

# Optimizing Global Injectivity for Constrained Parameterization

XINGYI DU, Washington University in St. Louis, USA

DANNY M. KAUFMAN, Adobe Research, USA

QINGNAN ZHOU, Adobe Research, USA

SHAHAR Z. KOVALSKY, Duke University, USA

YAJIE YAN, Facebook, USA

NOAM AIGERMAN, Adobe Research, USA

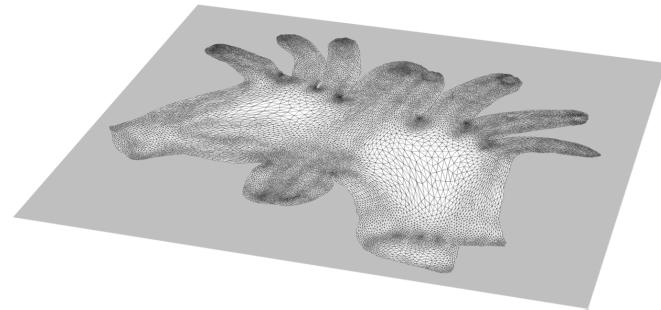
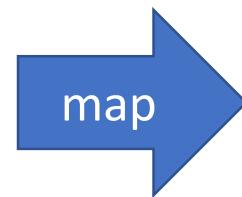
TAO JU, Washington University in St. Louis, USA



# Mapping Meshes to Plane

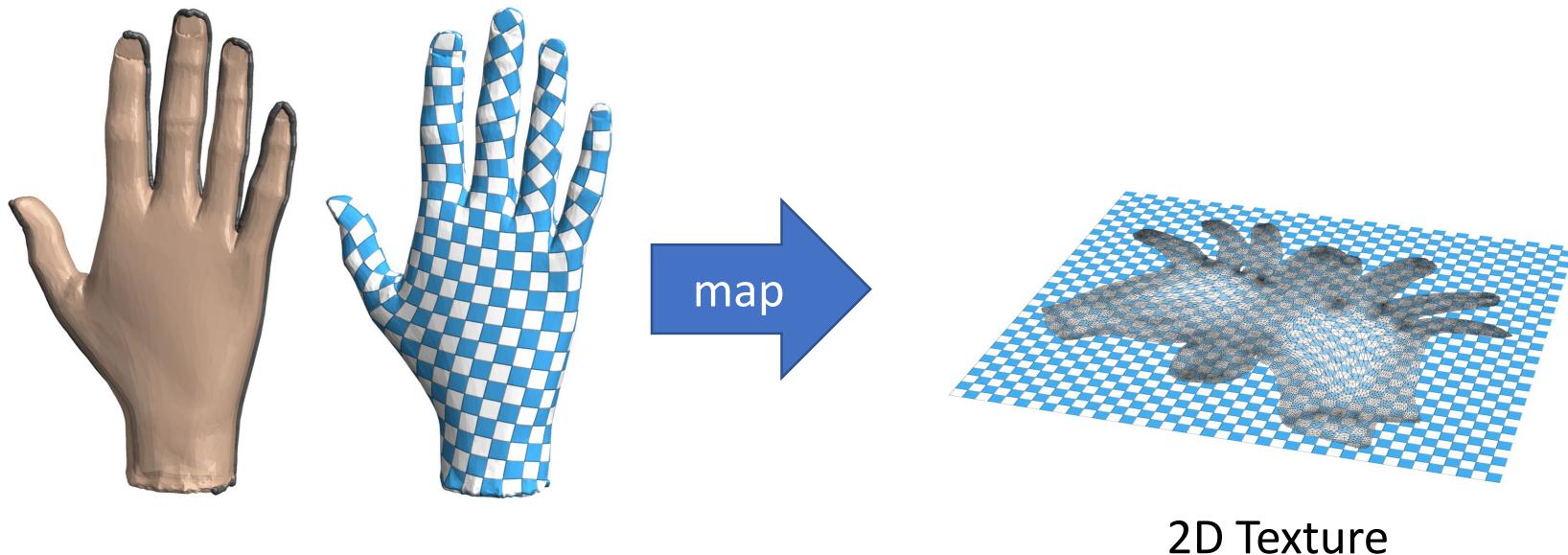


triangle mesh



2D plane

# Application: Texture Mapping

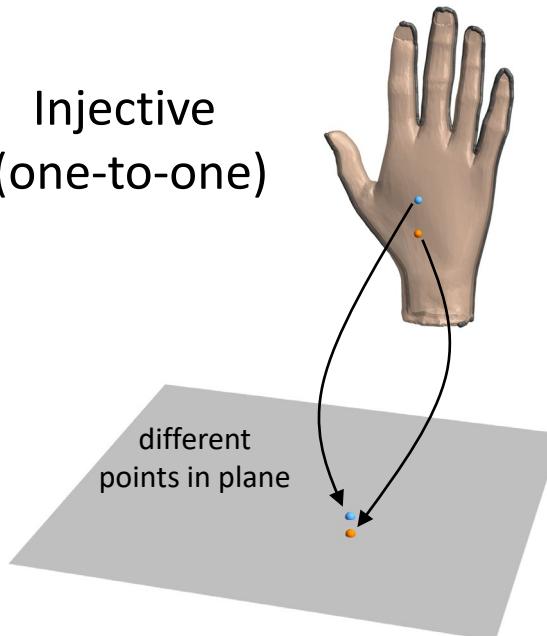




# Injective (one-to-one) Mapping



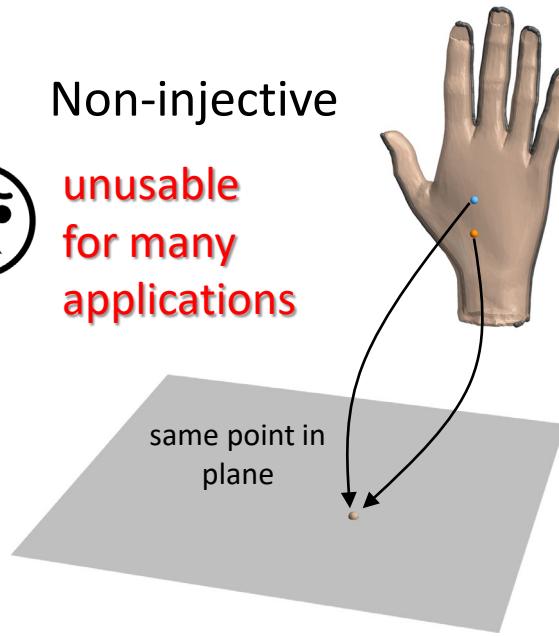
Injective  
(one-to-one)



Non-injective

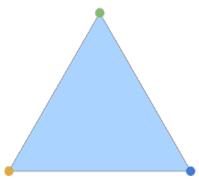


usable  
for many  
applications

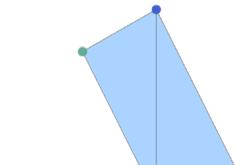


# Injectivity Criteria

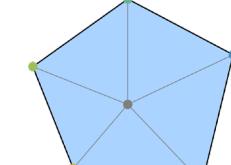
locally injective



✗ degenerate triangle

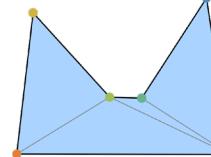


✗ inconsistent orientation



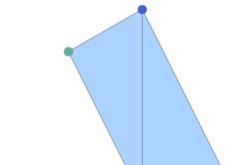
✗ overwound vertex

globally injective



✗ overlapping triangles

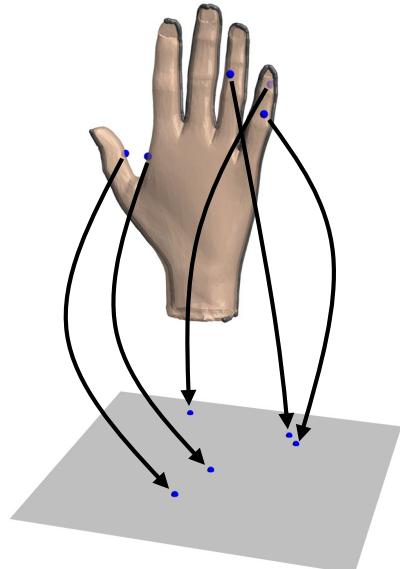
globally injective  
(inversion-free)



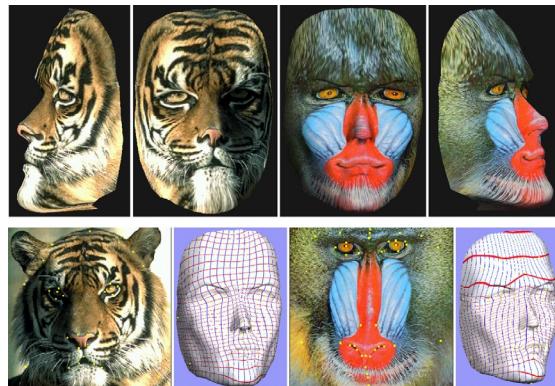
✗ entirely inverted



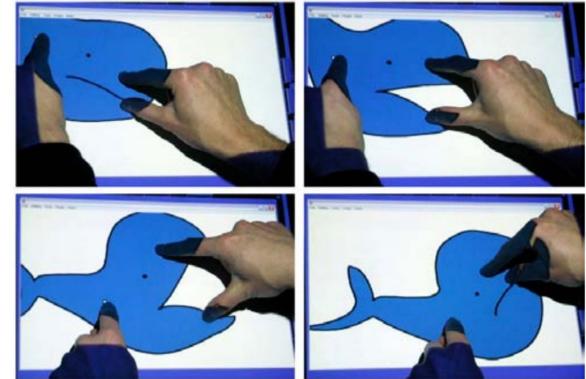
# Mapping under Positional Constraints



positional constraints

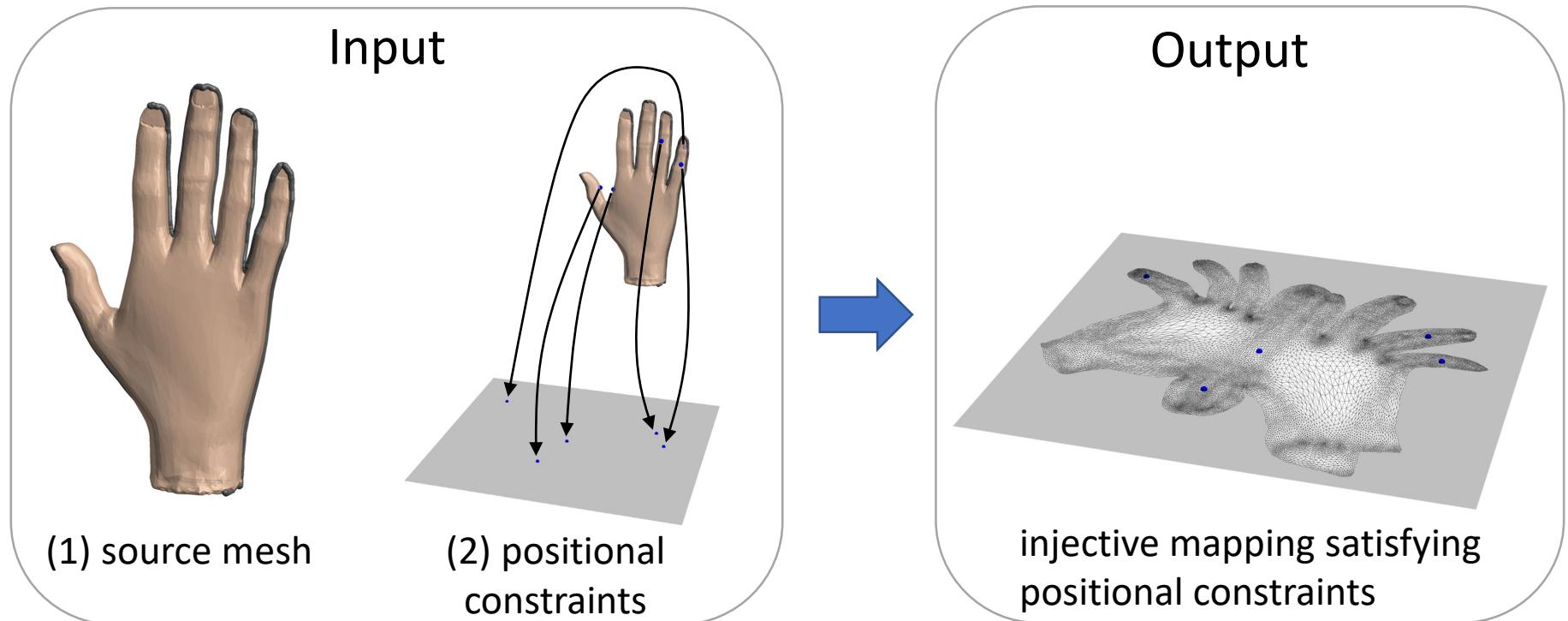


constrained texture mapping  
[Lévy 2001]



handle-based deformation  
[Igarashi et al. 2005]

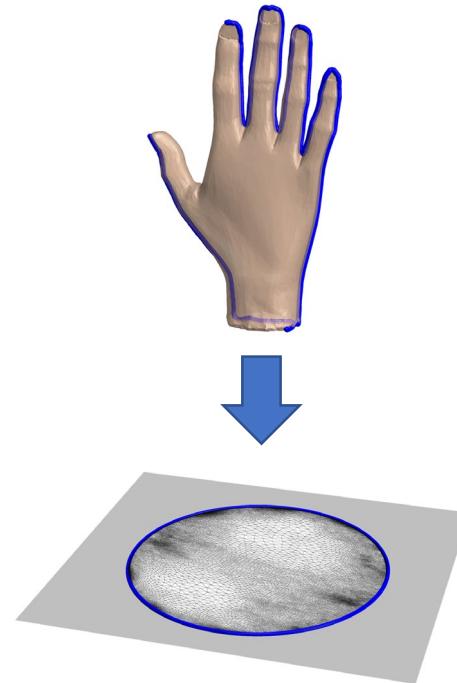
# Goal: Injective Mapping under Positional Constraints





# Injective Mapping

- Tutte embedding [Tutte 1963]
  - Convex boundary and no positional constraints

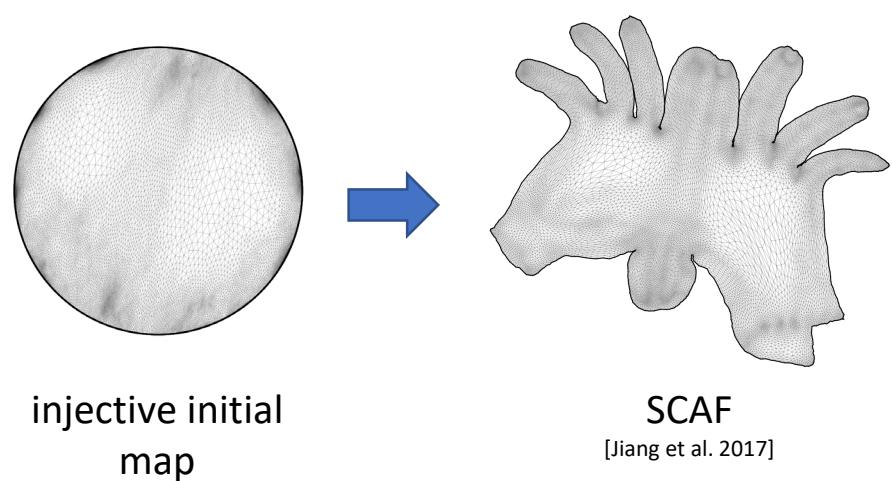


Tutte embedding  
[Tutte 1963]



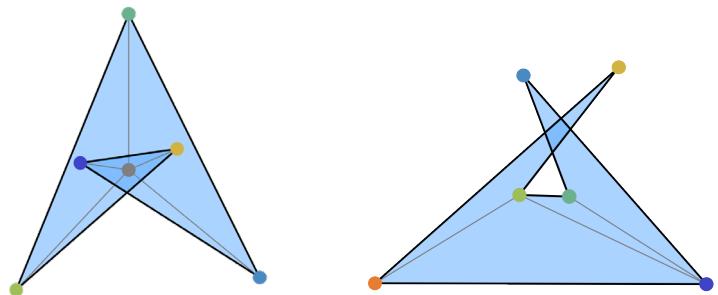
# Injective Mapping

- Tutte embedding [Tutte 1963]
  - Convex boundary and no positional constraints
- Maintenance methods
  - [Hormann and Greiner 2000], [Schüller et al. 2013], [Smith and Schaefer 2015], AMIPS [Fu et al. 2015], [Liu et al. 2016], SLIM [Rabinovich et al. 2017], CM [Shtengel et al. 2017], [Claici et al. 2017], SCAF [Jiang et al. 2017], BCQN [Zhu et al. 2018], [Liu et al. 2018], [Su et al. 2020], IDP [Fang et al. 2021]
  - Require injective maps to initialize



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  - Require injective maps to initialize
- Inversion-free methods
  - [Aigerman and Lipman 2013], LBD [Kovalsky et al. 2015], SA [Fu and Liu 2016], FF [Su et al. 2019], [Hefetz et al. 2019], TLC [Du et al. 2020]
  - Cannot avoid overlapping triangles

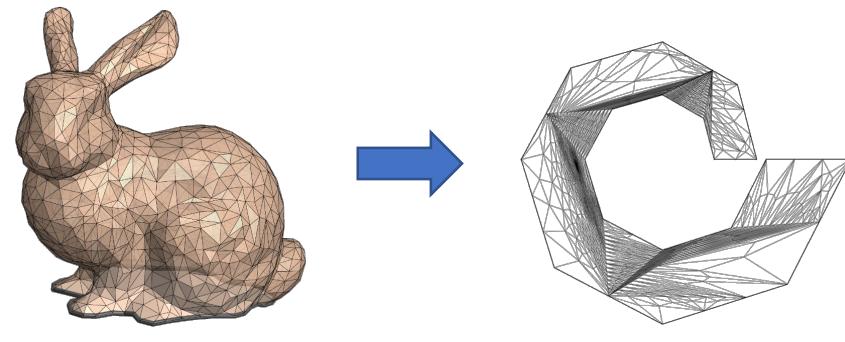


overlapping



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  - Require injective maps to initialize
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  - [Aigerman and Lipman 2013], LBD [Kovalsky et al. 2015], SA [Fu and Liu 2016], FF [Su et al. 2019], [Hefetz et al. 2019], TLC [Du et al. 2020]
  - Cannot avoid overlapping triangles
- Remeshing methods
  - [Eckstein et al. 2001], Matchmaker [Kraevoy et al. 2003], [Lee et al. 2008], [Agarwal et al. 2008], [Weber and Zorin 2014], [Gu et al. 2018], [Shen et al. 2019]
  - May change the mesh structure

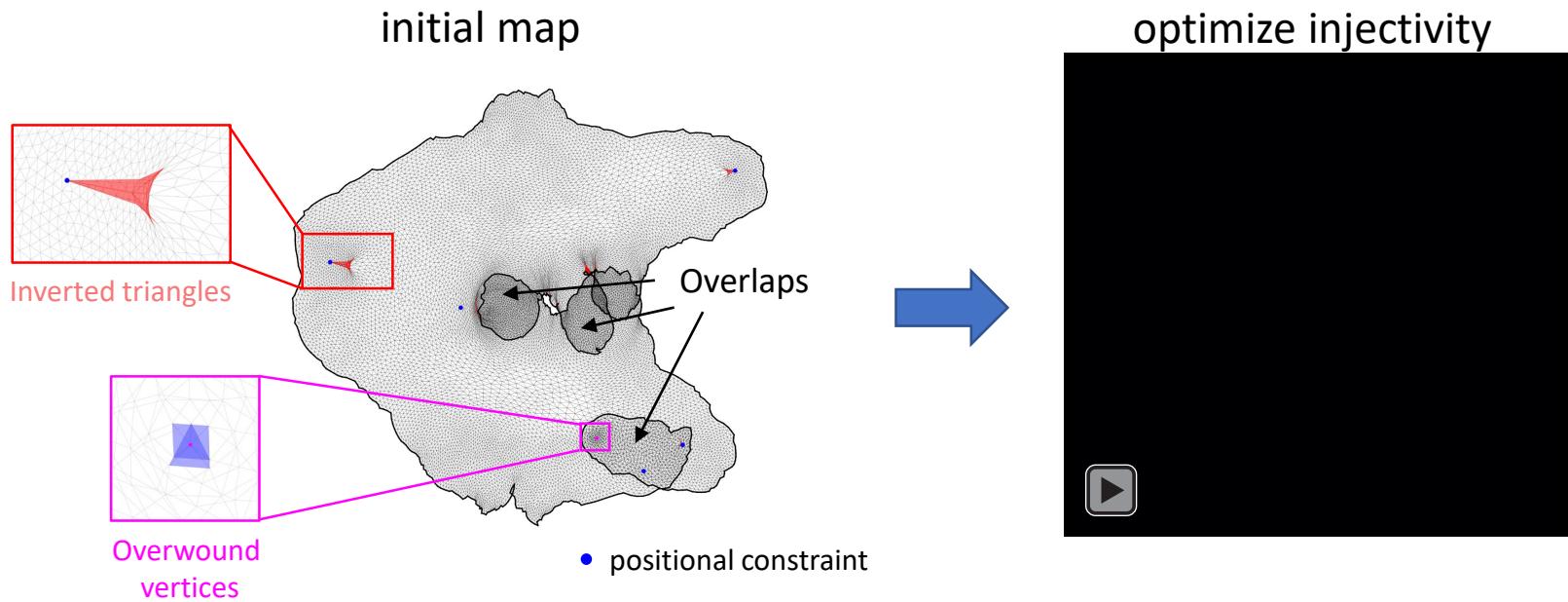


Progressive Embedding

[Shen et al. 2019]

# Contribution

- First method for computing globally injective maps under positional constraints (without changing the mesh structure)

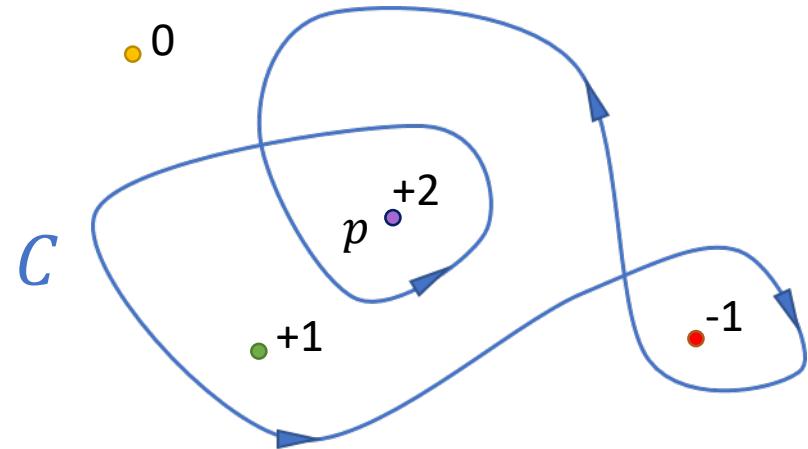


# Contribution

- New energy for promoting injectivity
  - Captures both inverted and overlapping areas
  - Smooth almost everywhere
  - Theoretical guarantees of injectivity at global minima

# Background: Winding Number

- Number of times a curve  $C$  travels CCW around a point  $p$

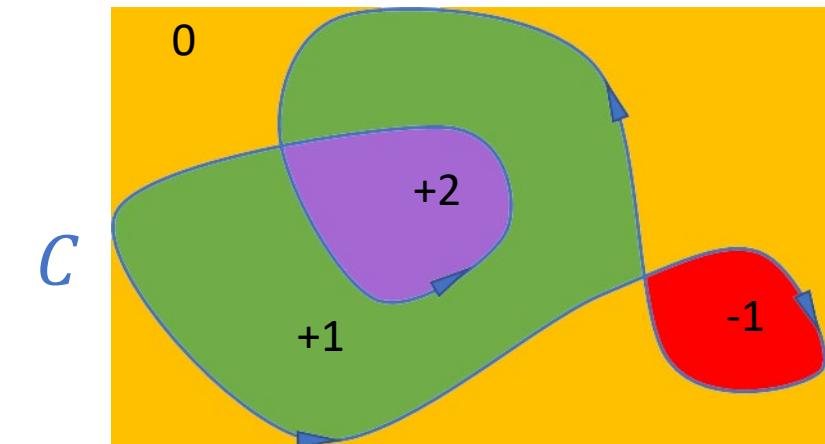




# Background: Winding Number



- Number of times a curve  $C$  travels CCW around a point  $p$

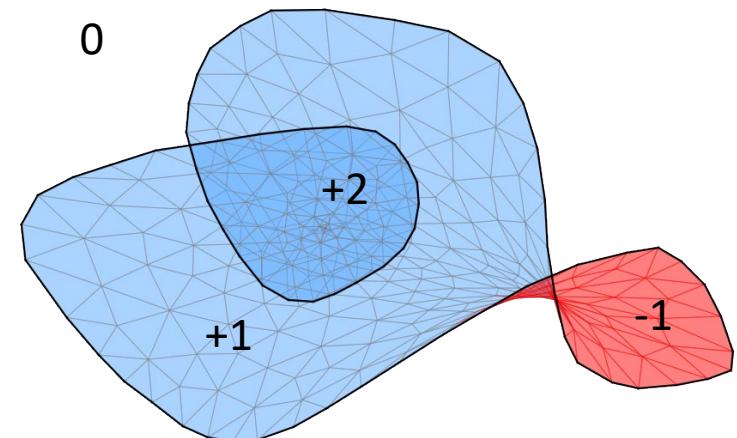




# Background: Winding Number

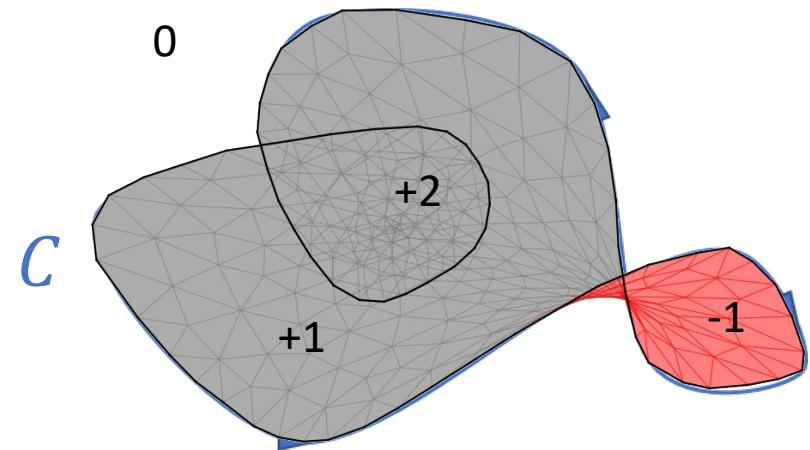
- Consider a mesh bounded by  $C$
- Winding number is the number of non-inverted triangles minus number of inverted triangles covering a point

$C$



# Occupancy

- The total area with positive winding numbers
- The area covered by at least one non-inverted triangle for any mesh bounded by  $C$



# Excess Area

- Mesh  $T$  with boundary  $\partial T$

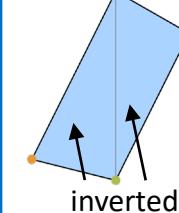
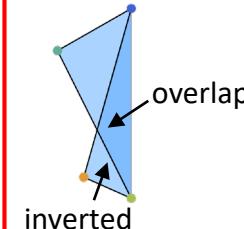
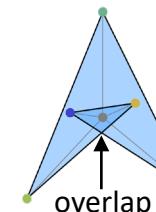
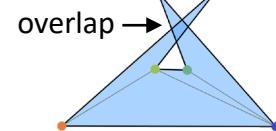
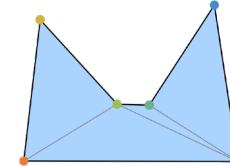
$$A_{excess}(T) = A_{unsigned}(T) - O(\partial T)$$

excess area      total unsigned area      occupancy

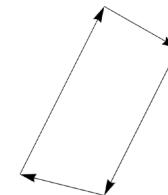
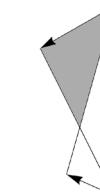
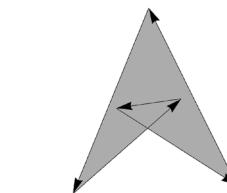
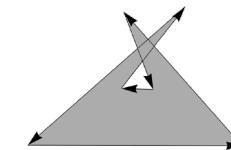
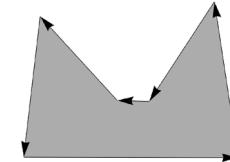
- Properties
  - $\max(A_{overlap}, A_{inverted}) \leq A_{excess} \leq A_{overlap} + A_{inverted}$
  - Zero if and only if  $T$  has no overlapping or inverted triangles

# Excess Area

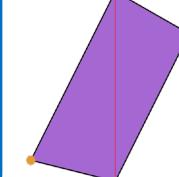
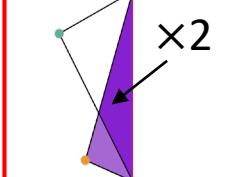
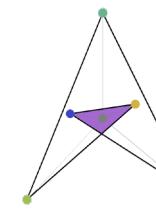
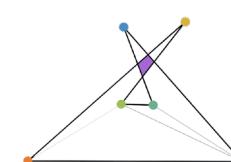
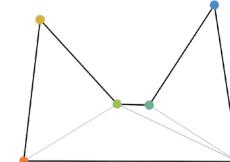
Total  
unsigned area



Occupancy



Excess Area



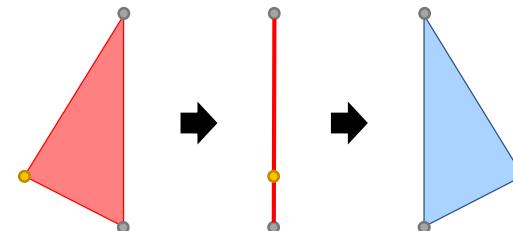
# Non-smoothness



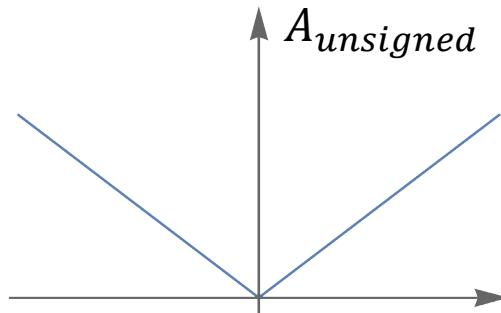
excess area      total unsigned area      occupancy

$$A_{excess}(T) = A_{unsigned}(T) - O(\partial T)$$

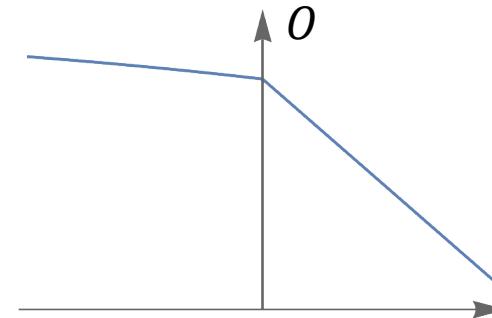
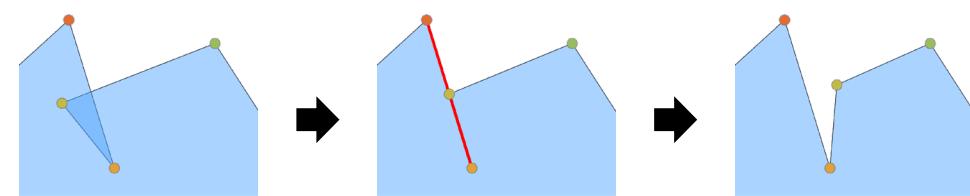
degenerate triangle



$A_{unsigned}$

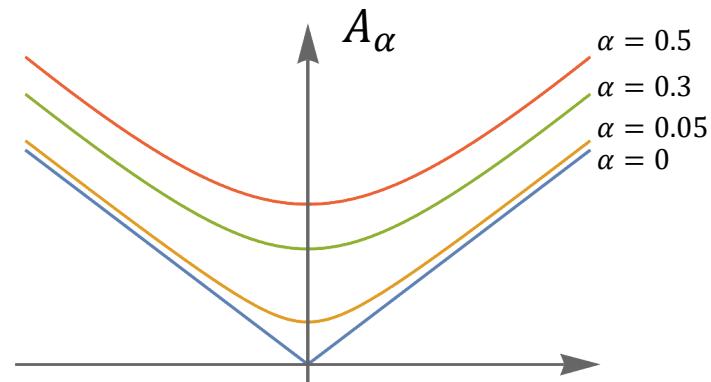
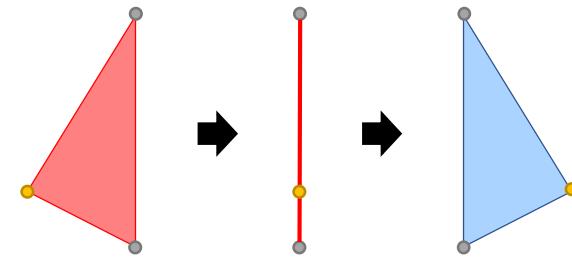
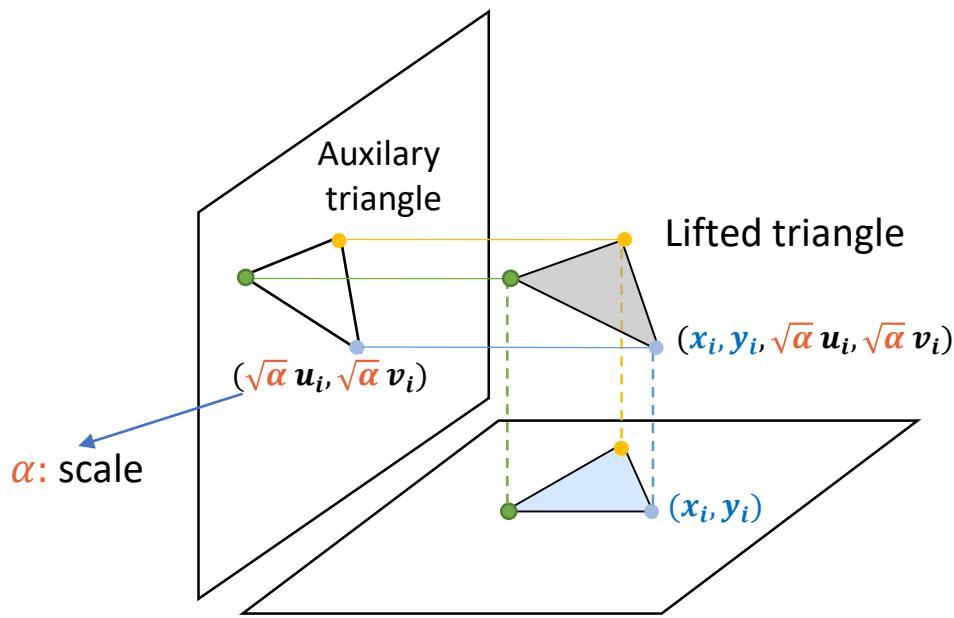


coinciding boundary



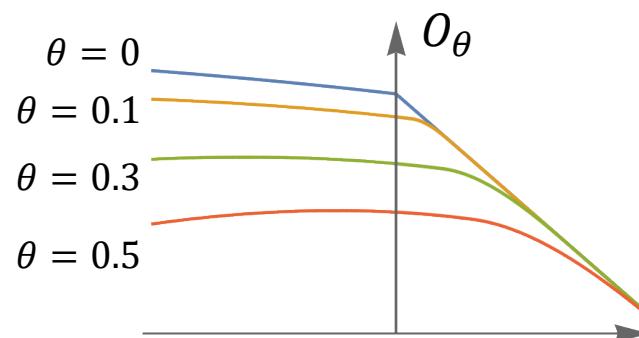
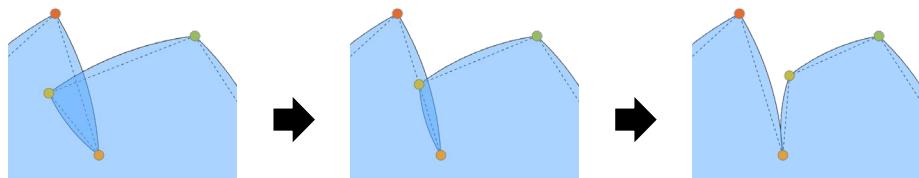
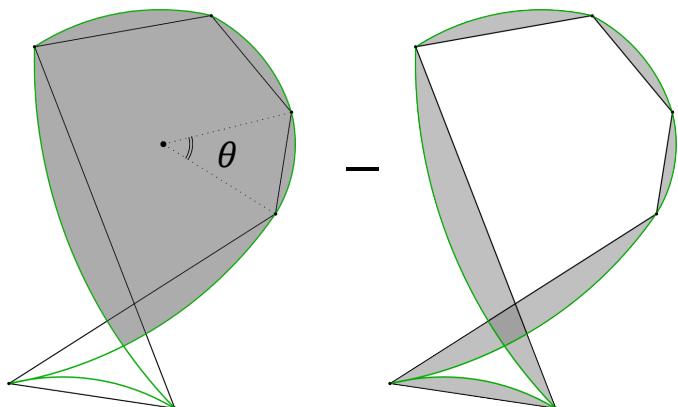
# Smoothing Unsigned Area

- Total lifted Content (TLC)  
• Area of lifted triangles



# Smoothing Occupancy

- Arc Occupancy



# Smooth Excess Area (SEA)

- Mesh  $T$  with boundary  $\partial T$

$$E_{\alpha,\theta}(T) = \boxed{\text{smooth excess area}} - \boxed{\text{total lifted content}} - \boxed{\text{arc occupancy}}$$
$$E_{\alpha,\theta}(T) = A_\alpha(T) - O_\theta(\partial T)$$

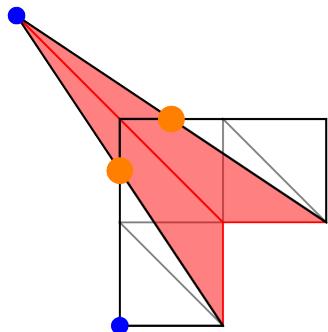
- Properties
  - For any  $\alpha > 0, \theta > 0$ ,  $\text{SEA} \geq \max(A_{overlap}, A_{inverted})$
  - For sufficiently small  $\alpha$  and  $\theta$ , the minima of SEA is *locally injective with arbitrarily small overlapping area*

# Optimizing SEA

- Quasi-Newton (L-BFGS)
- Termination Criteria
  - Map is globally injective
    - No inverted/degenerate triangles and no boundary intersection [Lipman 2014]
  - Reaches a max #iterations (e.g., 10 000)

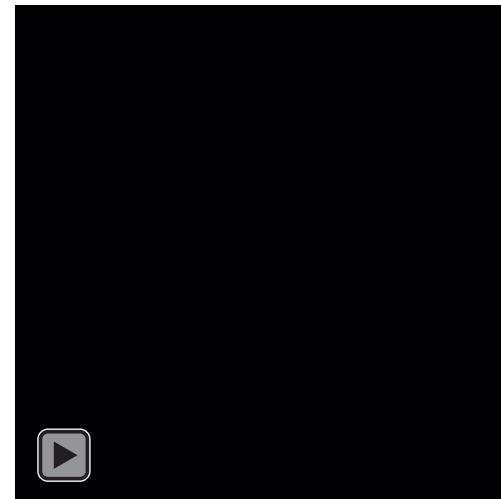
# Handcrafted Examples

initial map

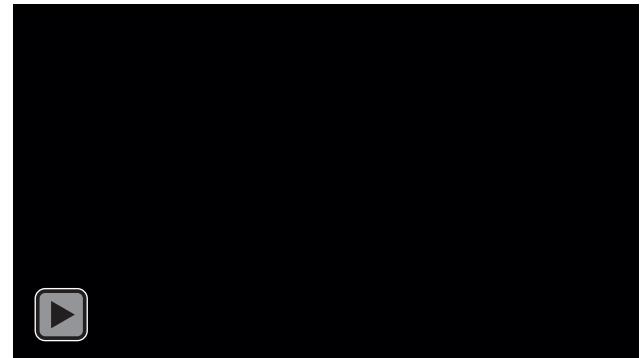


- positional constraint
- boundary intersection
- ▲ inverted triangle

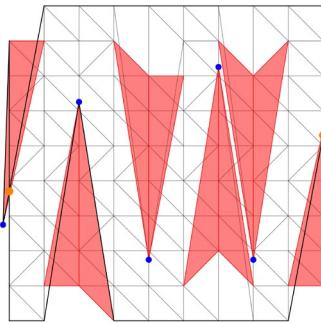
TLC – occupancy  
( $\alpha = 10^{-4}$ )



TLC – arc occupancy  
( $\alpha = 10^{-4}, \theta = 1.0$ )



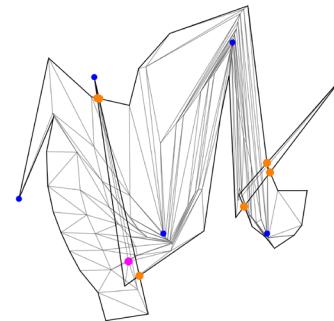
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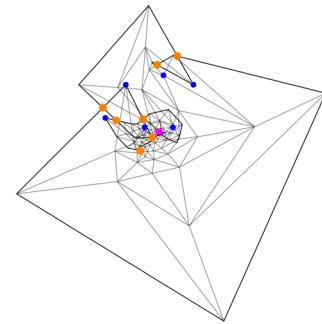
initial  
map



( $\alpha = 10^{-4}, \theta = 0.1$ )



Large-scale Bounded  
Distortion (LBD)  
[Kovalsky et al. 2016]



Simplex  
Assembly (SA)  
[Fu and Liu 2016]

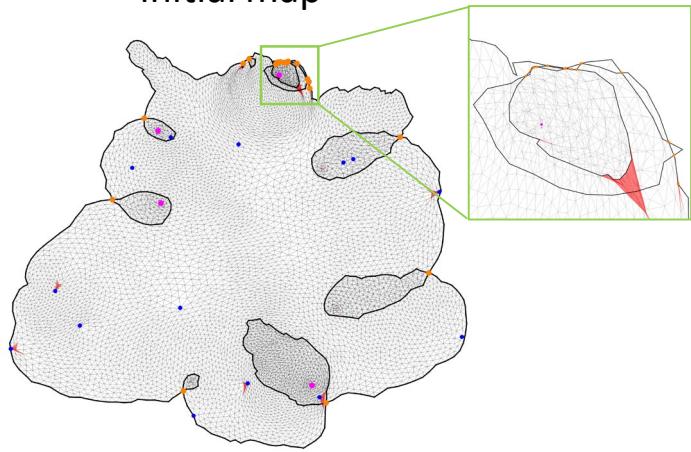
# Benchmark

- 1791 test examples
  - Up to 20 constraints
  - Non-injective initial map by ARAP
- Parameters
  - $\alpha = 10^{-4}, \theta = 0.1$
- Comparison with inversion-free methods
  - LBD [Kovalsky et al. 2016]
  - SA [Fu and Liu 2016]

source mesh



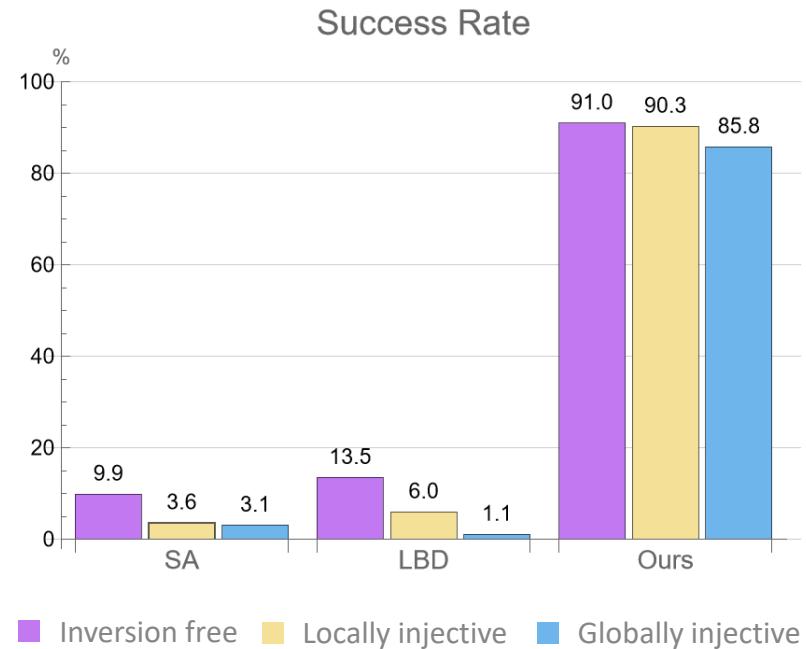
initial map



- positional constraint
- overwound vertex
- boundary intersection
- inverted triangle

# Benchmark

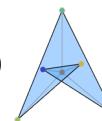
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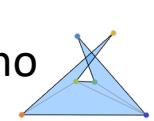
no



no



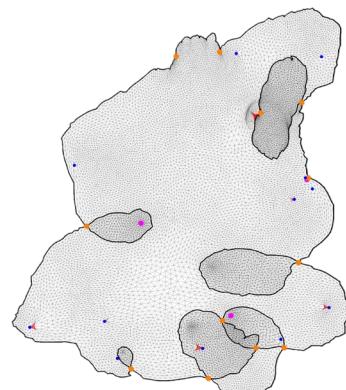
no



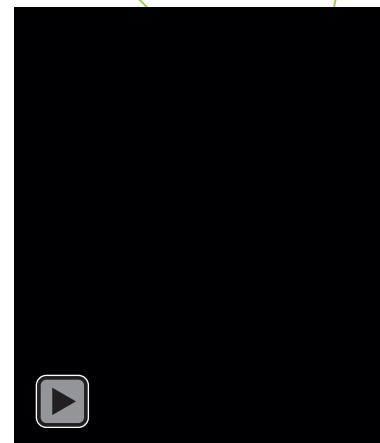
# Benchmark: Success



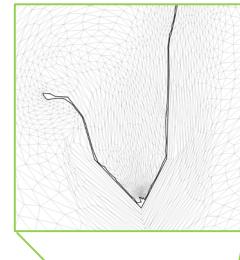
source  
mesh



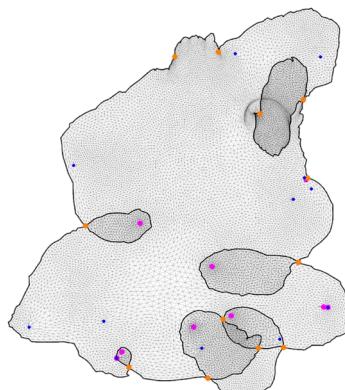
initial  
map



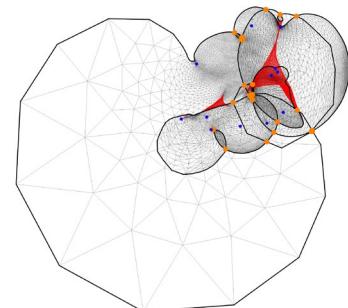
ours



100 inverted triangles  
3 overwound vertices  
12 boundary intersection



LBD

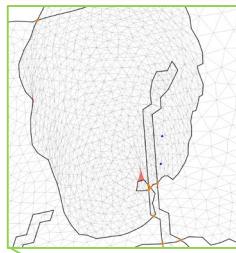


SA

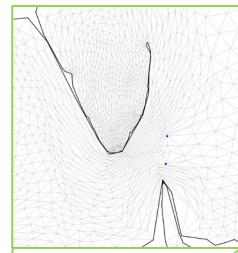
4 inverted triangles  
6 overwound vertices  
13 boundary intersection

2748 inverted triangles  
0 overwound vertices  
29 boundary intersection

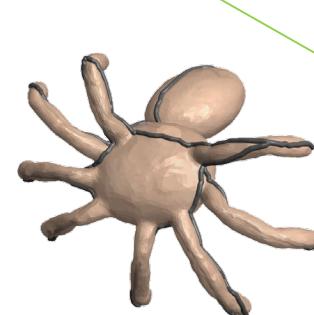
# Benchmark: Success



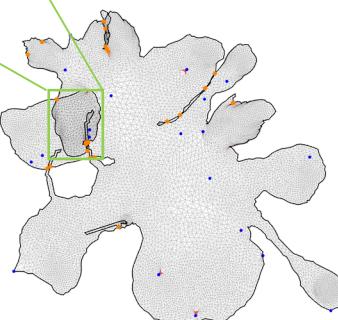
63 inverted triangles  
0 overwound vertices  
25 boundary intersection



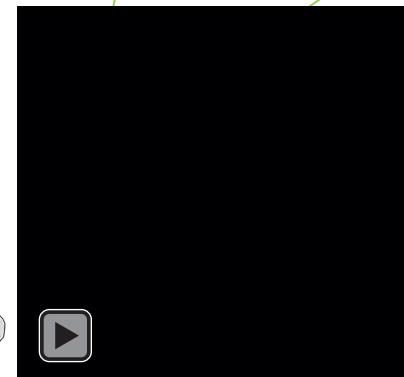
1 inverted triangles  
2 overwound vertices  
22 boundary intersection



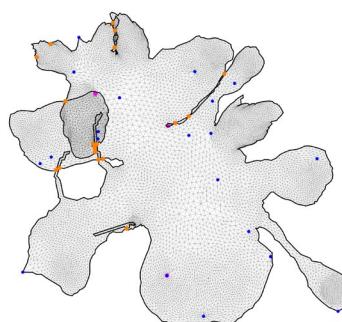
source  
mesh



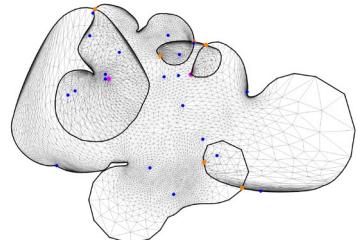
initial  
map



ours

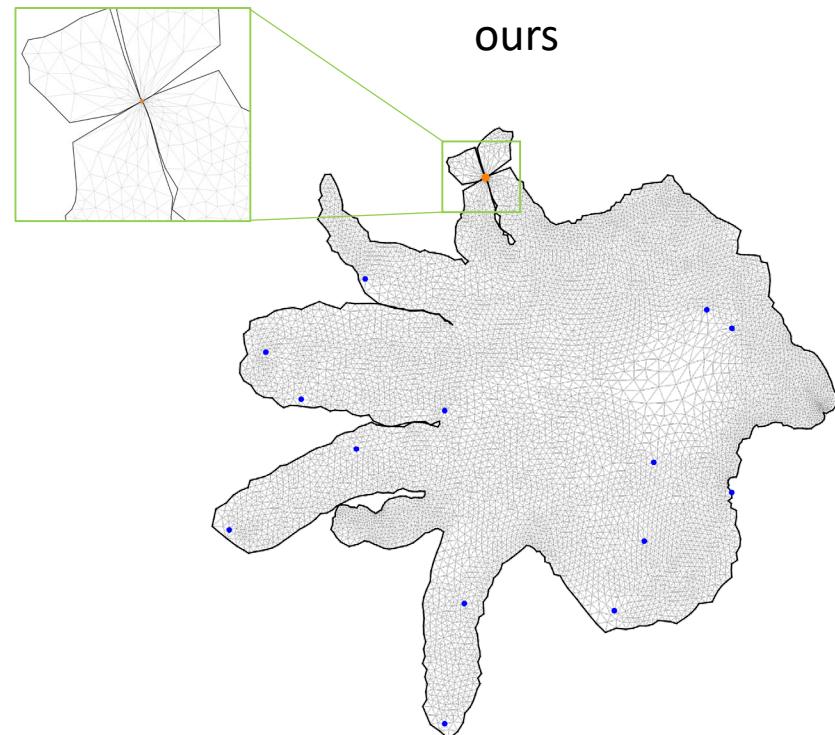
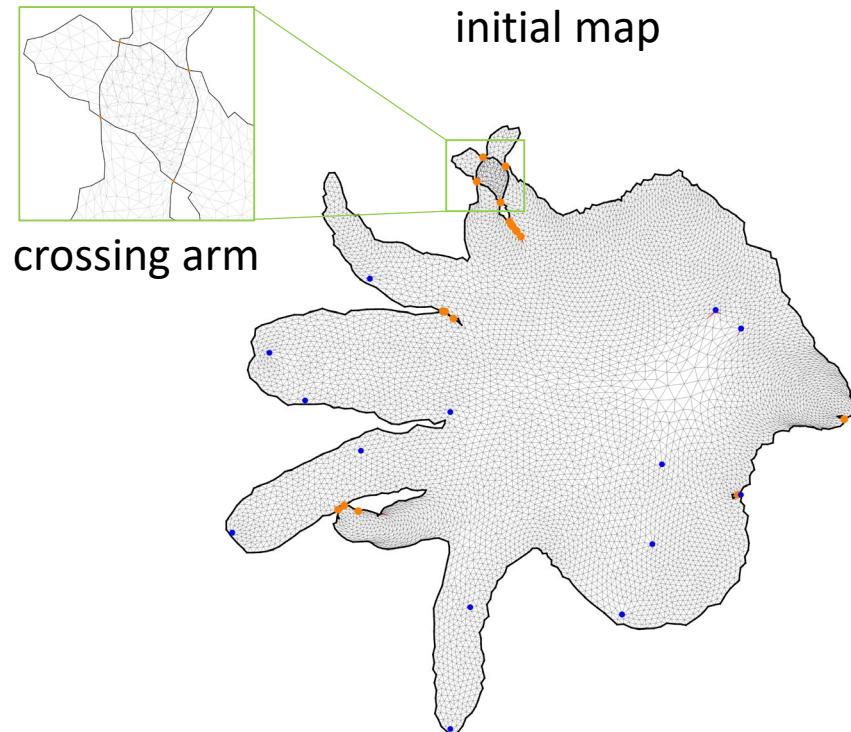


LBD



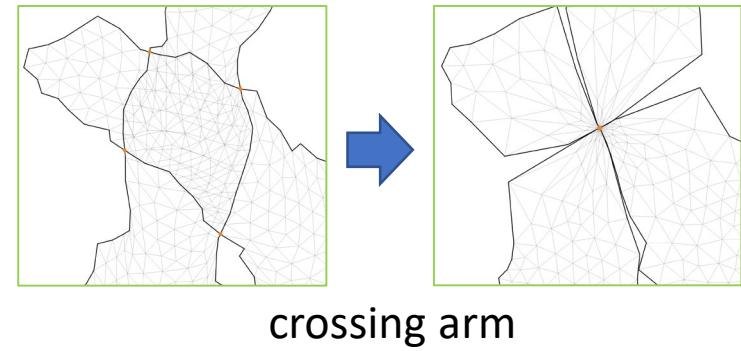
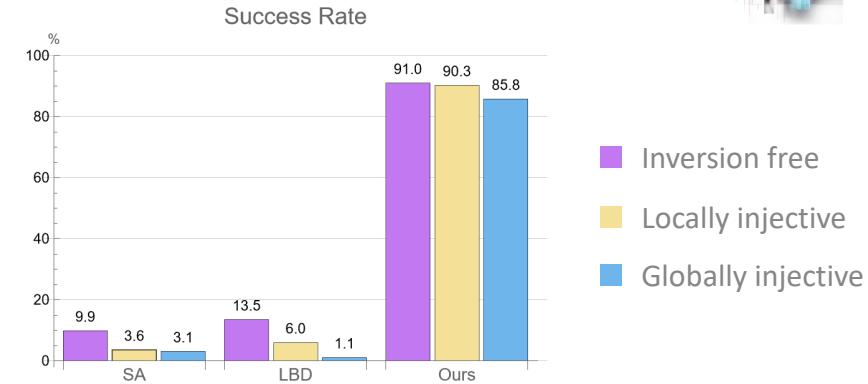
SA

# Benchmark: failure



# Summary

- First method to recover global injectivity under positional constraints
  - Significantly outperforms inversion-free methods
- Smooth Excess Area (SEA)
  - Captures both inverted and overlapping areas
  - Smooth almost everywhere
  - Theoretical guarantees of injectivity at global minima
- Future directions
  - Improve convergence rate
  - Higher order smoothness
  - Extend to 3D





# SIGGRAPH ASIA 2021 TOKYO

CONFERENCE

14 - 17 DECEMBER 2021

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