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The clustering coefficient of a scale-free random graph

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ABSTRACT

We consider a random graph process in which, at each time step, a new vertex is added with m out-neighbours, chosen with probabilities proportional to their degree plus a strictly positive constant. We show that the expectation of the clustering coefficient of the graph process is asymptotically proportional to $\frac{\log n}{n}$. Bollobás and Riordan have previously shown that when the constant is zero, the same expectation is asymptotically proportional to $\frac{(\log n)^2}{n}$.

1. Introduction

Recently there has been a great deal of interest in the structure of real-world networks, especially the Internet. Many mathematical models have been proposed: most of these describe graph processes in which new edges are added by some form of preferential attachment. There is a vast literature discussing empirical properties of these networks but there is also a growing body of more rigorous work. A wide-ranging account of empirical properties of networks can be found in [2]; a good survey of rigorous results can be found in [3] or in the recent book [9].

In [16] Watts and Strogatz defined 'small-world' networks to be those having small path length and being highly clustered, and discovered that many real-world networks are small-world networks, e.g. the power grid of the western USA and the collaboration graph of film actors.

There are conflicting definitions of the clustering coefficient appearing in the literature. See [3] for a discussion of the relationships between them. We define the clustering coefficient, $C(G)$, of a graph G as follows:

$$C(G) = \frac{3 \times \text{ number of triangles in } G}{\sum_{v \in V(G)} \binom{d(v)}{2}},$$

where $d(v)$ is the degree of vertex v.

The reason for the 3 in the numerator is to ensure that the clustering coefficient of a complete graph is 1. This is the maximum possible value for a simple graph. However our graphs will not be restricted to simple graphs and so the clustering coefficient can exceed 1. For instance if we take three vertices and join each pair by m edges then the clustering coefficient is $m^2/(2m-1)$. Note that the clustering coefficient of a graph with at most m edges joining any pair of vertices is at most m.

In this paper we establish rigorous results describing the asymptotic behaviour of the clustering coefficient for one class of model. Our graph theoretic notation is standard. Since our graphs are growing, we let $d_t(v)$ denote the total degree of vertex v at time t. Sometimes we omit t when the context is clear.

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The Barabási–Albert model (BA model) [1] is perhaps the most widely studied graph process governed by preferential attachment. A new vertex is added to the graph at each time step and is joined to m existing vertices of the graph chosen with probabilities proportional to their degrees. A key observation [1] is that in many large real-world networks, the proportion of vertices with degree d obeys a power law.

In [5] Bollobás et al. gave a mathematically precise description of the BA model and showed rigorously that for $d \le n^{\frac{1}{15}}$, the proportion of vertices with degree d asymptotically almost surely obeys a power law with exponent -3. Furthermore, in [4] Bollobás and Riordan proved that for $m \ge 2$, the graph is connected with high probability and the diameter is asymptotically $\log n/\log\log n$, while for $m = 1$ the diameter of the largest component is approximately $\log n$.

The most natural generalisation of the BA model is to take the probability of attachment to v at time $t + 1$ to be proportional to $d_t(v) + \beta$, where β is a constant representing the inherent attractiveness of a vertex. Buckley and Osthus [6] generalised the results in [5] to the case where the attractiveness is a positive integer. A much more general model was introduced by Cooper and Frieze in [7] and further results extending [5] were obtained. In particular they generalised the results of [6], by showing that for small d, the proportion of vertices with degree d asymptotically almost surely obeys a power law with exponent $-(3 + \beta)$. A similar result was obtained by Móri [15]. Many more results on these variations of the basic preferential model can be found in [3].

Bollobás and Riordan showed [3] that the expectation of the clustering coefficient of the model from [5] is asymptotically proportional to $(\log n)^2/n$. Bollobás and Riordan also considered in [3] a slight variant of the model from [5]. Their results imply that for this model the expectation of the clustering coefficient is also asymptotically proportional to $(\log n)^2/n$. We work with a model depending on two parameters β, m, which to the best of our knowledge was first studied rigorously by Móri in [14]. In a sense, that we make precise in the next section, Bollobás and Riordan's model is almost the special case of Móri's model corresponding to $\beta = 0$.

Our main result is to show that for $\beta > 0$, asymptotically the expectation of the clustering coefficient is proportional to $\log n/n$. The main strategy of our proof follows [3] and we use very similar notation. In Section 2 we give a definition of the model that we use and explain its relationship with the model studied in [3]. Section 3 contains results that give the probability of the appearance of a small subgraph. We obtain the expectation of the number of triangles appearing and of $\sum_v \binom{d(v)}{2}$ in Section 4. These two sections follow [3] quite closely. The overall aim is to express the expectation of the clustering coefficient as the quotient of the expectation of the number of triangles and the expectation of $\sum_v \binom{d(v)}{2}$. We justify doing this in Section 6 and make use of a concentration result proved in Section 5 using martingale methods. Bollobás and Riordan [3] used a similar strategy and mentioned that they also used martingale methods.

2. The model of Móri

We now describe in detail Móri's generalisation of the BA model [15]. Our definition involves a probability space finer than that described in [15] but the underlying graph processes $(G^n_{m,\beta})$ are identical. The process depends on two parameters: m, the out-degree of each vertex except the first, and $\beta \in \mathbb{R}$ such that $\beta > 0$. (In [15], Móri imposed the weaker condition that $\beta > -1$.)

We first define the process when $m = 1$. Let $G^1_{1,\beta}$ consist of a single vertex v_1 with no edges. The graph $G^{n+1}_{1,\beta}$ is formed from $G^n_{1,\beta}$ by adding a new vertex v_{n+1} together with a single directed edge e. The tail of e is v_{n+1} and the head is determined by a random variable f_{n+1}. We diverge slightly from [15] in our description of f_{n+1}.

Label the edges of $G^n_{1,\beta}$ with $e_2, \dots, e_n$ such that e_i is the unique edge whose tail is v_i. Now let

$$\Omega_{n+1} = \{(1, v), \dots, (n, v), (2, h), \dots, (n, h), (2, t), \dots, (n, t)\}.$$

We define f_{n+1} to take values in Ω_{n+1} such that for $1 \le i \le n$,

$$\Pr(f_{n+1} = (i, v)) = \frac{\beta}{(2+\beta)n - 2}$$

and for $2 \le i \le n$,

$$\Pr(f_{n+1} = (i, h)) = \Pr(f_{n+1} = (i, t)) = \frac{1}{(2+\beta)n - 2}.$$

The head of the new edge added to the graph at time $n + 1$ is called the *target vertex* of v_{n+1} and is determined as follows. If $f_{n+1} = (i, v)$ then the target vertex is v_i and we say that the choice of target vertex has been made *uniformly*. If $f_{n+1} = (i, h)$ then the target vertex is the head of e_i and if $f_{n+1} = (i, t)$ then the target vertex is the tail of e_i, that is v_i. When one of the last two cases occurs, we say that the choice of target vertex has been made *preferentially* by copying the head or tail, as appropriate, of e_i. Suppose we think of an edge as being composed of two half-edges such that each half-edge retains one endpoint of the original edge. Then the target vertex is chosen either by choosing one of the n vertices of $G^n_{1,\beta}$ uniformly at random or by choosing one of the $2n - 2$ half-edges of $G^n_{1,\beta}$ uniformly at random and selecting the vertex to which the half-edge is attached.

The definition implies that for $1 \le i \le n$, the probability that the target vertex of v_{n+1} is v_i is equal to

$$\frac{d_n(v_i) + \beta}{(2+\beta)n - 2}. \tag{2.1}$$

We might have defined f_{n+1} to be a random variable denoting the index of the target vertex of v_{n+1} and taking probabilities as given in (2.1). Indeed for much of the following we will abuse notation and assume that we did define f_{n+1} in this way. However it is useful to have the finer definition when we prove the concentration results in Section 5.

We extend this model to a random graph process $(G^n_{m,\beta})$ for $m > 1$ as follows: run the graph process $(G^t_{1,\beta})$ and form $G^n_{m,\beta}$ by taking $G^{nm}_{1,\beta}$ and merging the first m vertices to form v_1, the next m vertices to form v_2 and so on.

Notice that our definition will not immediately extend to the case $\beta = 0$ because when $n = 1$, the denominator of the expression in (2.1) is zero and so the process cannot start. One way to get around this problem is to define $G^2_{1,0}$ to be the graph with two vertices joined by a single edge and then let the process carry on from there. A second possibility, used in [3], is to attach an artificial half-edge to v_1 at the beginning. This half-edge remains present throughout the process, so the sum of the vertex degrees at time n is $2n - 1$ rather than $2n - 2$ as in the model that we use. However it turns out that the choice of which alternative to use makes no difference to the asymptotic form of the expectation of the clustering coefficient and so the results from [3] are directly comparable with ours.

In the following we only consider properties of the underlying undirected graph. However, it is helpful to have the extra notation and terminology of directed graphs to simplify the reading of some of the proofs.

3. Subgraphs of $G^n_{1,\beta}$

Let S be a labelled directed forest with no isolated vertices, in which each vertex has either one or no outgoing edge and each directed edge (v_i, v_j) has $i > j$. Moreover if v_1 belongs to S then this vertex has no outgoing edge. The restrictions on S are precisely those that ensure that S can occur as a subgraph of the evolving Móri tree with $m = 1$. We call such an S a *possible forest*.

In this section we generalise the calculation in [3] to calculate the probability that such a graph S is a subgraph of $G^n_{1,\beta}$ for $\beta > 0$. We will follow the method and notation of [3] closely.

We emphasise that we are not computing the probability that $G^n_{1,\beta}$ contains a subgraph isomorphic to S; the labels of the vertices of S must correspond to the vertex labels of $G^n_{1,\beta}$ for S to be considered to be a subgraph of $G^n_{1,\beta}$.

Denote the vertices of S by $v_{s_1}, \ldots, v_{s_k}$, where $s_j < s_{j+1}$ for $1 \le j \le k-1$. Furthermore, let

$$V^- = \{v_i \in V(S) : \text{ there is a } j > i \text{ such that } (v_j, v_i) \in E(S)\}$$

and

$$V^+ = \{v_i \in V(S) : \text{ there is a } j < i \text{ such that } (v_i, v_j) \in E(S)\}.$$

Let $d^{\text{in}}_S(v)$ ($d^{\text{out}}_S(v)$) denote the in-degree (out-degree) of v in S. In particular, $d^{\text{out}}_S(v)$ is either 0 or 1. For $t \ge i$, let $R_t(i) = |\{j > t : (v_j, v_i) \in E(S)\}|$. Observe that $R_i(i) = d^{\text{in}}_S(v_i)$. Moreover, let $c_S(i) = \sum_{k=1}^{i-1} R_{i-1}(k)$. Hence $c_S(i)$ is the number of edges in $E(S)$ from $\{v_i, \ldots, v_n\}$ to $\{v_1, \ldots, v_{i-1}\}$.

Lemma 3.1. *Let $\beta > 0$ and S be a possible forest. Then for $t \ge s_k$ the probability that S is subgraph of $G^t_{1,\beta}$ is given by*

$$\Pr\left(S \subset G^t_{1,\beta}\right) = \frac{\beta}{\beta + d^{\text{in}}_S(v_1)} \prod_{\substack{1 \le i \le t: \\ v_i \in V^-(S)}} \frac{\Gamma\left(1 + \beta + d^{\text{in}}_S(v_i)\right)}{\Gamma(1+\beta)} \cdot \prod_{\substack{1 < i \le t: \\ v_i \in V^+}} \frac{1}{(2+\beta)(i-1) - 2} \prod_{\substack{1 < i \le t: \\ v_i \notin V^+}} \left(1 + \frac{c_S(i)}{(2+\beta)(i-1) - 2}\right).$$

Proof. The proof is a generalisation of the proof for the analogous result in the case $\beta = 0$ in [3] but we include it for completeness.

Let S_t be the subgraph of S induced by the vertices $\{v_1, \ldots, v_t\} \cap V(S)$. We need to define the following random variables:

$$X_t = \prod_{(v_l, v_j) \in E(S_t)} I_{(v_l, v_j) \in E(G^t_{1,\beta})} \prod_{i \le t} \frac{\Gamma(d_t(v_i) + \beta + R_t(i))}{\Gamma(d_t(v_i) + \beta)}$$

and

$$Y_t = \prod_{(v_l, v_j) \in E(S_{t+1})} I_{(v_l, v_j) \in E(G^{t+1}_{1,\beta})} \prod_{i \le t} \frac{\Gamma(d_{t+1}(v_i) + \beta + R_{t+1}(i))}{\Gamma(d_{t+1}(v_i) + \beta)},$$

where I_A is the indicator of the event A.

Note that $d_t(v_j)$ for $1 \le j \le t$ and X_t are functions of the random variables $f_2, \dots, f_t$ while Y_t is a function of the random variables $f_2, \dots, f_{t+1}$. However, for all j, $R_t(j)$ is deterministic.

Observe that

$$X_{t+1} = \frac{\Gamma(d_{t+1}(v_{t+1}) + \beta + R_{t+1}(t+1))}{\Gamma(d_{t+1}(v_{t+1}) + \beta)} Y_t = \frac{\Gamma(1 + \beta + R_{t+1}(t+1))}{\Gamma(1+\beta)} Y_t.$$

First, assume that there is no $r \le t$ such that $(v_{t+1}, v_r) \in E(S)$ and so the new edge added at time $t+1$ cannot belong to S. This implies that for $i \le t$, $R_t(i) = R_{t+1}(i)$ and $\prod_{(v_l,v_j)\in E(S_t)} I_{(v_l,v_j)\in E(G^t_{1,\beta})} = \prod_{(v_l,v_j)\in E(S_{t+1})} I_{(v_l,v_j)\in E(G^{t+1}_{1,\beta})}$. Furthermore for all $i \le t$ with $i \ne f_{t+1}$, we have $d_{t+1}(v_i) = d_t(v_i)$. We also have $d_{t+1}(v_{f_{t+1}}) = d_t(v_{f_{t+1}}) + 1$.

For the moment fix $f_2, \dots, f_t$ so that X_t is completely determined. Now,

$$Y_t = \left(1 + \frac{R_t(f_{t+1})}{d_t(v_{f_{t+1}}) + \beta}\right) X_t.$$

Thus

$$\mathbf{E}\left[Y_t - X_t | f_2, \dots, f_t\right] = \sum_{r=1}^{t} \frac{R_t(r)}{d_t(v_r) + \beta} \Pr(f_{t+1} = r) X_t = \frac{\sum_{r=1}^{t} R_t(r)}{(2+\beta)t - 2} X_t.$$

By taking expectation with respect to $f_2, \dots, f_t$ we obtain

$$\mathbf{E}\left[Y_t\right] = \left(1 + \frac{\sum_{r=1}^{t} R_t(r)}{(2+\beta)t - 2}\right) \mathbf{E}\left[X_t\right] = \left(1 + \frac{c_S(t+1)}{(2+\beta)t-2}\right) \mathbf{E}\left[X_t\right]$$

and

$$\mathbf{E}\left[X_{t+1}\right] = \frac{\Gamma(1+\beta+R_{t+1}(t+1))}{\Gamma(1+\beta)} \left(1 + \frac{c_S(t+1)}{(2+\beta)t-2}\right) \mathbf{E}\left[X_t\right]. \tag{3.2}$$

Now suppose (v_{t+1}, v_r) is an edge of S for some $r \le t$. If $f_{t+1} \ne r$ then $X_{t+1} = 0$ so we will suppose that $f_{t+1} = r$. Then for all $i \le t$ with $i \ne r$, $d_{t+1}(v_i) = d_t(v_i)$, and $d_{t+1}(v_r) = d_t(v_r) + 1$. Furthermore for all $i \le t$ with $i \ne r$, $R_{t+1}(i) = R_t(i)$, but $R_{t+1}(r) = R_t(r) - 1$.

Hence providing $f_{t+1} = v_r$, we have

$$Y_t = \frac{1}{d_t(v_r) + \beta} X_t.$$

So

$$\mathbf{E}\left[Y_t | f_2, \dots, f_t\right] = \frac{d_t(v_r) + \beta}{(2+\beta)t-2} \frac{X_t}{d_t(v_r)+\beta} = \frac{X_t}{(2+\beta)t-2}.$$

Thus

$$\mathbf{E}\left[X_{t+1} | f_2, \dots, f_t\right] = \frac{1}{(2+\beta)t-2} \frac{\Gamma(1+\beta+R_{t+1}(t+1))}{\Gamma(1+\beta)} X_t.$$

So by taking expectation with respect to $f_2, \dots, f_t$,

$$\mathbf{E}\left[X_{t+1}\right] = \frac{1}{(2+\beta)t-2} \frac{\Gamma(1+\beta+R_{t+1}(t+1))}{\Gamma(1+\beta)} \mathbf{E}[X_t]. \tag{3.3}$$

Note that $X_1 = \frac{\Gamma(\beta+R_1(1))}{\Gamma(\beta)}$ and that for $t \ge s_k$, we have $\Pr(S \subset G^t_{1,\beta}) = \mathbf{E}\left[X_t\right]$. Using (3.2) and (3.3) and noting that $R_i(i) = 0$ for $v_i \notin V^-$, we have for $t \ge s_k$

$$\Pr(S \subset G^t_{1,\beta}) = \frac{\Gamma(\beta + R_1(1))}{\Gamma(\beta)} \prod_{\substack{1<i\le t:\\ v_i \in V^-}} \frac{\Gamma(1+\beta+R_i(i))}{\Gamma(1+\beta)} \cdot \prod_{\substack{1<i\le t:\\ v_i\in V^+}} \frac{1}{(2+\beta)(i-1)-2} \prod_{\substack{1<i\le t:\\ v_i \notin V^+}} \left(1 + \frac{c_S(i)}{(2+\beta)(i-1)-2}\right).$$

This is easily seen to be equivalent to the expression in the statement of the lemma. □

We now provide a more convenient form for the probability given in Lemma 3.1. This calculation is almost identical to the analogous one in [3] so we omit the proof.

Lemma 3.4. *Let $\beta > 0$ and S be a possible forest. Then for $t \geq s_k$ the probability that S is a subgraph of $G^t_{1,\beta}$ is given by*

$$\Pr(S \subset G^t_{1,\beta}) = \frac{\beta}{d^{\text{in}}_S(v_1) + \beta} \prod_{i:v_i \in V^-} \frac{\Gamma(1 + d^{\text{in}}_S(v_i) + \beta)}{\Gamma(1 + \beta)} \cdot \prod_{(v_i,v_j)\in E(S):i>j} \frac{1}{(2+\beta)(i^{1+\beta}j)^{1/(2+\beta)}} \exp\left(O\left(\sum_{j=2}^{k} c_S(s_j)^2/(j-1)\right)\right).$$

4. Calculation of expectations

Recall that the clustering coefficient $C(G)$ of a graph G is given by

$$C(G) = \frac{3 \times \text{ number of triangles in } G}{\sum_{v \in V(G)} \binom{d(v)}{2}}.$$

In this section we calculate the expectations of the numerator and denominator of this expression.

4.1. The expected number of triangles

We adapt the methods used in [3] to the case $\beta > 0$. For fixed $a < b < c$, we first calculate the expected number of triangles in $G^n_{m,\beta}$ on vertices v_a, v_b, v_c. Let $G^{mn}_{1,\beta}$ be the underlying tree used to form $G^n_{m,\beta}$. Label the vertices of the tree $v'_1, \ldots, v'_{mn}$. A triangle on v_a, v_b, v_c arises if there are vertices v'_{a_1}, v'_{a_2} with $(a-1)m+1 \leq a_1, a_2 \leq am$, v'_{b_1}, v'_{b_2} with $(b-1)m+1 \leq b_1, b_2 \leq bm$ and v'_{c_1}, v'_{c_2} with $(c-1)m+1 \leq c_1, c_2 \leq cm$ such that v'_{b_1} sends its outgoing edge to v'_{a_1}, v'_{c_1} sends its outgoing edge to v'_{a_2} and v'_{c_2} sends its outgoing edge to v'_{b_2}. For this to be possible, we need $c_1 \neq c_2$. Let S be the graph with vertices $v'_{a_1}, v'_{a_2}, v'_{b_1}, v'_{b_2}, v'_{c_1}, v'_{c_2}$ and edges (v'_{b_1}, v'_{a_1}), (v'_{c_1}, v'_{a_2}) and (v'_{c_2}, v'_{b_2}). Write $a_1 = am - l_1$, $a_2 = am - l_2$, $b_1 = bm - l_3$, $b_2 = bm - l_4$, $c_1 = cm - l_5$ and $c_2 = cm - l_6$. The cases where $a_1 = a_2$ and $a_1 \neq a_2$ are slightly different. We concentrate on the former to begin with.

We have $d^{\text{in}}_S(v_{a_1}) = 2$, $d^{\text{in}}_S(v_{b_2}) = 1$ and otherwise $d^{\text{in}}_S(v) = 0$. Suppose that $a_1 > 1$. Then applying Lemma 3.4 we see that

$$\Pr(S \subseteq G^{mn}_{1,\beta}) = \frac{\Gamma(3+\beta)\Gamma(2+\beta)}{(\Gamma(1+\beta))^2} \frac{1}{(2+\beta)^3} \left(\frac{1}{a_1a_2b_2(b_1c_1c_2)^{1+\beta}}\right)^{1/(2+\beta)} \exp(O(1/a)). \tag{4.1}$$

The same expression holds when $a_1 = 1$ because the extra multiplicative term of $\beta/(2+\beta)$ may be absorbed into the error term. Note that for $-1 \leq x \leq 1$, we have $\mathrm{e}^x = 1+O(x)$. Furthermore $1/a_i = 1/(am)(1+O(1/a))$, $1/b_i = 1/(bm)(1+O(1/a))$ and $1/c_i = 1/(cm)(1+O(1/a))$. So we may rewrite (4.1) as follows:

$$\Pr(S \subseteq G^{mn}_{1,\beta}) = \frac{(1+\beta)^2}{(2+\beta)^2} \frac{1}{m^3} \left(\frac{1}{a^2b^{2+\beta}c^{2+2\beta}}\right)^{1/(2+\beta)} (1+O(1/a)).$$

In this case where $a_1 = a_2$, there are $m^4(m-1)$ ways to choose $a_1, a_2, b_1, b_2, c_1, c_2$ such that there is a corresponding triangle on v_a, v_b, v_c in $G^n_{m,\beta}$.

Now we suppose that $a_1 \neq a_2$. We have $d^{\text{in}}_S(v_{a_1}) = d^{\text{in}}_S(v_{a_2}) = d^{\text{in}}_S(v_{b_2}) = 1$ and otherwise $d^{\text{in}}_S(v) = 0$. Applying Lemma 3.4 and carrying out calculations similar to those above we obtain

$$\Pr(S \subseteq G^{mn}_{1,\beta}) = \frac{(1+\beta)^3}{(2+\beta)^3} \frac{1}{m^3} \left(\frac{1}{a^2b^{2+\beta}c^{2+2\beta}}\right)^{1/(2+\beta)} (1+O(1/a)).$$

In this case there are $m^4(m-1)^2$ ways to choose $a_1, a_2, b_1, b_2, c_1, c_2$.

Let $N_{a,b,c}$ denote the number of triangles on v_a, v_b, v_c in $G^n_{m,\beta}$. From the calculations above, we see that

$$\mathbf{E}\left[N_{a,b,c}\right] = \left(m(m-1)\frac{(1+\beta)^2}{(2+\beta)^2} + m(m-1)^2\frac{(1+\beta)^3}{(2+\beta)^3}\right)\left(\frac{1}{a^2b^{2+\beta}c^{2+2\beta}}\right)^{1/(2+\beta)} \cdot (1+O(1/a)). \tag{4.2}$$

Now let N be the number of triangles in $G^n_{m,\beta}$. Then to calculate $\mathbf{E}[N]$ we merely sum (4.2) over all a, b, c with $a < b < c$. If we estimate this sum by integrating, we obtain the following.

Proposition 4.3. *For $\beta > 0$, the expected number of triangles in $G^n_{m,\beta}$ is*

$$\left(m(m-1)\frac{(1+\beta)^2}{\beta^2} + m(m-1)^2\frac{(1+\beta)^3}{\beta^2(2+\beta)}\right)\log n + O(1).$$

This result is very different from that obtained in [3] where it is shown that when $\beta = 0$ the expected number of triangles is $\Theta((\log n)^3)$. Unfortunately we do not have any intuition explaining the difference in the results. The term in Lemma 3.4 which governs the asymptotic order of the number of triangles is

$$\prod_{(v_i,v_j)\in E(S):i>j} \frac{1}{(i^{1+\beta}j)^{1/(2+\beta)}}.$$

The equivalent term

$$\prod_{(v_i,v_j)\in E(S):i>j} \frac{1}{\sqrt{ij}}$$

appears in the corresponding lemma in [3]. So when $\beta = 0$ the important term in (4.2) would become $1/(abc)$ and integrating now shows that the asymptotic number of triangles is $\Theta((\log n)^3)$.

4.2. The expectation of $\sum_{v\in V(G)}\binom{d(v)}{2}$

We begin by noting that if we regard each edge in the graph as consisting of two half-edges, with each half-edge retaining one endpoint of an edge, then $\sum_{v\in V(G^n_{m,\beta})}\binom{d_n(v)}{2}$ is the number of pairs of half-edges with the same endpoint. We say such a pair of half-edges is *adjacent*. Suppose that e_1 and e_2 are half-edges with endpoint v. If e_1 and e_2 form respectively half of the edges vu and vw with u, v, w pairwise distinct then we say that e_1 and e_2 form a *non-degenerate* pair of adjacent half-edges. Otherwise we say that they are *degenerate*.

Calculating the expected number of pairs of adjacent half-edges is slightly more complicated than calculating the expected number of triangles because there is less symmetry. We begin by counting the number of non-degenerate pairs of adjacent half-edges. Let $a < b < c$. We first calculate the expected number of pairs (v_b, v_a), (v_c, v_a) of adjacent half-edges in $G^n_{m,\beta}$ for $\beta > 0$. Just as in the previous section, there are two cases to consider, and calculations, using Lemma 3.4, similar to those above show that the number of such pairs of adjacent half-edges is

$$\left(m\frac{1+\beta}{2+\beta} + m(m-1)\frac{(1+\beta)^2}{(2+\beta)^2}\right)\left(\frac{1}{a^2b^{1+\beta}c^{1+\beta}}\right)^{1/(2+\beta)}(1+O(1/a)).$$

By integrating, we see that the total number of pairs of adjacent half-edges in $G^n_{m,\beta}$ for which the common vertex has the smallest index is

$$\left(m\frac{2+\beta}{\beta} + m(m-1)\frac{1+\beta}{\beta}\right)n + O(n^{2/(2+\beta)}).$$

Now the expected number of pairs (v_b, v_a), (v_c, v_b) of adjacent half-edges is

$$m^2\frac{(1+\beta)^2}{(2+\beta)^2}\left(\frac{1}{ab^{2+\beta}c^{1+\beta}}\right)^{1/(2+\beta)}(1+O(1/a)).$$

Again we integrate to derive that the total number of pairs of adjacent half-edges in $G^n_{m,\beta}$ for which the common vertex has the middle index is $m^2n + O(n^{2/(2+\beta)})$. This is not surprising because it can be shown that very few vertices either have loops or do not have m distinct out-neighbours. Each loopless vertex with m distinct loopless out-neighbours, that each have m distinct out-neighbours, is the vertex with greatest index in m^2 pairs of adjacent half-edges of this form.

Finally the expected number of pairs (v_c, v_a), (v_c, v_b) of adjacent half-edges is

$$m(m-1)\frac{(1+\beta)^2}{(2+\beta)^2}\left(\frac{1}{abc^{2+2\beta}}\right)^{1/(2+\beta)}(1+O(1/a)).$$

So the total number of pairs of adjacent half-edges in $G^n_{m,\beta}$ for which the common vertex has the largest index is $m(m-1)n/2 + O(n^{1/(2+\beta)})$. Again the result is not surprising because each loopless vertex with m distinct out-neighbours is the vertex of greatest index in $\binom{m}{2}$ pairs of adjacent half-edges of this form.

Note that the error term is slightly different in the last case. This is just a consequence of the approximations occurring when we integrate first over a. In each of the three cases, merely integrating the error term over a yields an $o(1)$ term. However this must be replaced by an $O(1)$ term to reflect the error inherent in approximating the sum by an integral. In the last of the three cases, this consequent increase in the order of the error term is smaller than in the first of the three cases and so there is a different error. (The error term for the second of the three cases is not as sharp as it might be because we have dealt crudely with a log factor introduced in the second integration.)

By carrying out calculations similar to those above, it can be shown that the number of degenerate pairs of adjacent half-edges is $O(n^{1/(2+\beta)})$.

Summing over all the possibilities we obtain the following result.

Proposition 4.4. *For $\beta > 0$, the expectation of $\sum_{v\in V(G)} \binom{d(v)}{2}$ in $G^n_{m,\beta}$ is*

$$\left(\frac{2+5\beta}{2\beta}m^2 + \frac{2-\beta}{2\beta}m\right) n + O(n^{2/(2+\beta)}).$$

Again the result is different from that obtained in [3] where it was shown that for the case $\beta = 0$ the expected number of pairs of adjacent edges is $\Theta(n \log n)$. Unfortunately we also do not have a good intuitive explanation of why there is a difference in the results but the following calculation, suggested by the referee, illustrates how the difference arises. In the case where $\beta = 0$, Bollobás et al. [5] show that for small d, the number of vertices of degree d is $\Theta(nd^{-3})$ and consequently the number of pairs of adjacent half-edges is roughly

$$\sum_d \Theta(nd^{-3})\Theta(d^2) = \Theta(n \log n).$$

However, when $\beta > 0$, the number of vertices of degree d is now $\Theta(nd^{-(3+\beta)})$ and so a similar calculation shows that there are now $\Theta(n)$ pairs of adjacent half-edges.

5. The concentration of $\sum_{v\in V(G)} \binom{d(v)}{2}$

In this section we show that the number of pairs of adjacent half-edges in $G^n_{m,\beta}$ is concentrated about its mean. This justifies obtaining the clustering coefficient by taking three times the quotient of the expected number of triangles and the expected number of pairs of adjacent half-edges. The main strategy is to apply a variant of the Azuma–Hoeffding inequality from [13], by making use of Móri's results [15] on the evolution of the maximum degree of $G^n_{m,\beta}$. (It is mentioned in [3] that martingale methods were used.) A key notion in the proof is to consider the mechanism by which edges incident with a fixed vertex are added.

Before we continue, we explain briefly why we follow this approach rather than the more elementary second-moment method. It is possible to apply the second-moment method to obtain some form of concentration. Certainly Lemma 3.1 may be applied to show that the leading order terms cancel in the usual way. However the concentration result that may be obtained is not tight enough for obtaining our final result without a considerable sharpening of the analysis in Section 6. It is far from clear whether this is possible. Furthermore the number of cases that need to be considered makes calculating the variance a gruesome proposition and therefore unlikely to be much shorter to describe than our approach.

Fix β and m. Let (H_t) be the graph process defined as follows. Run $(G^t_{1,\beta})$ and take H_n to be the graph formed from $G^n_{1,\beta}$ by merging groups of m consecutive vertices together until there are at most m left and finally merging the remaining unmerged vertices together. Note that H_n has $\lceil n/m\rceil$ vertices, which we denote by $v_1, \ldots, v_{\lceil n/m\rceil}$ in the obvious way, and $n-1$ edges. Furthermore, if $m|n$ and the graphs H_n and $G^{n/m}_{m,\beta}$ are formed from the same instance of the process $(G^t_{1,\beta})$, then H_n and $G^{n/m}_{m,\beta}$ are the same graph.

Let v_k be a vertex of H_s such that $km \le s$. For $t \ge s$, we define a partition $\Pi_{k,s}(t)$ of the half-edges incident with v_k. The partition always has $d_s(v_k) + 1$ blocks. When $t = s$, each block of the partition except for one contains one of the $d_s(v_k)$ half-edges incident with v_k; with a slight abuse of nomenclature the other block, which we call the *base* block, is initially empty. It follows that if v_k has a loop at time s then the two half-edges forming the loop are in separate blocks of $\Pi_{k,s}(s)$. As t increases and more edges are added to H, any newly added half-edge incident with v_k is added to the partition. If at time $t > s$ the target vertex of the newly added edge is not v_k then $\Pi_{k,s}(t) = \Pi_{k,s}(t-1)$. Suppose that at time $t > s$ the target vertex of the newly added edge f is v_k: if v_k is chosen preferentially by copying the half-edge $e \in A$, where A is a block of $\Pi_{k,s}(t-1)$, then we form $\Pi_{k,s}(t)$ from $\Pi_{k,s}(t-1)$ by adding the half-edge of f incident with v_k to A; if v_k is chosen uniformly then the half-edge of f incident with v_k is added to the base block.

Suppose that v_l is a vertex of H_s distinct from v_k such that $lm \le s$. Suppose further that we choose two distinct blocks from $\Pi_{k,s}(t)$ and $\Pi_{l,s}(t)$, such that neither is a base block. The joint distribution of the sizes of the two blocks is the same for any choice of blocks, whether they are both chosen from $\Pi_{k,s}(t)$, $\Pi_{l,s}(t)$ or one from each. Furthermore if we choose either base block from $\Pi_{k,s}(t)$ or $\Pi_{l,s}(t)$ and one other block that is not a base block, then again the joint distribution of the sizes of the blocks does not depend on our choice.

Lemma 5.1. *Let v_j and v_k be distinct vertices of H_s such that $\max\{jm, km\} \le s$. Let A (B) be respectively a block of $\Pi_{j,s}(t)$ ($\Pi_{k,s}(t)$) such that neither is a base block. Then*

$$\mathbf{E}[|A||B|] \le \mathbf{E}[|A|]\,\mathbf{E}[|B|] \le (t/s)^{2/(2+\beta)}(1 + O(1/s)).$$

Proof. Let e_1, e_2 be half-edges such that at time s, e_1 is incident with v_k and e_2 is incident with v_l. Then let a_t denote the size, at time t, of the block of $\Pi_{k,s}(t)$ containing e_1 and let b_t be defined similarly with respect to $\Pi_{l,s}(t)$ and e_2. We first establish the second inequality. We have $\mathbf{E}[a_s] = 1$ and for $t \ge s$,

$$\mathbf{E}\left[a_{t+1}|a_t\right] = a_t\left(1 + \frac{1}{(2+\beta)t-2}\right). \tag{5.2}$$

Hence

$$\mathbf{E}\left[a_{t+1}\right] = \frac{t-1/(2+\beta)}{t-2/(2+\beta)}\mathbf{E}\left[a_t\right].$$

Solving this recurrence, we obtain

$$\mathbf{E}\left[a_t\right] = \frac{\Gamma\left(t-\frac{1}{2+\beta}\right)\Gamma\left(s-\frac{2}{2+\beta}\right)}{\Gamma\left(t-\frac{2}{2+\beta}\right)\Gamma\left(s-\frac{1}{2+\beta}\right)}.$$

A standard result on the ratio of gamma functions [12] states that if a, b are fixed members of $\mathbb{R}$ then for all $x > \max\{|a|, |b|\}$,

$$\frac{\Gamma(x+b)}{\Gamma(x+a)} = x^{b-a}(1+O(1/x)).$$

Using this result, we obtain

$$\mathbf{E}\left[a_t\right] \le (t/s)^{1/(2+\beta)}(1+O(1/s)).$$

Since $|A|$ and $|B|$ are identically distributed, the second inequality in the lemma follows. We prove the first inequality by using induction on t. Observe that (a_{t+1}, b_{t+1}) can take the values (a_t+1, b_t), (a_t, b_t+1) and (a_t, b_t) with probabilities respectively $a_t/((2+\beta)t-2)$, $b_t/((2+\beta)t-2)$ and $1-(a_t+b_t)/((2+\beta)t-2)$. Therefore

$$\mathbf{E}\left[a_{t+1}b_{t+1}|a_tb_t\right] = a_tb_t + \frac{2a_tb_t}{(2+\beta)t-2}$$

and from (5.2) we get

$$\mathbf{E}\left[a_{t+1}\right]\mathbf{E}\left[b_{t+1}\right] = \mathbf{E}\left[a_t\right]\mathbf{E}\left[b_t\right]\left(1+\frac{1}{(2+\beta)t-2}\right)^2.$$

So

$$\mathbf{E}\left[a_{t+1}b_{t+1}\right] - \mathbf{E}\left[a_{t+1}\right]\mathbf{E}\left[b_{t+1}\right] \le \left(1+\frac{2}{(2+\beta)t-2}\right)\left(\mathbf{E}\left[a_tb_t\right] - \mathbf{E}\left[a_t\right]\mathbf{E}\left[b_t\right]\right)$$

and hence the result follows by induction. □

When the maximum degree of H_t becomes unusually large and the target vertex is chosen to be a vertex of maximum degree, the number of pairs of adjacent edges increases by an unusually large amount. The next result enables us to show that the probability of this happening is extremely small. Let $\Delta(G)$ denote the maximum degree of G. The following is a very slight reformulation of what Móri proves in [15, Theorem 3.1].

Theorem 5.3. *For any positive integer k, there exists $\tilde{M}_k \in \mathbb{R}$ such that for all n,*

$$\mathbf{E}\left[\left(\frac{\Delta(G^n_{1,\beta})+\beta}{n^{1/(2+\beta)}}\right)^k\right] \le \tilde{M}_k.$$

The following corollary is straightforward.

Corollary 5.4. *For any positive integers k, m, there exists $M_{k,m} \in \mathbb{R}$ such that for all positive integers $i_1, \dots, i_k$,*

$$\mathbf{E}\left[\frac{\Delta(H_{mi_1})}{(mi_1)^{1/(2+\beta)}}\cdots\frac{\Delta(H_{mi_k})}{(mi_k)^{1/(2+\beta)}}\right] \le M_{k,m}.$$

Proof. Since $\Delta(H_{mi_1}), \dots, \Delta(H_{mi_k})$ are all positive we have

$$\frac{\Delta(H_{mi_1})}{(mi_1)^{1/(2+\beta)}}\cdots\frac{\Delta(H_{mi_k})}{(mi_k)^{1/(2+\beta)}} \le \sum_{j=1}^{k}\left(\frac{\Delta(H_{mi_j})}{(mi_j)^{1/(2+\beta)}}\right)^k$$

and so

$$\mathbf{E}\left[\frac{\Delta(H_{mi_1})}{(mi_1)^{1/(2+\beta)}}\cdots\frac{\Delta(H_{mi_k})}{(mi_k)^{1/(2+\beta)}}\right] \le \sum_{j=1}^{k}\mathbf{E}\left[\left(\frac{\Delta(H_{mi_j})}{(mi_j)^{1/(2+\beta)}}\right)^k\right].$$

Recall that H_{mi} is formed by merging together blocks of m consecutive vertices in an instance of $G^{mi}_{1,\beta}$. So we have $\mathbf{E}\left[(\Delta(H_{mi}))^k\right] \leq \mathbf{E}\left[(m\Delta(G^{mi}_{1,\beta}))^k\right]$. Hence

$$\sum_{j=1}^{k} \mathbf{E}\left[\left(\frac{\Delta(H_{mi_j})}{(mi_j)^{1/(2+\beta)}}\right)^k\right] \leq m^k \sum_{j=1}^{k} \mathbf{E}\left[\left(\frac{\Delta(G^{mi_j}_{1,\beta})}{(mi_j)^{1/(2+\beta)}}\right)^k\right] \leq km^k\tilde{M}_k.$$

The result follows by taking $M_{k,m} = km^k\tilde{M}_k$. $\square$

Before we can state the large deviation result that we use, we need some more definitions. Recall that f_i is a random variable which determines the index of the target vertex of v_i and that the values taken by $f_2, f_3, \ldots, f_t$ together determine H_t. Furthermore the set of values that f_i can take is denoted by Ω_i and $f_2, \ldots, f_t$ are independent. Let $\Omega = \prod_{i=2}^{t} \Omega_i$.

Let $\mathbf{X} = (f_2, \ldots, f_t)$. We let $H_t(\mathbf{X})$ be the instance of H_t determined by the random variables $f_2, \ldots, f_t$. We will also use this notation both for other random variables associated with H_t and when some or all of the variables f_i are set to a particular value. The meaning should be clear from the context but we will generally use ω_i for a member of Ω_i and f_i for a random variable taking values in Ω_i.

Let $D(\mathbf{X}) = \sum_{v \in V(H_t(\mathbf{X}))} \binom{d_t(v)}{2}$ and let $F(\mathbf{X}) = D(\mathbf{X})t^{-2/(2+\beta)}$. Now let $g : \prod_{i=2}^{s} \Omega_i \to \mathbb{R}$ such that

$$(\omega_2, \ldots, \omega_s) \mapsto \mathbf{E}\left[F(\omega_2, \ldots, \omega_s, f_{s+1}, \ldots, f_t)\right]$$

and let ran : $\prod_{i=2}^{s-1} \Omega_i \to \mathbb{R}$ such that

$$(\omega_2, \ldots, \omega_{s-1}) \mapsto \sup\{|g(\omega_2, \ldots, \omega_{s-1}, x) - g(\omega_2, \ldots, \omega_{s-1}, y)| : x, y \in \Omega_s\}.$$

So $\mathrm{ran}(\omega_2, \ldots, \omega_{s-1})$ measures the maximum amount that the expected value of $F(\mathbf{X})$ changes by when the value of f_s is changed.

For $\boldsymbol{\omega} \in \Omega$, let

$$R^2(\boldsymbol{\omega}) = \sum_{k=2}^{t} \mathrm{ran}(\omega_2, \ldots, \omega_{k-1})^2.$$

Our aim is to bound $R^2(\boldsymbol{\omega})$ as $\boldsymbol{\omega}$ runs over all members of Ω with the possible exception of those belonging to some 'bad' subset $\mathcal{B}$ which we hope will have small probability. We specify $\mathcal{B}$ below, but for the moment let $\mathcal{B}$ be any subset of Ω. Let

$$r^2 = \sup\{R^2(\boldsymbol{\omega}) : \boldsymbol{\omega} \in \Omega \setminus \mathcal{B}\}.$$

Then Theorem 3.7 in [13] yields the following inequality. For all $x > 0$,

$$\Pr(|F(\mathbf{X}) - \mathbf{E}\left[F(\mathbf{X})\right]| \geq x) \leq 2(\mathrm{e}^{-2x^2/r^2} + \Pr(\mathbf{X} \in \mathcal{B})).$$

Fix $\delta > 0$. We let

$$\mathcal{B}_\delta = \left\{\mathbf{X} \in \Omega : \sum_{i=1}^{n} \left(\frac{\Delta(H_{mi}(\mathbf{X}))}{(mi)^{2/(2+\beta)}}\right)^2 \geq n^{\frac{\beta}{2+\beta}+\delta}\right\}.$$

Then we have the following.

Lemma 5.5. *For any $\delta > 0$ and $\gamma > 0$, there exists L such that $\Pr(\mathcal{B}_\delta) \leq L\frac{1}{n^\gamma}$, where L is a constant depending on δ, γ, β, m but not on n.*

Proof. For any positive integer k, Markov's inequality gives

$$\Pr(\mathcal{B}_\delta) \leq \frac{\mathbf{E}\left[\left(\sum_{i=1}^{n}\left(\frac{\Delta(H_{mi}(\mathbf{X}))}{(mi)^{2/(2+\beta)}}\right)^2\right)^k\right]}{n^{\frac{\beta k}{2+\beta}+k\delta}}.$$

The numerator of this fraction is equal to

$$\mathbf{E}\left[\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}\left(\frac{\Delta(H_{mi_1}(x))}{(mi_1)^{1/(2+\beta)}}\right)^2\cdots\left(\frac{\Delta(H_{mi_k}(x))}{(mi_k)^{1/(2+\beta)}}\right)^2\frac{1}{(m^k i_1\cdots i_k)^{2/(2+\beta)}}\right].$$

Using Corollary 5.4 this is at most

$$M_{2k,m}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}\left(\frac{1}{(m^k i_1\cdots i_k)^{2/(2+\beta)}}\right)=M_{2k,m}\left(\sum_{i=1}^{n}\frac{1}{(mi)^{\frac{2}{2+\beta}}}\right)^k$$
$$\leq M_{2k,m}\left(\frac{2+\beta}{\beta}\frac{n^{\frac{\beta}{2+\beta}}}{m^{\frac{2}{2+\beta}}}\right)^k.$$

Hence

$$\Pr(\mathcal{B}_\delta)\leq\frac{M_{2k,m}\left(\frac{2+\beta}{\beta}\frac{1}{m^{\frac{2}{2+\beta}}}\right)^k}{n^{k\delta}}$$

and so letting $k=\lceil\gamma/\delta\rceil$ gives the result. □

We can now state the main result of this section concerning the concentration of the number of pairs of adjacent half-edges around its expectation.

Theorem 5.6. *Let $\beta > 0$. For any $\epsilon > 0$, the number D of pairs of adjacent half-edges in $G^n_{m,\beta}$ is concentrated within $O(n^{(4+\beta)/(4+2\beta)+\epsilon})$ about its expected value. More precisely, for any $\epsilon > 0$ and $\gamma > 0$ there exists n^* such that for all $n \geq n^*$*

$$\Pr\left(|D-\mathbf{E}[D]|\geq n^{\frac{4+\beta}{4+2\beta}+\epsilon}\right)\leq\frac{1}{n^\gamma}.$$

Proof. Let $t = nm$, and fix $s \leq t$. Let $s' = m\lceil s/m\rceil$, so we have $s' \leq t$. Now let

$$\boldsymbol{\omega}_x=(\omega_2,\ldots,\omega_{s-1},x,\omega_{s+1},\ldots,\omega_t)\quad\text{and}\quad\boldsymbol{\omega}_y=(\omega_2,\ldots,\omega_{s-1},y,\omega_{s+1},\ldots,\omega_t),$$

where $\omega_i\in\Omega_i$ and $x, y\in\Omega_s$. For $z\in\{x,y\}$, let $d^z_t(v)$ denote the total degree of v at time t in $H_t(\boldsymbol{\omega}_z)$ and let e denote the edge added at time s. Suppose that in $H_t(\boldsymbol{\omega}_x)$ the target vertex of e is v_{k_1} and in $H_t(\boldsymbol{\omega}_y)$ the target vertex of e is v_{k_2}. Note that at any time, for every vertex v other than v_{k_1} or v_{k_2}, the degree of v is the same in $H_t(\boldsymbol{\omega}_x)$ and $H_t(\boldsymbol{\omega}_y)$. Therefore $F(\boldsymbol{\omega}_x)-F(\boldsymbol{\omega}_y)$ depends only on the degrees of v_{k_1} and v_{k_2} and is given by

$$F(\boldsymbol{\omega}_x)-F(\boldsymbol{\omega}_y)=t^{-2/(2+\beta)}\left(\binom{d^x_t(v_{k_1})}{2}+\binom{d^x_t(v_{k_2})}{2}-\binom{d^y_t(v_{k_1})}{2}-\binom{d^y_t(v_{k_2})}{2}\right).\tag{5.7}$$

From now on we will assume that $k_1\neq k_2$, because otherwise $F(\boldsymbol{\omega}_x)-F(\boldsymbol{\omega}_y)=0$. Consider the changes that occur to $H_{s'}$ if we replace $\boldsymbol{\omega}_y$ by $\boldsymbol{\omega}_x$. First the head of e is moved from v_{k_2} to v_{k_1}. Second it is possible that each of the at most $m-1$ edges that are added in the time interval $[s+1,s']$ also have an endpoint moved from v_{k_2} to v_{k_1}: this will happen if the target vertex of an edge added in the interval $[s+1,s']$ is chosen by preferentially copying the head of an edge which has been moved from v_{k_2} to v_{k_1}, in particular if the target vertex is chosen by preferentially copying the head of e. Consequently we have

$$d^y_{s'}(v_{k_1})+1\leq d^x_{s'}(v_{k_1})\leq d^y_{s'}(v_{k_1})+m$$

and furthermore

$$d^x_{s'}(v_{k_1})+d^x_{s'}(v_{k_2})=d^y_{s'}(v_{k_1})+d^y_{s'}(v_{k_2}).$$

Let $d=d^x_{s'}(v_{k_1})-d^y_{s'}(v_{k_1})$, $d_1=d^y_{s'}(v_{k_1})$ and $d_2=d^x_{s'}(v_{k_2})$. Note that both d_1 and d_2 and consequently also $|d_1-d_2|$ are at most $\Delta(H_{s-1}(\omega_1,\ldots,\omega_{s-1}))+m$.

Now let $A_0, A_1,\ldots,A_{d_1}$, $(B_0,B_1,\ldots,B_{d_2})$ denote the blocks of the partition $\Pi_{k_1,s'}(t)$ in $H_t(\boldsymbol{\omega}_y)$ ($\Pi_{k_2,s'}(t)$ in $H_t(\boldsymbol{\omega}_x)$) with A_0 (B_0) denoting the base block. The partition $\Pi_{k_1,s'}(t)$ in $H_t(\boldsymbol{\omega}_x)$ contains the blocks $A_0,\ldots,A_{d_1}$ but also d further blocks which we label $C_1,\ldots,C_d$. Then the partition $\Pi_{k_2,s'}(t)$ in $H_t(\boldsymbol{\omega}_y)$ contains the blocks $B_0,\ldots,B_{d_2},C_1,\ldots,C_d$. So using (5.7), we have

$$F(\boldsymbol{\omega}_x)-F(\boldsymbol{\omega}_y)=t^{-2/(2+\beta)}\left(\sum_{i=0}^{d_1}\sum_{j=1}^{d}|A_i||C_j|-\sum_{i=0}^{d_2}\sum_{j=1}^{d}|B_i||C_j|\right).\tag{5.8}$$

Now let

$$\boldsymbol{\omega}_x=(\omega_2,\ldots,\omega_{s-1},x,\omega_{s+1},\ldots,\omega_{s'},f_{s'+1},\ldots,f_t)$$

and

$$\boldsymbol{\omega}_y = (\omega_2, \ldots, \omega_{s-1}, y, \omega_{s+1}, \ldots, \omega_{s'}, f_{s'+1}, \ldots, f_t).$$

So both $H_t(\boldsymbol{\omega}_x)$ and $H_t(\boldsymbol{\omega}_y)$ evolve deterministically until time s' but randomly thereafter.

Recall that $d \le m$ and that $|d_1 - d_2|$ is at most $\Delta(H_{s-1}(\omega_2, \ldots, \omega_{s-1})) + m$. Hence from (5.8), Lemma 5.1 and the remarks immediately preceding the lemma, we see that

$$\left|\mathbf{E}\left[F(\boldsymbol{\omega}_x) - F(\boldsymbol{\omega}_y)\right]\right| \le (\Delta(H_{s-1}(\omega_2, \ldots, \omega_{s-1})) + m)m(1/s')^{2/(2+\beta)}(1 + O(1/s')).$$

Notice that this expression does not depend on x or y and holds for all $\omega_{s+1}, \ldots, \omega_{s'}$. Consequently

$$\operatorname{ran}(\omega_2, \ldots, \omega_{s-1}) \le (\Delta(H_{s-1}(\omega_2, \ldots, \omega_{s-1})) + m)m(1/s')^{2/(2+\beta)}(1 + O(1/s')).$$

Now let $\boldsymbol{\omega} \in \Omega \setminus \mathcal{B}_\delta$. Then

$$\begin{aligned} R^2(\boldsymbol{\omega}) &= \sum_{s=2}^{nm} (\Delta(H_{s-1}(\omega_2, \ldots, \omega_{s-1})) + m)^2 m^2 (1/s')^{4/(2+\beta)}(1 + O(1/s')) \\ &\le m^2 \sum_{s=2}^{nm} \left(\frac{2\Delta(H_{s'}(\omega_2, \ldots, \omega_{s'}))}{s'^{2/2+\beta}}\right)^2 (1 + O(1/s')) \\ &\le 4m^3 \sum_{i=1}^{n} \left(\frac{\Delta(H_{mi}(\omega_2, \ldots, \omega_{mi}))}{(mi)^{2/2+\beta}}\right)^2 (1 + O(1/i')) \\ &\le cn^{\frac{\beta}{2+\beta}+\delta}, \end{aligned}$$

where c is a constant.

Hence

$$\begin{aligned} \Pr\left(|D(\mathbf{X}) - \mathbf{E}\left[D(\mathbf{X})\right]| \ge n^{\frac{4+\beta}{4+2\beta}+\epsilon}\right) &= \Pr\left(|F(\mathbf{X}) - \mathbf{E}\left[F(\mathbf{X})\right]| \ge n^{\frac{\beta}{4+2\beta}+\epsilon}\right) \\ &\le 2\exp\left(\frac{-2n^{\frac{\beta}{2+\beta}+2\epsilon}}{cn^{\frac{\beta}{2+\beta}+\delta}}\right) + 2\Pr(\mathcal{B}_\delta). \end{aligned}$$

If we choose $\delta = \epsilon$ then the first term is at most $\frac{1}{2n^\gamma}$ for any $\gamma > 0$ and sufficiently large n. Applying Lemma 5.5 with any $\gamma^* > \gamma$ we see that for sufficiently large n we also have $2\Pr(\mathcal{B}_\epsilon) \le \frac{1}{2n^\gamma}$. Hence the result follows. $\square$

6. The expected clustering coefficient

In this section we finally state and prove our main result.

Theorem 6.1. *For any $\beta > 0$, the expected clustering coefficient of $G^n_{m,\beta}$ is given by*

$$\mathbf{E}[C(G^n_{m,\beta})] = \frac{3c_1 \log n}{c_2 n} + O(1/n),$$

where

$$c_1 = m(m-1)\frac{(1+\beta)^2}{\beta^2} + m(m-1)^2\frac{(1+\beta)^3}{\beta^2(2+\beta)}$$

and

$$c_2 = \frac{2+5\beta}{2\beta}m^2 + \frac{2-\beta}{2\beta}m.$$

Proof. Recall that $N = N(G^n_{m,\beta})$ and $D = D(G^n_{m,\beta})$ denote respectively the number of triangles and pair of adjacent edges in $G^n_{m,\beta}$. The expected clustering coefficient is given by $\mathbf{E}\left[C(G^n_{m,\beta})\right] = \mathbf{E}\left[3N/D\right]$.

Choose ϵ such that $0 < \epsilon < \frac{\beta}{4+2\beta}$ and let $\eta = \epsilon + \frac{4+\beta}{4+2\beta} < 1$. Let I denote the interval $[\mathbf{E}\left[D\right] - n^\eta, \mathbf{E}\left[D\right] + n^\eta]$. From Proposition 4.4 we have $\mathbf{E}\left[D\right] - n^\eta = c_2 n - (1 + o(1))n^\eta$ and $\mathbf{E}\left[D\right] + n^\eta = c_2 n + (1 + o(1))n^\eta$. Assume that $n \ge n^*$, the minimum value of n such that Theorem 5.6 may be applied with $\gamma = 4$. Since $C(G^n_{m,\beta}) \le m$, an upper bound for $\mathbf{E}\left[C(G^n_{m,\beta})\right]$ may be obtained as follows:

$$\begin{aligned}\mathbf{E}\left[C(G^n_{m,\beta})\right] &\le \sum_{j=1}^{\infty}\sum_{i\in I}\frac{3j}{i}\Pr(N=j, D=i) + m\Pr(D\notin I)\\ &\le \sum_{j=1}^{\infty}\frac{3j}{c_2 n-(1+o(1))n^{\eta}}\Pr(N=j) + m\Pr(D\notin I).\end{aligned}$$

Applying Theorem 5.6 with $\gamma = 1$ and then Proposition 4.3, we obtain

$$\begin{aligned}\mathbf{E}\left[C(G^n_{m,\beta})\right] &\le \sum_{j=1}^{\infty}\frac{3j}{c_2 n-(1+o(1))n^{\eta}}\Pr(N=j) + \frac{m}{n}\\ &= \frac{3c_1\log n}{c_2 n}(1+(1/c_2+o(1))n^{\eta-1}) + \frac{m}{n}\\ &= \frac{3c_1\log n}{c_2 n} + O(1/n).\end{aligned}$$

A lower bound for $\mathbf{E}(C(G^n_{m,\beta}))$ may be obtained as follows:

$$\begin{aligned}\mathbf{E}\left[C(G^n_{m,\beta})\right] &\ge \sum_{j=1}^{\infty}\sum_{i\in I}\frac{3j}{i}\Pr(N=j, D=i)\\ &\ge \sum_{j=1}^{\infty}\sum_{i\in I}\frac{3j}{c_2 n+(1+o(1))n^{\eta}}\Pr(N=j, D=i)\\ &= \frac{3\mathbf{E}[N]}{c_2 n+(1+o(1))n^{\eta}} - \sum_{j=1}^{\infty}\sum_{i\notin I}\frac{3j}{c_2 n+(1+o(1))n^{\eta}}\Pr(N=j, D=i).\end{aligned}$$

Now since there are at most n^3m^3 triangles in $G^n_{m,\beta}$,

$$\sum_{j=1}^{\infty}\sum_{i\notin I}\frac{3j}{c_2 n+(1+o(1))n^{\eta}}\Pr(N=j, D=i) \le \frac{3n^3m^3}{c_2 n+(1+o(1))n^{\eta}}\Pr(D\notin I).$$

Applying Theorem 5.6 with $\gamma = 4$ shows that this is $O(1/n)$. Finally

$$\frac{3\mathbf{E}[N]}{c_2 n+(1+o(1))n^{\eta}} = \frac{3c_1\log n}{c_2 n}(1-(1/c_2+o(1))n^{\eta-1}) = \frac{3c_1\log n}{c_2 n} + O(1/n). \quad \square$$

7. Conclusion

Our main result shows that for $\beta > 0$ the expectation of the clustering coefficient of the Móri graph is asymptotically proportional to $\log n/n$ and consequently that the Móri graphs do not have the small-world property. Bollobás and Riordan showed for an almost identical model that when $\beta = 0$, the expectation of the clustering coefficient is asymptotically proportional to $(\log n)^2/n$. An unexpected consequence, for which we do not yet have a good explanation, is that the clustering coefficient has a discontinuity at $\beta = 0$.

Relatively recent work by Leskovec et al. [10,11] studies real-world networks such as a physics citation graph, a patent citation graph and the graph formed by Internet routers. They show that the average degree of the graph grows with its order. Cooper and Prałat [8] have started an investigation of a modification of the BA model, in which a new vertex added at time t generates $[t^c]$ edges. It would be interesting to understand the behaviour of the clustering coefficient for these graphs and how it compares with the clustering coefficient for a classical random graph.

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References

[1] A.-L. Barabási, R. Albert, Emergence of scaling in random networks, Science 286 (1999) 509–512.
[2] A.-L. Barabási, R. Albert, Statistical mechanics of complex networks, Reviews of Modern Physics 74 (2002) 47–97.
[3] B. Bollobás, O.M. Riordan, Mathematical results on scale-free random graphs, in: S. Bornholdt, H.G. Schuster (Eds.), Handbook of Graphs and Networks: From the Genome to the Internet, Wiley-VCH, Berlin, 2003, pp. 1–34.
[4] B. Bollobás, O.M. Riordan, The diameter of a scale-free random graph, Combinatorica 24 (2004) 5–34.
[5] B. Bollobás, O.M. Riordan, J. Spencer, G. Tusnády, The degree sequence of a scale-free random graph process, Random Structures and Algorithms 18 (2001) 279–290.
[6] P.G. Buckley, D. Osthus, Popularity based random graph models leading to a scale-free degree sequence, Discrete Mathematics 282 (2004) 53–63.
[7] C. Cooper, A. Frieze, A general model of web graphs, Random Structures and Algorithms 22 (2003) 311–335.
[8] C. Cooper, P. Prałat, Scale-free graphs of increasing degree, Random Structures and Algorithms, in press (doi:10.1002/rsa.20318).
[9] R. Durrett, Random Graph Dynamics, Cambridge University Press, 2006.
[10] J. Leskovec, J. Kleinberg, C. Faloutsos, Graphs over time: densification laws, shrinking diameters and possible explanations, in: KDD'05: Proceedings of the eleventh ACM SIGKDD international conference on knowledge discovery in data mining, Chicago, 2005, pp. 177–187.
[11] J. Leskovec, J. Kleinberg, C. Faloutsos, Graph evolution: densification and shrinking diameters, in: ACMTKDD07, ACM Transactions on Knowledge Discovery from Data, vol. 1, 2007.
[12] O.I. Marichev, Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables, Ellis Horwood Limited, 1983.
[13] C. McDiarmid, Concentration, in: Probabilistic Methods for Algorithmic Discrete Mathematics, Springer, 1998, pp. 195–248.
[14] T.F. Móri, On random trees, Studia Scientiarum Mathematicarum Hungarica 39 (2002) 143–155.
[15] T.F. Móri, The maximum degree of the Barabási–Albert random tree, Combinatorics, Probability and Computing 14 (2005) 339–348.
[16] D.J. Watts, S.H. Strogatz, Collective dynamics of 'small-world' networks, Nature 393 (1998) 440–442.