

1) Matrix Factorization

$$\min_{U,V} \text{Obj} = \|X - UV^T\|_F^2 + \alpha \|U\|_F^2 + \beta \|V\|_F^2$$
$$\|A\|_F^2 = \text{Tr}(A^T A)$$
$$\text{Tr}(A) = \text{Tr}(A^T)$$
$$\text{Tr}(AB) = \text{Tr}(BA)$$
$$\frac{\partial \text{Tr}(AB)}{\partial A} = B^T$$
$$\frac{\partial \text{Tr}(A^T B)}{\partial A} = A(B^T + B)$$
$$\frac{\partial \text{Tr}(AA^T B)}{\partial A} = (B^T + B)A$$

Ex1:  $\frac{\partial \text{Obj}}{\partial U} = 2(UV^T V - XV + \alpha U)$

$\frac{\partial \text{Obj}}{\partial V} = 2(VU^T U - X^T U + \beta V)$

Obj is convex theo  $\begin{bmatrix} U \\ V \end{bmatrix}$

$u^t \leftarrow g(V, u^t)$   
 $v^t \leftarrow g(V, u^t)$   
 $\hookrightarrow$  convergence!

2) Non Negative MF

$$\min_{U,V} \text{Obj} = \|X - UV^T\|_F^2 + \alpha \|U\|_F^2 + \beta \|V\|_F^2$$

st:  $U \geq 0, V \geq 0$   
 $\forall i, j: u_{ij} \geq 0, v_{ij} \geq 0$

$\rightarrow \alpha, \beta = 0$   
$$\text{Obj} = \text{Tr}[(X - UV^T)^T (X - UV^T)]$$

$\hookrightarrow \frac{\partial \text{Obj}}{\partial U} = -2XV + 2U V^T V$   
 $\frac{\partial \text{Obj}}{\partial V^T} = -2U^T X + 2U^T U V$

$$u \leftarrow \frac{1}{\|V\|_F} (XV - UV^T V)$$
  
$$v^T \leftarrow \frac{1}{\|U\|_F} (U^T X - U^T U V)$$
  
 $u, v^T \geq 0$   
 $r_u = \frac{u}{\|u\|_F}$   
 $r_v = \frac{v^T}{\|v^T\|_F}$

$$\begin{cases} u \leftarrow u \left( \frac{XV}{u^T V} \right) \\ v^T \leftarrow v^T \left( \frac{u^T X}{u^T u v^T} \right) \end{cases}$$

int:  $u, v \geq 0$

cont:  $u \geq 0, v \geq 0$   
KKT

3) Non neg MF  
orthogonal non-negative MF



3)

$$\min \text{Obj} \|X - UV^T\|_F^2$$

st:  $U^T U = I$   
 $V^T V = I$   
 $U \geq 0$   
 $V \geq 0$

$\hookrightarrow$  KKT

$\hookrightarrow$  mối liên hệ với K-means

K-means:

$$X = (x_1, \dots, x_n) \in \mathbb{R}^{p \times n}$$

$$\min J_K = \sum_{k=1}^K \sum_{i \in C_k} \|x_i - m_k\|^2$$

$$= \sum_i \|x_i\|^2 - \sum_k \frac{1}{n_k} \sum_{i,j \in C_k} x_i^T x_j \quad (*)$$

$x_i \in C_k$

indicator vector:  $h_1, \dots, h_K$

$$h_k = (0, \dots, \underset{i}{1}, \dots, \underset{j}{1}, \dots, 0, 0) \quad / \quad \frac{1}{n_k}$$

$$\begin{aligned} x_i &\in C_k & h_k &\perp h_l \quad k \neq l \\ x_j &\in C_k & h_k^T h_l &= 0 \quad k \neq l \\ & & h_k^T h_l &= \delta_{kl} \end{aligned}$$

$$H = (h_1, \dots, h_K)$$

$$(*) \quad \max H^T W H = \text{Tr}(H^T W H)$$
  
 $H^T H = I$   
 $H \geq 0$   
 $\text{argmin}_H W = X^T X$

Consider:

$$W \succeq H H^T \quad H \geq 0$$

$$\min J_1 = \|XV - H H^T\|_F^2$$
  
 $H \geq 0$   
 $H^T H = I$

$$H = \text{argmin}_{H^T H = I, H \geq 0} \|W - H H^T\|$$
  
với  $\forall k \quad H^T H = I$

$$X \rightarrow U \quad V^T$$
  
 $\text{rank}(U) < \text{rank}(X)$

$$u > 0, u \geq 0$$