Calculus with Exercises A — Report 11

DO LE DUY

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Problem 10.1

(1) This integral is a definition of the *Gamma Function*.

- Because we have:

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty x^n e^{-x} dx \\
&= \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x} dx \\
&= \lim_{x\to\infty}\left(-x^n e^{-x}\right) - \left(-0^n e^{-0}\right) + n\int_0^\infty x^{n-1} e^{-x} dx \\
&= n\int_0^\infty x^{n-1} e^{-x} dx = n\Gamma(n) \\
\Gamma(1) &= \int_0^\infty x^{1-1} e^{-x} dx \\
&= \left[-e^{-x}\right]_0^\infty \\
&= \lim_{x\to\infty}\left(-e^{-x}\right) - \left(-e^{-0}\right) \\
&= 0-(-1) = 1.
\end{aligned}$$

We would expect the final result: $\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!$. We would get it using an induction proof argument:

- **Base Case**: We already had $\Gamma(1) = 0! = 1$.
- **Induction Step**: Assume that $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = (n-1)!$, then $\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!$. This was already demonstrated above.
- **Conclusion**: We could conclude that

$$\int_0^\infty x^n e^{-x} dx = n!$$

(2)

$$\int_0^1 \frac{1}{1-x^4} dx = \int_0^1 \left(\frac{a}{1-x} + \frac{b}{1+x} + \frac{cx+d}{1+x^2}\right) dx.$$

We have

$$\begin{cases} x^3 : a-b-c=0 \\ x^2 : a+b-d=0 \\ x : a-b+c=0 \\ 1 : a+b+d=1 \end{cases} \implies \begin{cases} a=b=0.25 \\ c=0 \\ d=0.5. \end{cases}$$

Then,

$$\int_0^1 \frac{1}{1-x^4}dx = \int_0^1 \left(\frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}\right)dx.$$

But

$$\int_0^1 \frac{1}{4(1-x)}dx = \lim_{x_1\to 1}\left(\int_0^{x_1} \frac{1}{1-x}dx\right) = \lim_{x_1\to 1}(-\ln(1-x_1)),$$

which diverges. We could conclude that the integral diverges.

Problem 10.2

Proposition. Let f be a positive continuous function on $[1,\infty)$ such that $\lim_{x\to\infty} f(x) = \alpha$. Show that if the integral

$$\int_1^\infty \frac{f(x)}{x}dx$$

converges, then α must be zero.

We will use contradiction to prove the above proposition. Assume that $\lim_{x\to\infty} f(x) = \alpha$, where α is an arbitrary positive number. Then there must exist an $x_0 \geq 1$ such that for all $x \geq x_0$, $f(x)$ is in the neighborhood $(\alpha - t, \alpha + t)$ for a positive $t : \alpha - t > 0$. Then, we have:

$$\int_1^\infty \frac{f(x)}{x} \geq \int_{x_0}^\infty \frac{f(x)}{x} \geq \int_{x_0}^\infty \frac{\alpha - t}{x} = (\alpha - t)(\ln\infty - \ln x_0)$$

, which does not converge. Thus, we have reached a contradiction. The argument is similar for $\alpha < 0$. We could conclude that α must be zero for $\int_1^\infty \frac{f(x)}{x}$ to converge.

Problem 10.3

We will find p in two ways.

- We will use Taylor expansion of $\sin x$ to find p.

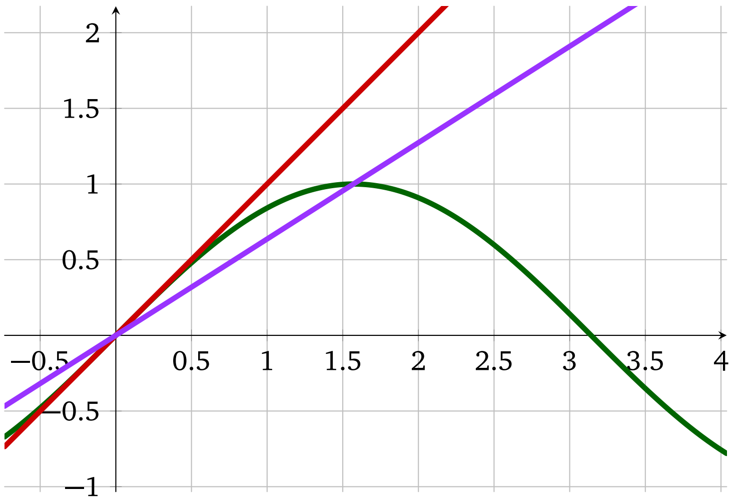
$$\begin{aligned}\sin(x) &= 0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

  Then:

$$\begin{aligned}&\int_0^1 \frac{\sin x}{x^p}dx \\ =&x^{1-p} - \frac{x^{3-p}}{3!} + \frac{x^{5-p}}{5!} - \frac{x^{7-p}}{7!} + \cdots\end{aligned}$$

  The integral is improper for $p > 1$. From the p-Series Theorem, we could conclude that for $1 < p < 2$ the integral is converges, and diverges for $p \geq 2$.

- It is easy to note that the integral is improper for $p > 1$. We will just use simple squeeze theorem to find p that the integral converges or diverges.



Because $\sin(x)$ is strictly concave on $[0, \pi/2]$. We have: $\dfrac{2}{\pi}x \leq \sin(x) \leq x$ for all $x \in [0, \pi/2]$. Thus,

$$\frac{(2/\pi)x}{x^p} = \frac{2}{\pi}x^{1-p} \leq \frac{\sin(x)}{x^p} \leq \frac{x}{x^p} = x^{1-p}$$

on this interval. This is followed by:

$$\frac{2}{\pi}\int_0^{\pi/2} x^{1-p}\mathrm{d}x \leq \int_0^{\pi/2} \frac{\sin(x)}{x^p}\mathrm{d}x \leq \int_0^{\pi/2} x^{1-p}\mathrm{d}x$$

For $p \geq 2$, $\int_0^{\pi/2} x^{1-p}\mathrm{d}x$ diverges, and therefore $\int_0^{\pi/2} \dfrac{\sin(x)}{x^p}\mathrm{d}x$ diverges. For $p < 2$, we have the integral being finite.