Linear Algebra with Exercises A - Report 01

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Diagonalizing a 2x2 Matrix

Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

1. The characteristic polynomial of A is:

$$p_A(x) = \det(A - xI) = \begin{vmatrix} 2-x & 1 \\ 1 & 1-x \end{vmatrix} = x^2 - 3x + 1$$

2. Let $\lambda_1 < \lambda_2$ be the roots of P. We have

$$\lambda_1 = \frac{3-\sqrt{5}}{2}$$

and

$$\lambda_2 = \frac{3+\sqrt{5}}{2}$$

3. Now we find the eigenspaces E_1 and E_2 corresponding to the two eigenvalues λ_1 and λ_2 of A. E_1 is the null space of the matrix

$$A - \lambda_1 I = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ is a vector in E_1. We will use elimination to find the null space:

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, we have

$$\frac{1+\sqrt{5}}{2}x + y = 0 \iff y = \frac{2}{1-\sqrt{5}}x$$

Therefore,

$$E_1 = \begin{bmatrix} 1 \\ \frac{2}{1-\sqrt{5}} \end{bmatrix} x$$

where x is an arbitrary constant.

4. Similarly, we find E_2:

$$E_2 = \begin{bmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{bmatrix} x$$

5. To show E_1 and E_2 are orthogonal, we will show that their dot product is equal zero:

$$\begin{bmatrix} 1 \\ \frac{2}{1-\sqrt{5}} \end{bmatrix}^T \begin{bmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{bmatrix} = 0$$

6. Using elimination

$$\begin{bmatrix} G & | & I \end{bmatrix} \rightarrow \begin{bmatrix} I & | & G^{-1} \end{bmatrix}$$

we find the inverse of G:

$$G^{-1} = \begin{bmatrix} -\frac{1}{2\sqrt{5}} & \frac{1+\sqrt{5}}{4\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & -\frac{1-\sqrt{5}}{4\sqrt{5}} \end{bmatrix}$$

7. We will check $AG = G\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with the following calculations:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{2}{1-\sqrt{5}} & \frac{2}{1+\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{4-2\sqrt{5}}{1-\sqrt{5}} & \frac{4+2\sqrt{5}}{1+\sqrt{5}} \\ \frac{3-\sqrt{5}}{1-\sqrt{5}} & \frac{3+\sqrt{5}}{1+\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{1-\sqrt{5}} & \frac{3+\sqrt{5}}{1+\sqrt{5}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ \frac{2}{1-\sqrt{5}} & \frac{2}{1+\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 0 \\ 0 & \frac{3+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{1-\sqrt{5}} & \frac{3+\sqrt{5}}{1+\sqrt{5}} \end{bmatrix}$$

8. Because G is invertible, we have:

$$(6) : AG = G\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \iff A = G\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} G^{-1}$$

9. We will prove this using induction:

Induction Hypothesis: $P(k)$: For any $k \in \mathbb{N}$, $A^k = G\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} G^{-1}$.

Base Case: We have already shown $P(1)$ is true.

Induction Step: Assume $P(k-1)$ is true, we will show $P(k)$ is also true.

$$A^k = A^{k-1}A = G\begin{bmatrix} \lambda_1^{k-1} & 0 \\ 0 & \lambda_2^{k-1} \end{bmatrix} G^{-1} G \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} G^{-1} = G\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} G^{-1}$$

10. We have:

$$A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$$

and

$$G\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} G^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{2}{1-\sqrt{5}} & \frac{2}{1+\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 0 \\ 0 & \frac{3+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2\sqrt{5}} & \frac{1+\sqrt{5}}{4\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & -\frac{1-\sqrt{5}}{4\sqrt{5}} \end{bmatrix}$$

$$G\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} G^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad G\begin{bmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{bmatrix} G^{-1} = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$$