Calculus with Exercises A — Report 05

DO LE DUY

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Problem 5.1

Proof using the $\varepsilon - \delta$ definition of functional limit:

Proof. Choose an arbitrary x_0, we will prove that for all $x \in V_\delta(x_0)$ different from x_0, it follows that $f(x) \in V_\varepsilon(L = 5 - 3x_0)$.
For all $\varepsilon > 0$, we could choose $\delta = \frac{1}{3}\varepsilon > 0$, then:

$$0 < |x - x_0| < \delta \implies |f(x) - L)| = |5 - 3x - 5 + x_0| = 3|x - x_0| < 3\delta = \varepsilon.$$

Thus, $\lim_{x \to x_0}(5 - 3x) = L$. $\square$

Proof using the sequential definition of functional limit:

Proof. Consider an arbitrary sequence (x_n) satisfying $x_n \neq x_0$ and $(x_n) \to x_0$. Then for all $\delta > 0$ there exists N such that $n > N : |x_n - x_0| < \delta$ Then for all $\varepsilon = 3\delta > 0$, it follows that:

$$|f(x_n) - L| = |5 - 3x_n - 5 + 3x_0| = 3|x_n - x_0| < \varepsilon.$$

Thus, $(f(x_n)) \to L$. We conclude that $\lim_{x \to x_0}(5 - 3x) = L$. $\square$

Problem 5.2

Following is an equivalent definition for infinite limit using the sequential approach:

Definition. Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:

- For all sequences $(x_n) \in A$ satisfying $x_n > x_0$ and $(x_n) \to c$, it follows that $f(x_n) \to \infty$.
- $\lim_{x \to x_0^+} f(x) = \infty$.

Proof. $(\to)$: Let's first assume that the $\delta - \varepsilon$ definition is satisfied and $\lim_{x \to x_0} f(x) = \infty$. For every L, there exists a neighborhood of c so that for every x in that neighborhood and greater than c, $f(x) > L$.

Consider an arbitrary sequence (x_n), which converges to x_0 and satisfied $x_n \neq c$. Choose and arbitrary $L > 0$, then there exists $V_\delta(x_0)$ that for all $x \in V_\delta(x_0)$ greater than c, $f(x) > L$. But because (x_n) converges to x_0, there exists N that (x_n) will eventually be in that neighborhood after $n \geq N$. It follows that $n \geq N$ implies $f(x_n) > L$. We have proved the forward implication.

$(\leftarrow)$: We will argue by contradiction. Assume that our sequential definition is true and $\delta - \varepsilon$ definition is false.

Therefore, there must exist at least one L for which no suitable $V_\delta(x_0)$. In other words, no matter what $\delta > 0$ we try, there will always be at least one point:

$$x \in V_\delta(x_0) \text{ with } x > x_0 \text{ for which } f(x) \leq L.$$

Let $\delta = \frac{1}{n}$, then in all of those n neighborhoods $V_\delta(x_0)$, we could find an x such that $f(x) \leq L$. But these x's make a sequence that converges to c, where the image sequence of it: $f(x_n) \leq L$. We have reached the contradiction. Thus, the backward implication is satisfied. □

Problem 5.3

Proof. Because $\lim_{x\to x_0} f(x) = -1$, there exists a $c > 0$ such that

$$|f(x) - (-1)| < \frac{1}{2}$$

whenever x is a point in domain of f differing from x_0 and satisfying $|x - x_0| < c$. Thus,

$$f(x) - (-1) \leq |f(x) - (-1)| < \frac{1}{2} \implies f(x) < -\frac{1}{2}$$

for all x in $(x_0 - c, x_0 + c)\,, x \neq x_0$ that are in the domain of f. This would complete the proof. □

Problem 5.4

(1) $\lim_{x\to+\infty}(6^x - 2^x) = \lim_{x\to+\infty}(2^x)(3^x - 1) = \lim_{x\to+\infty}(2^x)\lim_{x\to+\infty}(3^x - 1) = \infty$. Proving $\lim_{x\to\infty} C^x = \infty$ where $C > 1$: For all $N > 0$, there exists $n = \log_C(N + 1)$ that whenever $x > n$, $C^x = N + 1 > N$.

(2) $\lim_{x\to 0+} C^x = C^0 = 1$ for $C > 0$. Using the theorem that the exponential function C^x which $C > 0$ is continuous at every x.

(3)

$$\begin{aligned}
&\lim_{x\to+\infty} x^2\left(\frac{1}{x^3}\sin\frac{1}{x} - \cos\frac{1}{x}\right)\\
&= \lim_{x\to+\infty}\left(\frac{1}{x}\sin\frac{1}{x} - x^2\cos\frac{1}{x}\right)\\
&= \lim_{x\to+\infty}\left(\frac{1}{x}\sin\frac{1}{x}\right) - \lim_{x\to+\infty}(x^2)\lim_{x\to+\infty}\left(\cos\frac{1}{x}\right) = -\infty
\end{aligned}$$

Using squeeze theorem on the $\lim_{x\to+\infty}(\frac{1}{x}\sin\frac{1}{x})$, we found its limit is equal to zero. $\lim_{x\to+\infty}(x^2) = \infty$ and $\lim_{x\to+\infty}\cos\frac{1}{x} = 1$.