Homework Numerical Method in Data Science

Homework

Proportition 1 Tangent plane of a convex function lies below the curve.

It means that $f(x)+ < \nabla f(x), y - x > \leq f(y)$.

Proof

As $f(x)$ is a convex function, with all $\lambda \in [0, 1]$, we obtain:

$$f(\lambda y + (1 - \lambda)x) \leq \lambda f(y) + (1 - \lambda)f(x)$$

Therefore,

$$\begin{aligned}
f(\lambda y + (1 - \lambda)x) &\leq \lambda f(y) + (1 - \lambda)f(x) \\
f(\lambda y + (1 - \lambda)x) - f(x) &\leq \lambda(f(y) - f(x)) \\
\frac{f(\lambda y + (1 - \lambda)x) - f(x)}{\lambda} &\leq f(y) - f(x)
\end{aligned}$$

Set $\lambda \to 0$. Then

$$\begin{aligned}
\lim_{\lambda \to 0} \frac{f(\lambda y + (1 - \lambda)x) - f(x)}{\lambda} &\leq f(y) - f(x) \\
\lim_{\lambda \to 0} \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} &\leq f(y) - f(x) \\
\lim_{\lambda \to 0} \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)}(y - x) &\leq f(y) - f(x) \\
\nabla f^T(x)(y - x) &\leq f(y) - f(x) \\
< \nabla f(x), y - x > &\leq f(y) - f(x)
\end{aligned}$$