Tests of inflated zeros for censored Poisson regression model

1 Models

1.1 Poisson Model

Poisson Regression: $Y_i|X_i \sim i.d.$ Poisson(μ_i), $\log(\mu_i) = x_i^T\beta$

Poisson Model: $P(Y_i = y_i) = \frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!}$, $y_i \geq 0$

1.2 ZIP Model

ZIP Regression: $Y_i|X_i \sim i.d.$ ZIP(ω_i, μ_i), $\text{logit}(\omega_i) = u_i^T\beta_\omega$, $\log(\mu_i) = x_i^T\beta_\mu$

ZIP Model:

$$P(Y_i = y_i) = \begin{cases} \omega + (1-\omega)e^{-\mu_i}, & y_i = 0 \\ (1-\omega)\frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!}, & y_i > 0 \end{cases} \tag{1.1}$$

1.3 Censored Poisson Model

Censored Poisson Regression:

$$\begin{aligned} & Y_i|X_i \sim i.d.\text{Poisson}(\mu_i) \\ & Z_i = \min\{Y_i, L\} = \begin{cases} y_i & , y_i < L \\ L & , y_i \geq L \end{cases} ; \\ & Z_i|X_i \sim i.d.\ \text{CensoredPoisson}(\mu_i, L),\ \log(\mu_i) = x_i^T\beta \end{aligned} \tag{1.2}$$

Censored Poisson Model:

$$\begin{aligned} P(Z_i = z_i) &= P\{\min(Y_i, L) = z_i\} \\ &= \begin{cases} P(Y_i = z_i) & , z_i < L \\ P(Y_i \geq L) & , z_i = L \\ 0 & , z_i > L \end{cases} \\ &= \begin{cases} \frac{e^{-\mu_i}\mu_i^{z_i}}{z_i!} & , 0 \leq z_i < L \\ \sum_{y_i=L}^{\infty} P(Y_i = y_i) & , z_i = L \\ 0 & , z_i > L \end{cases} \end{aligned} \tag{1.3}$$

Likelihood function of censored Poisson:

$$d_i = I_{(z_i \geq L)}$$
$$c_i = P(Y_i \geq L) = \sum_{y_i=L}^{\infty} P(Y_i = y_i)$$
$$L_1 = \prod_{i=1}^{n} \left\{ \left[\frac{e^{-\mu_i} \mu_i^{z_i}}{z_i!} \right]^{(1-d_i)} \left[\sum_{y_i=L}^{\infty} P(Y_i = y_i) \right]^{d_i} \right\} \tag{1.4}$$

Log-likelihood function of censored Poisson:

$$l_1 = log(L_1) = \sum_{i=1}^{n} \left\{ (1-d_i) log \left[\frac{e^{-\mu_i} \mu_i^{z_i}}{z_i!} \right] + d_i log(c_i) \right\} \tag{1.5}$$

1.4 Censored ZIP Model

Censored ZIP Regression:

$$Y_i | X_i \sim i.d.\ \text{ZIP}(\omega_i, \mu_i)$$
$$Z_i = \min\{Y_i, L\} = \begin{cases} y_i & , y_i < L \\ L & , y_i \geq L \end{cases}$$
$$Z_i | X_i \sim i.d.\ \text{CensoredZIP}(\omega_i, \mu_i, L),\ \ \text{logit}(\omega_i) = u_i^T \beta_\omega,\ \ \log(\mu_i) = x_i^T \beta_\mu \tag{1.6}$$

Censored ZIP Model:

$$\begin{aligned} P(Z_i = z_i) &= P\{min(Y_i, L) = z_i\} \\ &= \begin{cases} P(Y_i = z_i) & , z_i < L \\ P(Y_i \geq L) & , z_i = L \\ 0 & , z_i > L \end{cases} \\ &= \begin{cases} \omega + (1-\omega) P(Y_i = 0) & , z_i = 0 \\ (1-\omega) P(Y_i = z_i) & , 0 < z_i < L \\ (1-\omega) \sum_{y_i=L}^{\infty} P(Y_i = y_i) & z_i = L \\ 0 & , z_i > L \end{cases} \\ &= \begin{cases} \omega + (1-\omega) e^{-\mu_i} & , z_i = 0 \\ (1-\omega) \frac{e^{-\mu_i} \mu_i^{z_i}}{z_i!} & , 0 < z_i < L \\ (1-\omega) \sum_{y_i=L}^{\infty} P(Y_i = y_i) & z_i = L \\ 0 & , z_i > L \end{cases} \end{aligned} \tag{1.7}$$

Likelihood function of censored ZIP model:

$$r_i = I_{(z_i = 0)}$$
$$d_i = I_{(z_i \geq L)}$$
$$c_i = P(Y_i \geq L) = \sum_{y_i=L}^{\infty} P(Y_i = y_i)$$

$$L_2 = \prod_{i=1}^{n} \left\{ \left[(1-\omega) \sum_{y_i=L}^{\infty} P(Y_i = y_i) \right]^{d_i} \left[\left(\omega + (1-\omega)e^{-\mu_i}\right)^{r_i} \left((1-\omega) \frac{e^{-\mu_i} \mu_i^{z_i}}{z_i!} \right)^{1-r_i} \right]^{1-d_i} \right\}$$

Log-likelihood function of censored ZIP model:

$$l_2 = log(L_2) = \sum_{i=1}^{n} \left\{ (1-d_i) \left[\left(r_i \bullet log(\omega + (1-\omega)e^{-\mu_i}) \right) + (1-r_i) log \left((1-\omega) \left(\frac{e^{-\mu_i} \mu_i^{z_i}}{z_i!} \right) \right) \right] \right.$$
$$\left. + d_i \bullet log((1-\omega) \bullet c_i) \right\}$$

2 Hypothesis

$$H_0 : \omega = 0 \;\; vs. \;\; H_1 : \omega > 0$$

3 Test

3.1 Wald Test

Wald statistics:

$$\frac{\hat{\omega}}{\hat{\sigma}} \sim N(0,1) \;\; or \;\; \frac{\hat{\omega}^2}{\hat{\sigma}^2} \sim \chi^2(1)$$

Method:

from MLE of the CZIP we can get $\hat{\omega}$; then use Fisher Information matrix to get estimate the var of $\hat{\omega}$; that we have Wald statistics $\frac{\hat{\omega}^2}{\hat{\sigma}^2}$ and it follows $\chi^2(1)$.

3.2 LR Test

LR statistics:

$$S_{LR} = 2 \left[l_2(\hat{\omega}, \hat{\beta}) - l_1(0, \hat{\beta}) \right] \sim \chi^2(1)$$

3.3 Score Test

Score statistics:

$$S_{score} \sim N(0,1)$$

Method: let$\theta = \frac{\omega}{1-\omega}$; then from MLE of the CZIP we can get$\hat{\theta}, \hat{\beta}$; then let $\theta = 0$equaling to $\omega = 0$, so we have $\hat{\theta_0}$; then use Fisher Information matrix to get estimate the var of$\hat{\theta_0}$; that we have Score statistics$\frac{\hat{\theta_0}}{\sigma_{\theta_0}} \sim N(0,1)$.

Details:

Firstly,$\theta = \frac{\omega}{1-\omega}$,$p_i = e^{-\mu_i}$;

Then, from MLE of the CZIP we can get $\hat{\theta}, \hat{\beta}$;

$$\frac{\partial l_2}{\partial \theta} = \sum_{i=1}^{n} \left\{ \frac{-1}{\theta+1} + (1-d_i)\left[\frac{r_i}{\theta+p_i}\right] \right\} \tag{3.1}$$

$$\frac{\partial l_2}{\partial \beta} = \sum_{i=1}^{n} \left\{ (1-d_i)\left[\frac{-r_i\mu_i p_i x_i^T}{\theta_i+p_i} + (1-r_i)(z_i-\mu_i)x_i^T\right] + \frac{d_i}{c_i}\frac{\partial c_i}{\partial \beta} \right\} \tag{3.2}$$

and,

$$\frac{\partial c_i}{\partial \beta} = \frac{\partial \sum_{y_i=L}^{\infty} P(Y_i=y_i)}{\partial \beta} \tag{3.3}$$

$$\begin{aligned}
&= \frac{\partial \left[1-\sum_{y_i=0}^{L-1} P(Y_i=y_i)\right]}{\partial \beta} \\
&= -\sum_{y_i=0}^{L-1} \frac{\partial P(Y_i=y_i)}{\partial \beta} \\
&= -\sum_{y_i=0}^{L-1} \frac{\partial \frac{e^{-\mu_i}\mu_i^{y_i}}{(y_i)!}}{\partial \beta} \\
&= -\sum_{y_i=0}^{L-1} (y_i-\mu_i)P(Y_i=y_i)x_i^T \\
&= \mu_i \sum_{k=0}^{L-1} P(Y_i=y_i) - y_i \sum_{k=0}^{L-1} P(Y_i=y_i) \\
&= \mu_i \sum_{k=0}^{L-1} P(Y_i=y_i) - \mu_i \sum_{k=0}^{L-2} P(Y_i=y_i) \\
&= \mu_i P(Y_i=L-1)
\end{aligned} \tag{3.4}$$

So,

$$\frac{\partial l_2}{\partial \beta} = \sum_{i=1}^{n} \left\{ (1-d_i)\left[\frac{-r_i\mu_i p_i}{\theta_i+p_i} + (1-r_i)(z_i-\mu_i)\right] + \frac{d_i}{c_i}\mu_i P(Y_i=L-1) \right\} x_i^T \tag{3.5}$$

Then let $\theta = 0$,

$$\frac{\partial l_2}{\partial \theta}|_{\theta=0} = \sum_{i=1}^{n} \left\{ \frac{(1-d_i)r_i - p_i}{p_i} \right\} \tag{3.6}$$

$$\frac{\partial l_2}{\partial \beta}|_{\theta=0} = \sum_{i=1}^{n} \left\{ (1-d_i)\left[-r_i\mu_i + (1-r_i)(z_i-\mu_i)\right] + \frac{d_i}{c_i}\mu_i P(Y_i=L-1) \right\} x_i^T \tag{3.7}$$

Next, use Fisher Information matrix to get estimate the var of$\hat{\theta_0}$. Finally, we have Score statistics $\frac{\hat{\theta_0}}{\sigma_{\theta_0}} \sim N(0,1)$.

3.4 He Test

He statistics:

$$\sqrt{n}\,(\hat{s}-0) \to N(0,\tau^2) \tag{3.8}$$

τ^2 is (1,1)term of $A^{-1}BA^{-T}$ $(1,1)$;

$$A(\gamma) = E\left[\frac{\partial}{\partial\gamma}\Psi_i(Y_i,\gamma)\right] \tag{3.9}$$

$$B(\gamma) = Var(\Psi_i(Y_i,\gamma) \tag{3.10}$$

Details:

$$\begin{aligned} E(r_i) &= P(z_i = 0) \\ &= P(Y_i = 0) \\ &= p_i \end{aligned} \tag{3.11}$$

$$\begin{aligned} s &= \frac{1}{n}\sum_{i=1}^{n}(r_i - E(r_i)) \\ &= \frac{1}{n}\sum_{i=1}^{n}(r_i - p_i) \end{aligned} \tag{3.12}$$

Estimation Equation(EE):

$$\psi_1 = \frac{1}{n}\sum_{i=1}^{n}(r_i - p_i - s) \tag{3.13}$$

$$\psi_2 = \frac{\partial l_1}{\partial\beta} = \frac{1}{n}\sum_{i=1}^{n}\left[(1-d_i)(z_i-\mu_i) + \frac{d_i}{c_i}\mu_i P(Y_i = L-1)\right]x_i^T \tag{3.14}$$

4 Simulations

4.1 QQ plot Setting

nism=100;n=c(50,100,200,500,1000);aa=c(-0.5,0.5,1,1.5)

case1:$\mu = e^{aa}$

case2:$\mu = e^{aa-1.45x}, x \sim N(0,1)$

case3:$\mu = e^{aa-1.45x}, x \sim U(0,1)$

cut=4:figure1,2,3;

cut=7:figure4,5,6;

cut=30:figure7,8,9;

4.2 Power plot Setting

nism=100;n=c(50,100,200,500,1000);aa=c(-0.5,0.5,1,1.5);lp=0.01*(seq(5,30,5))

case1:$\mu = e^{aa}, \omega = lp$

case2:$\mu = e^{aa-1.45x}, \omega = lp, x \sim N(0,1)$

case3:$\mu = e^{aa-1.45x}, \omega = lp, x \sim U(0,1)$

cut=4:figure10,11,12;

cut=7:figure13,14,15;

cut=30:figure16,17,18;

4.3 QQ plot

4.4 Power plot

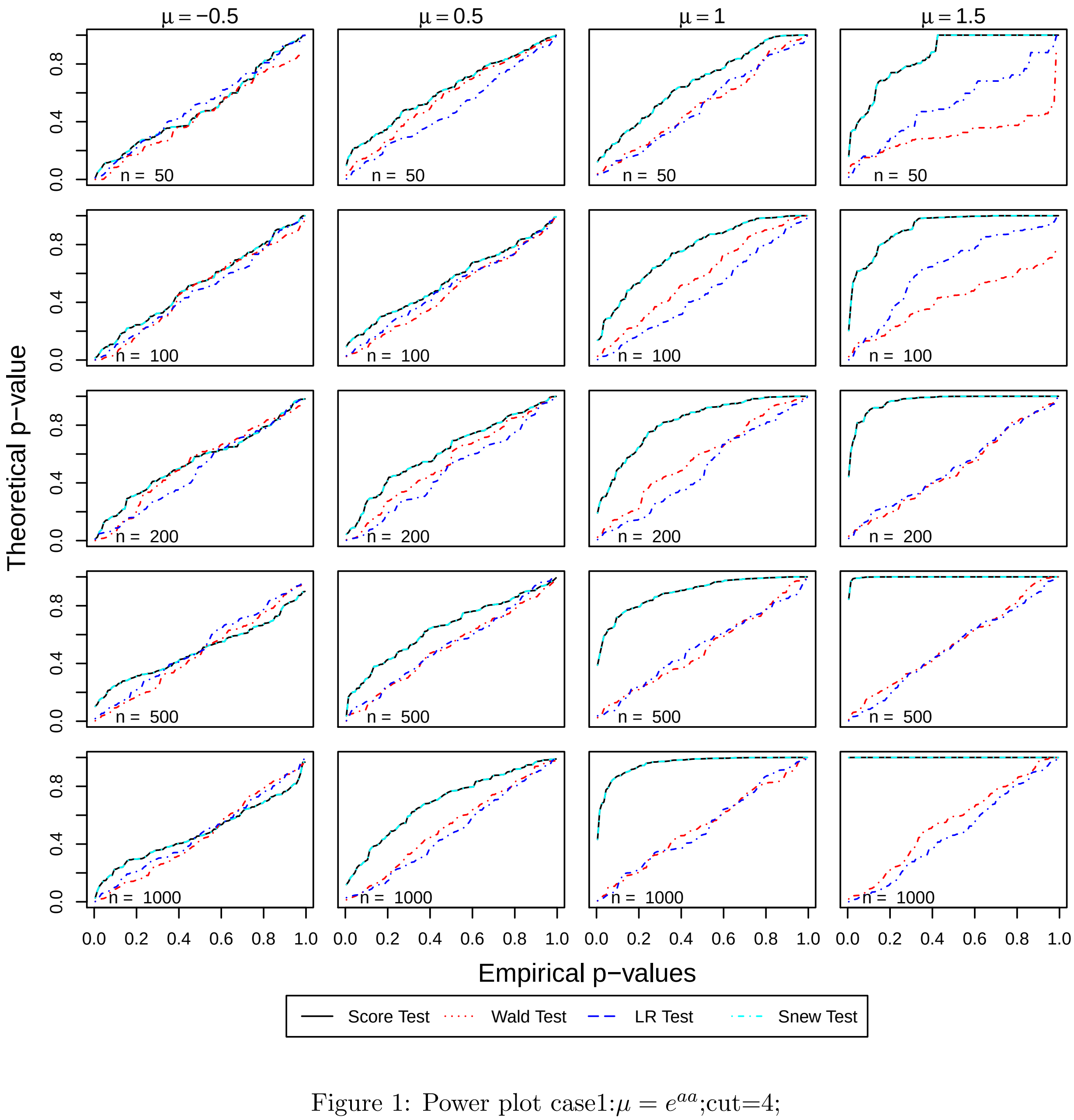

Figure 1: Power plot case1:$\mu = e^{aa}$;cut=4;

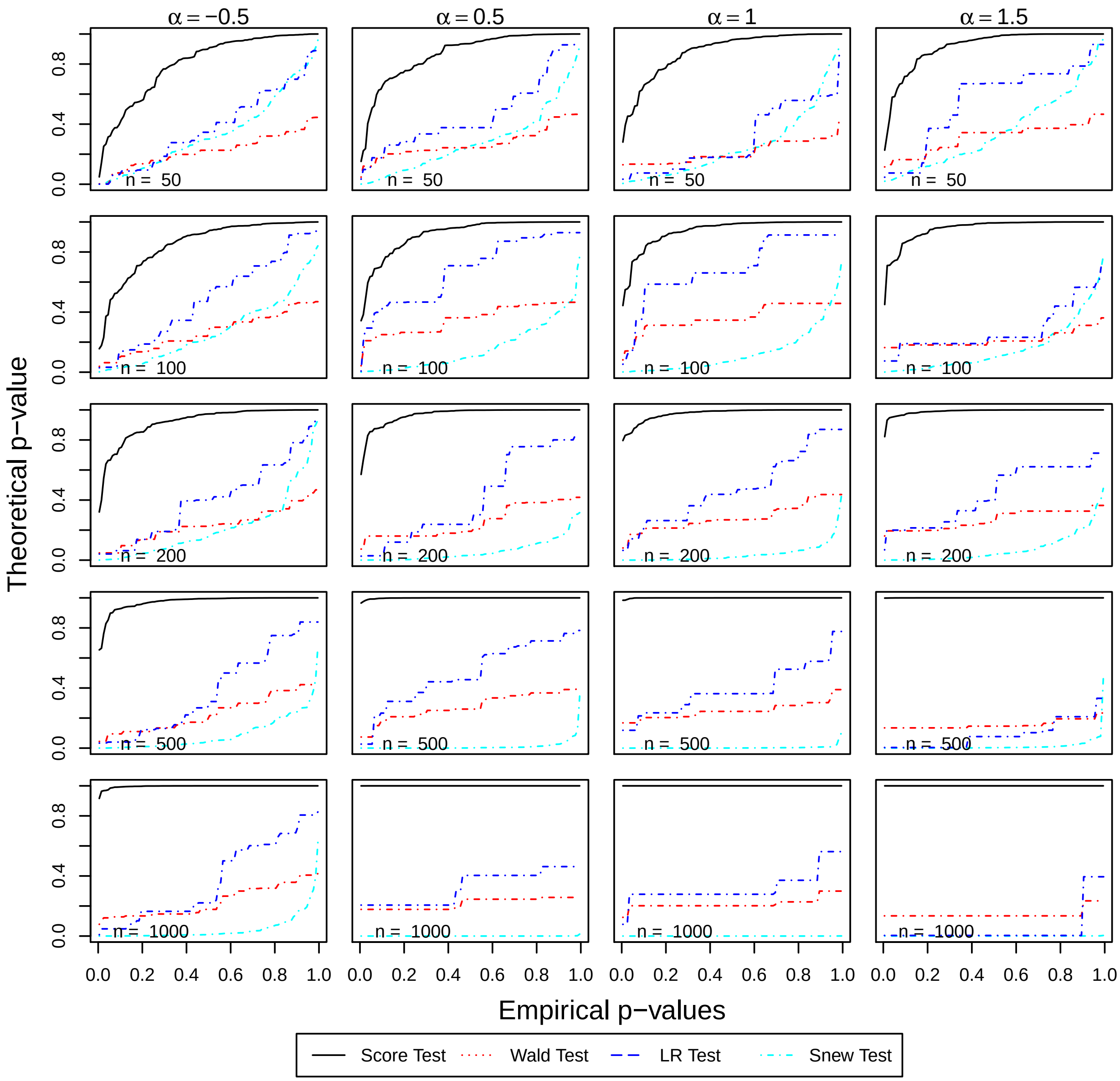

Figure 2: Power plot case2:$\mu = e^{aa-1.45x}, x \sim N(0,1)$;cut=4

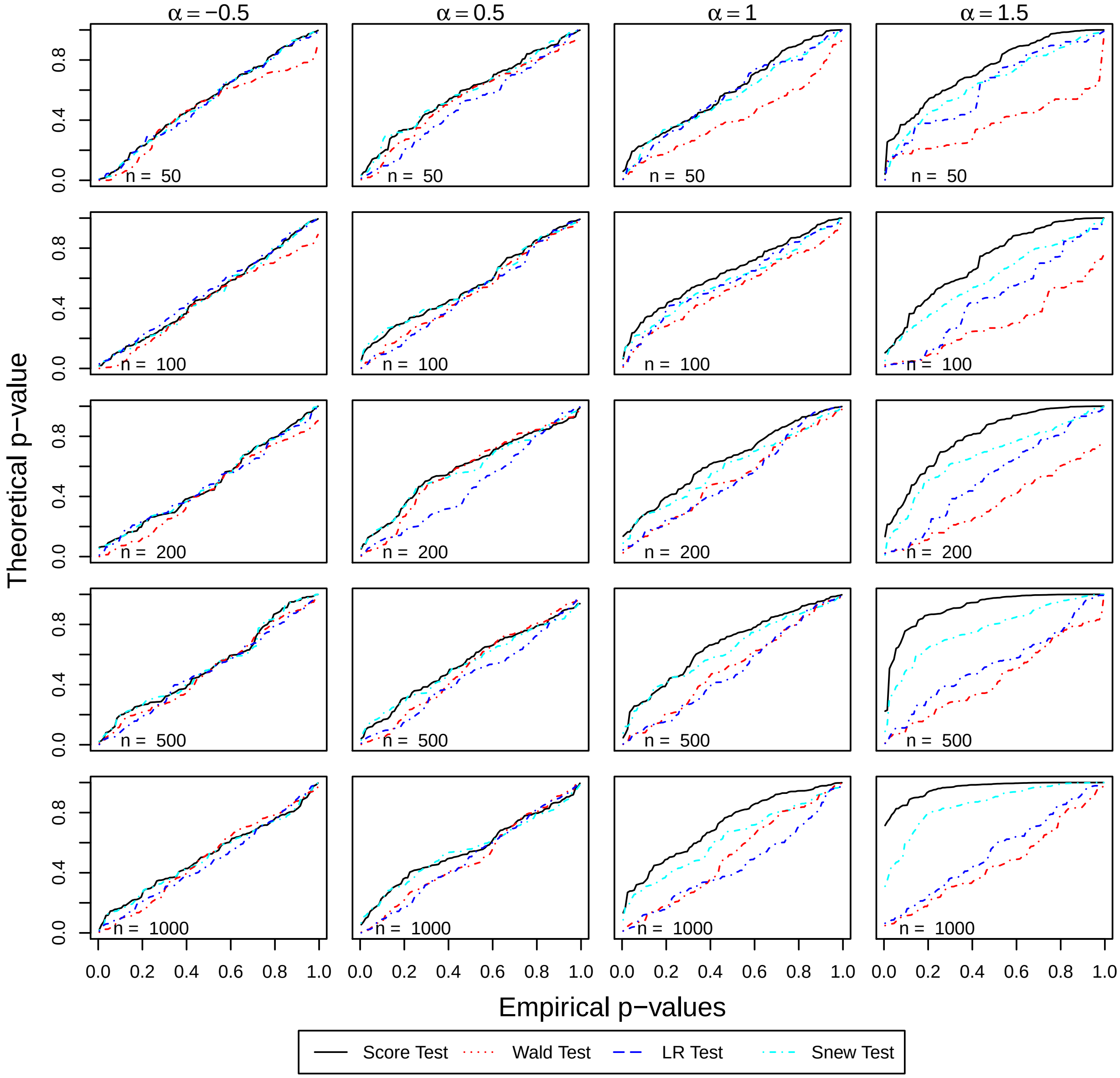

Figure 3: Power plot case3:$\mu = e^{aa-1.45x}, x \sim U(0,1)$;cut=4

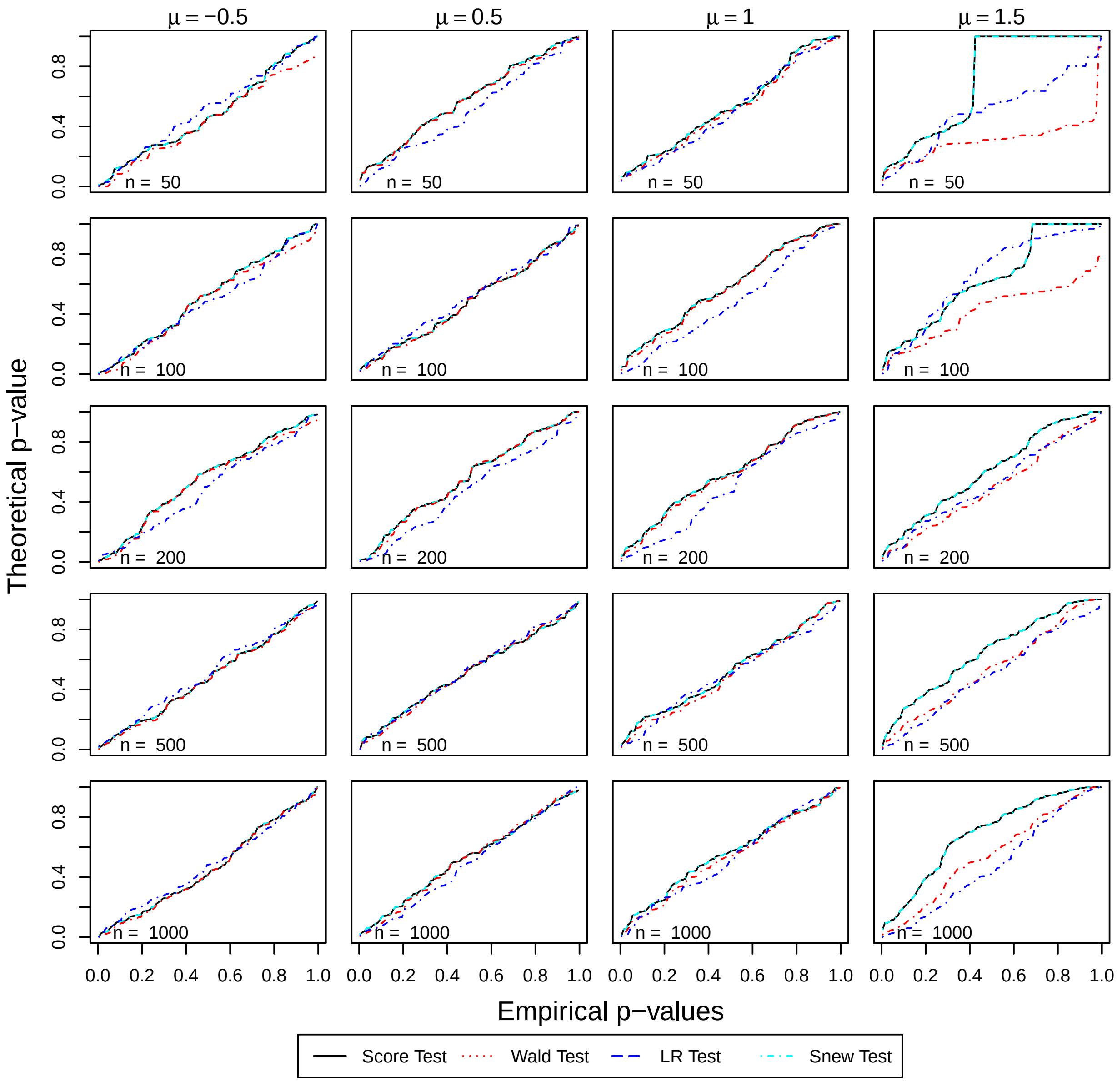

Figure 4: Power plot case1:$\mu = e^{aa}$;cut=7;

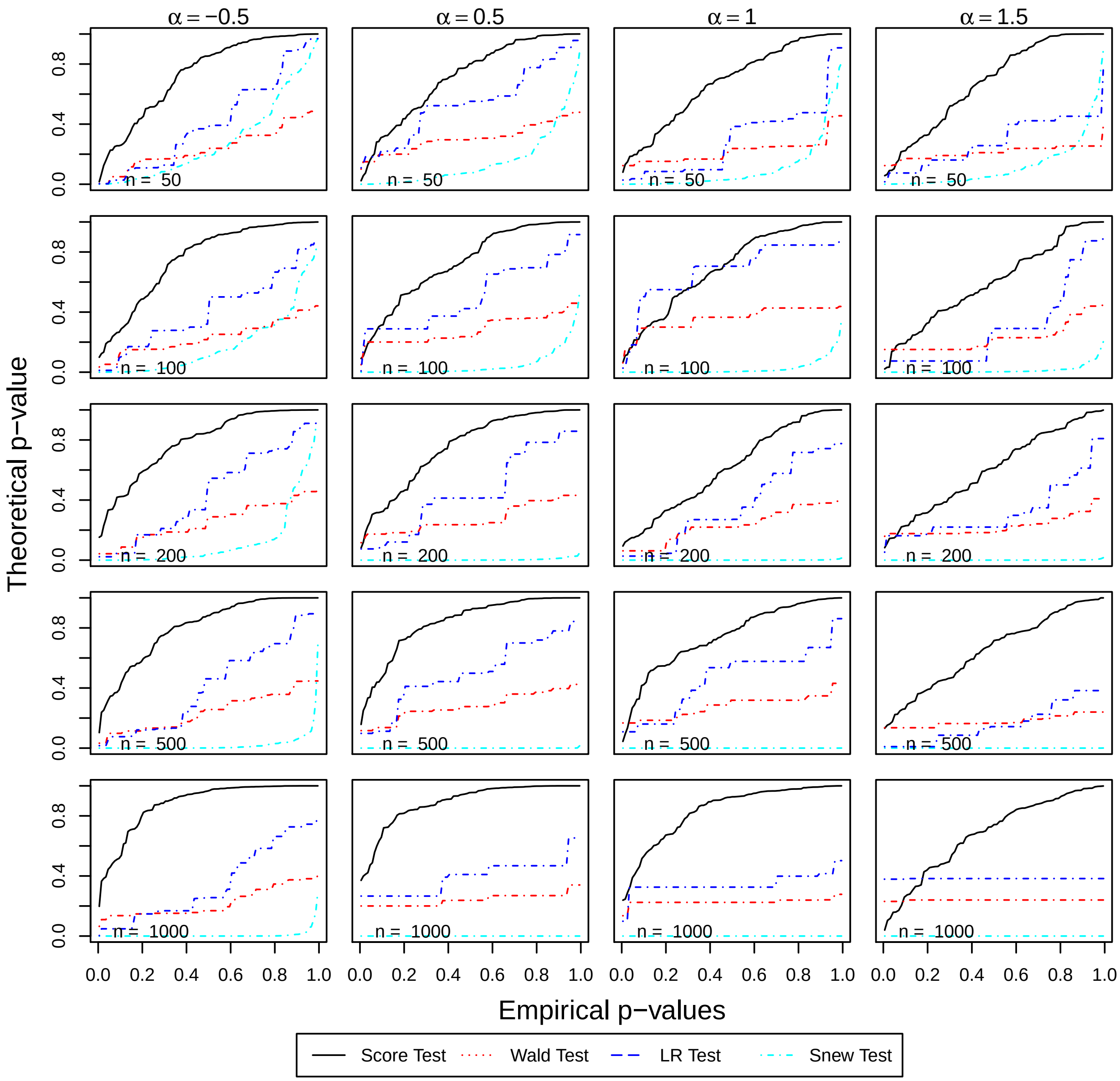

Figure 5: Power plot case2:$\mu = e^{aa-1.45x}, x \sim N(0,1)$;cut=7

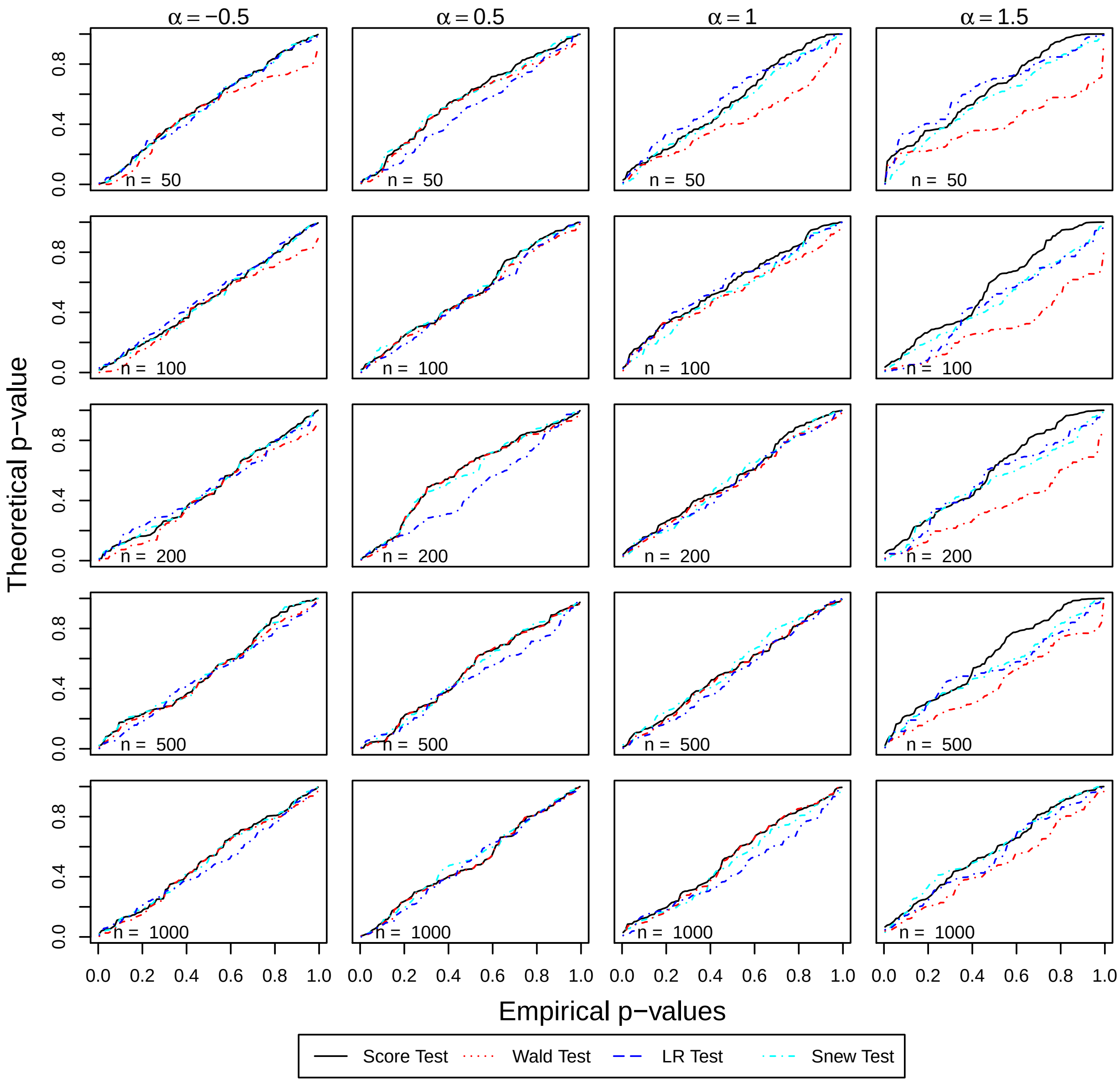

Figure 6: Power plot case3:$\mu = e^{aa-1.45x}, x \sim U(0,1)$;cut=7

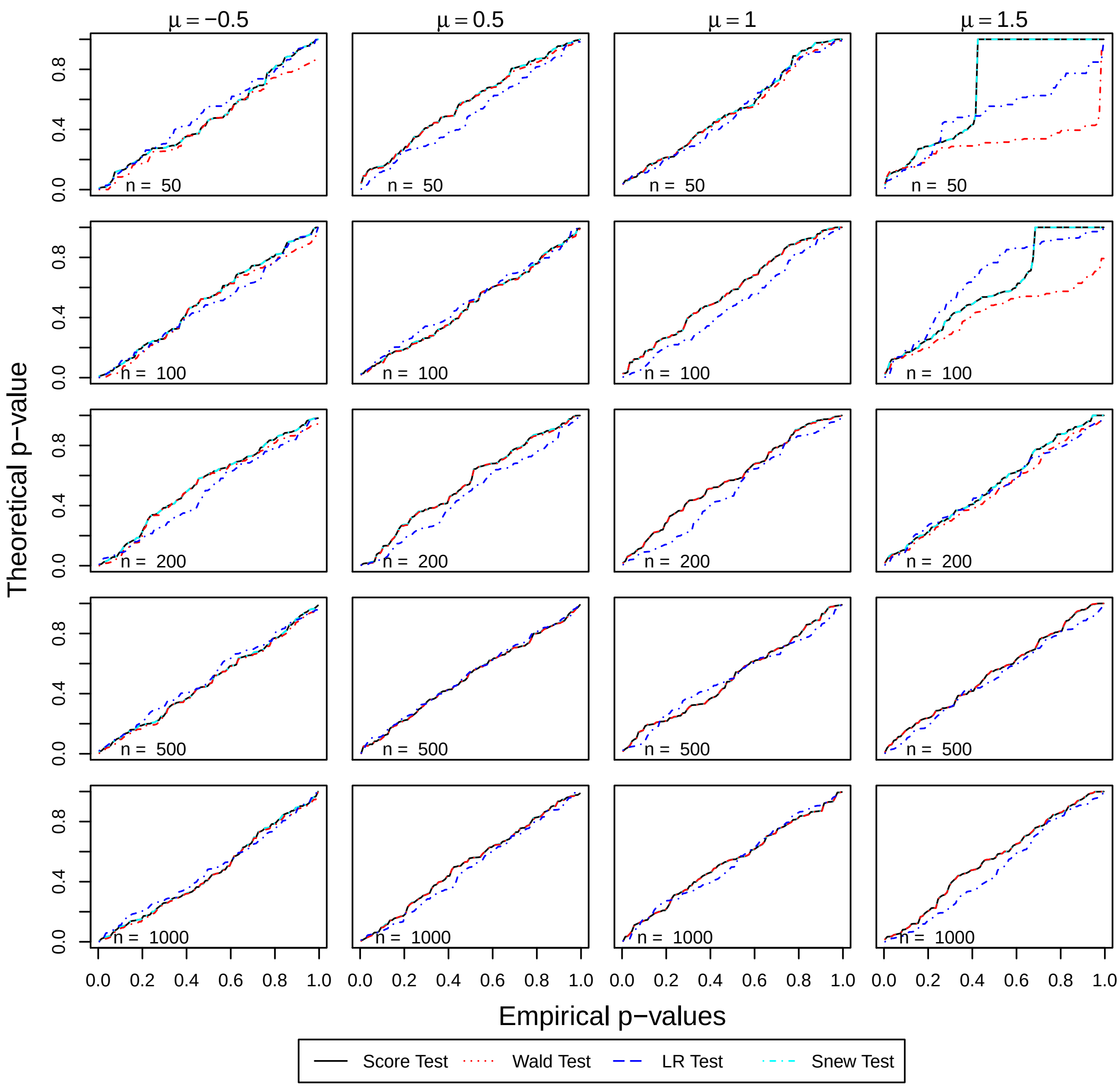

Figure 7: Power plot case1:$\mu = e^{aa}$;cut=30;

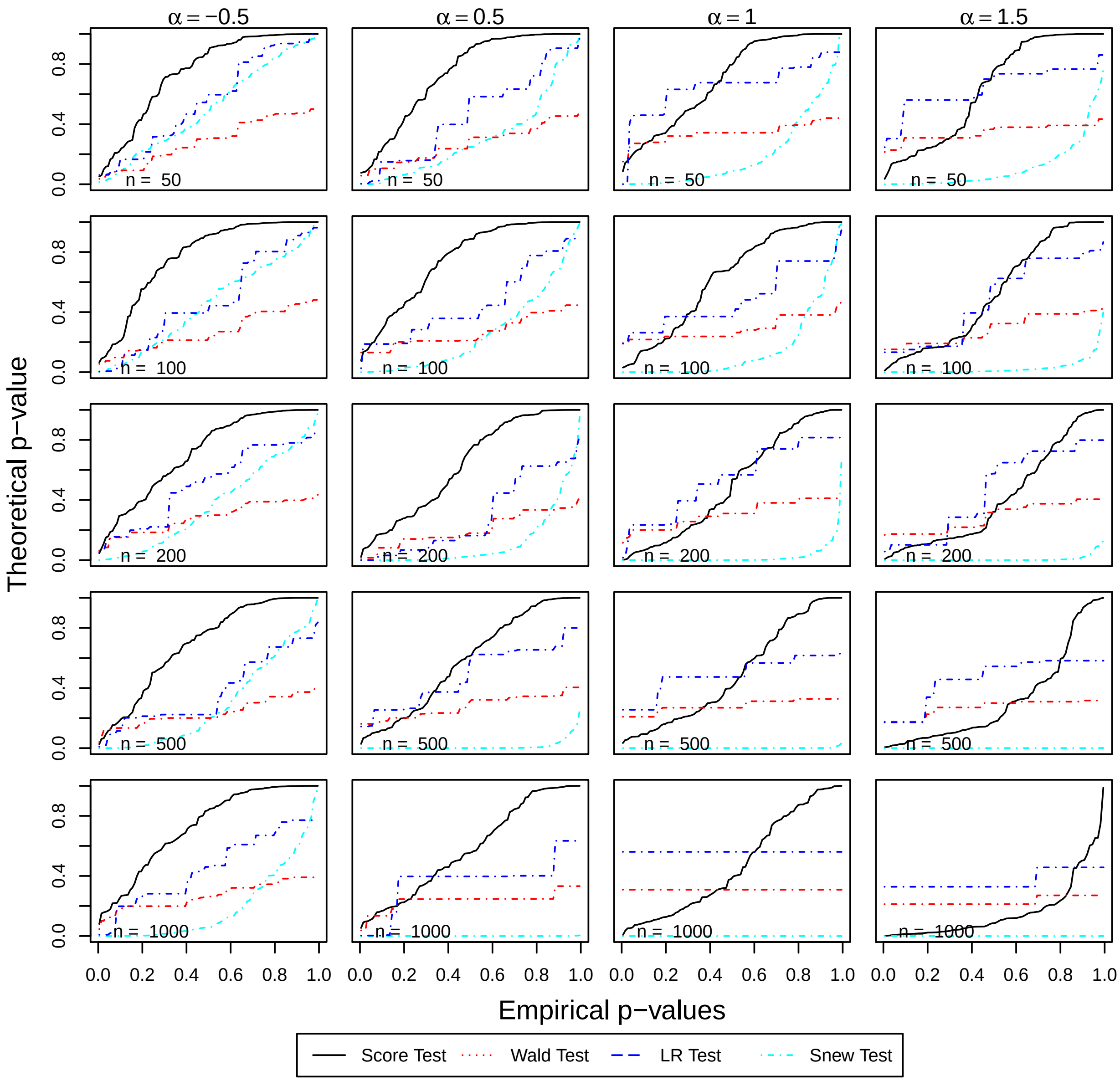

Figure 8: Power plot case2:$\mu = e^{aa-1.45x}, x \sim N(0,1)$;cut=30

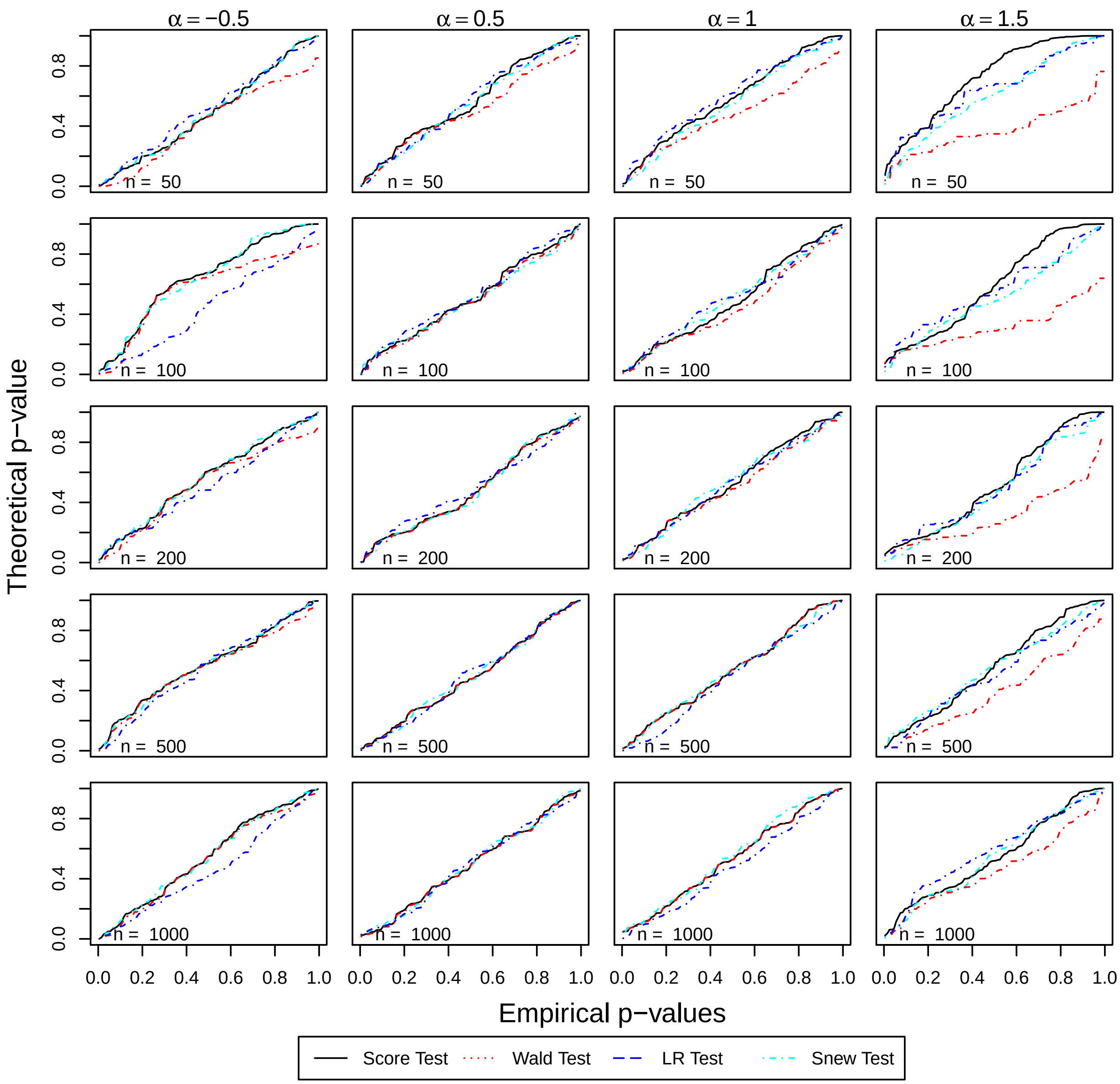

Figure 9: Power plot case3:$\mu = e^{aa-1.45x}, x \sim U(0,1)$;cut=30

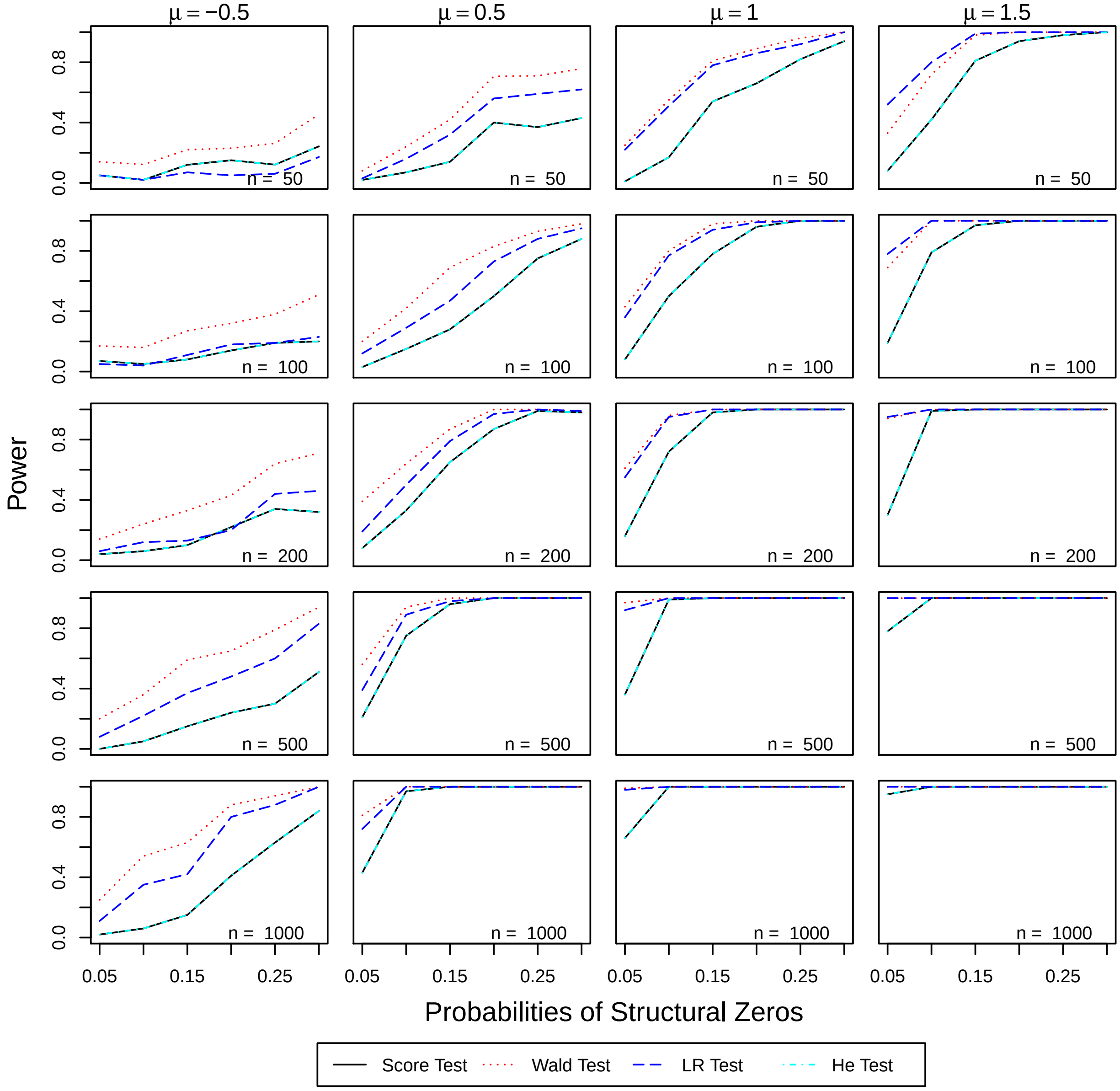

Figure 10: Power plot case1:$\mu = e^{aa}, \omega = lp$;cut=4;

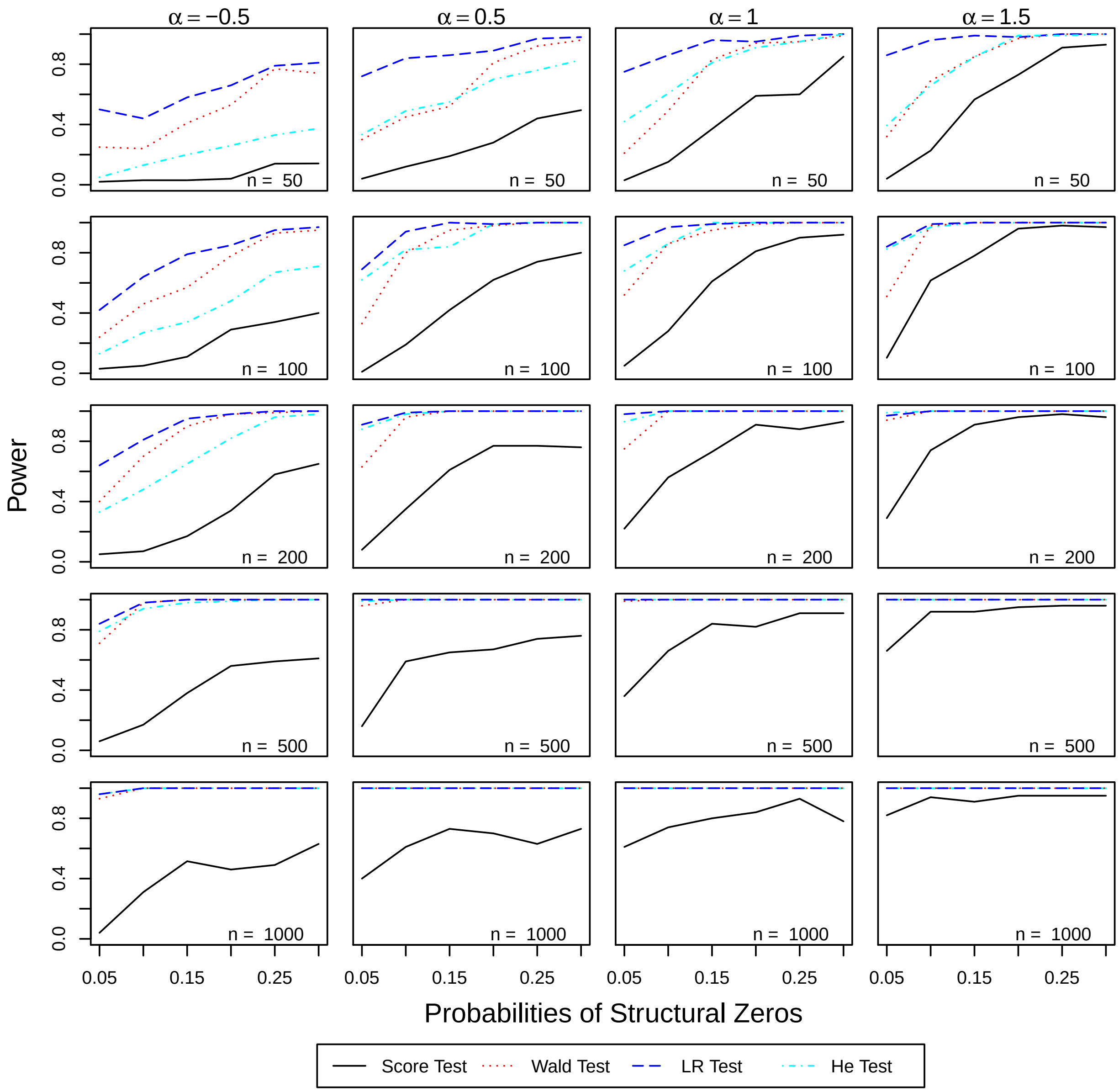

Figure 11: Power plot case2:$\mu = e^{aa-1.45x}, \omega = lp, x \sim N(0,1)$;cut=4

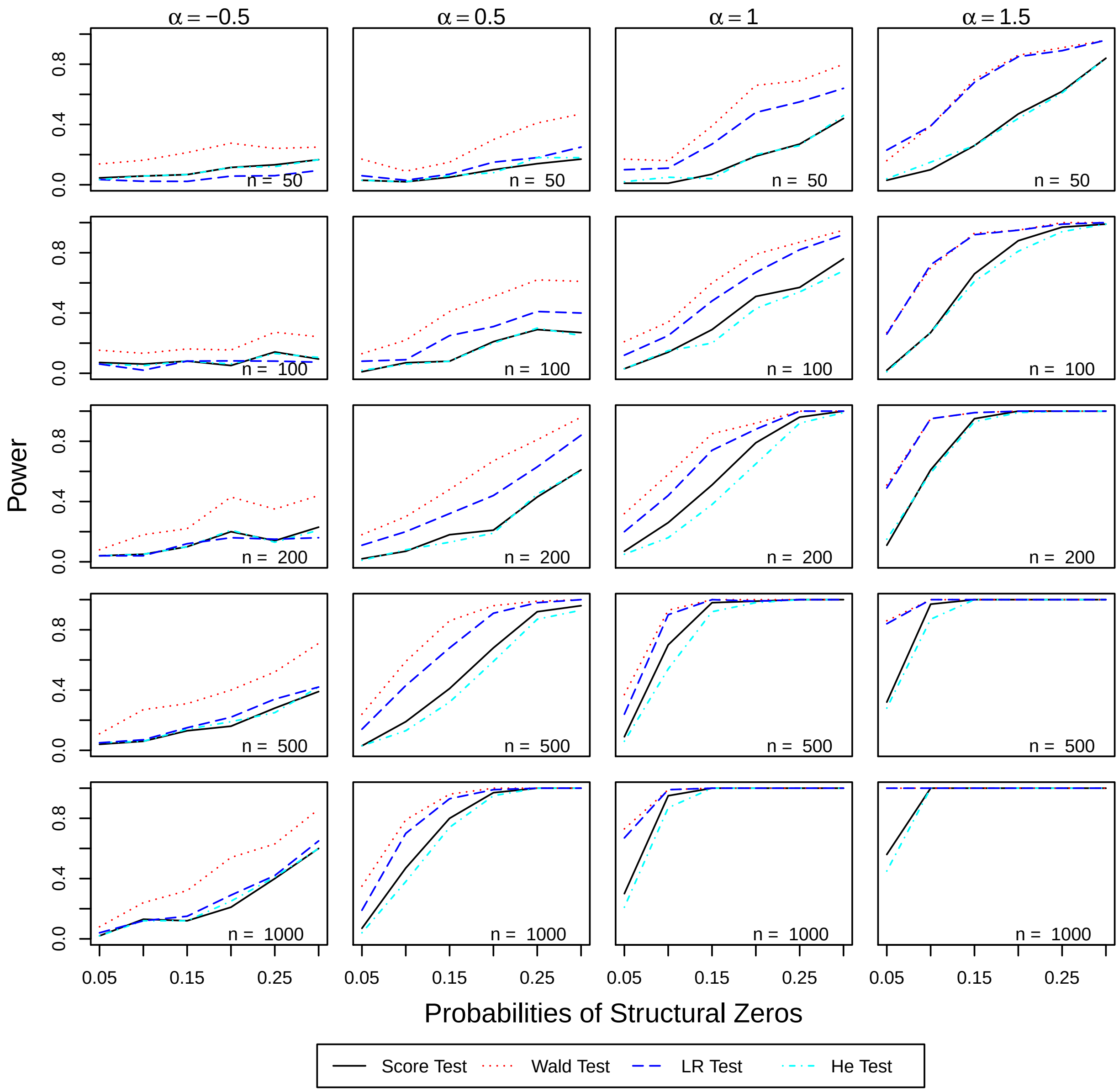

Figure 12: Power plot case3:$\mu = e^{aa-1.45x}, \omega = lp, x \sim U(0,1)$;cut=4

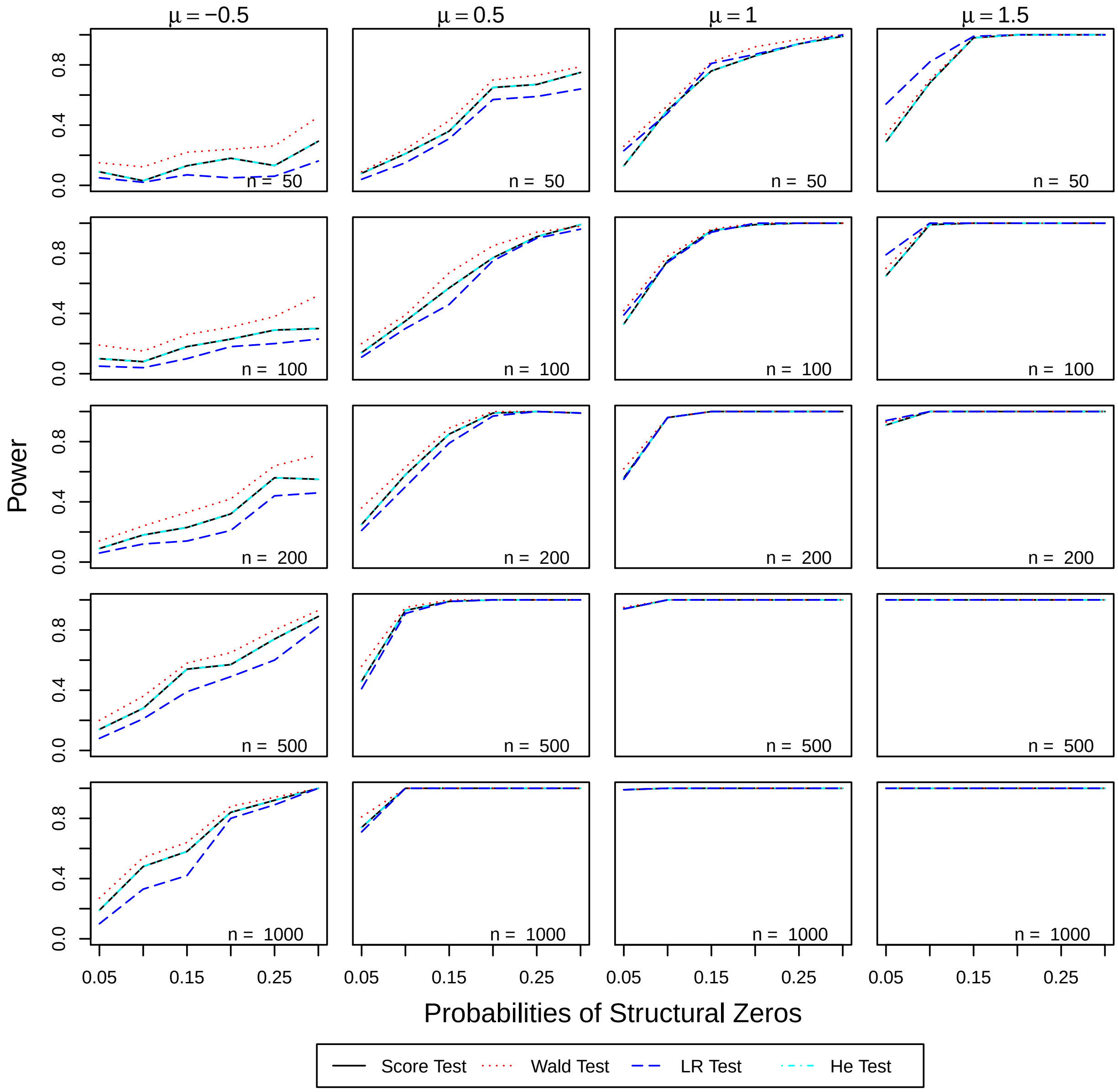

Figure 13: Power plot case1:$\mu = e^{aa}, \omega = lp$;cut=7;

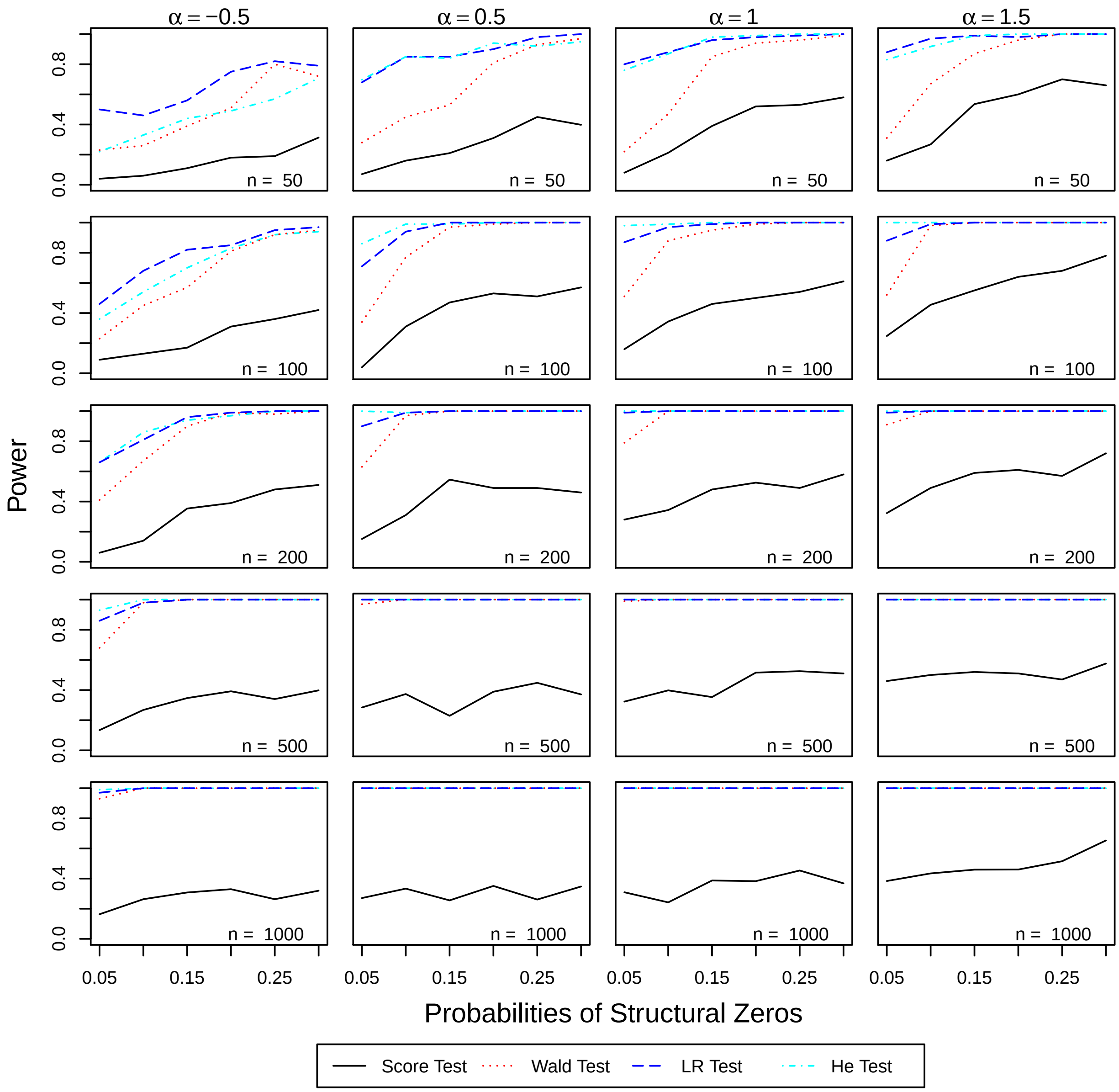

Figure 14: Power plot case2:$\mu = e^{aa-1.45x}, \omega = lp, x \sim N(0,1)$;cut=7

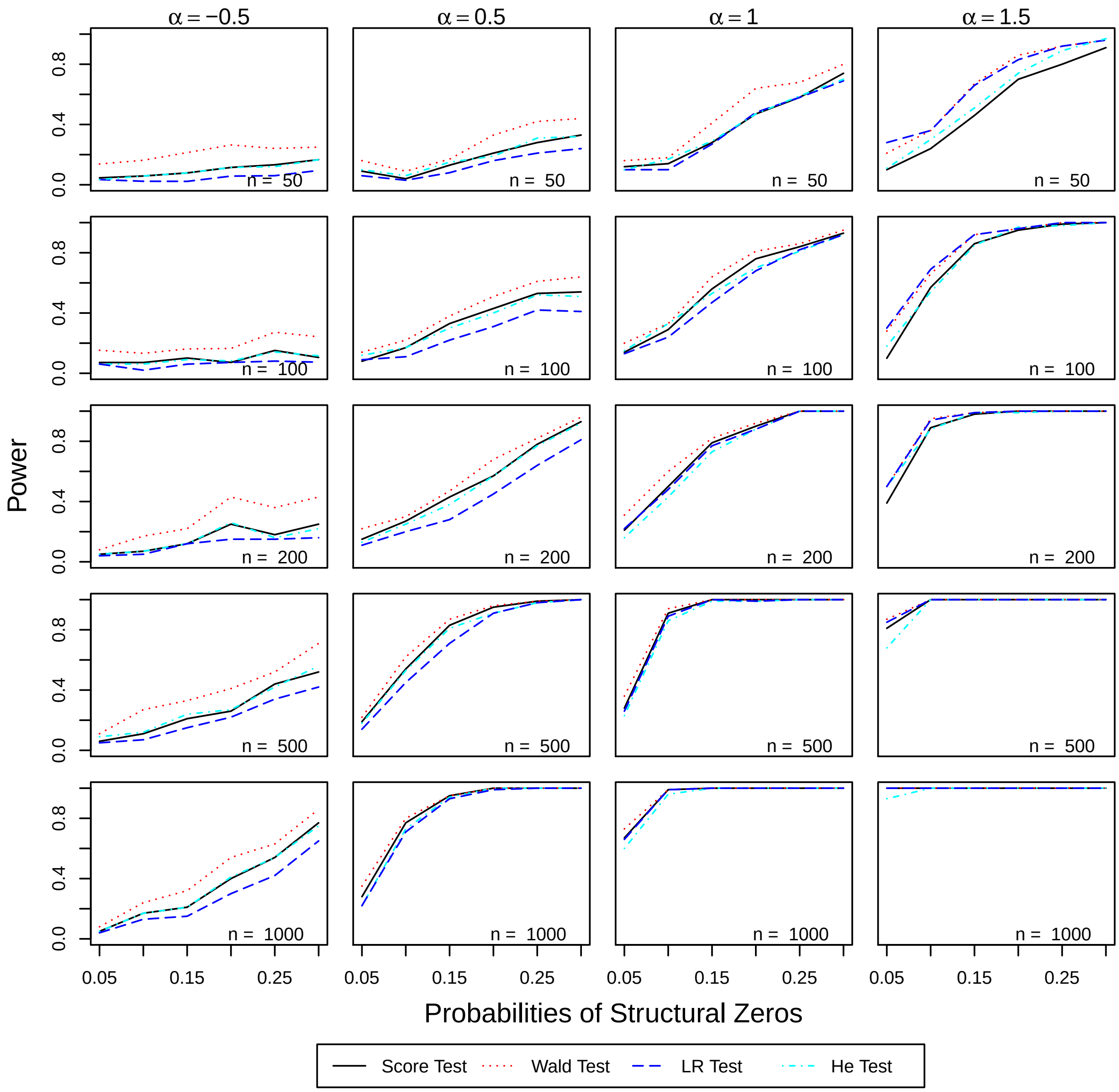

Figure 15: Power plot case3:$\mu = e^{aa-1.45x}, \omega = lp, x \sim U(0,1)$;cut=7

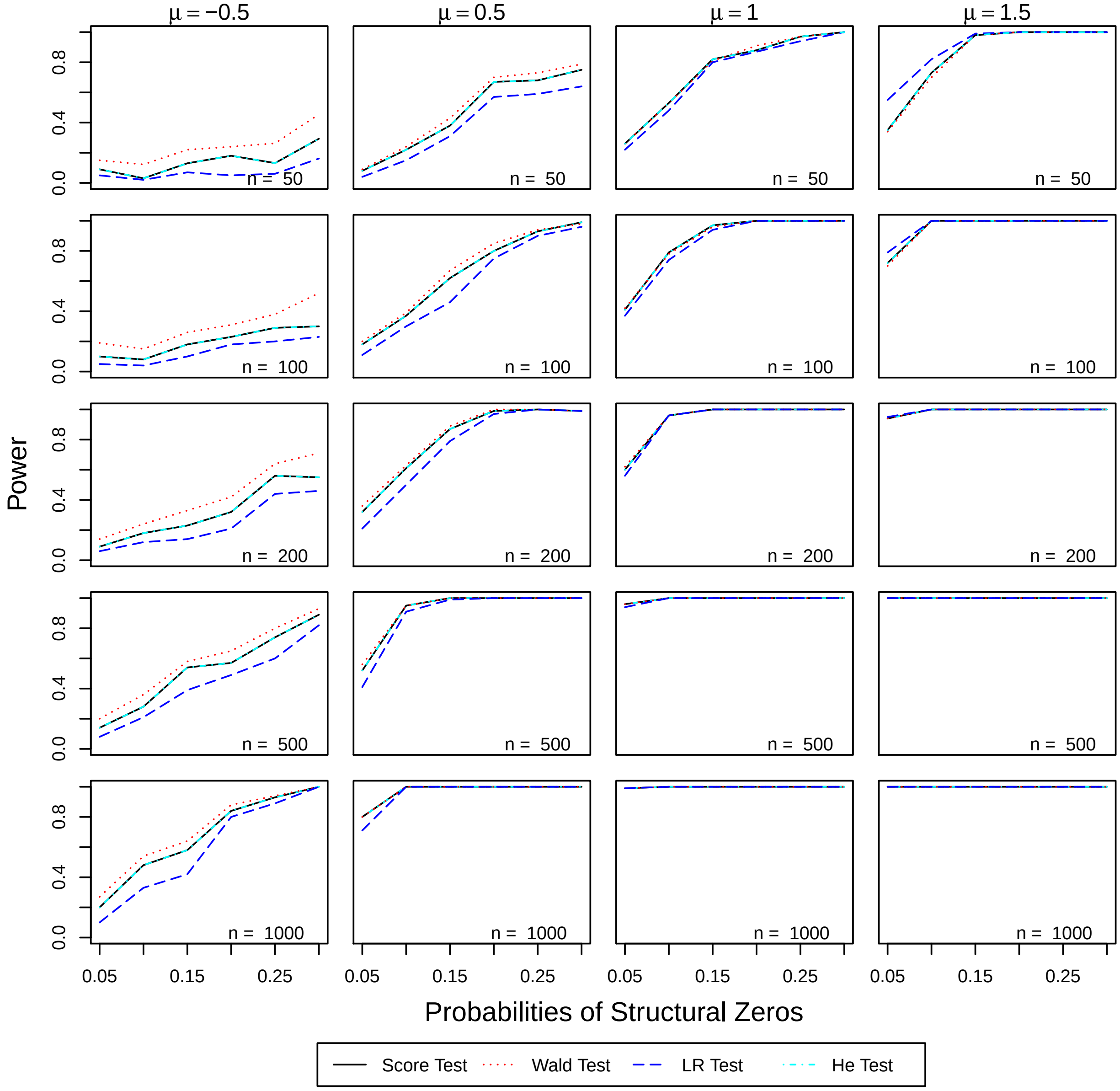

Figure 16: Power plot case1:$\mu = e^{aa}, \omega = lp$;cut=30;

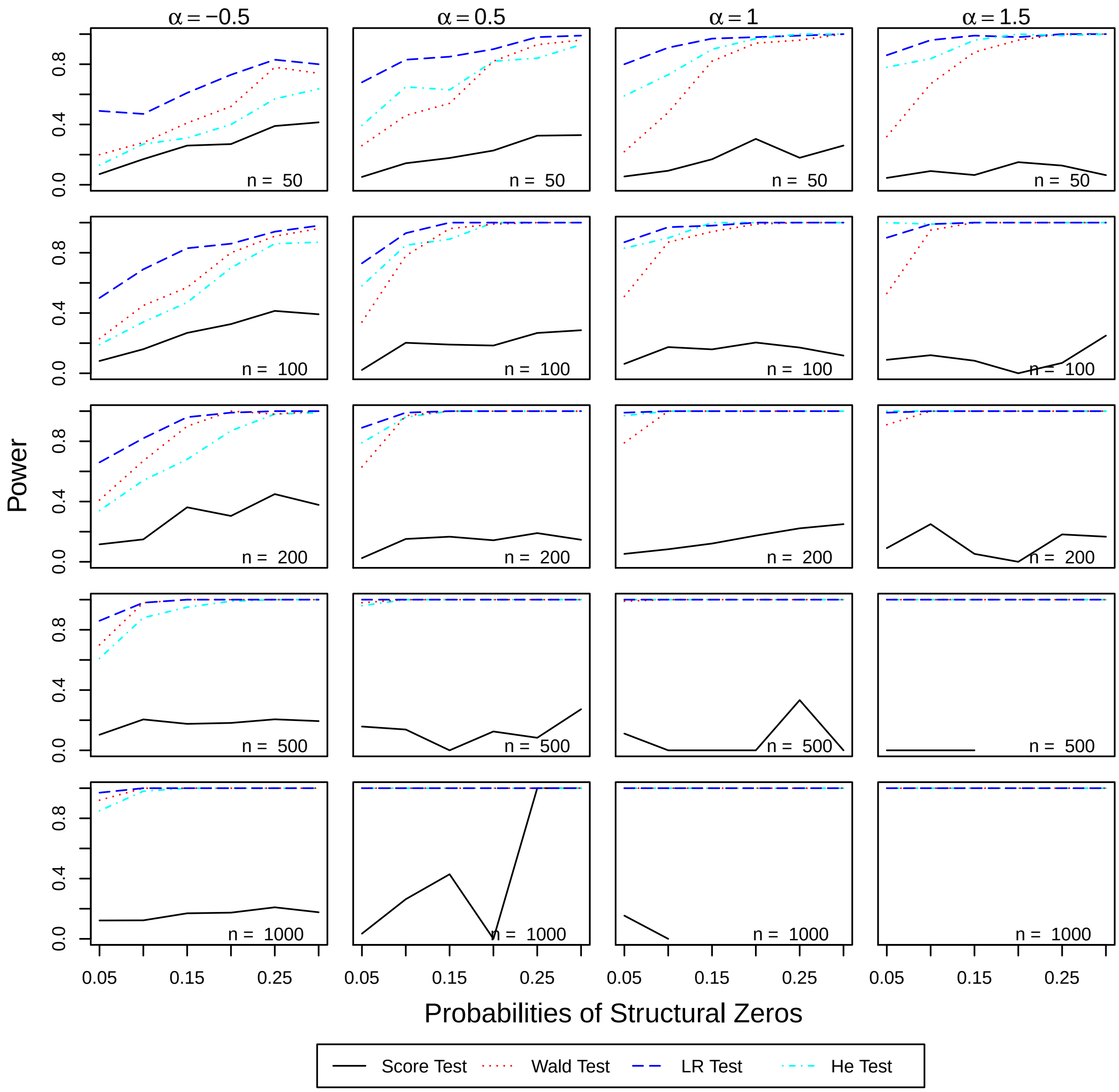

Figure 17: Power plot case2:$\mu = e^{aa-1.45x}, \omega = lp, x \sim N(0,1)$;cut=30

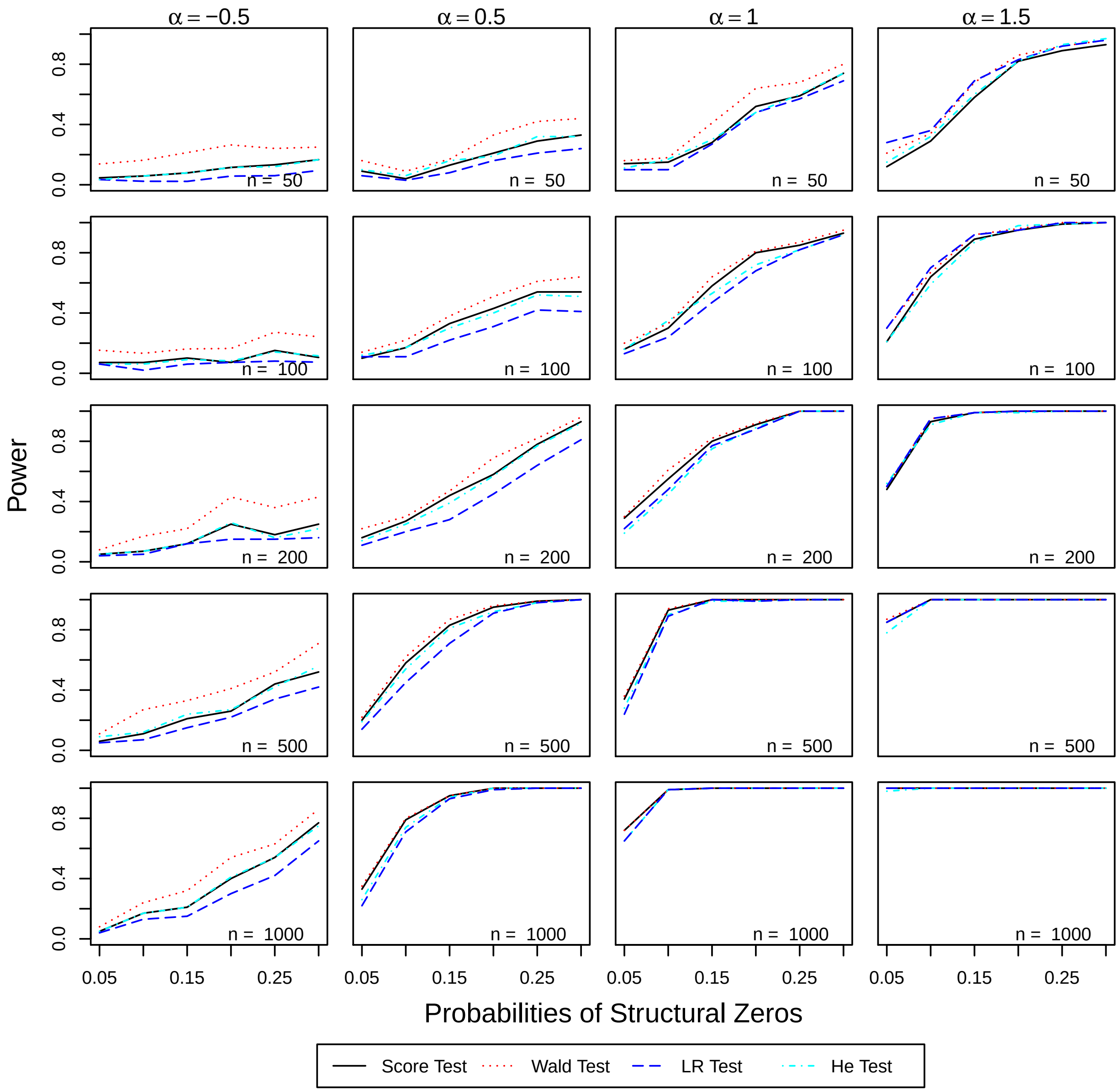

Figure 18: Power plot case3:$\mu = e^{aa-1.45x}, \omega = lp, x \sim U(0,1)$;cut=30