Assignment-4

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a) **Input:** Given set C of cities/towns in Turkey and a subset S_C of these (every) cities with a Turkey's route map.

Output: Shortest itinerary I_C from Istanbul to every city/town C in a given set C such that it also includes the subset S_C.

So, computational problem can be described as: "*As an input, given set C of cities/towns in Turkey and a subset S_C of these (every) cities with a Turkey's route map, it computes, as an output, the shortest itinerary I_C from Istanbul to every city/town C in a given set C such that it has also the subset S_C.*"

As it can be seen from the above output, this problem is a computation (optimization) problem, it is not a decision problem but, in NP-complete problems, we deal with the decision problem because it is easier. For this, we interpret this problem as a decision problem: *"Can we find a shortest itinerary I_C of length k from Istanbul to every city/town C in a given set C such that it also includes the subset S_C?"*

Input will be a positive integer k and a Turkey's route map.

Output will YES if there exist a shortest itinerary I_C of length k otherwise the output will be NO.

b) To show this problem, let's call this problem as L, is NP-complete we have to show two steps:

Step-1/Membership: L∈NP

Step-2/Hardness: L'$\leq_p$L where L' is a well known NP-complete problem (Note that both L' and L are decision problems and reduce L' to L).

Step-1/Membership: If there is there is a shortest itinerary I_C of length k from Istanbul to every city/town C in a given set C with the subset S_C, we can verify this shortest itinerary in polynomial time. We can check if it is correct by verifying if for every pair of cities (such as (u,v) where u and v are cities) in the itinerary, there is a path. We also need to check if it provides the conditions such as it provides $S_C$and it finishes the same city in the given set C. Thus, L ∈ NP.

Step-2/Hardness: For this step, we will use the Hamiltonian cycle for L' since it is a well known NP-complete problem. But for this purpose, we need to understand, what is a Hamiltonian cycle?

Definition: *A Hamiltonian cycle is a cycle in a graph G = (V, E) that visits each vertex exactly once.*

We need to show that: *"The Hamiltonian cycle (L') can be reduced in polynomial time to the given shortest itinerary problem (L)."*

For this:

1- We need to select any vertex u in graph G as Istanbul. We do not care which vertex will be Istanbul since G has a Hamiltonian cycle.

2- For any neighbour of u (u is for Istanbul), let's call v for u's neighbour, we have C = {v} and also we have S_C = V-{u, v} so that S_C has the remaining vertices in G. Normally, v is a neighbour of the vertex u so we can have a single path from u to v. However, since S_C has the remaining vertices in G, we also need to visit every remaining city in Turkey's route map exactly once for finding the shortest itinerary from Istanbul (u) to v.

3- For every pair of vertices in G, if there is an edge between these vertices, put the corresponding adjacent cities in Turkey's route map as these vertices and put the path as the edge between these vertices.

4- Put the distance from these adjacent cities to the path as a weight of this edge.

Our claim is L (shortest itinerary problem) has the shortest itinerary of length k (k = for all |V| sum their distance's) from Istanbul (u) to city v with visiting every remaining city in Turkey's route map exactly once <u>if and only if</u> graph G = (V, E) has a Hamiltonian cycle. "If and only if" implies that we have to show the reduction both ways.

We can prove this claim like that:

"=>" *L has the shortest itinerary of length k from Istanbul to city v with visiting every remaining city in Turkey's route map exactly once if graph G has a Hamiltonian cycle.*

We will prove this first condition with contradiction. Suppose that L does not have the shortest itinerary from Istanbul to city v. So we cannot guarantee that G has a Hamiltonian cycle since we do not need to visit all vertices exactly once because we do not have the shortest itinerary. Which creates a contradiction.

"<=" *Graph G has a Hamiltonian cycle if L has the shortest itinerary of length k from Istanbul to city v with visiting every remaining city in Turkey's route map exactly once.*

We will also prove this second condition with contradiction. Suppose that G does not have a Hamiltonian cycle. So we can clearly say that it can visit the same vertices more than once. But then, this situation creates a contradiction because L has the shortest itinerary of length k from Istanbul to city v with visiting every remaining city in Turkey's route map exactly once. So L does not repeat any cities.

Finally, we can clearly say that this reduction is correct as proven above and the Hamiltonian cycle is reduced in polynomial time to the given shortest itinerary problem. Hence, the shortest itinerary problem is NP-complete.