Calculus with Exercises A — Report 01

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June 13, 2020

Problem 1.1

- Let y be any solution of the equation a + x = b. Then,

$$\begin{aligned} a+y &= b && \\ -a+a+y &= -a+b && \text{by (A4): Existence of negative} \\ (-a+a)+y &= -a+b && \text{by (A2): Associative law} \\ 0+y &= b+(-a) && \text{by (A1): Commutative law} \\ y &= b+(-a) && \text{by (A3): Existence of 0} \end{aligned}$$

- Because the applications of axioms here go both ways, in this proof of **Uniqueness**, we have proved that $a+x=b \iff x=b+(-a)$. As the proof of **Existence** is equivalent to the proof of $(x=b+(-a) \Rightarrow a+x=b)$, we have demonstrated both the **Uniqueness** and **Existence** characteristic of the solution to the equation a + x = b.

Problem 1.2

- The following definition will be used.

  Bounded Set: Let E be a set of real numbers. A number M is said to be an upper bound for E if $x \leq M$ for all $x \in E$. A number m is said to be an lower bound for E if $m \leq x$ for all $x \in E$. A set that has an upper bound and a lower bound is called bounded.

- Prove: $\exists r>0 : |x|<r \ \ \forall x \in E \Rightarrow$ **E is bounded**. (1)

  $\exists r>0 : |x|<r \ \ \forall x \in E$

  $\Rightarrow \exists r>0 : -r<x<r \ \ \forall x \in E$

  $\Rightarrow$ -r and r are a lower bound and a upper bound of E

  $\Rightarrow$ **E is bounded**

- Prove: **E is bounded** $\Rightarrow \exists r>0 : |x|<r \ \ \forall x \in E$. (2)

  E is bounded $\Rightarrow \exists a,b : a<x<b \ \ \forall x \in E$. We will now prove (2) with case analysis:

  - If $a<-b \Rightarrow a<x<-a \ \ \forall x \in E \Rightarrow \exists r=|a| : |x|<r \ \ \forall x \in E$.
  - If $a>-b \Rightarrow -b<x<b \ \ \forall x \in E \Rightarrow \exists r=|b| : |x|<r \ \ \forall x \in E$.
  - If $a=-b \Rightarrow \exists r=|b|=|a| : |x|<r \ \ \forall x \in E$.

- (1) and (2): **E is bounded** $\iff \exists r>0 : |x|<r \ \ \forall x \in E.$.

Problem 1.3

The following definition will be used.

Supremum and Infimum: Let E be a nonempty set of real numbers that is bounded above. If M is the least of all the upper bounds, then M is said to be the least upper bound of E or the supremum of E, denoted by M = sup E. Similarly, let E be a nonempty set of real numbers that is bounded below. If m is the greatest of all the lower bounds, then m is said to be the greatest lower bound of E or the infimum of E, denoted by m = inf E.

3.1) **Collocation of Supremum:** Let A be a set of real numbers. Show that a real number x is the supremum of A if and only if $a \leq x$ for all $a \in A$ and for every positive number ε there is an element $a' \in A$ such that $x - \varepsilon < a'$.
Proof:

- Prove the forward of the collocation using contradiction:
  - Assume that x is the least upper bound of A. If exists a positive ϵ such that $x - \epsilon > a'$ for all $a' \in A$, $(x - \epsilon)$ will also be an upper bound of A and $(x - \epsilon) < x$. This contradicts with our assumption that x is the least upper bound.

- Prove the backward of the collocation using contradiction:
  - Assume that x is an upper bound of A and for every positive number ε there is an element $a' \in A$ such that $x - \varepsilon < a'$. If $x' \neq x$ is the least upper bound then $x' < x$ and there exists positive $\varepsilon < (x - x')$ such that $x - \varepsilon > a'$ for all $a' \in A$. This contradicts with our assumption that for every positive number ε there is an element $a' \in A$ such that $x - \varepsilon < a'$.

3.2) **Collocation of Infimum:** Let A be a set of real numbers. Show that a real number x is the infimum of A if and only if $a \geq x$ for all $a \in A$ and for every positive number ε there is an element $a' \in A$ such that $x + \varepsilon > a'$

Problem 1.4

4.1) $E = \{\sqrt[n]{n} : n \in \mathbb{N}\}$
The least upper bound is $\sqrt[3]{3}$ as it satisfies two conditions in the collocation about supremum of a set (Problem 1.3):

- $e \leq \sqrt[3]{3}$ for all $e \in E$
- for any $\varepsilon > 0$ there exists $e' = \sqrt[3]{3} \in E$ such that $e' > \sqrt[3]{3} - \varepsilon$

Similarly, the greatest lower bound is 1. The maximum is $\sqrt[3]{3}$ and minimum is 1.

4.2) $E = \{p \in \mathbb{Q} : p^2 \leq 7\}$
The least upper bound is $\sqrt{7}$ as it satisfies two conditions of the collocation of the supremum of a set E (Problem 1.3):

- $e \leq \sqrt{7}$ for all $e \in E$
- for any $\varepsilon > 0$ there exists $e' \in E$ such that $e' > \sqrt{7} - \varepsilon$

Similarly, the greatest lower bound is $-\sqrt{7}$. The maximum is $\sqrt{7}$ and minimum is $-\sqrt{7}$.

4.3) $E = \{n^{(-1)^n} : n \in \mathbb{N}\}$
The set is unbounded above so $\sup E = \infty$ and bounded below with $\inf E = 0$. There is no maximum and minimum.