Calculus with Exercises A — Report 08

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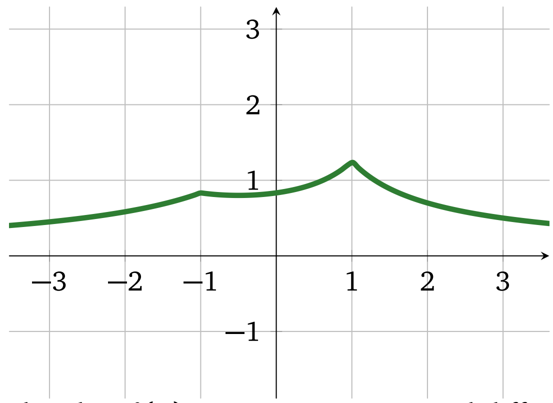
July 2, 2020

Problem 8.1

We rewrite the function in the following form:

$$f(x) = \begin{cases} \frac{1}{1-x} + \frac{1}{2-x} & \text{for } x \leq -1 \\ \frac{1}{3+x} + \frac{1}{2-x} & \text{for } -1 \leq x \leq 1 \\ \frac{1}{3+x} + \frac{1}{x} & \text{for } x \geq 1 \end{cases}$$



From the form of function, we could acknowledge that $f(x)$ is continuous on R and differentiable on the intervals $(\infty, -1), (-1, 1), (1, \infty)$. We could also easily prove that:

- $f(x)$ is increasing with the range $(0, \frac{5}{6})$ on the interval $(-\infty, -1)$.
- $f(x)$ is decreasing with the range $(0, \frac{5}{4})$ on the interval $(1, \infty)$.
- Following the theorem on the existence of extrema, we could find the maximum of $f(x)$ on the interval $[-1, 1]$ by comparing its values at three points at $x = -1,\ x = 1$ and x_0 where $f'(x_0) = 0$.

  We have $f'(x_0) = \dfrac{-1}{(3+x_0)^2} + \dfrac{1}{(2-x_0)^2} = 0 \iff x_0 = -0.5$.

  Comparing $f(-1),\ f(-0.5),\ f(1)$, we see that $f(1) = \frac{5}{4}$ is the maximum value of the function on the interval $[-1, 1]$.
  Thus, we could conclude that the maximum of the function on R is $\frac{5}{4}$.

Problem 8.2

For functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \quad \text{(l'Hospital' s Rule)}$$

In the following calculations, we will just implicitly consider the conditions to apply l'Hospital' s Rule as they are easily observable.

(1)

$$\begin{aligned}
A &= \lim_{x\to 0} \frac{e^x - \cos x - x}{x^2} \\
&= \lim_{x\to 0} \frac{e^x + \sin x - 1}{2x} \quad \text{(l'Hospital' s Rule)} \\
&= \lim_{x\to 0} \frac{e^x + \cos x}{2} \quad \text{(l'Hospital' s Rule)} \\
&= 1
\end{aligned}$$

(2)

$$\begin{aligned}
B &= \lim_{x\to \infty} x^p \ln x \\
&= \lim_{x\to \infty} \frac{\ln x}{x^{-p}} \\
&= \lim_{x\to \infty} \frac{1}{-px^{(-p)}} \quad \text{(l'Hospital' s Rule)} \\
&= 0 \quad \text{(p is negative)}
\end{aligned}$$

(3)

$$\begin{aligned}
C &= \lim_{x\to 0+} (e^x - \sin x - 1)\ln x \\
&= \lim_{x\to 0+} -(\sin x \ln x) + (e^x - 1)\ln x
\end{aligned}$$

We will consider the two limits: $\lim_{x\to 0+} \ln x(e^x - 1)$ and $\lim_{x\to 0+} \ln x \sin x$. Apply l'Hospital 's Rule continuously on the two functions, we have:

$$\begin{aligned}
&\lim_{x\to 0+} (e^x - 1)\ln x \\
&= \lim_{x\to 0+} \frac{\ln x}{1/(e^x - 1)} \\
&= \lim_{x\to 0+} \frac{-1}{xe^x/(e^x - 1)^2} \quad \text{(l'Hospital' s Rule)} \\
&= \lim_{x\to 0+} \frac{-(e^x - 1)^2}{xe^x} \quad \text{(l'Hospital' s Rule)} \\
&= \lim_{x\to 0+} \frac{2(e^x - 1)e^x}{xe^x + e^x} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
&\lim_{x\to 0+} \ln x \sin x \\
&= \lim_{x\to 0+} \frac{\ln x}{1/\sin x} \\
&= \lim_{x\to 0+} \frac{-\sin^2 x}{(x \cos x)} \quad \text{(l'Hospital' s Rule)} \\
&= \lim_{x\to 0+} \frac{2\sin x \cos x}{\cos x + x \sin x} \quad \text{(l'Hospital' s Rule)} \\
&= 0.
\end{aligned}$$

Because the two components exist, $C = 0$.

(4)

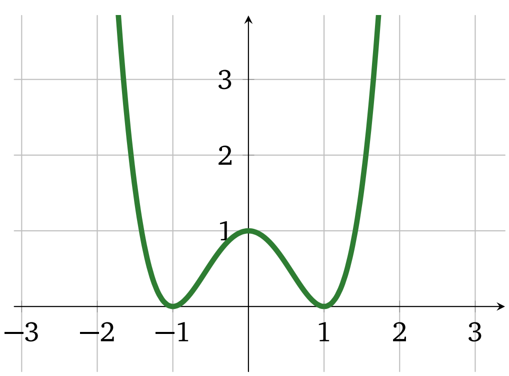
$$
\begin{aligned}
D &= \lim_{x\to 0+} (\sin x)^x \\
&= \lim_{x\to 0+} \exp\left(\ln(\sin x)^x\right) \\
&= \exp\left(\lim_{x\to 0+} x\ln(\sin x)\right) \\
&= \exp\left(\lim_{x\to 0+} \frac{\ln(\sin x)}{1/x}\right) \\
&= \exp\left(\lim_{x\to 0+} \frac{-x^2\cos x}{\sin x}\right) \quad \text{(l'Hospital' s Rule)} \\
&= \exp\left(\lim_{x\to 0+} \frac{-2x\cos x + x^2\sin x}{\cos x}\right) \quad \text{(l'Hospital' s Rule)} \\
&= \exp(0) = 1.
\end{aligned}
$$

In this question also, to be more precise we should have calculate the limit of $(\ln(\sin x)^x)$ to prove it exists first before move the limit into the exponential. The procedure should be the same like this when applying l'Hospital's Rule. But to be more concise, we will consider it implicit

Problem 8.3

To investigate monotonicity and convexity properties of $f(x) = (1-x^2)^2$, we will consider the first-order differential and the the second-order differential of $f(x)$ respectively:

$$
\begin{aligned}
f'(x) &= -4x(1-x)(1+x) \\
f''(x) &= -4 + 12x^2
\end{aligned}
$$



$$
\begin{cases}
f'(x) < 0 \text{ for } x < -1 : f(x) \text{ is decreasing.} \\
f'(x) = 0 \text{ for } x = -1 : f(x) \text{ reaches a minimum of } 0. \\
f'(x) > 0 \text{ for } -1 < x < 0 : f(x) \text{ is increasing.} \\
f'(x) = 0 \text{ for } x = 0 : f(x) \text{ reaches a maximum of } 1. \\
f'(x) < 0 \text{ for } 0 < x < 1 : f(x) \text{ is decreasing.} \\
f'(x) = 0 \text{ for } x = 1 : f(x) \text{ reaches a minimum of } 0. \\
f'(x) > 0 \text{ for } x > 1 : f(x) \text{ is increasing.}
\end{cases}
$$

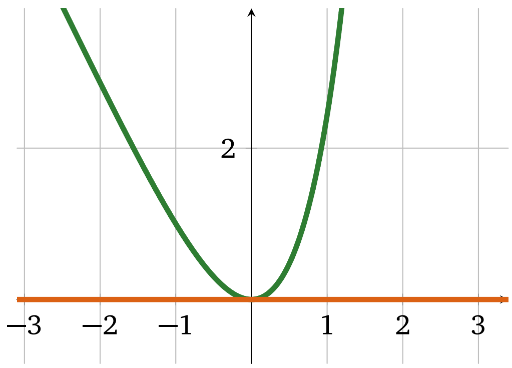
$$
\begin{cases}
f''(x) \geq 0 \text{ for } x \leq -1/\sqrt{3} : f(x) \text{ concave upward.} \\
f''(x) \leq 0 \text{ for } -1/\sqrt{3} \leq x \leq 1/\sqrt{3} : f(x) \text{ concave downward.} \\
f''(x) \geq 0 \text{ for } x \geq 1/\sqrt{3} : f(x) \text{ concave upward.}
\end{cases}
$$

Problem 8.4

We will prove $(1+x)e^x - 1 - 2x \geq 0$ by investigating monotonicity properties and extrema of the function $f(x) = (1+x)e^x - 1 - 2x$. We have:

$$f'(x) = 2e^x + xe^x - 2 \implies \begin{cases} f'(x) < 0 \text{ for } x < 0 \\ f'(x) = 0 \text{ for } x = 0 \\ f'(x) > 0 \text{ for } x > 0 \end{cases}$$

It is because when $x < 0$, we have $2e^x < 2$ and $xe^x < 0$; and when $x > 0$, $2e^x > 2$ and $xe^x > 0$.



Thus:

- $f(x)$ is decreasing on the interval $(-\infty, 0)$.
- $f(x)$ is increasing on the interval $(0, \infty)$.
- Because $f'(x)$ changes sign from negative to positive at $x = 0$, $f(x)$ reaches its minimum at $x = 0$.

We could then conclude that $f(x) \geq 0$ for all $x \in R$, in other words $(1+x)e^x \geq 1 + 2x$.