Calculus with Exercises A — Report 09

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Problem 9.1

Proposition. If f is continuous on an interval $[a,b]$ and $\int_a^b f(x)g(x) = 0$ for every continuous function g on $[a,b]$, then $f(x) = 0$ on $[a,b]$.

We will use contradiction to prove the above proposition. Assume that there is a continuous function $f(x) \neq 0$ on $[a,b]$ such that $\int_a^b f(x)g(x) = 0$ for every continuous function $g(x)$. Then for $g(x) = f(x)$ we have $\int_a^b f(x)g(x)dx = \int_a^b f(x)^2 dx = 0$. But $\int_a^b f(x)^2 dx > 0$ for every continuous function $f(x)$ that is not identically equal zero on $[a,b]$. We have reached a contradiction, thus, the proposition must be true.

Problem 9.2

(1) $\int_0^{\pi/4} \sin(2x)\sin(3x)dx = \frac{1}{2}\int_0^{\pi/4}(\cos x - \cos 5x)dx = \left[\frac{\sin x}{2} - \frac{\sin 5x}{10}\right]\Big|_0^{\pi/4} = \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$

(2) $\int_0^{\pi/2} \frac{\sin x dx}{2+\cos x} = -\int_0^{\pi/2} \frac{d(\cos x)}{2+\cos x} = -\ln(2+\cos x)\Big|_0^{\pi/2} = \ln\frac{3}{2}$

Problem 9.3

Theorem. Let f be a continuous function on an interval $[a,b]$. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=0}^{n-1} f\left(a + \frac{k}{n}(b-a)\right)$$

Using the theorem, we have:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{2^{1/n} + 2^{2/n} + \cdots + 2^{n-1/n} + 2^{n/n}}{n} \\
&= \lim_{n\to\infty} \frac{2^{0/n} + 2^{2/n} + \cdots + 2^{n-2/n} + 2^{n-1/n}}{n} \\
&= \lim_{n\to\infty} \frac{1-0}{n}\sum_{k=0}^{n-1} 2^{0+(k/n)(1-0)} \\
&= \int_0^1 2^x dx = \frac{2^x}{\ln 2}\Big|_0^1 = \frac{1}{\ln 2}
\end{aligned}$$

Problem 9.4

For the function f, there must exist two number m and M such that $m \leq f(x) \leq M$. We have:

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx$$

Let $\int_a^b g(x)dx = H$, since g is non-negative, we have:

$$m \leq \frac{1}{H}\int_a^b f(x)g(x) \leq M.$$

By the intermediate value theorem, function f reaches all value on the interval $[m, M]$, so there exists a ξ on (a, b) that:

$$\frac{1}{H}\int_a^b f(x)g(x) = f(\xi).$$

Which means:

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$