

Support Vector Machine

Hung Le

University of Victoria

January 30, 2019

Binary Classification

You are given a set of n data points $\mathcal{D} = \{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \dots, (\mathbf{x}_n, y_n)\}$ where each $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{-1, 1\}$. Find a **classifier** $f(\cdot) : \mathbb{R}^d \rightarrow \{0, 1\}$ such that:

$$f(\mathbf{x}_i) = \begin{cases} 1 & \text{if } y_i = 1 \\ -1 & \text{if } y_i = -1 \end{cases}$$

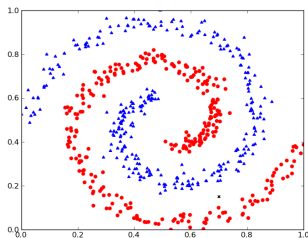


Figure: Spiral data¹

¹<https://www.classes.cs.uchicago.edu/archive/2013/winter/12200-1/>

Applications

- Spam email classification:
 - ▶ Each data point is (\mathbf{x}_i, y_i) where \mathbf{x}_i is a vector representation of i -th email, and $y_i = 1 / -1$ indicates the email is spam/non-spam.
- Testing disease: determine a person as a certain disease or not.
- Weather forecasting: tomorrow is rainy or not.

Support Vector Machine

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- $w^T \mathbf{x}_i + b > 0$ if $y_i = 1$
- $w^T \mathbf{x}_i + b < 0$ if $y_i = -1$

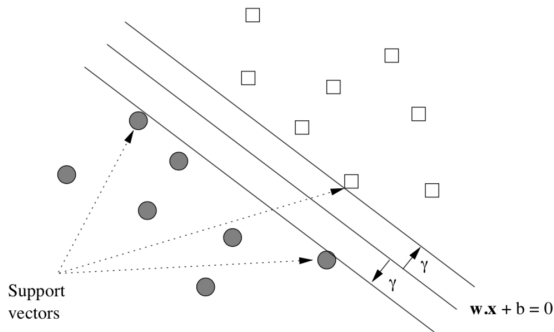
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We assume that our data is **linearly separable**, i.e, there exists such a separating hyperplane.

Support Vector Machine - An Example

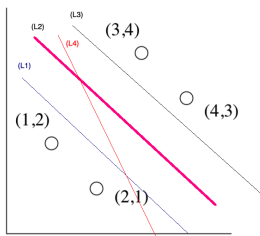


Support Vector Machine - A Toy Example

Given four points $(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, -1)$, $(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, -1)$, $(\begin{bmatrix} 3 \\ 4 \end{bmatrix}, 1)$, $(\begin{bmatrix} 4 \\ 3 \end{bmatrix}, 1)$. Find a separating line $w_1 \cdot x_1 + w_2 \cdot x_2 + b = 0$ for these points.



Support Vector Machine - A Toy Example



There are several possible lines:

$$\begin{aligned} (L_1) : w_1 + w_2 - 4 &= 0 & (L_2) : w_1 + w_2 - 5 &= 0 \\ (L_3) : w_1 + w_2 - 6 &= 0 & (L_4) : w_1 + 2w_2 - 6 &= 0 \end{aligned} \quad (1)$$

Which line should we choose? In theory, any line is acceptable.

SVM separating principle

Choose a line that that **maximizes** the **margin** of the point set to the separating line.

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Margin of a separating line (L) w.r.t the point set \mathcal{D} is the minimum distance of the point set to the line:

$$\gamma(L) = \min_{(\mathbf{x}_i, y_i) \in \mathcal{D}} d(\mathbf{x}_i, L) \quad (2)$$

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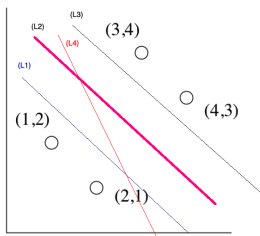
$$\gamma(L) = \min_{(\mathbf{x}_i, y_i) \in \mathcal{D}} d(\mathbf{x}_i, L) \quad (2)$$

Recall, distance from a point $\mathbf{x}_0 \in \mathbb{R}^d$ to the line (L) : $\mathbf{w}^T \mathbf{x} + b = 0$ is:

$$d(\mathbf{x}_0, L) = \frac{|\mathbf{w}^T \mathbf{x}_0 + b|}{\|\mathbf{w}\|_2} \quad (3)$$

where $\|\mathbf{w}\|_2 = \sqrt{\sum_{i=1}^d w[i]^2}$

Back to our Toy Example



$$\begin{aligned} (L_1) : w_1 + w_2 - 4 &= 0 & (L_2) : w_1 + w_2 - 5 &= 0 \\ (L_3) : w_1 + w_2 - 6 &= 0 & (L_4) : w_1 + 2w_2 - 6 &= 0 \end{aligned} \quad (4)$$

SVM will choose (L_2) with $\gamma(L_2) = \sqrt{2}$ (see the board calculation)

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- $w^T \mathbf{x}_i + b > 0$ if $y_i = 1$
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and the margin $\gamma(L)$ is **maximum** among all possible separating hyperplanes.

Points \mathbf{x}_j that have $d(L, \mathbf{x}_j) = \gamma(L)$ are called **support vectors**.

Support Vector Machine

The problem is equivalent to:

Find \mathbf{w} , b that:

$$\text{maximize}(\min_i \frac{|\mathbf{w}^T \mathbf{x}_i + b|}{\|\mathbf{w}\|_2}) \quad (5)$$

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Thus, we can assume that:

- $\mathbf{w}^T \mathbf{x}_j + b = 1$ for all support vectors \mathbf{x}_j of 1-class.
- $\mathbf{w}^T \mathbf{x}_j + b = -1$ for all support vectors \mathbf{x}_j of (-1) -class.

Support Vector Machine

The problem becomes (see the board calculation):

Find \mathbf{w} , b that :

$$\begin{aligned} & \text{minimize} \quad \frac{1}{2} \|\mathbf{w}\|_2^2 \\ & \text{subject to} \quad y_i(\mathbf{w}^T \mathbf{x}_i + b) \geq 1 \end{aligned} \tag{6}$$

Regularization Variant of SVM

Transform the constrained optimization problem from SVM to:

Find \mathbf{w} , b that :

$$\text{maximize} \quad f(\mathbf{w}, b) = \frac{1}{2} \|\mathbf{w}\|_2^2 + C \left(\sum_{i=1}^n \max(0, 1 - y_i(\mathbf{w}^T \mathbf{x}_i + b)) \right) \quad (7)$$

where C is a chosen positive number.

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- When C is sufficiently big, we force the optimization algorithm returning (\mathbf{w}, b) such that $y_i(\mathbf{w}^T \mathbf{x}_i + b) \geq 1$ for all i . This is only possible when the data is linearly separable.

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- When C is chosen appropriately, the optimization problem 7 has a **regularizing effect**.
 - ▶ We accept mis-classified points, but most other points are far away from the hyperplane.
 - ▶ The problem is well-defined even if the data is NOT linearly separable.

C is called the regularization parameter.

Optimization by SGD

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We have:

$$\frac{\partial L_i(\mathbf{w}, b)}{\partial w[j]} = \begin{cases} -y_i x_i[j] & \text{if } y_i(\mathbf{w}^T \mathbf{x} + b) \geq 1 \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

and

$$\frac{\partial L_i(\mathbf{w}, b)}{\partial b} = \begin{cases} -y_i & \text{if } y_i(\mathbf{w}^T \mathbf{x} + b) \geq 1 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

Optimization by SGD

Since:

$$f(\mathbf{w}, b) = \frac{1}{2} \|\mathbf{w}\|_2^2 + C \sum_{i=1}^n L_i(\mathbf{w}) \quad (11)$$

we have:

$$\frac{\partial f(\mathbf{w}, b)}{\partial w[j]} = w[j] + C \sum_{i=1}^n \frac{\partial L_i(\mathbf{w}, b)}{\partial w[j]} \quad (12)$$

and

$$\frac{\partial f(\mathbf{w}, b)}{\partial b} = C \sum_{i=1}^n \frac{\partial L_i(\mathbf{w}, b)}{\partial b} \quad (13)$$