Advanced Linear Algebra - Report 01

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Q3-2.

(2) $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

- For **Ax=b** to be solvable, **b** must be in the column space $C(A)$ of A. It is easy to see that A has the two column vectors $C_2 = 2C_1$. Therefore, $C(A)$ is a line in $\mathbb{R}^2$ that passes through $(1,2)$, and for **Ax=b** to be solvable **b** must be on that line.

- The nullspace $N(A)$ of A has the dimension 1 $(= n - r)$. To solve for

$$\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we will use elimination on the augmented matrix $[\ A \mid 0\ ]$ to have:

$$\left(\begin{array}{cc|c} 3 & 6 & 0 \\ 1 & 2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let C_2 be the free column and x_2 be the free variable:

$$\mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

The nullspace is a line in $\mathbb{R}^2$ that passes through $(-2,1)$. The nullspace $N(A)$ is orthogonal to the row space $C^T(A)$.

(3) $A = \begin{bmatrix} 3 & 6 & 3 & 6 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

- The columns of A in (3) are the same as in (2), thus the condition for **Ax=b** to be solvable is also the same as (2).

- The nullspace $N(A)$ of A has the dimension 3 $(= n - r)$. To solve for

$$\begin{bmatrix} 3 & 6 & 3 & 6 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we will use elimination on the augmented matrix $[\ A \mid 0\ ]$ to have:

$$\left(\begin{array}{cccc|c} 3 & 6 & 3 & 6 & 0 \\ 1 & 2 & 1 & 2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let C_2, C_3, C_4 be the free column and x_2, x_3, x_4 be the free variable:

$$\mathbf{x} = \begin{bmatrix} -2x_2 - x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

The nullspace is a hyper-plane with dim(3) in $\mathbb{R}^4$ that is spanned by the above three vectors. The nullspace $N(A)$ is orthogonal to the row space $C^T(A)$.

(4) $A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$

- It is easy to realize that the three columns of A are independent: $C_3 = C_2 - C_1$. So $C(A)$ is a subspace of $\mathbb{R}^3$ that is spanned by C_2 and C_1. For **Ax=b** to be solvable, **b** must be in the column space.

- The nullspace $N(A)$ of A has the dimension 1 $(= n - r)$. To solve for

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we will use elimination on the augmented matrix $[\ A \mid 0\ ]$ to have:

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 8 & 8 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 2 & 4 & 2 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let C_3 be the free column and x_3 be the free variable:

$$\mathbf{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_3$$

The nullspace $N(A)$ is a line in $\mathbb{R}^3$ that is spanned by the above vector.

Q3-3.

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

  Using elimination on A, we have $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. So the rank of A is 2. The dimension of the nullspace N(A) is n - r = 0. We conclude that N(A) is the zero vector in $\mathbb{R}^2$.

- $B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}$

  The row space of B is spanned by the same vectors as A, so its dimension is still 2. As the row space of B still belongs to $\mathbb{R}^2$, the dimension of its nullspace is the same as A being $n - r = 0$. We conclude that N(B) is the zero vector in $\mathbb{R}^2$.

- $C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$

It is easy to see that the rank of C is 2 (same column space as A). But the row space of C is a subspace of $\mathbb{R}^4$, so the dimension of its nullspace is $n-r=2$. To solve for

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we will use elimination on the augmented matrix $\begin{bmatrix} C & | & 0 \end{bmatrix}$ to have:

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 0 \\ 3 & 8 & 6 & 16 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 0 \\ 0 & 2 & 0 & 4 & 0 \end{array}\right)$$

Let C_3, C_4 be the free column and x_3, x_4 be the free variable:

$$\mathbf{x} = \begin{bmatrix} -2x_3 \\ -2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} x_4$$

The nullspace $N(A)$ is a plane in $\mathbb{R}^4$ that is spanned by the above vectors.