Advanced Linear Algebra

Q9-2:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad . \quad (A - \lambda I) = \begin{bmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda(\lambda - 1) - 2 = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = -1$$

To find the eigenvectors x_1, x_2. Let

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} c$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} d$$

$$A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix} \quad . \quad (A^{-1} - \lambda^{-1} I) = \begin{bmatrix} -1/2 - \lambda^{-1} & 1 \\ 1/2 & -\lambda^{-1} \end{bmatrix} = 0$$

$$\Rightarrow \lambda^{-1}\left(\frac{1}{2} + \lambda^{-1}\right) - \frac{1}{2} = 0 \Rightarrow \lambda^{-2} + \frac{1}{2}\lambda^{-1} - \frac{1}{2} = 0$$

$$\Rightarrow \lambda_1^{-1} = 1/2, \quad \lambda_2^{-1} = -1$$

The eigenvector of A^{-1} : x_1^{-1}, x_2^{-1}. Then:

$$\begin{bmatrix} -1 & 1 \\ 1/2 & -1/2 \end{bmatrix} x_1^{-1} = 0 \Rightarrow x_1^{-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} c$$

$$\begin{bmatrix} 1/2 & 1 \\ 1/2 & 1 \end{bmatrix} x_2^{-1} = 0 \Rightarrow x_2^{-1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} d$$

Checking: $\lambda_1 + \lambda_2 = \operatorname{tr} A = 1$; $\lambda_1 \lambda_2 = \det A = -2$

$$\lambda_1^{-1} + \lambda_2^{-1} = \operatorname{tr} A^{-1} = -1/2 \; ; \; \lambda_1^{-1} . \lambda_2^{-1} = \det A^{-1} = -1/2$$

$$\lambda_1 = \frac{1}{\lambda_1^{-1}} \; ; \; \lambda_2 = \frac{1}{\lambda_2^{-1}} \; ; \; x_1 = x_1^{-1} ; \; x_2 = x_2^{-1}$$

NO.

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Q 10-1:

(1) $A = \begin{bmatrix}1&1\\1&1\end{bmatrix}$; *) Find eigen values: $(1-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \begin{matrix}\lambda_1 = 0\\ \lambda_2 = 2\end{matrix}$

*) Find eigenvectors : $\begin{bmatrix}1&1\\1&1\end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix}-1\\1\end{bmatrix} c$

$\begin{bmatrix}-1&1\\1&-1\end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix}1\\1\end{bmatrix} d$

$A = \begin{bmatrix}-1&1\\1&1\end{bmatrix}\begin{bmatrix}0&0\\0&2\end{bmatrix}\begin{bmatrix}-1/2&1/2\\1/2&1/2\end{bmatrix}$. There is no A^{-1}. $A^3 = \begin{bmatrix}-1&1\\1&1\end{bmatrix}\begin{bmatrix}0&0\\0&8\end{bmatrix}\begin{bmatrix}-1/2&1/2\\1/2&1/2\end{bmatrix}$

$= \begin{bmatrix}0&8\\0&8\end{bmatrix}\begin{bmatrix}-1/2&1/2\\1/2&1/2\end{bmatrix}$

$= \begin{bmatrix}4&4\\4&4\end{bmatrix}$

(2) $A = \begin{bmatrix}2&1\\0&0\end{bmatrix}$ $\quad \lambda_1 = 0$, $\lambda_2 = 2$

$\begin{bmatrix}2&1\\0&0\end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix}-1\\2\end{bmatrix} c$, $\begin{bmatrix}0&1\\0&-2\end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix}1\\0\end{bmatrix} d$

$A = \begin{bmatrix}-1&1\\2&0\end{bmatrix}\begin{bmatrix}0&0\\0&2\end{bmatrix}\begin{bmatrix}0&1/2\\1&1/2\end{bmatrix}$ There is no A^{-1}

$A^3 = \begin{bmatrix}-1&1\\2&0\end{bmatrix}\begin{bmatrix}0&0\\0&8\end{bmatrix}\begin{bmatrix}0&1/2\\1&1/2\end{bmatrix}$

$= \begin{bmatrix}0&8\\0&0\end{bmatrix}\begin{bmatrix}0&1/2\\1&1/2\end{bmatrix} = \begin{bmatrix}8&4\\0&0\end{bmatrix}$.

Q 10-4; $A = \begin{bmatrix}5&1\\0&4\end{bmatrix}$ we have $\lambda_1 = 5$, $\lambda_2 = 4$

• Find eigenvectors: $\begin{bmatrix}0&1\\0&-1\end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix}1\\0\end{bmatrix} c$; $\begin{bmatrix}1&1\\0&0\end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix}-1\\1\end{bmatrix} d$

$A = \begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}5&0\\0&4\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}$; $A^k = \begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}5^k&0\\0&4^k\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}$

$= \begin{bmatrix}5^k&-4^k\\0&4^k\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix} = \begin{bmatrix}5^k&5^k-4^k\\0&4^k\end{bmatrix}$

Q 10-5:

$A - \lambda I = \begin{bmatrix}1-\lambda&1&1\\1&-\lambda&0\\1&0&-\lambda\end{bmatrix}$ • Find eigenvalues:

$(1-\lambda)\lambda^2 + \lambda + \lambda = 0$

$\Rightarrow -\lambda^3 + \lambda^2 + 2\lambda = 0$

$\Rightarrow \lambda(-\lambda^2 + \lambda + 2) = 0 \Rightarrow \begin{matrix}\lambda_0 = 0\\ \lambda_1 = 2\\ \lambda_2 = -1\end{matrix}$

• Find eigen vectors

$\begin{bmatrix}1&1&1\\1&0&0\\1&0&0\end{bmatrix} x_0 = 0 \Rightarrow x_0 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$; $\begin{bmatrix}-1&1&1\\1&-2&0\\1&0&-2\end{bmatrix} x_1 = 0 \Rightarrow \begin{bmatrix}-1&1&1\\0&-1&1\\0&1&-1\end{bmatrix} x_1 = 0$

$\left.\begin{bmatrix}2&1&1\\1&1&0\\1&0&1\end{bmatrix} x_2 = 0 \Rightarrow \begin{bmatrix}2&1&1\\0&1/2&-1/2\\0&-1/2&1/2\end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix}-1\\1\\1\end{bmatrix} d \right\} \Rightarrow x_1 = \begin{bmatrix}2\\1\\1\end{bmatrix} c$

$S = \begin{bmatrix}0&2&-1\\-1&1&1\\1&1&1\end{bmatrix}$; $\Lambda = \begin{bmatrix}0&0&0\\0&2&0\\0&0&-1\end{bmatrix}$

$$[S \mid I] \xrightarrow[R_2 = R_1]{R_1 = R_2} \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2/2]{R_3 = R_3 + R_1} \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - 2R_2]{R_1 = -R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3/3} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 + R_3]{R_2 = R_2 + R_3/2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & 1/6 & 1/6 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/3 & 1/6 & 1/6 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right]$$

Then $A = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1/3 & 1/6 & 1/6 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}$

$$A^k = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 2^k & \\ & & (-1)^k \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1/3 & 1/6 & 1/6 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}$$