

# CONCRETE DUALITY FOR STRICT $\infty$ -CATEGORIES

G.V. KONDRATIEV

**ABSTRACT.** An elementary theory of strict  $\infty$ -categories with application to concrete duality is given. New examples of first and second order concrete duality are presented.

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## 0. Introduction

*"One should jump over not just theorems but whole theories as well"*

V.I. Arnold

This work is an analysis of the concept duality being used in modern mathematics. The main theorem 6.1.1 is a generalization for strict  $\infty$ -categories of Porst-Tholen criterion of existence of a concrete natural dual adjunction for usual 1-categories.

### 0.1. Why duality is effective

Duality is one of the fundamental recurring ideas in all of mathematics, and category theory provides the appropriate framework for defining and analyzing the idea that “opposite structures can reflect one another”. Some of the most famous theorems in mathematics are duality theorems. We are thinking in particular of Pontryagin duality, Gelfand -Naimark duality and Stone duality.

The computational power of such dualities increases since one can choose that side which works in the simplest and most effective manner in a given situation. Most mathematicians, when they develop practical techniques, use some kind of duality (such as distributions in functional analysis or flows in differential topology) which simplifies the main ideas and formulations. It would probably not be a sufficient reason for using duality if everything was exactly mirrored. Some properties are not preserved under categorical equivalence. This gives rise to an additional dimension for those new constructions which are not reducible to either of the opposites. Notions such as schemes arise in this way. Historically, the abstract concept of duality was introduced much later than numerous (famous) concrete examples. A deep categorical analysis of first order duality was given in [P-Th].

### 0.2. Not everything is preserved under duality. Bifurcation theory

As an example, this phenomenon is well-known in the qualitative theory of differential equations when small changes of parameter cause catastrophes in the solution space (under the general duality of differential equations to their solution spaces). This is the subject of bifurcation theory. The same phenomenon holds for algebraic equations and for any type of equations and their deformations in the previous sense. The reason of this phenomenon is that duality between ‘big’ categories of equations and their solution spaces is always 1-duality and ‘never’ 2-duality. The exact criterion for the order of duality can be translated to a criterion for bifurcations.

### 0.3. Development of Modern Geometry

**Duality** plays a central role in the principal steps of the development of Modern Geometry. It is now a standard tool to talk about spaces which are unknown but which are well representable by their dual objects. All the development of modern algebraic geometry can be regarded as a sequence of extensions of algebraic duals, which can be seen from the following diagram:

$$\begin{array}{ccccc}
\text{AlgVar}^{op} & \xrightarrow{\sim} & \text{FinGenComAlg} & & \\
\downarrow & & \downarrow & & \\
\text{AffSchemes}^{op} & \xrightarrow{\sim} & \text{ComAlg}^c & \longrightarrow & \text{AntiComALg}^c \longrightarrow \text{DiffAntiComAlg} \\
\uparrow & & \uparrow & & \downarrow \\
\text{Diff}^{op} & \xrightarrow{\sim} & \text{FinGenSmoothComAlg} & & \text{NonComALg}^c \longrightarrow \text{DiffNonComAlg} \\
\downarrow & & \downarrow & & \uparrow \\
\text{SmAffSchemes}^{op} & \xrightarrow{\sim} & \text{SmComAlg} & & \text{NonComSp}^{op} \hookrightarrow \text{SolSpNCDiffEq}^{op}
\end{array}$$

It is still a compact diagram, some steps are omitted, some extensions are not unique (e.g., for algebraic geometry, it is better to regard commutative algebras as anticommutative ones concentrated in degree 0. But for algebraic topology it is natural to regard graded commutative algebras as graded anticommutative ones with degrees of all elements doubled). One of the key ideas of this thesis is that the  $\infty$ -category setting allows us to expand this diagram in a new (homotopical) dimension. So that (monoidal)  $\infty$ -categories give an appropriate framework for Homotopical Algebraic Geometry.

#### 0.4. Low-dimensional and $\infty$ -dimensional Approaches to Homotopy Theory

Higher dimensional functors preserve “homotopy” invariants but not in a canonical way, i.e. they usually do not preserve  $\pi_*$ ,  $H_*$ ,  $H^*$ , etc. This is because these “homotopy” invariants are not formulated **internally** in a category. For example, regard the classifying space functor  $B : \mathbf{wTopGrp} \rightarrow \mathbf{Top}$ . In  $\mathbf{wTopGrp}$  (category of weak topological groups) there are two (noncomparable in general) 2-categorical structures: when 2-cells are conjugations and when they are homotopy classes of homotopies; [the last structure is weaker if we restrict ourselves to a subcategory of path connected weak topological groups]. We note:

**Proposition 0.4.1.** *The classifying space functor  $B : \mathbf{wTopGrp} \rightarrow \mathbf{Top}$  is*

- a 2-functor with respect to conjugations in  $\mathbf{wTopGrp}$ ,
- a 2-functor and 2-equivalence (not 1-equivalence) with respect to homotopy classes of homotopies in  $\mathbf{wTopGrp}$ . Its quasiinverse is the loop space functor  $\Omega : \mathbf{Top} \rightarrow \mathbf{wTopGrp}$ .  $\square$

One would expect that there are many relations between conjugation invariants and homotopy invariants for  $\mathbf{wTopGrp}$  and  $\mathbf{Top}$ . Indeed, there are some such, but they are not straightforward. For example,  $H^*(BG)$  is a conjugation-invariant commutative anticommutative algebra. We would expect that it is a subalgebra (or, maybe a quotient algebra) of  $\mathbf{AdInvPol}(g)$  (the algebra of polynomials on the Lie algebra  $g$  invariant under conjugations) but this is not true in general, although  $\mathbf{AdInvPol}(g)$  is isomorphic to  $H^*(BG)$  for compact Lie groups  $G$  (Chern-Weil homomorphism). The relation between  $H^*(G)$  and  $H^*(BG)$  is rather complicated: it is given by the Eilenberg-Moore spectral sequence  $H^*(G) \otimes H^*(BG) \rightarrow 0$ . Why does such a nice equivalence  $B : \mathbf{wTopGrp} \xrightarrow{\sim} \mathbf{Top} : \Omega$  give such complicated relations between homotopy invariants? Because these homotopy invariants are not defined internally.

The typical definitions of homotopy invariants, such as the functor  $\pi_n(X) = [S^n, X]$ , are not invariant under 2(and higher order)-functors, because the  $n$ -spheres  $S^n$  are not traditionally determined categorically. One of the goals of this thesis is to introduce a new notion of homotopy group, which are invariant under higher-order categorical equivalences Our reinterpretation of homotopy is as follows:

**Definition 0.4.1.**

- For an  $\infty$ -category  $\mathbf{C}$ ,  $I \in Ob \mathbf{C}$ , and a point  $x : I \rightarrow X$  (**formal homotopy groups** of  $X$  are defined as follows  $\tilde{\pi}_n^I(X, x) := \mathbf{Aut}(e^{n-1}x)/\sim$  (where  $e^{n-1}$  is  $n - 1$  times application of the identity operation  $e$ ).
- When functors  $\mathbf{Aut}(e^{n-1}(-))/\sim$  are **representable** the representing objects  $\tilde{S}^n$  are called (**formal spheres** (in this case we have  $\tilde{\pi}_n^I(X, x) := (\mathbf{Aut}(e^{n-1}(x))/\sim) \xrightarrow{\sim} [\tilde{S}^n, X]$ ).  $\square$

For  $\infty$ -**Top** when  $I = \mathbf{1}$  there is a homomorphism of the usual homotopy groups into our formal ones  $\pi_n(X, x) \rightarrow \tilde{\pi}_n^I(X, x)$  (induced by the quotient map  $I^n/(I^{n-1} \times 0) \cup (I^{n-1} \times 1) \rightarrow S^n$ ). For a category  $\infty$ -**TopMan**<sub>b</sub> of topological manifolds with boundary and homotopies relative to the boundary, formal homotopy groups coincide with the usual ones  $\tilde{\pi}_n^I = \pi_n$  when  $I = \mathbf{1}$ . For  $\infty$ -**Top**<sub>\*</sub> (pointed spaces and maps)  $\tilde{\pi}_n^I(X, x) = [\mathbf{1}, X]$  are trivial for all  $n$  although  $[S^n, X]$  gives the usual homotopy groups.

Both functors  $B$  and  $\Omega$  preserve the homotopy type of  $\mathbf{1}$ . So,  $\tilde{\pi}_n^I(G) = \tilde{\pi}_n^I(BG)$ . But these groups are trivial and they give no information (if we change  $\mathbf{1}$  to a more complicated object  $I \in Ob \mathbf{Top}$  then the information can be very nontrivial).

**Proposition 0.4.2.** *If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is an  $\infty$ -equivalence between full topological subcategories of  $\infty$ -**TopMan**<sub>b</sub> such that  $\mathbf{1} \sim F(\mathbf{1})$  then  $F$  preserves the usual homotopy groups.*  $\square$

## 0.5. Concrete Duality

The underlying philosophy of our theory of concrete duality is that the world is nonlinear and opposites converge rather than diverge.

**Definition 0.5.1.** Two  $n$ -categories  $\mathcal{A}$  and  $\mathcal{B}$  are called **concretely dual** if there exists a “schizophrenic” object  $D$  living in both of these categories such that hom-functors  $\mathcal{A}(-, D) : \mathcal{A} \rightarrow n\text{-Cat}$  and  $\mathcal{B}(-, D) : \mathcal{B} \rightarrow n\text{-Cat}$  factor through the other category, i.e.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ & \searrow \mathcal{A}(-, D) & \downarrow V \\ & & n\text{-Cat} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{G} & \mathcal{A} \\ & \searrow \mathcal{B}(-, D) & \downarrow U \\ & & n\text{-Cat} \end{array}$$

where  $F, G$  are equivalences quasiinverse to each other.  $\square$

The higher order the duality is, the more (homotopy) invariants are preserved.

If  $\mathcal{A} \hookrightarrow n\text{-Top}$  then the forgetful functor  $U : \mathcal{A} \rightarrow n\text{-Cat}$  is usually the composite of inclusion and the  $n$ -groupoid functor  $\mathcal{A} \hookrightarrow n\text{-Top} \xrightarrow{n\text{-Top}(\mathbf{1}, -)} n\text{-Cat}$  [by Grothendieck’s hypothesis  $\infty\text{-Top}(\mathbf{1}, -) : \infty\text{-Top} \rightarrow \infty\text{-Cat}$  is an equivalence with its image].

The above factorization (lifting) of hom-functors is frequently **initial**. For first order categories it was proven by Porst and Tholen [P-Th] that initial means maximal and any other concrete duality factors through the initial (natural) one; for higher order categories, the analogous statement has not been proven yet. We hope that the structures introduced here will be useful in extending this result to the higher order case.

**Proposition 0.5.1.**

- Every (weak) duality (adjunction)  $\mathcal{A} \begin{array}{c} \xrightarrow{F} \\[-1ex] \xleftarrow{G} \end{array} \mathcal{B}$  is concrete (over  $\mathcal{C}$ ) if there are **representable** forgetful functors  $U : \mathcal{A} \rightarrow \mathcal{C}$  and  $V : \mathcal{B} \rightarrow \mathcal{C}$ . The dualizing object  $D$  is both  $FI$  and  $GJ$  in a sense to be made precise, where  $I, J$  are representing objects for  $U, V$ .
- If  $\mathcal{A}$  and  $\mathcal{B}$  have representable forgetful functors over  $\mathcal{C}$  and a dualizing object  $D$  such that the corresponding hom-functors  $\mathcal{A}(-, D)$ ,  $\mathcal{B}(-, D)$  satisfy the **initial lifting condition** (essentially, the arrow  $f^n : VX \rightarrow \mathcal{A}(Y, D)$  is a  $\mathcal{B}$ -arrow iff the composite  $VX \xrightarrow{f^n} \mathcal{A}(Y, D) \xrightarrow{ev_{x^n}} \mathcal{A}(I, D)$  is a  $\mathcal{B}$ -arrow  $\forall x^n : I \rightarrow Y$ , and similarly for  $\mathcal{B}(-, D)$ ) then there exists a concrete dual adjunction between  $\mathcal{A}$  and  $\mathcal{B}$  which is natural and strict.
- Concrete natural duality is a strict adjunction. [Higher order duality need not be an adjunction at all]  $\square$

Point 2 of the above proposition is a generalization (for  $n$ -categories) of the Porst-Tholen theorem about concrete duality for first order categories.

The main and most interesting interplay for duality is between algebra and geometry. Certain complicated colimits in algebraic categories are often easily viewed via duality as geometric limits (e.g. the notion of tensor product of algebras is more understandable via the notion of product of manifolds).

Examples of **well-known dualities** are those between algebraic varieties and finitely-generated commutative algebras, between affine schemes and commutative rings (Grothendieck), compact abelian groups and abelian groups (Pontryagin), Boolean algebras and Boolean spaces (Stone), commutative  $C^*$ -algebras and compact Hausdorff spaces (Gelfand-Naimark), and others.

In this paper several new examples of concrete duality are introduced. These include duality for differential equations (introducing anticommutative geometry of solution spaces), Vinogradov duality (formalizing the well-known duality between modules of linear differential operators and jet modules of sections), Gelfand-Naimark 2-duality (extending the usual one to homotopy classes of homotopies), Pontryagin-Lukacs duality (Lukacs' extension of Pontryagin duality to locally precompact abelian groups).

## 1. Categories, functors, natural transformations, modifications

**Definition 1.1.**

- **$\infty$ -precategory** is a (big) set  $L$  endowed with

- (1) a grading  $L = \coprod_{n \geq 0} L^n$
- (2) unary operations  $d, c : \coprod_{n \geq 1} L^n \rightarrow \coprod_{n \geq 0} L^n$ ,  $\deg(d) = \deg(c) = -1$ ,  $dc = d^2$ ,  $cd = c^2$
- (3) a unary operation  $e : \coprod_{n \geq 0} L^n \rightarrow \coprod_{n \geq 0} L^n$ ,  $\deg(e) = 1$ ,  $de = 1$ ,  $ce = 1$
- (4) partial binary operations  $\circ_k$ ,  $k = 1, 2, \dots$ , of degree 0.  $f \circ_k g$  is determined iff  $d^k f = c^k g$

such that each **hom-set**  $L(a, a') := \{f \in L \mid \exists k \in \mathbb{N} \text{ } d^k f = a, c^k f = a'\}$ ,  $\deg(a) = \deg(a')$ , inherits all properties (1)-(4).

- $\forall a, a', a'' \in L^m$  there are maps  $\mu_{a, a', a''} : \coprod_{n \geq 0} L^n(a', a'') \times L^n(a, a') \rightarrow L(a, a'')$  such that if the

bottom composite is determined then

$$\begin{array}{ccc} \coprod_{n \geq 0} L^n(a', a'') \times L^n(a, a') & \xrightarrow{\mu_{a,a',a''}} & L(a, a'') \\ i \times i \uparrow & & \uparrow i \\ L^n(a', a'') \times L^n(a, a') & \xrightarrow{\circ_{n+1}} & L^n(a, a'') \end{array}$$

$\mu_{a,a',a''}$  are called **horizontal composites** on level  $\deg(a)$ ; all composites inside of  $L(a, a')$  are **vertical**.  $\square$

### Remarks.

- Our definition of  **$\infty$ -precategory** coincides with what Penon calls a *magma*; essentially it is a reflexive globular set with all possible binary composites [Lei].
- If  $\alpha^n, \beta^n \in L^n, n > 0$ , such that  $d\alpha^n \neq d\beta^n$  or  $c\alpha^n \neq c\beta^n$ , then  $L(\alpha^n, \beta^n) = \emptyset$  (because of  $d^2 = dc, c^2 = cd$ ). So,  $\mu_{a,a',a''}$  can be the empty map  $\emptyset : \emptyset \rightarrow \emptyset$ .
- It is convenient to use a letter with appropriate superscript, like  $x^m, \alpha^k$ , etc., as an element (or sometimes as a variable) with domain  $L^m, L^k$ , etc. respectively (or with domain  $L^m(a, b), L^k(x, y)$ , etc.) Also, the grading can be taken to range over  $\mathbb{Z}$  under the assumption that  $L^{-m} := \emptyset, m > 0$ .
- Call elements  $a \in L^0$  of degree 0 **objects** of  $L$ , elements  $f^n \in L^n(a, a'), a, a' \in L^0$ , **arrows of degree  $n + 1$  from  $a$  to  $a'$** .
- Denote **horizontal composites** by  $*$ , and extend it over arrows of different degrees by the rule  $* : L(b, c) \times L(a, b) \rightarrow L(a, c) : (g^n, f^m) \mapsto \mu_{a,b,c}(e^{\max(m,n)-n}g^n, e^{\max(m,n)-m}f^m) =: g^n * f^m$  ( $f^m \in L^m(a, b), g^n \in L^n(b, c)$ ).  $\square$

The following definition of equivalence is given “coinductively” (see [J-R])

**Definition 1.2.** For  $a, b \in L^n$   $a \sim b$  iff  $\exists \underset{g}{\underbrace{a \xrightarrow{f} b}}$  such that  $e(a) \sim g \circ_1 f$  and  $f \circ_1 g \sim e(b)$

(it means that there exists an  $f \in L^0(a, b), g \in L^0(b, a)$  and two infinite sequences of arrows of higher order, one in  $L(a, a)$  and the other in  $L(b, b)$ ; all this data we will call *arrows representing the given equivalence*).  $\square$

$\sim$  is reflexive and symmetric, but may be not transitive.

**Lemma 1.1.** If  $L$  is an  $\infty$ -category such that

$\circ_1$  is weakly associative:  $f \circ_1 (g \circ_1 h) \sim (f \circ_1 g) \circ_1 h$  (for composable arrows),

$\circ_1$  satisfies the weak unit law:  $\forall f \in \coprod_{n \geq 1} L^n \left\{ \begin{array}{l} f \circ_1 edf \sim f \\ ecf \circ_1 f \sim f \end{array} \right.$ ,

$\sim$  is compatible with  $\circ_1$ , i.e.  $(f \sim g) \& (h \sim k) \Rightarrow (f \circ_1 h) \sim (g \circ_1 k)$  (for composable arrows),

$\sim$  is transitive in higher orders: i.e. there exists  $m > 0$  such that if  $\sim$  is transitive for  $\coprod_{n \geq m} L^n$ , then  $\sim$  is transitive in all orders.

*Proof.* Let  $a \underset{g}{\underbrace{\xrightarrow{f} b}} \underset{g'}{\underbrace{\xrightarrow{f'} c}}$  be the given equivalences, i.e.  $ea \sim g \circ_1 f, eb \sim f \circ_1 g, eb \sim g' \circ_1 f',$

$ec \sim f' \circ_1 g'$ . Then  $a \underset{g \circ_1 g'}{\underbrace{\xrightarrow{f' \circ_1 f} c}}$  is the required equivalence since  $ea \sim g \circ_1 f \sim g \circ_1 (eb \circ_1 f) \sim g \circ_1 ((g' \circ_1 f') \circ_1 f) \sim (g \circ_1 g') \circ_1 (f' \circ_1 f)$  and similarly  $ec \sim (f' \circ_1 f) \circ_1 (g \circ_1 g')$ .

□

**Remarks.**

- Transitivity in higher orders trivially holds for  $n$ -categories (starting from level  $n$ ), taking  $\sim$  as the identity. For proper  $\infty$ -categories it is better to make the assumption “ $\sim$  is transitive in all orders” from the beginning.
- This lemma shows that although transitivity of  $\sim$  is not automatic for  $\infty$ -precategories, it is indeed consistent with (weak) associativity, the unit law, and compatibility of  $\sim$  with composites. □

**Definition 1.3.** An  $\infty$ -precategory  $L$  with relation  $\sim$  as above is called a (weak)  **$\infty$ -category** iff

- $\sim$  is transitive:  $\alpha \sim \beta \sim \gamma \Rightarrow \alpha \sim \gamma$ ,
- $\sim$  is compatible with all composites:  $(f \sim g) \& (h \sim k) \Rightarrow (f \circ_n h) \sim (g \circ_n k)$  (when they are defined),
- horizontal composites preserve properties (1)-(2) and weakly preserve properties (3)-(4) of  $\infty$ -precategories in the following sense:
  - (1) grading  $\deg_{L(a,a'')}( \mu_{a,a',a''}(f,g) ) = \deg_{L(a',a'')}(f) = \deg_{L(a,a')}(g)$
  - (2)  $\mu_{a,a',a''}(df,dg) = d\mu_{a,a',a''}(f,g)$ ,  $\mu_{a,a',a''}(cf,cg) = c\mu_{a,a',a''}(f,g)$
  - (3)  $\mu_{a,a',a''}(ef,eg) \sim e\mu_{a,a',a''}(f,g)$
  - (4)  $\mu_{a,a',a''}(f \circ_k f', g \circ_k g') \sim \mu_{a,a',a''}(f,g) \circ_k \mu_{a,a',a''}(f',g')$  (“interchange law”)
- each  $\circ_k, k \in \mathbb{N}$ , is **weakly associative**:  $(f \circ_k g) \circ_k h \sim f \circ_k (g \circ_k h)$  (for composable elements),
- The **weak unit law** holds:  $e^k c^k f \circ_k f \sim f$ ,  $f \circ_k e^k d^k f \sim f$  (when all operations are defined). □

**Remarks.**

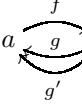
- It is instructive to see what goes wrong if we attempt to consider a bicategory as an instance of this definition. One would think that we could obtain an example by defining  $\sim$  on 1-cells as isomorphism of 1-cells and as equality for 2-cells. However, the problem lies in the horizontal composition of 2-cells which would be required to be strictly associative, whereas in general the horizontal composite of 2-cells is not.
- By lemma 1.1, for  $n$ -categories, the transitivity condition on  $\sim$  follows from the others.
- Hom-sets in an  $\infty$ -category  $L$  are  $\infty$ -categories themselves, and horizontal composites  $* : L(b,c) \times L(a,b) \rightarrow L(a,c)$ , are  $\infty$ -functors.
- Since strict functors preserve the equivalences  $\sim$  for categories in which horizontal composites preserve identity and composites strictly, the compatibility condition on  $\sim$  with composites holds automatically. □

A category is called **strict** if the associativity and unit laws hold for elements (not just for  $\sim$ -equivalence classes) and horizontal composites preserve identities and composites strictly. Note that  $\sim$  still makes sense for strict categories.

**Proposition 1.1.** In a strict  $\infty$ -category  $L$ , arrows of degree  $n$  (i.e.,  $L^n$ ) form a 1-category with objects  $L^0$ , arrows  $L^n$ , domain function  $d^n$ , codomain function  $c^n$ . Observe that  $d, c : L^n \rightarrow L^{n-1}$  are 1-functors. □

**Lemma 1.2.**

- In the strict  $\infty$ -category  $L$   $e^k(f \circ_n g) = e^k f \circ_{n+k} e^k g$  (when either side is defined).
- $\sim$  is preserved under  $\sim$ , i.e., if  $a \xrightarrow{f} a'$  is an equivalence with  $a' \xrightarrow{g} a$ , its quasiinverse (i.e.  $ea \sim g \circ f$ ,  $ea' \sim f \circ g$ ), and if  $f' \sim f$  then  $g$  is quasiinverse of  $f'$  as well.

- A quasiinverse is determined up to  $\sim$ , i.e. if  and  $g' \circ_1 f \sim ea \sim g \circ_1 f$  and  $f \circ_1 g' \sim eb \sim f \circ_1 g$  then  $g' \sim g$ .
- All  $n+1$  composites in  $\mathbf{End}(e^n a) := L^0(e^n a, e^n a)$ ,  $n \geq 0$  coincide up to equivalence  $\sim$ .

*Proof.*

- Assume  $f, g \in L^m$ ,  $m \geq n$ . Then  $f \circ_n g = \mu_{d^n g, c^n g, c^n f}(f, g)$ , which preserves  $e$ .
- $ea \sim g \circ f \sim g \circ f'$ ,  $ea' \sim f \circ g \sim f' \circ g$ .
- $g' = g' \circ_1 eb \sim g' \circ_1 f \circ_1 g \sim g \circ_1 f \circ_1 g \sim g \circ_1 eb = g$ .
- $f \circ_{n+1} g = \mu_{a,a,a}(f, g) \sim \mu_{a,a,a}(f \circ_k e^{n+1} a, e^{n+1} a \circ_k g) \sim \mu_{a,a,a}(f, e^{n+1} a) \circ_k \mu_{a,a,a}(e^{n+1} a, g) \sim f \circ_k g$ ,  $1 \leq k \leq n+1$ .  $\square$

**Definition 1.4.** An arrow  $(f : a \rightarrow a') \in L^0(a, a')$ ,  $\deg(a) = \deg(a') = m \geq 0$ , is called

- **monic** if  $\forall g, h : z \rightarrow a$  if  $f \circ_1 g \sim f \circ_1 h$  then  $g \sim h$
- **epic** if  $\forall g', h' : a' \rightarrow w$  if  $g' \circ_1 f \sim h' \circ_1 f$  then  $g' \sim h'$
- an **equivalence** if there exists  $f' : a' \rightarrow a$  such that  $edf \sim f' \circ_1 f$  and  $edf' \sim f \circ_1 f'$   $\square$

**Proposition 1.2.** For composable arrows

- If  $f, g$  are monics then  $f \circ_1 g$  is monic. If  $f \circ_1 g$  is monic then  $g$  is monic
- If  $f, g$  are epis then  $f \circ_1 g$  is epic. If  $f \circ_1 g$  is epic then  $f$  is epic
- If  $f, g$  are equivalences then  $f \circ_1 g$  is an equivalence  $\square$

**Proposition 1.3.** All arrows representing equivalence  $a \sim b$  are equivalences.  $\square$

**Definition 1.5.** An  $\infty$ -functor  $F : L \rightarrow L'$  is a function which strictly preserves the following properties (1)-(2) of precategories:

- (1) if  $a \in L^n$  then  $F(a) \in L'^n$
- (2)  $F(da) = dF(a), F(ca) = cF(a)$

and weakly preserves the following properties (3)-(4):

- (3)  $F(ea) \sim eF(a)$
- (4)  $F(a \circ_k b) \sim F(a) \circ_k F(b)$   $\square$

**Remark.**

- We do not require the functor  $F$  to preserve equivalences  $\sim$  because it is not automatic and can be too restrictive. However, the functors preserving  $\sim$  are very important (e.g., see point 1.2).
- The inverse map  $F'$  for a bijective weak functor  $F$  is not a functor, in general. If  $F$  preserves  $\sim$  then to say the inverse map  $F'$  is a (weak) functor is equivalent to saying  $F'$  preserves  $\sim$ . The inverse of a strict functor is always a strict functor.  $\square$

**Lemma 1.3.**

- Strict functors preserve equivalences  $\sim$ .
- If functor  $F : L \rightarrow L'$  is such that each restriction on hom-sets  $F_{a,b} : L(a, b) \rightarrow L'(F(a), F(b))$ ,  $a, b \in L^0$ , preserves equivalences  $\sim$ , then  $F$  preserves equivalences  $\sim$ .
- If  $F : L \rightarrow L'$  is an embedding (injective map) such that  $\forall a, b \in L^0 F_{a,b} : L(a, b) \rightarrow L'(F(a), F(b))$  is a strict isomorphism and inverse  $F'$  to codomain restriction of  $F : L \xrightarrow[\substack{F \\ | \\ Im(F)}]{} Im(F) \hookrightarrow L'$  is a functor, then  $F$  reflects  $\sim$ .

*Proof.*

- Each arrow presenting a given equivalence  $x \sim y$  is between a domain and a codomain which are constructed in a certain way only by composites and identity operations from arrows of smaller degree presenting the given equivalence and from elements  $x$  and  $y$ . A strict functor keeps the structure of the domains and codomains of arrows presenting the equivalence  $x \sim y$ . So, the image of arrows presenting an equivalence  $x \sim y$  will be a family of arrows presenting an equivalence  $F(x) \sim F(y)$ .

- For arrows of degree  $> 0$  equivalences are preserved by assumption. Let  $a \xrightarrow[\sim]{f,g} b$ ,  $a, b \in L^0$ ,

be an equivalence for objects in  $L$ , i.e.  $ea \sim g \circ_1 f$ ,  $eb \sim f \circ_1 g$ . Then there are two

opposite arrows  $F(a) \xrightarrow[\sim]{F(f),F(g)} F(b)$ . By assumption,  $F(ea) \sim F(g \circ_1 f)$ ,  $F(eb) \sim F(f \circ_1 g)$ . So,

$eF(a) \sim F(ea) \sim F(g \circ_1 f) \sim F(g) \circ_1 F(f)$  and  $eF(b) \sim F(eb) \sim F(f \circ_1 g) \sim F(f) \circ_1 F(g)$ .

Therefore,  $F(a) \xrightarrow[\sim]{F(f),F(g)} F(b)$  is an equivalence.

- The inverse of a strict isomorphism is a strict isomorphism, i.e. preserves equivalences. So,  $F'$  is a functor which preserves equivalences in all hom-sets and, consequently, preserves all equivalences. Preservation of equivalences for  $F'$  is exactly reflection of equivalences for  $F$ .  $\square$

#### Lemma 1.4.

- $x = y$  iff  $ex \sim ey$  [in particular,  $=$  is definable via  $\sim$ ].
- Functors preserving  $\sim$  strictly preserve all composites  $\circ_k$ ,  $k \geq 1$ .
- Functors weakly preserving  $e^2$  strictly preserve  $e$ , i.e.  $e^2F(a) \sim F(e^2a) \Rightarrow eF(a) = F(ea)$ .
- Quasiequal functors (i.e.  $F(f^n) \sim G(f^n)$  for all  $f^n \in L^n$ ,  $n \geq 0$ ) are equal.

*Proof.*

- $x = y \Rightarrow ex = ey \Rightarrow ex \sim ey$ . Conversely,  $ex \sim ey \Rightarrow dex = dey \Rightarrow x = y$ .
- Sufficient to prove  $eF(f \circ_k g) \sim e(F(f) \circ_k F(g))$ , but it holds  $eF(f \circ_k g) \sim F(e(f \circ_k g)) \sim (F \text{ preserves } \sim) F((ef) \circ_{k+1} (eg)) \sim F(ef) \circ_{k+1} F(eg) \sim eF(f) \circ_{k+1} eF(g) \sim e(F(f) \circ_k F(g))$ .
- $e^2F(a) \sim F(e^2a) \Rightarrow de^2F(a) = dF(e^2a) \Rightarrow eF(a) = F(ea)$ .
- Again, it is sufficient to prove  $eF(f^n) \sim eG(f^n)$ .  
 $eF(f^n) \sim F(ef^n) \sim (\text{by assumption}) G(ef^n) \sim eG(f^n)$ .  $\square$

**Corollary.**  $\infty$ -categories in the sense of definition 1.1.3 are almost strict, namely, with strict associativity, identity, and interchange laws.

*Proof.* Strict associativity and strict identity laws hold because, by the axioms, the functors  $L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f^n, g^n, h^n) \mapsto (h^n * g^n) * f^n$  and  $L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f^n, g^n, h^n) \mapsto h^n * (g^n * f^n)$ ,  $\deg(x) = \deg(y) = \deg(z) = \deg(t)$ , are quasiequal, and, respectively, functors  $L(x, y) \rightarrow L(x, y) : f \mapsto f$  and  $L(x, y) \rightarrow L(x, y) : f \mapsto ey * f$ ,  $\deg(x) = \deg(y)$  (similarly for the right identity), are quasiequal. The strict interchange law holds because the functor  $L(x, y) \times L(y, z) : (f, g) \mapsto g * f$  preserves  $\sim$ .  $\square$

**Definition 1.6.** For two given functors  $F, G$ , an  $\infty$ -natural transformation  $\alpha : F \rightarrow G$  is a function  $\alpha : L^0 \rightarrow L'^1 : a \mapsto (F(a) \xrightarrow{\alpha(a)} G(a))$  such that

$$\mu_{F(a), F(b), G(b)}(e^k \alpha(b), F(f)) \sim \mu_{F(a), G(a), G(b)}(G(f), e^k \alpha(a))$$

for all  $f \in L^k(a, b)$ ,  $k = 0, 1, \dots$   $\square$

**Definition 1.7.** For two given functors  $F, G$  and two natural transformations  $F \xrightarrow[\beta]{\alpha} G$

an  **$\infty$ -modification**  $\lambda : \alpha \rightarrow \beta$  is a function  $\lambda : L^0 \rightarrow L'^2 : a \mapsto (\alpha(a) \xrightarrow{\lambda(a)} \beta(a))$  such that

$$\mu_{F(a), F(b), G(b)}(e^k \lambda(b), F(f)) \sim \mu_{F(a), G(a), G(b)}(G(f), e^k \lambda(a))$$

for all  $f \in L^{k+1}(a, b)$ ,  $k = 0, 1, \dots$   $\square$

Analogously, modifications of higher order are introduced. We call modifications 1-modifications, natural transformations 0-modifications.

**Definition 1.8.** Given two functors  $F, G$ , two 0-modifications  $F \xrightarrow[\alpha_2^0]{\alpha_1^0} G$ ,

two 1-modifications  $\alpha_1^0 \xrightarrow[\alpha_2^1]{\alpha_1^1} \alpha_2^0, \dots$ , two  $n - 1$ -modifications  $\alpha_1^{n-2} \xrightarrow[\alpha_2^{n-1}]{\alpha_1^{n-1}} \alpha_2^{n-2}$

**$\infty$ -n-modification**  $\alpha^n : \alpha_1^{n-1} \rightarrow \alpha_2^{n-1}$  is a function  $\alpha^n : L^0 \rightarrow L'^{n+1}$ :

$a \mapsto (\alpha_1^{n-1}(a) \xrightarrow{\alpha^n(a)} \alpha_2^{n-1}(a))$  such that

$$\mu_{F(a), F(b), G(b)}(e^k \alpha^n(b), F(f)) \sim \mu_{F(a), G(a), G(b)}(G(f), e^k \alpha^n(a))$$

for all  $f \in L^{k+n}(a, b)$ ,  $k = 0, 1, \dots$   $\square$

**Corollary.** All  $n$ -modifications in the sense of Definition 1.1.8 are strict, i.e. all naturality squares commute strictly.

*Proof.* By the conditions in Definition 1.1.8, two functors  $\alpha^n(b)*F(-) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$  and  $G(-)*\alpha^n(a) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$  are quasiequal and, so, equal.  $\square$

**Definition 1.9.**  **$\infty$ -CAT** is an  $\infty$ -category consisting of

- A graded set  $C = \coprod_{n \geq 0} C^n$ , where  $C^0$  are categories,  $C^1$  functors,  $C^n$  ( $n - 2$ )-modifications
- if  $\alpha^n : \alpha_1^{n-1} \rightarrow \alpha_2^{n-1} \in C^n$  then  $d\alpha^n = \alpha_1^{n-1}, c\alpha^n = \alpha_2^{n-1}$
- $e\alpha^n \in C^{n+1}$  is the map  $L^0 \rightarrow L'^{(n+1)} : a \mapsto e(\alpha^n(a))$
- for given two  $n$ -modifications  $\alpha_1^n, \alpha_2^n$  such that  $d^k\alpha_1^n = c^k\alpha_2^n$

$$\alpha_1^n \circ_k \alpha_2^n := \begin{cases} a \mapsto (\alpha_1^n(a) \circ_k \alpha_2^n(a)) & \text{if } k < n + 2 \\ a \mapsto (\alpha_1^n(F'(a)) \circ_{(n+1)} G(\alpha_2^n(a))) & \text{if } k = n + 2, F' = c^{(n+1)}\alpha_2^n, G = d^{(n+1)}\alpha_1^n \end{cases}$$

The first composite works when  $\alpha_1^n, \alpha_2^n \in \infty\text{-CAT}(L, L')$ , the second when  $\alpha_1^n \in \infty\text{-CAT}(L', L'')$   $\alpha_2^n \in \infty\text{-CAT}(L, L')$ , where  $L, L', L''$  are categories.  $\square$

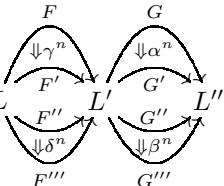
**Lemma 1.5.** In  $\infty\text{-CAT}$  there are two ways of taking horizontal composites (and they are equal):  $\alpha^n * \beta^n := \alpha^n F' \circ_{n+1} G \beta^n = G' \beta^n \circ_{n+1} \alpha^n F$  (where  $F := d^{n+1}\beta^n, F' := c^{n+1}\beta^n, G := d^{n+1}\alpha^n, G' := c^{n+1}\alpha^n$ ).

$$\begin{array}{ccc}
 & G'F(a) \xrightarrow{G'(\beta^n(a))} G'F'(a) & \\
 \text{Proof follows from the naturality square for } \alpha^n & \uparrow \alpha^n(F(a)) & \uparrow \alpha^n(F'(a)) \quad \square \\
 GF(a) \xrightarrow{G(\beta^n(a))} GF'(a) & &
 \end{array}$$

**Proposition 1.4.** *Categories, functors, natural transformations, modifications, etc. form the  $\infty$ -category  $\infty\text{-CAT}$  of  $\infty$ -categories.*  $\square$

*Proof* is similar to that for  $2\text{-CAT}$ .

- Horizontal composites preserve grading (obvious).
- $d, c, e$  are preserved for a similar reason, e.g., take  $d: (d\alpha^n) * (d\beta^n)(a) := (d\alpha^n)(F'(a)) \circ_n G(d\beta^n(a)) = d(\alpha^n(F'(a))) \circ_n d(G(\beta^n(a))) = d(\alpha^n(F'(a)) \circ_{n+1} G(\beta^n(a))) = d((\alpha^n * \beta^n)(a)) = (d(\alpha^n * \beta^n))(a)$  (where  $F' := c^{n+1}\beta^n$ ,  $G := d^{n+1}\alpha^n$ ).

- (interchange law)  (by condition  $d^k\delta^n = c^k\gamma^n$ ,  $d^k\beta^n = c^k\alpha^n$ ,  $k < n + 2$ ,

all  $F$ 's and  $G$ 's are functors)

$$(\beta^n \circ_k \alpha^n) * (\delta^n \circ_k \gamma^n)(a) := (\beta^n \circ_k \alpha^n)(F'''(a)) \circ_{n+1} G((\delta^n \circ_k \gamma^n)(a)) = (\beta^n(F'''(a)) \circ_k \alpha^n(F'''(a))) \circ_{n+1} (G(\delta^n(a)) \circ_k G(\gamma^n(a))) =$$

$$\begin{cases} (\beta^n(F'''(a)) \circ_{n+1} G(\delta^n(a))) \circ_k (\alpha^n(F'''(a)) \circ_{n+1} G(\gamma^n(a))) = (*) \\ (\beta^n(F'''(a)) \circ_{n+1} (\alpha^n(F'''(a)) \circ_{n+1} G(\delta^n(a)))) \circ_{n+1} G(\gamma^n(a)) = (**) \\ \{(*) = ((\beta^n * \delta^n) \circ_k (\alpha^n * \gamma^n))(a) \text{ if } k < n + 1 \text{ (in this case } G = G'', G' = G''', F = F'', F' = F''') \\ \{(**) = \beta^n(F'''(a)) \circ_{n+1} (G'(\delta^n(a)) \circ_{n+1} \alpha^n(F''(a))) \circ_{n+1} G(\gamma^n(a)) = (**) \\ \{(**) = ((\beta^n * \delta^n) \circ_{n+1} (\alpha^n * \gamma^n))(a) \text{ if } k = n + 1 \text{ (in this case } F' = F'', G' = G'') \end{cases}$$

- The associativity law for vertical composites and the identity law hold essentially because of the componentwise definition of vertical composites. The associativity law for horizontal composites is due to the interchange law and lemma 1.5.  $\square$

**Definition 1.10.** A category  $L$  is called an  $\infty$ - $n$ -category if  $L^{j+1} = e(L^j)$  for  $j \geq n$ .  $\square$

A quotient  $L/\sim$  is not a category in general since  $\sim$  is not compatible with  $e$ . However, if we take the quotient only on a fixed level  $n$  and make all higher arrows identities we get  $\infty$ - $n$ -category  $L^{(n)}$ ,  $n$ -th approximation of  $L$ . Generally there are no functors  $L^{(n)} \hookrightarrow L$ ,  $L \twoheadrightarrow L^{(n)}$  (except for the last surjection if  $L$  is a weak  $\infty$ -( $n + 1$ )-category and all  $(n + 1)$ -arrows are isomorphisms's).

### 1.a. Weak categories, functors, natural transformations, modifications.

As we saw above, using a weak language (substituting  $\sim$  for  $=$ ) does not give a weak category theory. The only advantage was that we could deal with  $\sim$  instead of  $=$ , which is important for the classification problem (that still makes sense for strict  $\infty$ -categories). All known definitions of weak categories [C-L, Lei, Koc] are nonelementary (at least, they use functors, natural transformations, operads, monads just for the very definition). Probably, this is a fundamental feature of weak categories. To introduce them we also need the whole universe  $\infty\text{-PreCat}$  of  $\infty$ -precategories.

**Definition 1.a.1.**  $\infty\text{-PreCat}$  consists of

- **$\infty$ -precategories** (definition 1.1) together with  $\sim$ -relation in each [ $\sim$  may be not transitive],
- **$\infty$ -functors** (definition is like 1.5 for  $\infty$ -categories), i.e. functions  $F : L \rightarrow L'$  of degree 0 preserving  $d$  and  $c$  strictly, and  $e$  and  $\circ_k$ ,  $k \geq 1$ , weakly,
- **lax  $\infty$ -n-modifications**,  $n \geq 0$ , i.e. **total** maps  $\alpha^n : L \rightarrow L'$  (with variable degree on different elements, but  $\leq n+1$ , more precisely, the induced map  $\mathbb{N} \rightarrow \mathbb{N} : \deg(x) \mapsto (\deg(\alpha^n(x)) - \deg(x))$  is an antimonotone map, decreasing by 1 at each step from  $n+1$  at  $\deg(x) = 0$  to 1 at  $\deg(x) = n$  and remaining constant 1 after) being defined for a given sequence of two functors  $F, G : L \rightarrow L'$ , two 0-modifications (natural transformations)  $\alpha_1^0, \alpha_2^0 : F \rightarrow G$ , ..., two ( $n-1$ )-modifications  $\alpha_1^{n-1}, \alpha_2^{n-1} : \alpha_1^{n-2} \rightarrow \alpha_2^{n-2}$  as  $\alpha^n :=$

$$\left\{ \begin{array}{ll} (\alpha^n(x) : \alpha_1^{n-1}(x) \rightarrow \alpha_2^{n-1}(x)) \in L'^n(F(x), G(x)) & x \in L^0 \\ \alpha^n(x) := \alpha^n(e^{n+1-k}x) \in L'^{n+1}(F(d^k x), G(c^k x)) & x \in L^k \\ (\alpha^n(x) : \alpha^n(c^{n+1}x) \circ_{n+1} F(x) \rightarrow G(x) \circ_{n+1} \alpha^n(d^{n+1}x)) \in & x \in L^{n+1} \\ \quad \in L'^{n+1}(F(d^{n+1}x), G(c^{n+1}x)) & \\ (\alpha^n(x) : \alpha^n(cx) \circ_1 (e\alpha^n(c^{n+2}x) \circ_{n+2} F(x)) \rightarrow & x \in L^{n+2} \\ (G(x) \circ_{n+2} e\alpha^n(d^{n+2}x)) \circ_1 \alpha^n(dx)) \in L'^{n+2}(F(d^{n+2}x), G(c^{n+2}x)) & \\ \alpha^n(x) : \alpha^n(cx) \circ_1 (e\alpha^n(c^2x) \circ_2 (e^2\alpha^n(c^{n+3}x) \circ_{n+3} F(x))) \rightarrow & x \in L^{n+3} \\ ((G(x) \circ_{n+3} e^2\alpha^n(d^{n+3}x)) \circ_2 e\alpha^n(d^2x)) \circ_1 \alpha^n(dx) \in L'^{n+3}(F(d^{n+3}x), G(c^{n+3}x)) & \\ \dots & \\ \alpha^n(x) : & x \in L^{n+m} \\ \alpha^n(cx) \circ_1 \dots \circ_{m-2} (e^{m-2}\alpha^n(c^{m-1}x) \circ_{m-1} (e^{m-1}\alpha^n(c^{n+m}x) \circ_{n+m} F(x)) \dots) \rightarrow & \\ \underbrace{\dots}_{m-1} & \\ (\dots (G(x) \circ_{n+m} e^{m-1}\alpha^n(d^{n+m}x)) \circ_{m-1} e^{m-2}\alpha^n(d^{m-1}x)) \circ_{m-2} \dots \circ_1 \alpha^n(dx) \in & \\ \in L'^{n+m}(F(d^{n+m}x), G(c^{n+m}x)) & \\ \dots & \\ d\alpha^n := \alpha_1^{n-1}, \quad c\alpha^n := \alpha_2^{n-1} \quad [(d\alpha^n)(x) \neq d(\alpha^n(x)), \quad (c\alpha^n)(x) \neq c(\alpha^n(x)) \text{ if } \deg(x) > 0]. & \end{array} \right. \quad \square$$

### Remarks.

- $\infty$ -n-modifications look terrible but they are the weakest form of naturality (infinite sequences of naturality squares arising by considering naturality squares given by equations  $e_1(x) \sim e_2(x)$  which express  $\sim$ -naturality in  $x$ . This leads to an infinite sequence of naturality squares). To deal with such entities a kind of operad is needed.
- To give an  $n$ -modification  $\alpha^n$  is the same as to give a map  $\alpha^n|_{L^0} : L^0 \rightarrow L'$  of degree  $n+1$  and  $\forall a, b \in L^0$  a natural transformation  $\nu_{a,b}^{\alpha^n} : \alpha^n(b) * F(-) \rightarrow G(-) * \alpha^n(a) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ , where  $F = d^{n+1}\alpha^n$ ,  $G = c^{n+1}\alpha^n$ .
- When  $\alpha^n(x)$ ,  $\deg(x) > 0$ , are all identities (of the required types)  $\infty$ -n-modifications are called **strict**. They are the usual modifications and composable as in definition 1.1.9 when the universe  $\infty\text{-CAT}$  is strict (in that case strict modifications are weak as well). In a weak universe  $\infty\text{-CAT}$  strict modifications need not to be weak (i.e. to be modifications at all).
- **$\infty\text{-PreCAT}$**  is not an  $\infty$ -precategory itself because there are no identities and composites for weak  $n$ -modifications. The problem here with identities and composites is not clear, for example if they exist at all without making either naturality condition or  $\infty$ -categories stricter.

- In general, these two sides “categories and functors” and “ $n$ -modifications” form a strange pair. If we weaken one of these sides, the other one becomes stricter (under condition that  $\infty\text{-CAT}$  is a (let it be very weak) **category**). So, the following **hypothesis** holds:

*There is no  $\infty\text{-CAT}$  with simultaneously weak categories, functors, and  $n$ -modification.*

For example, if we want weak modifications and want them to be composable we need to introduce several axioms on categories, one of which is like ‘ $\forall a, b \in L^0$  and  $\forall$  functors  $F, G : L \rightarrow L'$  if  $\exists$  natural transformations  $\alpha : f_1 * F(-) \rightarrow G(-) * g_1 : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$  and  $\beta : f_2 * F(-) \rightarrow G(-) * g_2 : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$  and  $n+1$ -cells  $f_1, f_2$  and  $g_1, g_2$  are  $\circ_k$ -composable then  $\exists$  a natural transformation ( $k$ -composite)  $\gamma : (f_1 \circ_k f_2) * F(-) \rightarrow G(-) * (g_1 \circ_k g_2) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ ’. But such axioms make very special categories. From the other side, if we want categories to be weak we need to make stricter (maybe, strict)  $n$ -modifications in order that they would be composable. The problem is in existence of composites (and units) for weak  $n$ -modifications.

- Instead of lax  $n$ -modifications we could use modifications with  $\alpha^n(x)$  being  $\sim$  for  $\deg(x) > 0$  in  $L'$ . In both cases in order to make horizontal composites (at least,  $F * \alpha^n := F \circ_{\text{SET}} \alpha^n$ ) we need functors preserving composites (or composites and  $\sim$ ), i.e. ‘weak modifications’  $\Rightarrow$  ‘strict functors’.
- If the above hypothesis was true it would be nice, e.g. a universe where  $\infty\text{-Top}$  lives would contain only strict  $n$ -modifications.  $\square$

**Definition 1.a.2.** A **weak  $\infty$ -category**  $L$  is an  $\infty$ -precategory (see definition 1.1) such that

- $\sim$  is transitive  $x \sim y \sim z \Rightarrow x \sim z$ ,
- horizontal composites  $*$  strictly preserve properties (1)-(2) of precategories
  - (1)  $\deg(x * y) = \deg(x) = \deg(y)$  if  $\deg(x) = \deg(y)$  (interchange law for degree)
  - (2)  $d(x * y) = (dx) * (dy)$ ,  $c(x * y) = (cx) * (cy)$  if  $\deg(x) = \deg(y)$  (interchange law for domain and codomain)
 and weakly preserve properties (3)-(4) of precategories
  - (3)  $e(x * y) \sim (ex) * (ey)$  if  $\deg(x) = \deg(y)$  (interchange law for identity)
  - (4)  $(x \circ_k y) * (z \circ_k t) \sim (x * z) \circ_k (y * t)$  if  $\deg(x) = \deg(y) = \deg(z) = \deg(t)$  (interchange law for composites) [ $\circ_k$  has smaller ‘deepness’  $k$  than the given  $* = \circ_n$ ,  $n > k$ ],
- (**weak associativity**)  
 $\forall x, y, z, t \in L^n$  for two functors  $l_{x,y,z,t} : L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f, g, h) \mapsto (h * g) * f$  and  $r_{x,y,z,t} : L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f, g, h) \mapsto h * (g * f)$   $\exists$  natural transformation  $\alpha_{x,y,z,t} : l_{x,y,z,t} \rightarrow r_{x,y,z,t}$ ,
- (**weak unit**)  
 $\forall x, y \in L^n$  and functors  $u_{x,y}^l : L(x, y) \rightarrow L(x, y) : f \mapsto ey * f$  and  $u_{x,y}^r : L(x, y) \rightarrow L(x, y) : f \mapsto f * ex$   $\exists$  natural transformations  $\epsilon_{x,y}^l : u_{x,y}^l \rightarrow Id$  and  $\epsilon_{x,y}^r : Id \rightarrow u_{x,y}^r$ .  $\square$

### Remarks.

- We do not introduce a universe  $\infty\text{-CAT}$  with weak categories, functors and  $n$ -modifications because there are no (at least, obvious) units and composites for  $n$ -modifications (however, identity natural transformations exist if only the vertical composites of natural transformations are defined, for if  $F : L \rightarrow L'$  is a functor take  $(eF)(a) := e(F(a))$ ,  $a \in L^0$  and by the weak unit law  $\forall a, b \in L^0 \exists$  a natural transformation  $\nu_{a,b} : e(F(b)) * F(-) \rightarrow F(-) * e(F(a)) : L^{\geq 0}(a, b) \rightarrow L'^{\geq 0}(F(a), F(b))$ , take  $\nu_{a,b} := (\epsilon_{F(a), F(b)}^{u^r} \circ_1 \epsilon_{F(a), F(b)}^{u^l}) * F := (\epsilon_{F(a), F(b)}^{u^r} \circ_1 \epsilon_{F(a), F(b)}^{u^l}) \circ_{\text{SET}} F$ ). The problem is what are the weakest conditions on categories, functors and  $n$ -modifications in order that they form a category. Maybe there are several independent such conditions and, so, several categories  $\infty\text{-CAT}$  with weakest entities.

- To keep a usual form of (weak) associativity and (weak) unit we could introduce relations  $\sim_k$  for elements of images of two functors  $F, G : L \rightarrow L'$  connected by a natural transformation  $\alpha : F \rightarrow G$ , namely,  $x \sim_k y$  if  $\exists z \in L^k$  such that  $x = F(z), y = G(z)$ . These relations are not reflexive, symmetric or transitive. Then we could write associativity and unit laws as  $(x \circ_k y) \circ_k z \sim_{k-1} x \circ_k (y \circ_k z)$  and  $e^k c^k x \circ_k x \sim_{k-1} x, x \sim_{k-1} x \circ_k e^k d^k x$ . Under assumption that composites and units exist in an  $\infty\text{-CAT}$  we could choose more sensible piece of  $\infty\text{-CAT}$  with categories in which  $\sim_0 \equiv \sim$  and all  $\sim_k$  are symmetric and transitive by the requirement that  $\alpha_{x,y,z,t}, \epsilon_{x,y}^l, \epsilon_{x,y}^r$  are equivalences.  $\square$

### Examples

1. **2-Top** is a strict  $\infty$ -2-category with 2-cells, as homotopy classes of homotopies, and just identities in higher order ( $\sim$  on the level of objects means homotopy equivalence of spaces, on the level of 1-arrows homotopies of maps, and on the level  $\geq 2$  coincidence). **2-Cat** is similar.
2. It is widely believed that  $\infty\text{-Top}$  is a (weak)  $\infty$ -category with homotopies between homotopies as higher order cells. It is hoped that this notion of (weak)  $\infty$ -category (as above) will be useful in clarifying this issue. Assuming this, we can give two further examples, as follows.
3.  $\infty\text{-Diff}$  is an  $\infty$ -category of differentiable manifolds in the same way as  $\infty\text{-Top}$ .
4.  $\infty\text{-TopAlg}$  is an  $\infty$ -category of topological algebras in the same way as  $\infty\text{-Top}$  where each instance of homotopy is a homomorphism of topological algebras.
5.  $\infty\text{-Compl}$  is an  $\infty$ -category of (co)chain complexes with (algebraic) homotopies for homotopies as higher order cells (see [Lei] ).
6. For a 1-category  $A$ ,  $A_{equiv}$  is a strict  $\infty$ -2-category such that  $A_{equiv}^0 = A^0$ ,  $A_{equiv}^1 = \{f \in$

$$A \mid f \begin{array}{c} \xrightarrow{\exists H} \\ \sim \\ \forall h \end{array} \left. \begin{array}{c} \bullet \\ \bullet \end{array} \right\}, A_{equiv}^2 = \left\{ \text{isomorphisms's} \mid \forall f, g \in A_{equiv}^1 \exists! f \xrightarrow[\sim]{\gamma} g \text{ iff } f \begin{array}{c} \xrightarrow{\exists H} \\ \sim \\ \forall h \end{array} \left. \begin{array}{c} \bullet \\ \bullet \end{array} \right\}. \right. \begin{array}{c} \xrightarrow{\exists H} \\ \sim \\ \forall h \end{array} \left. \begin{array}{c} \bullet \\ \bullet \end{array} \right\}.$$

$A_{equiv}$  contains all equivariant maps  $f : X \rightarrow Y$  with respect to a group homomorphism  $\rho : \mathbf{Aut}(X) \rightarrow \mathbf{Aut}(Y)$ .

7. The (weak) covariant  $\infty\text{-Hom}$ -functor  $L(a, -) : L \rightarrow \infty\text{-CAT}$ :

$$\left\{ \begin{array}{ll} b \mapsto L(a, b) & b \in L^0 \\ (f : b \rightarrow b') \mapsto (L(a, f) : g \mapsto \mu(e^k f, g)) & f \in L^0(b, b'), g \in L^k(a, b) \\ (\alpha : f \rightarrow f') \mapsto (L(a, \alpha) : x \mapsto \mu(\alpha, ex)) & \alpha \in L^1(b, b'), x \in L^0(a, b) \\ (\delta : \alpha \rightarrow \alpha') \mapsto (L(a, \delta) : x \mapsto \mu(\delta, e^2 x)) & \delta \in L^2(b, b'), x \in L^0(a, b) \\ \dots & \\ (\alpha^n : \alpha_1^{(n-1)} \rightarrow \alpha_2^{(n-1)}) \mapsto (L(a, \alpha^n) : x \mapsto \mu(\alpha^n, e^n x)) & \alpha^n \in L^n(b, b'), x \in L^0(a, b) \\ \dots & \end{array} \right.$$

8. The **opposite category**  $L^{op}$  is an  $\infty$ -category such that

- $(L^{op})^n = L^n, n \geq 0$
- $d^{op}(\alpha^n) = \begin{cases} d(\alpha^n) & \text{if } n \geq 2 \\ c(\alpha^n) & \text{if } n = 1 \end{cases}$
- $c^{op}(\alpha^n) = \begin{cases} c(\alpha^n) & \text{if } n \geq 2 \\ d(\alpha^n) & \text{if } n = 1 \end{cases}$
- $e^{op} = e$
- $\beta^n \circ_k^{op} \alpha^n = \begin{cases} \beta^n \circ_k \alpha^n & \text{if } \alpha^n, \beta^n \in L^n, k < n \\ \alpha^n \circ_k \beta^n & \text{if } \alpha^n, \beta^n \in L^n, k = n \end{cases}$  (for composable elements)

9. The (weak) **contravariant  $\infty$ -Hom-functor**  $L(-, b) : L^{op} \rightarrow \infty\text{-CAT}$  :

$$\left\{ \begin{array}{ll} a \mapsto L(a, b) & a \in L^0 \\ (f : a \rightarrow a') \mapsto (L(f, b) : g \mapsto \mu(g, e^k f)) & f \in L^0(a, a'), g \in L^k(a', b) \\ (\alpha : f \rightarrow f') \mapsto (L(\alpha, b) : x \mapsto \mu(ex, \alpha)) & \alpha \in L^1(a, a'), x \in L^0(a', b) \\ (\delta : \alpha \rightarrow \alpha') \mapsto (L(\delta, b) : x \mapsto \mu(e^2 x, \delta)) & \delta \in L^2(a, a'), x \in L^0(a', b) \\ \dots & \\ (\alpha^n : \alpha_1^{(n-1)} \rightarrow \alpha_2^{(n-1)}) \mapsto (L(\alpha^n, b) : x \mapsto \mu(e^n x, \alpha^n)) & \alpha^n \in L^n(a, a'), x \in L^0(a', b) \\ \dots & \end{array} \right.$$

10. The **Yoneda embedding**  $\mathbf{Y} : L \rightarrow \infty\text{-CAT}^{L^{op}} : \alpha \mapsto L(-, \alpha)$ ,  $\alpha \in L$ , where  $L$  is an  $\infty$ -category.

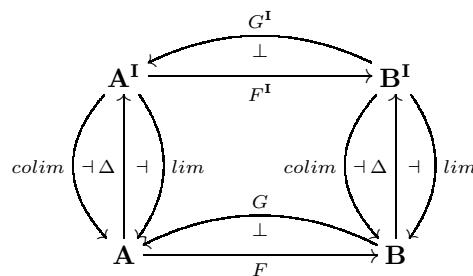
11. **Set** is simultaneously an object and a full subcategory of  $\infty\text{-CAT}$ .

12. A (big) set  $L_\sim := \coprod_{n \geq 0} L_\sim^n$ , where  $L_\sim^n$  are defined recursively as  $L_\sim^0 := L^0$  and  $L_\sim^n$  are all equivalences from  $L^n$  with domain and codomain in  $L_\sim^{n-1}$ , is a subcategory of  $L$ . Similarly,  $L_{k\sim} := \coprod_{n \geq 0} L_{k\sim}^n$ ,  $k \geq 0$ , where  $L_{k\sim}^n := \begin{cases} L^n & n \leq k \\ \text{equivalences from } L^n \text{ with dom and codom in } L_{k\sim}^{n-1} & n > k \end{cases}$  is a subcategory of  $L$ . From this point  $L_\sim = L_{0\sim}$ . Such categories are most important for the classification problem (up to  $\sim$ ). Sometimes, 'invariants' can be constructed only for  $L_\sim$  (see point 1.2.1).

13. **Higher order concepts** can simplify proof of first order facts. E.g., each strict 2-functor  $\Phi : 2\text{-CAT} \rightarrow 2\text{-CAT}$ , where  $2\text{-CAT}$  is the usual strict category of categories, functors, and natural transformations, preserves adjunction (indeed, triangle identities  $\begin{cases} \varepsilon G \circ G\eta = 1_G \\ F\varepsilon \circ \eta F = 1_F \end{cases}$  are respected by  $\Phi \begin{cases} \Phi(\varepsilon)\Phi(G) \circ \Phi(G)\Phi(\eta) = 1_{\Phi(G)} \\ \Phi(F)\Phi(\varepsilon) \circ \Phi(\eta)\Phi(F) = 1_{\Phi(F)} \end{cases}$ ). It gives short proofs of the following results.

a) *Right adjoints preserve limits (left adjoints preserve colimits).*

PROOF.



where  $(-)^I \equiv 2\text{-CAT}(I, -) : 2\text{-CAT} \rightarrow 2\text{-CAT}$  is a hom-2-functor.

Now,  $G^I \circ \Delta = \Delta \circ G$  (obvious). Taking right adjoints of both sides completes the proof  $lim \circ F^I \simeq F \circ lim$  (for colimits the same argument works  $F^I \circ \Delta = \Delta \circ F \Rightarrow colim \circ G^I \simeq G \circ colim$ ).  $\square$

b) *Each 1-Cat-valued presheaf admits a sheafification (1-Cat is a category of small categories and functors between them).*

PROOF. 1-Cat-valued presheaf on  $\mathbf{C}$  is the same as an internal category object in  $\mathbf{Set}^{\mathbf{C}^{op}}$ .

There is an adjoint situation  $\mathbf{Sh}(\mathbf{C}) \xrightarrow{\perp} \mathbf{Set}^{\mathbf{C}^{op}}$  in  $\mathbf{LEX}$ , where  $\mathbf{LEX} \hookrightarrow 2\text{-CAT}$  is a 2-category of finitely complete categories, functors preserving finite limits, and (arbitrary)

natural transformations. There is a 2-functor  $\mathbf{CAT}(-) : \mathbf{LEX} \rightarrow \mathbf{2-CAT}$  assigning to each category in  $\mathbf{LEX}$  the category of its internal category objects and to each functor and natural transformation the induced ones. Then  $\exists$  an adjunction  $\mathbf{CAT}(\mathbf{Sh}(\mathbf{C})) \xrightleftharpoons[\quad]{\perp} \mathbf{CAT}(\mathbf{Set}^{\mathbf{C}^{op}})$  which means that each 1-Cat-valued presheaf can be sheafified by the top curved arrow.  $\square$

### 1.1. Fractal organization of the new universe.

**Fractal Principle.** Object  $A$  with properties  $\{P_i\}_I$  has fractal structure if there are subobjects  $\{A_j\}_J$  which relate to each other in a certain way (express it by additional property  $P =$  'to have  $|J|$  subobjects which relate in the certain way') and each  $A_j$  inherits all properties  $\{P_i\}_I \& P$ .  $\square$

In spite of its complicated structure, each  $\infty$ -category and even  $\infty$ -**CAT** itself, has a regular structure which is repeated for certain arbitrary small pieces. Such pieces are, of course, the hom-sets  $L(a, b)$  which inherit all properties (1)-(4), associativity and identity laws, and each piece of which still has the same structure. In particular,  $L(a, b)(c, d) = L(c, d)$ . An  $\infty$ -functor restricted to such a piece is again an  $\infty$ -functor. Moreover, each  $\infty$ -category can be regarded as a hom-set of a little bit bigger category if we formally attach two distinct elements  $\alpha, \beta \in L^{-1}$  with their identities of higher order  $e^n(\alpha), e^n(\beta)$ ,  $n \geq 1$  (such that  $d(L^0) = \alpha, c(L^0) = \beta$  and composites with these identities of other elements hold strictly). Other natural pieces of  $L$  which inherit all properties and are  $\infty$ -categories are  $L^{\geq n}, L^{\geq n}(a, b)$  (elements of degree not lower than  $n$ ).

### 1.2. Notes on Coherence Principle.

This principle is an axiom to deal with the equivalence relation  $\sim$ . It is not logically necessary for higher order category theory itself. There can be categories in which it does not hold.

**Coherence Principle.** For a given set of cells  $\{a_i\}_I$  and a given set of base equivalences  $\{t_j(\{a_i\}_I) \sim s_j(\{a_i\}_I)\}_J$  for any two constructions  $F_1(\{a_i\}_I)$  and  $F_2(\{a_i\}_I)$  and any two derived equivalences  $\varepsilon_i^0 : F_1(\{a_i\}_I) \sim F_2(\{a_i\}_I)$ ,  $i = 1, 2$  there are derived equivalences  $\varepsilon_m^1 : \varepsilon_1^0 \sim \varepsilon_2^0$ ,  $m \in M^1$ , such that for any two of them  $\varepsilon_{m_1}^1, \varepsilon_{m_2}^1$  there are derived equivalences  $\varepsilon_m^2 : \varepsilon_{m_1}^1 \sim \varepsilon_{m_2}^1$ ,  $m \in M^2$  again such that for any pair of them  $\varepsilon_{m_1}^2, \varepsilon_{m_2}^2$  there are derived equivalences of higher order, etc.  $\square$

Here constructions mean application of composites, functors, natural transformations,.. to  $\{a_i\}_I$ . Derived equivalences mean equivalences obtained from base ones by virtue of the categorical axioms.

## 2. $(m, n)$ -invariants

### Definition 2.1.

- **Equivalence**  $x^k \sim y^k$ ,  $x^k, y^k \in L^k$ ,  $k \geq 0$ , is called **of degree**  $l$ ,  $\deg(\sim) := l$ ,  $l \geq 0$ , if all arrows representing it (starting from order  $k + l + 1$  and higher) are identities and for  $l > 0$  there is at least one nonidentity arrow on level  $k + l$ . If there is no such  $l \in \mathbb{N}$ ,  $\deg(\sim) := \infty$ . Denote  $\sim$  of degree  $l$  by  $\sim_l$ .
- A pair of equivalent elements  $x^k \sim y^k$ ,  $k \geq 0$ , is called **of degree**  $l$ ,  $\deg(x^k \sim y^k) := l$ ,  $l \geq 0$ , if the lowest degree of equivalences existing between  $x^k$  and  $y^k$  is  $l$ .
- An  $\infty$ -category  $L$  is called **of degree 1**,  $\deg(L) = l$ ,  $l \geq 0$ , if for any pair of equivalent objects  $a \sim a'$ ,  $a, a' \in L^0$ , there exists an equivalence  $a \sim_k a'$  of degree  $k \leq l$  and there exists at least one pair of equivalent objects from  $L$  of degree  $l$ .
- A **Functor**  $F : L \rightarrow L'$  is called  **$(m, n)$ -invariant** if  $F$  preserves equivalences  $\sim$ ,  $m = \deg(L)$ ,  $0 \leq n \leq \deg(L')$  and  $F$  maps every pair of equivalent objects of degree  $\leq m$  to a pair of

equivalent objects of degree  $\leq n$ , i.e.  $\deg(a \sim a') \leq m \Rightarrow \deg(F(a) \sim F(a')) \leq n$ , and boundary  $n$  is actually achieved on a pair of equivalent objects of  $L$ .  $\square$

### Remarks.

- $(m, n)$ -invariants are important for the classification problem (up to  $\sim$ ). If  $n < m$  an  $(m, n)$ -invariant decreases complexity of the equivalence relation, i.e. partially resolves it.
- There can be trivial invariants which do not distinguish anything and do not carry any information such as constant functors  $c : L \rightarrow L'$  (although they are  $(\deg(L), 0)$ -invariants).  $\square$

### Examples

1.  $\deg(ea) = 0$ ;  $\deg(f : a \xrightarrow[\text{isomorphisms}]{} a') = 1$ ;  $\deg(\mathbf{Set}) = 1$ ;  $\deg(\infty\text{-}\mathbf{Top}) = 2$ ;  $\deg(\infty\text{-}\mathbf{CAT}) = \infty (?)$ .
2. Homology and cohomology functors  $H_*, H^* : \infty\text{-}\mathbf{Top} \rightarrow \mathbf{Ab}$  (trivially extended over higher order cells) are  $(2, 1)$ -invariants.
3.  $\tilde{\pi}_n^I / \sim : L_{1\sim}^* \rightarrow \mathbf{Grp}$  is an  $(\infty, 1)$ -invariant (see proposition 2.1.2).
4. Let  $X$  be a smooth manifold with Lie group action  $\rho : G \times X \rightarrow X$ ,  $L$  be a category with  $L^0$ , the set of submanifolds of  $X$ ,  $L^1(a, b) := \{(a, g, b) \in L^0 \times G \times L^0 \mid \rho(g, a) = b\}$ ,  $L^n := eL^{n-1}$  for  $n \geq 2$ ,  $L'$  be a category with  $L'^0 := C^\infty(X, \mathbb{R})$  (smooth functions),  $L'^1(f, h) := \{(f, g, h) \in L'^0 \times G \times L'^0 \mid f \circ \rho(g^{-1}, -) = h\}$ ,  $L'^n := eL'^{n-1}$  for  $n \geq 2$ . If  $F : L \rightarrow L'$  is a construction (functor) assigning invariant functions to objects from  $L$  then  $F$  is a  $(1, 0)$ -invariant.
5. Each equivalence  $L \xrightarrow{\sim} L'$  is  $(\deg(L), \deg(L'))$ -invariant with  $\deg(L) = \deg(L')$ .

### 2.1. Homotopy groups associated to $\infty$ -categories.

Let  $L$  be an  $\infty$ -category in which  $*$  strictly preserves  $e$  and  $\sim$  (i.e.  $*$  is a strict functor). Denote by  $eqL := \{f \in L \mid \exists g. edf \sim g \circ_1 f, edg \sim f \circ_1 g\}$  the subset of equivalences of the  $\infty$ -category  $L$ . It may not be a category (because it is not closed under  $d, c$ , in general).

**Definition 2.1.1.** Assume,  $L(I, -) : L \rightarrow \infty\text{-}\mathbf{CAT}$ ,  $x \in L^0(I, a)$ . Then  $\tilde{\pi}_n^I(a, x) :=$

$$\begin{cases} (L^0(I, a), x) & \text{if } n=0 \\ \mathbf{Aut}_{L(I, a)}(e^{n-1}x) := eqL(I, a)(e^{n-1}x, e^{n-1}x) \cap (L(I, a))^0(e^{n-1}x, e^{n-1}x) = \\ = eqL(e^{n-1}x, e^{n-1}x) \cap L^{n+1} & \text{if } n > 0 \end{cases}$$

are (weak) **homotopy groups** of object  $a$  at point  $x$  with representing object  $I \in L^0$ .  $\square$

$\tilde{\pi}_0^I(a, x)$  or  $\tilde{\pi}_0^I(a, x)/\sim$  are just pointed sets,  $\tilde{\pi}_n^I(a, x)/\sim, n > 0$  are strict groups.

### Remarks.

- If  $L = \infty\text{-}\mathbf{Top}$ ,  $I = \mathbf{1}$  then  $\tilde{\pi}_n^I(X, x) = [I^n / (I^{n-1} \times 0) \cup (I^{n-1} \times 1), X]$ . The quotient map  $(I^{n-1} \times 0) \cup (I^{n-1} \times 1) \rightarrow S^n$  induces a group homomorphism  $\pi_n(X, x) \rightarrow \tilde{\pi}_n^I(X, x)$ . If  $L = \infty\text{-}\mathbf{TopMan}_b$  (the infinity category of topological manifolds with boundary as objects, and homotopies relative to the boundary as higher order cells),  $I = \mathbf{1}$  then  $\tilde{\pi}_n^I(X, x) = [I^n / (I^{n-1} \times 0) \cup (I^{n-1} \times 1), X] \text{ rel } (\partial I^n) = [S^n, X] = \pi_n(X, x)$  (i.e. formal homotopy groups coincide with the usual ones).
- In the case when functors  $\tilde{\pi}_n^I$  are **representable** (by certain cogroup objects  $\tilde{S}_I^n$ ) we call the representing objects  $\tilde{S}_I^n$  **(formal) spheres**. It makes sense to define (as usual)  $\tilde{\pi}_n^I(a) := [\tilde{S}_I^n, a]$ , but these two definitions will not always be equivalent. The first one is more internal, and the only external parameter is  $I$ .

- Any  $\infty$ -functor  $F : L \rightarrow L'$ , preserving  $\sim$ , induces (weak) **group homomorphisms**  $F_{I,a} : \tilde{\pi}_n^I(a) \rightarrow \tilde{\pi}_n^{F^I}(Fa)$ . So, for example, every  $\infty$ -equivalence between full subcategories of  $\infty\text{-}\mathbf{TopMan}_b$ , preserving the homotopy type of  $\mathbf{1}$ , will preserve the (usual) homotopy groups.

□

**Definition 2.1.2.** For a map  $f : a \rightarrow b$  such that  $f \circ x = y$ ,  $x \in L^0(I, a)$ ,  $y \in L^0(I, b)$  the **induced map**  $f_* \equiv \tilde{\pi}_n^I(f) : \tilde{\pi}_n^I(a, x) \rightarrow \tilde{\pi}_n^I(b, y)$  is determined by restriction of the functor  $L(I, f) :$

$$\begin{cases} L^0(I, a) \rightarrow L^0(I, b) : x' \mapsto f \circ_1 x' & \text{if } n = 0 \\ \mathbf{Aut}_{L(I, a)}(e^{n-1}x) \rightarrow \mathbf{Aut}_{L(I, b)}(e^{n-1}y) : g \mapsto \mu_{I, a, b}(e^n f, g) & \text{if } n > 0 \end{cases} \quad \square$$

**Remark.** To be correctly defined, induced maps  $\tilde{\pi}_n^I(f)$  for  $n > 1$  need commutativity of  $*$  with  $e$ . The first two “groups”  $\tilde{\pi}_0^I(a, x), \tilde{\pi}_1^I(a, x)$  always make sense and depend functorially on objects. □

**Proposition 2.1.1 (homotopy invariance of homotopy groups).** If  $x : I \rightarrow a$ ,  $f \sim f' \in L^0(a, b)$  such that  $f \circ_1 x \sim f' \circ_1 x$  is a trivial equivalence (all arrows for  $\sim$  are identities) then  $\tilde{\pi}_n^I(f)/\sim = \tilde{\pi}_n^I(f')/\sim : \tilde{\pi}_n^I(a, x)/\sim \rightarrow \tilde{\pi}_n^I(b, f \circ x)/\sim$ .

*Proof* is immediate. □

**Proposition 2.1.2.**  $\tilde{\pi}_n^I/\sim : L_{1\sim}^* \rightarrow \mathbf{Grp}$  is an  $(\infty, 1)$ -invariant, where  $L_{1\sim}^* := \coprod_{n \geq 0} L_{1\sim}^{*n}$ ,  $L_{1\sim}^{*n} :=$

$$\begin{cases} L^{*n} \text{ (pointed objects and maps)} & n = 0, 1 \\ \text{equivalences from } L^n \text{ with dom and codom in } L_{1\sim}^{*(n-1)} & n > 1 \end{cases}.$$

*Proof.* The partial functor  $\tilde{\pi}_n^I/\sim : L^{*0} \coprod L^{*1} \rightarrow \mathbf{Grp}$  is trivially extendable starting from equivalences on level 2 (because of proposition 2.1.1). □

### Example (Fundamental Group)

Let  $2\text{-}\mathbf{Top}$  be the usual  $\mathbf{Top}$  with homotopy classes of homotopies as 2-cells. Define the **fundamental groupoid** 2-functor as the representable  $\Pi(-) := \mathbf{Hom}_{2\text{-}\mathbf{Top}}(\mathbf{1}, -) : 2\text{-}\mathbf{Top} \rightarrow 2\text{-}\mathbf{Cat}$ :

$$\begin{cases} X \rightarrow \Pi(X) & \mathbf{Ob}(\Pi(X)) \text{ are its points, } \mathbf{Ar}(\Pi(X)) \text{ are homotopy classes of paths} \\ (X \xrightarrow{f} Y) \mapsto \Pi(f) & \text{transformation of fundamental groupoids, } \Pi(f) : \begin{cases} x \mapsto f(x) \\ [\gamma] \mapsto [f \circ \gamma] \end{cases} \\ (f \xrightarrow{[H]} f') \mapsto \Pi([H]) & \text{nat. trans. } \Pi([H]) = \{[H] * i_x\}_{x \in X} : \mathbf{Hom}_{2\text{-}\mathbf{Top}}(\mathbf{1}, f) \xrightarrow{\sim} \mathbf{Hom}_{2\text{-}\mathbf{Top}}(\mathbf{1}, f') \\ (\text{where } \{[H] * i_x\}_{x \in X} = \{[H(x, -)]\}_{x \in X}) & \text{are homotopy classes of paths between } f(x) \text{ and } f'(x) \text{ natural} \\ \text{in } x \in X. & \end{cases}$$

$\pi_1(X, x_0) := \mathbf{Aut}_{\Pi(X)}(x_0) \hookrightarrow \Pi(X)$  is the **fundamental group** of the space  $X$  at point  $x_0 \in X$ ,  $\pi_1((X, x_0) \xrightarrow{f} (Y, y_0)) := \mathbf{Aut}_{\Pi(X)}(x_0) \xrightarrow{\Pi(f)} \mathbf{Aut}_{\Pi(Y)}(y_0)$ .

### Proposition 2.1.3.

- If  $[H] : f \xrightarrow{\sim} f' : X \rightarrow Y$  is a 2-cell in  $2\text{-}\mathbf{Top}$  then  $\pi_1(f')([\gamma]) = [H(x_0, -)] \circ \pi_1(f)([\gamma]) \circ [H(x_0, -)]^{-1}$  for all  $[\gamma] \in \pi_1(X, x_0)$ .
- In the case  $[H] : f \xrightarrow{\sim} f' : (X, x_0) \rightarrow (Y, y_0)$  is a pointed 2-cell ( $[H(x_0, -)] = 1_{f(x_0)} : f(x_0) \rightarrow f(x_0) = f'(x_0)$ ) then  $\pi_1(f) = \pi_1(f')$ .

$$\begin{array}{ccc}
 & f(x_0) \xrightarrow[\sim]{[H(x_0, -)]} f'(x_0) & \\
 \text{Proof follows from the naturality square} & \downarrow \Pi(f)([\gamma]) & \downarrow \Pi(f')([\gamma]) \\
 f(x_0) & \xrightarrow[\sim]{[H(x_0, -)]} & f'(x_0)
 \end{array}
 \quad \square$$

## 2.2. Duality and Invariant Theory.

**Proposition 2.2.1.** Let  $\mathbf{K}$  be **Set**, **Top** or **Diff**<sup>+</sup> (spectra of smooth completion (see 2.3) of commutative algebras with Zariski topology),  $G$  a group. Then there exists a concrete natural

dual adjunction  $\mathbf{ComAlg}^{op} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{H} \end{array} G\mathbf{-K}$  with  $k$  ( $\mathbb{R}$  or  $\mathbb{C}$ ), its schizophrenic object, such that

$k \in Ob G\mathbf{-K}$  has trivial action of  $G$ , and  $F \circ H : G\mathbf{-K} \rightarrow G\mathbf{-K}$  is a functor “taking the quotient space generated by the equivalence relation  $x \sim y$  iff  $x, y \in \text{Closure}(\text{the same orbit})$ ” (it is essentially the orbit space).  $\square$

### Definition 2.2.1.

- The adjoint object  $\mathcal{A}_X = HX$  for an object  $X$  in  $G\mathbf{-K}$  is called the **algebra of invariants**.
- If  $U : G\mathbf{-K} \rightarrow G\mathbf{-K}$  is an endofunctor then  $\mathcal{A}_{U(X)}$  is called the **algebra of  $U$ -invariants** of the object  $X$ .  $\square$

### Remarks.

- For  $U = (-)^n$ , the  $n$ -fold Cartesian product,  $\mathcal{A}_{U(X)}$  is the **algebra of  $n$  point invariants**.
- For  $\mathbf{K} = \mathbf{Diff}$ ,  $U = \mathbf{Jet}^n$ ,  $\mathbf{Jet}^n(X) := \{j_0^n f \mid f \in \mathbf{Diff}(k, X)\}$ , the set of all  $n$ -jets of all maps from  $k$  to  $X$  at point 0 (with a certain manifold structure obtained from local trivializations), we get **differential invariants**.
- The functor  $U = \mathbf{Jet}^\infty : \mathbf{Diff} \rightarrow \mathbf{Diff}^+$  does not fit into the above scheme, but everything is still correct if  $U : G\mathbf{-K} \rightarrow G\mathbf{-K}_1$  is an extension to  $G\mathbf{-K}_1$ , a category concretely adjoint to **ComAlg**.
- $G$  can be, of course, **Aut**( $X$ ).

According to Klein’s Erlangen Program, every group acting on a space determines a geometry and, conversely, every geometry hides a group of transformations. Properties of geometric objects which are invariant under all transformations are called *geometric* (or invariant or absolute) for the given  $G$ -space and a class of geometric objects.

**The equivalence problem** [Car1, Car2, Vas, Olv, Gar] consists of a  $G$ -space  $X$  and two “geometric objects”  $S_1, S_2$  of the same type on the space  $X$ . It is required to determine if these two objects can be mapped to one another by an element of  $G$ . An approach is to find a (complete) system of invariants of each object.

#### 2.2.1. Classification of covariant geometric objects.

By *covariant geometric objects* we mean objects like submanifolds, foliations or systems of differential equations, i.e., objects which behave contravariantly (!) from the categorical viewpoint and which can be described by a **differential ideal**  $I$  ( $dI \subset I$ ) in  $\Lambda(X)$ , the exterior differential algebra of  $X$ .

**Proposition 2.2.1.1.** Let  $G$  be a Lie-like group (i.e., there exists an algebra of invariant forms on  $G$ ) [A-V-L, Car1, Car2]. Then any  $G$ -equivariant map  $\sigma : G \rightarrow X$  ( $G$  is given with left shift action and  $X$  is a left  $G$ -space) produces a system of invariants of the differential ideal  $I \subset \Lambda(X)$  (with generators of degree 0 and 1) in the following way:

- Take the image  $\bar{\Lambda}_{inv} := \text{Im}(\Lambda_{inv}(G) \hookrightarrow \Lambda(G) \twoheadrightarrow \Lambda(G)/\sigma^*(I))$ , where  $\Lambda_{inv}(G)$  is a subalgebra of left-invariant forms on  $G$ ,  $\sigma^* : \Lambda(X) \rightarrow \Lambda(G)$  is the induced map of exterior differential algebras,  $\sigma^*(I)$  is the smallest differential ideal in  $\Lambda(G)$  containing the image of  $I$  under  $\sigma^*$ .
- Take the module  $\Lambda^0(G) \cdot \bar{\Lambda}_{inv}^1$  generated by 1-forms in  $\bar{\Lambda}_{inv}$  over  $\Lambda^0(G)$ . There is an open set  $\mathcal{O} \subset G$  and a basis  $\{\omega_{inv}^\alpha\}_{\alpha \in A} \subset \bar{\Lambda}_{inv}^1$  for the module  $\Lambda^0(G) \cdot \bar{\Lambda}_{inv}^1$  restricted to  $\mathcal{O}$ , i.e.,  $\forall \omega_{inv}^i \in \bar{\Lambda}_{inv}^1 \exists! \text{ functions } f_\alpha^i \in C^\infty(\mathcal{O}) \text{ such that } \omega_{inv}^i = \sum_\alpha f_\alpha^i \omega_{inv}^\alpha$ . Form set  $J_0 := \{f_\alpha^i\}$ .
- Take the expansion of differentials  $df_\alpha^i = \sum_\beta f_{\alpha\beta}^i \omega_{inv}^\beta$  (over  $\mathcal{O}$ ). Form the set  $J_1 := \{f_{\alpha\beta}^i\}$ .
- Continue this process to get  $J_2 := \{f_{\alpha\beta\gamma}^i\}, \dots, J_n := \{f_{\alpha_1\dots\alpha_{n+1}}^i\} \dots$  Form the set  $J := \bigcup_n J_n$ . Its elements are relative invariants of the differential ideal  $I \subset \Lambda(X)$ .
- Take the algebra  $\mathcal{A}_J \subset C^\infty(\mathcal{O})$ , generated by  $J$ , and take its smooth completion  $\overline{\mathcal{A}_J}$  (see 3.2). Then the ideal  $\text{Rel}(\mathcal{A}_J) \longrightarrow \overline{\text{Alg}(J)} \longrightarrow \overline{\mathcal{A}_J}$ , of all relations of  $\mathcal{A}_J$ , gives absolute invariants of the differential ideal  $I \subset \Lambda(X)$ , where  $\overline{\text{Alg}(J)}$  is the smooth completion of the free algebra generated by  $J$ .

*Proof* follows from the diagrams

$$\begin{array}{ccc} G & \xrightarrow{l_g} & G \\ \sigma \downarrow & & \downarrow \sigma \\ X & \xrightarrow{l_g} & X \end{array} \quad \begin{array}{ccc} \Lambda_{inv}(G) & \xleftarrow{id} & \Lambda_{inv}(G) \\ \sigma^* \uparrow & & \uparrow \sigma^* \\ \Lambda(X) & \xleftarrow{l_g^*} & \Lambda(X) \end{array}$$

and equations  $\omega_{inv}^i = \sum_\alpha f_\alpha^i \omega_{inv}^\alpha \text{ mod}(\sigma^*(I))$ .  $\square$

**Remark.**  $G\text{-Diff}(G, X)$  is in 1-1-correspondence with all sections of the orbit space  $X_G$ . So, if  $X$  is homogeneous then it is exactly the set of all points of  $X$  and  $\sigma : G \rightarrow X = G \xrightarrow{\sim} G \times \{x_0\} \xrightarrow{1 \times i_{x_0}} G \times X \xrightarrow{\rho} X$  is a  $G$ -equivariant map corresponding to the point  $x_0 \in X$ , where  $\rho$  is the given  $G$ -action on  $X$ .

The following result can be found in [Lap]. Although not well-known, it is a fundamental classification of analytic geometric objects.

**Proposition 2.2.1.2 (Exterior differential algebra associated to a group of analytic automorphisms).** Let  $X$  be an analytic  $n$ -dimensional manifold,  $\text{An}(X)$ , its group of automorphisms,  $H^\infty(X) := \{j_0^\infty f \mid f \in \text{Diff}(k^n, X), X \text{ is analytic, Jacobian}(f) \neq 0\}$ , the  $\infty$ -frame bundle over  $X$  (with a usual topology and manifold structure). Then there is an exterior differential  $k$ -algebra  $\Lambda_{inv}(H^\infty(X))$  of invariant forms on  $H^\infty(X)$  freely generated by elements of degree 1 obtained by the following process:

- $\omega^i := x_j^i dx^j$  are any “shift” forms on  $X$
- $\omega_j^i$  are the most general solutions of Maurer-Cartan equations  $d\omega^i = \omega_j^i \wedge \omega^j$
- $\omega_{jk}^i$  are the most general solutions of Maurer-Cartan equations  $d\omega_j^i = \omega_k^i \wedge \omega_j^k + \omega_{jk}^i \wedge \omega^k$
- $\omega_{jkl}^i, \dots, \omega_{i_1\dots i_n}^i, \dots$

All forms are symmetric in the lower indices. They characterize the underlying space of  $\text{An}(X)$  uniquely up to analytic isomorphisms.  $\square$

**Remark.** At each point  $x_0 \in X$ ,  $\omega^i = 0$ , and forms  $\bar{\omega}_{i_1\dots i_n}^i := \omega_{i_1\dots i_n}^i|_{\omega^i=0}$ ,  $n \geq 1$ , are free generators of the exterior differential algebra of the **differential group** acting simply transitively on each fiber of  $H^\infty(X)$ .

### 2.2.2. Classification of smooth embeddings into a Lie group.

This is often the last step of smooth classification of geometric objects [Car2, Fin, Kob]. The process of finding differential invariants is similar to that in Proposition 2.2.1.1. The following is essentially in [Vas0, Vas, Lap].

**Proposition 2.2.2.1.** *For a smooth embedding  $f : X \rightarrow G$  of a smooth manifold  $X$  into a Lie group  $G$ , a complete system of differential invariants of  $f$  can be obtained in the following way:*

- *$Im(f^* : \Lambda_{inv}^1(G) \rightarrow \Lambda(X))$  is locally free, so, has as its basis  $\omega_{inv}^i, i = 1, \dots, n, n = dim(X)$ , near each point.*
- *Coefficients of linear combinations  $\omega_{inv}^I = \sum_{i=1}^n a_i^I \omega_{inv}^i, I = n+1, \dots, dim(G)$ , are differential invariants of first order (of the map  $f$ ).*
- *Coefficients of differentials of invariants of first order  $da_i^I = \sum_{j=1}^n a_{ij}^I \omega_{inv}^j$  are differential invariants of second order (of the map  $f$ ).*
- *... Coefficients of differentials of invariants of  $(k-1)$  order  $da_{i_1 \dots i_{k-1}}^I = \sum_{i_k=1}^n a_{i_1 \dots i_k}^I \omega_{inv}^{i_k}$  are differential invariants of order  $k$  ...*

Such calculated invariants characterize an orbit  $G \cdot f$  uniquely up to “changing the parameter space”  $X \xrightarrow{\sim} X'$ .  $\square$

### 2.3. Tangent functor for smooth algebras.

This is an example of the dual construction for the main functor of Differential Geometry (which suggests how it can be extended over spectra of commutative algebras).

Let  $T : \mathbf{Diff} \rightarrow \mathbf{Diff}$  be the **tangent** functor on the category of real  $\infty$ -smooth manifolds. In local coordinates it is of the form 
$$\begin{cases} X \rightarrow TX : (x^i) \rightarrow (x^i, \xi^j) & X \in Ob \mathbf{Diff} \\ f \rightarrow Tf : (f^i(x)) \rightarrow (f^i(x), \frac{\partial f^j}{\partial x^k} \xi^k) & f \in Ar \mathbf{Diff} \end{cases}$$

$\mathbf{Diff} \hookrightarrow \mathbb{R}\text{-Alg}^{op}$  is a subcategory of the opposite of the category of real commutative algebras. Working in  $\mathbf{Diff}$ , it is hard (if possible at all) to give a coordinate-free characterization of  $T$ . The question is how to characterize the image in  $\mathbb{R}\text{-Alg}$ ?

**Definition 2.3.1.** Let  $\mathcal{A} \in Ob \mathbb{R}\text{-Alg}$ .

- $\rho : \mathcal{A} \rightarrow \mathbf{Top}(\mathbf{Spec}_{\mathbb{R}}(\mathcal{A}), \mathbb{R})$  is called **functional representation** homomorphism of  $\mathcal{A}$ , where  $\mathbf{Spec}_{\mathbb{R}}(\mathcal{A}) = \mathbb{R}\text{-Alg}(\mathcal{A}, \mathbb{R})$  has the initial topology with respect to all functions  $\rho(a), a \in \mathcal{A}$ ,  $\rho(a)(f) := ev(f, a) := |f|(a)$ .
- $\mathcal{A}$  is called **smooth** if  $\forall a_1, a_2, \dots, a_n \in \mathcal{A}$  and  $\forall f : \mathbb{R}^n \rightarrow \mathbb{R} \in C^\infty(\mathbb{R}^n)$  the composite  $f \circ \langle \rho(a_1), \rho(a_2), \dots, \rho(a_n) \rangle \in Im(\rho)$ .  $\square$

Denote by  $\mathbb{R}\text{-Sm-Alg} \hookrightarrow \mathbb{R}\text{-Alg}$  full subcategory of smooth algebras.

**Lemma 2.3.1.**  $\mathbb{R}\text{-Sm-Alg} \hookrightarrow \mathbb{R}\text{-Alg}$  is a reflective subcategory, i.e. the inclusion has a left adjoint  $\mathbf{Sm} : \mathbb{R}\text{-Alg} \rightarrow \mathbb{R}\text{-Sm-Alg}$ , **smooth completion** of  $\mathbb{R}$ -algebras.

*Proof.* Just take for each  $\mathbb{R}$ -algebra  $\mathcal{A}$   $\mathbb{R}$ -algebra  $\mathbf{Sm}(\mathcal{A})$  of all terms  $\{f(a_1, \dots, a_n) \mid f : \mathbb{R}^n \rightarrow \mathbb{R}, a_1, \dots, a_n \in \mathcal{A}\}$  (all smooth operations are admitted). Each morphism  $f$  from an  $\mathbb{R}$ -algebra  $\mathcal{A}$  to a smooth algebra  $\mathcal{B}$  is uniquely extendable to  $\tilde{f} : \mathbf{Sm}(\mathcal{A}) \rightarrow \mathcal{B}$ .  $\square$

Let **Sym-Alg** be the category of symmetric partial differential algebras.  $Ob(\mathbf{Sym-Alg})$  are graded commutative algebras over commutative  $\mathbb{R}$ -algebras with a differential  $d : \mathcal{A}^0 \rightarrow \mathcal{A}^1$  of degree 1 determined only on elements of degree 0 ( $d$  is  $\mathbb{R}$ -linear and satisfies the Leibniz rule).  $Ar(\mathbf{Sym-Alg})$  are graded degree 0 algebra homomorphisms which respect  $d$ .

**Lemma 2.3.2.** *There is an adjunction  $\mathbb{R}\text{-Alg} \xrightleftharpoons[\substack{\perp \\ p_0}]{\text{Sym}} \text{Sym}\text{-Alg}$ ,*

where:  $p_0$  is the projection onto the 0-component  $\begin{cases} p_0(\mathcal{A}) := \mathcal{A}^0 \\ p_0(\mathcal{A} \xrightarrow{f} \mathcal{B}) := (\mathcal{A}^0 \xrightarrow{f^0} \mathcal{B}^0) \end{cases}$ ,

**Sym** is the functor forming the graded symmetric algebra over the module of differentials of the given algebra

$$\begin{cases} \text{Sym}(\mathcal{C}) := \text{Sym}(\Lambda^1(\mathcal{C})) \\ \text{Sym}(\mathcal{C} \xrightarrow{h} \mathcal{D}) := (\text{Sym}(\mathcal{C}) \xrightarrow{\tilde{h}} \text{Sym}(\mathcal{D})) \\ \tilde{h}(\sum c_{i^1 \dots i^k} (dc_1)^{i^1} \dots (dc_k)^{i^k}) := \sum h(c_{i^1 \dots i^k}) (dh(c_1))^{i^1} \dots (dh(c_k))^{i^k} \end{cases}$$

□

**Lemma 2.3.3.**  $\mathbb{R}\text{-Alg} \xrightarrow{\text{Sm}} \mathbb{R}\text{-Sm-Alg}$

(smooth completion does not change spectrum).

*Proof.*  $\forall \alpha : \mathcal{A} \rightarrow \mathbb{R} \exists!$  an extension  $\tilde{\alpha} : \text{Sm}(\mathcal{A}) \rightarrow \mathbb{R} : f(a_1, \dots, a_n) \mapsto f(\alpha(a_1), \dots, \alpha(a_n))$ . And conversely, each such  $\tilde{\alpha}$  is restricted uniquely to  $\alpha$ . Initial topology on  $\mathbb{R}\text{-Alg}((\text{Sm})(\mathcal{A}), \mathbb{R})$  does not change because new functions are functionally (continuously) dependent on old ones. □

**Remark.** With the Zariski topology on spectra, the smooth completion yields the same set with a weaker topology. For  $C^\infty(X)$ ,  $X \in \text{Ob } \mathbf{Diff}$  the Zariski and initial topologies coincide.

**Proposition 2.3.1.** • The tangent functor  $T : \mathbb{R}\text{-Sm-Alg} \rightarrow \mathbb{R}\text{-Sm-Alg}$  is equal to the composite  $\mathbb{R}\text{-Sm-Alg} \hookrightarrow \mathbb{R}\text{-Alg} \xrightarrow{\text{Sym}} \text{Sym}\text{-Alg} \xrightarrow{U} \mathbb{R}\text{-Alg} \xrightarrow{\text{Sm}} \mathbb{R}\text{-Sm-Alg}$ , where  $U$  forgets the differential  $d$  and grading.

$$\bullet \text{ To the canonical projection } p_X : TX \downarrow X \text{ there corresponds a canonical embedding } i_{C^\infty(X)} : C^\infty(X) \hookrightarrow T(C^\infty(X)).$$

*Proof.*

- If  $X \in \text{Ob } \mathbf{Diff}$   $TX \sim \text{Spec}_{\mathbb{R}}(U \circ \text{Sym}(C^\infty(X))) \sim \text{Spec}_{\mathbb{R}}(\text{Sm} \circ U \circ \text{Sym}(C^\infty(X)))$ .
- This is immediate. □

**Remark.** It is reasonable to define  $T$  on  $\mathbb{R}\text{-Alg}$  as  $T := U \circ \text{Sym}$  and transfer it to spectra via duality  $\mathbb{R}\text{-Alg}^{op} \xrightleftharpoons[\substack{\perp \\ G}]{F} \text{Spec}_{\mathbb{R}}$  (as  $F \circ T^{op} \circ G$ ).

### 3. Representable $\infty$ -functors

**Definition 3.1.**  $\infty$ -categories  $L$  and  $L'$  are **equivalent** if  $L \sim L'$  in  $\infty\text{-CAT}$ . □

If equivalence  $L \sim L'$  is given by functors  $L \xrightleftharpoons[\substack{\sim \\ G}]{F} L'$  then  $\forall a \in L^0 a \sim G \circ F(a)$ ,  $\forall b \in L'^0 b \sim F \circ G(b)$  naturally in  $a$  and  $b$ .

**Definition 3.2.**  $\infty$ -functor  $F : L \rightarrow L'$  is (weakly)

- **faithful** if  $\forall a, a' \in L^0 \forall f^n, g^n \in L^n(a, a') F(f^n) \sim F(g^n) \Rightarrow f^n \sim g^n$ ,
- **full** if  $\forall a, a' \in L^0 \forall h^n \in L'^n(F(a), F(a')) \exists f^n \in L^n(a, a')$  such that  $F(f^n) \sim h^n$ ,
- **surjective on objects** if  $\forall b \in L'^0 \exists a \in L^0$  such that  $F(a) \sim b$ .  $\square$

Unlike first order equivalence, there is no simple criterion of higher order equivalence.

**Proposition 3.1.** *If the functor  $L \xrightarrow{\sim} L'$  is an equivalence then  $F$  is (weakly) faithful, full and surjective on objects.*

*Proof.* "⇒" Regard the diagram (where  $G$  is a quasiinverse of  $F$ )

$$\begin{array}{ccc} & e^n \rho_a & \\ a & \swarrow \sim \quad \searrow \sim & G \circ F(a) \\ f^n \downarrow & e^n \theta_a & \downarrow G(F(f^n)) \\ a' & \swarrow \sim \quad \searrow \sim & G \circ F(a') \\ & e^n \theta_{a'} & \end{array}$$

where:  $f^n \in L^n(a, a')$ ,  $e^n \rho_a \in L^n(a, G(F(a)))$ ,  $e^n \theta_a \in L^n(G(F(a)), a)$ ,  $n \geq 0$ .

Take  $f^n, g^n : a \rightarrow a' \in L^n(a, a')$  such that  $F(f^n) \sim F(g^n)$ . Then  $f^n \sim e^n \theta_{a'} \circ_{n+1} G(F(f^n)) \circ_{n+1} e^n \rho_a \sim e^n \theta_{a'} \circ_{n+1} G(F(g^n)) \circ_{n+1} e^n \rho_a \sim g^n$ , i.e.,  $F$  is faithful ( $G$  is faithful by symmetry).

Take  $\alpha^n : F(a) \rightarrow F(a') \in L'^n(F(a), F(a'))$ . Then  $\beta^n := e^n \theta_{a'} \circ_{n+1} G(\alpha^n) \circ_{n+1} e^n \rho_a : a \rightarrow a' \in L^n(a, a')$  is such that  $G(F(\beta^n)) \sim G(\alpha^n)$ . So,  $F(\beta^n) \sim \alpha^n$  because  $G$  is faithful. Therefore,  $F$  is full ( $G$  is full by symmetry).

$F$  and  $G$  are obviously surjective on objects.  $\square$

**Remark.** The inverse direction "⇐" for the above proposition works only partially. Namely,

for each  $b \in L'^0$  choose  $G(b) \in L^0$  and equivalence  $b \xleftarrow[\theta_b]{\sim} F(G(b))$  (which is possi-

ble since  $F$  is surjective on objects), moreover, if  $b = F(a)$  choose  $G(b) = a$ ,  $\rho_b = eb$ ,  $\theta_b = e(F(G(b))) = eb$ . For each  $f^n : b \rightarrow b' \in L'^n(b, b')$  choose an element  $G(f^n) \in L^n(G(b), G(b'))$  such that  $e^n \rho_{b'} \circ_{n+1} f^n \circ_{n+1} e^n \theta_b \sim F(G(f^n))$  (which is possible since  $F$  is fully faithful). Then  $G : L' \rightarrow L$  is obviously a (weak) functor.  $a = G(F(a))$  is natural in  $a$  by construction, but  $b \sim F(G(b))$  is natural in  $b$  for only first order arrows  $\rho_b, \theta_b$  presenting  $\sim$ . So,  $F$  should be somehow 'naturally surjective on objects' which does not make sense yet when the functor  $G$  is not defined.  $\square$

**Definition 3.3.** An  $\infty$ -functor  $F : L \rightarrow L'$  is called

- an **isomorphism** if it is a bijection (on sets  $L, L'$ ) and the inverse map is a functor,
- a **quasiisomorphism** if there exists a functor  $G : L' \rightarrow L$  such that  $\forall a^n \in L^n \quad G(F(a^n)) \sim a^n$  and  $\forall b^n \in L'^n \quad F(G(b^n)) \sim b^n$ ,  $n \geq 0$ .  $\square$

**Proposition 3.2.** *The notions of (functor) isomorphism and quasiisomorphism coincide.*

*Proof.* Each isomorphism is a quasiisomorphism. Conversely, if  $L \xleftarrow[G]{\sim} L'$  is a quasiisomorphism then  $\forall a^n \in L^n$ ,  $n \geq 0$ ,  $G(F(ea^n)) \sim ea^n$ . So,  $d(G(F(ea^n))) = dea^n$ , i.e.  $G(F(dea^n)) =$

$de a^n$  and  $G(F(a^n)) = a^n$  (instead of  $d, c$  could be used). The same,  $\forall b^n \in L'^n, n \geq 0, F(G(b^n)) = b^n$ .  $\square$

Denote (quasi)isomorphism (equivalence) relation by  $\simeq$ .

### Examples (isomorphic $\infty$ -categories)

1. Assume,  $f^n \xrightarrow[\alpha]{\simeq} g^n$  are isomorphic elements of degree  $n$  (in a strict category  $L$ ) then  $L(f^n, f^n) \simeq L(g^n, g^n)$  are isomorphic  $\infty$ -categories. Indeed, there is an isomorphism  $F : L(f^n, f^n) \rightarrow L(g^n, g^n) : x \mapsto \alpha * (x * \alpha^{-1})$ , where  $*$  means a horizontal composite.  $F$  is a functor. Its inverse is  $G : L(g^n, g^n) \rightarrow L(f^n, f^n) : y \mapsto \alpha^{-1} * (y * \alpha)$ . [For  $\alpha$  just an equivalence, it is not true]
2.  $\infty\text{-CAT}(L(-, a), F) \simeq F(a)$  (see below, the Yoneda Lemma).  $\square$

**Definition 3.4.** Two  $n$ -modifications  $\alpha^n, \beta^n : L \rightarrow \infty\text{-CAT}$ ,  $n \geq 0$ , are called **quasiequivalent** of depth  $k$ ,  $0 \leq k \leq n+1$ , (denote it by  $\alpha^n \approx_k \beta^n$ ) if their corresponding components are quasiequivalent of depth  $k-1$ , i.e.  $\forall a \in L^0 \alpha^n(a) \approx_{k-1} \beta^n(a)$ .  $\approx_0$  means  $\sim$  by definition. [In other words,  $\alpha^n \approx_k \beta^n$  if all their components of components on depth  $k$  are equivalent, i.e.  $\alpha^n \approx_0 \beta^n$  if they are equivalent  $\alpha^n \sim \beta^n$ ;  $\alpha^n \approx_1 \beta^n$  if their components are equivalent  $\forall a \in L^0 \alpha^n(a) \sim \beta^n(a)$ ;  $\alpha^n \approx_2 \beta^n$  if components of all components are equivalent; etc.]. If  $\alpha^n, \beta^n : L \rightarrow L'$  are proper  $n$ -modifications (living in  $\infty\text{-CAT}$ ) for them only  $\approx_0$  and  $\approx_1$  make sense.  $\square$

### Lemma 3.1.

- $\approx_k$  is an equivalence relation.
- $\approx_{k_1} \Rightarrow \approx_{k_2}$  if  $k_1 \leq k_2$ .
- If  $\alpha^n \approx_k \beta^n$  then  $d\alpha^n = d\beta^n$ ,  $c\alpha^n = c\beta^n$ .
- If  $(L_1, \approx_{k_1}), (L_2, \approx_{k_2})$  are two  $\infty$ -categories (not necessarily proper, i.e. living in  $\infty\text{-CAT}$ ) for which given equivalence relations make sense for all elements, and  $F : L_1 \rightarrow L_2$ ,  $G : L_2 \rightarrow L_1$  are maps (not necessarily functors) such that  $\forall l_1 \in L_1 G(F(l_1)) \approx_{k_1} l_1$  and  $\forall l_2 \in L_2 F(G(l_2)) \approx_{k_2} l_2$ , and  $F, G$  both preserve  $d$  (or  $c$ ) then  $F, G$  are bijections inverse to each other.
- For  $L, L' \in Ob(\infty\text{-CAT})$  and  $a \in L^0$  the map  $ev_a : \infty\text{-CAT}(L, L') \rightarrow L' : f^n \mapsto f^n(a)$  is a strict functor. [Similar statement holds when  $L, L'$  are not proper, e.g.  $\infty\text{-CAT}$ , but we need to formulate it for a bigger universe containing  $\infty\text{-CAT}$ ]

*Proof.* The first two statements are obvious. The third one follows from the fact  $x \sim y \Rightarrow dx = dy$ ,  $cx = cy$  and that  $d, c$  are taken componentwise. The fourth statement follows by the same argument as in the proof of proposition 1.3.2. The last statement holds because, again, all operations in  $\infty\text{-CAT}(L, L')$  are taken componentwise.  $\square$

**Remark.** For the proof of the Yoneda lemma, a double evaluation functor is needed. For two functors  $F, G : L \rightarrow \infty\text{-CAT}$  take the restriction of the evaluation functor  $ev_a$  on the hom-set between  $F$  and  $G$ , i.e.  $ev_{a,F,G} : \infty\text{-CAT}(L, \infty\text{-CAT})(F, G) \rightarrow \infty\text{-CAT}(F(a), G(a)) : f^n \mapsto f^n(a)$ , where  $\infty\text{-CAT}$  is a bigger (and weaker) universe containing  $\infty\text{-CAT}$  as an object. Now, take a second evaluation functor  $ev_x : \infty\text{-CAT}(F(a), G(a)) \rightarrow G(a) : g^n \mapsto g^n(x)$ ,  $x \in (F(a))^0$ . Then the double evaluation functor is the composite  $ev_x \circ_1 ev_{a,F,G} : \infty\text{-CAT}(L, \infty\text{-CAT})(F, G) \rightarrow G(a) : f^n \mapsto f^n(a)(x)$ . It is a strict functor.  $\square$

$\infty\text{-CAT}$ -valued functors, natural transformations and modifications live now in a bigger universe  $\infty\text{-CAT}$ , and we do not yet have for them appropriate definitions.

**Definition 3.5.**  $\infty$ -CAT-valued functors, natural transformations and modifications are introduced in a similar way as the usual ones by changing all occurrences of  $\sim$  with (one degree weaker relation)  $\approx_1$ , i.e.

- a map  $F : L \rightarrow \infty\text{-CAT}$  of degree 0 is a **functor** if  $F$  strictly preserves  $d$  and  $c$ ,  $Fdx = dFx$ ,  $Fcx = cFx$ , and weakly up to  $\approx_1$  preserves  $e$  and composites,  $Fex \approx_1 eFx$ ,  $F(x \circ_k y) \approx_1 F(x) \circ_k F(y)$ ,
- For a given sequence of two functors  $F, G : L \rightarrow \infty\text{-CAT}$ , ..., two  $(n-1)$ -modifications  $\alpha_1^{n-1}, \alpha_2^{n-1} : \alpha_1^{n-2} \rightarrow \alpha_2^{n-2}$  strict (or weak)  $n$ -modification  $\alpha^n : \alpha_1^{n-1} \rightarrow \alpha_2^{n-1}$  is a map  $\alpha^n : L^0 \rightarrow \infty\text{-CAT}^{n+1}$  such that  $\forall a, b \in L^0 \alpha^n(b) * F(-) \approx_1 G(-) * \alpha^n(a) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$  (components of values of functors are equivalent).  $\square$

**Definition 3.6.** A covariant (contravariant) functor  $F : L \rightarrow \infty\text{-CAT}$  is

- **weakly representable** if  $\exists a \in L^0$  such that  $L(a, -) \sim F$  ( $L(-, a) \sim F$ ). It means there is an equivalence of two  $\infty$ -categories  $L(a, b) \sim F(b)$  ( $L(b, a) \sim F(b)$ ) natural in  $b$ ,
- **strictly representable** if there exists  $a \in L^0$  such that  $L(a, -) \simeq F$  ( $L(-, a) \simeq F$ ), i.e.  $\forall b \in L^0 \exists$  an isomorphism  $L(a, b) \simeq F(b)$  ( $L(b, a) \simeq F(b)$ ) natural in  $b$ .  $\square$

**Lemma 3.2.** For given representable  $L(-, a) : L^{op} \rightarrow \infty\text{-CAT}$  and functor  $F : L^{op} \rightarrow \infty\text{-CAT}$

- all natural transformations  $\tau^0 : L(-, a) \rightarrow F$  are of the form  $\forall b \in Ob L$  the  $b$ -component is a functor  $\tau_b^0 : L(b, a) \rightarrow F(b)$ ,  $\tau_b^0(f^m) \sim F(f^m)(\tau_a^0(ea))$ ,  $f^m \in L^m(b, a)$ ,
- all  $n$ -modifications  $\tau^n : L(-, a) \rightarrow F$ ,  $n \geq 1$ , are of the form  $\forall b \in Ob L$  the  $b$ -component is a  $(n-1)$ -modification  $\tau_b^n : L(b, a) \rightarrow F(b)$ ,  $\tau_b^n(f^0) \sim F(f^0)(\tau_a^n(ea))$ ,  $f^0 \in L^0(b, a)$ .

$$\begin{array}{ccc} & a & \\ & \uparrow f^m & \\ b & & \\ \text{Proof : follows from the naturality square} & L(a, a) & \xrightarrow{\tau_a^n} F(a) \\ & \downarrow L(f^m, a) & \downarrow F(f^m) \\ & L(b, a) & \xrightarrow{\tau_b^n} F(b) \end{array} \quad n \geq 0 \quad \square$$

**Lemma 3.3.** For a given  $n$ -cell  $\beta^n \in (F(a))^n$ ,  $n \geq 0$ ,  $n$ -modification  $\tau^n : L(-, a) \rightarrow F$  such that  $\tau_a^n(ea) = \beta^n$  exists and unique up to  $\approx_2$ .

*Proof.*: Uniqueness follows from lemma 1.3.2, existence from the definition of  $n$ -modification  $\tau_b^n(f^m) := F(f^m)(\beta^n)$  (for  $n > 0$ ,  $m = 0$  only) and the naturality square showing correctness of

$$\begin{array}{ccc} & b & \\ & \uparrow g^k & \\ c & & \\ \text{the definition} & L(b, a) & \xrightarrow{\tau_b^n} F(b) \\ & \downarrow L(g^k, a) & \downarrow F(g^k) \\ & L(c, a) & \xrightarrow{\tau_c^n} F(c) \end{array} \quad (\mu_{c,b,a}(f^m, g^k) := \mu_{c,b,a}(e^{\max(m,k)-m} f^m, e^{\max(m,k)-k} g^k)) \quad \square$$

**Corollary 1.** All  $n$ -modifications  $\tau^n : L(-, a) \rightarrow F$ ,  $n \geq 0$ , have strict form  $\tau_b^n(f^0) = F(f^0)(\tau_a^n(ea))$ ,  $f^0 \in L^0(b, a)$ .  $\square$

**Corollary 2 (criterion of representability).** A  $\infty$ -CAT-valued presheaf  $F : L^{op} \rightarrow \infty\text{-CAT}$  is

- **strictly representable** (with representing object  $a \in L^0$ ) iff there exists an object  $\beta^0 \in (F(a))^0$  such that  $\forall \gamma^n \in (F(b))^n$ ,  $n \geq 0$ ,  $\exists!$   $n$ -arrow  $(f^n : b \rightarrow a) \in L^n(b, a)$  with  $\gamma^n = F(f^n)(\beta^0)$ ,

- **weakly representable** (with representing object  $a \in L^0$ ) iff there exists an object  $\beta^0 \in (F(a))^0$  such that  $\forall b \in Ob L$  the functor  $L(b, a) \rightarrow F(b) : f^n \mapsto F(f^n)(\beta^0)$  is an equivalence of categories.

(Similar statements hold for a covariant presheaf  $F : L \rightarrow \infty\text{-CAT}$ )  $\square$

**Proposition 3.3 (Yoneda Lemma).** *For the functor  $F : L^{op} \rightarrow \infty\text{-CAT}$  and the object  $a \in L^0$ , there is a strict isomorphism  $\infty\text{-CAT}(L(-, a), F) \simeq F(a)$  natural in  $a$  and  $F$ .*

*Proof.* Strict functoriality of the correspondence  $\tau^n \mapsto \tau_a^n(ea)$  is straightforward (because it is a double evaluation functor). The map  $\beta^n \mapsto F(-)(\beta^n)$  is quasiinverse to the first map (with respect to  $\approx_2$  and  $=$  equivalence relations in  $\infty\text{-CAT}(L(-, a), F)$  and  $F(a)$  respectively), and it strictly preserves  $d$  and  $c$ . So, these both maps are strict isomorphisms.

$$\begin{array}{ccccc}
 & a & F & \infty\text{-CAT}(L(-, a), F) & \xrightarrow{\simeq} F(a) \\
 \text{Naturality is given by} & \uparrow f^m & \downarrow \alpha^k & \downarrow \infty\text{-CAT}(L(-, f^m), \alpha^k) & \downarrow \alpha^k(f^m) \\
 & b & G & \infty\text{-CAT}(L(-, b), G) & \xrightarrow{\simeq} G(b)
 \end{array}$$

(where  $\alpha^k(f^m) := \mu_{F(a), F(b), G(b)}(e^{\max(k, m)-k}\alpha_b^k, e^{\max(k, m)-m+1}F(f^m))$ ,  $k, m \geq 0$ )  $\square$

**Remark.** The Yoneda lemma for  $\infty$ -categories is similar to the one for first order categories with the difference that elements  $\beta^n \in (F(a))^n$  of degree  $n$  now determine higher degree arrows ( $n$ -modifications)  $\beta^n : L(-, a) \rightarrow F$  in a  $\infty\text{-CAT}$ -valued presheaf category.  $\square$

**Proposition 3.4 (Yoneda embedding).** *There is a Yoneda embedding  $\mathbf{Y} : L \rightarrow \infty\text{-CAT}^{L^{op}}$ :  $\alpha \mapsto L(-, \alpha)$ ,  $\alpha \in L$ , which is an extension of the isomorphisms from the Yoneda lemma determined on hom-sets  $L(a, b)$ ,  $a, b \in L^0$ . The Yoneda embedding preserves and reflects equivalences  $\sim$ .*

*Proof.* By the Yoneda isomorphism for a given  $f^n \in L^n(a, b)$ , the corresponding  $n$ -modification is  $L(-, b)(f^n) : L(-, a) \rightarrow L(-, b)$  which is the same as  $L(-, f^n) : L(-, a) \rightarrow L(-, b)$ ; i.e. the functor  $\mathbf{Y} : L \rightarrow \infty\text{-CAT}^{L^{op}} : \alpha \mapsto L(-, \alpha)$ ,  $\alpha \in L$ , locally coincides with isomorphisms from the Yoneda lemma. By lemma 1.1.3 this functor preserves and reflects equivalences  $\sim$ .  $\square$

**Remark.** Under the assumption that the category  $\infty\text{-CAT}$  of **weak** categories, functors and  $n$ -modifications exists, all the above reasons remain essentially the same, i.e. the Yoneda lemma and embedding seem to hold in a weak situation.  $\square$

#### 4. (Co)limits

**Definition 4.1.** An  **$\infty$ -graph** is a graded set  $G = \coprod_{n \geq 0} G^n$  with two unary operations  $d, c : \coprod_{n \geq 1} G^n \rightarrow \coprod_{n \geq 0} G^n$  of degree  $-1$  such that  $d^2 = dc$ ,  $c^2 = cd$ .  $\square$

**Definition 4.2.** An  **$\infty$ -diagram**  $D : G \rightarrow L$  from  $\infty$ -graph  $G$  to  $\infty$ -category  $L$  is a function of degree 0 which preserves operations  $d, c$ .  $\square$

**Proposition 4.1.** All diagrams from  $G$  to  $L$ , natural transformations, modifications form an  $\infty$ -category  $\mathbf{Dgrm}_{G, L}$  in the same way as the functor category  $\infty\text{-CAT}(L', L)$ .  $\square$

For a given object  $a \in L^0$  the **constant diagram** to  $a$  is  $\Delta(a) : G \rightarrow L : g \mapsto e^n a$  if  $g \in G^n$ .  $\Delta : L \rightarrow \mathbf{Dgrm}_{G, L}$  is an  $\infty$ -functor.

Denote  $\{e\}\alpha := \{\alpha, e\alpha, e^2\alpha, \dots, e^n\alpha, \dots\}$ ,  $\alpha \in L$ .

**Definition 4.3.** Diagram  $D : G \rightarrow L$  has

- a limit if the functor  $\mathbf{Dgrm}_{G,L}(\Delta(-), D) : L^{op} \rightarrow \infty\text{-CAT}$  is representable.

If  $\nu : L(-, a) \xrightarrow{\sim} \mathbf{Dgrm}_{G,L}(\Delta(-), D)$  is an equivalence then

$\nu_a(\{e\}ea) \subset \mathbf{Dgrm}_{G,L}(\Delta(a), D)$  is called a **limit cone** over  $D$ ,  $a$  is its **vertex** (or diagram **limit**  $\lim D$ ),  $\nu_a(ea)$  are its **edges**,  $\nu_a(e^k a)$ ,  $k > 1$ , are identities

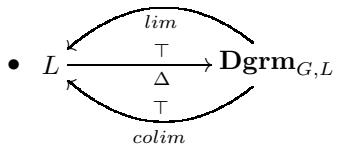
- colimit if functor  $\mathbf{Dgrm}_{G,L}(D, \Delta(-)) : L \rightarrow \infty\text{-CAT}$  is representable.

If  $\nu : L(a, -) \xrightarrow{\sim} \mathbf{Dgrm}_{G,L}(D, \Delta(-))$  is the equivalence then

$\nu_a(\{e\}ea) \subset \mathbf{Dgrm}_{G,L}(D, \Delta(a))$  is called **colimit cocone** over  $D$ ,  $a$  is its **vertex** (or diagram **colimit**  $\text{colim } D$ ),  $\nu_a(ea)$  are its **edges**,  $\nu_a(e^k a)$ ,  $k > 1$ , are identities  $\square$

**Remark.** The conditions on equivalence  $\nu$  in the above definition can be strengthened. If it is a (natural) isomorphism then (co)limits are called **strict** and as a rule they are different from **weak** ones [Bor1].

**Proposition 4.2.** *For strict (co)limits the following is true*



- Strict right adjoints preserve limits (strict left adjoints preserve colimits).

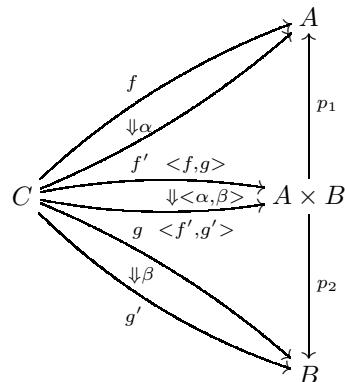
*Proof.*

- It is immediate from definition 4.3 and proposition 5.1.
  - The argument is the same as for first order categories (see example 13, point **1.a**) [the essential thing is that a strict adjunction is determined by (triangle) identities which are preserved under  $\infty$ -functors].  $\square$

## Examples

1. (strict binary products in **2-Top** and **2-CAT**) They coincide with '1-dimensional' products.

The mediating 2-cell arrow is given componentwise



2. (“equalizer” of a 2-cell in 2-CAT) [Bor1] For a given 2-cell  $A \begin{array}{c} F \\ \Downarrow \alpha \\ G \end{array} B$  in 2-CAT its strict limit is a subcategory  $\mathbf{E} \hookrightarrow \mathbf{A}$  such that  $F(A) = G(A)$  and  $\alpha_A = 1_{F(A)} : F(A) \rightarrow G(A)$  (on objects), and  $F(f) = G(f)$  (on arrows).

3. (strict and weak pullbacks in 2-CAT) [Bor1] Let  $\mathcal{P}$  be a “2-dimensional” graph  $1 \xrightarrow{x} 0 \xleftarrow{y} 2$  with trivial 2-cells,  $F : \mathcal{P} \rightarrow \text{2-CAT}$  be a 2-functor. Then its limit is a pullback diagram

$$\begin{array}{ccc} F(1) \times_{F(0)} F(2) & \xrightarrow{p_2} & F(2) \\ p_1 \downarrow & \searrow p_3 & \downarrow F(y) . \\ F(1) & \xrightarrow{F(x)} & F(0) \end{array} \quad \text{When the limit is taken strictly } F(1) \times_{F(0)}$$

$F(2)$  coincides with the “1-dimensional” pullback, i.e.  $F(1) \times_{F(0)} F(2) \hookrightarrow F(1) \times F(2)$  is a subcategory consisting of objects  $(A, B)$ ,  $A \in \text{Ob } F(1)$ ,  $B \in \text{Ob } F(2)$ ,  $F(x)(A) = F(y)(B)$  and arrows  $(f, g)$ ,  $f \in \text{Ar } F(1)$ ,  $g \in \text{Ar } F(2)$ ,  $F(x)(f) = F(y)(g)$ . When the limit is taken **weakly**  $F(1) \times_{F(0)} F(2)$  is not a subcategory of product  $F(1) \times F(2)$ . It consists of 5-tuples  $(A, B, C, f, g)$ ,  $A \in \text{Ob } F(1)$ ,  $B \in \text{Ob } F(2)$ ,  $C \in \text{Ob } F(0)$ ,  $f : F(x)(A) \xrightarrow{\sim} C$ ,  $g : F(y)(B) \xrightarrow{\sim} C$  are isomorphisms, with arrows  $(a, b, c)$ ,  $a : A \rightarrow A'$ ,  $b : B \rightarrow B'$ ,  $c : C \rightarrow C'$  such that  $c \circ f = f' \circ F(x)(a)$ ,  $c \circ g = g' \circ F(y)(b)$ . Projections  $p_1, p_2, p_3$  are obvious. The pullback square commutes up to isomorphisms  $f : F(x) \circ p_1 \Rightarrow p_3$ ,  $g : F(y) \circ p_2 \Rightarrow p_3$ .  $\square$

## 5. Adjunction

**Definition 5.1.** The situation  $L \begin{array}{c} \xleftarrow{F} \\[-1ex] \perp \\[-1ex] \xrightarrow{G} \end{array} L'$  (where  $L, L'$  are  $\infty$ -categories,  $F, G$  are  $\infty$ -functors) is called

- **weak  $\infty$ -adjunction** if there is an equivalence  $L(-, G(+)) \sim L'(F(-), +) : L^{\text{op}} \times L' \rightarrow \infty\text{-CAT}$  (i.e.  $L(a, G(b)) \sim L'(F(a), b)$  natural in  $a \in L^0, b \in L'^0$ ),
- **strict  $\infty$ -adjunction** if there is an isomorphism  $L(-, G(+)) \xrightarrow{\sim} L'(F(-), +) : L^{\text{op}} \times L' \rightarrow \infty\text{-CAT}$  (i.e.  $L(a, G(b)) \simeq L'(F(a), b)$  natural in  $a \in L^0, b \in L'^0$ ).  $\square$

**Proposition 5.1.** *The following are equivalent*

1.  $L \begin{array}{c} \xleftarrow{F} \\[-1ex] \perp \\[-1ex] \xrightarrow{G} \end{array} L'$  *is a strict  $\infty$ -adjunction*
2.  $\forall b \in L'^0 \ L'(F(-), b)$  *is strictly representable*
3.  $\forall a \in L^0 \ L(a, G(-))$  *is strictly representable*

*Proof.*

- 1.  $\implies$  2., 3. is immediate
- 2.  $\implies$  1. From the criterion of strict representability (see point 1.3) it follows that  $\forall b \in L'^0$  there exists a “universal element”  $(\beta_b^0 : F(G(b)) \rightarrow b) \in L'^0(F(G(b)), b)$  such that  $\forall (f^n : F(c) \rightarrow b) \in L'^n(F(c), b) \exists! n\text{-arrow } (g^n : c \rightarrow G(b)) \in L^n(c, G(b))$  with  $f^n =$

$$\mu_{F(c), F(G(b)), b}(e^n \beta_b^0, F(g^n)) \quad \begin{array}{ccc} G(b) & & F(G(b)) \xrightarrow{e^n \beta_b^0} b \\ \uparrow \exists! g^n | & & \uparrow F(g^n) \\ c & & F(c) \end{array}$$

$$\begin{array}{ccc} & & b \\ & \nearrow \forall f^n & \\ & & \end{array}$$

$$\begin{array}{ccc} G(b) & & F(G(b)) \xrightarrow{e^n \beta_b^0} b \\ \uparrow & & \uparrow \\ G(b') & & F(G(b')) \xrightarrow{e^n \beta_{b'}^0} b' \\ \uparrow G(f^n) & & \uparrow F(G(f^n)) \\ & & f^n \end{array}$$

Consequently,  $\forall (f^n : b' \rightarrow b) \in L'^n(b', b)$  the diagram holds

It shows that assignment  $Ob L' \ni b \mapsto G(b) \in Ob L$  is extendable to a functor  $G : L' \rightarrow L$  (in an essentially unique way) and that  $\beta^0 : FG \rightarrow 1_{L'}$  is a natural transformation (**counit**  $\varepsilon$  of the adjunction  $F \dashv G$ ).

$$\text{Isomorphism } \varphi_{c,b} : L'(F(c), b) \rightarrow L(c, G(b)) \text{ such that } F(\varphi_{c,b}(f^n)) \begin{array}{c} \xrightarrow{e^n \beta_b^0} \\ \uparrow \\ F(c) \end{array} b \text{ is natural in } f^n$$

$c \in Ob L, b \in Ob L'$  because of the naturality square

$$\begin{array}{ccc} c & & b \\ \uparrow g^n & & \downarrow f^n \\ L'(F(c), b) & \xrightarrow{\varphi_{c,b}} & L(c, G(b)) \\ \downarrow L'(F(g^n), f^n) & & \downarrow L(g^n, G(f^n)) \\ L'(F(c'), b') & \xrightarrow{\varphi_{c',b'}} & L(c', G(b')) \end{array}$$

(indeed,  $\forall h^n \in L'(F(c), b)$   $G(f^n) * \varphi_{c,b}(h^n) * g^n \sim \varphi_{c',b'}(f^n * h^n * F(g^n))$ , where  $*$  is the horizontal composite, since  $e^n \beta_{b'}^0 * F(G(f^n)) * \varphi_{c,b}(h^n) * g^n \sim f^n * e^n \beta_b^0 * F(\varphi_{c,b}(h^n)) * F(g^n) \sim f^n * h^n * F(g^n)$ )

- 3.  $\Rightarrow$  1. is similar to 2.  $\Rightarrow$  1.  $\square$

**Remark.** The analogous statement for a weak  $\infty$ -adjunction is not true. In the above proof “universal elements” were used in an essential way.  $\square$

**Definition 5.2.** For a given **strict**  $\infty$ -adjunction  $L \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} L'$

- universal elements  $\varepsilon_b : F(G(b)) \rightarrow b$  representing functors  $L'(F(-), b)$  ( $b \in Ob L'$  is a parameter) form a natural transformation  $\varepsilon : FG \rightarrow 1_{L'}$  which is called the **counit** of the adjunction,
- Universal elements  $\eta_a : a \rightarrow G(F(a))$  representing functors  $L(a, G(-))$  ( $a \in Ob L$  is a parameter) form a natural transformation  $\eta : 1_L \rightarrow GF$  which is called the **unit** of the adjunction.  $\square$

**Remark.** For a **weak**  $\infty$ -adjunction no useful unit and counit exist.  $\square$

**Proposition 5.2.**

- For both weak and strict adjunctions: the composition of left adjoints is a left adjoint (the composition of right adjoints is a right adjoint).
- For a weak (strict) adjunction, a right or left adjoint is determined uniquely up to equivalence  $\sim$  (up to isomorphism  $\simeq$ ).

*Proof.*

- If  $L \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} L' \begin{array}{c} \xrightarrow{F'} \\ \perp \\ \xleftarrow{G'} \end{array} L''$  then  $L''(F'Fl, l'') \sim L'(Fl, G'l'') \sim L(l, GG'l'')$  (composite of natural equivalences). [For a strict adjunction the same reason works]
- Assume,  $L'(a, G'b) \sim L(Fa, b) \sim L'(a, Gb)$  are natural equivalences then  $L'(-, G'b) \sim L'(-, Gb)$  is a natural transformation (equivalence) natural in  $b$ . Then, by the Yoneda embedding,  $G'b \sim Gb$  naturally in  $b$ , i.e.  $G' \sim G$ . [Again, changing  $\sim$  with  $\simeq$  still works].  $\square$

**Proposition 5.3.** For a strict adjunction  $L \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} L'$  the Kan definition and the definition via “unit-counit” coincide, i.e. the following are equivalent

- $\varphi_{a,b} : L(a, G(b)) \simeq L'(F(a), b) : \varphi_{a,b}^*$  natural in  $a \in L^0, b \in L'^0$ ,
- $\exists$  natural transformations  $\eta : 1_L \rightarrow GF$  and  $\varepsilon : FG \rightarrow 1_{L'}$  satisfying the triangle identities  $\varepsilon F \circ_1 F\eta = 1_F$  and  $G\varepsilon \circ_1 \eta G = 1_G$ .

*Proof.* For a strict adjunction, the same proof as for first order categories works.

- Universal elements  $\eta_a, \varepsilon_b$  for functors  $L(a, G(-)), L'(F(-), b)$  mean that they are images of

$$1_{F(a)}, 1_{G(b)} \text{ under functors } \varphi_{a,F(a)}^*, \varphi_{G(b),b}, \text{ i.e.}$$

$$\begin{array}{ccc} FGFa & \xrightarrow{\varepsilon_{Fa}} & Fa \\ \uparrow F\eta_a & \nearrow 1_{Fa} & \downarrow \\ Fa & & \end{array} \quad \begin{array}{ccc} Gb & \xrightarrow{\eta_{Gb}} & GFGb \\ \downarrow 1_{Gb} & \searrow & \downarrow G\varepsilon_b \\ Gb & & \end{array}$$

(strict equalities)

- Define maps  $\begin{cases} \varphi_{a,b}(f^n) := e^n(\varepsilon_b) \circ_{n+1} F(f^n), & f^n \in L^n(a, G(b)) \\ \varphi_{a,b}^*(g^n) := G(g^n) \circ_{n+1} e^n(\eta_a), & g^n \in L'^n(F(a), b) \end{cases}$

They are functors  $\begin{cases} \varphi_{a,b} := \varepsilon_b * F(-) : L(a, G(b)) \rightarrow L'(F(a), b) \\ \varphi_{a,b}^* := G(-) * \eta_a : L'(F(a), b) \rightarrow L(a, G(b)) \end{cases}$  and inverses to each other:

$$\varphi_{a,b}^*(\varphi_{a,b}(f^n)) = \varphi_{a,b}^*(e^n \varepsilon_b \circ_{n+1} F(f^n)) = G(e^n \varepsilon_b \circ_{n+1} F(f^n)) \circ_{n+1} e^n \eta_a = e^n G(\varepsilon_b) \circ_{n+1} (GF(f^n) \circ_{n+1} e^n \eta_a) = e^n G(\varepsilon_b) \circ_{n+1} (e^n \eta_{G(b)} \circ_{n+1} f^n) = e^{n+1} G(b) \circ_{n+1} f^n = f^n, \text{ and similar,}$$

$$\varphi_{a,b}(\varphi_{a,b}^*(g^n)) = g^n.$$

$$\begin{array}{ccc} L(a, G(b)) & \xrightarrow{\varphi_{a,b}} & L'(F(a), b) \\ \text{Naturality (e.g., of } \varphi_{a,b} \text{) follows from the square} & \downarrow L(x^m, G(y^m)) & \downarrow L'(F(x^m), y^m) \\ L(a', G(b')) & \xrightarrow{\varphi_{a',b'}} & L'(F(a'), b') \end{array}$$

$$(\varphi_{a',b'}(L(x^m, G(y^m))(f^n)) = \varphi_{a',b'}(G(y^m) * f^n * x^m) = \varepsilon_{b'} * FG(y^m) * F(f^n) * F(x^m) = y^m * \varepsilon_b * F(f^n) * F(x^m) = L'(F(x^m), y^m)(\varphi_{a,b}(f^n)), \text{ where } n = 0 \text{ or } m = 0). \quad \square$$

### Examples of higher order adjunctions

1. Every usual 1-adjunction  $A \rightleftarrows B$  is an  $\infty$ -1-adjunction for trivial  $\infty$ -extensions of  $A$  and  $B$ .
2. Gelfand-Naimark dual 1-adjunction  $\mathbf{C}^*\mathbf{Alg}^{op} \rightleftarrows \mathbf{CHTop}$  is extendable to  $\infty$ -2-adjunction (see 9).
3. Quillen theorem [Mac]. Let  $\Delta$  be a category of finite linearly ordered sets,  $\mathbf{Set}^{\Delta^{op}}$  the category of simplicial sets,  $Ho(\mathbf{Top}) := (2\text{-}\mathbf{Top})^{(1)}$ ,  $Ho(\mathbf{Set}^{\Delta^{op}}) := (2\text{-}\mathbf{Set}^{\Delta^{op}})^{(1)}$ . Then

$$\begin{array}{ccccc} \Delta & \downarrow & & & \\ & \searrow \text{Yoneda} & & & \\ & & \mathbf{Set}^{\Delta^{op}} & \dashrightarrow & \mathbf{Top} \\ & \downarrow & & \swarrow & \downarrow \\ & & Ho(\mathbf{Set}^{\Delta^{op}}) & \xrightarrow{\perp} & Ho(\mathbf{Top}) \end{array}$$

$\square$

So, the top adjunction is actually a 2-adjunction (or  $\infty$ -2-adjunction).

All the above adjunctions are strict.

## 6. Concrete duality for $\infty$ -categories

Duality preserves all categorical properties. It is significant that concrete duality for  $\infty$ -categories behaves the same as for 1-categories.

### Definition 6.1.

- **Duality** is an equivalence  $L^{op} \sim L'$ .

- A **Concrete duality** over  $\mathbb{B} \hookrightarrow \infty\text{-CAT}$  is a duality  $L^{op} \begin{array}{c} G \\ \sim \\ F \end{array} L'$  such that there exist (faithful) forgetful functors  $U : L \rightarrow \mathbb{B}$ ,  $V : L' \rightarrow \mathbb{B}$  and objects  $\tilde{A} \in L^0$ ,  $\tilde{B} \in L'^0$  such that
  - $U(\tilde{A}) \sim V(\tilde{B})$ ,
  - $V \circ_1 G \sim L(-, \tilde{A})$ ,  $U \circ_1 F^{op} \sim L'(-, \tilde{B})$

$$\begin{array}{ccc} L^{op} & \xrightarrow{G} & L' \\ & \searrow & \downarrow V \\ L(-, \tilde{A}) & & \mathbb{B} \end{array} \quad \begin{array}{ccc} L'^{op} & \xrightarrow{F^{op}} & L \\ & \searrow & \downarrow U \\ L'(-, \tilde{B}) & & \mathbb{B} \end{array}$$

Representing objects  $\tilde{A} \in L^0$ ,  $\tilde{B} \in L'^0$  are called **dualizing** or **schizophrenic objects** for the given concrete duality [P-Th].

[for a **concrete dual adjunction** the definition is similar]  $\square$

**Proposition 6.1 (representable forgetfuls  $\Rightarrow$  concrete dual adjunction).** Let  $(L, U)$ ,

$(L', V)$  be (weakly) dually adjoint  $\infty$ -categories  $L^{op} \begin{array}{c} G \\ \sim \\ F \end{array} L'$  with representable forgetful functors  $U \sim L(A_0, -) : L \rightarrow \mathbb{B}$ ,  $V \sim L'(B_0, -) : L' \rightarrow \mathbb{B}$  (where  $\mathbb{B} \hookrightarrow \infty\text{-CAT}$  is a subcategory). Then this adjunction is concrete over  $\mathbb{B}$  with dualizing object  $(\tilde{A}, \tilde{B})$ , where  $\tilde{A} := F(B_0)$ ,  $\tilde{B} := G(A_0)$ , i.e.

- $U(\tilde{A}) \sim V(\tilde{B})$

$$\begin{array}{ccc} L^{op} & \xrightarrow{G} & L' \\ & \searrow & \downarrow V \\ L(-, \tilde{A}) & & \mathbb{B} \end{array} \quad \begin{array}{ccc} L'^{op} & \xrightarrow{F^{op}} & L \\ & \searrow & \downarrow U \\ L'(-, \tilde{B}) & & \mathbb{B} \end{array}$$

*Proof.*

- $U(\tilde{A}) = UF(B_0) \sim L(A_0, FB_0) \sim L'(B_0, GA_0) \sim V(GA_0) = V\tilde{B}$
- $VG(-) \sim L'(B_0, G(-)) \sim L(-, FB_0) = L(-, \tilde{A})$  (and similar,  $UF(-) \sim L'(-, \tilde{B})$ )  $\square$

### Remarks.

- Concrete duality as above should be called **weak**. **Strict** variants of definition 6.1 and proposition 6.1 also exist (by changing  $\sim$  to isomorphism  $\simeq$  and weak dual adjunction to the strict one).
- (Weak or strict) concrete duality (dual adjunction) is given essentially by hom-functors which admit lifting along forgetful functors (to obtain proper values). Representing objects of these functors have equivalent (or isomorphic) underlying objects.
- For the usual 1-dimensional categories  $\mathbb{B} = \mathbf{Set} \hookrightarrow \infty\text{-CAT}$  ( $\infty$ -1-subcategory). For dimension  $n$ , as a rule,  $\mathbb{B} = n\text{-Cat} \hookrightarrow \infty\text{-CAT}$  ( $\infty$ - $n$ -subcategory of small  $(n-1)$ -categories).  $\square$

### 6.1. Natural and non natural duality.

#### Definition 6.1.1.

- For hom-set  $L(A, \tilde{A})$  and element  $(x : A_0 \rightarrow A) \in L^0(A_0, A)$  the **evaluation functor** at the point  $x$  is  $ev_{A,x} := L(x, \tilde{A}) : L(A, \tilde{A}) \rightarrow L(A_0, \tilde{A})$  ( $ev_{A,x} \in \mathbb{B}^1 \hookrightarrow \infty\text{-CAT}^1$ ). Similarly, the **evaluation  $(n - 1)$ -modification**  $ev_{A,x^n}$ ,  $n = 1, 2, \dots$ , for  $x^n \in L^n(A_0, A)$  is  $L(x^n, \tilde{A}) \in \mathbb{B}^n(L(A, \tilde{A}), L(A_0, \tilde{A}))$ .
- For a forgetful functor  $V : L' \rightarrow \mathbb{B}$  an arrow  $f^n : V(Y) \rightarrow V(Y') \in \mathbb{B}^n(V(Y), V(Y'))$  is called an  **$L'$ -arrow** if  $\exists \Phi^n : Y \rightarrow Y' \in L'^n(Y, Y')$  such that  $V(\Phi^n) = f^n$ .
- A lifting of hom-functor  $V \circ G \sim L(-, \tilde{A})$  is called **initial** [A-H-S] if  $\forall A \in L^0 \forall Y \in L'^0 \forall f^n : V(Y) \rightarrow L(A, \tilde{A}) \in \mathbb{B}^n(V(Y), L(A, \tilde{A}))$   $f^n$  is an  $L'$ -arrow iff  $\forall (x^n : A_0 \rightarrow A) \in L^n(A_0, A)$   $ev_{A,x^n} \circ_{n+1} f^n : V(Y) \rightarrow L(A_0, \tilde{A}) \in \mathbb{B}^n(V(Y), L(A_0, \tilde{A}))$  is an  $L'$ -arrow.
- If liftings of hom-functors  $V \circ G \sim L(-, \tilde{A})$ ,  $U \circ F \sim L'(-, \tilde{B})$  are both initial, then the concrete dual adjunction  $L^{op} \begin{array}{c} \xrightarrow{\quad G \quad} \\[-1ex] \xleftarrow{\quad F \quad} \end{array} L'$ , if it exists, is called **natural** [Hof, P-Th], and otherwise, non-natural.  $\square$

Even if  $U\tilde{A} \sim V\tilde{B}$  and  $\forall A \in L^0, B \in L'^0$   $\mathbb{B}$ -objects  $L(A, \tilde{A})$ ,  $L'(B, \tilde{B})$  can be lifted to  $L', L$ , the hom-functors  $L(-, \tilde{A})$ ,  $L'(-, \tilde{B})$  need not (which happens only if lifting of the assignments  $A \mapsto L(A, \tilde{A})$ ,  $B \mapsto L'(B, \tilde{B})$  can be extended functorially over all cells).

We introduce the following concept. **The initial lifting condition for the evaluation cones**

$\{ev_{A,x^n} \in \mathbb{B}^n(L(A, \tilde{A}), L(A_0, \tilde{A}))\}_{x^n \in L^n(A_0, A)}^{n \in \mathbb{N}}$ ,  $\{ev_{B,y^n} \in \mathbb{B}^n(L'(B, \tilde{B}), L'(B_0, \tilde{B}))\}_{y^n \in L'^n(B_0, B)}^{n \in \mathbb{N}}$  consists of the following requirements:

- hom-categories of the form  $L(A, \tilde{A})$ ,  $L'(B, \tilde{B}) \in Ob(\mathbb{B})$  can be lifted to  $L', L$
- evaluation cones  
 $\{ev_{A,x^n} \in \mathbb{B}^n(L(A, \tilde{A}), L(A_0, \tilde{A}))\}_{x^n \in L^n(A_0, A)}^{n \in \mathbb{N}}$ ,  $\{ev_{B,y^n} \in \mathbb{B}^n(L'(B, \tilde{B}), L'(B_0, \tilde{B}))\}_{y^n \in L'^n(B_0, B)}^{n \in \mathbb{N}}$  can be lifted to  $\{ev_{A,x^n} \in L'^n(G(A), \tilde{B})\}_{x^n \in L^n(A_0, A)}^{n \in \mathbb{N}}$ ,  $\{ev_{B,y^n} \in L^n(F(B), \tilde{A})\}_{y^n \in L'^n(B_0, B)}^{n \in \mathbb{N}}$  in  $L', L$
- $\forall f^n \in \mathbb{B}^n(VX, L(A, \tilde{A}))$   $f^n$  is  $L'$ -arrow iff  $\forall x^n \in L^n(A_0, A)$   $\mu(ev_{A,x^n}, f^n) \in \mathbb{B}^n(VX, L(A_0, \tilde{A}))$  is  $L'$ -arrow (and, symmetrically,  $\forall g^n \in \mathbb{B}^n(UY, L'(B, \tilde{B}))$   $g^n$  is an  $L$ -arrow iff  $\forall y^n \in L'^n(B_0, B)$   $\mu(ev_{B,y^n}, g^n) \in \mathbb{B}^n(UY, L'(B_0, \tilde{B}))$  is an  $L$ -arrow)  $\square$

In the following proof, we denote lifted evaluation maps by  $ev_{A,x}$  (or something similar) and underlying evaluation maps in  $\mathbb{B}$  by  $|ev_{A,x}|$ .

**Proposition 6.1.1.** *If two strict  $\infty$ -categories  $L, L'$  concrete over  $\mathbb{B} \hookrightarrow \infty\text{-CAT}$  with representable (strictly faithful) forgetful functors  $U = L(A_0, -)$ ,  $V = L'(B_0, -)$  have objects  $\tilde{A} \in L^0$ ,  $\tilde{B} \in L'^0$  such that*

- $U\tilde{A} \sim V\tilde{B}$
- the hom-functors  $L(-, \tilde{A}) : L^{op} \rightarrow \mathbb{B}$ ,  $L'(-, \tilde{B}) : L'^{op} \rightarrow \mathbb{B}$  satisfy the **initial lifting condition for evaluation cones**

then there exists a natural **strict** concrete dual adjunction  $L^{op} \begin{array}{c} \xrightarrow{\quad G \quad} \\[-1ex] \xleftarrow{\quad F \quad} \end{array} L'$   $L(A, FB) \underset{\text{nat. iso}}{\sim}$

$$\begin{array}{ccc}
 L'^{op} & \xrightarrow{G} & L' \\
 L'(B, GA) & \searrow L(-, \tilde{A}) & \downarrow V \\
 & \mathbb{B} &
 \end{array}
 \quad
 \begin{array}{ccc}
 L'^{op} & \xrightarrow{F^{op}} & L \\
 L'(-, \tilde{B}) & \searrow & \downarrow U \\
 & \mathbb{B} &
 \end{array}
 \quad \text{with } (\tilde{A}, \tilde{B}) \text{ its schizophrenic object.}$$

*Proof.*

- $L(A, \tilde{A}), L'(B, \tilde{B})$  are lifted to  $L', L$  by the assumed condition.
- Let  $f^n \in L^n(A, A')$ , then  $L(f^n, \tilde{A}) : L(A', \tilde{A}) \rightarrow L(A, \tilde{A})$  is an  $L'$ -arrow since  $ev_{A, a^n} \circ_{n+1} L(f^n, \tilde{A}) := L(a^n, \tilde{A}) \circ_{n+1} L(f^n, \tilde{A}) = L(f^n \circ_{n+1} a^n, \tilde{A}) = ev_{A', f^n \circ_{n+1} a^n}$ , which is liftable  $\forall a^n \in L^n(A_0, \tilde{A})$ . Therefore,  $L(f^n, \tilde{A})$  is an  $L'$ -arrow, and similarly,  $L'(g^n, \tilde{B})$  is an  $L$ -arrow, i.e., there exist maps  $L'^{op} \xrightleftharpoons[F]{G} L'$ , which are obviously functorial.

Why do they give an adjunction?

- (unit and counit) 1-arrow (unit)  $\eta_B : B \rightarrow GFB$  is given by  $|\eta_B| =: V\eta_B : |B| \rightarrow |GFB| : b \mapsto [ev_{B, b} : FB \rightarrow \tilde{A}]$ ,  $b \in |B| = L'(B_0, B)$ ,  $|GFB| = L(FB, \tilde{A})$ ,  $|ev_{B, b}| : |FB| \rightarrow |\tilde{A}|$ ,  $|FB| = L'(B, \tilde{B})$ ,  $|\tilde{A}| = L(A_0, \tilde{A}) \sim L'(B_0, \tilde{B})$ . Why can  $|\eta_B|$  be lifted to  $L'$ ? Take the composite with evaluation maps  $|ev_{FB, c}| \circ_1 |\eta_B|(b) = |ev_{FB, c}|(ev_{B, b}) = |ev_{B, b}|(c) = |c|(b)$ , where  $c \in |FB|^0 = L'^0(B, \tilde{B}) = L^0(A_0, FB)$ ,  $b \in |B|^n$ . So,  $|ev_{FB, c}| \circ_1 |\eta_B| = |c|$  is an  $L'$ -arrow. Therefore,  $|\eta_B|$  is an  $L'$ -arrow. The counit is given symmetrically  $\varepsilon_A : FGA \rightarrow A$ ,  $|\varepsilon_A| : |A| \rightarrow |FGA| : a \mapsto [ev_{A, a} : GA \rightarrow \tilde{B}]$ ,  $|A| = L(A_0, A)$ ,  $|FGA| = L'(GA, \tilde{B})$ ,  $|ev_{A, a}| : |GA| \rightarrow |\tilde{B}|$ ,  $|GA| = L(A, \tilde{A})$ ,  $|\tilde{B}| = L'(B_0, \tilde{B}) \sim L(A_0, \tilde{A})$ . By the same argument  $|\varepsilon_A|$  is an  $L$ -arrow.
- (triangle identities)  $G\varepsilon_A \circ_1 \eta_{GA} = 1_{GA}$ ,  $F\eta_B \circ_1 \varepsilon_{FB} = 1_{FB}$ . It is sufficient to prove them for underlying maps. Since forgetful functors are faithful this follows.

$|G\varepsilon_A| \circ_1 |\eta_{GA}| \stackrel{?}{=} |1_{GA}|$ , where  $|\eta_{GA}| : |GA| \rightarrow |GFGA|$ ,  $|GA| = L(A, \tilde{A})$ ,  $|GFGA| = L(FGA, \tilde{A})$ ,  $\varepsilon_A : A \rightarrow FGA$ ,  $|G\varepsilon_A| : |GFGA| \rightarrow |GA|$ .

Take  $(f^n : A \rightarrow \tilde{A}) \in |GA| = L^n(A, \tilde{A})$ ,  $a^m \in |A| = L^m(A_0, A)$ . Two cases are possible

$$\begin{cases}
 (a) (f^n, n > 0) \& (a^0) : ||G\varepsilon_A| \circ_1 |\eta_{GA}|(f^n)|(a^0) = |L(\varepsilon_A, \tilde{A})(ev_{GA, f^n})|(a^0) = |ev_{GA, f^n \circ_{n+1}} \\
 (b) (f^0) \& (a^n, n \geq 0) : ||G\varepsilon_A| \circ_1 |\eta_{GA}|(f^0)|(a^n) = |L(\varepsilon_A, \tilde{A})(ev_{GA, f^0})|(a^n) = |ev_{GA, f^0 \circ_1} \\
 (a) e^n \varepsilon_A|(a^0) = |ev_{GA, f^n} \circ_{n+1} e^n \varepsilon_A|(a^0) = |ev_{GA, f^n}|(ev_{A, e^n a^0}) = |ev_{A, e^n a^0}|(f^n) = |f^n|(a^0) \\
 (b) \varepsilon_A|(a^n) = |ev_{GA, f^0} \circ_1 \varepsilon_A|(a^n) = |ev_{GA, f^0}|(ev_{A, a^n}) = |ev_{A, a^n}|(f^0) = |f^0|(a^n) \\
 (a) =: \mu_{A_0, A, \tilde{A}}^L(f^n, e^n a^0) = ||1_{GA}|(f^n)|(a^0) \\
 (b) =: \mu_{A_0, A, \tilde{A}}^L(e^n f^0, a^n) = ||1_{GA}|(f^0)|(a^n)
 \end{cases}$$

The second triangle identity holds similarly.

- (naturality of  $\eta_B, \varepsilon_A$ ) Again, it is sufficient to prove naturality for underlying maps

$$\begin{array}{ccc}
 |B| & \xrightarrow{|\eta_B|} & |GFB| \\
 |f| \downarrow & |GFf| = L(Ff, \tilde{A}) \downarrow & \\
 |B'| & \xrightarrow{|\eta_{B'}|} & |GFB'|
 \end{array}
 \quad \text{Two cases are } \begin{cases}
 (a) (b^n \in |B|^n, n \geq 0) \& (f^0 \in L'^0(B, B')) \\
 (b) (b^0 \in |B|^0) \& (f^n \in L'^n(B, B'))
 \end{cases}$$

$$\begin{array}{c}
(a) \quad \begin{array}{ccc}
|B| & \xrightarrow{|\eta_B|} & |GFB| \\
|f^0| \downarrow & & \downarrow |GFf^0| =_{L(Ff^0, \tilde{A})} \\
|B'| & \xrightarrow{|\eta_{B'}|} & |GFB'|
\end{array} \\
\begin{array}{ccc}
b^n & \xrightarrow{\hspace{2cm}} & ev_{B,b^n} \\
\downarrow & & \downarrow \\
|f^0|(b^n) & \xrightarrow{\hspace{2cm}} & ev_{B',|f^0|(b^n)}
\end{array} \\
\begin{array}{ccc}
b^0 & \xrightarrow{\hspace{2cm}} & ev_{B,e^n b^0} \\
\downarrow & & \downarrow \\
|f^n|(b^0) & \xrightarrow{\hspace{2cm}} & ev_{B',|f^n|(b^0)}
\end{array} \\
(b) \quad \begin{array}{ccc}
|B| & \xrightarrow{e^n |\eta_B|} & |GFB| \\
|f^n| \downarrow & & \downarrow |GFf^n| =_{L(Ff^n, \tilde{A})} \\
|B'| & \xrightarrow{e^n |\eta_{B'}|} & |GFB'|
\end{array} \\
\begin{array}{ccc}
ev_{B,b^n} \circ_{n+1} e^n(Ff^0) & & ev_{B',|f^0|(b^n)} \\
\parallel & & \parallel \\
ev_{B,e^n b^0} \circ_{n+1} (Ff^n) & & ev_{B',|f^n|(b^0)}
\end{array}
\end{array}$$

(recall  $|f^n|(b^0) \equiv \mu(f^n, e^n b^0)$ ,  $|f^0|(b^n) \equiv \mu(e^n f^0, b^n)$ )

Why  $\begin{cases} (a) \ ev_{B,b^n} \circ_{n+1} e^n(Ff^0) = ev_{B',|f^0|(b^n)} \\ (b) \ ev_{B,e^n b^0} \circ_{n+1} (Ff^n) = ev_{B',|f^n|(b^0)} \end{cases}$  ?

Take underlying maps:

$$\begin{cases} (a) \ |ev_{B,b^n}| \circ_{n+1} e^n |Ff^0|(h^n) = |ev_{B,b^n}|(h^n \circ_{n+1} e^n f^0) = |h^n \circ_{n+1} e^n f^0|(b^n) = \\ (b) \ |ev_{B,e^n b^0}| \circ_{n+1} |Ff^n|(h^0) = |ev_{B,e^n b^0}|(e^n h^0 \circ_{n+1} f^n) = |e^n h^0 \circ_{n+1} f^n|(e^n b^0) = \\ \begin{cases} (a) = |h^n| \circ_{n+1} |e^n f^0|(b^n) = |ev_{B',|f^0|(b^n)}|(h^n), \ h^n \in L'^n(B', \tilde{B}) \\ (b) = e^n |h^0| \circ_{n+1} |f^n|(e^n b^0) = |ev_{B',|f^n|(b^0)}|(h^0), \ h^0 \in L'^0(B', \tilde{B}) \end{cases} \end{cases}$$

(the types of the above arrows are  $Ff : FB' \rightarrow FB$ ,  $ev_{B,b} : FB \rightarrow \tilde{A}$  ( $L$ -map),  $ev_{B',|f|(b)} : FB' \rightarrow \tilde{A}$  ( $L$ -map),  $|ev_{B,b}| : L'(B, \tilde{B}) \rightarrow |\tilde{B}| = L'(B_0, \tilde{B})$ ,  $|ev_{B',|f|(b)}| : L'(B', \tilde{B}) \rightarrow |\tilde{B}| = L'(B_0, \tilde{B})$ ,  $|Ff| : L'(B', \tilde{B}) \rightarrow L'(B, \tilde{B})$ ,  $|Ff| = L'(f, \tilde{B})$ ).

Therefore,  $\eta_B$  is natural. Similarly,  $\varepsilon_A$  is natural.

- (isomorphisms-functors  $L(A, FB) \xrightleftharpoons[\theta_{A,B}^*]{\theta_{A,B}}$   $L'(B, GA)$  )

Define  $\begin{cases} \theta_{A,B}(f^n) := G(f^n) \circ_{n+1} e^n(\eta_B), \ f^n \in L^n(A, FB) \\ \theta_{A,B}^*(g^n) := F(g^n) \circ_{n+1} e^n(\varepsilon_A), \ g^n \in L'^n(B, GA) \end{cases}$

Let  $g^n \in L'^n(B, GA)$ . Then  $\theta_{A,B}(\theta_{A,B}^*(g^n)) := G(Fg^n \circ_{n+1} e^n(\varepsilon_A)) \circ_{n+1} e^n(\eta_B) = e^n(G\varepsilon_A) \circ_{n+1} GFg^n \circ_{n+1} e^n(\eta_B) \stackrel{\text{nat. of } \eta_B}{=} e^n(G\varepsilon_A) \circ_{n+1} e^n(\eta_{GA}) \circ_{n+1} g^n \stackrel{\text{triangle id.}}{=} e^n(1_{GA}) \circ_{n+1} g^n = e^{n+1}(GA) \circ_{n+1} g^n = g^n$ . Similarly,  $\theta_{A,B}^*(\theta_{A,B}(f^n)) = f^n$ ,  $f^n \in L^n(A, FB)$ .  $\theta_{A,B}$ ,  $\theta_{A,B}^*$  are obviously functors. So, they are isomorphisms.

- (naturality of  $\theta_{A,B}$ ,  $\theta_{A,B}^*$ ) We need to prove the diagram

$$\begin{array}{ccccc}
A & & B & & L(A, FB) \xrightarrow{e^n \theta_{A,B}} L'(B, GA) \\
\uparrow x^n & & \uparrow y^n & & \downarrow L(x^n, FY^n) \\
A' & & B' & & L(A', FB') \xrightarrow{e^n \theta_{A',B'}} L'(B', GA')
\end{array}$$

$\downarrow L'(y^n, Gx^n)$  commutes.

$$L'(y^n, Gx^n) \circ_{n+1} e^n \theta_{A,B} \stackrel{?}{=} e^n \theta_{A',B'} \circ_{n+1} L(x^n, Fy^n)$$

Two cases are:  $\begin{cases} (a) (f^0 \in L(A, FB)) \& (x^n, y^n, n > 0) \\ (b) (f^n \in L(A, FB), n \geq 0) \& (x^0, y^0) \end{cases}$

$$\begin{array}{ccc}
f^0 & \xrightarrow{\quad} & e^n G(f^0) \circ_{n+1} e^n(\eta_B) \\
\downarrow & & \downarrow \\
& & Gx^n \circ_{n+1} (e^n G(f^0) \circ_{n+1} e^n(\eta_B)) \circ_{n+1} y^n \\
(a) & & = \parallel ? \\
Fy^n \circ_{n+1} e^n f^0 \circ_{n+1} x^n & \xrightarrow{\quad} & G(Fy^n \circ_{n+1} e^n f^0 \circ_{n+1} x^n) \circ_{n+1} e^n(\eta_{B'}) = \parallel (\eta_B \text{ is nat.}) \\
\downarrow & & \downarrow \\
& & Gx^n \circ_{n+1} e^n Gf^0 \circ_{n+1} GFy^n \circ_{n+1} e^n(\eta_{B'}) \\
\\
f^n & \xrightarrow{\quad} & G(f^n) \circ_{n+1} e^n(\eta_B) \\
\downarrow & & \downarrow \\
& & e^n Gx^0 \circ_{n+1} (G(f^n) \circ_{n+1} e^n(\eta_B)) \circ_{n+1} e^ny^0 \\
(b) & & = \parallel ? \\
e^n Fy^0 \circ_{n+1} f^n \circ_{n+1} e^nx^0 & \xrightarrow{\quad} & G(e^n Fy^0 \circ_{n+1} f^n \circ_{n+1} e^nx^0) \circ_{n+1} e^n(\eta_{B'}) = \parallel (\eta_B \text{ is nat.}) \\
\downarrow & & \downarrow \\
& & e^n Gx^0 \circ_{n+1} Gf^n \circ_{n+1} e^n GFy^0 \circ_{n+1} e^n(\eta_{B'}) \\
\end{array}$$

Therefore,  $L$  and  $L'$  are concretely dually adjoint. This correspondence is natural (by condition) and strict ( $\theta_{A,B}$  and  $\theta_{A,B}^*$  are isomorphisms).  $\square$

**Corollary.** Concrete natural duality is a **strict** adjunction.  $\square$

### Well-known dualities [P-Th, Bel, A-H-S]

All dualities below are of first order, natural [P-Th], and obtained by restriction of appropriate dual adjunctions.

1.  $\mathbf{Vec}_k$  is dually equivalent to itself  $\mathbf{Vec}_k^{op} \xrightleftharpoons[\mathbf{Vec}_k(-, k)]{\perp} \mathbf{Vec}_k$ , where  $\mathbf{Vec}_k$  is a category of vector spaces over field  $k$

2.  $\mathbf{Set}^{op} \sim \mathbf{Complete\ Atomic\ Boolean\ Algebras}$

3.  $\mathbf{Bool}^{op} \sim \mathbf{Boolean\ Spaces}$  (Stone duality), where  $\mathbf{Bool}$  is a category of Boolean rings (every

element is idempotent). It is obtained from the dual adjunction  $\mathbf{CRing} \xrightleftharpoons[\mathbf{Top}(-, \mathbf{2})]{\perp} \mathbf{Top}$ ,

where  $\mathbf{2}$  is two-element ring and discrete topological space.  $\mathbf{CRing}(A, \mathbf{2}) \hookrightarrow \mathbf{2}^A$  (subspace in Tychonoff topology)

4.  $\hom(-, \mathbb{R}/\mathbb{Z}) : \mathbf{CompAb}^{op} \sim \mathbf{Ab}$  (Pontryagin duality), where **CompAb**, **Ab** are categories of compact abelian groups and abelian groups respectively
5.  $\hom(-, \mathbb{C}) : \mathbf{C^*Alg}^{op} \sim \mathbf{CHTop}$  (Gelfand-Naimark duality), where **C\*Alg**, **CHTop** are categories of commutative  $\mathbb{C}^*$ -algebras and compact Hausdorff spaces.  $\mathbf{C^*Alg}(A, \mathbb{C}) \hookrightarrow \mathbb{C}^A$  (subspace in Tychonoff topology)

## 7. Vinogradov duality

Let  $K$  be a commutative ring,  $A$  a commutative algebra over  $K$ ,  $A\text{-Mod} \hookrightarrow K\text{-Mod}$  be the categories of modules over  $A$  and  $K$  respectively.

**Definition 7.1.** [V-K-L] For  $P, Q \in Ob(A\text{-Mod})$

- $K$ -linear maps  $l(a) := a \cdot -, r(a) := - \cdot a, \delta(a) := l(a) - r(a) : K\text{-Mod}(P, Q) \rightarrow K\text{-Mod}(P, Q)$  are called **left, right multiplications** and **difference operator** (by element  $a \in A$ ),
- A  $K$ -linear map  $\Delta : P \rightarrow Q$  is a **differential operator of order  $\leq r$**  if  $\forall a_0, a_1, \dots, a_r \in A$   $\delta_{a_0, a_1, \dots, a_r}(\Delta) = 0$ , where  $\delta_{a_0, a_1, \dots, a_r} := \delta_{a_0} \circ \delta_{a_1} \circ \dots \circ \delta_{a_r}$ .  $\square$

**Lemma 7.1.**

- If  $\Delta_1 \in K\text{-Mod}(P, Q), \Delta_2 \in K\text{-Mod}(Q, R)$  are differential operators of order  $\leq r$  and  $\leq s$  respectively, then  $\Delta_2 \circ \Delta_1 : K\text{-Mod}(P, R)$  is a differential operator of order  $\leq r+s$ ,
- $\forall a \in A, P \in Ob(A\text{-Mod})$  module multiplication (by  $a$ )  $l_a : P \rightarrow P : p \mapsto ap$  is a differential operator of order 0.  $\square$

The differential operators between  $A$ -modules form the arrows of a category  $A\text{-Diff}$ , such that  $A\text{-Mod} \hookrightarrow A\text{-Diff} \hookrightarrow K\text{-Mod}$ , and the first two categories have the same objects.  $A\text{-Diff}$  is enriched in  $(K\text{-Mod}, \otimes_K)$  and enriched in two different ways in  $(A\text{-Mod}, \otimes_K)$ , except that composition is not an  $A$ -module map. Module multiplication for the first enrichment  $A\text{-Diff}$  in  $(A\text{-Mod}, \otimes_K)$  is given by  $A \times A\text{-Diff}(P, Q) \rightarrow A\text{-Diff}(P, Q) : (a, \Delta) \mapsto l_a \circ \Delta$ , for the second enrichment by  $A \times A\text{-Diff}(P, Q) \rightarrow A\text{-Diff}(P, Q) : (a, \Delta) \mapsto \Delta \circ l_a$ . Denote  $A\text{-Diff}$  with left module multiplication in hom-sets  $l_a \circ -$  by the same name  $A\text{-Diff}$  and with right multiplication in hom-sets  $- \circ l_a$  by  $A\text{-Diff}^+$ .

**Proposition 7.1.**

- $\forall P, Q \in Ob(A\text{-Mod}) A\text{-Diff}(P, Q) = \bigcup_{s=0}^{\infty} \mathbf{Diff}_s(P, Q), A\text{-Diff}^+(P, Q) = \bigcup_{s=0}^{\infty} \mathbf{Diff}_s^+(P, Q)$  are filtered  $A$ -modules by submodules of differential operators of order  $\leq s$ ,  $s = 0, 1, \dots$ ,
- $\forall P \in Ob(A\text{-Mod}) A\text{-Diff}(P, P)$  is an associative  $K$ -algebra.  $\square$

**Proposition 7.2.**

- $\mathbf{Diff}_s(P, -), \mathbf{Diff}_s^+(-, P) : A\text{-Mod} \rightarrow A\text{-Mod}$  are  $A$ -linear functors,
- $\forall P \in Ob(A\text{-Mod})$  functor  $\mathbf{Diff}_s^+(-, P)$  is representable by object  $\mathbf{Diff}_s^+(P) := \mathbf{Diff}_s^+(A, P)$ , i.e.  $\forall Q \in Ob(A\text{-Mod}) A\text{-Mod}(Q, \mathbf{Diff}_s^+(P)) \xrightarrow{\sim} \mathbf{Diff}_s^+(Q, P)$ ,
- $\forall P \in Ob(A\text{-Mod})$  functor  $\mathbf{Diff}_s(P, -)$  is representable by object  $\mathbf{Jet}^s(P) := A \otimes_K P \text{ mod } \mu^{s+1}$ , where  $\mu^{s+1}$  is a submodule of  $A \otimes_K P$  generated by elements  $\delta^{a_0} \circ \dots \circ \delta^{a_{s+1}}(a \otimes p)$  [ $\delta^b(a \otimes p) := ab \otimes p - a \otimes bp$ ], i.e.  $\forall Q \in Ob(A\text{-Mod}) A\text{-Mod}(\mathbf{Jet}^s(P), Q) \xrightarrow{\sim} \mathbf{Diff}_s(P, Q)$ ,
- inclusion  $A\text{-Mod} \hookrightarrow A\text{-Diff}^+$  is an (enriched) left adjoint with counit  $ev : \mathbf{Diff}^+(P) \rightarrow P : \Delta \mapsto \Delta(1)$ , i.e.  $\forall \Delta \in \mathbf{Diff}^+(Q, P) \exists! f_\Delta \in A\text{-Mod}(Q, \mathbf{Diff}^+(P))$  such that

$$\begin{array}{ccc} \mathbf{Diff}^+(P) & \xrightarrow{ev} & P \\ \uparrow f_\Delta & \nearrow \Delta & \\ Q & & \end{array}$$

and this correspondence is  $A$ -linear,  $f_\Delta : q \mapsto (a \mapsto \Delta(aq))$ ,

- inclusion  $A\text{-Mod} \hookrightarrow A\text{-Diff}$  is an (enriched) right adjoint with unit  $j^\infty : P \rightarrow \mathbf{Jet}^\infty(P) : p \mapsto 1 \otimes p \bmod \mu^\infty$  [ $\mu^\infty := \bigcap_{s=0}^{\infty} \mu^s$ ], i.e.  $\forall \Delta \in \mathbf{Diff}(P, Q) \exists! f^\Delta \in A\text{-Mod}(\mathbf{Jet}^\infty(P), Q)$  such that

$$\begin{array}{ccc} P & \xrightarrow{j^\infty} & \mathbf{Jet}^\infty(P) \\ & \searrow \Delta & \downarrow f^\Delta \\ & & Q \end{array}$$

and this correspondence is  $A$ -linear,  $f^\Delta : (a \otimes p) \bmod \mu^\infty \mapsto a\Delta(p)$ ,

- subcategory  $A\text{-Mod}$  is reflective and coreflective in  $A\text{-Diff}$  (enriched in  $K\text{-Mod}$ ).  $\square$

$\forall s \in \mathbb{N}$  introduce two full subcategories of  $A\text{-Mod}$ :

- $A\text{-Mod-Diff}_s$ , consisting of all  $A$ -modules of type  $\mathbf{Diff}_s(P, A)$ ,  $P \in Ob(A\text{-Mod})$ ,
- $A\text{-Mod-Jet}^s$ , consisting of all  $A$ -modules of type  $\mathbf{Jet}^s(P)$ ,  $P \in Ob(A\text{-Mod})$ .

**Proposition 7.3 (Vinogradov Duality).** *For a commutative algebra  $A$  there is a concrete natural dual adjunction  $A\text{-Mod-Diff}_s^{op} \rightleftarrows A\text{-Mod-Jet}^s$ ,  $s \in \mathbb{N}$ , obtained by restriction*

of  $A\text{-Mod}^{op} \rightleftarrows A\text{-Mod}$ .  $A$  is a schizophrenic object.  $\square$

### Remarks.

- The above duality theorem is not stated explicitly in [V-K-L] but the result is implicitly there.
- The above proposition states a formal analogue of duality between differential operators and jets over a fixed manifold  $X$ . Geometric modules of sections of vector bundles over  $X$  correspond to modules  $P$  over  $C^\infty(X)$  with the property  $\bigcap_{x \in X} \mu_x P = 0$ , where  $\mu_x$  is a maximal ideal at point  $x \in X$ . Functors  $\mathbf{Diff}_s(-, A)$  and  $\mathbf{Jet}^s(-)$  preserve the module property to be geometric [V-K-L].
- This duality is an alternative (algebraic) way to introduce jet-bundles in Geometry (instead of the classical approach due to Grothendieck and Ehresmann as equivalence classes of maps which tangent of order  $s$  at a point). When  $A = C^\infty(X)$  and  $P$  is a geometric module realizable as a vector bundle  $V(P)$  over  $X$ , then  $\mathbf{Jet}^s(P)$  is realizable as  $\mathbf{Jet}^s(V(P))$  over  $X$  in the classical sense [V-K-L, Vin1, Vin2].  $\square$

## 8. Duality for differential equations

**Proposition 8.1.** *Let  $\mathbf{UAlg}$  be a category of universal algebras with a representable forgetful functor. Then every topological algebra  $\mathfrak{A}$  is a schizophrenic object (see [P-Th]), and so yields a natural dual adjunction between  $\mathbf{UAlg}$  and  $\mathbf{Top}$ .*

*Proof.*

- The initial topology on  $\mathbf{UAlg}(B, \mathfrak{A})$  gives the initial lifting with respect to evaluation maps  $ev_{B, b} : \mathbf{UAlg}(B, \mathfrak{A}) \rightarrow |\mathfrak{A}|$ ,  $b \in |B|$ .
- The algebra of continuous functions  $\mathbf{Top}(X, \mathfrak{A})$  is initial with respect to the evaluation maps  $ev_{X, x} : \mathbf{Top}(X, \mathfrak{A}) \rightarrow |\mathfrak{A}|$ ,  $x \in |X|$  (which are obviously homomorphisms) since operations in  $\mathbf{Top}(X, \mathfrak{A})$  are pointwise and each arrow  $f \in \mathbf{Top}(X, \mathfrak{A})$  is completely determined by all its

values  $ev_{X,x}(f) = |f|(x)$ ,  $x \in |X|$ . Hence, if  $g : |B| \rightarrow \mathbf{Top}(X, \mathfrak{A})$  is a **Set**-map such that  $\forall x \in |X|$   $ev_{X,x} \circ g$  is a homomorphism,  $(\omega_n(ev_{X,x} \circ g)b_1, \dots, (ev_{X,x} \circ g)b_n = ev_{X,x} \circ g\omega_n b_1, \dots, b_n = ev_{X,x}\omega_n g b_1, \dots, g b_n)$ , where  $\omega_n$  is an  $n$ -ary operation. The first equality holds because  $ev_{X,x} \circ g$  is a homomorphism, the second equality because  $ev_{X,x}$  is a homomorphism), then  $g$  is a homomorphism since two maps whose values coincide at each point coincide themselves.  $\square$

**Corollary.** Take  $\mathbf{UAlg} = k\text{-}\Lambda\text{-Alg}$ , the category of exterior differential algebras over a field  $k$  ( $\mathbb{R}$  or  $\mathbb{C}$ ). These are thought of as presenting “generalized differential equations”. Take  $\mathfrak{A} = \Lambda(C^\infty(\mathbb{R}^n))$  or  $\Lambda(C^\omega(\mathbb{C}^n))$  (which acts as a parameter space) with a topology not weaker than  $jet^\infty$ . Then there exists a natural dual adjunction  $k\text{-}\Lambda\text{-Alg}^{op} \rightleftarrows \mathbf{Top}$  (between differential equations and their solution spaces).  $\square$

**Remark.** If we regard the category  $k\text{-}\Lambda\text{-Alg}$  whose forgetful functor is representable, we will get a lot of extra “points” which do not have geometric sense. Only graded maps of degree 0 to  $\mathfrak{A}$  have geometric sense (they present integral manifolds of dimension not bigger than  $n$ ). In this case, the representation of exterior differential algebras, when it exists, will not be via their solution spaces but via much bigger spaces. If we restrict  $k\text{-}\Lambda\text{-Alg}$  to only graded morphisms of degree 0 then the forgetful functor is not representable. But the notion of “schizophrenic object” still makes sense and the theorem for natural dual adjunction [P-Th] still holds. So, there is a representation of exterior differential algebras via their usual solution spaces.  $\square$

We denote concrete subcategories of **Top** dual to categories  $k\text{-Alg}$  (algebras over  $k$ ) and  $k\text{-}\Lambda\text{-Alg}$  (exterior differential algebras over  $k$  with graded degree 0 morphisms) by **alg-Sol** and **diff-Sol** respectively, i.e.,  $k\text{-Alg}^{op} \sim \mathbf{alg-Sol}$ ,  $k\text{-}\Lambda\text{-Alg}^{op} \sim \mathbf{diff-Sol}$ . In particular, **alg-Sol** contains all algebraic and all smooth  $k$ -manifolds ( $k = \mathbb{R}$  or  $\mathbb{C}$ ), **diff-Sol** contains all spaces of the form  $\mathbf{alg-Sol}(k^n, X)$  (with representing object  $\mathfrak{A} = \Lambda(C^\infty(k^n))$ ).

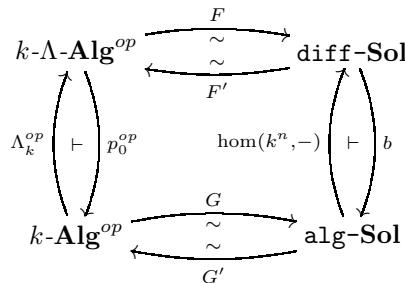
**Lemma 8.1** (rough structure of **diff-Sol**).

- $Ob(\mathbf{diff-Sol})$  are pairs  $(X, \coprod_{i=1}^n \mathcal{F}_i)$  where  $X := k\text{-}\Lambda\text{-Alg}(D, k) = k\text{-Alg}(D, k) \in Ob(\mathbf{alg-Sol})$ ,  $\mathcal{F}_i \subset \mathbf{alg-Sol}(k^i, X)$ ,  $1 \leq i \leq n$  [ $\mathcal{F}_i$  are not arbitrary subspaces of  $\mathbf{alg-Sol}(k^i, X)$ ].
- $Ar(\mathbf{diff-Sol})$  are pairs  $(f, \coprod_{i=1}^n \mathbf{alg-Sol}(k^i, f)) : (X, \coprod_{i=1}^n \mathcal{F}_i) \rightarrow (X', \coprod_{i=1}^n \mathcal{F}'_i)$  where  $f : X \rightarrow X' \in Ar(\mathbf{alg-Sol})$ ,  $\mathbf{alg-Sol}(k^i, f) : \mathcal{F}_i \rightarrow \mathcal{F}'_i$ ,  $1 \leq i \leq n$ .  $\square$

**Proposition 8.2.** There are the following adjunctions

- $k\text{-Alg} \rightleftarrows_{p_0}^{p_1} k\text{-}\Lambda\text{-Alg}$  where  $\Lambda_k$  is the free exterior differential algebra functor,  $p_0$  is the projection onto the subalgebra of degree-0 elements,
- $\mathbf{alg-Sol} \rightleftarrows_b \mathbf{diff-Sol}$  where  $b$  is the base space functor,

*such that*



*Proof.*

- $k\text{-}\Lambda\text{-Alg}(\Lambda_k(A), D) \xrightarrow{\sim} k\text{-Alg}(A, p_0(D))$  (natural in  $A$  and  $D$ )

$$\rho \xrightarrow{\sim} \rho_0$$

where  $\rho_0$  is the 0-component of graded degree 0 homomorphism  $\rho = \bigoplus_{i \geq 0} \rho_i$ .

- $\text{diff-Sol}(S, \hom(k^n, X)) \xrightarrow{\sim} \text{alg-Sol}(b(S), X)$  (natural in  $S$  and  $X$ )

$$\begin{array}{ccc} \in & & \in \\ f & \xrightarrow{\sim} & f \end{array}$$

where:  $S$  is a pair  $(b(S), \coprod_{i=1}^n \mathcal{F}_i)$ ,  $\mathcal{F}_i \subset \text{hom}(k^i, b(S))$ ,  $1 \leq i \leq n$ , right  $f : b(S) \rightarrow X$  is a usual map, and left  $f := (f, \coprod_{i=1}^n \text{hom}(k^i, f)) : (b(S), \coprod_{i=1}^n \mathcal{F}_i) \rightarrow (X, \coprod_{i=1}^n \text{hom}(k^i, X))$ .

The above square of adjunctions is immediate.

### 8.1. Cartan involution.

For systems in Cartan involution (as defined below) a (single) solution can be calculated recursively beginning from smallest 0 dimension. By Cartan's theorem [BC3G, Car1, Fin, Vas] every system can be made into such a form by a sufficient number of differential prolongations [BC3G, Car1, Fin, Vas]. There is a cohomological criterion for systems to be in involution.

**Definition 8.1.1.** Let  $\mathcal{A} \in Ob(k\text{-}\Lambda\text{-Alg})$ ,  $\mathfrak{A}_n$  be  $\Lambda_{\mathbb{R}}(C^\infty(\mathbb{R}^n))$  or  $\Lambda_{\mathbb{C}}(C^\omega(\mathbb{C}^n))$ ,  $n \geq 0$ .

- Any (differential homomorphism of degree 0)  $\rho : \mathcal{A} \rightarrow \mathfrak{A}_n$  is called an **integral manifold** of  $\mathcal{A}$  (of dimension not bigger than  $n$ ).
  - $\deg(\rho : \mathcal{A} \rightarrow \mathfrak{A}_n) = m$ ,  $0 \leq m \leq n$ , iff  $\rho$  can be factored through a  $\gamma : \mathcal{A} \rightarrow \mathfrak{A}_m$ , i.e.,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\gamma} & \mathfrak{A}_m \\ & \searrow \rho & \downarrow \\ & & \mathfrak{A}_n \end{array}$$

and  $m$  is the smallest such number.

- $\deg(\mathcal{A}) = n$  iff maximal degree of integral manifolds of  $\mathcal{A}$  is  $n$ .
  - $\mathcal{A}$ ,  $\deg(\mathcal{A}) = n$ , is in **Cartan involution** iff for each  $m$ -dimensional integral manifold  $\rho : \mathcal{A} \rightarrow \mathfrak{A}_m$ ,  $m < n$ , there exists an  $(m+1)$ -dimensional integral manifold  $\beta : \mathcal{A} \rightarrow \mathfrak{A}_{m+1}$  which

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\exists \beta} & \mathfrak{A}_{m+1} \\
 \text{contains } \rho, \text{ i.e.,} & \searrow_{\forall \rho} & \downarrow \\
 & & \mathfrak{A}_m
 \end{array}
 \quad \square$$

### Remarks.

- $\mathfrak{A}_0$  is just  $k$  ( $\mathbb{R}$  or  $\mathbb{C}$ ) with the trivial differential.  $\rho : \mathcal{A} \rightarrow \mathfrak{A}_0$  corresponds to a point  $b(\rho) : b(\mathcal{A}) \rightarrow k$ . Each point of  $\mathcal{A}$  is a 0-dimensional integral manifold.
- The original Cartan definition was for classical algebras (quotient algebras of  $\Lambda_{\mathbb{R}}(C^{\omega}(\mathbb{R}^N))$ ) and in terms of 'infinitesimal integral elements' (nondifferential homomorphisms of degree 0  $f : \mathcal{A} \rightarrow \Lambda_k(d\tau^1, \dots, d\tau^N)$ ) [BC3G, Car1, Fin]. For that case, the two definitions coincide.
- By a number of differential prolongations (adding new jet-variables with obvious relations), every classical system can be put into Cartan involution form (E. Cartan's theorem).
- The integration step (constructing an integral manifold of 1 higher dimension) is done by the method of "Cauchy characteristics".  $\square$

**Proposition 8.1.1.** Let  $\mathcal{A}$  be a quotient algebra of  $\Lambda_{\mathbb{R}}(C^{\omega}(\mathbb{R}^N))$ ,  $\deg(\mathcal{A}) = n$ , corresponding

$$\begin{array}{ccc}
 \mathcal{E}^q & \longrightarrow & X \equiv b(F(\mathcal{A})) = \text{Jet}^q(\mathbb{R}^{n+k}) \\
 & \searrow & \downarrow \pi \qquad \swarrow \\
 & \mathbb{R}^n &
 \end{array}, \quad \dim(X) = N.$$

Then  $\mathcal{A}$  is in Cartan involution iff the following **Spencer  $\delta$ -complex** is acyclic:

$$0 \longrightarrow g^{(r)} \xrightarrow{\delta} g^{(r-1)} \otimes \Lambda^1(\mathbb{R}^n) \xrightarrow{\delta} g^{(r-2)} \otimes \Lambda^2(\mathbb{R}^n) \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} g^{(r-n)} \otimes \Lambda^n(\mathbb{R}^n) \longrightarrow 0$$

where  $g^{(r)} := T(\text{Jet}^r(\mathcal{E}^q)) \cap V\pi_{q+r-1}^{q+r} \hookrightarrow S_{q+r}(T_*\mathbb{R}^n) \otimes V\pi$  is  $r$ -th prolongation of symbol  $g$ ,  $\pi_{q+r-1}^{q+r} : \text{Jet}^{q+r}(\mathbb{R}^{n+k}) \rightarrow \text{Jet}^{q+r-1}(\mathbb{R}^{n+k})$  is a natural projection of jet-bundles,  $V$  is the "vertical" subbundle,  $S_p$  is the  $p$ -th symmetric power,

$$\delta(\alpha_1 \cdots \alpha_{q+r-l} \otimes v \otimes \beta_1 \wedge \cdots \wedge \beta_l) := \sum_{i=1}^{q+r-l} \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_{q+r-l} \otimes v \otimes \alpha_i \wedge \beta_1 \wedge \cdots \wedge \beta_l, \quad v \text{ is a section of } V\pi.$$

*Proof.* See [A-V-L, Sei, Vin2, V-K-L, Ver]  $\square$

The original Cartan involutivity test was in terms of certain dimensions of "infinitesimal integral elements". The above theorem is due to J.P. Serre [A-V-L, La-Se].

## 9. Gelfand-Naimark 2-duality

Let  $\mathbf{C}^*\mathbf{Alg}^{\text{op}} \xrightleftharpoons[\substack{\perp \\ G}]{} \mathbf{CHTop}$  be the usual Gelfand-Naimark duality between commutative

$C^*$ -algebras and compact Hausdorff spaces. Both categories are strict 2-categories with homotopy classes of homotopies as 2-cells (homotopy of  $C^*$ -algebras is a homotopy in  $\mathbf{Top}$  each instance of which is a  $C^*$ -algebra homomorphism). The reasonable question is: can it be extended to a 2-duality? The answer is yes.

By definition

$$\begin{array}{ccc} \mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{\text{ev}} & |B| \\ f \times 1 \uparrow & \nearrow \bar{f} & \\ |I| \times |A| & & \end{array} \quad \begin{array}{ccc} \mathbf{C}^*\mathbf{Alg}(B, C) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A,B,C}} & \mathbf{C}^*\mathbf{Alg}(A, C) \\ 1 \times f \uparrow & \nearrow F(\bar{f}) & \\ \mathbf{C}^*\mathbf{Alg}(B, C) \times |I| & & \end{array}$$

So that, if  $f : |I| \times |A| \rightarrow |B|$  is a homotopy in  $\mathbf{C}^*\mathbf{Alg}$ , then its image in  $\mathbf{CHTop}$  is  $F(\bar{f}) : |F(B)| \times |I| \rightarrow |F(A)|$  (where  $| \cdot |$  denotes underlying set or map).

We need to prove that such extended  $F$  preserves 2-categorical structure (for  $G$  proof is symmetric).

### Preserving homotopies

**Lemma 9.1.** *If  $B$  is locally compact then  $\mathbf{Top}(B, C) \times \mathbf{Top}(A, B) \xrightarrow{c_{A,B,C}} \mathbf{Top}(A, C)$  is continuous (with compact-open topology in all hom-sets).*

*Proof* is standard. Let  $f = g \circ h = c_{A,B,C}(g, h)$ . Take  $U^K$  be a (subbase) nbhd of  $f$ . Sufficient to show that  $\exists$  (subbase) nbhds  $U_1^{K^1} \ni g$ ,  $U_2^{K^2} \ni h$ , s.t.  $U_1^{K^1} \circ U_2^{K^2} = c_{A,B,C}(U_1^{K^1}, U_2^{K^2}) \subset U^K$ . Take  $U_1 = U$ ,  $K_2 = K$ ,  $K_1$  be a compact nbhd of  $h(K)$ , s.t.  $K_1 \subset g^{-1}(U)$  ( $K_1$  exists by local compactness of  $B$ ),  $U_2 = \text{int}(K_1)$ .  $\square$

**Corollary.** *If  $A$  is locally compact then  $\text{ev}_{A,B} : \mathbf{Top}(A, B) \times |A| \rightarrow |B|$  is continuous.*

*Proof.* Each space  $A$  is homeomorphic to  $\mathbf{Top}(1, A)$  (with compact-open topology), and  $\text{ev}_{A,B}$  corresponds to  $c_{1,A,B}$ .  $\square$

**Lemma 9.2.** • Initial topology on  $|F(A)| = \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})$  w.r.t. evaluation maps  $\forall a \in A$   $\mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \times 1 \xrightarrow{1 \times a} \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \times |A| \xrightarrow{\text{ev}} |\mathbb{C}|$  is point-open.

• Initial topology on  $|G(X)| = \mathbf{CHTop}(X, \mathbb{C})$  w.r.t. evaluation maps  $\forall x \in X$   $\mathbf{CHTop}(X, \mathbb{C}) \times 1 \xrightarrow{1 \times x} \mathbf{CHTop}(X, \mathbb{C}) \times |X| \xrightarrow{\text{ev}} |\mathbb{C}|$  is compact-open.

*Proof.* See [P-Th], [Joh], [Eng].  $\square$

**Lemma 9.3.** *If  $\mathcal{A}, \mathcal{B} \subset \mathbf{LCTop}$  are naturally dual subcategories of locally compact spaces (let  $D$  be a dualizing object) then if  $\mathcal{A}(X, D)$  has compact-open topology (as initial topology w.r.t. evaluation maps) then initial topology of  $|X| \cong \mathcal{B}(\mathcal{A}(X, D), D)$  is compact-open as well.*

*Proof.* Evaluation map  $\text{ev} : \mathcal{A}(X, D) \times |X| \rightarrow |D|$  is continuous (since  $X$  is locally compact and  $\mathcal{A}(X, D)$  has compact-open topology). It implies that initial (point-open) topology on  $|X| \cong \mathcal{B}(\mathcal{A}(X, D), D)$  is actually compact-open [by assumption, topology of  $|X|$  is initial w.r.t. all maps  $'f' : |X| \xrightarrow{\sim} 1 \times |X| \xrightarrow{f \times 1} \mathcal{A}(X, D) \times |X| \xrightarrow{\text{ev}} |D|$ . It means that topology on  $|X| \cong \mathcal{B}(\mathcal{A}(X, D), D)$  is point-open since subbase open sets in point-open and initial topologies are the same  $U'^f := \{x \in |X| \mid 'f'(x) \in \underset{\text{open}}{U} \subset D\} = 'f'^{-1}(U)\}$ .

We need to show that  $\{x \in |X| \mid \forall f \in \underset{\text{compact}}{K} \subset \mathcal{A}(X, D). 'f'(x) \in \underset{\text{open}}{U} \subset D\} = \bigcap_{f \in K} 'f'^{-1}(U)$

is open in point-open topology on  $|X|$ .

Take  $x \in \bigcap_{f \in K} 'f'^{-1}(U)$ , then  $\text{ev}(K, x) \subset U$ . By continuity of  $\text{ev}$ ,  $\forall y \in K. \exists V_y \underset{\text{open}}{\ni} y$ .

$\exists W_y \underset{\text{open}}{\ni} x$ , s.t.  $\text{ev}(V_y, W_y) \subset U$ .  $\bigcup_{y \in K} V_y \supset K$ , so, by compactness,  $\bigcup_{j=1, \dots, n} V_{y_j} \supset K$ . Therefore,

$ev(V_{y_j}, \bigcap_{j=1,\dots,n} W_{y_j}) \subset U$ ,  $ev(\bigcup_{j=1,\dots,n} V_{y_j}, \bigcap_{j=1,\dots,n} W_{y_j}) \subset U$ ,  $ev(K, \bigcap_{j=1,\dots,n} W_{y_j}) \subset U$ , i.e.,  $x$  is internal.  $\square$

**Corollary.** *Gelfand-Naimark duality preserves homotopies.*

*Proof.*  $|A| = \mathbf{CHTop}(X, \mathbb{C})$  has compact-open topology.  $|X| = \mathbf{C^*Alg}(A, \mathbb{C})$  has point-open topology, so, by **Lemma 9.3** compact-open topology.

Multiplication  $c_{A,B,C}$  is continuous (since all hom-sets have compact-open topology). Therefore,  $F(\bar{f})$  is continuous.

[In inverse direction  $G : \mathbf{CHTop} \rightarrow \mathbf{C^*Alg}$  there is no problem because  $\mathbf{CHTop}(X, \mathbb{C})$  has compact-open topology. See also [Loo]].  $\square$

### Preserving homotopy relation between homotopies

Let  $\bar{\bar{f}} : |I| \times |I| \times |A| \rightarrow |B|$  be continuous, s.t.  $\bar{\bar{f}}(0, t, a) = \bar{f}_0(t, a)$ ,  $\bar{\bar{f}}(1, t, a) = \bar{f}_1(t, a)$ .

$$\begin{array}{ccc}
 \mathbf{C^*Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\
 \bar{f}^T \times 1_{|A|} \uparrow & \nearrow \bar{f} & \downarrow \bar{f}_0, \bar{f}_1 \\
 |I| \times |I| \times |A| & & \\
 0 \times 1_{|I| \times |A|} \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) 1 \times 1_{|I| \times |A|} & & \\
 1 \times |I| \times |A| & & \\
 \sim \quad \quad \quad & & \\
 \quad \quad \quad & & \\
 & & \\
 \mathbf{C^*Alg}(B, C) \times \mathbf{C^*Alg}(A, B) & \xrightarrow{c_{A,B,C}} & \mathbf{C^*Alg}(A, C) \\
 1 \times \bar{f}^T \uparrow & \nearrow F(\bar{f}) & \downarrow F(\bar{f}_0), F(\bar{f}_1) \\
 \mathbf{C^*Alg}(B, C) \times |I| \times |I| & & \\
 1 \times ((0 \times 1_{|I|}) \circ \sim, 1_{|I|}) \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) 1 \times ((1 \times 1_{|I|}) \circ \sim, 1_{|I|}) & & \\
 \mathbf{C^*Alg}(B, C) \times |I| & & 
 \end{array}$$

So,  $F(\bar{\bar{f}})$  is a homotopy from  $F(\bar{f}_0)$  to  $F(\bar{f}_1)$ .  $F(\bar{\bar{f}})$  is continuous since  $c_{A,B,C}$  is continuous in compact-open topology.  $\mathbf{C^*Alg}(B, C)$  has compact-open topology by **Lemma 9.3**.

### Preserving unit 2-cells $i_f$

$$\begin{array}{ccc}
 \mathbf{C^*Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\
 f' \times 1 \uparrow & & \uparrow f \\
 1 \times |A| & \xrightarrow[p_2]{\sim} & |A| \\
 ! \times 1 \uparrow & \nearrow p_2 & \\
 |I| \times |A| & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{C^*Alg}(B, C) \times \mathbf{C^*Alg}(A, B) & \xrightarrow{c_{A,B,C}} & \mathbf{C^*Alg}(A, C) \\
 1 \times f' \uparrow & \nearrow F(f \circ p_2) & \downarrow - \circ f \\
 \mathbf{C^*Alg}(B, C) \times 1 & \xrightarrow[p_1]{\sim} & \mathbf{C^*Alg}(B, C) \\
 1 \times ! \uparrow & \nearrow F(i_f) & \downarrow p_1 \\
 \mathbf{C^*Alg}(B, C) \times |I| & & 
 \end{array}$$

So, if  $i_f = f \circ p_2 \circ (! \times 1_{|A|}) = f \circ p_2$ , then  $F(i_f) = F(f) \circ p_1 = i_{F(f)}$ .

**Preserving composites**  $i_g * \bar{f} : |I| \times |A| \xrightarrow{\bar{f}} |B| \xrightarrow{g} |C|$   
**and**  $\bar{f} * i_h : |I| \times |A'| \xrightarrow[1 \times h]{\sim} |I| \times |A| \xrightarrow{\bar{f}} |B|$

$$\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(A, C) \times |A| & \xrightarrow{\text{ev}} & |C| \\
(g \circ -) \times 1 \uparrow & & \uparrow g \\
\mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{\text{ev}} & |B| \\
f \times 1 \uparrow & \nearrow \bar{f} & \\
|I| \times |A| & & 
\end{array}
\quad
\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(C, \mathbb{C}) \times |I| & & \\
1 \times (\mathbf{C}^*\mathbf{Alg}(A, g) \circ f) \downarrow & \searrow F(g \circ \bar{f}) & \\
\mathbf{C}^*\mathbf{Alg}(C, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, C) & \xrightarrow{c_{A, C, \mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
\mathbf{C}^*\mathbf{Alg}(g, \mathbb{C}) \times 1 \uparrow & & \uparrow \sim \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A, B, \mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
1 \times f \uparrow & & \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & & 
\end{array}$$

$g \circ \bar{f}$  is a homotopy corresponding to  $\mathbf{C}^*\mathbf{Alg}(A, g) \circ f$ . Outer perimeter of the right diagram commutes because of definition of  $F(\bar{f})$ ,  $F(g \circ \bar{f})$  and associativity low [if  $(s, t) \in \mathbf{C}^*\mathbf{Alg}(C, \mathbb{C}) \times |I|$  then  $s \circ (g \circ f(t)) = (s \circ g) \circ f(t)$ ]. So,  $F(g \circ \bar{f}) = F(\bar{f}) \circ (F(g) \times 1_{|I|})$ , i.e.,  $F(i_g * \bar{f}) = F(\bar{f}) * i_{F(g)}$ .

$$\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{\text{ev}} & |B| \\
f \times 1 \uparrow & \nearrow \bar{f} & \uparrow \\
|I| \times |A| & & \\
1 \times h \uparrow & \nearrow \bar{f} \circ (1 \times h) & \\
|I| \times |A'| & & \\
(\bar{f} \circ (1 \times h))^T \times 1 \downarrow & \stackrel{\bar{f} \circ (1 \times h) =}{\stackrel{\text{evo}(f \times 1) \circ (1 \times h) =}{\stackrel{\text{evo}(f \times h) =}{}} \nearrow & \\
\mathbf{C}^*\mathbf{Alg}(A', B) \times |A'| & \xrightarrow{\text{ev}} & |B|
\end{array}
\quad
\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A, B, \mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
1 \times f \uparrow & \nearrow F(\bar{f}) & \downarrow \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & & \mathbf{C}^*\mathbf{Alg}(h, \mathbb{C}) \\
1 \times (\bar{f} \circ (1 \times h))^T \downarrow & \nearrow F(\bar{f} \circ (1 \times h)) & \downarrow \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A', B) & \xrightarrow{c_{A', B, \mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A', \mathbb{C})
\end{array}$$

Right internal triangle of the right diagram commutes since if  $(g, t) \in \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$  then  $\mathbf{C}^*\mathbf{Alg}(h, \mathbb{C}) \circ c_{A, B, \mathbb{C}} \circ (1 \times f)(g, t) = (g \circ f(t)) \circ h = g \circ (f(t) \circ h) = c_{A', B, \mathbb{C}}(g, f(t) \circ h) = c_{A', B, \mathbb{C}}(g, (\bar{f} \circ (1 \times h))^T(t)) = c_{A', B, \mathbb{C}} \circ (1 \times (\bar{f} \circ (1 \times h))^T)(g, t)$ . So,  $F(\bar{f} * i_h) = F(\bar{f} \circ (1 \times h)) = F(h) \circ F(f) = i_{F(h)} * F(f)$ .

### Preserving vertical composites

We need to show if  $\bar{f} : \bar{f} \circ i_0 \simeq \bar{f} \circ i_1$  and  $\bar{g} : \bar{g} \circ i_0 \simeq \bar{g} \circ i_1$  are homotopies in  $\mathbf{C}^*\mathbf{Alg}$  s.t.  $\bar{f} \circ i_1 = \bar{g} \circ i_0$  then  $F(\bar{g} \odot \bar{f}) = F(\bar{g}) \odot F(\bar{f})$ .

By definition, vertical composite  $\bar{g} \odot \bar{f}$  is

$$\begin{array}{ccc}
|A| \times |[0, \frac{1}{2}]| & \xleftarrow[\sim]{1 \times \alpha} & |A| \times |I| \\
& \swarrow \bar{f} & \\
& \nearrow 1 \times i & \\
|A| \times |I| & \xrightarrow[\sim]{\exists! \bar{g} \odot \bar{f}} & |B| \\
& \swarrow \bar{g} & \\
|A| \times |[\frac{1}{2}, 1]| & \xleftarrow[\sim]{1 \times \beta} & |A| \times |I|
\end{array}$$

The left diagram shows a commutative square with vertices  $\mathbf{C}^*\mathbf{Alg}(A, B) \times |A|$ ,  $|B|$ ,  $\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B)$ , and  $\mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})$ . The horizontal arrows are  $ev: \mathbf{C}^*\mathbf{Alg}(A, B) \times |A| \rightarrow |B|$  and  $c_{A, B, \mathbb{C}}: \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) \rightarrow \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})$ . The vertical arrows are  $1 \times (\bar{g} \odot \bar{f})^T: \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) \rightarrow \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})$  and  $F(\bar{f}): |B| \rightarrow \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})$ . The right diagram is similar, showing a commutative square with vertices  $\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$ ,  $\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$ ,  $\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$ , and  $\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$ . The horizontal arrows are  $F(\bar{g} \odot \bar{f}): \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| \rightarrow \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$  and  $F(\bar{f}): |I| \rightarrow \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$ . The vertical arrows are  $1 \times (j \circ \beta)^T: \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| \rightarrow \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$  and  $1 \times (i \circ \alpha): \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| \rightarrow \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$ .

By uniqueness  $f \equiv \bar{f}^T = (\bar{g} \odot \bar{f})^T \circ i \circ \alpha$ ,  $g \equiv \bar{g}^T = (\bar{g} \odot \bar{f})^T \circ j \circ \beta$ .

So,  $\begin{cases} F(\bar{g} \odot \bar{f}) \circ (1 \times (i \circ \alpha)) = F(\bar{f}) \\ F(\bar{g} \odot \bar{f}) \circ (1 \times (j \circ \beta)) = F(\bar{g}) \end{cases}$ . It means  $F(\bar{g} \odot \bar{f}) = F(\bar{g}) \odot F(\bar{f})$ .

**Preserving horisontal composites**  $A \xrightarrow[\substack{f_1 \\ f_0}]{} B \xrightarrow[\substack{g_1 \\ g_0}]{} C$

$\bar{g} * \bar{f} := (\bar{g} * i_{f_1}) \odot (i_{g_0} * \bar{f}) \simeq (i_{g_1} * \bar{f}) \odot (\bar{g} * i_{f_0})$  (homotopic homotopies).  
 $F(\bar{g} * \bar{f}) = F(\bar{g} * i_{f_1}) \odot F(i_{g_0} * \bar{f}) = (i_{F(f_1)} * F(\bar{g})) \odot (F(\bar{f}) * i_{F(g_0)}) \simeq F(\bar{f}) * F(\bar{g})$ .

Proposition 9.1 completes the proof of Gelfand-Naimark 2-duality  $\mathbf{C}^*\mathbf{Alg}^{\text{op}} \xrightleftharpoons[\substack{\perp \\ G}]{} \mathbf{CHTop}$ .

**Proposition 9.1.** If  $\mathbf{C} \xrightleftharpoons[\substack{F \\ G}]{} \mathbf{D}$  are two strict  $n$ -categories and two strict  $n$ -functors in the opposite directions such that the restriction  $\mathbf{C}^{\leq 1} \xrightleftharpoons[\substack{\perp \\ G^{\leq 1}}]{} \mathbf{D}^{\leq 1}$  is an adjunction with unit  $\eta: 1_{\mathbf{C}^{\leq 1}} \rightarrow$

$G^{\leq 1} F^{\leq 1}$  and counit  $\varepsilon: F^{\leq 1} G^{\leq 1} \rightarrow 1_{\mathbf{D}^{\leq 1}}$  which are still natural transformations for the extension (i.e.  $\eta: 1_{\mathbf{C}} \rightarrow GF$  and  $\varepsilon: FG \rightarrow 1_{\mathbf{D}}$  are natural transformations) then the extended situation

$\mathbf{C} \xrightleftharpoons[\substack{\perp \\ G}]{} \mathbf{D}$  is a strict adjunction.

*Proof.* A strict adjunction is completely determined by its 'unit-counit' (proposition 5.3).  $\eta: 1_{\mathbf{C}} \rightarrow GF$  and  $\varepsilon: FG \rightarrow 1_{\mathbf{D}}$  are natural transformations and satisfy triangle identities  $\varepsilon F \circ_1 F \eta = 1_F$  and  $G \varepsilon \circ_1 \eta G = 1_G$  (because, e.g.  $\varepsilon F = \varepsilon F^{\leq 1}$ ,  $1_F = 1_{F^{\leq 1}}$  (set-theoretically), etc.)

□

**Corollary.** Any 1-adjunction between a category of topological algebras and a subcategory of topological spaces is a 2-adjunction if it can be extended functorially over 2-cells in the way that each instance of the image of a homotopy is the image of this instance of the preimage-homotopy.

*Proof.* Under given conditions unit and counit of 1-adjunction are automatically natural trans-

formations for the extension. E.g., take unit  $\eta$ . Naturality square  $\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ f^1 \downarrow & & \downarrow GFF^1 \\ B & \xrightarrow{\eta_B} & GFB \end{array}$ , where

$f^1 : A \times I \rightarrow B$  is a homotopy, holds because each instance of it holds (since  $\eta$  is a unit of 1-adjunction), i.e.  $\forall t \in I \eta_B \circ f^1(-, t) = GF(f^1(-, t)) \circ \eta_A$ , it means  $\eta_B \circ f^1 = GF(f^1) \circ (\eta_A \times I)$ , i.e.  $\eta_B * f^1 = GF(f^1) * \eta_A$ .  $\square$

Gelfand-Naimark case is one of the above corollary. End of proof of Gelfand-Naimark 2-duality.  $\square$

**Remark.** There are 'forgetful' functors  $\mathbf{C}^*\mathbf{Alg} \rightarrow \mathbf{2-Set}$  and  $\mathbf{CHTop} \rightarrow \mathbf{2-Set}$  (where  $\mathbf{2-Set}$  is the usual  $\mathbf{Set}$  with just one iso-2-cell for each pair of maps with the same domain and codomain) but they are not faithful and forget too much in order  $\mathbf{2-Set}$  could be an underlying category of Gelfand-Naimark 2-duality.  $\square$

**Proposition 9.2.** • Gelfand-Naimark 2-duality is concrete over  $\mathbf{2-Cat}$  ( $\mathbf{2-Cat}$  is the usual 2-category of (small) categories, functors and natural transformations), i.e.  $\exists$  (faithful) forgetful

functors  $U : \mathbf{C}^*\mathbf{Alg} \rightarrow \mathbf{2-Cat}$  and  $V : \mathbf{CHTop} \rightarrow \mathbf{2-Cat}$  such that  $\begin{array}{ccc} \mathbf{C}^*\mathbf{Alg}^{\text{op}} & \xrightarrow{F} & \mathbf{CHTop} \\ & \searrow & \downarrow V \\ & \mathbf{C}^*\mathbf{Alg}(-, \mathbb{C}) & \rightarrow \mathbf{2-Cat} \end{array}$  and

$\begin{array}{ccc} \mathbf{CHTop}^{\text{op}} & \xrightarrow{G^{\text{op}}} & \mathbf{C}^*\mathbf{Alg} \\ & \searrow & \downarrow U \\ & \mathbf{CHTop}(-, \mathbb{C}) & \rightarrow \mathbf{2-Cat} \end{array}$  where  $U$  and  $V$  are composites of inclusion and fundamental groupoid

functors  $(U : \mathbf{C}^*\mathbf{Alg} \hookrightarrow \mathbf{2-Top} \xrightarrow{\text{2-Top}(1, -)} \mathbf{2-Cat} \text{ and } V : \mathbf{CHTop} \hookrightarrow \mathbf{2-Top} \xrightarrow{\text{2-Top}(1, -)} \mathbf{2-Cat})$ .

- This duality is natural, i.e. lifting of hom-functors  $\mathbf{C}^*\mathbf{Alg}(-, \mathbb{C})$ ,  $\mathbf{CHTop}(-, \mathbb{C})$  along  $V$  and  $U$  is initial.  $\square$

**Remark.** 2-duality allows us to transfer (co)homology theories from one side to another. Under a reasonable assumption that K-theory was determined in a universal way we could get **M. Atiyah theorem** that *K-groups of commutative  $C^*$ -algebras and compact Hausdorff spaces coincide*. The problem, however, is that K-groups were determined technically (not universally). But, there is a theorem by J. Cuntz [Weg] that K-theory is universally determined on a large subcategory of  $C^*$ -algebras.  $\square$

## 10. Lukacs' extension of Pontryagin duality

The following is a new and recent example of a concrete duality, due to G.Lukacs [Luk]. The extension is natural with the same dualizing object  $\mathbb{R}/\mathbb{Z}$ , and establishes a concrete duality for abelian locally precompact groups.

### Definition 10.1.

- The set  $X$  in a topological group  $G$  is called **precompact** if  $\forall U \ni e$  (neighbourhood of identity)  $\exists$  a finite subset  $F \subset G$  such that  $X \subset FU$ .
- The group  $G$  is **locally precompact** if it contains a precompact neighbourhood of the identity.

□

(Locally) precompact groups are very closed to (locally) compact ones. Namely, their two-sided uniformity completions give (locally) compact groups, and conversely, dense subgroups of (locally) compact groups are (locally) precompact.

**Proposition 10.1 (Pontryagin-Lukacs).** *There are the following natural dualities*

$$\begin{array}{ccc}
 \mathbf{Ab}^{op} & \begin{matrix} \xrightarrow{\sim} \\ \xleftarrow{\sim} \end{matrix} & \mathbf{CompAb} \\
 \downarrow & & \downarrow \\
 \mathbf{locCompAb}^{op} & \begin{matrix} \xrightarrow{\sim} \\ \xleftarrow{\sim} \end{matrix} & \mathbf{locCompAb} \\
 \downarrow & & \downarrow \\
 \mathbf{locPreCompAb}^{op} & \begin{matrix} \xrightarrow{\sim} \\ \xleftarrow{\sim} \end{matrix} & \mathbf{locCompAb}^{\Rightarrow}
 \end{array}$$

where  $\mathbf{locCompAb}^{\Rightarrow}$  is a category of dense embeddings of locally compact abelian groups into compact abelian groups (with commutative squares in  $\mathbf{locCompAb}$  as arrows). □

### Remarks.

- The main idea of this extension is that every locally precompact group  $G$  can be represented as a dense injective  $\mathbf{locCompAb}$ -morphism  $G_d \rightarrow \text{compl}(G)$ , where  $G_d$  is the same group with discrete topology, and  $\text{compl}(G)$  is its completion with respect to two-sided uniformity on  $G$ . After that, the usual Pontryagin duality is used [Luk].

- The dualizing object in  $\mathbf{locCompAb}^{\Rightarrow}$  is  $\begin{array}{c} \mathbb{R}/\mathbb{Z} \\ \downarrow id \\ \mathbb{R}/\mathbb{Z} \end{array}$ . □

## 11. Differential algebras as a dual to Lie calculus

For Lie groups there is an equivalent alternative calculus via exterior differential algebras. For Lie groups of transformations, it turns out to be more powerful than via Lie algebras. It was developed by E. Cartan and after him by the Russian School in Differential Geometry, mainly, by A.M. Vasiliev [Vas0, Vas].

### Definition 11.1.

- The exterior differential algebra  $\Lambda \in Ob(k\text{-}\Lambda\text{-Alg})$ ,  $k = \mathbb{C}$  or  $\mathbb{R}$ , is called **linear** if it finitely generated by elements of degree 1 (with possible linear (resolvable) relations between them over  $k$ ).
- The exterior differential algebra  $\Lambda \in Ob(k\text{-}\Lambda\text{-Alg})$ ,  $k = \mathbb{C}$  or  $\mathbb{R}$ , is called **quasilinear** if it is finitely generated by elements of degree 0 and 1 with relations between either elements of degree 0 or linear relations on elements of degree 1 with coefficients in  $\Lambda^0$ .
- A smooth map  $f : X \rightarrow Y$  is called **quasialgebraic** if there exist quasilinear subalgebras  $\Lambda_1 \hookrightarrow k\text{-}\Lambda(X)$  and  $\Lambda_2 \hookrightarrow k\text{-}\Lambda(Y)$  such that  $f^*(\Lambda_2) := k\text{-}\Lambda(f)(\Lambda_2) \hookrightarrow \Lambda_1$ . □

Quasialgebraic maps admit an effective description. All homomorphisms of Lie groups are quasialgebraic.

**Proposition 11.1.** *There are equivalences  $\mathbf{locLieGrp} \sim \mathbf{LieAlg} \sim k\text{-}\Lambda\text{-}\mathbf{Alg}_{lin}^{op}$  (local Lie groups  $\sim$  Lie algebras  $\sim$  (opposite of the category of) linear exterior differential algebras).  $\square$*

**Lemma 11.1.**

- The functor  $\mathbf{Diff}^{op} \xrightarrow{C^\infty} \mathbf{ComAlg} \xrightarrow{k\text{-}\Lambda} k\text{-}\Lambda\text{-}\mathbf{Alg} \xrightarrow{compl} k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$  is monoidal with respect to Cartesian product  $\times$  in  $\mathbf{Diff}$  and exterior product  $\wedge$  in  $k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$ , where  $k\text{-}\Lambda$  is a free exterior differential algebra functor over  $k$ ,  $compl$  is a smooth (or analytic) completion of exterior differential algebras.
- Analogously, the functor  $\mathbf{LieGrp}^{op} \xrightarrow{k\text{-}\Lambda_{inv}} k\text{-}\Lambda\text{-}\mathbf{Alg}$  (assigning the algebra of (left)invariant forms) is monoidal with respect to cartesian product  $\times$  in  $\mathbf{LieGrp}$  and exterior product  $\wedge$  in  $k\text{-}\Lambda\text{-}\mathbf{Alg}$ .  $\square$

**Remarks.**

- The smooth (analytic) completion functor  $compl : k\text{-}\Lambda\text{-}\mathbf{Alg} \rightarrow k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$  is a left adjoint to the inclusion (of the subcategory of smooth (analytic) exterior differential algebras)  $k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl} \hookrightarrow k\text{-}\Lambda\text{-}\mathbf{Alg}$  (it is given essentially by the smooth (analytic) completion of the algebra of coefficients of an exterior differential algebra).
- The exterior product  $\wedge$  in  $k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$  is bigger than in  $k\text{-}\Lambda\text{-}\mathbf{Alg}$  and is equal to the smooth (analytic) completion of (the usual algebraic) exterior product in  $k\text{-}\Lambda\text{-}\mathbf{Alg}$ .  $\square$

**Definition 11.2.**

- The exterior differential algebra  $\mathcal{A}$  is called **smoothly realizable** if there exists a manifold  $Y \in Ob \mathbf{Diff}$  and an embedding  $\mathcal{A} \hookrightarrow \Lambda(Y)$ . It is **fully smoothly realizable** if  $\mathcal{A}^1$  (locally) generates  $T^*Y$ .
- A **Geometric triple** is a (locally trivial) fibre bundle  $(G \times Y \rightarrow X) \in Ar \mathbf{Diff}$ , equivariant with respect to a (left) action of (Lie group)  $G$  on  $X$  and  $\rho : G \times G \times Y \rightarrow G \times Y : (g, h, y) \mapsto (gh, y)$  an action on  $G \times Y$ . A **morphism of geometric triples** is a morphism of fibre bundles

$$\begin{array}{ccc} G_1 \times Y_1 & \xrightarrow{\sigma \times F} & G_2 \times Y_2 \\ \rho_1 \downarrow & & \downarrow \rho_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array} \quad \text{where } \sigma : G_1 \rightarrow G_2 \text{ is a Lie group homomorphism. A geometric triple}$$

$\rho : G \times Y \rightarrow X$  is **local** if  $G$  is a local Lie group and  $X$  is a local  $G$ -space (admits a local group of transformations).

- An **algebraic triple** is an exterior product of two differential algebras  $\mathcal{A} \wedge \mathcal{B}$ , where  $\mathcal{A}$  is linear, with a differential ideal  $I \subset \mathcal{A} \wedge \mathcal{B}$  generated by elements of degree 1. A **morphism of algebraic triples**  $(\mathcal{A}_1 \wedge \mathcal{B}_1, I_1) \rightarrow (\mathcal{A}_2 \wedge \mathcal{B}_2, I_2)$  is a differential homomorphism  $\alpha \wedge \beta : \mathcal{A}_1 \wedge \mathcal{B}_1 \rightarrow \mathcal{A}_2 \wedge \mathcal{B}_2$  such that the differential ideal generated by the image  $\alpha \wedge \beta(I_1)$  is  $I_2$ . An algebraic triple  $(\mathcal{A} \wedge \mathcal{B}, I)$  is **smoothly realizable** if  $\mathcal{B}$  is a smoothly realizable algebra.  $\square$

Lie groups of transformations are particular cases of geometric triples when  $X = Y$  and the projection  $\rho : G \times Y \rightarrow X$  coincides with the action of  $G$  on  $X$ .

**Proposition 11.2.** [Vas] *The smooth manifold  $X$  admits a left action of the finite dimensional Lie group  $G$  iff there exists a smooth manifold  $Y$ , smoothly realizable algebra  $\mathcal{B} \hookrightarrow \Lambda(Y)$ , and differential ideal  $I \subset \Lambda_{inv}(G) \wedge \mathcal{B}$  generated by 1-forms such that the foliation in  $G \times Y$  determined by  $I$  is a (locally trivial) fibre bundle  $G \times Y \rightarrow X$  with the base  $X$ .  $\square$*

**Proposition 11.3.**  $\mathbf{locGeomTriple} \sim \mathbf{realAlgTriple}^{op}$  (local geometric triples  $\sim$  (opposite to) smoothly realizable algebraic triples).  $\square$

**Remark.** By proposition 11.2. Lie groups of transformations are in duality with a certain full subcategory of **realAlgTriple**.

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*E-mail address:* gennadii@hotmail.com