Lesson 1: Introduction to stochastic processes

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- The theory of random processes was first developed in connection with the study of fluctuations and noise in physical systems.

- A random process is the mathematical model of an empirical process whose development is governed by probability laws.

- Random processes provides useful models for the studies of such diverse fields as statistical physics, communication and control, time series analysis, population growth, and management sciences.

Definition

A **random process** is a family of r.v.'s $\{X(t), t \in T\}$ defined on a given probability space, indexed by the parameter t, where t varies over an index set T.

In a random process $\{X(t), t \in T\}$, the index set T is called the parameter set of the random process. The set of possible values of the process, denoted by I , is called **state space**.

For a fixed outcome $\omega \in \Omega$, a plot of a stochastic process $X(t, \omega)$ as a function of t is called a path (**sample path**) or a realization of $X(t)$.

The **ensemble** of a stochastic process is the set of all possible sample paths that can result from an experiment. That is the set of functions $\{X(t,\omega_1),\ldots,X(t,\omega_N)\}$ corresponding to the N outcomes of experiments.

An example of random process is the river flow. It includes both regular changes, such as the seasonal alternations, and irregular variations.

Another example of stochastic process is the number of active calls at a switch at time. The number of active calls is measured for every second over one 15 minute interval. This measurement is taken every day starting at 10AM. An **ensemble average** can be obtained from all measurements for after 10AM. Or a time average can be obtained over a 15-minute interval based on one-day's measurements.

- If the index set T of a random process is discrete, then the process is called a discrete-time process (discrete-parameter process). A discrete-parameter process is also called a random sequence and is denoted by $\{X_n, n = 1, 2, \dots\}$.
- If T is continuous, then we have a continuous-time process (continuous-parameter process).
- If the state space I of a random process is discrete, then the process is called a discrete-state process, often referred to as a chain.
- If the state space I is continuous, then we have a continuous-state process.

Classification of Random Processes

Depending on the continuous or discrete nature of the state space Ω and parameter set T, a random process can be classified into four types:

1. If both T and I are discrete, the random process is called a discrete random sequence.

For example, if X_n represents the outcome of the nth toss of a fair dice, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, \ldots\}$ and $I = \{1, 2, 3, 4, 5, 6\}$.

2. If T is discrete and I is continuous, the random process is called a continuous random sequence.

For example, if X_n represents the temperature at the end of the nth hour of a day, then $\{X_n, n = \overline{1,24}\}$ is a continuous random sequence, since temperature can take any value in an interval and hence continuous.

3. If T is continuous and I is discrete, the random process is called a discrete random process.

For example, if $X(t)$ represents the number of telephone calls received in the interval $(0, t)$ then $\{X(t)\}$ is a discrete random process, since $I = \{0, 1, 2, 3, \dots\}$.

4. If both T and I are continuous, the random process is called a continuous random process. For example, if $X(t)$ represents the maximum temperature at a place in the interval $(0, t)$, $\{X(t)\}$ is a continuous random process.

♣ In the names given above, the word "discrete" or "continuous" is used to refer to the nature of I and the word "sequence" or "process" is used to refer to the nature of T.

A basic very important stochastic process in discrete time is discrete white noise $\{e(t), t \in Z\}$ such that

$$\begin{aligned} E[e(t)] &= 0 \\ Var[e(t)] &= \sigma^2 \\ Cov[e(s), e(t)] &= 0 \text{ for } s \neq t. \end{aligned}$$

White noise is commonly used in the production of electronic music. It is also used to obtain the impulse response of an electrical circuit, in particular of amplifiers and other audio equipment.

Discrete case

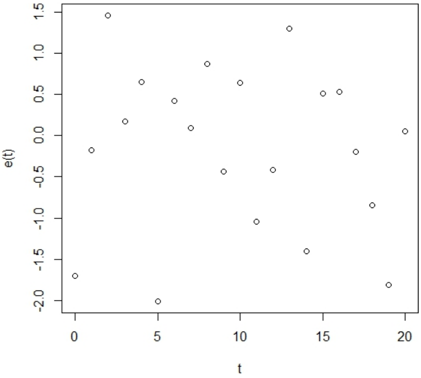

Figure: A sample path of a discrete white noise.

Continuous case

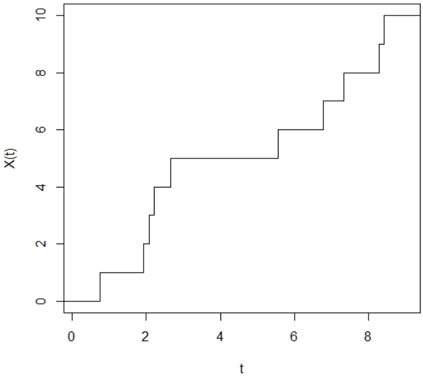

Figure: A sample path of a Poisson process with rate $\lambda = 1$.

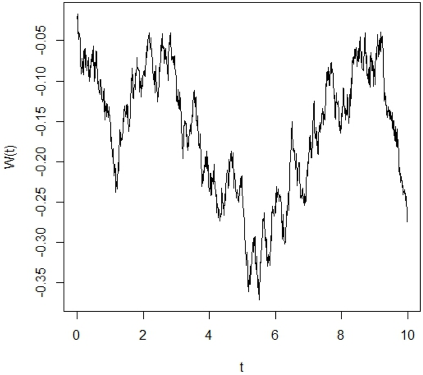

Figure: A sample path of a Wiener process.

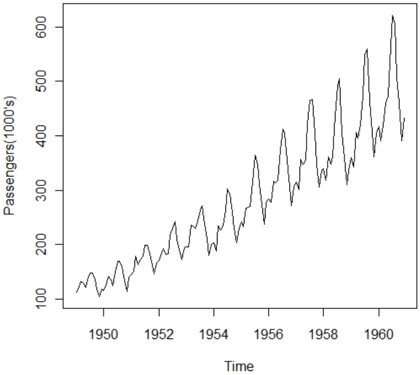

Figure: International air passenger bookings in the United States for the period 1949-1960.

Consider a random process $\{X(t)\}$. For a fixed time t_1, $X_{t_1} = X_1$ is a r.v and its cdf $F_X(x_1, t_1)$ is defined as

$$F_X(x_1; t_1) = P(X(t_1) \leq x_1).$$

- $F_X(x_1, t_1)$ is known as **first order distribution** of X(t).
- Similarly, given t_1 and t_2, $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two r.v.'s. Their joint distribution is known as the **second-order distribution** of $X(t)$ and is given by

$$F_X(x_1, x_2; t_1, t_2) = P(X(t_1) \leq x_1, X(t_2) \leq x_2).$$

- In general, we define the **nth-order** distribution of $X(t)$ by

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n). \quad (1)$$

- If the stochastic process $X(t)$ is discrete-valued, then a collection of probability mass functions can be used to specify the stochastic process:

$$p_X(x_1, \dots, x_n; t_1, \dots, t_n) = P(X(t_1) = x_1, \dots, X(t_n) = x_n) \quad (2)$$

- If the stochastic process $X(t)$ is continuous-valued, then $X(t)$ is specified by

$$f(x_1, \dots, x_n; t_1, \dots, t_n) = \frac{\partial^n F_X(x_1, \dots, x_n; t_1, \dots, t_n)}{\partial x_1 \dots \partial x_n} \quad (3)$$

The **mean** of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)]. \tag{4}$$

where $X(t)$ is treated as a random variable for a fixed value of t.

♣ In general, $\mu(t)$ is a function of time, and it is often called the **ensemble average** of $X(t)$.

The **autocorrelation function** is defined by

$$R_X(t,s) = E[X(t)X(s)] \tag{5}$$

Note that

- $R_X(t,s) = R_X(s,t)$
- $R_X(t,t) = E[X^2(t)]$

The **autocovariance function** of $X(t)$ is defined by

$$\begin{aligned} K_X(t,s) = Cov[X(t),X(s)] &= E\{[X(t)-\mu_X(t)][X(s)-\mu_X(s)]\} \\ &= R_X(t,s)-\mu_X(t)\mu_X(s). \end{aligned} \tag{6}$$

The **correlation coefficient** of the process $X(t)$ is the ratio

$$r(t_1,t_2) = \frac{K_X(t_1,t_2)}{\sqrt{K_X(t_1,t_1)K_X(t_2,t_2)}} = \frac{Cov[X(t_1),X(t_2)]}{\sqrt{var[X(t_1)]var[X(t_2)]}} \tag{7}$$

Note that

- if $E[X(t)]=0$ then $K_X(t,s)=R_X(t,s)$
- the variance of $X(t)$ is

$$\sigma_X^2(t) = Var[X(t)] = E[X(t)-\mu_X(t)]^2 = K_X(t,t) \tag{8}$$

Example 1. Suppose that $X(t)$ is a process with

$$\mu_X(t) = E[X(t)] = 3, R_X(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$$

We will determine the mean, the variance and the covariance of the r.v.'s $Z = X(5)$ and $W = X(8)$. Clearly,

$$E(Z) = 3, \text{ and } E(W) = 3$$

$$E(Z^2) = R_X(5,5) = 13 \text{ and } E(W^2) = R_X(8,8) = 13$$

$$E(ZW) = R_X(5,8) = 9 + 4e^{-0.6} = 11.195$$

Thus $Var(W) = Var(Z) = 13 - 3^2 = 4$ and their covariance $K_X(5,8) = R_X(5,8) - \mu_X(5)\mu_X(8) = 11.195 - 9 = 2.195$.

- The cross-correlation of two processes $X(t)$ and $Y(t)$ is the function

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] \tag{9}$$

- The cross-covariance of two processes $X(t)$ and $Y(t)$ is

$$K_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2) \tag{10}$$

- Two processes $X(t)$ and $Y(t)$ are called orthogonal if

$$R_{XY}(t_1, t_2) = 0 \text{ for every } t_1 \text{ and } t_2. \tag{11}$$

- They are called uncorrelated if

$$K_{XY}(t_1, t_2) = 0 \text{ for every } t_1 \text{ and } t_2. \tag{12}$$

Definition

A random process $\{X(t), t \in T\}$ is said to be **stationary** or strict-sense stationary if, for all n and for every set of time instants $\{t_i \in T, i = 1, 2, \ldots, n\}$,

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau). \quad (13)$$

for any τ.

♣ Hence, the distribution of a stationary process will be unaffected by a shift in the time origin, and $X(t)$ and $X(t + \tau)$ will have the same distributions for any τ. In other words, a process is stationary if, in choosing any fixed point s as the origin, the ensuing process has the same probability law.

♣ However, strongly stationary processes are never seen in practice and are discussed only for their mathematical properties.

- Loosely speaking a stationary process is one whose statistical properties do not change over time.
- This is an extremely strong definition: it means that all moments of all degrees (expectations, variances, third order and higher) of the process, anywhere are the same.

For the first order distribution,

$$F_X(x,t) = F_X(x,t+\tau) = F_X(x) \tag{14}$$

and

$$f_X(x,t) = f_X(x) \tag{15}$$

Then

$$\mu_X(t) = E[X(t)] = \mu \tag{16}$$

$$Var[X(t)] = \sigma^2. \tag{17}$$

where μ and σ^2 are constants.

Similarly, for the second-order distribution,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_2 - t_1). \tag{18}$$

and

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1). \tag{19}$$

♣ Nonstationary processes are characterized by distributions depending on the points $t_1, \ldots, t_n$.

Example 2. If the white noise process values $\ldots, e(-1), e(0), e(1), \ldots,$ for discrete white noise are independent and identically distributed, then $\{e(t), t \in Z\}$ is a stationary process, because

$$\begin{aligned}
P[e(t_1+h) \le x_1 &, \quad \ldots, e(t_n+h) \le x_n] \\
&= P[e(t_1+h) \le x_1] \times \ldots \times P[e(t_n+h) \le x_n] \\
&= P[e(t_1) \le x_1] \times \ldots \times P[e(t_n) \le x_n] \\
&= P[e(t_1) \le x_1, \ldots, e(t_n) \le x_n].
\end{aligned}$$

Example 3. The process $X(t) = \xi$ for $t \in Z$, where ξ is one single r.v., is a stationary, because

$$P[X(t_1 + h) \leq x_1, \ldots, X(t_n + h) \leq x_n]) = P(\xi \leq \min\{x_1, \ldots, x_n\})$$

does not depend on the value of $h \in Z$.

Definition

If stationary condition (13) of a random process X(t) does not hold for all n but holds for $n \leq k$, then we say that the process $X(t)$ is stationary to order k.

Definition

If $X(t)$ is stationary to order 2, then $X(t)$ is said to be wide-sense stationary (WSS) or weak stationary.

If $X(t)$ is a WSS random process, then we have

- $E[X(t)] = \mu$
- $R_X(t, s) = E[X(t)X(s)] = R_X(|s - t|)$.

♣ Note that a strict-sense stationary process is also a WSS process, but, in general, the converse is not true.

Two processes $X(t)$ and $Y(t)$ are called jointly WSS if each is WSS and their cross-correlation depends only on $\tau = t_1 - t_2$

$$R_{XY}(\tau) = E[X(t+\tau)Y(t)]. \tag{20}$$

Example 4. Suppose that $X(t)$ is a WSS process with autocorrelation $R(\tau) = Ae^{-\alpha|\tau|}$. We now determine the second moment of a r.v $X(8) - X(5)$. Clearly

$$\begin{aligned} E[X(8) - X(5)]^2 &= E[X(8)^2] + E[X(5)^2] - 2E[X(8)X(5)] \\ &= R(0) + R(0) - 2R(3) \\ &= 2A - 2Ae^{-3\alpha}. \end{aligned}$$

Example 5. Discrete white noise $\{e(t), t \in Z\}$ is weakly stationary, as

$$\mu_e(t) = E[e(t)] = 0$$

$$R_e(t, t+\tau) = Cov[e(t), e(t+\tau)] = \begin{cases} \sigma^2 & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}$$

do not depend on t.

Definition

In a random process $X(t)$, if $X(t_i)$ for $i = 1, 2, \ldots, n$ are independent r.v.'s, so that for $n = 2, 3, \ldots$, then we call $X(t)$ an independent random process.

Thus, a first-order distribution is sufficient to characterize an independent random process $X(t)$.

A random process $\{X(t), t \geq 0\}$ is said to have **independent increments** if whenever $0 < t_1 \leq t_2 \leq \ldots \leq t_n$,

$$X(0), X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1}) \quad (21)$$

are independent.

Definition

If $\{X(t), t \geq 0\}$ has independent increments and $X(t) - X(s)$ has the same distribution as $X(t+h) - X(s+h)$ for all $s, t, h \geq 0, s < t$, then the process $X(t)$ is said to have **stationary independent increments**.

If $X(t)$ is a stationary independent increments and if $X(0) = 0$, then

$$E[X(t)] = \mu_1 t \tag{22}$$

$$Var[X(t)] = \sigma_1^2 t \tag{23}$$

$$Var[X(t) - X(s)] = \sigma_1^2 (t - s), \ t > s \tag{24}$$

$$K_X(s, t) = Cov[X(t), X(s)] = \sigma_1^2 \min(t, s) \tag{25}$$

where $\mu_1 = \mu_X(1), \sigma_1^2 = Var[X(1)]$.

♣ From (22), we see that processes with stationary independent increments are nonstationary. Examples of processes with stationary independent increments are Poisson processes and Wiener processes, which are discussed later.