Calculus with Exercises A — Report 06

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Problem 6.1

Proof. To prove that $f(x) = x^2$ is continuous at every point of $\mathbb{R}$ we will prove that $f(x)$ is continuous at an arbitrary point of c.
Choose an arbitrary ε, we will show that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \sqrt{\varepsilon + c^2} - |c|$.
We have:

$$\begin{aligned} |x^2 - c^2| = |x-c||x+c| &\leq |x-c|(|x-c| + |2c|) \\ &< \left(\sqrt{\varepsilon + c^2} - |c|\right)\left(\sqrt{\varepsilon + c^2} + |c|\right) \\ &= \varepsilon + c^2 - c^2 = 0 \end{aligned}$$

$\square$

Problem 6.2

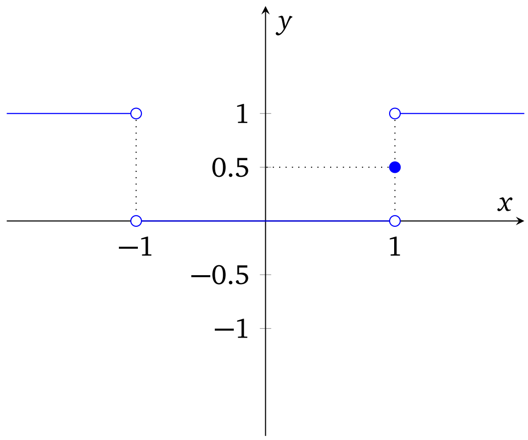
(1)

$$f(x) = \lim_{x \to \infty} \frac{x^n}{1 + x^n}$$

We will inspect $f(x)$ with x at some interval of value.

- $x > 1$: $f(x) = \lim_{n\to\infty} \dfrac{x^n}{1+x^n} = 1 - \lim_{x\to\infty} \dfrac{1}{1+x^n} = 1 - 0 = 1$
- $x = 1$: $f(x) = f(x) = \lim_{n\to\infty} \dfrac{x^n}{1+x^n} = 1 - \lim_{x\to\infty} \dfrac{1}{1+x^n} = 1 - 0 = 1$
- $-1 < x < 1$: $f(x) = f(x) = \lim_{n\to\infty} \dfrac{x^n}{1+x^n} = 1 - \lim_{x\to\infty} \dfrac{1}{1+x^n} = 1 - 0 = 1$
- $x = -1$: $f(x) = f(x) = \lim_{n\to\infty} \dfrac{x^n}{1+x^n} = 1 - \lim_{x\to\infty} \dfrac{1}{1+x^n} = 1 - 0 = 1$. $f(x)$ is undefined at $x = -1$ because $\dfrac{-1}{1+(-1)^n}$ is $\dfrac{1}{2}$ for n even and $\dfrac{-1}{1-1}$ for n odd.
- $x < -1$: $f(x) = f(x) = \lim_{n\to\infty} \dfrac{x^n}{1+x^n} = 1 - \lim_{x\to\infty} \dfrac{1}{1+x^n} = 1 - 0 = 1$

(2) The graph for $f(x)$.



(3) Points where f(x) is not continuous: {-1,1} are the points where f(x) is not continuous.

Problem 6.3

(1) If $f+g$ is continuous then at least on of the functions f and g must be continuous.
False - Both of the two could be discontinuous.

$$f(x)=\begin{cases} 1 \text{ for } x \geqslant 0 \\ 0 \text{ for } x<0 \end{cases} \text{ and } g(x)=\begin{cases} 0 \text{ for } x \geqslant 0 \\ 1 \text{ for } x<0 \end{cases}$$

Here $f(x)+g(x)=1$ for every $x \in \mathbb{R}$ is continuous at $f(0)+g(0)$ but both $f(x)$ and $g(x)$ is not.

(2) If f is continuous then $|f|$ is continuous. True.
f continuous at point c then $\forall \varepsilon>0$ we could find $\delta>0$ so that whenever $|x-c|<\delta$ then $|F(x)-f(c)|<\varepsilon$ But

$$|f(x)|-|f(c) \leqslant f(x)-f(c) \quad \text{(triangle inequality))}$$

$$\Rightarrow \quad ||f(x)|-|f(c)||<\varepsilon$$

We could conclude that $|f(x)|$ is also continuous at point c.

(3) If $|f|$ is continuous then f is continuous. False
$f(x)=\begin{cases} 1 \text{ for } x \geqslant 0 \\ -1 \text{ for } x<0 \end{cases}$ $\Rightarrow |f(x)|=1$ for $x \in \mathbb{R}$ If $|f(x)|$ / is continuous at 0 but $f(x)$ is discontinuous at 0.

(4) If f is continuous then $f(f(x))$ us continous. True.
This follows Theorem 5.42. Because f is continuous on $\mathbb{R}$, it is continuous at x and $f(x)$. So $f(f(x))$ is continuous on $\mathbb{R}$.

(5) If $f(f(x))$ is continuous on $\mathbb{R}$ then f is continuous on $\mathbb{R}$. False.

$$f(x)=\begin{cases} 2 & \text{for } x \geq 0 \\ 1 & \text{for } x<0 \end{cases}$$

Then

$$f(f(x)) = \begin{cases} 2 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$$

Problem 6.4

Generally speaking, a function is uniformly continuous on a set if there is a bound on its rate of change. Now for x^p we could easily observe that for $p > 1$ rate of change of x^p will approach infinity as p approaches infinity, and for $p < 0$ rate of change of x^p will approach infinity as p approaches zero. x^p would be uniformly continuous otherwise. We have three approaches to tackle the proof: sequence, $\varepsilon - \delta$, and Mean Value Theorem. It is clear that differentiation would be helpful as uniformly continuous is concerned with the rate of change of function. Actually, Mean Value Theorem would be quite easy to applied in all.

CASE 1: $p > 1$. We will use sequence in this case. Personally, sequence is the best approach in not-satisfied proof. We just need to choose a pair of strong enough sequences.
Set $x_n = n$ and $y_n = n + \dfrac{1}{n^{p-\lfloor p \rfloor}}$ then $|y_n - x_n| \to 0$ as $n \to \infty$ as required. We have:

$$\begin{aligned} |f(y_n) - f(x_n)| &= \left| \left(n + \frac{1}{n^{p-\lfloor p \rfloor}}\right)^p - n^p \right| = \left(n + \frac{1}{n^{p-\lfloor p \rfloor}}\right)^p - n^p \\ &= \left(n + \frac{1}{n^{p-\lfloor p \rfloor}}\right)^{\lfloor p \rfloor} \cdot \left(n + \frac{1}{n^{p-\lfloor p \rfloor}}\right)^{p-\lfloor p \rfloor} - n^{\lfloor p \rfloor} n^{p-\lfloor p \rfloor} \\ &\geq n^{p-\lfloor p \rfloor} \left(\left(n + \frac{1}{n^{p-\lfloor p \rfloor}}\right)^{\lfloor p \rfloor} - n^{\lfloor p \rfloor} \right) = A \end{aligned}$$

If $1 < p < 2$ then $A = 1 > 0$.
If $p \geq 2$, it is clear that A would be bigger than 0.

CASE 2: $p = 1$. Given $\varepsilon > 0$. $|f(y) - f(x)| = |y - x| < \varepsilon$ whenever $|y - x < \delta = \varepsilon$. So $p = 1$ satisfies.

CASE 3: $0 < p < 1$. We will use the *Mean Value Theorem* in this case. Let $y \geq x > 0$, there would exist a x_0 such that $< 0x_0 < x$. We have:

$$|f(y) - f(x)| = p(x^*)^{p-1}|y - x| \leq p(x_0)^{p-1}|y - x| < \epsilon$$
$$\text{whenever } |y - x| < \delta = \frac{x_0^{1-p}}{p}\epsilon$$

$x = 0$ would satisfy as well because x^p is continuous.

CASE 4: $p = 0$: Trivially satisfies.

CASE 5: $p < 0$: *We could just use the sequences $\dfrac{1}{n}$ and $\dfrac{1}{n^2}$(or any two sequences with different negative power) to prove x^p is not uniformly continuous. Nevertheless, I will attempt to use the $\varepsilon - \delta$ in this case.*

Suppose x^p is uniformly continuous. Pick $\varepsilon = 1$. Then $\exists \delta > 0$ so that $\forall x, y \in (0, \infty)$ with $|x - y| < \delta$, we have $|x^p - y^p| < \varepsilon = 1$. We will try to reach a contradiction.
Pick $x \in (0, 1)$ with $x < \delta$ and set $y = \dfrac{x}{2^{(-1/p)}}$. Then $|x - y| < \delta$. We have:

$$\begin{aligned}
|x^p - y^p| &= \left| x^p - \left(\frac{x}{2^{(-1/p)}} \right)^p \right| \\
&= |x^p (1-2)| \\
&= x^p > 1 \quad \forall x \in (0,1) \quad \text{and} \quad p < 0.
\end{aligned}$$

We have reached a contradiction.