Physics of Wave and Oscillation
Assignment 3

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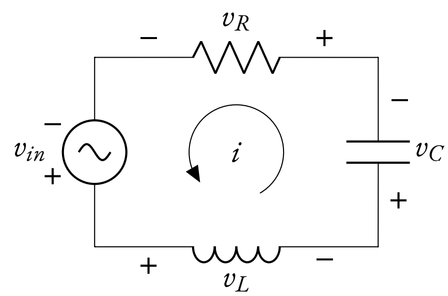
1 System Equation of Driven RLC Circuit

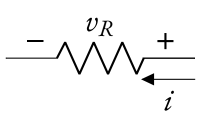


Consider RLC circuit where the three components are all in series with the voltage source. The direction of voltage and current between each component are assumed to be like in the figure. We will derive the differential equation of the system starting from the Kirchhoff's Voltage Law:

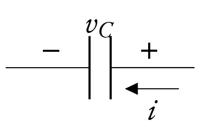
$$v_R + v_L + v_C = v_{in} \tag{1}$$

First we analyze each element of the circuit below:

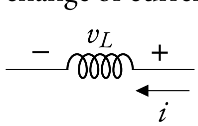
- For R: $v_R = iR$.

- For C: $\dfrac{Cdv_C}{dt} = \dfrac{dq}{dt} = i$, where q is the electrostatic charge of the capacitor. By Coulomb's Law, we have $q = v_C C$. The instant change of q is equal the current that passes through the capacitor.

- For L: $v_L = \dfrac{d\phi}{dt} = L\dfrac{di}{dt}$, where $\phi = Li$ is the magnetic flux through the coil. By Faraday's law of induction, the voltage will be induced in a way that hindered the change in magnetic flux through the circuit. If we assume the positive sign of voltage and direction of current like in the figure, the positive change of current will induce a voltage with the assumed sign that hinders the current source.

We let v_{in} be the typical sine wave input: $E_0 \cos \omega t$. To obtain a differential equation in terms of q, we substitute $v_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$, $v_R = R\frac{dq}{dt}$, $v_C = \frac{1}{C}q$ into (1):

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E_0\cos(\omega t)$$
$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{E_0}{L}\cos(\omega t)$$

Let

$$\frac{1}{\sqrt{LC}} = \omega_0, \frac{R}{L} = 2\gamma,$$

we obtain a familiar differential equation:

$$\frac{d^2q}{dt^2} + 2\gamma\frac{dq}{dt} + \omega_0^2 q = \frac{E_0}{L}\cos(\omega t) \tag{2}$$

The equation for a series RLC circuit and a spring pendulum looks really similar. We will attempt to create an analogy table to map between the two system.

Spring Pendulum		RLC Circuit	
Displacement	x	Charge	q
Damping	Γ	Resistance	R
Mass	m	Inductance	L
Compliance	$1/k$	Capacitance	C
Force	F	Voltage	E_0
Velocity	v	Current	i
Momentum	$p = mv$	Flux	$\Phi = Li$

2 Solution to the System Equation

By following the lectures, we have known that there are three steps to solve the differential equation:

- Find the homogeneous solution $q_H(t)$
- Find the particular solution $q_P(t)$
- The total solution is then the sum of the homogeneous solution and the particular solution as follows

$$q(t) = q_{\mathrm{H}}(t) + q_{\mathrm{P}}(t)$$

The homogeneous solution would have three different forms:

- $\gamma < \omega_0 \quad \Rightarrow \quad$ under-damped dynamics: $q_H = Ae^{-\gamma t}\cos(\sqrt{\omega_0^2 - \gamma^2}t + \alpha)$.
- $\gamma = \omega_0 \quad \Rightarrow \quad$ critically-damped dynamics: $q_H = x(t) = (at + b)e^{-\gamma t}$
- $\gamma > \omega_0 \quad \Rightarrow \quad$ over-damped dynamics: $q_H = e^{-\gamma t}\left(R_1 e^{+\sqrt{\gamma^2-\omega_0^2}t} + R_2 e^{-\sqrt{\gamma^2-\omega_0^2}t}\right)$

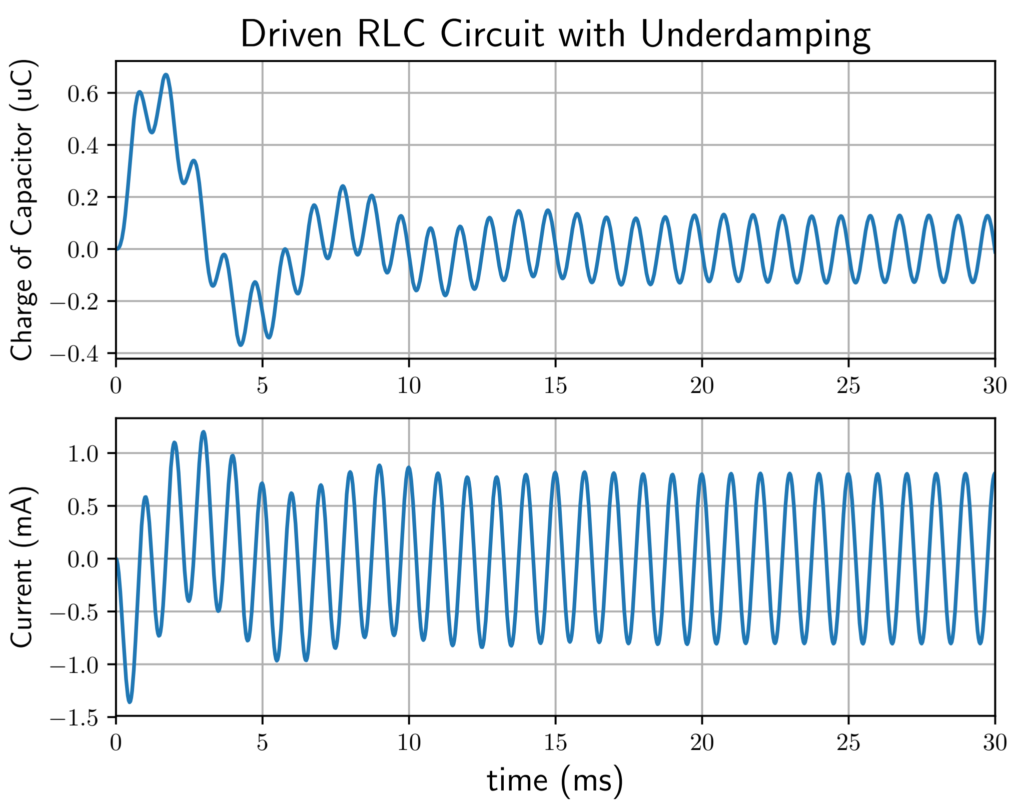
Similar to the spring pendulum, the particular solution is:

$$q_P = Q\cos(\omega_0 - \beta)$$

where

$$Q = \frac{E_0/L}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$
$$\beta = \arctan(\frac{2\omega_0\gamma}{\omega_0^2 - \omega^2})$$

The homogeneous term will fade out as time passes. The second term of particular solution will last as long as there is the external voltage. Just for fun we will include here a graph of a driven RLC circuit with $E_0 = 5$V, $f = 10$kHz, $R = 500\Omega$, $L = 1$H, $C = 1\mu$C.



We would like to make some adjustment to A and β to match with convention in circuit theory.

$$\begin{aligned}
Q &= \frac{E_0/L}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} \\
&= \frac{E_0}{L\sqrt{(1/LC-\omega^2)^2+\omega^2(R/L)^2}} \\
&= \frac{E_0}{\omega\sqrt{((1/\omega C)-\omega L)^2+R^2}} \\
&= \frac{E_0}{\omega\sqrt{(Z_L-Z_C)^2+R^2}} \\
&= \frac{E_0}{\omega Z} \\
\beta &= \arctan(\frac{2\omega_0\gamma}{\omega_0^2-\omega^2}) \\
&= \arctan(\frac{R}{1/\omega C-\omega L}) \\
&= \arctan(\frac{R}{Z_C-Z_L})
\end{aligned}$$

The quality Z_C and Z_L are called the impedance of capacitor and inductor respectively, whereas Z is called the impedance of the circuit. It consists of a resistance and R a reactance $Z_L - Z_C$. Why should we denote this way? Let's check the current:

$$i = \frac{dq}{dt} = A\omega\cos(\omega_0-\beta+\pi/2) = \frac{E_0}{Z}\cos(\omega_0-\beta+\pi/2) = I_0\cos(\omega_0-\beta+\pi/2) \tag{3}$$

Later in other part of our derivation, we may see more convenient points of this denotation.
On the other hand, we would like to think about the complex plane a bit. We could remember from the previous lectures, Q is the real part of a complex number $q*$, which came from the solver to find $q(t)$:

$$q*\left(-\omega^2+2\gamma j\omega+\omega_0^2\right)e^{j\omega t} = \frac{E_0}{L}e^{j\omega t}$$

We will use j instead of i to denote the complex part. Now, for purely reason of convention, we would like inspect the current element i by differentiate the above equation in term of time:

$$\begin{aligned}
q^*a\left(-\omega^2+2\gamma j\omega+\omega_0^2\right)\omega je^{j\omega t} &= j\omega\frac{E_0}{L}e^{j\omega t} \\
i^*\left(-\omega^2+2\gamma j\omega+\omega_0^2\right)e^{j\omega t} &= j\omega\frac{E_0}{L}e^{j\omega t}
\end{aligned}$$

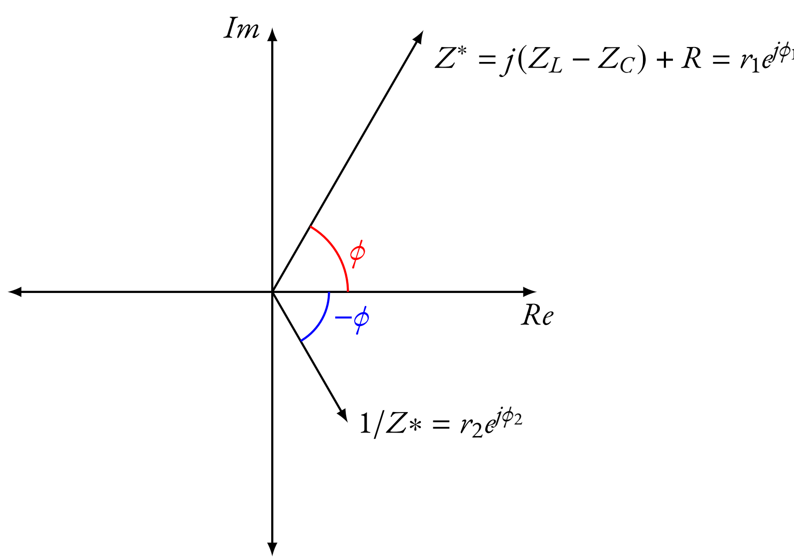
Now we could write $i*$ in terms of R, Z_C, Z_L:

$$\begin{aligned} i^* &= \frac{j\omega E_0/L}{\left(-\omega^2 + 2j\gamma\omega + \omega_0^2\right)} \\ &= \frac{j\omega E_0}{L(1/LC - \omega^2) + j\omega R} \\ &= \frac{E_0}{j(Z_L - Z_C) + R} \\ &= \frac{E_0}{Z^*} \end{aligned}$$

We could then obtain

$$\begin{aligned} I_0 &= \frac{E_0}{\sqrt{(Z_L - Z_C)^2 + R^2}} \\ \beta &= \arctan\left(\frac{R}{Z_C - Z_L}\right) \end{aligned}$$

from just some observation in the complex plane.
We have $Z^* = r_1 e^{j\phi_1}$ and $1/Z^* = r_2 e^{j\phi_2}$. But $Z^*(1/Z^*) = 1$ so $r_1 = 1/r_2$ and $\phi_1 = -\phi_2 = -\beta + \pi/2$. We could find I_0 and β without much calculation.



3 Energy State in Driven RLC Circuit

Let's analyze the RLC energy state like in a spring pendulum system. As we already mapped the elements of the mechanical system to the electrical system, we could actually deduce the corresponding energy also:

Spring Pendulum		RLC Circuit	
Energy supplied	$E = \int Fudt$	Energy supplied	$E = \int vidt$
Power supplied	$P = Fu$	Power supplied	$P = vi$
Power by damper	$P = u^2\Gamma$	Power by resistor	$P = i^2R$
Kinetic energy	$E = Mu^2/2$	Conductor's magnetic energy	$E = Li^2/2$
Potential energy	$E = kx^2/2$	Capacitor's electrostatic energy	$E = q^2/2C$

The energy in the capacitor at time t is:

$$w_C = \frac{q^2}{2C} = \frac{E_0^2}{2C\omega^2 Z^2}\cos(\omega t - \beta)^2$$

The energy in the inductor at time t is:

$$w_L = \frac{1}{2}Li^2 = \frac{LE_0^2}{2Z^2}\sin(\omega t - \beta)^2$$

The total energy is the sum

$$\begin{aligned}
w = w_C + w_L &= \frac{E_0^2}{2Z^2\omega}(\frac{1}{\omega C}\cos(\omega t - \beta)^2 + \omega L\sin(\omega t - \beta)^2) \\
&= \frac{E_0^2}{2Z^2\omega}(\frac{1}{\omega C}\cos(\omega t - \beta)^2 + \omega L - \omega L\cos(\omega t - \beta)^2) \\
&= \frac{E_0^2}{2Z^2\omega}(\omega L + (\frac{1}{\omega C} - \omega L)\cos(\omega t - \beta)^2) \\
&= \frac{E_0^2}{2Z^2\omega}(Z_L + (Z_C - Z_L)\cos(\omega t - \beta)^2) \\
(&= \frac{1}{2}L\omega^2 A^2 + \frac{1}{2}L(\omega_0^2 - \omega^2)A^2\cos^2(\omega t - \beta))
\end{aligned}$$

As the voltage varies with time, the instant energy state is also a function of time. We will find the average energy through a cycle:

$$\begin{aligned}
W &= \frac{1}{T}\int_0^T \left\{ \frac{E_0^2}{2Z^2\omega}(Z_L + (Z_C - Z_L)\cos(\omega t - \beta)^2) \right\} \\
&= \frac{E_0^2(Z_L + Z_C)}{4Z^2\omega} = \frac{E_0^2(Z_L + Z_C)}{4Z^2}\frac{T}{2\pi} \\
(&= \frac{1}{4}\left(\omega_0^2 + \omega^2\right)LA^2)
\end{aligned}$$

We see that the average energy in every cycle is a constant. In the steady state, similar to a spring pendulum, the work W_X generated by the voltage will be transferred into heat W_R by the resistor. We will show that W_R and W_X are equivalent.

$$\begin{aligned}
W_X &= \int_0^T v_{in} i dt = \int_0^T E_0 \cos(\omega t)\frac{E_0}{Z}(-\sin(\omega t - \beta))dt \\
&= \frac{E_0^2}{Z}\int_0^T \cos\omega t(-\sin\omega t\cos\beta + \sin\beta\cos\omega t)dt \\
&= \frac{E_0^2}{2Z} T\sin\beta \\
(\sin\beta &= \frac{R}{\sqrt{R^2 + (Z_L - Z_C)^2}} = \frac{R}{Z}) \\
W_R &= \int_0^T i^2 R dt = \int_0^T \frac{E_0^2 R}{Z^2}\sin^2(\omega t - \beta)dt \\
&= \frac{E_0^2 R}{2Z^2} T = \frac{I_0^2 R}{2} T
\end{aligned}$$

4 Resonance Phenomenon

Let's calculate the power of the voltage:

$$\bar{P} = \frac{\Delta W_X}{\Delta t}$$

As we already got the amount of work in a cycle of T, the power is simply:

$$\bar{P} = \frac{E_0^2 R}{2Z^2} = \frac{E_0^2 R}{2(R^2 + (Z_L - Z_C)^2)}$$

In the above equation, we can see that only the term $(Z_L - Z_C)^2$ is dependent of ω. The power would reach maximum value if:

$$Z_L = Z_C \quad \text{or} \quad \omega = \omega_0$$

And in other cases of $\omega \longrightarrow \infty$ and $\omega \longrightarrow 0$; $\quad \bar{P} \longrightarrow 0$. The maximum power is:

$$\bar{P}_{\max} = \frac{E_0^2}{2R}$$

At a certain frequency the power dissipated by the resistor is half of the maximum power which as mentioned occurs at $\omega = \omega_0$ The half power occurs at the frequencies for which:

$$\frac{1}{\sqrt{2}} = \frac{R}{Z} = \frac{R}{\sqrt{(Z_L - Z_C)^2 + (R)^2}} \implies \pm R = \omega L - \frac{1}{\omega C}$$

The equation has two roots:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$
$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

where $\Delta\omega = \omega_2 - \omega_1 = 2\gamma$ is the bandwidth, ω_1 is the lower half-power frequency and ω_2 is the upper half-power frequency. By multiplying the two roots we also realize that $\omega_0 = \sqrt{\omega_1\omega_2}$.
The resonance phenomenon and bandwidth is applied into filtering in electrical engineering. The rapid change in impedance near resonance can be used to pass or block signals close to the resonance frequency.