PSYCHOMETRIKA—VOL. 31, NO. 4
DECEMBER, 1966

THE CENTRALITY INDEX OF A GRAPH

GERT SABIDUSSI

MCMASTER UNIVERSITY

1. *Introduction*

It appears that widespread use is made in the sociological and psychological literature of an index which purports to measure the degree of centralization of a graph. This so-called *Bavelas centrality index* was introduced by Leavitt ([10], p. 47), and is based on the *relative centralities* (defined by Bavelas in [1] and [2]) which describe the extent to which a vertex of a graph is centrally located. Relative centralities were independently introduced by Harary [6], the only difference being that he considers directed graphs instead of undirected ones. Most of Harary's definitions can already be found in [1 and 2]. A discussion of the Bavelas index and relative centralities (there called distance sums) can also be found in ([9], ch. 6, pp. 185–191).

Recently some questions have been raised as to the real significance of Bavelas's index. That the correspondence between index and intuitive centrality is not very satisfactory has been noted by Flament ([5], pp. 51–52). He gives two examples (n-circuits and complete n-graphs) which obviously differ greatly in centrality (the complete n-graph being highly centralized, the n-circuit hardly at all) but have the same Bavelas index. A brief analysis is then given to show that all homogeneous graphs, i.e., graphs all of whose vertices are automorphic, have minimal Bavelas index irrespective of their actual structure, thus confirming that the index reflects intuition but poorly. Through this discussion Flament is then led to think that what the index really measures is "the degree of disparity between the points of a graph" or "a degree of automorphism." We shall show (section 2) that Flament's criticism has not gone nearly far enough. Although homogeneity of a graph implies minimality of the index, the converse is by no means true. In fact, we shall give an example (and infinitely many others can be given) of a graph with minimum index and not a single nontrivial automorphism. Thus minimum index may be attained even in the case of extreme non-homogeneity.

Beauchamp [3] has gone a step beyond Flament by actually defining a new index which, it is claimed, is an improvement over the Bavelas index. This claim seems to be based on the fact that the new index distinguishes, in a manner compatible with intuition, between graphs all of which have minimum Bavelas index. But there is, of course, no guarantee that a reversal of this situation does not occur for other types of graphs.

Any claim that one index is better than another is meaningless unless one has defined what one means by saying that one graph is more centralized than another. So far, all authors who have dealt with this matter have been content to give a few examples and to appeal directly to intuition. There is no doubt that an appeal to intuition must be made at some level, but it has to be made in a mathematically precise fashion. The main purpose of this paper is to give such a precise definition, and then to test to what extent the known indices satisfy the requirements of that definition. Regrettably, but not surprisingly, they do not survive the test.

The axiomatic development is carried out in section 4. Sections 2 and 3 contain the relevant facts about the relative centralities which are needed in section 4.

For the sake of simplicity we shall only consider *finite undirected* graphs (in contrast to [6] where all graphs are directed). Thus a *graph* X consists of a finite set $V(X)$ (the vertex set of X) together with a set $E(X)$ (the edges of X) of unordered pairs to vertices of X. Unordered pairs will be indicated by brackets. We shall usually write $x \in X$ for $x \in V(X)$. For $x \in X$ we denote by $V(x; X)$ the set of vertices adjacent to x, i.e.,

$$V(x; X) = \{y \in V(X) : [x, y] \in E(X)\}.$$

By $d(x; X)$ or d_x we denote the number of elements of $V(x; X)$. $d(x; X)$ is called the *degree* of x in X.

A *path* joining two vertices $x, y \in X$ is a sequence $(x_0, \cdots, x_n)$ of distinct vertices of X such that $x_0 = x$, $x_n = y$, and $[x_{i-1}, x_i] \in E(X)$, $i = 1, \cdots, n$. n is called the *length* of the path. X is *connected* if there exists a path joining any two vertices of X. In general, a maximal connected subgraph of X is called a *component* of X.

If e is an edge we shall denote by (e) the graph consisting of e and its two endpoints. If $e \in E(X)$, then $X - e$ will denote the graph with

$$V(X - e) = V(X), E(X - e) = E(X) - \{e\}.$$

Similarly, if $x \in X$, $X - x$ is the largest subgraph of X with $V(X - x) = V(X) - \{x\}$. If Y is a subgraph of X, then $X \backslash Y$ will denote the smallest subgraph of X with $E(X \backslash Y) = E(X) - E(Y)$.

An *automorphism* of X is a one-one function η of $V(X)$ onto $V(X)$ such that $[x, y] \in E(X)$ implies $[\eta x, \eta y] \in E(X)$. The set of all automorphisms of X forms a group with composition of functions as multiplication. This group, the *automorphism group* of X, will be denoted by $G(X)$.

If A is any set, $|A|$ will denote the number of elements of A. In particular, if X is a graph, $|X|$ is an abbreviation of $|V(X)|$, and is called the *order* of X.

2. *Peer Groups*

Let X be a connected graph, $x, y \varepsilon X$. By $\rho(x, y)$ we denote the *distance* of x and y in X, i.e., the length of a shortest path in X joining x and y. For $x \varepsilon X$, and $i = 0, 1, \cdots$ put

$$A_i(x) = \{y \varepsilon X : \rho(x, y) = i\}, \tag{1}$$

and let $\alpha_i(x) = |A_i(x)|$. Since X is finite, all but finitely many of the sets $A_i(x)$ are empty. The sets $A_i(x)$, $i = 0, 1, \cdots$, partition the vertex-set of X, hence

$$\sum_{i=0}^{\infty} \alpha_i(x) = n.$$

Here and in the sequel it is convenient to use infinity as the upper limit of summation, although, of course, all these sums are finite.

By the *point-centrality* of a vertex x of the graph X we mean the number

$$s(x) = \sum_{z \varepsilon X} \rho(x, z).$$

This is at slight variance with ([2], p. 727), where the point-centrality (= relative centrality) of a vertex x is defined by

$$\sum_{y \varepsilon X} s(y)/s(x).$$

However, if one wants to compare the centralities of various vertices of the same graph, nothing is gained by using the more complicated expression.

Using the numbers $\alpha_i(x)$ it follows immediately that for any $x \varepsilon X$,

$$s(x) = \sum_{i=1}^{\infty} i\alpha_i(x) \tag{2}$$

For the directed case see ([6], Theorem 1).

In terms of the point-centralities the *Bavelas index* γ is defined by

$$\gamma = \sum_{x, y \varepsilon X} \frac{s(x)}{s(y)}.$$

We shall now consider graphs of diameter $\leqq 2$. The *diameter* of a graph X is

$$\operatorname{diam} X = \max_{x, y \varepsilon X} \rho(x, y).$$

If $\operatorname{diam} X \leqq 2$, and $|X| = n$, then for any $x \varepsilon X$,

$$\alpha_1(x) = d_x, \qquad \alpha_2(x) = n - d_x - 1,$$

and $\alpha_i(x) = 0$ for all $i \geqq 3$. Substitution in (2) yields

$$s(x) = 2(n - 1) - d_x, \qquad x \varepsilon X. \tag{3}$$

Now it is known ([5], p. 51) that γ is a minimum if and only if

$$s(x) = s(y) \quad \text{for any} \quad x, y \in X, \tag{4}$$

i.e., if and only if X is a *peer group* ([6], p. 38). By virtue of (3) this means that γ is minimal if and only if $d_x = d_y$ for any x, $y \in X$, i.e., if X is *regular*. Thus we have

PROPOSITION 1. *A connected graph X of diameter 2 has minimum Bavelas index if and only if X is regular.*

The class of all regular graphs of diameter $\leqq 2$ is extremely large, and structural differences between its members are great. In particular, if X is a graph of order n such that $d_x \geqq \frac{1}{2}(n - 1)$ for all $x \in X$, then X is connected and diam $X \leqq 2$. A proof of the connectedness of X can be found in ([4], p. 370, property 3), and almost the same argument establishes the statement about the diameter. Combining this observation with Proposition 1 we obtain

COROLLARY. *Let X be a graph of order n such that $d_x \geqq \frac{1}{2}(n - 1)$ for all $x \in X$, then X has minimum Bavelas index if and only if X is regular.*

Another convenient subclass of the graphs of diameter $\leqq 2$ is formed by the so-called *joins* of graphs introduced by Zykov [11].

DEFINITION. Let X_0, X_1 be two disjoint graphs. The *join* of X_0 and X_1 is the graph $X_0 \vee X_1$ given by

$$V(X_0 \vee X_1) = V(X_0) \cup V(X_1)$$

$$E(X_0 \vee X_1) = E(X_0) \cup E(X_1) \cup \{[x, y] : x \in X_0 , y \in X_1\}.$$

It is immediate from this definition that X_0 and X_1 are subgraphs of $X_0 \vee X_1$, and that in $X_0 \vee X_1$ every vertex of X_0 is adjacent to every vertex of X_1. Hence if X_0 and X_1 are non-empty, then diam $X_0 \vee X_1 \leqq 2$.

For $x \in X_i$,

$$d(x; X_0 \vee X_1) = d(x; X_i) + n_{1-i} ,$$

where $n_i = |X_i|$, $i = 0, 1$. Hence $X_0 \vee X_1$ is regular if and only if X_0, X_1 are regular, and

$$r_0 + n_1 = r_1 + n_0 , \tag{5}$$

where r_i is the degree of X_i, $i = 0, 1$.

An example of a pair of regular graphs X_0, X_1 satisfying (5) is given in Fig. 1. $X_0 \vee X_1$ is regular, hence by Proposition 1 it has minimum Bavelas index. It can be checked that both X_0 and X_1 have trivial automorphism group. We shall use this fact to prove that $X_0 \vee X_1$ likewise has no non-trivial automorphisms.

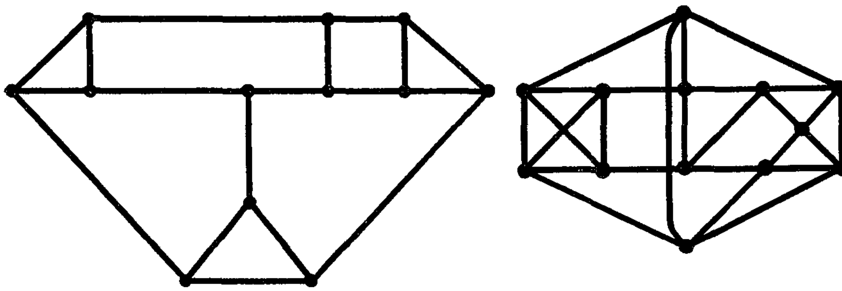

FIGURE 1
A Pair of Regular Graphs (X_0 is shown on the left. X_1 is shown on the right.)

We shall now consider the automorphism group of $X_0 \vee X_1$. Given $\varphi_i \in G(X_i)$, $i = 0, 1$, we define a mapping φ of $V(X_0 \vee X_1)$ onto itself by setting $\varphi x = \varphi_i x$ if $x \in V(X_i)$, $i = 0$ or 1. Then φ is an automorphism of $X_0 \vee X_1$ and will be denoted by (φ_0, φ_1). The set H of all mappings of the form (φ_0, φ_1), where $\varphi_i \in G(X_i)$, $i = 0, 1$, is a subgroup of $G(X_0 \vee X_1)$. It was shown by Harary ([7], p. 32) that $G(X_0 \vee X_1) = H$ if and only if no component of $-X_0$ is isomorphic to a component of $-X_1$. Here $-X$ denotes the graph given by

$$V(-X) = V(X),$$

$$E(-X) = \{[x, y] : x, y \in V(-X), x \neq y, [x, y] \notin E(X)\}.$$

Call a graph X *join-indecomposable* if there do not exist two non-empty disjoint graphs Y, Z such that X is isomorphic to $Y \vee Z$. Then Harary's result immediately yields the following trivial corollary: *If X_0, X_1 are non-isomorphic join-indecomposable graphs, then $G(X_0 \vee X_1) = H$.*

PROPOSITION 2. *If X_0, X_1 are non-empty and $X_0 \vee X_1$ is d-regular, then $|X_0 \vee X_1| \leqq 2d$.*

PROOF. By (5), $d = r_0 + n_1 = r_1 + n_0$. Hence $2d = r_0 + r_1 + n_0 + n_1$, so that $n_0 + n_1 \leqq 2d$. But $n_0 + n_1 = |X_0 \vee X_1|$.

By Proposition 2 both X_0 and X_1 of Fig. 1 are join-indecomposable. Hence by the corollary to Harary's result, $G(X_0 \vee X_1) = H = 1$. Thus we have: *There exist graphs with minimal Bavelas index and trivial automorphism group.* (Compare ([6] p. 39) for the directed case. The network given there (Fig. 13) does, however, have a non-trivial automorphism.)

In view of Proposition 1 it is tempting to conjecture that if a graph X has minimum index, then X is regular. Perhaps this, rather than dissimilarity with respect to automorphisms, is the meaning of Flament's phrase that the index measures "the degree of disparity between points." However, even this weak conjecture is not generally true: *For all diameters $\geqq 3$ there*

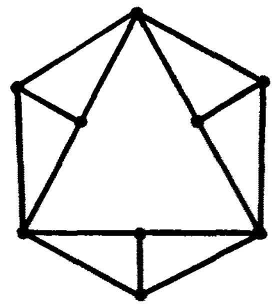

FIGURE 2
Non-regular Graph with Diameter 3 and Minimum Bavelas Index

exist non-regular graphs with minimal Bavelas index. For diameter 3 the simplest example is the graph of Fig. 2. Equally simple examples can be given for all odd diameters >3. For even diameters it appears that a more complicated construction is necessary; e.g., in the case of diameter 4 the smallest non-regular graph of minimum index we have been able to find has 56 vertices.

The following is a straightforward generalization of Proposition 1.

PROPOSITION 3. *Let X be a connected graph with* diam $X = m$, *and* mesh $X \geqq 2m - 1$, *i.e.,* mesh $X = 2m - 1$ *or* $2m$. *Then X has minimum Bavelas index if and only if X is regular.*

Here mesh X denotes the order of a shortest circuit of X.

PROOF. Since mesh $X \geqq 2m - 1$,

$$\alpha_1(x) = d_x , \qquad \alpha_i(x) = d_x(d_x - 1)^{i-1}, \qquad i = 2, \cdots , m - 1.$$

Hence

$$\begin{aligned} s(x) = \sum_{i=1}^{m-1} i d_x(d_x - 1)^{i-1} + m\left(n - \sum_{i=1}^{m-1} d_x(d_x - 1)^{i-1}\right) \\ = mn - \sum_{i=1}^{m-1} (m - i) d_x(d_x - 1)^{i-1}. \end{aligned}$$

For $t \geqq 2$ the polynomial $f(t) = \sum_{i=1}^{m-1} (m - i)t(t - 1)^{i-1}$ is strictly monotone increasing, hence $s(x) = s(y)$ implies $d_x = d_y$, and conversely.

PROPOSITION 4. *If X is a connected graph of order n, then*

$$|s(x) - s(y)| \leqq (n - 2)\rho(x, y) \tag{6}$$

for any x, $y \varepsilon X$. Moreover, if $[x, y] \varepsilon E(X)$, then $s(x) - s(y) = n - 2$ if and only if $d_x = 1$.

PROOF. By the triangle inequality for the distance,

$$|\rho(x, z) - \rho(y, z)| \leqq \rho(x, y)$$

for all $z \in X$. Hence

$$|s(x) - s(y)| \leqq \sum_{z \neq x, y} |\rho(x, z) - \rho(y, z)| \leqq (n - 2)\rho(x, y).$$

Now suppose that $[x, y] \in E(X)$, and that $d_x = 1$. Then

$$\rho(x, z) = \rho(y, z) + 1$$

for every $z \neq x$. Hence $s(x) - s(y) = n - 2$.

Conversely, suppose that $s(x) - s(y) = n - 2$. Then

$$\sum_{z \neq x, y} (\rho(x, z) - \rho(y, z)) = n - 2. \tag{7}$$

Since $e = [x, y] \in E(X)$, $|\rho(x, z) - \rho(y, z)| \leqq 1$ for every $z \neq x, y$. Hence none of the summands in (7) can be $\leqq 0$, i.e., $\rho(x, z) - \rho(y, z) = 1$ for all $z \neq x$. This implies that e cannot belong to a circuit of X. Otherwise there would exist a circuit C of smallest order k such that $e \in E(C)$. Let z be the vertex of C for which $\rho(x, z) = [(k - 1)/2]$. Then $\rho(y, z) = [k/2]$, a contradiction. Hence e is a *bridge*, i.e., $X - e$ consists of two components K_x, K_y, containing x and y, respectively. For any $z \in K_x$, $\rho(y, z) = \rho(x, z) + 1$, hence $K_x = \{x\}$, i.e., $d_x = 1$.

PROPOSITION 5. *Let X_0, X_1 be two connected subgraphs of X such that*

$$E(X_i) \neq \emptyset, \qquad i = 0, 1, \qquad X_0 \cup X_1 = X \quad \text{and} \quad X_0 \cap X_1 = w.$$

Then for any $x \in X_i$

$$s(x) = s_i(x) + s_{1-i}(w) + \rho(x, w)(n_{1-i} - 1), \qquad i = 0, 1, \tag{8}$$

where $n_i = |X_i|$, and $s_i(x)$ is the point-centrality of x relative to X_i, $i = 0, 1$.

PROOF. For $x \in X_i$, and $j = 0, 1, \cdots$ put

$$A_{ij}(x) = \{y \in X_i : \rho_i(x, y) = j\}, \qquad i = 0, 1,$$

and let $\alpha_{ij}(x) = |A_{ij}(x)|$. Here ρ_i denotes distance in X_i; note, however, that $\rho_i(x, y) = \rho(x, y)$ for any $x, y \in X_i$, where ρ denotes distance in X.

Since w is a cut vertex of X it follows that

$$\rho(x, y) = \rho_i(x, w) + \rho_{1-i}(w, y) = \rho(x, w) + \rho(w, y)$$

for any $x \in X_i$, $y \in X_{1-i}$. Hence for $x \in X_i$,

$$\alpha_j(x) = \begin{cases} \alpha_{ij}(x), & \text{if} \quad j \leqq \rho(x, w), \\ \alpha_{ij}(x) + \alpha_{1-i, j-\rho(x,w)}(w), & \text{if} \quad j \geqq \rho(x, w). \end{cases}$$

$$\begin{aligned} s(x) &= \sum_{j=1}^{\infty} j\alpha_j(x) = \sum_{j=1}^{\rho(x,y)} j\alpha_{ij}(x) + \sum_{j=\rho(x,y)+1}^{\infty} j(\alpha_{ij}(x) + \alpha_{1-i,j-\rho(x,w)}(w)) \\ &= s_i(x) + \sum_{j=\rho(x,y)+1}^{\infty} j\alpha_{1-i,j-\rho(x,w)}(w) \\ &= s_i(x) + \sum_{k=1}^{\infty} (k + \rho(x, y))\alpha_{1-i,k}(w) \\ &= s_i(x) + s_{1-i}(w) + \rho(x, y) \sum_{k=1}^{\infty} \alpha_{1-i,k}(w), \end{aligned}$$

so that

$$s(x) = s_i(x) + s_{1-i}(w) + \rho(x, w)(n_{1-i} - 1), \qquad x \in X_i .$$

Using standard terminology we shall say that a connected graph X is 2-*connected* if it contains no *cut-vertex,* i.e., no vertex x such that $X - x$ is disconnected. We can then state the

COROLLARY. *If X has minimum Bavelas index, then X is 2-connected.*

PROOF. If X has a cut vertex w, then $X = X_0 \cup X_1$ where X_0 , X_1 are as in Proposition 5. Hence by (8)

$$s(w) = s_0(w) + s_1(w). \tag{9}$$

Since X has minimum index $s(x) = s(w)$ for all $x \in X_i$, $i = 0, 1$. By (8) and (9) this means

$$s_i(w) - s_i(x) = \rho(x, w)(n_{1-i} - 1)$$

for all $x \in X_i$, $i = 0, 1$. In particular, if $[x, w] \in E(X_i)$, then

$$s_i(w) - s_i(x) = n_{1-i} - 1.$$

Hence by Proposition 4, $n_{1-i} - 1 \leqq n_i - 2$, i.e., $n_{1-i} \leqq n_i - 1$, $i = 0, 1$. Thus $n_1 \leqq n_0 - 1 \leqq n_1 - 2$, a contradiction.

It is an immediate consequence of this Corollary and (4) that if X has minimum Bavelas index and contains a vertex of degree 1, then $|X| = 2$, i.e., X consists of a single edge. In other words, if $|X| \geqq 3$ and X has minimum index, then $d_x \geqq 2$ for every $x \in X$. This also follows from the second part of Proposition 4. The case $d_x = 2$ for all $x \in X$ is possible, since circuits have minimum index.

3. *The Center of Separable Graphs*

In contradistinction to the Bavelas index the point centralities $s(x)$ make good geometrical sense. If one wishes, one may interpret the distance $\rho(x, y)$ of two vertices as the cost of travel from x to y, or, alternatively, as the time required to travel from x to y. $s(x)$ may then be interpreted as the total cost (or time) of access from x to all other vertices of the graph. If for a given vertex this total cost is minimal, the vertex may reasonably be said to lie in the center of the graph. We are thus led to define the *center* of a connected graph X as follows:

$$C = \{x \in X : s(x) \leqq s(y) \text{ for all } y \in X\}.$$

Clearly every connected graph has a non-empty center. In this section we derive a few properties of the center of *separable* graphs, i.e., graphs which possess a cut-vertex. For trees these results allow one to determine the center without calculating the point-centralities $s(x)$. All results in this section are based on (8).

Throughout, n will denote the order of the graph X.

Proposition 6. *If X is the union of two connected edge-disjoint graphs X_0, X_1 such that $X_0 \cap X_1 = w$, then the center of X is contained either in X_0 or in X_1. Moreover, if $C \subset V(X_0)$, then $C = \{w\}$ or $|X_0| \geqq n/2 + 1$.*

Proof. Let $x_i \in X_i$, $i = 0, 1$, and suppose that $s(x_0) = s(x_1)$ is minimal. Then by (8),

$$0 \geqq s(x_i) - s(w) = s_i(x_i) - s_i(w) + \rho(x_i, w)(n_{1-i} - 1).$$

Hence by Proposition 4,

$$\rho(x_i, w)(n_i - 2) \geqq s_i(w) - s_i(x_i) \geqq \rho(x_i, w)(n_{1-i} - 1).$$

If $\rho(x_i, w) > 0$ for $i = 0, 1$, then $n_i - 2 \geqq n_{1-i} - 1$, so that

$$2n_i \geqq n + 2, \qquad i = 0, 1. \tag{10}$$

Hence $n_0 + n_1 \geqq n + 2$, a contradiction. Thus $x_0 = w$ or $x_1 = w$, and hence $C \subset V(X_1)$ or $C \subset V(X_0)$.

If $C \subset V(X_0)$ and $|C| \geqq 2$, then (10) holds for $i = 0$, and hence $n_0 \geqq n/2 + 1$. This proves the second part of the proposition.

An immediate consequence of Proposition 6 is

Corollary 1. *The center of a graph X is contained in some block of X.*

A *block* is a maximal non-separable subgraph of X.

Corollary 2. *If X is a tree, the center of X consists of at most two (adjacent) vertices.*

Proof. In a tree every block consists of two adjacent vertices.

Note the similarity between Proposition 6 and its two corollaries and Lemma 1 of [8], although there the notion of center is quite different from the one used here (cf. the remark at the end of this section).

By an argument similar to that in the proof of Proposition 6 one can show the following:

PROPOSITION 7. *If x_0 is a cut-vertex of a connected graph X such that every component of $X - x_0$ has order $\leqq n/2$, then $x_0 \in C$.*

PROOF. Given a component K of $X - x_0$, let X_0 be the largest subgraph of X with $V(X_0) = V(K) \cup \{x_0\}$, and put $X_1 = X \backslash X_0$. Then $X_0 \cap X_1 = x_0$. Take $x \in X_0$. By (8), and Proposition 4,

$$\begin{aligned} s(x) - s(x_0) &= s_0(x) - s_0(x_0) + \rho(x, x_0)(n_1 - 1) \\ &\geqq \rho(x, x_0)(n_1 - n_0 + 1), \end{aligned}$$

where $n_i = |X_i|$, $i = 0, 1$. Note that $n_0 = |K| + 1$, and $n_0 + n_1 = n + 1$. By hypothesis, $n_0 \leqq n/2 + 1$, hence $n_1 \geqq n/2$, and $n_1 - n_0 + 1 \geqq 0$. Thus $s(x) \geqq s(x_0)$ for every $x \in X$, i.e., $x_0 \in C$.

For any connected graph X define D to be the set of all vertices x of X such that every component of $X - x$ has order $\leqq n/2$. The following properties of the set D are straightforward: (i) Either $n = 2$ or every $x \in D$ is a cut-vertex of X; (ii) D consists of at most two vertices (it may be empty); (iii) if D consists of exactly two vertices, x_0, x_1, then n is even, x_0 and x_1 are adjacent, and $e = [x_0, x_1]$ is a bridge of X (i.e., $X - e$ is disconnected); (iv) D is contained in the center of X (this is Proposition 7).

DEFINITION. A graph is called *tree-like* if $C = D$.

To justify this definition we have to show that every tree is tree-like. For the proof of this we make the following preliminary remarks. Let X_0, X_1 be two disjoint non-empty connected graphs. Choose $x_i \in X_i$, $i = 0, 1$, and form a graph X by joining x_0 and x_1 by a new edge $e = [x_0, x_1]$. Then X is a connected graph of order $n = n_0 + n_1$, where $n_i = |X_i|$, $i = 0, 1$, and x_0, x_1 are cut-vertices of X. Hence if we put $Y_i = X_i \cup (e)$, $i = 0, 1$, then by (8) and Proposition 4,

$$s(x_i) = s_{X_i}(x_i) + s_{Y_{1-i}}(x_i) = s_{X_i}(x_i) + s_{X_{1-i}}(x_{1-i}) + (n_{1-i} - 1), \quad i = 0, 1,$$

and hence

$$s(x_1) - s(x_0) = n_0 - n_1 = n - 2n_1 . \tag{11}$$

PROPOSITION 8. *Let X be a connected graph, $x_0 \in C$, and suppose that $[x_0, x_1]$ is a bridge of X. Then the component of $X - x_0$ which contains x_1 has at most $n/2$ vertices.*

PROOF. Suppose this component has order $n_1 > n/2$. Then by (11), $s(x_1) < s(x_0)$, contrary to the minimality of $s(x_0)$.

COROLLARY. *Every tree is tree-like.*

Proof. Every edge of a tree is a bridge.

For use in section 4 we include here

Proposition 9. *Let* $X = X_0 \cup X_1$, *where* X_0, X_1 *are connected graphs with* $E(X_i) \neq \emptyset$, $i = 0, 1$, *and* $X_0 \cap X_1 = w$. *If* w *belongs to the centers of both* X_0 *and* X_1, *then* w *belongs to the center of* X.

This is an immediate consequence of (8).

It is perhaps worth pointing out that the notion of center introduced at the beginning of this section is different from the one generally used in graph theory [e.g. 8], and introduced for directed graphs in ([9], pp. 161–164). The customary procedure is to attach to each vertex x of a connected graph X the number

$$m(x) = \max_{y \in X} \rho(x, y), \tag{12}$$

and to define the center C_m as the set of all vertices x for which $m(x)$ is minimal. That C and C_m are quite different sets can be seen from the following example. Let $(x_0, \cdots, x_{2n+1})$ be a path of length $2n + 1$ and form X by joining to x_0 a set of $2n + 1$ vertices of degree 1, distinct from $x_0, \cdots, x_{2n+1}$. Then by Proposition 7 and the Corollary to Proposition 8, $C = \{x_0\}$. On the other hand, it is trivial to verify that $C_m = \{x_n\}$. Thus the distance between the two centers can be made arbitrarily large.

4. *Axioms for Centrality*

In this section we propose to give a tentative definition of what should be understood by a centrality index of a graph. No attempt will be made to measure centrality in any absolute sense. We believe that the statement that one graph is more centralized than another is meaningful only relative to a given point-centrality function σ. Whether a centrality index is intuitively adequate will largely depend on the ability of the underlying function σ to describe the center of a graph, particularly in the case of graphs where the center is easily recognizable in the intuitive sense (e.g., in stars). Our discussion is guided to a considerable extent by the assumption that the point-centrality $s(x)$ (defined by equation (2)) is intuitively adequate (c.f. the first paragraph of section 3). We shall not try to compare the centralities of graphs of different orders.

Our axioms will be phrased in terms of two simple graph-theoretic operations which may be called (a) adding an edge, and (b) switching an edge.

(a) Let x_0, x_1 be two distinct vertices of a graph X such that $e = [x_0, x_1] \notin E(X)$. Form $Y = X \cup (e)$. We shall say that Y has been obtained from X by *adding* an edge incident with x_0 (or x_1). Clearly, adding an edge to a connected graph produces a connected graph.

(b) Let x, x_0, x_1 be three distinct vertices of a connected graph X such that

$$e_0 = [x, x_0] \in E(X) \quad \text{and} \quad e_1 = [x, x_1] \notin E(X).$$

Form a graph Y by deleting e_0 and adding e_1, i.e., $Y = (X - e_0) \cup (e_1)$. We shall say that the edge e_0 has been *switched* from x_0 to x_1. Such a switch will be called *admissible* if and only if

$$\rho(x_0, x_1) \leqq \rho(x_0, x),$$

i.e., if the *pivot*, x, does not lie between x_0 and x_1. Otherwise the switch will be called *inadmissible*. In the case of an admissible switch the resulting graph Y will always be connected. This need not be the case for inadmissible switches. Unless otherwise stated the phrase "switching an edge" will always mean that the switch in question is admissible.

Let $\mathfrak{G}$ be the class of all finite connected graphs, and for any positive integer n let $\mathfrak{G}_n$ be the class of all $X \in \mathfrak{G}$ with $|X| = n$. Similarly, if $\mathfrak{A}$ is any class of graphs, $\mathfrak{A}_n$ will denote the class of all $X \in \mathfrak{A}$ with $|X| = n$.

Suppose we are given a function σ on $\mathfrak{G}_n$ which assigns to every graph $X \in \mathfrak{G}_n$ a vector $(a_1, \cdots, a_n)$ of non-negative real numbers. If $x_1, \cdots x_n$ are the vertices of X, we shall denote a_i by $\sigma_X(x_i)$, $i = 1, \cdots, n$. We can then define the *center* of X with respect to σ by

$$C_\sigma(X) = \{x \in X : \sigma_X(x) \leqq \sigma_X(y) \text{ for all } y \in V(X)\}.$$

Convention: The center of X with respect to $s(x)$ will always be denoted by $C(X)$ or, when the context is clear, simply by C.

Note that at this point, i.e., without any conditions imposed on σ, the word "center" is merely a convenient name for a certain set of vertices.

DEFINITION 1. Given a class $\mathfrak{A}$ of graphs and a function σ on $\mathfrak{G}_n$ as described above, $\mathfrak{A}_n$ will be called *σ-admissible*, and σ a *point-centrality* on $\mathfrak{A}_n$ if and only if the following conditions are satisfied.

(A1) $\mathfrak{A}_n$ is closed under isomorphisms, i.e., if $X \in \mathfrak{A}_n$ and η is an isomorphism of X onto Y, then $Y \in \mathfrak{A}_n$.

(A2) If $X \in \mathfrak{A}_n$, if $x \in C_\sigma(X)$, and if Y is obtained by switching an edge to x, or by adding an edge incident with x, then $Y \in \mathfrak{A}_n$. We shall express this by saying that $\mathfrak{A}_n$ is closed under the operations of switching and adding an edge to vertices of the center.

(A3) If $X \in \mathfrak{A}_n$ and η is an isomorphism of X onto Y, then

$$\sigma_Y(\eta x) = \sigma_X(x)$$

for every $x \in V(X)$.

(A4) If $X \in \mathfrak{A}_n$, if $x \in V(X)$, and if $Y \in \mathfrak{A}_n$ is obtained by adding an edge incident with x, then

$$\sigma_Y(x) < \sigma_X(x), \quad \text{and} \quad \sigma_Y(y) \leqq \sigma_X(y)$$

for every $y \varepsilon V(Y)(= V(X))$.

(A5) If $X \varepsilon \mathfrak{A}_n$, if $x \varepsilon C_\sigma(X)$, and if Y is obtained either by switching an edge to x, or by adding an edge incident with x, then

$$\sigma_Y(x) < \sigma_X(x), \quad \text{and} \quad x \varepsilon C_\sigma(Y).$$

If σ is a point-centrality on $\mathfrak{A}_n$, if $X \varepsilon \mathfrak{A}_n$, and $x \varepsilon V(X)$, then the non-negative number $\sigma_X(x)$ will be called the point-centrality of x in X.

DEFINITION 2. Let σ be a function on $\mathfrak{G}_n$ as described above, and let $X, Y \varepsilon \mathfrak{G}_n$. Then (i) $XA_\sigma Y$ if and only if Y has been obtained from X by adding an edge incident with a vertex of $C_\sigma(X)$; (ii) $XS_\sigma Y$ if and only if Y has been obtained from X by switching an edge to a vertex of $C_\sigma(X)$; (iii) $X <_\sigma Y$ if and only if there is a sequence of graphs $X_0, \cdots, X_r$ such that $X_0 = X$, $X_r = Y$, and $X_{i-1}R_\sigma X_i$, $i = 1, \cdots, r$, where $R_\sigma = A_\sigma$ or S_σ. If $\mathfrak{A}$ is a σ-admissible class of graphs, if $X \varepsilon \mathfrak{A}_n$, and if $X <_\sigma Y$ we shall say that X *is less σ-centralized than* Y. For the function $s(x)$ we shall omit the subscript s, i.e., we shall write A for A_s, etc.

Conditions A1 and A2 provide a natural setting for conditions A3 and A5, respectively. One might have imposed a third condition on $\mathfrak{A}_n$, corresponding to A4, viz., that $\mathfrak{A}_n$ be closed under the addition of arbitrary edges, but we have not done this, partly because of the strongly ad hoc nature of condition A4, partly because classes satisfying the extra condition would be unnecessarily large. For example, if $\mathfrak{A}_n$ contains all trees (of order n), satisfies A1, A2, and is closed under addition of arbitrary edges, then $\mathfrak{A}_n$ is the class of all connected graphs of order n.

Certain natural classes of graphs fail to satisfy condition A2, e.g., the class of all trees is closed under switching of edges to vertices of the center (relative to the function $s(x)$), but not under addition of edges. The reverse is true for the class of all graphs of diameter $\leqq 2$. In cases like these, the class $\mathfrak{A}$ may be replaced by $\mathfrak{A}^*$, the smallest class which contains $\mathfrak{A}$ and satisfies A1 and A2. That such a class $\mathfrak{A}^*$ always exists is obvious.

Condition A3 expresses the natural requirement that a point-centrality must be invariant under isomorphisms. In particular, this condition implies that if x and y are automorphic vertices of X (i.e., if there is an automorphism η of X such that $y = \eta x$), then

$$\sigma_X(y) = \sigma_X(x).$$

Condition A4 was included mainly because it is satisfied by our prototype, $s(x)$. Apart from that, it seems reasonable to say that adding an edge, i.e., a direct channel of communication, will serve to centralize, rather than decentralize a network, although one might argue that this would depend to some extent on the location of the new edge in the network.

Condition A5 is the most significant one. If the word "center" is to have anything like its intuitive meaning, then giving a member of the center a new direct channel of communication must not remove it from the center. On the contrary, it ought to strengthen its position. This strengthening may be achieved at the expense of weaking some other part of the network (by switching an edge), or without such weaking (by simply adding an edge). This vague argument would seem to carry weight for arbitrary switches to the center, i.e., admissible as well as inadmissible ones. The following example shows, however, that for inadmissible switches, $s(x)$ would not satisfy A5 nor would the axiom be intuitively acceptable. Let $(x_0, \cdots, x_{2n})$ be a $(2n+1)$-circuit and let X be formed by joining to x_0 a set of $2m$ vertices $y_1, \cdots, y_{2m}$ of degree 1. It then follows from Proposition 9 that $C(X) = \{x_0\}$, regardless of the values of m and n. Now consider the edge $e_0 = [x_{2n-2}, x_{2n-1}]$, and switch it to x_0, pivoting about x_{2n-1}. Note that here the pivot lies between x_0 and x_{2n-2}. For the resulting graph X' one obtains that if $i \geqq 1$, then $X' - x_i$ has exactly two components of orders $i + 2m + 2$ and $2n - i - 2$, respectively. By Proposition 7, $x_i \varepsilon C'$ if both of these numbers are $\leqq n + m$. This gives $i = n - m - 2$, and hence by Proposition 8, $C' = \{x_{n-m-2}\}$. This agrees well with intuition, but not with the unrestricted axiom A5 according to which $C(X')$ would have to be $\{x_0\}$.

The axioms we have given are not intended to, and do not, characterize $s(x)$, although, of course, such a characterization would be highly desirable. In fact, we do not know whether $s(x)$ is a point-centrality on the class of all finite connected graphs (the difficulty is the verification of A5; it is clear, however, that $s(x)$ satisfies that part of A5 which says that if XAY or XSY, then $s_Y(x) < s_X(x)$ for the distinguished $x \varepsilon C(X)$). On the other hand, a simple example of a point-centrality on $\mathfrak{G}_n$ is provided by the function

$$\sigma_X(x) = n - d(x; X). \tag{13}$$

Here $C_\sigma(X)$ consists of the vertices of maximum degree, and the verification of the five axioms is trivial.

A function which, perhaps somewhat surprisingly, fails to be a point-centrality is the function $m(x)$ defined by (12). This is shown by the following simple example. Let $X = (x_0, \cdots, x_{2n})$ be a path of length $2n$. Then $C_m(X) = \{x_n\}$. Now form X' by adding to X the edge $[x_n, x_{2n}]$. Then XA_mX', but $C_m(X') = \{x_{n-1}\}$, i.e., the operation of adding an edge does not preserve the center. Similarly it can be seen that the operation of switching an edge likewise does not preserve the center.

In connection with axioms A1–A5 it is natural to ask whether any additional conditions should be imposed on a point-centrality σ. For example, one might consider it desirable that members of the center be in some sense "close together", or at least that any two members of the center can communicate with each other by a chain consisting only of members of the center (i.e., that the center be connected). These conditions, however, are not even satisfied by $s(x)$, as is shown by the example of the complete bipartite graph $K_{r,s}$ with $r < s$. $K_{r,s}$ has vertices $x_1, \cdots, x_r, y_1, \cdots, y_s$, and edges $[x_i, y_j]$, $i = 1, \cdots, r$, $j = 1, \cdots, s$. It is therefore of diameter 2, and hence its center consists of $x_1, \cdots, x_r$. Thus (1) the center is totally disconnected, and (2) any two central vertices are at maximum distance from each other. Moreover, this is no special property of the function $s(x)$; any point-centrality σ either satisfies (1) and (2), or else it is constant on $K_{r,s}$. This follows from axiom A3 and the fact that $V(K_{r,s})$ consists of exactly two classes of automorphic vertices. Thus for any point-centrality (in fact for any function invariant under isomorphisms),

$$C_\sigma(K_{r,s}) = \{x_1, \cdots, x_r\}, \qquad \text{or} \qquad C_\sigma(K_{r,s}) = \{y_1, \cdots, y_s\},$$

$$\text{or} \qquad C_\sigma(K_{r,s}) = V(K_{r,s}).$$

In the first two cases we have properties (1) and (2); as regards to the third case, the adequacy of a point-centrality which does not distinguish between the vertices x_i and y_j may at least be doubted. In views of difficulties such as these we have refrained from imposing any connectedness conditions on the center.

We now give three examples of classes of graphs which are admissible with respect to $s(x)$: (a) The class of all tree-like graphs. (b) The class $\mathfrak{A}$ consisting of all graphs X such that X contains a unique vertex x_0 of maximum degree, and

$$\max_{y \in X} \rho(x_0, y) \leqq 2.$$

(c) The class $\mathfrak{B}$ consisting of the class $\mathfrak{D}_2$ of all graphs of diameter $\leqq 2$ together with the class $\mathfrak{A}$ of example (b). For these classes axiom $\overline{\text{A1}}$ is satisfied. A3 and A4 are satisfied for any class satisfying A1. Hence it remains to verify A2 and A5.

Example (a). If $n = 2$ there is nothing to show. Hence we may assume that $n \geqq 3$. By section 3, if X is tree-like, then $|C| \leqq 2$. Let $x_0 \in C$. Since $n \geqq 3$, x_0 is a cut-vertex, hence $X = X_0 \cup X_1$, where $E(X_i) \neq \emptyset$, $i = 0, 1$, X_0, X_1 are edge-disjoint, connected, and $X_0 \cap X_1 = x_0$. Let X' be obtained from X by switching an edge to x_0, or by adding an edge incident with x_0. Without loss of generality we may assume that in either case the new edge joins x_0 with a vertex y_1 in X_1. By Y_1 denote the largest subgraph of X' with $V(Y_1) = V(X_1)$. It is immediately obvious that $x_0 \in D'$, and hence $x_0 \in C'$ (primed quantities refer to X'). This verifies A5, and in the case where $C = \{x_0\}$, also A2.

Now consider the case where C consists of two vertices x_0, x_1. Let K be the component of $X' - x_1$ which contains x_0. Then K contains X_0 as well as y_1. Hence

$$|K| \geqq n_0 + 1 = n/2 + 1,$$

where $n_0 = |X_0|$. Thus $x_1 \notin D'$; so in order to show that X' is tree-like we have to show that $x_1 \notin C'$. By (7) and (8),

$$s'(x_1) - s'(x_0) = s_{Y_1}(x_1) - s_{Y_1}(x_0) + (n_0 - 1) > (2 - n_1) + (n_0 - 1) = 0,$$

where $n_1 = |X_1| = |Y_1|$. The inequality is strict (by the second part of Proposition 4) because $d(x_0\ ; Y_1) \geqq 2$ (it is possible that $d(x_1\ ; Y_1) = 1$, but this causes no difficulty since then $s_{Y_1}(x_1) - s_{Y_1}(x_0) = n_1 - 2$, and $n_1 \geqq 3$ because Y_1 contains x_0 , x_1 , y_1). Thus indeed $x_1 \notin C'$. This completes the verification of A2.

Note that the class of tree-like graphs contains $\mathfrak{T}^*$, where $\mathfrak{T}$ is the class of all trees.

Example (b). It is clear that if $X \varepsilon \mathfrak{A}$, then switching or adding an edge to the vertex of maximum degree, x_0 , produces another graph in $\mathfrak{A}$ whose vertex of maximum degree is again x_0 . To verify A2 and A5 it therefore is sufficient to show that $C(X) = \{x_0\}$.

From

$$\max_{y \varepsilon X} \rho(x_0\ , y) \leqq 2$$

it follows immediately that diam $X \leqq 4$. Hence by (3),

$$s(x) = 2(n - 1) - d_x + r(x), \qquad x \varepsilon X,$$

where $r(x) = \alpha_3(x) + 2\alpha_4(x) \geqq 0$. Since $d_{x_0} > d_x$ for all $x \neq x_0$, it follows that

$$s(x_0) = 2(n - 1) - d_{x_0} < 2(n - 1) - d_x \leqq s(x)$$

for all $x \neq x_0$. Thus $C = \{x_0\}$.

Example (c). Take any $X \varepsilon \mathfrak{D}_2$. C consists of the vertices of maximum degree. Let $x_0 \varepsilon C$ and let X' be obtained by switching an edge to x_0 . Then x_0 is the unique vertex of maximum degree of X' and clearly

$$\max_{y \varepsilon X'} \rho'(x_0\ , y) \leqq 2.$$

Thus either $X' \varepsilon \mathfrak{A}$ or $X' \varepsilon \mathfrak{D}_2$. If, on the other hand XAX', then trivially $X' \varepsilon \mathfrak{D}_2$. Thus in either case $X' \varepsilon \mathfrak{B}$, and hence A2 is satisfied. The proof of A5 is the same as in example (b), except that the following additional case has to be considered. Let XAX', where $[x_0\ , x_1]$ is the new edge and $x_0 \varepsilon C$. Then

$$d'_{x_i} = d_{x_i} + 1, \qquad i = 0, 1, \quad \text{and} \quad d'_x = d_x \quad \text{for all} \quad x \neq x_0\ , x_1\ .$$

Hence $x_0 \varepsilon C'$; in fact either $C' = \{x_0\}$ or $C' = \{x_0\ , x_1\}$ according as $x_1 \notin C$ or $x_1 \varepsilon C$.

Note that $\mathfrak{B}$ contains $\mathfrak{D}_2^*$.

After these preliminaries we can now define what we mean by a centrality index.

Definition 3. Let $\mathfrak{A}$ be a class of graphs which is admissible with respect to a point-centrality σ. A *centrality index* on $\mathfrak{A}$ with respect to σ is a non-negative real-valued function ζ on $\mathfrak{A}$ such that

(I1) $X \in \mathfrak{A}_n$ and $X \cong Y$ implies $\zeta(X) = \zeta(Y)$;
(I2) $X \in \mathfrak{A}_n$, $X <_\sigma Y$ implies $\zeta(X) < \zeta(Y)$.

Condition I2 simply requires that if Y is more σ-centralized than X (in the sense of Definition 2), then the centrality index of Y is greater than that of X.

Given a point-centrality σ on an admissible class $\mathfrak{A}$ of graphs there always exists a centrality index on $\mathfrak{A}$ in the sense of the above definition. For any $X \in \mathfrak{A}$ define

$$\epsilon(X) = 1/\min_{x \in X} \sigma_X(x). \tag{14}$$

Then if $X \in \mathfrak{A}_n$ and $XA_\sigma Y$ or $XS_\sigma Y$, it follows from A5 that

$$\min_{y \in Y} \sigma_Y(y) = \sigma_Y(x_0) < \sigma_X(x_0) = \min_{x \in X} \sigma_X(x),$$

where x_0 is the vertex of $C_\sigma(X)$ to which an edge has been added or switched. Hence $\epsilon(X) < \epsilon(Y)$. ϵ may be called the *trivial centrality index*.

Note that if ζ_1, ζ_2 are centrality indices on $\mathfrak{A}$ then so are

$$a_1\zeta_1 + a_2\zeta_2 \quad \text{and} \quad \zeta_1^{a_1}\zeta_2^{a_2}$$

for any a_1, $a_2 \geqq 0$, not both zero. Thus if $\zeta_1, \cdots, \zeta_r$ are centrality indices, then any polynomial in $\zeta_1, \cdots, \zeta_r$ with non-negative coefficients, not all zero, is again a centrality index.

For a given point-centrality σ we shall consider the following three index functions.

(i) The *Beauchamp index* β ([3], p. 161):

$$\beta(X) = \sum_{x \in X} \frac{1}{\sigma_X(x)}.$$

(ii) The *Bavelas index* γ:

$$\gamma(X) = \sum_{x,y \in X} \frac{\sigma_X(x)}{\sigma_X(y)}$$

(iii) The *dispersion* ([1], p. 21; [2], p. 727):

$$\delta(X) = \Big(\sum_{x \in X} \sigma_X(x)\Big)^{-1}$$

Note that δ is essentially the same as the *gross status* introduced in ([6], p. 32) for directed graphs. From these definitions one has

$$\beta = \gamma\delta \tag{15}$$

and hence

PROPOSITION 10. *The Beauchamp index β is a centrality index on every σ-admissible class on which both γ and δ are centrality indices.*

PROPOSITION 11. *Let $\mathfrak{A}$ be a σ-admissible class, $X \in \mathfrak{A}$. Then $XA_\sigma Y$ implies $\beta(X) < \beta(Y)$ and $\delta(X) < \delta(Y)$.*

PROOF. By A4, $\sigma_Y(x) \leqq \sigma_X(x)$ for every $x \in V(Y) = V(X)$, and the inequality is strict for at least one x. Hence $\beta(X) < \beta(Y)$ and $\delta(X) < \delta(Y)$.

There exist certain point-centrality functions σ such that the Beauchamp index is a centrality index on any σ-admissible class. For example, let σ be the function defined by (13). Take any graph X of order n, and let X' be obtained by switching an edge from $x_1 \in X$ to $x_0 \in C_\sigma(X)$. Abbreviate σ_X by σ, $\sigma_{X'}$ by σ'. Then

$$\sigma'(x_0) = \sigma(x_0) - 1, \qquad \sigma'(x_1) = \sigma(x_1) + 1, \qquad \sigma'(x) = \sigma(x)$$

for all $x \neq x_0, x_1$. Hence

$$\beta(X') - \beta(X) = [\sigma(x_0)(\sigma(x_0) - 1)]^{-1} - [\sigma(x_1)(\sigma(x_1) + 1)]^{-1} > 0$$

since $\sigma(x_0) \leqq \sigma(x_1)$.

Note, incidentally, that for the point-centrality σ defined by (13) the dispersion is not a centrality index on any σ-admissible class. This comes from the fact that $XS_\sigma Y$ implies $\delta(X) = \delta(Y)$. Since on the class of all graphs of diameter $\leqq 2$, $s(x)$ differs from $\sigma_X(x)$ only by a constant it follows that the dispersion is not a centrality index relative to $s(x)$ on any class containing $\mathfrak{D}_2$.

PROPOSITION 12. *If $X, X' \in \mathfrak{D}_2$, and XSX', then $\beta(X) < \beta(X')$, and $\gamma(X) < \gamma(X)$. However, all three indices β, γ, δ fail to be centrality indices on $\mathfrak{D}_2^*$.*

PROOF. The proof that $\beta(X) < \beta(X')$ is contained in the remark following Proposition 11. We have also just noted that $\delta(X) = \delta(X')$. Hence by (15), $\gamma(X) < \gamma(X')$. For β and δ the failure on $\mathfrak{D}_2^*$ is shown by the following example. Let K_m be a complete m-graph, $m \geqq 2$, (x_0, y_0, y_1, x_1) a path disjoint from K_m. Form X by joining every vertex in K_m to both x_0 and x_1. Then $X \in \mathfrak{D}_2$ and the center of X consists of x_0, x_1 and all vertices of K_m. Switch $[x_1, y_1]$ to x_0, pivoting at x_1. This is admissible. The resulting graph Y has diameter 3, and $\beta(Y) < \beta(X)$, $\delta(Y) < \delta(X)$. To show that γ fails on $\mathfrak{D}_2^*$ (in fact, even on $\mathfrak{D}_2$ itself) take the same X as before and add the edge $[x_0, y_1]$. Note that the new graph X' belongs to $\mathfrak{D}_2$. It is straightforward to calculate that

$$\gamma(X) = m^2 + 9m + 9 + R_1(m),$$

$$\gamma(X') = m^2 + 9m + \tfrac{33}{4} + R_2(m),$$

where $R_i(m) \to 0$ as $m \to \infty$. Hence for all sufficiently large m, $\gamma(X') < \gamma(X)$.

Next, we investigate the behavior of the three indices on $\mathfrak{T}^*$, where $\mathfrak{T}$ is the class of all trees.

PROPOSITION 13. *If X is a tree and XSX', then X' is a tree, and $\delta(X) < \delta(X')$.*

PROOF. That X' is a tree is obvious.

Now suppose that X, X_0, X_1, w are as in Proposition 5. Let $S = 1/\delta(X)$, $S_i = 1/\delta(X_i)$, $i = 0, 1$. Then by summing (8) over all $x \in X$,

$$S = S_0 + S_1 + 2(n_0 - 1)s_1(w) + 2(n_1 - 1)s_0(w). \tag{16}$$

This holds for any separable graph X with cut-vertex w. Now let X be a tree, and let $x_0 \in C$. Let X' be obtained by switching an edge $e = [x_1, x_2]$ to x_0, pivoting about x_2. Let X_0 be the component of $X - e$ which contains x_1 (and x_0), and put $X_1 = X \backslash X_0$. Then the only edge of X_1 incident with x_1 is e, and $X_0 \cap X_1 = x_1$. Thus (16) holds with w replaced by x_1. Similarly, X' can be decomposed into X_0', X_1' with cut-vertex x_0, where $X_0' = X_0$, and $\underline{X_1'} \cong X_1$. Hence

$$S_1' = S_1, \qquad s_0'(x_0) = s_0(x_0), \qquad s_1'(x_0) = s_1(x_1),$$

so that by (14),

$$S' = S_0 + S_1 + 2(n_0 - 1)s_1(x_1) + 2(n_1 - 1)s_0(x_0).$$

Hence

$$S - S' = 2(n_1 - 1)[s_0(x_1) - s_0(x_0)].$$

Now by (8),

$$s(x_1) - s(x_0) = s_0(x_1) - s_0(x_0) - \rho(x_0, x_1)(n_1 - 1) \geqq 0,$$

the inequality being a consequence of $x_0 \in C$. Hence

$$s_0(x_1) - s_0(x_0) \geqq \underline{\rho}(x_0, x_1)(n_1 - 1) > 0,$$

i.e., $S > S'$.

In spite of being well-behaved for trees the dispersion fails to be a centrality index on $\mathfrak{T}^*$. Consider the three graphs of Fig. 3. It is clear that

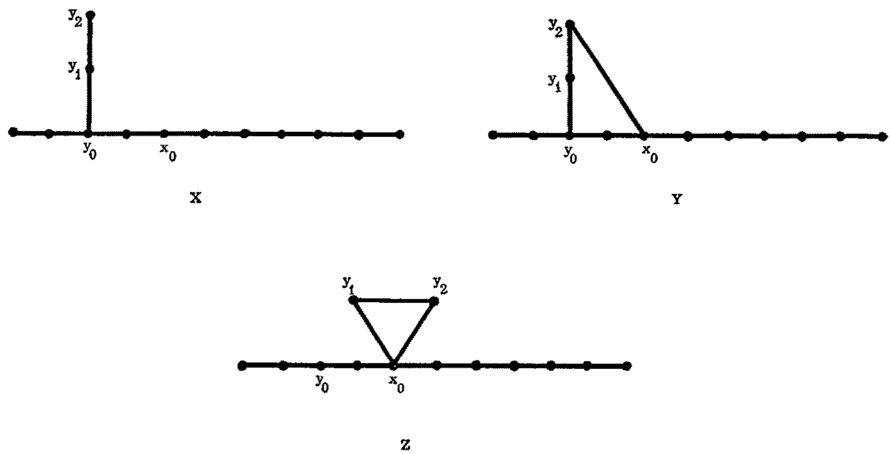

FIGURE 3
Graphs X, Y, and Z ($\delta(Z) < \delta(Y)$)

$XAYSZ$, and hence both Y and Z belong to $\mathfrak{T}^*$. But, as can easily be checked by direct calculation, $\delta(Z) < \delta(Y)$.

In ([3], p. 162) Beauchamp states a theorem which would imply that if $X \epsilon \mathfrak{T}$ and XSX', then $\beta(X) < \beta(X')$. Regrettably, Beauchamp's proof is incorrect, although, at least in the case of trees, the theorem itself may be true. At any rate, β fails to be a centrality index on $\mathfrak{T}^*$, a counterexample being the graphs Y and Z of Fig. 3, just as in the case of δ. It is perhaps of interest to note that Beauchamp's theorem is true for the dispersion; the proof is essentially the same as that of Proposition 13.

Of the three indices, γ shows the most unsatisfactory behavior on $\mathfrak{T}^*$.

PROPOSITION 14. *There exist trees X and X' such that XSX' and $\gamma(X) > \gamma(X')$. Thus γ is not a centrality index on $\mathfrak{T}^*$.*

PROOF. There are many ways of proving this. The particular counterexample we have chosen shows that stars are not graphs of maximum Bavelas index.

A *star* of order n is a graph with vertices $x_0, \cdots, x_{n-1}$ and edges $[x_0, x_i]$, $i = 1, \cdots, n-1$. In other words, the star of order n is the complete bipartite graph $K_{1,n-1}$. (Cf. the remarks preceding Examples (a), (b), (c)). Given an arbitrary tree of order n it is clear that by making a certain number of switches to the center one obtains $K_{1,n-1}$. Hence by axiom A5, $T < K_{1,n-1}$ for every tree $T \epsilon \mathfrak{T}_n$. If γ were a centrality index, then by I2, $\gamma(T) < \gamma(K_{1,n-1})$ for every $T \epsilon \mathfrak{T}_n$. We now show that this is not the case. Let $T_{\alpha,n+1}$ be the tree of order $n + 1 = \alpha\beta + 1$ which consists of β paths of length α, $P_i = (x_0, x_{i1}, \cdots, x_{i\alpha})$, $i = 1, \cdots, \beta$, having precisely the vertex x_0 in common. Then $s(x_0) = \frac{1}{2}n(\alpha + 1)$,

$$s(x_{ij}) = (1 + \cdots + j) + (\beta - 1)((j + 1) + \cdots + (j + \alpha)) + (1 + \cdots + (\alpha - j))$$
$$= \tfrac{1}{2}(n(2j + \alpha + 1) + 2j(j - 2\alpha)).$$

Using these values, and writing $\gamma(T_{\alpha,n+1})$ in the form

$$\gamma(T_{\alpha,n+1}) = an^2 + bn + c + R(n),$$

where R is a rational function with $R(n) \to 0$ as $n \to \infty$, the coefficient of n^2 is easily calculated:

$$a = \frac{2(\alpha + 1)}{\alpha} \sum_{j=1}^{\alpha} \frac{1}{\alpha + 2j + 1}.$$

By a very crude estimate

$$a > \frac{2(\alpha + 1)}{\alpha} \sum_{j=1}^{\alpha} \int_j^{j+1} \frac{dt}{\alpha + 2t + 1} = \frac{\alpha + 1}{\alpha} \log \frac{3(\alpha + 1)}{\alpha + 3},$$

and this converges to log 3 as $\alpha \to \infty$. Hence $a > 1$ for all sufficiently large α, whereas for stars ($\alpha = 1$), $a = 1$. This in turn means that for all sufficiently large n which are multiples of α, the graph $T_{\alpha,n+1}$ has a higher Bavelas index than the star of the same order.

5. *Conclusion*

The various propositions and examples given in section 4 constitute a rather devastating verdict on the three indices β, γ, δ. It must be emphasized that we have tested these indices only on classes of graphs for which we can actually prove their $s(x)$-admissibility. Thus our inability to prove that $s(x)$ is a point-centrality on the class of all graphs has no bearing on the fact that β, γ, δ fail to be centrality indices on $\mathfrak{D}_2^*$ and $\mathfrak{T}^*$.

Of the three indices, β is the most acceptable one (cf. the remark following Proposition 11), while the Bavelas index is decidedly the worst. Most unfortunately, all three fail on the class $\mathfrak{T}^*$, although δ, and perhaps also β, is a useful index on $\mathfrak{T}$ itself (Proposition 13).

One may, of course, blame this wholesale failure on our axiom system. There is little doubt, however, that the three indices (except possibly δ) would not survive even a more sophisticated system of axioms. In view of this we strongly suggest that β, γ, δ be discarded and that centrality be measured by the trivial index ϵ defined by (14). ϵ is more easily calculated than any of the other indices, and, whatever its intuitive shortcomings, it has the decided advantage of satisfying a well-defined system of axioms. Also, ϵ has the highly desirable property of satisfying a weak version of Beauchamp's theorem where one of the two graphs is a tree.

THEOREM. *Let Y be a connected graph, T a tree disjoint from Y, w an* *arbitrary vertex of Y, w_0, w_1 two distinct vertices of T such that w_0 lies on the path joining w_1 with $C(T)$ (i.e., with the vertex of $C(T)$ nearest w_1). Form X_i by identifying w with w_i, $i = 0, 1$. Then $\epsilon(X_0) > \epsilon(X_1)$.*

PROOF. Let us denote by z_i the vertex of X_i obtained by identifying w with w_i. We will assume first that $e = [w_0, w_1] \varepsilon E(T)$. Since w_0 lies on the path of T joining w_1 with $C(T)$ it follows from (11) that $s_T(w_0) < s_T(w_1)$. By s_i denote point-centrality in X_i, $i = 0, 1$. Then by (8) for any $x \varepsilon T$,

$$s_i(x) = s_T(x) + s_Y(w) + \rho_T(x, w_i)(n - 1) \tag{17}$$

where $n = |Y|$. Let T_i be the component of $T - e$ which contains w_i, $i = 0, 1$. Then for $x \varepsilon T_i$,

$$\rho_T(x, \underline{w_{1-i}) =} \rho_T(x, w_i) + 1,$$

and hence

$$s_i(x) < s_{1-i}(x), \qquad i = 0, 1. \tag{18}$$

Now consider

$$m_i = \min_{x \varepsilon X_i} s_i(x), \qquad i = 0, 1.$$

Suppose m_1 is attained in Y. For $y \varepsilon Y$,

$$s_i(y) = s_Y(y) + s_T(w_i) + \rho_Y(y, w)(k - 1), \qquad i = 0, 1,$$

where $k = |T|$. Hence for any $y \varepsilon Y$, $s_0(y) < s_1(y)$. It follows that

$$m_0 = \min_{x \varepsilon X_0} s_0(x) \leqq \min_{y \varepsilon Y} s_0(y) < \min_{y \varepsilon Y} s_1(y) = m_1 .$$

Suppose next that m_1 is attained in T_1. Since w_1 is closer to $C(T)$ than any other vertex of T_1, $s_1(x) \geqq s_1(w_1)$ for all $x \varepsilon T_1$, by (17). Hence $m_1 = s_1(z_1)$. Again by (17),

$$\underline{m_0} \leqq s_0(z_0) = s_1(z_1) + (s_T(w_0) - s_T(w_1)) < s_1(z_1) = m_1 .$$

Alternatively, this case can be argued by saying that $m_1 = s_1(z_1)$ implies that m_1 is attained in Y (z_1 also being a vertex of Y), and then the previous case applies.

Finally, suppose that m_1 is attained in T_0, say $m_1 = s_1(x_0)$ for $x_0 \varepsilon T_0$. Then by (18),

$$m_1 = s_1(x_0) > s_0(x_0) \geqq \min_{x \varepsilon X_0} s_0(x) = m_0 .$$

Thus in all cases $m_1 > m_0$, i.e., $\epsilon(X_0) > \epsilon(X_1)$.

If w_0 and w_1 are not adjacent, then we apply the above argument successively to the edges of the path joining w_0 and w_1.

That in spite of its various desirable properties E is not entirely without fault can be seen from the following fact: $\epsilon(X) = 1/(n - 1)$ for any graph X of order n in which one vertex is adjacent to all other vertices. Thus a complete graph has the same index as the star of the same order. One feels that this should not be so, although apparently it is a matter of debate which of the two graphs should be considered more centralized. However, no index will ever be free from such inconsistencies, simply because it is an artificial device which imposes a total quasi-order on a system which cannot be totally ordered in structural terms.

REFERENCES

[1] Bavelas, A. A mathematical model for group structures. *Appl. Anthrop.*, 1948, **7**, 16–30.

[2] Bavelas, A. Communication patterns in task-oriented groups. *J. acoust. Soc. Amer.*, 1950, **22**, 725–730.

[3] Beauchamp, M. A. An improved index of centrality. *Behav. Sci.*, 1965, **10**, 161–163.

[4] Beatty, J. C. and Miller, R. E. On equi-cardinal restrictions of a graph. *Canad. math. Bull.*, 1964, **7**, 369–376.

[5] Flament, C. Applications of graph theory to group structure. Englewood, Cliffs, N. J.: Prentice-Hall, 1963.

[6] Harary, F. Status and contrastatus. *Sociometry*, 1959, **22**, 23–43.

[7] Harary, F. The group of the composition of two graphs. *Duke math. J.*, 1959, **26**, 29–34.

[8] Harary, F. and Norman, R. Z. The dissimilarity characteristic of Husimi trees. *Ann. Math.*, 1953, **58**, 134–141.

[9] Harary, F., Norman, R. Z., and Cartwright, D. *Structural models: An introduction to the theory of directed graphs*. New York: Wiley, 1965.

[10] Leavitt, H. S. Some effects of certain patterns on group performance. *J. abnorm. soc. Psychol.*, 1951, **46**, 38–50.

[11] Zykov, A. A. On some properties of linear complexes. *Mat. Sbornik, N. S.*, 1949, **24** (66), 163–188. *Amer. math. Soc.* translation No. 79, 1952.

Manuscript received 3/2/65
Revised manuscript received 1/26/66