_{t-1} (\varepsilon_t)] = E [\sigma_t^2]$$

and show that is asymptotically constant in time (it does not depend upon time).

A process  $\{\varepsilon_t\}$  which satisfies a GARCH(p,q) model with positive coefficient  $\omega \geq 0$ ,  $\alpha_i \geq 0$   $i = 1, \dots, q$ ,  $\beta_i \geq 0$   $i = 1, \dots, p$  is covariance stationary if and only if:

$$\alpha(1) + \beta(1) < 1$$

This is a sufficient but non necessary conditions for strict stationarity. Because ARCH processes are thick tailed, the conditions for covariance stationarity are often more stringent than the conditions for strict stationarity.

# Strict Stationarity GARCH(1,1)

Nelson shows that when  $\omega > 0$ ,  $\sigma_t^2 < \infty$  a.s. and  $\{\varepsilon_t, \sigma_t^2\}$  is strictly stationary if and only if  $E [\ln (\beta_1 + \alpha_1 z_t^2)] < 0$

$$E [\ln (\beta_1 + \alpha_1 z_t^2)] \leq \ln [E (\beta_1 + \alpha_1 z_t^2)] = \ln (\alpha_1 + \beta_1)$$

when  $\alpha_1 + \beta_1 = 1$  the model is strictly stationary.  $E [\ln (\beta_1 + \alpha_1 z_t^2)] < 0$  is a weaker requirement than  $\alpha_1 + \beta_1 < 1$ .

## Example

ARCH(1), with  $\alpha_1 = 1$ ,  $\beta_1 = 0$ ,  $z_t \sim N(0, 1)$

$$E [\ln (z_t^2)] \leq \ln [E (z_t^2)] = \ln (1)$$

It's strictly but not covariance stationary. The ARCH(q) is covariance stationary if and only if the sum of the positive parameters is less than one.

# Forecasting volatility

Forecasting with a GARCH(p,q) (Engle and Bollerslev 1986):

$$\sigma_{t+k}^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t+k-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t+k-i}^2$$

we can write the process in two parts, before and after time  $t$ :

$$\sigma_{t+k}^2 = \omega + \sum_{i=1}^n [\alpha_i \varepsilon_{t+k-i}^2 + \beta_i \sigma_{t+k-i}^2] + \sum_{i=k}^m [\alpha_i \varepsilon_{t+k-i}^2 + \beta_i \sigma_{t+k-i}^2]$$

where  $n = \min \{m, k-1\}$  and by definition summation from 1 to 0 and from  $k > m$  to  $m$  both are equal to zero. Thus

$$E_t [\sigma_{t+k}^2] = \omega + \sum_{i=1}^n [(\alpha_i + \beta_i) E_t (\sigma_{t+k-i}^2)] + \sum_{i=k}^m [\alpha_i \varepsilon_{t+k-i}^2 + \beta_i \sigma_{t+k-i}^2] .$$



In particular for a GARCH(1,1) and  $k > 2$ :

$$\begin{aligned}
 E_t [\sigma_{t+k}^2] &= \sum_{i=0}^{k-2} (\alpha_1 + \beta_1)^i \omega + (\alpha_1 + \beta_1)^{k-1} \sigma_{t+1}^2 \\
 &= \omega \frac{[1 - (\alpha_1 + \beta_1)^{k-1}]}{[1 - (\alpha_1 + \beta_1)]} + (\alpha_1 + \beta_1)^{k-1} \sigma_{t+1}^2 \\
 &= \sigma^2 \frac{[1 - (\alpha_1 + \beta_1)^{k-1}]}{[1 - (\alpha_1 + \beta_1)]} + (\alpha_1 + \beta_1)^{k-1} \sigma_{t+1}^2 \\
 &= \sigma^2 + (\alpha_1 + \beta_1)^{k-1} [\sigma_{t+1}^2 - \sigma^2]
 \end{aligned}$$

When the process is covariance stationary, it follows that  $E_t [\sigma_{t+k}^2]$  converges to  $\sigma^2$  as  $k \rightarrow \infty$ .

# The IGARCH(p,q) model

The GARCH(p,q) process characterized by the first two conditional moments:

$$E_{t-1} [\varepsilon_t] = 0$$

$$\sigma_t^2 \equiv E_{t-1} [\varepsilon_t^2] = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2$$

where  $\omega \geq 0$ ,  $\alpha_i \geq 0$  and  $\beta_i \geq 0$  for all  $i$  and the polynomial

$$1 - \alpha(x) - \beta(x) = 0$$

has  $d > 0$  unit root(s) and  $\max\{p, q\} - d$  root(s) outside the unit circle is said to be:

- Integrated in variance of order  $d$  if  $\omega = 0$
- Integrated in variance of order  $d$  with trend if  $\omega > 0$ .

# The IGARCH(p,q) model

The Integrated GARCH(p,q) models, both with or without trend, are therefore part of a wider class of models with a property called *persistent variance* in which the current information remains important for the forecasts of the conditional variances for all horizon. So we have the Integrated GARCH(p,q) model when (necessary condition)

$$\alpha(1) + \beta(1) = 1$$

# The IGARCH(p,q) model

To illustrate consider the IGARCH(1,1) which is characterised by

$$\alpha_1 + \beta_1 = 1$$

$$\begin{aligned}\sigma_t^2 &= \omega + \alpha_1 \varepsilon_{t-1}^2 + (1 - \alpha_1) \sigma_{t-1}^2 \\ \sigma_t^2 &= \omega + \sigma_{t-1}^2 + \alpha_1 (\varepsilon_{t-1}^2 - \sigma_{t-1}^2) \quad 0 < \alpha_1 \leq 1\end{aligned}$$

For this particular model the conditional variance  $k$  steps in the future is:

$$E_t [\sigma_{t+k}^2] = (k - 1) \omega + \sigma_{t+1}^2$$

# The EGARCH(p,q) Model

- GARCH models assume that only the magnitude and not the positivity or negativity of unanticipated excess returns determines feature  $\sigma_t^2$ .
- There exists a negative correlation between stock returns and changes in returns volatility, i.e. volatility tends to rise in response to "bad news", (excess returns lower than expected) and to fall in response to "good news" (excess returns higher than expected).

# The EGARCH(p,q) Model

If we write  $\sigma_t^2$  as a function of lagged  $\sigma_t^2$  and lagged  $z_t^2$ , where  $\varepsilon_t^2 = z_t^2 \sigma_t^2$

$$\sigma_t^2 = \omega + \sum_{j=1}^q \alpha_j z_{t-j}^2 \sigma_{t-j}^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2$$

it is evident that the conditional variance is invariant to changes in sign of the  $z_t$ 's. Moreover, the innovations  $z_{t-j}^2 \sigma_{t-j}^2$  are not *i.i.d.*

- The nonnegativity constraints on  $\omega^*$  and  $\phi_k$ , which are imposed to ensure that  $\sigma_t^2$  remains nonnegative for all  $t$  with probability one. These constraints imply that increasing  $z_t^2$  in any period increases  $\sigma_{t+m}^2$  for all  $m \geq 1$ , ruling out random oscillatory behavior in the  $\sigma_t^2$  process.

# The EGARCH(p,q) Model

- The GARCH models are not able to explain the observed covariance between  $\varepsilon_t^2$  and  $\varepsilon_{t-j}$ . This is possible only if the conditional variance is expressed as an asymmetric function of  $\varepsilon_{t-j}$ .
- In GARCH(1,1) models, shocks may persist in one norm and die out in another, so the conditional moments of GARCH(1,1) may explode even when the process is strictly stationary and ergodic.
- GARCH models essentially specify the behavior of the square of the data. In this case a few large observations can dominate the sample.

# The EGARCH(p,q) Model

In the  $EGARCH(p,q)$  model (*Exponential GARCH(p,q)*) put forward by Nelson the  $\sigma_t^2$  depends on both size and the sign of lagged residuals. This is the first example of asymmetric model:

$$\ln(\sigma_t^2) = \omega + \sum_{i=1}^p \beta_i \ln(\sigma_{t-i}^2) + \sum_{i=1}^q \alpha_i [\phi z_{t-i} + \psi (|z_{t-i}| - E|z_{t-i}|)]$$

$\alpha_1 \equiv 1$ ,  $E|z_t| = (2/\pi)^{1/2}$  when  $z_t \sim NID(0, 1)$ , where the parameters  $\omega$ ,  $\beta_i$ ,  $\alpha_i$  are not restricted to be nonnegative. Let define

$$g(z_t) \equiv \phi z_t + \psi [|z_t| - E|z_t|]$$

by construction  $\{g(z_t)\}_{t=-\infty}^{\infty}$  is a zero-mean, i.i.d. random sequence.



# The EGARCH(p,q) Model

- The components of  $g(z_t)$  are  $\phi z_t$  and  $\psi [|z_t| - E|z_t|]$ , each with mean zero.
- If the distribution of  $z_t$  is symmetric, the components are orthogonal, but not independent.
- Over the range  $0 < z_t < \infty$ ,  $g(z_t)$  is linear in  $z_t$  with slope  $\phi + \psi$ , and over the range  $-\infty < z_t \leq 0$ ,  $g(z_t)$  is linear with slope  $\phi - \psi$ .
- The term  $\psi [|z_t| - E|z_t|]$  represents a magnitude effect.

# The EGARCH(p,q) Model

- If  $\psi > 0$  and  $\phi = 0$ , the innovation in  $\ln(\sigma_{t+1}^2)$  is positive (negative) when the magnitude of  $z_t$  is larger (smaller) than its expected value.
- If  $\psi = 0$  and  $\phi < 0$ , the innovation in conditional variance is now positive (negative) when returns innovations are negative (positive).
- A negative shock to the returns which would increase the debt to equity ratio and therefore increase uncertainty of future returns could be accounted for when  $\alpha_i > 0$  and  $\phi < 0$ .

# The EGARCH(p,q) Model

Nelson assumes that  $z_t$  has a GED distribution (exponential power family). The density of a GED random variable normalized is:

$$f(z; v) = \frac{v \exp \left[ - \left( \frac{1}{2} \right) |z/\lambda|^v \right]}{\lambda 2^{(1+1/v)} \Gamma(1/v)} \quad -\infty < z < \infty, 0 < v \leq \infty$$

# The EGARCH(p,q) Model

where  $\Gamma(\cdot)$  is the gamma function, and

$$\lambda \equiv \left[ 2^{(-2/v)} \Gamma(1/v) / \Gamma(3/v) \right]^{1/2}$$

$v$  is a tail thickness parameter.

$z$ 's distribution

$v = 2$       standard normal distribution

$v < 2$       thicker tails than the normal

$v = 1$       double exponential distribution

$v > 2$       thinner tails than the normal

$v = \infty$       uniformly distributed on  $[-3^{1/2}, 3^{1/2}]$

With this density,  $E|z_t| = \frac{\lambda 2^{1/v} \Gamma(2/v)}{\Gamma(1/v)}$ .

# Non linear ARCH(p,q) Model

The *Non linear ARCH(p,q)* model (Engle - Bollerslev 1986):

$$\sigma_t^\gamma = \omega + \sum_{i=1}^q \alpha_i |\varepsilon_{t-i}|^\gamma + \sum_{i=1}^p \beta_i \sigma_{t-i}^\gamma$$

$$\sigma_t^\gamma = \omega + \sum_{i=1}^q \alpha_i |\varepsilon_{t-i} - k|^\gamma + \sum_{i=1}^p \beta_i \sigma_{t-i}^\gamma$$

for  $k \neq 0$ , the innovations in  $\sigma_t^\gamma$  will depend on the size as well as the sign of lagged residuals, thereby allowing for the leverage effect in stock return volatility.

# GJR model

The Glosten - Jagannathan - Runkle model (1993):

$$\sigma_t^2 = \omega + \sum_{i=1}^p \beta_i \sigma_{t-i}^2 + \sum_{i=1}^q (\alpha_i \varepsilon_{t-1}^2 + \gamma_i S_{t-i}^- \varepsilon_{t-i}^2)$$

where

$$S_t^- = \begin{cases} 1 & \text{if } \varepsilon_t < 0 \\ 0 & \text{if } \varepsilon_t \geq 0 \end{cases}$$

# Asymmetric GARCH(p,q)

The *Asymmetric GARCH*(p,q) model (Engle, 1990):

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i (\varepsilon_{t-i} + \gamma)^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2$$

The *QGARCH* by Sentana (1995):

$$\sigma_t^2 = \sigma^2 + \Psi' x_{t-q} + x_{t-q}' A x_{t-q} + \sum_{i=1}^p \beta_i \sigma_{t-i}^2$$

when  $x_{t-q} = (\varepsilon_{t-1}, \dots, \varepsilon_{t-q})'$ . The linear term  $(\Psi' x_{t-q})$  allows for asymmetry. The off-diagonal elements of  $A$  accounts for interaction effects of lagged values of  $x_t$  on the conditional variance. The QGARCH nests several asymmetric models.

# The APARCH model

The proliferation of GARCH models has inspired some authors to define families of GARCH models that would accommodate as many individual as models as possible. The *Asymmetric Power ARCH* (Ding, Engle and Granger, 1993)

$$r_t = \mu + \epsilon_t$$

$$\epsilon_t = \sigma_t z_t \quad z_t \sim N(0, 1)$$

$$\sigma_t^\delta = \omega + \sum_{i=1}^q \alpha_i (|\epsilon_{t-i}| - \gamma_i \epsilon_{t-i})^\delta + \sum_{j=1}^p \beta_j \sigma_{t-j}^\delta$$

where

$$\begin{array}{llll} \omega > 0 & \delta \geq 0 & \alpha_i \geq 0 & i = 1, \dots, q \\ -1 < \gamma_i < 1 & i = 1, \dots, q & \beta_j \geq 0 & j = 1, \dots, p \end{array}$$



# The APARCH model

This model imposes a Box-Cox transformation of the conditional standard deviation process and the asymmetric absolute residuals. The Box-Cox transformation for a positive random variable  $Y_t$ :

$$Y_t^{(\lambda)} = \begin{cases} \frac{Y_t^\lambda - 1}{\lambda} & \lambda \neq 0 \\ \log Y_t & \lambda = 0 \end{cases}$$

The asymmetric response of volatility to positive and negative "shocks" is the well known leverage effect.

This generalized version of ARCH model includes seven other models as special cases.

# The APARCH model

- ➊ ARCH( $q$ ) model, just let  $\delta = 2$  and  $\gamma_i = 0, i = 1, \dots, q, \beta_j = 0, j = 1, \dots, p$ .
- ➋ GARCH( $p, q$ ) model just let  $\delta = 2$  and  $\gamma_i = 0, i = 1, \dots, q$ .
- ➌ Taylor/Schwert's GARCH in standard deviation model just let  $\delta = 1$  and  $\gamma_i = 0, i = 1, \dots, q$ .
- ➍ GJR model just let  $\delta = 2$ .

# The GARCH-*in-mean* Model

The GARCH-in-mean (GARCH-M) proposed by Engle, Lilien and Robins (1987) consists of the system:

$$y_t = \gamma_0 + \gamma_1 x_t + \gamma_2 g(\sigma_t^2) + \epsilon_t$$

$$\sigma_t^2 = \beta_0 + \sum_{i=1}^q \alpha_i \epsilon_{t-1}^2 + \sum_{i=1}^p \beta_i \sigma_{t-1}^2$$

$$\epsilon_t \mid \Phi_{t-1} \sim N(0, \sigma_t^2)$$

When  $y_t \equiv (r_t - r_f)$ , where  $(r_t - r_f)$  is the risk premium on holding the asset, then the GARCH-M represents a simple way to model the relation between risk premium and its conditional variance.

This model characterizes the evolution of the mean and the variance of a time series simultaneously.

The GARCH-M model therefore allows to analyze the possibility of time-varying risk premium.

It turns out that:

$$y_t \mid \Phi_{t-1} \sim N(\gamma_0 + \gamma_1 x_t + \gamma_2 g(\sigma_t^2), \sigma_t^2)$$

In applications,  $g(\sigma_t^2) = \sqrt{\sigma_t^2}$ ,  $g(\sigma_t^2) = \ln(\sigma_t^2)$  and  $g(\sigma_t^2) = \sigma_t^2$  have been used.

# The News Impact Curve

- The news have asymmetric effects on volatility.
- In the asymmetric volatility models good news and bad news have different predictability for future volatility.
- The news impact curve characterizes the impact of past return shocks on the return volatility which is implicit in a volatility model.
- Holding constant the information dated  $t - 2$  and earlier, we can examine the implied relation between  $\varepsilon_{t-1}$  and  $\sigma_t^2$ , with  $\sigma_{t-i}^2 = \sigma^2$   $i = 1, \dots, p$ .
- This impact curve relates past return shocks (news) to current volatility.
- This curve measures how new information is incorporated into volatility estimates.

For the GARCH model the News Impact Curve (NIC) is centered on  $\epsilon_{t-1} = 0$ .

**GARCH(1,1):**

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

The news impact curve has the following expression:

$$\sigma_t^2 = A + \alpha \epsilon_{t-1}^2$$

$$A \equiv \omega + \beta \sigma^2$$

In the case of EGARCH model the curve has its minimum at  $\epsilon_{t-1} = 0$  and is exponentially increasing in both directions but with different parameters.

**EGARCH(1,1):**

$$\sigma_t^2 = \omega + \beta \ln(\sigma_{t-1}^2) + \phi z_{t-1} + \psi (|z_{t-1}| - E|z_{t-1}|)$$

where  $z_t = \epsilon_t / \sigma_t$ . The news impact curve is

$$\sigma_t^2 = \begin{cases} A \exp \left[ \frac{\phi + \psi}{\sigma} \epsilon_{t-1} \right] & \text{for } \epsilon_{t-1} > 0 \\ A \exp \left[ \frac{\phi - \psi}{\sigma} \epsilon_{t-1} \right] & \text{for } \epsilon_{t-1} < 0 \end{cases}$$

$$A \equiv \sigma^{2\beta} \exp \left[ \omega - \alpha \sqrt{2/\pi} \right]$$

$$\phi < 0 \quad \psi + \phi > 0$$

- The EGARCH allows good news and bad news to have different impact on volatility, while the standard GARCH does not.
- The EGARCH model allows big news to have a greater impact on volatility than GARCH model. EGARCH would have higher variances in both directions because the exponential curve eventually dominates the quadrature.



## The Asymmetric GARCH(1,1) (Engle, 1990)

$$\sigma_t^2 = \omega + \alpha (\epsilon_{t-1} + \gamma)^2 + \beta \sigma_{t-1}^2$$

the NIC is

$$\sigma_t^2 = A + \alpha (\epsilon_{t-1} + \gamma)^2$$

$$A \equiv \omega + \beta \sigma^2$$

$$\omega > 0, 0 \leq \beta < 1, \sigma > 0, 0 \leq \alpha < 1.$$

is asymmetric and centered at  $\epsilon_{t-1} = -\gamma$ .

## The Glosten-Jagannathan-Runkle model

$$\sigma_t^2 = \omega + \alpha \epsilon_t^2 + \beta \sigma_{t-1}^2 + \gamma S_{t-1}^- \epsilon_{t-1}^2$$

$$S_{t-1}^- = \begin{cases} 1 & \text{if } \epsilon_{t-1} < 0 \\ 0 & \text{otherwise} \end{cases}$$

The NIC is

$$\sigma_t^2 = \begin{cases} A + \alpha \epsilon_{t-1}^2 & \text{if } \epsilon_{t-1} > 0 \\ A + (\alpha + \gamma) \epsilon_{t-1}^2 & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

$$A \equiv \omega + \beta \sigma^2$$

$$\omega > 0, 0 \leq \beta < 1, \sigma > 0, 0 \leq \alpha < 1, \alpha + \beta < 1$$

is centered at  $\epsilon_{t-1} = -\gamma$ .