**HOMEWORK 5**

Section 4.5:

29) a) Use proof by division into cases to prove that the square of any integer has the form 3k or 3k+1 for some integer k.

b) Use the mod notation to rewrite the result of part (a).

Solution:

1. Suppose n is an arbitrary integer, we prove that n2 has the f form 3k or 3k+1 for some integer k.

Since n is an integer, n has form one of the following forms

3q or 3q + 1 or 3q + 2

Case 1: n = 3q

n2 = (3q)2  by substitution

= 9q2

= 3(3q2)

Let k = 3q2. Then, k is an integer (a product of integers). So n2 = 3k by substitution.

Case 2: n = 3q + 1

n2 = (3q + 1)2  by substitution

= 9q2 + 6q + 1

= 3(3q2 + 2q) + 1

Let k = 3q2 + 2q. Then, k is an integer (a sum of products of integers). So n2 = 3k + 1 by substitution.

Case 3: n = 3q + 2

n2 = (3q + 2)2  by substitution

= 9q2 + 12q + 4

= 3(3q2 + 4q + 1) + 1

Let k = 3q2 + 4q + 1. Then, k is an integer (a sum of products of integers). So n2 = 3k + 1 by substitution.

Through three cases, n2 always has the form 3k or 3k+1 for some integer k.

1. ∀ integer n, if n is a square, then n mod 3 = 0 or n mod 3 = 1.

Section 4.7:

4) Use proof by contradiction to show that for every integer m, 7m + 4 is not divisible by 7.

Solution:

We suppose that the negation of the statement is true. That is, suppose that there is an integer m such that 7m + 4 is divisible by 7.

By definition of divisibility, 7m + 4 = 7k, for some integer k.

Then, 4 = 7k – 7m = 7(k – m)

Let q = k – m, then q is an integer (a subtraction of integers).

So, 4 = 7q by substitution. By definition of divisibility, 4 is divisible by 7. This is a contradiction since 7 does not divide 4. [This contradiction shows that the supposition is false and, hence, that the proposition is true.]

Section 4.9:

7) Graph with four vertices of degrees 1, 1, 1, and 4.

Solution:

There is no such graph. Because the total degree of the graph is 1 + 1 + 1 + 4 = 7, which is odd. This contradicts Corollary 4.9.2 which shows that the total degree of a graph is even.

15) A small social network contains three people who are network friends with six other people in the network, one person who is network friend with five other people in the network, and five people who are network friends with four other people in the network. The rest are network friends with three other people in the network. The network contains 41 pairs of network friends.

a) How many people are network friends with three other people in the network?

b) How many people are in the network?

Solution:

1. Define a graph G by letting each vertex represent a person in the social network and letting each edge represent one network-friend relationship between two people in the network. Let x be the number of people who are network friends with three other people in the network. Then,

The total degree of G = 3.6 + 1.5 + 5.4 + x.3 = 43 + 3x (1)

Because three people are network friends with six other people in the network, one person is network friend with five other people in the network, five people are network friends with four other people in the network, and x people are network friends with three other people in the network.

Besides, since the network contains 41 pairs of network friends, graph G has 41 edges. By the handshake theorem, the total degree is twice the number of edges.

Hence,

The total degree of G = 2.41 = 82 (2)

From (1) and (2),

43 + 3x = 82

3x = 82 – 43 =39

x = 39 / 3

x = 13

Therefore, there are 13 people who are network friends with three other people in the network.

1. The number of people in the network is the sum of the number who is network friends with six other people, plus the number who is network friend with five other people, plus the number who is network friends with four other people, and plus the number who is network friends with three other people.

Hence, the number of people in the network = 3 + 1 + 5 +13 = 22.

So there are 22 people in the network.