Calculus with Exercises A — Report 10

DO LE DUY

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Problem 10.1

(1)

$$\int e^{\sqrt{x}}dx = \int 2\sqrt{x}e^{\sqrt{x}}d(\sqrt{x})$$

$$\text{Set } \sqrt{x} = t, \text{ then :}$$

$$\begin{aligned}\int e^{\sqrt{x}}dx &= \int 2te^t dt \\ &= 2te^t - 2\int e^t dt \\ &= 2te^t - 2e^t \\ &= 2e^{\sqrt{x}}(\sqrt{x}-1)\end{aligned}$$

(2)

$$\int \frac{1}{\sqrt{\sqrt{x}+1}}dx, \text{ set } u = \sqrt{x}+1.$$

Then $x = u^2 - 2u + 1$, and $dx = 2(u-1)du$. We have:

$$\begin{aligned}\int \frac{1}{\sqrt{\sqrt{x}+1}}dx &= 2\int \frac{1}{\sqrt{u}}(u-1)du \\ &= 2\int u^{1/2}du - 2\int u^{-1/2}du \\ &= \frac{4}{3}u^{3/2} - 4u^{1/2} \\ &= \frac{4}{3}(\sqrt{x}-2)(\sqrt{x}+1)^{1/2} + C\end{aligned}$$

(3)

$$\int \sqrt{1-x^2}dx, \text{ set } x = \sin\theta.$$

Then $\sqrt{1-x^2} = \cos\theta$,and $dx = \cos\theta d\theta$. We have:

$$\begin{aligned}\int \sqrt{1-x^2}dx &= \int (\cos\theta)^2 d\theta \\ &= \int \frac{\cos(2\theta)+1}{2}d\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C \\ &= \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right) + C\end{aligned}$$

(4)

$$\int x(\ln x)^2 dx = \int (\ln x)(x \ln x dx), \text{set } u = \ln x \text{ and } dv = (x \ln x dx)$$

We will find $v : v = \int x \ln x dx$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx$$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right), \text{ therefore:}$$

$$\int x(\ln x)^2 dx = \ln x\left(\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)\right) - \int \frac{1}{x}\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)dx$$

$$= \ln x\left(\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)\right) - \int \frac{x}{2}\left(\ln x - \frac{1}{2}\right)dx$$

$$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{4}\ln x - \frac{1}{2}\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C$$

$$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

(5)

We will prove a general reduction formula:

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx, n \neq 1. \text{ We have:}$$

$$\int \sec^n x dx = \int \sec^{n-2}(\tan^2 x + 1)dx = \int \sec^{n-2} x + \int \tan x \sec^{n-3} d(\sec x)$$

$$= \int \sec^{n-2} x + \frac{\sec^{n-2} x \tan x}{n-2} - \frac{1}{n-2}\int \sec^{n-2}\sec^2 x dx. \text{ Therefore:}$$

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx. \text{ Thus, we have:}$$

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\tan x + \sec x|$$

(6)

$$\int \frac{1}{\sqrt{x^2-1}}dx, \text{ set } x = \sec\theta. \text{ So } dx = \tan\theta\sec\theta d\theta, \text{ we have:}$$

$$\int \frac{1}{\sqrt{x^2-1}}dx = \int \frac{1}{\sqrt{(\sec\theta)^2-1}}\tan\theta\sec\theta d\theta$$

$$= \int \sec\theta d\theta = \ln|\tan\theta + \sec\theta| + C$$

$$= \ln|\sqrt{x^2-1} + x| + C$$

(7)

$$\begin{aligned}
&\int \left(\sqrt{x^2+1}\right) dx, \text{ set } x = \tan\theta. \text{ So } dx = (\sec\theta)^2 d\theta, \text{ we have:}\\
&\int \left(\sqrt{x^2+1}\right) dx = \int \sqrt{(\tan\theta)^2+1}\sec^2\theta d\theta\\
&\qquad = \int \sec^3\theta d\theta\\
&\qquad = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\tan\theta+\sec\theta| \text{ from (5)}\\
&\qquad = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x+\sqrt{x^2+1}|
\end{aligned}$$

(8)

$$\begin{aligned}
\int \frac{2x^2+x+1}{(x+3)(x-1)^2} dx &= \int \left(\frac{1}{(x-1)^2} + \frac{1}{x-1} + \frac{1}{x+3}\right) dx\\
&= \frac{-1}{x-1} + \ln(x-1) + \ln(x+3)
\end{aligned}$$