Advanced Macroeconomics II

Handout 5 - Integration

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Short recap

Prototypical DP problem:

$$V(k,z) = \max_{\{c,k'\}} u(c) + \beta E\left[V\left(k', z'\right)|z\right]$$
$$\text{s.t.} c + k' = f(k,z)$$
$$z' = h(z,\eta); \eta \text{ stochastic}$$

- We are looking for functions $\mathrm{V}, \mathrm{g}^{\mathrm{c}}, \mathrm{g}^{\mathrm{k}}$: We cannot solve this.

We need to solve an approximate problem:

- Approximate continuous function: **Interpolation**
  - Requires "exact" solution of maximization problem: **Optimization**
  - Requires computing expectations: **Integration**

Integration - Many options

1. Monte Carlo integration
2. Quadrature methods
3. Discretize state space
   - Tauchen (1986)
   - Tauchen & Hussey (1991)
   - Rouwenhorst (2008)
   - Gaussian mixture (i.a. Civale, Diez-Catalan & Fazilet, 2017; Guvenen, McKay, Ryan, 2022)

Monte Carlo Integration

Monte Carlo integration

- Idea: Exploit the law of large numbers

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = E[f(x)] = \int f(x)\,dG(x)$$

- Approximate your integral with a sum
- Key: Where to evaluate it... we need draws from $x \sim G$
  - Actually, we need a lot of draws
  - Monte Carlo relies on large numbers to get relative frequencies right
  - If density of $x \sim G$ at a is higher, there will be more draws x_i close to a
- Monte Carlo is generally costly, requires too many function evaluations.

Monte Carlo integration - Expectations

Algorithm 1: Expectation by Monte Carlo

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input : Number of seeds (N_0) and number of candidates ($N^\star$)
output: $E\left[V\left(k', z'\right)|z\right]$ with $z' = h\left(z, \eta\right)$ and $\eta \sim G$

1. Generate N random draws for $\eta \sim G$. Call them $\{\eta_i\}_{i=1}^N$
   Note: Do this once at the beginning of the code ;

for *i=1:N* **do**
   2. Evaluate $f_i = V\left(k', h\left(z, \eta_i\right)\right)$
      Note: This step requires interpolation of V in the z direction ;

3. Return average: $E\left[V\left(k', z'\right)|z\right] \approx \frac{1}{N}\sum_i f_i$

Monte Carlo integration - When to use it?

- This is rarely the best way to go for expectations
  - Requires high N to provide good approximation
  - That is a lot of interpolations for each evaluation of the expectation
- Sometimes evaluations are not that expensive
  - Then Monte Carlo is cheap and easy
- It is a good tool for other integrals
  - Model simulation (with and without heterogeneity)
- It is the easiest method to parallelize
  - Depends on your computational resources

Quadrature Methods

Gaussian Quadrature methods

- Idea: Approximate $\int$ with $\sum$, like in Monte Carlo, but with fewer points

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} \omega_i f(x_i)$$

- Key: Choose where to evaluate $\{x_i\}$ and appropriate weights $\{\omega_i\}$

Warning: GQ only works well if f is well approximated by an N-degree polynomial

Note: For equally spaced grid points and non-smooth functions use Romberg integration (see Numerical Recipes, Sec. 4.3)

Gaussian Quadrature methods

Objective: Get a method with exact results for integrals of the type:

$$\int_a^b f(x)\,dx = \int_a^b W(x)\,h(x)\,dx = \sum_{i=1}^{N} w_i h(x_i)$$

where $h(\cdot)$ is a polynomial and $W(\cdot)$ is a weighting function

- This method will give approximate results for functions (f) that are well approximated by a polynomial (h) times a weighting function (W)

$$f(x) \approx W(x)\,h(x)$$

- We can always choose $W(x) = 1$ and then $h(x) = f(x)$
- We don't actually need to know h. We can define $h(x) = {}^{f(x_i)}/_{W(x_i)}$:

$$\int_a^b f(x)\,dx = \int_a^b W(x)\,h(x)\,dx \approx \sum_{i=1}^{N} \omega_i f(x_i) \qquad \text{where } \omega_i = \frac{w_i}{W(x_i)}$$

Gaussian Quadrature methods

Algorithm 2: Gaussian Quadrature

input : Number of points N, integrand f, weighting function W
output: Points $\{x_i\}$, weights $\{\omega_i\}$, integral $\int_a^b f(x)dx \approx \sum_i \omega_i f(x_i)$

1. Choose a weighting function W;
2. Construct the family of orthonormal polynomials wrt W up to degree N;
3. Obtain roots of the polynomial of degree N in $[a, b]$
   These roots are the points $\{x_i\}$;
4. Evaluate the auxiliary weights $w_i = \frac{\langle p_{N-1}|p_{N-1}\rangle}{p_{N-1}(x_i)p'_N(x_i)} = \frac{\int_a^b W(x)p_{N-1}(x)^2 dx}{p_{N-1}(x_i)p'_N(x_i)}$
   The weights we look for are $\omega_i = \frac{w_i}{W(x_i)}$;
5. Evaluate the integral: $\int_a^b f(x)dx \approx \sum_i \omega_i f(x_i)$

Gaussian Quadrature methods

We happen to know the solution for a bunch of weighting functions:

- Gauss-Legendre: $W(x) = 1$ for $x \in [-1, 1]$
  - Polynomial recursion: $(i+1) P_{i+1} = (2i+1) x P_i - i P_{i-1}$
- Gauss-Chebyshev: $W(x) = {}^1\!/\!\sqrt{1-x^2}$ for $x \in [-1, 1]$
  - Polynomial recursion: $T_{i+1} = 2x T_i - T_{i-1}$
- Gauss-Hermite: $W(x) = e^{-x^2}$ for $x \in [-\infty, \infty]$
  - Polynomial recursion: $H_{i+1} = 2x H_i - 2i H_{i-1}$

See Numerical Recipes for more results (including weights)

Note: Use Gauss-Hermite for integrating Gaussian shocks.
Let $\tilde{h}(x) = \frac{1}{\sqrt{\pi}} V\left(k', h\left(z, \sqrt{2}\sigma_\eta x + \mu_\eta\right)\right)$ and $W(x) = e^{-x^2} \propto \Phi(x)$
Careful with extrapolation... $x \in (-\infty, \infty)$

Gaussian Quadrature - Problems

- How good is your approximation? God knows!
  - No practical error estimate...
  - There are theoretical estimates, they can be very large
- From NR: "[$W(x)$ is] ready to give high-order accuracy to integrands of the form polynomials times $W(x)$, and ready to *deny* high order accuracy to integrands that are otherwise perfectly smooth and well behaved."
- Methods are not nested: going from N to $N+1$ changes all $\{x_i, w_i\}$
- Bad performance when function has kinks, or doesn't look polynomial

Gauss-Kronrod Quadrature

- This should be your go-to method
- Uses nested Gaussian quadrature to iteratively evaluate the integral
  - The nested part helps by reusing old function evaluations
- Provides a practical error bound from the change in the integral
- Better than Gaussian quadrature if function is not polynomial

Discretizing the State Space

General idea

- Instead of approximating the integral approximate the stochastic process
  - Discretize z (and $h(z,\eta)$) instead of η
  - Approximate process for z with a discrete Markov process
- Markov process characterized by:
  - Discrete state space: $z \in \{z_1, \ldots, z_N\}$
  - Transition matrix: $\Pi = [\pi_{ij}]$, s.t. $\Pr\left(z^{'} = z_j | z = z_i\right) = \pi_{ij}$, $\sum_j \pi_{ij} = 1$
- Compute expectation:

$$E\left[V\left(k^{'}, z^{'}\right)|z = z_i\right] = \sum_{j=1}^{N} \pi_{ij} V\left(k^{'}, z_j\right)$$

Note: No approximation. No interpolation.

Key problem: How to get transition matrix?

Two approaches:

1. Full blown estimation
   - Set a grid for z: $\{z_1, \ldots, z_N\}$
   - Estimate transition matrix to match desired moments
   - Moments of z or moments of the model
   - $N(N-1)$ numbers to estimate
2. Parametrize process for z
   - Typical assumption is AR(1): $z' = h(z, \eta) = \rho z + \eta$
   - Use a method to choose Π to match properties of AR(1)
   - Only have to choose ρ and σ_η

Tauchen (1986)

$$z' = \rho z + \eta \qquad \eta \sim N(0, \sigma_\eta)$$

- Note that $E[z] = 0$ and that shocks are symmetric
- Start from an equally spaced grid centered at 0: $\{z_1, \ldots, z_N\}$
  - Heuristic: Extend grid Ω standard deviations around mean (recall $\sigma_z = \sigma_\eta / \sqrt{1-\rho^2}$)

    $$z_1 = -\Omega\sigma_z, \ldots, z_n = z_{n-1} + \Delta_z, \ldots, z_N = \Omega\sigma_z \text{ where: } \Delta_z = \Omega\sigma_z/(N-1)$$

  - Usually $\Omega = 3$, but this depends on what you are modeling
- Fill in transition probabilities from normal distribution:

$$\pi_{ij} = \begin{cases} \Phi\left(\frac{z_j - \rho z_i + \Delta_z/2}{\sigma_\eta}\right) & \text{if } j = 1 \\ \Phi\left(\frac{z_j - \rho z_i + \Delta_z/2}{\sigma_\eta}\right) - \Phi\left(\frac{z_j - \rho z_i - \Delta_z/2}{\sigma_\eta}\right) & \text{if } j = 2, \ldots, N-1 \\ 1 - \Phi\left(\frac{z_j - \rho z_i - \Delta_z/2}{\sigma_\eta}\right) & \text{if } j = N \end{cases}$$

Tauchen (1986)

Algorithm 3: Tauchen (1986)

input : Number of points N, width of grid Ω, process parameters ρ, σ_η

output: Discrete approximation of $z^{'} = \rho z + \eta$, $\eta \sim N(0, \sigma_\eta)$: (z grid , Π)

1. Construct grid: $z = range(-\Omega\sigma_z, \Omega\sigma_z, length = N)$, $\Delta_z = {}^{2\Omega\sigma_z}/_{N-1}$;
   where $\sigma_z = {}^{\sigma_\eta}/_{sqrt1-\rho^2}$;

for *i=1:N,j=1:N* **do**

2. Fill in π_{ij} as:

$$\pi_{ij} = \begin{cases} \Phi\left(\frac{z_j - \rho z_i + \Delta z/2}{\sigma_\eta}\right) & \text{if } j = 1 \\ \Phi\left(\frac{z_j - \rho z_i + \Delta z/2}{\sigma_\eta}\right) - \Phi\left(\frac{z_j - \rho z_i - \Delta z/2}{\sigma_\eta}\right) & \text{if } j = 2, \dots, N-1 \\ 1 - \Phi\left(\frac{z_j - \rho z_i - \Delta z/2}{\sigma_\eta}\right) & \text{if } j = N \end{cases}$$

Tauchen & Hussey (1991)

Exploit Gaussian Quadrature

- Because shocks are normally distributed use Guass-Hermite
- Without persistence (say $z^{'} \sim N(\mu, \sigma)$) we get:

$$E\left[V\left(k^{'}, z^{'}\right)\right] = \int V\left(k^{'}, z^{'}\right)\phi\left(\frac{z^{'} - \mu}{\sigma}\right)dz^{'} \approx \sum_{j=1}^{N}\frac{w_j}{\sqrt{\pi}}V\left(k^{'}, \sqrt{2}\sigma x_j + \mu\right)$$

  with $\{x_i\}$ the roots of the Hermite polynomials and $\{w_i\}$ the weights
- With persistence we get: $E\left[V\left(k^{'}, z^{'}\right)|z\right] = \int V\left(k^{'}, z^{'}\right)\phi\left(\frac{z^{'} - \rho z}{\sigma_\eta}\right)dz^{'}$

Issue: Conditional mean of $z^{'}$ varies with z.

- We would have to evaluate objective for different values with each z

Tauchen & Hussey (1991)

Solution: Express integral wrt unconditional normal, apply formula

$$E\left[V\left(k^{'},z^{'}\right)|z\right]=\int\left(V\left(k^{'},z^{'}\right)\frac{\phi\left(\frac{z^{'}-\rho z}{\sigma_\eta}\right)}{\phi\left(\frac{z^{'}}{\sigma_\eta}\right)}\right)\phi\left(\frac{z^{'}}{\sigma_\eta}\right)dz^{'}$$

$$\approx\sum_{j=1}^{N}\frac{w_j}{\sqrt{\pi}}V\left(k^{'},\sqrt{2}\sigma_\eta x_j\right)\frac{\phi\left(\frac{\sqrt{2}\sigma_\eta x_j-\rho z}{\sigma_\eta}\right)}{\phi\left(\frac{\sqrt{2}\sigma_\eta x_j}{\sigma_\eta}\right)}$$

- Fixed grid points: $z_i=\sqrt{2}\sigma_\eta x_i$, where $\{x_i\}$ are Gauss-Hermite nodes
  - Define $\omega_i={}^{w_i}/_{\sqrt{\pi}}$, where $\{w_i\}$ are Gauss-Hermite weights
- Probabilities: $\pi_{ij}=\frac{\phi\left(\frac{z_j-\rho z_i}{\sigma_\eta}\right)}{\phi\left(\frac{z_j}{\sigma_\eta}\right)}\frac{\omega_i}{s_i}$, where $s_i=\sum_n\frac{\phi\left(\frac{z_n-\rho z_i}{\sigma_\eta}\right)}{\phi\left(\frac{z_n}{\sigma_\eta}\right)}\omega_i$

Tauchen & Hussey (1991)

Algorithm 4: Tauchen & Hussey (1991)

input : Number of points N, process parameters ρ, σ_η
output: Discrete approximation of $z^{'} = \rho z + \eta$, $\eta \sim N(0, \sigma_\eta)$: (z grid , Π)

1. Obtain Gauss-Hermite nodes and weights {x,w} ;

2. Define grid as $z_i = \sqrt{2}\sigma_\eta x_i$;

for *i=1:N,j=1:N* **do**

3. Fill in $\pi_{ij} = \frac{\phi\left(\frac{z_j - \rho z_i}{\sigma_\eta}\right)}{\phi\left(\frac{z_j}{\sigma_\eta}\right)} \frac{\omega_i}{s_i}$ where $\omega_i = {}^{w_i}/\sqrt{\pi}$ and $s_i = \sum_n \frac{\phi\left(\frac{z_n - \rho z_i}{\sigma_\eta}\right)}{\phi\left(\frac{z_n}{\sigma_\eta}\right)} \omega_i$;

Rouwenhorst (1995)

Objective: Construct a Markov Process that matches moments of z'

- Instead of starting from a grid and getting transition probabilities

Steps:

1. Construct a particular type of Markov Process
   - Construction is recursive
2. Derive properties of the Markov Process
   - Find stationary distribution and conditional moments
3. Match moments from AR(1) process

Rouwenhorst (1995) - Markov Process

- Pick size of the grid: $N \geq 2$
- Let $\{z_1, \dots, z_N\}$ be an equally spaced grid such that $z_1 = -\psi$ and $z_N = \psi$
  - We need to find ψ. Clearly $\Delta_z = \frac{2\psi}{N-1}$.
- Construct transition matrix recursively:

$$\Pi_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

$$\Pi_N = p\begin{bmatrix} \Pi_{N-1} & \vec{0} \\ \vec{0}^T & 0 \end{bmatrix} + (1-p)\begin{bmatrix} \vec{0} & \Pi_{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + (1-q)\begin{bmatrix} \vec{0}^T & 0 \\ \Pi_{N-1} & \vec{0} \end{bmatrix} + q\begin{bmatrix} 0 & \vec{0}^T \\ \vec{0} & \Pi_{N-1} \end{bmatrix}$$

  where $\vec{0}$ is an $(N-1) \times 1$ zero vector. We need to find p and q.
- Divide all rows by 2 to ensure they sum to 1 (except top and bottom)

Rouwenhorst (1995) - Moments

Results from Kopecky & Suen (2010)

Conditional Mean	$E\left[z^{'} \vert z = z_i\right]$	$(q-p)\,\psi + (p+q-1)\,z_i$
Conditional Var	$V\left[z^{'} \vert z = z_i\right]$	$\frac{4\psi^2}{(N-1)^2}\left[(N-i)(1-p)\,p + (i-1)\,q\,(1-q)\right]$
Unconditional Mean	$E[z]$	$\frac{q-p}{2-(p+q)}\psi$
Unconditional Var	$V[z] = E[z^2]$	$\psi^2\left[1 - 4s(1-s) + \frac{4s(1-s)}{N-1}\right]$; where $s = \frac{1-q}{2-(p+q)}$
Autocovariance	$Cov\left[z^{'}, z\right]$	$(p+q-1)\,V[z]$
Autocorrelation	$Corr\left[z^{'}, z\right]$	$p+q-1$

Moreover, the stationary distribution is Binomial $(N-1, 1-s)$.

Rouwenhorst (1995) - Matching the AR(1)

$$z' = \rho z + \eta \qquad \text{where } \eta \sim N\left(0, \sigma_\eta\right)$$

- Conditional moments are immediate:

$$E\left[z'|z = z_i\right] = \rho z_i \qquad V\left[z'|z = z_i\right] = \sigma_\eta^2$$

  Also the autocorrelation: $Corr\left[z', z\right] = \rho$
- Unconditional moments: $E\left[z\right] = 0 \qquad V\left[z\right] = \frac{\sigma_\eta^2}{1-\rho^2}$
- Matching moments gives:

$$p = q \qquad p + q - 1 = \rho \longrightarrow p = q = \frac{1+\rho}{2}$$

$$\sigma_\eta^2 = \frac{4\psi^2}{\left(N-1\right)^2}\left[\left(N-i\right)\left(1-p\right)p + \left(i-1\right)q\left(1-q\right)\right] \longrightarrow \psi = \sqrt{N-1}\frac{\sigma_\eta}{\sqrt{1-\rho^2}}$$

Rouwenhorst (1995)

Algorithm 5: Rouwenhorst (1995): Discretize AR(1)

Function `Rouwenhorst(`N,ρ, σ_η`)`:

1. Define $p = {}^{1+\rho}/_{2}$ and $\psi = \sigma_\eta \sqrt{{}^{N-1}/_{1-\rho^2}}$
2. Construct grid: $z = range(-\psi, \psi, length = N)$, $\Delta_z = {}^{2\psi}/_{N-1}$

if $N{=}{=}2$ **then**

3.1. Define $\Pi_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$

else

3.2.1. $\Pi_{N-1} = \text{Rouwenhorst}(N-1, \rho, \sigma_\eta)$

3.2.2. $\Pi_N = p\begin{bmatrix} \Pi_{N-1} & \vec{0} \\ \vec{0}^T & 0 \end{bmatrix} + (1-p)\begin{bmatrix} \vec{0} & \Pi_{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + (1-q)\begin{bmatrix} \vec{0}^T & 0 \\ \Pi_{N-1} & \vec{0} \end{bmatrix} + q\begin{bmatrix} 0 & \vec{0}^T \\ \vec{0} & \Pi_{N-1} \end{bmatrix}$

3.2.3. Ajust intermediate rows to sum to 1

4. Bonus: Return stationary distribution $G = \text{Binomial}(N-1, {}^{1}/_{2})$

Gaussian Mixtures

Issue: Our AR(1) processes do not match higher order moments of data

- Papers by Guvenen and Song show earnings data is far from "Normal"
- Income process presents left skewness and high kurtosis
- Bloom, Guvenen & Salgado show importance of skewness for firms

Solution: Use Gaussian Mixture Models

- Shocks come from a mixture of gaussian sources
- More sources of variation allow us to capture higher order moments

Matching higher moments

Results from Civale, Diez-Catalan & Fazilet (2015)

$$z^{'} = \rho z + \eta \qquad \text{where } \eta \sim \begin{cases} N(\mu_1, \sigma_1) & \text{with prob. } p \\ N(\mu_2, \sigma_2) & \text{with prob. } 1-p \end{cases}$$

- This process is flexible enough to generate skewness and kurtosis in η
  - These properties are inherited by z
- The process imposes constraints
  - Parameter ρ is key
  - Given ρ and moments of z we get moments of $\Delta_k z$
  - Moments of $\Delta_k z$ are often the target in the data

How to use Gaussian mixtures

Good news: Tauchen's method applies!

- Let F_η be the CDF of the Gaussian mixture η
- Let $\{z_1, \ldots, z_N\}$ be a grid for z
- All formulas apply using the new distribution

Bad news: Results are sensitive to choice of state space grid

- Civale, Diez-Catalan & Fazilet (2015) propose optimizing over grid
  - They use the method of moments
- Process is incompatible with persistence of Skewness and Kurtosis
  - See Appendix B.2. where they propose a method to address this

Gaussian mixtures - Very popular

Gaussian mixtures used widely, mostly for income fluctuation problems

- Housing Wealth Effects: The Long View (Guren, McKay, Nakamura, Steinsson ,2020)
- Time-Varying Idiosyncratic Risk and Aggregate Consumption Dynamics (McKay, 2017)
- Countercyclical Labor Income Risk and Portfolio Choices over the Life-Cycle (Catherine, 2020)
- Nonlinear household earnings dynamics, self-insurance, and welfare (DeNardi, Fella, Paz-Pardo, 2020)
- Monetary policy according to HANK (Kaplan, Moll, Violante, 2018)
  - Continuous time methods mixing a jump process for kurtosis

Final Words on Methods

Literature's take: Use Rouwenhorst

- Three separate papers get to the same conclusion
  - Kopecky & Suen (2010)
  - Galindev & Lkhagvasuren (2010)
  - Fella, Gallipoli & Pan (2019)
- Rouwenhorst's method outperforms all competition

"A 5 point grid Rouwenhurst approximation is generally as accurate as a 25 grid point approximation with other methods"

  - Other methods suffer when $\rho \to 1$. Rouwenhorst suffers much less.
  - Rouwenhorst's design to match moments makes it better
  - Rouwenhorst does not target higher order moments, but still outperforms other methods at low grid sizes.

An example

- Discretize $z' = \rho z + \eta$, where $\eta \sim N\left(\mu_\eta, \sigma_\eta^2\right)$
- Choose $\rho = 0.95$, $\mu_\eta = 0$ and $\sigma_\eta = 0.2$
- Simulate 10.000 periods of the Markov Chain

	Exact Moments		Tauchen N=5	Tauchen N=11	Tauchen N=21	Rouwenhorst N=5	Rouwenhorst N=11	Rouwenhorst N=21
$E[z]$	$\frac{\mu_\eta}{1-\rho}$	0	0.05	-0.03	0.03	-0.03	0.00	0.01
$\sqrt{V[z]}$	$\frac{\sigma_\eta}{\sqrt{1-\rho^2}}$	0.64	0.87	0.73	0.67	0.65	0.63	0.63
$corr\left(z, z'\right)$	ρ	0.95	0.99	0.95	0.95	0.95	0.95	0.95

Relevant Extensions

- Correlated AR(1) process (like a VAR)
  - Galindev & Lkhagvasuren (2010)
  - Method reduces to decomposing covariance matrix to get independent shocks
  - Then apply a modified version of Rouwenhorst
- Non-stationary process - Designed for life cycle models
  - Fella, Gallipoli & Pan (2019)
  - Methods are extensions of Tauchen or Rouwenhorst
- Many other methods...Read the papers!

Application: GE capitalist/union economy

The economy: Two agents

Capitalists:

- Infinitively lived derive utility from consumption: $u(c) = {}^{c^{1-\gamma}}/_{1-\gamma}$
- Produce output with capital and labor (CRS technology): $y = zk^{\alpha}\ell^{1-\alpha}$
- Use their own capital (no borrowing), hire labor in market at wage w

Union:

- Infinitively lived derive utility from consumption: $u(c) = {}^{c^{1-\gamma}}/_{1-\gamma}$
- Weird union preferences: Demand constant wage
- Union internalizes effect on wages and controls labor to adjust price
- Hand-to-mouth (no borrowing or savings)

Capitalists

$$
\begin{aligned}
V(z,k;w) = \max_{\{c,k'\}}\ & u(c) + \beta E\left[V\left(z^{'},k^{'};w^{'}\right)|z\right] \\
\text{s.t. }\ & c + k^{'} \leq \pi(z,k;w) \\
& \pi(z,k;w) = \max_{\ell} zk^{\alpha}\ell^{1-\alpha} - w\ell + (1-\delta)k \\
& \log z^{'} = \rho \log z + \eta; \qquad \eta \sim N\left(0,\sigma_{\eta}^{2}\right) \\
& \text{Law of motion for wages (more on this later)}
\end{aligned}
$$

- Note that capitalists have to take w as given
  - We will talk how to deal with this explicitly later
  - For now: leap of faith

Capitalists - Profits

$$\pi\left(z,k:w\right)=\max_{\ell}\, zk^{\alpha}\ell^{1-\alpha}-w\ell+\left(1-\delta\right)k$$

Optimal labor choice:

$$\ell^{\star}=\left(\frac{1-\alpha}{w}z\right)^{\frac{1}{\alpha}}k$$

Optimal profits:

$$\pi\left(z,k;w\right)=\left[\underbrace{\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}z^{\frac{1}{\alpha}}+\left(1-\delta\right)}_{\Gamma(z;w)}\right]k$$

Capitalists - Homothetic-Homogeneous DP

We are in luck!

- Capitalist problem is that of maximizing a homothetic objective subject to homogenous constraint.
- Particularly constraint is homogenous of degree 1

$$V\left(z,k;w\right)=\max_{\left\{c,k'\right\}}\ u\left(c\right)+\beta E\left[V\left(z^{'},k^{'};w^{'}\right)|z\right]$$
$$\text{s.t. } c+k^{'}\leq\Gamma\left(z;w\right)k$$

- We can guess that the problem is separable in z and k

$$V\left(z,k;w\right)=\upsilon\left(z;w\right)u\left(k\right)$$

Capitalists - Guess and verify

$$V\left(z,k;w\right)=\max_{k'}\frac{\left(\Gamma\left(z;w\right)k-k^{'}\right)^{1-\gamma}}{1-\gamma}+\underbrace{\beta E\left[\upsilon\left(z^{'};w^{'}\right)|z\right]}_{\Upsilon(z)}\frac{\left(k^{'}\right)^{1-\gamma}}{1-\gamma}$$

First order condition:

$$\left(\Gamma\left(z;w\right)k-k^{'}\right)^{-\gamma}=\Upsilon\left(z\right)\left(k^{'}\right)^{-\gamma}$$

$$\Gamma\left(z;w\right)k=\left(1+\left(\Upsilon\left(z\right)\right)^{\frac{-1}{\gamma}}\right)k^{'}$$

Policy function: Save a fraction of income

$$k^{'}=\underbrace{\frac{\Upsilon\left(z\right)^{\frac{1}{\gamma}}}{1+\Upsilon\left(z\right)^{\frac{1}{\gamma}}}}_{s(z;w)}\underbrace{\Gamma\left(z;w\right)k}_{\pi(z,k;w)}$$

Capitalists - Guess and verify

$$
\begin{aligned}
\upsilon(z;w)\frac{k^{1-\gamma}}{1-\gamma} &= \left((1-s(z;w))\Gamma(z;w)\right)^{1-\gamma}\frac{k^{1-\gamma}}{1-\gamma} + \Upsilon(z)\left(s(z;w)\Gamma(z;w)\right)^{1-\gamma}\frac{k^{1-\gamma}}{1-\gamma} \\
\upsilon(z;w) &= \left((1-s(z;w))\Gamma(z;w)\right)^{1-\gamma} + \Upsilon(z)\left(s(z;w)\Gamma(z;w)\right)^{1-\gamma} \\
\upsilon(z;w) &= \left[(1-s(z;w))^{1-\gamma} + \Upsilon(z)\, s(z;w)^{1-\gamma}\right]\Gamma(z;w)^{1-\gamma} \\
\upsilon(z;w) &= \left[1+\Upsilon(z)^{\frac{1}{\gamma}}\right]\left(\frac{\Gamma(z;w)}{1+\Upsilon(z)^{\frac{1}{\gamma}}}\right)^{1-\gamma} \\
\upsilon(z;w) &= \left[1+\Upsilon(z)^{\frac{1}{\gamma}}\right]^{\gamma}\left(\Gamma(z;w)\right)^{1-\gamma} \\
\upsilon(z;w) &= \left[1+\left(\beta E\left[\upsilon\left(z';w'\right)|z\right]\right)^{\frac{1}{\gamma}}\right]^{\gamma}\left(\Gamma(z;w)\right)^{1-\gamma}
\end{aligned}
$$

Workers' Union

$$W(z,K) = u\left(\overline{w}\ell^s(z,K)\right) + \beta E\left[W\left(z^{'}, K^{'}(z,K)\right)|z\right]$$

Not much to do here... sorry

- Union dictates labor supply of workers according to some rule ℓ^s
  - Rule is set to maintain a constant wage of $\overline{w}$
- Agents need to know the equilibrium law of motion for aggregate capital K
  - We know we will get it from the capitalists

General equilibrium

Market clearing:

- Union sets $\ell^s(z,k)$ such that:

$$w^\star = (1-\alpha)\, z \left(\frac{k}{\ell^s(z,k)} \right)^\alpha = \overline{w}$$

- Labor depends on z and k, but not wages
- Capitalists do not take into account their effect on aggregate prices
- However, **we know** how capital and productivity affect prices

Dynamic Programming: Final

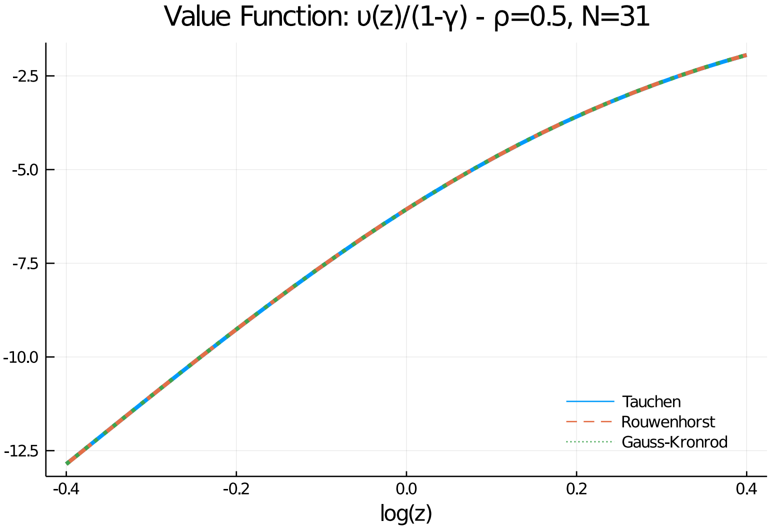
$$v(z) = \left[1 + \left(\beta E\left[v\left(z'\right)|z\right]\right)^{\frac{1}{\gamma}}\right]^{\gamma} (\Gamma(z))^{1-\gamma}$$

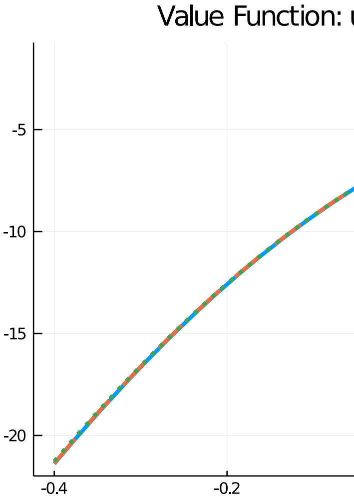
where

$$\Gamma(z) = \alpha \left(\frac{1-\alpha}{\overline{w}}\right)^{\frac{1-\alpha}{\alpha}} z^{\frac{1}{\alpha}} + (1-\delta)$$

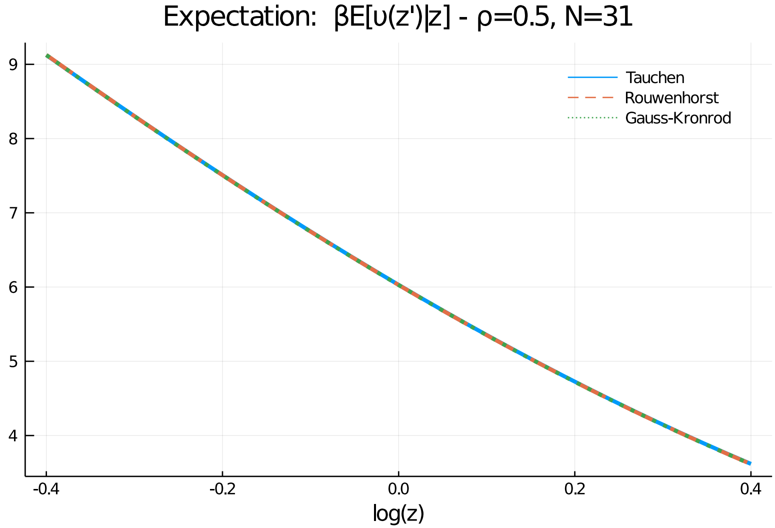
- Looks the same... but we dropped w as it is constant in equilibrium
- To solve the dynamic programming problem we need to integrate
  - No max involved, only integrals

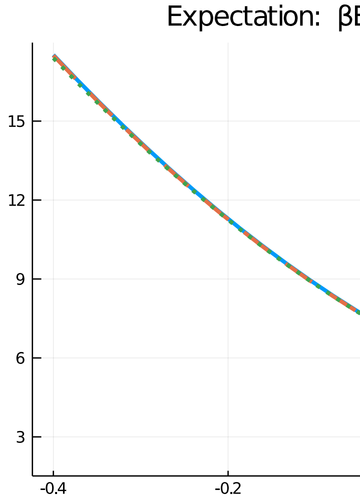
Dynamic Programming: Result





Differences in expectations





What to make of these graphs?

- We do not have a closed form solution... so hard to claim success
- Tauchen and Rouwenhorst give similar resuls:
  - But we were using 31 nodes... kind of cheating
- Rouwenhorst vs Gauss-Kronrod
  - Hard to tell because GK uses a lot of extrapolation
- Rouwenhorst seems like the best option
  - Computationally feasible
  - Reliable for high persistence values