Applications of Markov chain approximation methods to optimal control problems in economics*

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November 20, 2020

PRELIMINARY. COMMENTS WELCOME.

Abstract

In this paper we explore some of the benefits of using the finite-state Markov chain approximation (MCA) method of Kushner et al. (2001) to solve continuous-time optimal control problems. We first show that for certain choices of the approximating chain and solution method for the resulting system of equations, the MCA method reduces to the implicit finite-difference scheme of Achdou et al. (2017). We then illustrate the benefits of departing from each of these choices by means of two examples taken from the economics literature. In the first, we employ variations of modified policy function iteration to solve income fluctuation problems in two and three dimensions. In the second, we apply the MCA method to portfolio problems with highly correlated state variables, a setting that commonly occurs within general equilibrium models with financial frictions and for which it is difficult to construct monotone (and convergent) finite-difference schemes. In both cases the MCA method results in an increase in speed of more than an order of magnitude.

1 Introduction

Dynamic optimization problems are ubiquitous in economics, and since closed-form expressions for such problems are available only in isolated, special cases, quantitative work requires the use of

*We appreciate comments from Ben Craig, Jesus Fernandez-Villaverde, Pablo Guerron, Andres Victor Luduvice, Sergio Ocampo and seminar participants at The University of New South Wales and Fall 2019 Midwest Macro meetings. All remaining errors are our own.

numerical methods for their solution. In this paper we solve a number of dynamic optimization problems that arise naturally in economic applications by employing the Markov-chain approximation (MCA) method of Harold Kushner and Paul Dupuis. Although expositions and applications of this method already exist in the economics literature,[1] the method has several advantages over alternative approaches to continuous-time optimization problems that remain unexploited. To the best of our knowledge this paper is the first to outline such advantages by means of examples taken from the economics literature.

The most common approach to solving continuous-time optimization problems is the method of finite-differences, which has recently been applied to a number of economic environments by Achdou et al. (2017). In this method, one first establishes that value function is a (viscosity) solution of a partial differential equation known as the Hamilton-Jacobi-Bellman equation, before replacing derivatives with quotients and solving the ensuing finite system of equations. In contrast, the MCA method approximates the solution to the continuous-time control problem by replacing it with a simpler problem in which the state evolves according to a Markov chain assuming finitely many values. The method formalizes the intuitive idea that if the discrete-time process is "close" to the original process then the value function of the discrete-state problem should also be close to the original value function.

The criteria necessary for the convergence of the value function of the latter problem to that of the original problem are referred to as *local consistency* conditions. These amount to the requirement that the increments of the chain possess the first and second-order conditional moments of the original process, at least up to a term that is second-order in the time increment. One benefit to proceeding in this manner is that arguments from discrete-time dynamic programming already familiar to economists — such as the contraction mapping theorem and Blackwell's conditions — are applicable to this new problem and ensure the convergence of the numerical algorithms. Further, even in the presence of non-convexities and in multiple dimensions, the Markov chain may often be chosen so as to eliminate the need for costly root-finding, without sacrificing the global convergence of the algorithm.

We first establish a connection between the above approaches by showing that a limiting case of

[1] See e.g. Barczyk and Kredler (2014), and Golosov and Lucas Jr (2007).

one widely used finite-difference scheme is equivalent to a particular case of MCA method. Formally, we show that a limiting version of the implicit finite-difference scheme of Achdou et al. (2017) is equivalent to using the MCA method for a certain chain with negligible timestep and solving the resulting Bellman equation using policy function iteration. Establishing this connection is useful since it means that the former algorithm amounts to making two choices, a choice of chain and a choice of solution method, neither of which may be optimal for a given problem. The two classes of examples given in this paper illustrate the benefits of departing from each of these choices.

The first example shows the benefits of departing from policy function iteration. It is well known that policy function iteration converges at a quadratic rate near the solution, and so typically requires a small number of iterations for convergence. However, updating the value function using this algorithm requires solving a linear system of equations, which becomes very costly computationally as either the number of gridpoints increases or the sparsity structure of the matrix becomes more complex. One therefore expects that the implicit finite-difference method slows down rapidly as one increases the number of gridpoints or the dimension of the state variable. Section 3 confirms this point by considering variations of a problem common in economics, in which an infinitely-lived risk-averse agent makes a consumption-savings choice in the presence of idiosyncratic risk and/or discrete-choices of over a durable good. We show that for standard parameters and moderate grid sizes, versions of the *modified* policy function iteration of Puterman and Shin (1978) can lead to an increase in the speed of convergence of more than an order of magnitude relative to policy function iteration, with no loss of accuracy.

The second example we consider illustrates the flexibility granted to the practitioner when constructing chains by applying the MCA method to a problem that arises in many general equilibrium models with financial frictions, such as those outlined in Brunnermeier and Sannikov (2016). In these environments is often the case that the agents face a portfolio problem with highly correlated state variables. This property poses difficulties for the finite-differences approach, as it becomes challenging to choose schemes that retain the monotonicity requirements necessary for convergence. Although this high correlation complicates the construction of approximating Markov chains, the flexibility of the local consistency requirements allows for an intuitive and systematic approach to overcoming this difficulty. First, one may allow the timestep to depend upon the location in the grid, and second, one may allow the chain to place probability on points not immediately adja-

cent to the given one (*non-local* points). This search is reminiscent of the construction of monotone schemes by Bonnans et al. (2004) and D'Avernas and Vandeweyer (2019). However, it appears simpler and more general, and Section 4.3 shows it may be performed in a systematic fashion requiring only pointwise maximization over a small set of points. Importantly, once we have constructed this chain the solution to the individual portfolio problem may be found rapidly using policy function iteration, even for arbitrarily small timesteps.

Associated with each guess of prices, law of motion for the aggregate state, and continuation values of the agent, we have optimal policies for consumption and investment. Given these policy functions, we then impose market-clearing requirements and consistency between individual and aggregate laws. In this manner we are essentially finding a kind of "static equilibrium" for a population of agents facing given (potentially non-equilibrium) continuation values. Once we solve the individual problems for these new aggregate quantities, we have a map from the space of continuation values to itself, which we iterate upon until convergence. As with the rest of the literature, we are unable to establish convergence of our algorithm to the competitive equilibrium. However, we establish convergence of one component (the individual problem) and in practice the search for equilibrium quantities converges and does so rapidly. As an example, we compute the equilibrium of a model similar to that given in D'Avernas and Vandweyer (2019) in less than one tenth of the time.

For clarity and to aid the reader in the implementation of details, Section A tests the speed and accuracy of all methods considered in the paper by applying them to linear-quadratic problems, for which policy and value functions are attainable in closed-form.

2 Motivating example

In this section we outline the MCA method in the context of the stochastic one-sector neoclassical growth model. Although this example may be easily solved via a number of different numerical methods, it serves to give an intuitive account of how the method works and to contrast it with the finite-difference method. As we noted in the introduction, the basic idea here is to approximate the solution by solving a simpler problem in which the state evolves according to a chain that assumes only finitely-many values. The value function associated with this simpler problem will be a good

approximation to the original value function if for any given control vector, the increments of the chain share the same first and second conditional moments as the original process.

2.1 Setup

Suppose that a social planner wishes to maximize the expected lifetime utility of an infinitely-lived representative agent with preferences over consumption

$$U(c) = \mathbb{E}\bigg[\rho \int_0^\infty e^{-\rho t} u(c_t) dt\bigg].$$

We assume that capital and consumption goods may be costlessly transformed into one another, and so the sole state variable is the capital stock, which evolves according to the law of motion

$$dk_t = [f(k_t) - \delta k_t - c_t]dt + \sigma(k_t)dZ_t, \tag{1}$$

for some smooth function f satisfying the Inada conditions, constant $\delta > 0$, Brownian motion $(Z_t)_{t\geq 0}$ and smooth function σ. For simplicity suppose σ vanishes outside of some interval $[\underline{k}, \overline{k}]$ and that at these boundary points we impose $c(\underline{k}) \leq f(\underline{k}) - \delta\underline{k}$ and $c(\overline{k}) \geq f(\overline{k}) - \delta\overline{k}$, respectively, where $f(\underline{k}) > \delta\underline{k}$. Given any $k_0 \in [\underline{k}, \overline{k}]$ a natural way to solve this problem is to replace the law of motion (1) with the discrete counterpart

$$k_{t+\Delta_t} = k_t + (f(k_t) - \delta k_t - c_t)\Delta_t + \sqrt{\Delta_t}\sigma(k_t)X_t \tag{2}$$

for some $\Delta_t > 0$, where $(X_t)_{t=0}^\infty$ is an i.i.d. sequence of random variables with mean zero assuming the values ± 1. Using standard dynamic programming arguments,[2] one may show the value function for the discretized problem is the unique fixed point of the functional equation $BV = V$, where

$$BV(k) = \max_{c\geq 0} \Delta_t u(c) + e^{-\rho\Delta_t}\mathbb{E}\Big[V\Big(k + (f(k) - \delta k - c)\Delta_t + \sqrt{\Delta_t}\sigma(k)X\Big)\Big] \tag{3}$$

subject to $c(\underline{k}) \leq f(\underline{k}) - \delta\underline{k}$ and $c(\overline{k}) \geq f(\overline{k}) - \delta\overline{k}$. One may show T is a contraction on the space of continuous functions on $[\underline{k}, \overline{k}]$, so the fixed-point may be found by applying it repeatedly to an arbitrary initial guess.

The finite-state Markov chain method of Kushner et al. (2001) approximates the original problem in a fundamentally different way. Instead of (2), we consider an optimal control problem in which the capital stock assumes values in a finite grid $S := \{\underline{k}, \underline{k} + \Delta_k, \ldots, \overline{k} - \Delta_k, \overline{k}\}$, where

[2] As outlined in e.g. Stokey and Lucas.

$\Delta_k = (\overline{k} - \underline{k})/N$ for some $N > 1$. We construct the chain such that that the increments possess the same conditional mean and variance as (2). Specifically, if $c_t = c$, then the transition probabilities of the Markov chain given $k_t = k$ are supported on $\{k - \Delta_k, k, k + \Delta_k\}$ and given by

$$\begin{aligned} p(k, k + \Delta_k, c) &= \frac{\Delta_t}{\Delta_k^2}\left(\frac{\sigma^2(k)}{2} + \Delta_k[f(k) - c - \delta k]^+\right) \\ p(k, k - \Delta_k, c) &= \frac{\Delta_t}{\Delta_k^2}\left(\frac{\sigma^2(k)}{2} + \Delta_k[f(k) - c - \delta k]^-\right) \\ p(k, k, c) &= 1 - p(k, k - \Delta_k, c) - p(k, k + \Delta_k, c) \end{aligned} \tag{4}$$

where $x^{\pm} := \max\{\pm x, 0\}$ for $x \in \mathbb{R}$. Associated with (4) we have the Bellman operator

$$\tilde{B}V(k) = \max_{c \geq 0} \Delta_t u(c) + e^{-\rho \Delta_t} \mathbb{E}\big[V(k')\big] \tag{5}$$

where again we impose the requirements $c(\underline{k}) \leq f(\underline{k}) - \delta \underline{k}$ and $c(\overline{k}) \geq f(\overline{k}) - \delta \overline{k}$. Using the stochastic formulation of dynamic programming given in Stokey and Lucas, it is easy to show that the operator in (5) is a contraction and so the (unique) fixed point in the space of continuous functions on $[\underline{k}, \overline{k}]$ may be found simply by iterating successively on an arbitrary guess. Now note that the conditional mean of $\Delta k_t \equiv k_{t+\Delta_t} - k_t$ is

$$\mathbb{E}[k_{t+\Delta_t} - k_t | k_t = k] = -p(k - \Delta_k, c)\Delta_k + p(k + \Delta_k, c)\Delta_k = \Delta_t(f(k) - c - \delta k)$$

where for brevity we omit the dependence of probabilities on the original point. Further, the conditional variance $\mathbb{E}[(k_{t+\Delta_t} - k_t)^2 | k_t = k] - \mathbb{E}[(k_{t+\Delta_t} - k_t) | k_t = k]^2$ of Δk is

$$\Delta_t\Big(\sigma^2 k^2 + \Delta_k |f(k) - c - \delta k| - \Delta_t (f(k) - c - \delta k)^2\Big).$$

As $\Delta_t, \Delta_k \to 0$, the law of motion for this problem converges to the law of motion (2). One may then use weak convergence arguments to show formally that fixed points of the operators defined in (3) and (5) also converge to one another (and the value function of the original problem).

Why is this construction useful, given that both (3) and (5) lead to a discrete Bellman equation? The main point here is that in the second discretization, when the agent in the contemplates how varying the control (consumption) alters tomorrows payoffs, she need only compare *local* continuation payoffs, so the shape or regularity of the value function is irrelevant. Using (4), the functional equation may be written

$$\frac{(1 - e^{-\rho \Delta_t})}{\Delta_t} V(k) = \max_{c \geq 0} \rho u(c) + e^{-\rho \Delta_t}\big([f(k) - c - \delta k]^+ V^F - [f(k) - c - \delta k]^- V^B\big) \tag{6}$$

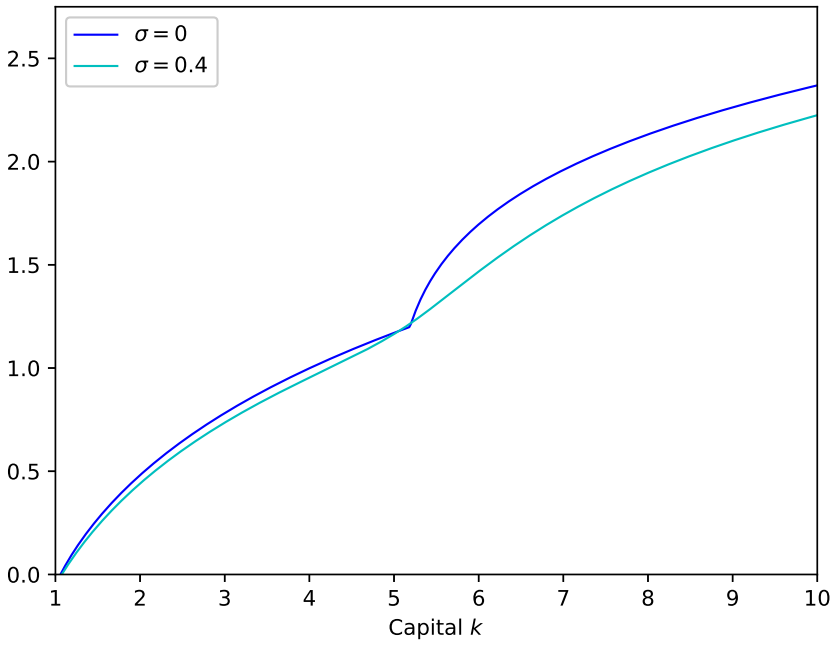

Figure 1: Value functions in one-sector growth model with non-concave technology

where V^F and V^B are approximate forward and backward derivatives. The crucial difference between (3) and (6) is that in the latter optimal consumption is available in closed-form, regardless of the shape of either the value function or production function and the non-concavity in the problem. To illustrate, suppose that utility is logarithmic and the production function is $f(k) = \max\left\{\sqrt{k}, 5\sqrt{k-5}\right\}$. Figure 1 shows the computed value functions for the parameters ($\rho = 1, \delta = 0.05, \underline{k} = 1, \overline{k} = 20, N = 1000$), for both a deterministic ($\sigma = 0$) and stochastic ($\sigma = 0.4$ in interior, vanishing at boundaries) case, and Figure 2 shows the corresponding policy functions. We used policy function iteration and imposed a tolerance of maximum error between successive iterations of 10^{-8}. In both case convergence occurs in less than 0.03 seconds, beginning with an initial guess of zero net saving.

We now compare the above approach with a class of finite-differences method that have been applied to a number of economic problems of interest by Achdou et al. (2017). We first (heuristically) derive the appropriate partial differential equation in order to motivate the algorithm.[3]

[3]For details we refer to Achdou et al. (2017) and further references therein. Our goal is not to recapitulate the theory of finite-difference methods but to instead relate to Markov-chain approximation methods.

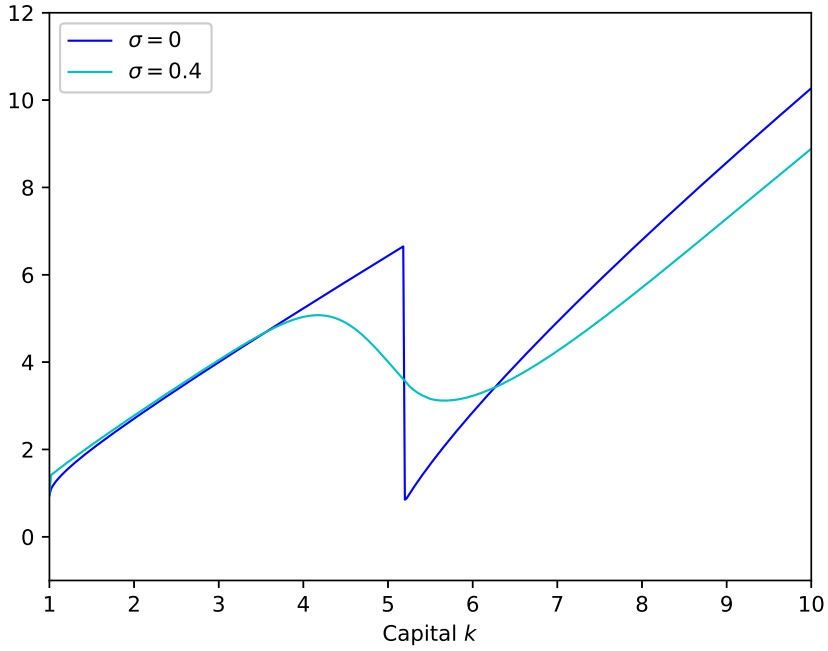

Figure 2: Policy functions in one-sector growth model with non-concave technology

Using the discrete-time version of the Principle of Optimality, for any $t, h > 0$ we have

$$V(k,t) = \max_c \left[\int_t^{t+h} u(c(s))ds + V(k(t+h), t+h) \right].$$

Subtracting $V(k,t)$ from both sides, dividing by h and using Ito's lemma gives

$$0 = \max_c u(c) + \frac{\partial V}{\partial t} + [f(k) - c - \delta k]\frac{\partial V}{\partial k} + \frac{\sigma(k)^2}{2}\frac{\partial^2 V}{\partial k^2}.$$

A common approach to solving the above partial differential equation is to approximate the partial derivatives with various choices of difference quotients and solve the resulting finite system of equations. Achdou et al. (2017) proceed in this manner and consider two formulations of the method, which they term *explicit* and *implicit*. In this section we focus only on the latter. To understand the algorithm, fix a rectangular grid S for $[\underline{k}, \overline{k}] \times [0, T]$ for some $T > 0$ and imagine we are given a terminal value $V(k,T)$. For each $(k_j, t_n) \in S$ write $V^n(k_j) = V(k_j, t_n)$ and define V^{n+1} as the solution to

$$\begin{aligned}\left(\frac{1}{\Delta_t} + \rho\right)V^{n+1} = \frac{V^n}{\Delta_t} + u(c^n) + [f(k_j) - c_j^n - \delta k_j]^+(V^{n+1})^F \\ - [f(k) - c - \delta k]^-(V^{n+1})^B + \frac{\sigma(k)^2}{2}(V^{n+1})^C\end{aligned} \tag{7}$$

where c^n solves $\max_{c\geq 0} \Delta_t u(c^n) + [f(k) - c - \delta k]^+ (V^n)^F - [f(k) - c - \delta k]^- (V^n)^B$ and F, B and C superscripts denote forward, backward and central differences, respectively. In practice, for small grids convergence is rapid and insensitive to changes in the timestep when the latter is large. To understand why, note we may write the fixed point of (7) as $\max_{c\in\Gamma(k)} u(c) + T_{\text{IFD}}(c)V$, where

$$T_{\text{IFD}}(c) = [f(k) - c - \delta k]^+ V^F(k) - [f(k) - c - \delta k]^- V^B(k) + \frac{\sigma(k)^2}{2} V^C(k) - \rho V(k). \tag{8}$$

If we set $\Delta_t = \infty$ then the implicit method may be written as follows: fix V_0; find c_0 solving $\max_{c\in\Gamma(k)} u(c) + T_{\text{IFD}}(c)V_0$; find V_1 solving $0 = u(c_0) + T(c_0)V_1$; replace V_0 with V_1 and repeat until convergence. The method fits within the framework of Puterman and Brumelle (1979), who establish the convergence of algorithms of this form when $T(c)^{-1} \leq 0$. Finally, if we denote the operator associated with the Markov chain defined by (4) by $T(c;\Delta_t) = [e^{-\rho\Delta_t} P(c) - I]/\Delta_t$, then the following allows us to understand both the convergence properties of the implicit method and its relationship with the Markov chain approximation method.

Lemma 2.1. *For any policy c we have* $\lim_{\Delta_t\to 0} T(c;\Delta_t) = T_{\text{IFD}}(c)$.

Proof. Simply compare the right-hand side of (8) with

$$T(c;\Delta_t)V = -\frac{(1 - e^{-\rho\Delta_t})}{\Delta_t} V + e^{-\rho\Delta_t}\left([f(k) - c - \delta k]^+ V^F - [f(k) - c - \delta k]^- V^B + \frac{\sigma(k)^2}{2} V^C\right)$$

from which the result follows by taking limits. □

Lemma 2.1 shows that the implicit finite-difference method of Achdou et al. (2017) amounts to solving the original problem by considering a *particular Markov chain* and a *particular solution method* (policy function iteration) for the resulting system of equations. For a general optimal control problem, neither of these choices may be optimal for either speed or accuracy, and the remainder of this document is devoted to illustrating the benefits of departing from implicit finite-difference schemes for a number of economic problems. We first outline the method formally before turning to these examples.

2.2 Formal overview

This section provides a general overview of the theory developed in Kushner et al. (2001). We focus only on fixing consistent notation and stating the relevant results and definitions necessary to understand subsequent examples and refer the reader to the text for details. We are interested in continuous-time control problems of the following form.

Definition 2.1. Let $B := (B_t)_{t\geq 0}$ be a standard n-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, and denote by $(\mathcal{F}_t)_{t\geq 0}$ the associated natural filtration. For a fixed compact set $U \in \mathbb{R}^m$ define $\mathcal{C}$ to be the set of *admissible controls*, the set of stochastic processes $(u_t)_{t\geq 0}$ adapted to $(\mathcal{F}_t)_{t\geq 0}$ such that $u_t \in U$ for all $t \geq 0$. For some functions $F : \mathbb{R}^n \times U \to \mathbb{R}$, $\mu : \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\mu : \mathbb{R}^n \times U \to \mathbb{R}^{n\times n}$ we consider the optimal control problem

$$\begin{aligned} V(x) &= \max_{u\in\mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} F(x_t, u_t)dt\right] \\ dx_t &= \mu(x_t, u_t)dt + \sigma(x_t, u_t)dB_t \\ x_0 &= x. \end{aligned}$$

We refer to x and u as the *state* and *control* variables, F as the *payoff* function, and μ and σ as the *drift* and *diffusion* functions.

The law of motion for the state is a shorthand for

$$x_t = x_0 + \int_0^t \mu(x_s, u_s)ds + \int_0^t \sigma(x_s, u_s)dB_s \tag{9}$$

where the final term is an Ito integral. For the examples of this paper we make the following assumption.

Assumption 2.1. The functions F, μ and σ are continuous, bounded and Lipschitz.

Assumption 2.1 can be weakened without affecting the validity of the approach. However, it covers many examples of interest to us and ensures that a weak solution to (9) exists and is unique for any admissible control, so that the value function in Definition 2.1 is well-defined. To reduce a problem of the form Definition 2.1 to a finite-state problem we must specify how to approximate the underlying *state* and *objective* function. The following introduces the notion of a locally consistent approximating Markov chain, which captures the requirement that the first and second condition moments coincide with drift and diffusion coefficients up to a term that vanishes with the timestep.

Definition 2.2. A finite-state Markov chain approximation to the processes $(x_t)_{t\geq 0}$ satisfying (9) for some admissible control $(u_t)_{t\geq 0}$ consists of a family of Markov chains $(\xi^h)_{h>0}$ with finite state spaces $(S_h)_{h>0}$, together with a family of time increment functions $(\Delta^h t(x,u))_{h>0}$ satisfying $\lim_{h\to 0} \sup_{x,u} \Delta^h t(x,u) = 0$ and $\inf_{x,u} \Delta^h t(x,u) > 0$ for all $h > 0$. Defining $\Delta_n^h x \text{“} = \xi_{n+1}^h - \xi_n^h$ and $t_n^h := \sum_{i=1}^n \Delta_i^h$, the approximation is *locally consistent* if

$$\begin{aligned} \mathbb{E}^h_{x,n,u}[\Delta_n^h x] &= \Delta^h t(x)\mu(x,u) + o(\Delta^h t(x,u)) \\ \mathbb{E}^h_{x,n,u}[(\Delta_n^h x - \mathbb{E}^h_{x,n,u}[\Delta_n^h x])^2] &= \Delta^h t(x,u)\sigma(x,u)\sigma(x,u)^t + o(\Delta^h t(x,u)) \end{aligned} \tag{10}$$

where $\mathbb{E}^h_{x,u,n}$ denotes the conditional expectation of the chain ξ^h at time t_n given $(\xi^h_n, u^h_n) = (x, u)$, where $u^h_n := u_{t^h_n}$. In what follows we refer to (10) as the *mean* and *covariance* consistency requirements, respectively.

We will drop superscripts and subscripts for expectations, since the appropriate operator will be obvious from the context. For each ξ^h we approximate the objective in Definition 2.1 as

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} F(x_t, u_t)dt\right] \approx \mathbb{E}\left[\sum_{n=0}^\infty e^{-\rho t^h_n} \Delta^h_n F(\xi^h_n, u^h_n)\right]. \tag{11}$$

For each $h > 0$, associated with the Markov chains given in Definition 2.2 we also define the continuous-time processes $\overline{\xi}^h$ and $\overline{u}^h$ that coincide with the above chains at the discrete times $(t_n)_{n\geq 0}$, are right-continuous and piecewise constant. The sum on the right-hand side of (11) is approximately $\mathbb{E}\left[\int_0^\infty e^{-\rho t} F(\overline{\xi}^h_t, \overline{u}^h_t)dt\right]$ with the only difference being the continuous discounting on the intervals $[t_n, t_{n+1})$, which necessarily vanishes as $h \to 0$. The weak convergence arguments of Kushner et al. (2001) are applied to these continuously interpolated processes, so that all approximations to the original process are defined on the same path space. However, for each $h > 0$ the value function we solve numerically corresponds to the control problem with objective (11) and state evolving according to x^h, and so may be solved with discrete-time techniques.

Definition 2.3. Given a family of Markov chains $\{\xi^h\}_{h>0}$ locally consistent with (9) for each control, define the approximate value functions

$$V^h(x) = \max_{u\in\mathcal{C}} \mathbb{E}\left[\sum_{n=0}^\infty e^{-\rho t^h_n} F(\xi^h_n, u^h_n)\right]$$
$$\xi^h_0 = x$$

for any $h > 0$, where the maximum is once again over the set of all admissible controls.

The finite-state Markov chain approach applies discrete-time dynamic programming arguments problems of the form in Definition 2.3 rather than the original problem in Definition 2.1. This leads to the Bellman equation for the controlled Markov process being given by

$$V^h(x) = \max_{u\in U} \Delta_t F(x, u) + e^{-\rho\Delta_t} \mathbb{E}^u[V^h(x')] \tag{12}$$

where x evolves according to the given approximating Markov chain. We will consider problems with discounting and uniformly bounded payoff functions, so that there are no subtleties regarding the applicability of the Principle of Optimality, and the fixed-point of (12) coincides with the

sequence problem given in Definition 2.3. Familiar arguments, such as those outlined in Stokey and Lucas, show that the right-hand side defines a contraction on the space of continuous functions defined on some compact subset of the state space. Finally, Kushner et al. (2001) show that under standard assumptions on the functions defined in the original problem Definition 2.1, local consistency ensures convergence of the approximate value functions to the true value function. The following is Theorem 5.2 on page 293 of Kushner et al. (2001).

Theorem 2.2. *Under Assumption 2.1 we have* $V^h(x) \to V(x)$ *as* $h \to 0$.

The MCA method may be used to solve problems in which there are jumps in the state variable. Although we do not strive for the most general framework possible, we outline here the theory necessary to solve a problem of particular interest to economists, in which a risk-averse consumer faces a consumption-savings problem with fluctuating income and may consume discrete amounts of a durable consumption good. The state variable for the agent will consist of their wealth, their income, and the current value of the durable good. This necessitates a discussion of jump processes, since the purchase of the durable good will coincide with a fall in wealth that does not vanish with the length of the time interval. We therefore consider a jump-diffusion process of the form

$$dx_t = \mu(x_t, u_{Dt})dt + \sigma(x_t)dZ_t + dJ_t(u_{Jt}) \tag{13}$$

where $(J_t)_{t\geq 0}$ is a jump process defined by

$$J_t = \int_0^t \int_\Gamma q(x_{s-}, u_{Js}, \rho)N(dsd\rho). \tag{14}$$

In (14) one interprets the integrand $q(x_{s-}, u_{Js}, \rho)$ as the size of the jumps, with ρ denoting the realization of uncertainty and supported in some compact set Γ. The quantity N is a Poisson random measure with intensity density $h(dtd\rho) = \lambda dt \times \Pi(d\rho)$, meaning $\mathbb{E}[N(A)] = \int_A h(dtd\rho)$ for all Borel A. The control vector is written $u_t := (u_{Dt}, u_{Jt})$ to illustrates that some components affect only the drift and others affect only the jumps. For the case of interest to us we may assume that ρ is supported at a single point and that the jumps in (13) corresponds to purchases of the durable good. It may help to imagine that the "jumps" are arriving at a constant exogenous rate $\lambda > 0$, but that they coincide with a movement in the state variable only when $q_t \neq 0$.

To construct a locally consistent Markov chain for (13) one begins with a locally consistent Markov chain for the diffusion component and obtain the approximation by independently drawing

from this and the jump component. The definition of a locally consistent finite-state Markov chain now includes an additional component, q_h, representing the jumps of the Markov chain. The manner in which the transition probability given in Definition 2.4 is constructed from the probabilities associated with the diffusion term has an intuitive interpretation. We may be interpret it as arising by drawing from a jump process with probability $\lambda\Delta_t$ and drawing from the continuous part with probability $1 - \lambda\Delta_t$.[4]

Definition 2.4 (Local consistency with jumps)**.** A family of finite-state Markov chains $\{\xi^h\}_{h>0}$ with state spaces $\{S_h\}_{h>0}$ is locally consistent with the jump diffusion (13) if for each $h > 0$ there exist transition probabilities $\{p_D^h(x, x')\}_{x,x' \in S_h}$ and functions q_h such that:

1. The family of Markov chains defined by $\{p_D^h(x, x')\}_{x,x' \in S_h}$ is locally consistent with the diffusion process $(z_t)_{t \geq 0}$ defined by $dz_t = \mu(z_t, u_t)dt + \sigma(z_t)dZ_t$;

2. The functions $(q_h)_{h>0}$ satisfy $|q_h(x, u_J) - q(x, u_J)| \to 0$ as $h \to 0$ uniformly in (x, u_J); and

3. For some $\delta_h(x, u) = o(\Delta_t(x, u))$,

$$p^h(x, x', u) = (1 - \lambda\Delta_t - \delta_h(x, u))p_D^h(x, x', u) + (\lambda\Delta_t + \delta_h(x, u))\chi(x, x', u)$$

   where $\chi(x, x', u) := 1_{x+q_h(x,u_J)=x'}$.

The analogue of Theorem 2.2 for the case of controlled jump processes is outlined in Chapter 13 of Kushner et al. (2001). To solve control problems of the form in Definition 2.1 we therefore need only solve the Bellman equation (12) for some choice of locally consistent approximating Markov chain. This is important because the literature on dynamic programming with finite state spaces contains a wealth of techniques for solving finite-state Markov decision problems. Section 3 illustrates the benefits of this viewpoint by solving an income fluctuation problem using modified policy function iteration. Section 4 deals with a more involved point that the MCA method is well-suited to solving problems with high correlation among multiple state variables, for which the construction of convergent finite-difference schemes is particularly difficult, and not considered in the analysis of Achdou et al. (2017).

[4]The following definition is less general than that in Kushner et al. (2001) but is sufficient to cover the example in Section 3.3.

3 Alternative solution methods for finite-state problems

In the one-sector growth model of Section 2, we constructed a locally consistent chain for the capital process and solved the resulting system of equations using policy function iteration. The algorithm converged in a small number of iterations, which is unsurprising given that policy function iteration is known to converge locally at a quadratic rate. However, the updating step in policy function iteration requires solving a linear system of equations. The computational cost of this operation grows rapidly when the number of gridpoints increases, or the sparsity structure of the transition matrix becomes more complex, both of which occur naturally as the dimension grows.

In our first application we therefore illustrate the benefits of employing the *modified* policy function iteration algorithm of Puterman and Shin (1978). This algorithm generalizes value function iteration by updating the value function a fixed number of times between successive updates of the policy function. In this case convergence is known to occur only at a linear rate, and so will typically require more iterations than policy function iteration. However, crucially, at no point in the algorithm do we need to solve a linear system of equations. Further, we show that an analogue of modified policy function remains applicable even when the timestep vanishes, and so is well suited to settings in which the state variable may change by large amount instantaneously, as is the case in problems in which one must choose consumption in a fixed, finite set.

The construction of the approximating chain in the one-sector growth model of Section 2 required no delicate choices, since in one direction a chain can only move up, down or stay at the same point. In higher dimensions, one may need to consider transitions to diagonal directions, or even non-local transitions. We return to this point in Section 4. For ease of exposition this section isolates the benefits of modified policy function iteration relative to policy function iteration, and so considers a problem for which a chain which requires no subtle construction. Section 3.2 applies modified policy function iteration to an income fluctuation problem in which income is the product of two diffusion processes. Section 3.3 considers a variation of an income fluctuation problem with discrete choices over a durable good and uses the normalization used prior to Lemma 2.1 to establish that an analogue of modified policy function iteration is applicable even when the timestep is sent to zero. The appendix verifies the accuracy of these methods for linear-quadratic problems.

3.1 Modified and generalized policy function iteration

We briefly recapitulate here the arguments and algorithms of Puterman and Brumelle (1979) and Puterman and Shin (1978) in order to fix ideas and to explain our generalization. Suppose we have a finite-state Markov chain with state space S of cardinality $|S|$, time increment $\Delta_t \in \mathbb{R}^{|S|}$ and that at each point $x \in S$ the control u may assume values in some subset U of Euclidean space, with the associated transition probabilities $P : S^2 \times U \to [0, 1]$. The dynamic programming equation for the discrete-state problem with flow payoff function f and discount rate $e^{-\rho\Delta_t}$ is

$$V(x) = \max_{u \in U} \Delta_t(x) f(x, u) + e^{-\rho\Delta_t(x)} \sum_{x' \in S} P(x, x', u) V(x') \qquad x \in S. \tag{15}$$

Defining $\beta(x) := e^{-\rho\Delta_t(x)}$ and $F(x, u) = \Delta_t(x) f(x, u)$, we can (with some abuse of notation) write this in matrix form as $V = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + \beta P(\hat{u}) V$, where dependence of F, β and V on the state is omitted and $P(\hat{u}) \in \mathbb{R}^{|S| \times |S|}$ is the matrix of transition probabilities. For any $\hat{u} \in U^{|S|}$ define $T(\hat{u}) := e^{-\rho\Delta_t} P(\hat{u}) - I$ and write the Bellman equation as

$$0 = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + T(\hat{u}) v =: B(v) \tag{16}$$

where the second equality defines $B : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$. Policy function iteration algorithm may then be summarized as follows:

- Choose v_0 arbitrarily.
- Choose $\hat{u}(v_0)$ to solve $B(v_0) = F(\hat{u}(v_0)) + T(\hat{u}(v_0)) v_0 = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + T(\hat{u}) v_0$.
- Given $\hat{u}(v_0)$, define v_1 as the value of adhering to $\hat{u}(v_0)$, $v_1 = -T(\hat{u}(v_0))^{-1} F(\hat{u}(v_0))$.
- Repeat the above steps with v_1 in place of v_0 and continue until convergence.

In what follows we will write $\hat{u}(v)$ for the $\hat{u} \in U^{|S|}$ that attains the maximum in (16), and further abbreviate $\hat{u}_n := \hat{u}(v_n)$. The updating law may be written more parsimoniously as

$$v_{n+1} = -T(\hat{u}(v_n))^{-1} F(\hat{u}(v_n)) = v_n - T(\hat{u}(v_n))^{-1} B(v_n). \tag{17}$$

A key insight of Puterman and Brumelle (1979) is that policy function iteration is essentially an abstract version of Newton's method, and inherits some of the same properties, such as rapid (quadratic) convergence near the solution. However, as we noted earlier, the updating step in policy function iteration requires the solution of a linear system of equations of size as large as the

number of gridpoints. Computational time therefore grows very rapidly as one increases the grid size or dimension, motivating the search for alternatives to policy function iteration.

To this end, note that if $T = \beta P - I$ then $-T(\hat{u}_n)^{-1} = \sum_{j=0}^{\infty}(\beta P(\hat{u}_n))^j$ and (17) becomes

$$v_{n+1} = v_n + \sum_{j=0}^{\infty}(\beta P(\hat{u}_n))^j B(v_n). \tag{18}$$

Modified policy function iteration simply truncates this sum at a finite integer k,

$$v_{n+1} = v_n + \sum_{j=0}^{k}(\beta P(\hat{u}_n))^j B(v_n). \tag{19}$$

One can interpret modified policy function iteration with k steps as beginning with guesses for the policy function $\hat{u}_n$ and value function v_n and iterating $k+1$ times. Note that the case $k = 0$ is value function iteration and that policy function iteration arises as $k \to \infty$. Puterman and Shin (1978) focus upon the case in which $T(\hat{u}) = \beta P(\hat{u}) - I$ for some $\beta \in (0,1)$ and transition matrix P, and establish rates of convergence for modified policy function iteration in this case. However, for our purposes it is useful to note that there exist analogues of (19) for a general T that retain the monotonicity properties of modified policy function iteration. For an arbitrary T and normalizing function $C : S \times U^{|S|} \to \mathbb{R}$ we define a *generalized* modified policy function iteration algorithm by

$$v_{n+1} = v_n + \sum_{j=0}^{k}(I + \tilde{T}(\hat{u}_n))^j \tilde{B}(v_n) \tag{20}$$

where $\tilde{T}(\hat{u}_n)(x,x') := T(\hat{u}_n)(x,x')/C(x,\hat{u}_n)$ and $\tilde{B}(v_n)(x) := B(v_n)/C(x,\hat{u}_n)$ for all $x, x' \in S$. The rule (20) is based on the idea of applying modified policy function iteration to a Bellman equation of the form (16) in which all quantities are divided by $C(\cdot,\hat{u}_n)$ at the nth stage (which leaves the fixed-point unchanged). The following establishes monotonicity of the above algorithm under certain conditions.

Lemma 3.1. *If C is chosen so $I + \tilde{T}(\hat{u}_n) \geq 0$ for all $n \geq 0$, then $B(v_n) \geq 0$ implies $B(v_{n+1}) \geq 0$ for all $n \geq 0$. Consequently, the sequence defined by (20) is monotone increasing if $B(v_0) \geq 0$.*

Proof. For any $v, w \in \mathbb{R}^{|S|}$ we have $B(w) = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + T(\hat{u})w \geq F(\hat{u}(v)) + T(\hat{u}(v))(w)$ and

so $B(w) \geq B(v) + T(\hat{u}(v))(w - v)$. For fixed $n \geq 1$ and $(v, w) = (v_n, v_{n+1})$ we then have

$$\begin{aligned} B(v_{n+1}) &\geq B(v_n) + T(\hat{u}_n)(v_{n+1} - v_n) \\ &= B(v_n) + T(\hat{u}_n) \sum_{j=0}^{k} (I + \tilde{T}(\hat{u}_n))^j \tilde{B}(v_n) \\ &= (I + \tilde{T}(\hat{u}_n))^{k+1} B(v_n) \geq 0 \end{aligned}$$

as claimed. □

If $C \equiv 1$ and $T(\hat{u}) := \beta P(\hat{u}) - I$ for some transition matrices $P(\hat{u})$ and $\beta \in (0, 1)$, then the non-negativity requirement of Lemma 3.1 holds. This is the original setting of Puterman and Shin (1978), who further derive rates of convergence in this case. The possibility of applying a version of the above algorithm to more general operators than those of the form $T = \beta P - I$ is mentioned Puterman and Brumelle (1979) (although we cannot obtain the references that elaborate on this point). Our interest in the above generalization arises because we will apply the algorithm to a *normalized* version of the transition operator similar to that appearing in Lemma 2.1. Specifically, if $P(\Delta_t) : U^{|S|} \to \mathbb{R}^{|S| \times |S|}$ denotes the transition probability functions arising from a discretization from the MCA method with constant timestep Δ_t, then we apply the algorithm (20) to the operator defined by $T(\hat{u}) = \lim_{\Delta_t \to 0} (e^{-\rho \Delta_t} P(\hat{u}, \Delta_t) - I)/\Delta_t$ for $\hat{u} \in U^{|S|}$. This is convenient because it will avoid the need to choose the timestep so that the transition probabilities remain in the unit interval through the iterative process.

3.2 Income fluctuation problem

We first consider the problem of an infinitely-lived agent who may borrow and lend at a risk-free rate and who faces idiosyncratic income risk. As emphasized by Achdou et al. (2017), this is a natural application for an economist as such problems are an integral component of Bewley-Huggett-Aiyagari incomplete markets models which form the backbone of much of modern macroeconomics. Suppose that preferences over consumption are given by

$$U(c) = \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right] \tag{21}$$

and that wealth evolves according to

$$da_t = [r a_t - c_t + y_t] dt \tag{22}$$

where $r > 0$ is fixed and $(y_t)_{t\geq 0}$ denotes the income of the agent. We will also assume that the agent faces a borrowing constraint of the form $a_t \geq \underline{a}$ for all $t \geq 0$ and some $\underline{a}$.[5] We assume $y_t = \exp(z_t)$ where $z_t := z_{1t} + z_{2t}$ for z_1 and z_2 satisfying

$$dz_{it} = -\theta_i z_{it} dt + \sigma_i dZ_{it} \tag{23}$$

for $i = 1, 2$, where $Z := (Z_{1t}, Z_{2t})_{t\geq 0}$ is standard two-dimensional Brownian motion. When numerically approximating (23) we assume the volatility vanishes near the boundary, but omit this from the notation for brevity. To construct our approximating chain, we must specify the state space, the transition probabilities, and the (possibly state-dependent) timestep. Write $\overline{a}$ for the maximum level of wealth in the discretization, and $\underline{z}_1, \overline{z}_1, \underline{z}_2$ and $\overline{z}_2$ for the lower and upper bounds for the income processes. For a vector of integers $N = (N_a, N_1, N_2)$, define the state increments

$$(\Delta_a, \Delta_1, \Delta_2) = ((\overline{a} - \underline{a})/N_a, (\overline{z}_1 - \underline{z}_1)/N_1, (\overline{z}_2 - \underline{z}_2)/N_2)$$

and the individual grids $S_a = \{\underline{a} + \Delta_a, \ldots, \overline{a} - \Delta_a\}$ and $S_i = \{\underline{z}_i + \Delta_i, \ldots, \overline{z}_i - \Delta_i\}$ for $i \in \{1, 2\}$ and define the state space $S_h := S_a \times S_1 \times S_2$. We choose our transition probabilities so that if the chain is at point $x := (a, z_1, z_2) \in S_h$ at time t then the possible values at time $t + \Delta_t$ lie in the set

$$\Delta(x) := \{(a, z_1, z_2), (a \pm \Delta_a, z_1, z_2), (a, z_1 \pm \Delta_1, z_2), (a, z_1, z_2 \pm \Delta_2)\}.$$

One may check that the following defines a locally consistent chain for any c,

$$\begin{aligned} p(a \pm \Delta_a, z_1, z_2) &= \frac{\Delta_t}{\Delta_a}[ra - c + \exp(z_1 + z_2)]^{\pm} \\ p(a, z_1 \pm \Delta_1, z_2) &= \frac{\Delta_t}{\Delta_1^2}\left(\frac{\sigma_1^2}{2}\chi_1(z_1) + \Delta_1[-\theta_1 z_1]^{\pm}\right) \\ p(a, z_1, z_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_2^2}\left(\frac{\sigma_2^2}{2}\chi_2(z_2) + \Delta_2[-\theta_2 z_2]^{\pm}\right) \end{aligned} \tag{24}$$

where $\chi_i(z_i) := 1_{z_i \notin \{\underline{z}_i + \Delta_i, \overline{z}_i - \Delta_i\}}$ and $i = 1, 2$, provided the timestep is chosen so that the above probabilities remain within the unit interval. The borrowing constraint may be imposed by requiring $c \leq ra + y$ when $a = \underline{a} + \Delta_a$. To ensure the process remains on the grid we also impose $c \geq ra + y$ for $a = \overline{a} - \Delta_a$, although for a sufficiently large upper limit this will not be binding. The Bellman equation is then

$$V(x) = \max_{c\geq 0} \Delta_t \frac{c^{1-\gamma}}{1-\gamma} + e^{-\rho\Delta_t}\mathbb{E}[V(x')] \tag{25}$$

[5]The state-dependence of the constraint set means this does not fit into the framework of Section 2.2. This may be dealt with by introducing the notion of a reflecting boundary, or by assuming the consumer obtains a large negative payoff upon hitting the boundary. For brevity of notation we omit these details.

for all $x \in S_h$, where the expectation operator is defined by (24). Optimal consumption solves

$$\max_{c \geq 0} \frac{c^{1-\gamma}}{1-\gamma} + e^{-\rho \Delta_t} \big([ra - c + y]^+ V^{Fa} - [ra - c + y]^- V^{Ba} \big)$$

which again requires no nonlinear root-finding.

Numerical illustration: we now compare computational times for the above algorithms for a fixed set of parameters. To the extent possible we adopt the parameters of Achdou et al. (2017), who consider a two-dimensional problem and compare the performance of their implicit finite-difference scheme with the endogenous grid method of Carroll (2006). Arguments analogous to those provided for the one-sector neoclassical growth model in Section 2 reveal that the implicit finite-difference method of Achdou et al. (2017) is asymptotically equivalent to using policy function iteration for the probabilities (24) as the timestep vanishes. To contrast the method with the literature we therefore compare policy function iteration with the algorithms of Section 3.1.

We set $\gamma = 2$, $\rho \approx 0.0526$ and $r = 0.03$. With a single income state variable Achdou et al. (2017) target an annual autocorrelation of 0.95, which implies $\theta \approx 0.0513$. Since the stationary solution to (23) is Gaussian with mean zero and variance $\nu^2 := \sigma^2/(2\theta)$, their choice of $\nu = 0.2$ implies $\sigma \approx 0.064$. To illustrate the effect of changing the sparsity structure of the transition matrix on the performance of different methods, we consider two choices for the income process. In each case we choose the parameters of Achdou et al. (2017) for the first component of income. In the first case we set $z_2 = 0$ so that the problem becomes two-dimensional, while in the second we choose $(\theta_2, \sigma_2) = (\theta_1, \sigma_1)$. For each choice we solve the above problem using the modified and generalized modified policy function iteration algorithms of Section 3.1 for a number of grid sizes and relaxation steps. In the former case we must also specify a timestep with the property that the probabilities in (24) remain bounded within the unit interval. This can either be found via experimentation or by imposing *a priori* bounds on consumption and checking ex-post that they do not bind. We adopt the latter approach, and guess-and-verify that consumption never exceed two and a half times interest and labor income. Note this experimentation is unnecessary in the case of generalized modified policy function iteration, in which the timestep has been sent to zero.

We now record the speed of convergence for for policy function iteration, value function iteration, and modified value function iteration with $k = \{10, 50, 100\}$, and four standard deviations in each

	PFI	VFI	k = 10	k = 50	k = 100
(200, 10)	0.083422	13.345196	1.100620	0.339219	0.177748
(300, 15)	0.156383	12.876851	1.555979	0.468378	0.308606
(400, 20)	0.215893	21.555077	2.887941	0.993074	0.638374
(500, 25)	0.320404	41.644928	5.516093	1.702569	1.115530

Table 1: Time until convergence: $2D$, MPFI

	PFI	VFI	k = 10	k = 50	k = 100
(200, 10)	0.065006	1.889994	0.185716	0.073545	0.075748
(300, 15)	0.109626	4.953089	0.428060	0.131680	0.102389
(400, 20)	0.250566	10.068502	0.946442	0.306341	0.219811
(500, 25)	0.408300	19.648714	1.732262	0.498475	0.336704

Table 2: Time until convergence: $2D$, Generalized MPFI

dimension of the grid and a tolerance between successive iterations of 10^{-6}. All figures are in second and all calculations were performed using Python and a Intel Core i7-8650U processor with 1.9 GHz. Table 3 gives the time in seconds. Note that for the third and fourth choices of grids, modified policy function iteration is easily an order of magnitude faster than policy function iteration (which is essentially the implicit finite-difference method of Achdou et al. (2017)). Table 4 gives the analogous results for the case of generalized modified policy function iteration. The generalized modified policy function iteration appears to be slightly slower for this example. However, it is simpler as one need not worry about the timestep being chosen such that the probabilities lie in the unit interval, which in more general settings than the above may be non-trivial to ensure. As such, it does not appear that one approach always dominates the other and we believe both are of interest. We again emphasize that Section A tests the speed and accuracy of all methods considered in the paper by applying them to linear-quadratic problems, for which policy and value functions are attainable in closed-form.

	PFI	VFI	k = 10	k = 50	k = 100
(45, 15, 15)	1.641302	20.132271	2.531317	0.777083	0.418868
(60, 20, 20)	16.465262	50.869687	7.735266	1.802173	0.936221
(75, 25, 25)	54.635781	147.478504	15.291911	4.326966	2.631439
(90, 30, 30)	138.588922	357.339421	42.479975	12.173866	7.508772

Table 3: Time until convergence: $3D$, MPFI

	PFI	VFI	k = 10	k = 50	k = 100
(45, 15, 15)	4.073966	15.302004	1.605577	0.603225	0.861970
(60, 20, 20)	27.281565	67.305161	6.053501	1.696085	2.165526
(75, 25, 25)	97.109719	184.874694	17.400236	4.465696	3.496366
(90, 30, 30)	259.249338	357.462360	34.493084	9.910862	5.767009

Table 4: Time until convergence: $3D$, Generalized MPFI

3.3 Durable consumption and discrete choice

One interesting property of the generalized modified policy function iteration given in Section 3.1 is that it remains applicable even as the timestep vanishes. This is useful for discrete-choice models in which there are large and instantaneous changes in wealth. To illustrate, we now consider a variation of the income fluctuation problem of Section 3.2 in which the agent has preferences over non-durable consumption that may assume a continuum of values, as well as a durable good that may assume only finitely-many values. This is similar to a continuous-time version of the model of Fella (2014), who extends the endogenous grid method of Carroll (2006) to allow for both adjustment costs and discrete choices. The MCA method may be applied to this case without any delicate choices of grids, or any need for interpolation of the function. In this case the amount by which that wealth changes upon purchase of the durable good does not vanish as the size of the grid tends to zero, and so we are not able to restrict attention to adjacent transitions. However, as we shall see, this does not cause any major problems.

Assume the agent has preferences over non-durable and durable consumption given by

$$U(c, D) := \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} u(c_t, D_t) dt\right] \tag{26}$$

for some u and denote the values of durable consumption by $S_D := \{\underline{D}, \underline{D} + \Delta_D, \ldots, \underline{D} + N_D \Delta_D\}$ for some $\underline{D}, N_D$ and Δ_D. Note that in contrast with previous examples this grid S_D is a primitive of the problem, and not a choice made in the discretization. We again suppose income is of the form $y_t = \exp(z_t)$ for some mean-reverting $(z_t)_{t \geq 0}$, and model the choice of the durable good as follows. At any instant the agent makes a binary choice indicating whether they wish to change the durable good. However, the opportunities to change the durable good only arrive stochastically at some constant rate $\lambda > 0$. As $\lambda \to \infty$ this approximates a situation in which the durable good may change instantaneously. If $\overline{p}$ denotes the price of the durable good, then for some constants θ and σ with $\theta > 0$ the laws of motion for assets a_t, log income z_t and durable consumption D_t are

$$\begin{aligned} da_t &= [ra_t + y_t - c_t]dt - \overline{p} dD_t(q_t) \\ dz_t &= -\theta z_t dt + \sigma dZ_t \\ dD_t(q_t) &= dJ_t(q_t) \end{aligned} \tag{27}$$

where $(J_t)_{t \geq 0}$ is a jump process with arrival rate λ and $q_t \in \{0, \overline{q}\}$ indicates the desired change conditional on the jump arrival. We first define the discretized problem for a positive timestep before considering operators that arise in the limit as we send this quantity to zero, as per the discussion following Lemma 3.1.

First define the equispaced grids $S := S_a \times S_z \times S_D$, where $S_a := \{\underline{a} + \Delta_a, \ldots, \overline{a} - \Delta_a\}$ and $S_z := \{\underline{z} + \Delta_y, \ldots, \overline{z} - \Delta_y\}$ for some integers $N_a, N_z \geq 1$ and bounds $\underline{a}, \overline{a}, \underline{z}$ and $\overline{z}$, where $(\Delta_a, \Delta_y) := ((\overline{a} - \underline{a})/N_a, (\overline{z} - \underline{z})/N_z)$. To ensure income remains on the grid we impose $\overline{p}\Delta_D = K\Delta_a$ for some $K \geq 1$. We first define the transition probabilities for wealth and income,

$$\begin{aligned} p(a \pm \Delta_a, z, D) &= \frac{\Delta_t}{\Delta_a}[ra + y - c]^{\pm} \\ p(a, z \pm \Delta_z, D) &= \frac{\Delta_t}{\Delta_y^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^{\pm}\right) \end{aligned}$$

where $\chi(z) := 1_{z \notin \{\underline{z} + \Delta_z, \overline{z} - \Delta_z\}}$ and define the transitions for the durable good

$$p(a - \overline{p}\Delta_D, z, D + \Delta_a) = \lambda 1_{q_t = \overline{q}} \Delta_t$$

where $q_t := (q_{at}, q_{zt}, q_{Dt}) \in \{0, \overline{q}\}$ for $\overline{q} = (-\overline{p}\Delta_a, 0, \Delta_D)$. The Bellman equation is then

$$0 = \max_{\substack{c \geq 0 \\ q \in \{0, \overline{q}\}}} u(c, D) + T(c, q; \Delta_t)V$$

where

$$T(c, q; \Delta_t) = u(c, D) + \frac{1}{\Delta_t}\big(e^{-r\Delta_t}\mathbb{E}[V(a', z', D')] - V(a, z, D)\big).$$

The optimal policy for the durable good is $q := \overline{q}(V(a - K\Delta_a, z, D + \Delta_D) > V(a, z, D))$. For our illustration we assume preferences of the form

$$u(c, D) = \ln c + \eta \ln(D + \iota) \tag{28}$$

for some $\eta, \iota > 0$, so that the problem of finding optimal consumption is identical to the problem of Section 3.2 with $\gamma = 1$. We now define $\tilde{T}(c, q) = \lim_{\Delta_t \to 0} T(c, q; \Delta_t)$. Simplification gives

$$\begin{aligned}
\tilde{T}(c, q) = u(c, D) &+ \frac{1}{\Delta_a}[ra + y - c]^+[V(a + \Delta_a, z, D) - V(a, z, D)] \\
&+ \frac{1}{\Delta_a}[ra + y - c]^-[V(a - \Delta_a, y, D) - V(a, y, D)] \\
&+ \frac{1}{\Delta_z^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^+\right)[V(a, z + \Delta_z, D) - V(a, z, D)] \\
&+ \frac{1}{\Delta_z^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^-\right)[V(a, z - \Delta_z, D) - V(a, z, D)] \\
&+ \lambda(V(a - q_a, z + q_z, D + q_D) - V(a, z, D)) - rV(a, z, D).
\end{aligned} \tag{29}$$

Numerical illustration: For simplicity we choose the parameters for the process for income as in Section 3.2, and the parameters η, ι and r from Fella (2014). We also follow Fella (2014) in choosing the upper bound for durable consumption to be roughly 10 times the unconditional average of income. Table 3 gives the time until convergence for generalized modified policy function iteration, with a tolerance between successive iterations of 10^{-6}. Once again we find that modified policy function is significantly faster than policy function iteration. Further, note that although there is no closed-form solution against which to check the accuracy of these algoritms, since we have fixed the same discretization throughout all of the above and they converge with 10^{-6} tolerance, they all have essentially the same level of accuracy.

4 Problems with highly correlated state variables

This section applies the MCA method to a general equilibrium model with financial frictions in the spirit of Brunnermeier and Sannikov (2014). There is a unit mass continuum of infinitely lived

	PFI	VFI	k = 10	k = 50	k = 100
(50, 10, 10)	0.382562	2.896908	0.349048	0.168027	0.159010
(100, 20, 10)	4.794657	21.105528	2.114040	0.608535	0.567014
(150, 30, 10)	16.679804	80.534670	9.213652	2.320960	1.716259
(200, 40, 10)	44.446097	253.265637	25.383114	7.502274	4.233294

Figure 3: Time until convergence for discrete choice problem

agents with preferences over a single consumption good. Agents may be one of two types, termed *experts* and *households*. Experts have access to a linear production technology subject to aggregate depreciation shocks, and all agents have access to a risk-free technology with exogenous return (a storage technology). Capital may be transformed one-for-one into consumption and the sole tradable asset for all agents is a risk-free bond in zero net supply. To illustrate the flexibility of the MCA approach, we enrich the setting of Brunnermeier and Sannikov (2014) to allow for time-varying volatility correlated with the depreciation shocks. A large and growing number of models in this literature possess this feature and it poses difficulties for finite-difference approaches.

Section 4.1 outlines an environment and Section 4.2 formulates the problem and equilibrium notions recursively. Section 4.3 outlines a general approach for solving a single decision problem with perfect correlation among multiple state variables. Section 4.4 uses the policy functions of Section 4.3 and imposes the requirements that the law of motion of the wealth share be consistent with individual decisions and bond market-clearing. Section 4.5 ties the above observations together to summarize the algorithm and computes an example.

4.1 Setup

Agents may be one of two types, indexed $i \in \{E, H\}$, referred to as *experts* and *households*, respectively. There is a unit mass of each type indexed by $j \in [0, 1]$. Both types of agents are infinitely-lived with the same flow utility function but differ in their discount rates, with preferences over sequences of a single consumption good represented by

$$U_i(c) = \mathbb{E}\left[\rho_i \int_0^\infty e^{-\rho_i t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right]$$

for some $\rho_H, \rho_E > 0$ and $\gamma \in (0,1)$. Aggregate capital in the economy at time $t \geq 0$ is denoted k_t, and the amount held by the jth agent of type i is denoted k_{it}^j. When the jth agent of type i invests a fraction ι_{it}^j of their capital in new capital, the flow output of consumption produced is

$$(\Pi_i - \iota_{it}^j) k_{it}^j dt$$

and the law of motion of their capital stock is

$$dk_{it}^j = \iota_{it}^j k_{it}^j dt + \sigma_t k_{it}^j dZ_t \tag{30}$$

where $Z = (Z_t)_{t \geq 0}$ is a Brownian motion common to all agents. The increments of Brownian motion in (30) may then be thought of as representing stochastic depreciation shocks. Note that the case in which households have no access to a risky technology may be interpreted as one with $\Pi_H = -\infty$. Further, the linearity in the investment production technology implies that the price of capital is constant (and here is unity). In this case, changing the coefficient of the drift term in (30) has the same implications as changing Π_i. We also assume that agents have access to a risk-free storage technology with exogenous and constant real return $\underline{r}$. The volatility $(\sigma_t)_{t \geq 0}$ in (30) is common to all agents and evolves over time according to

$$d\sigma_t = \theta(\overline{\sigma} - \sigma_t)dt + \sigma_\sigma dZ_t \tag{31}$$

for some positive $\theta, \overline{\sigma}$ and σ_σ. We emphasize that the Brownian motions in (30) and (31) coincide.

Agents may trade a risk-free bond in zero net supply with (endogenously determined) return denoted $(r_t)_{t \geq 0}$. An agent with wealth a_t must choose capital k_t, bond b_t and storage h_t holdings satisfying $k_t + b_t + h_t = a_t$. When an agent of type $i \in \{E, H\}$ adheres to choices $(c_i, h_i, k_i) = (c_{it}, h_{it}, k_{it})_{t \geq 0}$ for consumption, storage and capital their wealth evolves according to

$$da_t = [r_t a_t + (\underline{r} - r_t) h_{it} - c_{it} + (\Pi_i - r_t) k_{it}] dt + \sigma_t k_t dZ_t. \tag{32}$$

Note that market clearing for investment requires the price of capital in terms of consumption to be unity, with investment ultimately determined by the consumption-savings decision. It is for this reason that investment is omitted from (32). We then have the following.

Definition 4.1. The problem of an agent of type $i \in \{E, H\}$ with wealth a at time $t \geq 0$ with

volatility σ is

$$\begin{aligned} V_{it}(a,\sigma) &= \max_{c,k,h\geq 0} \mathbb{E}\left[\rho_i \int_t^\infty e^{-\rho_i(\tau-t)} \frac{c_\tau^{1-\gamma}}{1-\gamma} d\tau\right] \\ da_\tau &= [r_\tau a_\tau + (\underline{r} - r_\tau)h_\tau - c_\tau + (\Pi_i - r_\tau)k_\tau]d\tau + \sigma_\tau k_\tau dZ_\tau \\ d\sigma_\tau &= \theta(\overline{\sigma} - \sigma_\tau)d\tau + \sigma_\sigma dZ_\tau \\ (a_t, \sigma_t) &= (a, \sigma). \end{aligned} \tag{33}$$

The notion of competitive equilibrium in this economy is standard: agents maximize utility and markets clear. We first formulate this in a sequential manner.

Definition 4.2 (Competitive equilibrium; sequential formulation)**.** Given an initial distribution of wealth (a_{i0}^j) for $i \in \{E, W\}$ and $j \in [0,1]$ and initial volatility σ_0, a competitive equilibrium consists of a stochastic process $r = (r_t)_{t\geq 0}$ for the risk-free rate, stochastic processes $(V_{it}^j)_{t\geq 0}$ and $(c_{it}^j, h_{it}^j, k_{it}^j)_{t\geq 0}$ for $i \in \{E, W\}$ and $j \in [0,1]$ such that V_i^j solves (33) with associated policy functions (c_i^j, h_i^j, k_i^j) given r and (a_i^j, σ_0), and the markets for consumption and bonds clear at all dates almost surely. The goods market clearing condition is

$$\int_0^1 c_{Et}^j dj + \int_0^1 c_{Ht}^j dj = \int_0^1 [\Pi_E - \iota_{Et}^j] k_{Et}^j dj + \int_0^1 [\Pi_H - \iota_{Ht}^j] k_{Ht}^j dj$$

and the bond market clearing condition is

$$0 = \int_0^1 (1 - h_{Et}^j - k_{Et}^j) dj + \int_0^1 (1 - h_{Ht}^j - k_{Ht}^j) dj.$$

The homotheticity of flow utility and the log-linearity of the law of motion for wealth then give the following, which motivates our subsequent search for a recursive formulation of both the individual problem and competitive equilibrium.

Lemma 4.1 (Homogeneity)**.** *For any process $(r_t)_{t\geq 0}$ and $i \in \{E, W\}$ there exist processes $(\overline{V}_{it})_{t\geq 0}$ and $(\overline{c}_{it}, \overline{h}_{it}, \overline{k}_{it})_{t\geq 0}$ such that $V_{it}(a,\sigma) = \overline{V}_{it}(\sigma)a^{1-\gamma}/(1-\gamma)$ and $c_{it}(a,\sigma) = \overline{c}_{it}(\sigma)a, h_{it}(a,\sigma) = \overline{h}_{it}(\sigma)a$ and $k_{it}(a,\sigma) = \overline{k}_{it}(\sigma)a$, respectively, for all $t, a, \sigma \geq 0$.*

Proof. First define $\overline{V}_{it}(\sigma) = V_{it}(1,\sigma)(1-\gamma)$ for the normalized utility process, and

$$(\overline{c}_{it}(\sigma), \overline{h}_{it}(\sigma), \overline{k}_{it}(\sigma)) = (c_{it}(1,\sigma), h_{it}(1,\sigma), k_{it}(1,\sigma))$$

for the normalized policy processes. It suffices to note that if $c_{it}(1,\sigma), h_{it}(1,\sigma)$ and $k_{it}(1,\sigma)$ solve (33) with $a = 1$ then scaling all policy functions by a solves (33) for general a. □

In what follows we abuse notation slightly and write V_i for $\overline{V}_i$. Using the linearity of policy functions in Lemma 4.1, aggregate consumption, storage and capital demand are

$$\begin{aligned} C_t &= \overline{c}_{Et} \int_0^1 a_{Et}^j dj + \overline{c}_{Ht} \int_0^1 a_{Ht}^j dj \\ H_t &= \overline{h}_{Et} \int_0^1 a_{Et}^j dj + \overline{h}_{Ht} \int_0^1 a_{Ht}^j dj \\ K_t &= \overline{k}_{Et} \int_0^1 a_{Et}^j dj + \overline{k}_{Ht} \int_0^1 a_{Ht}^j dj. \end{aligned}$$

These aggregate quantities may be written as functions of the type-specific policy functions and the wealth share of experts, defined as

$$x_t := \frac{\int_0^1 a_{Et}^j dj}{\int_0^1 a_{Et}^j dj + \int_0^1 a_{Ht}^j dj}. \tag{34}$$

Section B.1 shows that the wealth share evolves according to the law of motion

$$dx_t = \mu_x(x_t, \sigma_t) x_t dt + \sigma_x(x_t, \sigma_t) x_t dZ_t$$

for some μ_x and σ_x depending on the policy functions of each agent. We will use this to calculate a *Markov equilibrium* in which the policy functions, price functions and law of motion of the wealth share are time-independent functions of (x, σ).

4.2 Recursive formulation

We now suppose that $r_t = r(x_t, \sigma_t)$, $\mu_{xt} = \mu_x(x_t, \sigma_t)$ and $\sigma_{xt} = \sigma_x(x_t, \sigma_t)$ for all $t \geq 0$ almost surely for some functions r, μ_x and σ_x. The problem (33) may be written in the following form.

Definition 4.3. For any functions r, μ_x and σ_x, state (a, x, σ) and $i \in \{E, W\}$, the problem of the ith type of agent may be written

$$\begin{aligned} W_i(a, x, \sigma) = \max_{c_t, k_t \geq 0} \mathbb{E}\left[\int_0^\infty \rho_i e^{-\rho_i t} \frac{c_t^{1-\gamma}}{1-\gamma}\right] dt \\ da_t &= [r(x_t, \sigma_t) + (\underline{r} - r(x_t, \sigma_t))\overline{h}_t - \overline{c}_t] a_t dt + \overline{k}_t a_t dR_t \\ dx_t &= \mu_x(x_t, \sigma_t) dt + \sigma_x(x_t, \sigma_t) x_t dZ_t \\ d\sigma_t &= \theta(\overline{\sigma} - \sigma_t) dt + \sigma_\sigma Z_t \\ (a_0, x_0, \sigma_0) &= (a, x, \sigma) \end{aligned}$$

where $dR_t = (\Pi_i - r(x_t, \sigma_t))dt + \sigma_t dZ_t$.

Lemma 4.1 implies that the value functions in Definition 4.3 assume the form

$$W_i(a,x,\sigma) = V_i(x,\sigma)\frac{a^{1-\gamma}}{1-\gamma}$$

for some function V_i, with associated policy functions of the form

$$(c_i(a,x,\sigma), h_i(a,x,\sigma), k_i(a,x,\sigma)) = \big(\overline{c}_i(x,\sigma)a, \overline{h}_i(x,\sigma)a, \overline{k}_i(x,\sigma)a\big)$$

for all (a,x,σ).

Definition 4.4 (Markov equilibrium)**.** A Markov equilibrium consists of functions for the risk-free rate and drift and diffusion for the wealth share, together with value functions V_i and policy functions $(\overline{c}_i, \overline{h}_i, \overline{k}_i)$ for $i \in \{E,H\}$ solving (4.3), such that for all (x,σ) we have

$$0 = \big(1-\overline{k}_E(x,\sigma) - \overline{h}_E(x,\sigma)\big)x + \big(1-\overline{k}_H(x,\sigma) - \overline{h}_H(x,\sigma)\big)[1-x]$$

and the law of motion for the wealth share is consistent with individual policy functions.

We now outline how to solve for the Markov equilibrium. The Hamilton-Jacobi-Bellman equation for the value function $W = W(x,a,\sigma)$ of each agent is of the form

$$\begin{aligned}\rho W = \max_{\overline{c},\overline{h},\overline{k}\geq 0}\ & \frac{\rho(\overline{c}a)^{1-\gamma}}{1-\gamma} + \big[r + (\underline{r}-r)\overline{h} - \overline{c} + (\Pi - r)\overline{k}\big]aW_1 + \frac{\sigma^2\overline{k}^2}{2}a^2W_{11} + \mu_x x W_2 \\ & + \frac{\sigma_x^2 x^2}{2}W_{22} + \sigma_x x\sigma\overline{k}aW_{12} + \sigma\overline{k}a\sigma_\sigma W_{13} + \sigma_x x\sigma_\sigma W_{23} + \theta(\overline{\sigma}-\sigma)W_3 + \frac{\sigma_\sigma^2}{2}W_{33}.\end{aligned} \tag{35}$$

We impose $r \geq \underline{r}$ and henceforth omit storage choice from the agent problems. We compute equilibria by solving a discretized version of the above problem for an arbitrary function for the risk-free rate on a grid and then iterate over this function until convergence. Prior to elaborating upon the search for equilibria we explain how to solve the individual problem (35). This poses some difficulties, as standard constructions fail to work when there is high correlation between the state variables. To see why, suppose that $(X_t)_{t\geq 0}$ is a two-dimensional process satisfying

$$dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t$$

where $\mu : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2\times m}$ and $(Z_t)_{t\geq 0}$ is m-dimensional Brownian motion. Now define coefficients $a_{ij}(X) = (\sigma\sigma^T)_{ij}$ for $i,j = 1,2$. Suppose that μ and σ vanish outside of some domain $[-M,M]^2$, and set $S := S_1 \times S_2$ where S_1 and S_2 are arbitrary uniform grids with increments Δ_1 and Δ_2, respectively. Denote an arbitrary element of S by $x = (x_1,x_2)$ and consider a Markov chain such that if the chain is at point x at time t then the possible values at time $t+\Delta_t$ are

$$\Delta(x) := \{(x_1,x_2), (x_1 \pm \Delta_1, x_2), (x_1, x_2 \pm \Delta_2), (x_1 \pm \Delta_1, x_2 \pm \Delta_2), (x_1 \pm \Delta_1, x_2 \mp \Delta_2)\}. \tag{36}$$

The set (36) is simply the set of *adjacent* points in $\mathbb{R}^2$. It is easy to check that if

$$a_{ii} - \sum_{j \neq i} |a_{ij}| \Delta_i / \Delta_j \geq 0, \tag{37}$$

for $i = 1, 2$, then for sufficiently small $\Delta_t > 0$ the following define a locally consistent chain

$$\begin{aligned}
p(x_1 \pm \Delta_1, x_2) &= \frac{\Delta_t}{\Delta_1^2}\left(\frac{1}{2}[a_{11} - |a_{12}|\Delta_1/\Delta_2] + \Delta_1 \max\{\pm\mu_1(x), 0\}\right) \\
p(x_1, x_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_2^2}\left(\frac{1}{2}[a_{22} - |a_{12}|\Delta_2/\Delta_1] + \Delta_2 \max\{\pm\mu_2(x), 0\}\right) \\
p(x_1 \pm \Delta_1, x_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_1 \Delta_2}\frac{1}{2}\max\{a_{12}, 0\} \\
p(x_1 \pm \Delta_1, x_2 \mp \Delta_2) &= \frac{\Delta_t}{\Delta_1 \Delta_2}\frac{1}{2}\max\{-a_{12}, 0\}.
\end{aligned}$$

However, the above construction will fail to work whenever (37) fails for some point in the domain, since the expressions for probabilities may be negative, leading to the inapplicability of standard dynamic programming arguments and instability in the associated numerical algorithms. Indeed, when (37) fails it may be impossible to exactly match the moments of the increments of the underlying process using only local transitions. To illustrate, consider a drift-free diffusion process of the form $(dx_{1t}, dx_{2t}) = (\sigma_1(x)dZ_t, \sigma_2(x)dZ_t)$ for some functions σ_1 and σ_2, where $(Z_t)_{t \geq 0}$ is one-dimensional Brownian motion. In this case we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1(x)^2 & \sigma_1(x)\sigma_2(x) \\ \sigma_1(x)\sigma_2(x) & \sigma_2(x)^2 \end{bmatrix}.$$

In order for the above expressions for probabilities to be non-negative we need $\sigma_1(x)^2/\Delta_1 \geq |\sigma_1(x)\sigma_2(x)|/\Delta_2$ and $\sigma_2(x)^2/\Delta_2 \geq |\sigma_1(x)\sigma_2(x)|/\Delta_1$. This is only true if $|\sigma_1(x)|/\Delta_1 = |\sigma_2(x)|/\Delta_2$ and so cannot be assured to hold everywhere for arbitrary σ_1 and σ_2. A different construction is necessary for such a process, one that may call for non-local transitions. Section 4.3 is devoted to this construction, explaining the general process before turning to the particular problem in (35) and obtaining the optimal policies of the agent, given continuation values, prices and functions for the law of motion of wealth.

4.3 Chain construction with high correlation

We first illustrate the construction of a locally consistent chain for a drift-free process of the form $(dx_{1t}, dx_{2t}) = (\sigma_1(x)dZ_t, \sigma_2(x)dZ_t)$ on a domain of the form $[0, M_1] \times [0, M_2]$, for functions σ_1, σ_2 and constants $M_1, M_2 > 0$. We fix integers $N_1, N_2 \geq 1$ and define $\Delta_i = M_i/N_i$ for $i = 1, 2$ and

$$S_h = \{\Delta_1, \ldots, M_1 - \Delta_1\} \times \{\Delta_2, \ldots, M_2 - \Delta_2\}. \tag{38}$$

We also write $(\overline{\sigma}_1, \overline{\sigma}_2) := (\sigma_1(x_t)/\Delta_1, \sigma_2(x_t)/\Delta_2)$ and $w := \overline{\sigma}_1/\overline{\sigma}_2$. An arbitrary member of S_h is of the form $(i\Delta_1, j\Delta_2)$ for $i \in \{1, \ldots, N_1 - 1\}$ and $j \in \{1, \ldots, N_2 - 1\}$. We impose two restrictions on the possible transitions (m_1, m_2) from a point $(i\Delta_1, j\Delta_2)$: the state may only leave the grid from a point adjacent to the boundary; and the number of increments that the state may move in either direction cannot exceed a fixed integer $\overline{m} \geq 1$. This translates into the restriction

$$\begin{aligned} |m_1| &\leq \min\{\overline{m}, \min\{i - 1, N_1 - 1 - i\}\} \\ |m_2| &\leq \min\{\overline{m}, \min\{j - 1, N_2 - 1 - j\}\}. \end{aligned} \tag{39}$$

The set of non-zero integer pairs satisfying (39) is denoted $\Gamma(i, j)$. In constructing our chain we consider two cases that differ in the number of points to which the state may travel: a three-point approximation and a five-point approximation. First suppose that at any $(x_1, x_2) = (\Delta_1 i, \Delta_2 j) \in S_h$, the transitions assume only three values:

$$(\Delta x_1, \Delta x_2) \in \{(0, 0), (\Delta_1 m_1(i, j), \Delta_2 m_2(i, j)), (-\Delta_1 m_1(i, j), -\Delta_2 m_2(i, j))\}$$

for some m_2 and m_1. If the non-zero values occur with equal probability $\overline{p}$ then the mean consistency requirement is satisfied for all (m_1, m_2), and the covariance consistency requirements are

$$\begin{aligned} 2\overline{p}\Delta_1^2 m_1^2 &= \Delta_t \Delta_1^2 \overline{\sigma}_1^2 + o(\Delta_t) \\ 2\overline{p}\Delta_1 \Delta_2 m_1 m_2 &= \Delta_t \Delta_1 \Delta_2 \overline{\sigma}_1 \overline{\sigma}_2 + o(\Delta_t) \\ 2\overline{p}\Delta_2^2 m_2^2 &= \Delta_t \Delta_2^2 \overline{\sigma}_2^2 + o(\Delta_t). \end{aligned} \tag{40}$$

First note that if we can find non-zero integers satisfying $m_2 = m_1 \overline{\sigma}_2/\overline{\sigma}_1$ then (40) becomes

$$\begin{aligned} 2\overline{p}\Delta_1^2 m_1^2 &= \Delta_t \Delta_1^2 \overline{\sigma}_1^2 + o(\Delta_t) \\ 2\overline{p}\Delta_1 \Delta_2 m_1^2 \overline{\sigma}_2/\overline{\sigma}_1 &= \Delta_t \Delta_1 \Delta_2 \overline{\sigma}_1 \overline{\sigma}_2 + o(\Delta_t) \\ 2\overline{p}\Delta_2^2 m_1^2 \overline{\sigma}_2^2/\overline{\sigma}_1^2 &= \Delta_t \Delta_2^2 \overline{\sigma}_2^2 + o(\Delta_t) \end{aligned}$$

which will be satisfied with zero o terms if $\Delta_t = 2\overline{p}m_1^2/\overline{\sigma}_1^2 = 2\overline{p}m_2^2/\overline{\sigma}_2^2$. Motivated by this observation, in general we choose a non-zero integer pair to minimize $|m_2 - m_1\overline{\sigma}_2/\overline{\sigma}_1|$ and adjust Δ_t such that either the first or third requirement in (40) holds. We proceed on a case-by-case basis to ensure the timestep never vanishes: if $m_1 \geq m_2$, set $\Delta_t = 2\overline{p}m_1^2/\overline{\sigma}_1^2$ and if $m_2 > m_1$, set $\Delta_t = 2\overline{p}m_2^2/\overline{\sigma}_2^2$.

Figure 4 depicts the above process for selecting integer transitions. The slope of the line in the figure is $\overline{\sigma}_2/\overline{\sigma}_1$, the red dots represent the possible transitions satisfying (39), and the black dot

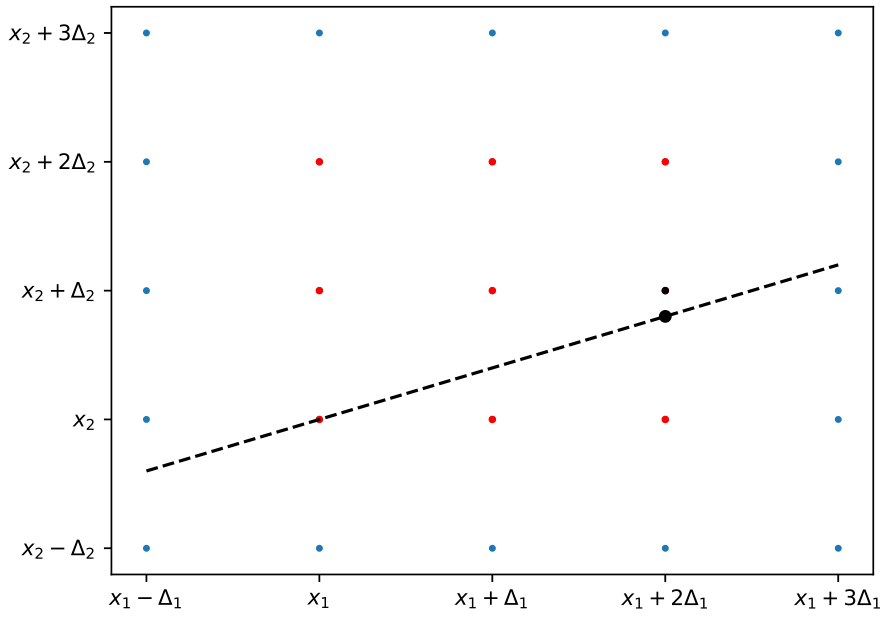

Figure 4: Optimal transition selection

on the grid represents the (m_1, m_2) selected. This picture also suggests that the approximation may be made more accurate by placing some probability on a point adjacent to the black dot. To formalize this idea, first define the candidate o terms in (40)

$$e_1 = 2\overline{p}\Delta_1^2 m_1^2 - \Delta_t \Delta_1^2 \overline{\sigma}_1^2$$
$$e_2 = 2\overline{p}\Delta_1\Delta_2 m_1 m_2 - \Delta_t \Delta_1 \Delta_2 \overline{\sigma}_1 \overline{\sigma}_2$$
$$e_3 = 2\overline{p}\Delta_2^2 m_2^2 - \Delta_t \Delta_2^2 \overline{\sigma}_2^2$$

and expand the possible transitions to five points:

$$(\Delta x_1, \Delta x_2) \in \{(0,0), \pm(\Delta_1 m_{11}, \Delta_2 m_{12}), \pm(\Delta_1 m_{21}, \Delta_2 m_{22})\}$$

for some quadruple $m = (m_{11}, m_{12}, m_{21}, m_{22})$. For some $\overline{p} \in (0, 1/2)$, we declare the probability of $(\Delta x_1, \Delta x_2) \in \{(\Delta_1 m_{11}, \Delta_2 m_{12}), (\Delta_1 m_{21}, \Delta_2 m_{22})\}$ equal to $\overline{p}$, and the probability of $(\Delta x_1, \Delta x_2) = (0,0)$ equal to $1 - 2\overline{p}$. Given $(i\Delta_1, j\Delta_2) \in S_h$ and recalling $w := \overline{\sigma}_1/\overline{\sigma}_2$, define

$$z = \min(1, w)(m_2 - m_1/w)$$
$$(m_1, m_2) = \operatorname{argmin}\left\{|j' - i'/w| \mid (i', j') \in \Gamma(i, j)\right\}$$

The first integer pair will the minimizer $m_1 = (m_1, m_2)$ specified in the above three-point approximation. Geometrically, one may interpret z as the distance between the two black dots in Figure 4. The selection of our second point m_2 will depend upon the signs of z and $w-1$, as this determines if the desired adjacent point lies to the left, right, above or below the original point:

- if $w > 1$ and $z > 0$ then $(m_{21}, m_{22}) = (m_{11}, m_{12} - 1)$;
- if $w > 1$ and $z \leq 0$ then $(m_{21}, m_{22}) = (m_{11}, m_{12} + 1)$;
- if $w \leq 1$ and $z > 0$ then $(m_{21}, m_{22}) = (m_{11} + 1, m_{12})$; and
- if $w \leq 1$ and $z \leq 0$ then $(m_{21}, m_{22}) = (m_{11} - 1, m_{12})$.

This selection process may be summarized as

$$m_{21} = m_{11} + (w \leq 1)(2(z > 0) - 1)$$
$$m_{22} = m_{12} + (w > 1)(2(z \leq 0) - 1).$$

We place probability $\overline{p}(1 - |z|)$ on m_1 and $\overline{p}|z|$ on m_2, the timestep given by

$$\Delta_t = 2\overline{p}\left((w > 1)\frac{(m_{11})^2}{\sigma_1^2/\Delta_1^2} + (w \leq 1)\frac{(m_{12})^2}{\sigma_2^2/\Delta_2^2}\right). \tag{41}$$

The description and accuracy of this five-point approximation is summarized in the following.

Lemma 4.2 (Five-point approximation)**.** *Given $\overline{p} \in (0, 1/2)$, define an approximation to the process $(dx_{1t}, dx_{2t}) = (\sigma_1(x_t)dZ_t, \sigma_2(x_t)dZ_t)$ by declaring, for $x \in S_h$, $p(x_1, x_2) = 1 - 2\overline{p}$ and*

$$p(x_1 \pm m_{11}\Delta_1, x_2 \pm m_{12}\Delta_2) = \overline{p}|z|$$
$$p(x_1 \pm m_{12}\Delta_1, x_2 \pm m_{22}\Delta_2) = \overline{p}(1 - |z|)$$

where $z = \min(1, w)(m_{12} - m_1^{(1)}/w)$ for $w = (\sigma_1/\Delta_1)/(\sigma_2/\Delta_2)$, the transitions satisfy

$$(m_{11}, m_{12}) = \operatorname{argmin}\left\{|j' - i'/w| \mid (i', j') \in \Gamma(i, j)\right\}$$
$$(m_{21}, m_{22}) = (m_{11} + (w \leq 1)(2(z > 0) - 1), m_{12} + (w > 1)(2(z \leq 0) - 1))$$

and Δ_t is given by (41)*. Two of the three local consistency requirements may be satisfied exactly, and the remaining requirement has relative error proportional to $2\overline{p}\Delta_2^2 z(1 - z)/m_1^2$.*

Proof. We proceed on a case-by-case basis, depending upon the sign of $w-1$. The errors are

$$\begin{aligned} e_{11} &= \mathbb{E}[(\Delta x_1)^2] - \Delta_t \sigma_1^2 \\ e_{12} &= \mathbb{E}[(\Delta x_1)(\Delta x_2)] - \Delta_t \sigma_1 \sigma_2 \\ e_{22} &= \mathbb{E}[(\Delta x_2)^2] - \Delta_t \sigma_2^2. \end{aligned} \tag{42}$$

Condition $w > 1$ is equivalent to $\sigma_1/\Delta_1 > \sigma_2/\Delta_2$ or $\sigma_1/\sigma_2 > \Delta_1/\Delta_2$. In this case $m_{11} = m_{21} = m_1$ and $\Delta_t = 2\overline{p}\Delta_1^2 m_1^2/\sigma_1^2$, so the first term in (42) is $e_{11} = 2\overline{p}\Delta_1^2 m_1^2 - \Delta_t \sigma_1^2 = 0$. We also have $m_{12} = m_2$ and $m_{22} = m_{12}^{(1)} + 2(z \leq 0) - 1$, so the second term in (42) becomes

$$\begin{aligned} e_{12} &= 2\overline{p}\Delta_1\Delta_2 m_1((1-|z|)m_2 + |z|[m_2 + 2(z \leq 0) - 1]) - \Delta_t \sigma_1 \sigma_2 \\ &= 2\overline{p}\Delta_1\Delta_2 m_1(|z|[2(z \leq 0) - 1] + z) = 0. \end{aligned}$$

Finally, the third error term in (42) simplifies to

$$\begin{aligned} e_{22} &= 2\overline{p}\Delta_2^2\big(\eta m_2^2 + (1-\eta)[m_2 + 2(z \leq 0) - 1]^2\big) - \Delta_t \sigma_2^2 \\ &= 2\overline{p}\Delta_2^2\big[m_2^2 + (1-\eta)[2(2(z \leq 0) - 1)m_2 + (2(z \leq 0) - 1)^2] - \sigma_2^2(\Delta_1^2/\Delta_2^2)m_1^2/\sigma_1^2\big] \\ &= 2\overline{p}\Delta_2^2\big[|z|[2(2(z \leq 0) - 1)m_2 + (2(z \leq 0) - 1)^2] + m_2^2 - m_1^2/w^2\big]. \end{aligned}$$

Using $z = m_2 - m_1/w$ we write $(m_1/w)^2 = (m_2 - z)^2 = m_2^2 - 2m_2 z + z^2$ to note

$$e_{22} = 2\overline{p}\Delta_2^2\big[|z|[(4(z \leq 0) - 2)m_2 + 1] + 2m_2 z - z^2\big] = 2\overline{p}\Delta_2^2(|z| - |z|^2)$$

as claimed. The case with $w \leq 1$ is symmetric. □

The accuracy of the the above method is verified for a number of linear-quadratic problems in Section A.1. Prior to applying this algorithm to the problem (35), we outline two simplifications. By Lemma 4.1 the value function of each agent satisfies

$$\begin{aligned} aW_{12}(a,x,\sigma) &= V_1(x,\sigma)a^{1-\gamma} = (1-\gamma)W_2(a,x,\sigma) \\ aW_{13}(a,x,\sigma) &= V_2(x,\sigma)a^{1-\gamma} = (1-\gamma)W_3(a,x,\sigma). \end{aligned}$$

Substituting into (35) then gives

$$\begin{aligned} \rho W = \max_{\overline{c},\overline{k} \geq 0} \; & \frac{\rho(\overline{c}a)^{1-\gamma}}{1-\gamma} + \big[r - \overline{c} + (\Pi - r)\overline{k}\big]aW_1 + \frac{\sigma^2\overline{k}^2}{2}a^2 W_{11} + [\mu_x x + \sigma_x x \sigma \overline{k}(1-\gamma)]W_2 \\ & + \frac{\sigma_x^2 x^2}{2}W_{22} + [\theta(\overline{\sigma} - \sigma) + \sigma\overline{k}\sigma_\sigma(1-\gamma)]W_3 + \sigma_x x \sigma_\sigma W_{23} + \frac{\sigma_\sigma^2}{2}W_{33}. \end{aligned}$$

For convenience we define $y_t := \ln a_t$ and note that Ito's lemma implies

$$dy_t = \Big(r - \overline{c}_t + (\Pi - r)\overline{k}_t - \sigma^2\overline{k}_t^2/2\Big)dt + \sigma\overline{k}_t dZ_t.$$

We obtain a control problem with state (y, x, σ), controls (c, k), flow payoffs $\rho\overline{c}^{1-\gamma}e^{(1-\gamma)y}/(1-\gamma)$ and law of motion of the state

$$\begin{aligned} dy_t &= \Big(r - \overline{c}_t + (\Pi - r)\overline{k}_t - \sigma_t^2\overline{k}_t^2/2\Big)dt + \sigma_t\overline{k}_t dZ_t^{(1)} \\ dx_t &= \big(\mu_x + \sigma_t\sigma_x(1-\gamma)\overline{k}_t\big)x_t dt + \sigma_x x_t dZ_t^{(2)} \\ d\sigma_t &= \big(\theta(\overline{\sigma} - \sigma_t) + \sigma_t\sigma_\sigma(1-\gamma)\overline{k}_t\big)dt + \sigma_\sigma dZ_t^{(2)} \end{aligned} \tag{43}$$

where $(Z^{(1)}, Z^{(2)})$ are independent Brownian motions. We emphasize that this system is not the original one faced by the agent, but as the above homogeneity arguments show, it leads to the same value function. This does not allow us to completely eliminate the high degree of correlation between the state variables and so the solution of the associated portfolio problems requires non-local transitions. However, crucially, the diffusion terms exhibiting high correlation are not controlled by the agent. We now outline our approximation to (43). First define the infinite grid $S_y = \{\dots, -\Delta_y, 0, \Delta_y, \dots\}$ for log wealth, and the finite grids

$$\begin{aligned} S_x &= \{0, \Delta_x, \dots, 1 - \Delta_x, 1\} \\ S_\sigma &= \{\underline{\Sigma}, \underline{\Sigma} + \Delta_\sigma, \dots, \overline{\Sigma} - \Delta_\sigma, \overline{\Sigma}\} \end{aligned} \tag{44}$$

for the aggregate state, where $\Delta_x = (1-0)/N_x$ and $\Delta_\sigma = (\overline{\Sigma} - \underline{\Sigma})/N_\sigma$, and then define $S = S_y \times S_x \times S_\sigma$. To ensure the process remains on (44) we alter (31) so that σ_σ vanishes at $\underline{\Sigma}$ and $\overline{\Sigma}$. For clarity, we write the transition probabilities as the sum of $p^{(1)}, p^{(2)}$ and $p^{(3)}$, associated with the transitions for individual wealth, and the drift and diffusion for the aggregate state, respectively. The transition probabilities at any point $X' = (y', x', \sigma') \neq X = (y, x, \sigma)$ are then

$$p(X') = p^{(1)}(X') + p^{(2)}(X') + p^{(3)}(X')$$

with $p(X)$ chosen such that probabilities sum to unity. The transitions in log wealth are

$$\begin{aligned} p^{(1)}(y + \Delta_y, x, \sigma) &= \frac{\Delta_t}{\Delta_y^2}\Big(\sigma^2\overline{k}^2/2 + \Delta_y\big[r + (\Pi - r)\overline{k}\big]\Big) \\ p^{(1)}(y - \Delta_y, x, \sigma) &= \frac{\Delta_t}{\Delta_y^2}\Big(\sigma^2\overline{k}^2/2 + \Delta_y\Big[\overline{c} + \sigma^2\overline{k}^2/2\Big]\Big) \end{aligned} \tag{45}$$

for some Δ_t specified below. For the drift terms for the aggregate state we have

$$
\begin{aligned}
p^{(2)}(y, x + \Delta_x, \sigma) &= \frac{\Delta_t}{\Delta_x}\big([\mu_x x]^+ + \sigma_x x \sigma(1-\gamma)\overline{k}_t\big) \\
p^{(2)}(y, x - \Delta_x, \sigma) &= \frac{\Delta_t}{\Delta_x}[\mu_x x]^- \\
p^{(2)}(y, x, \sigma + \Delta_\sigma) &= \frac{\Delta_t}{\Delta_\sigma}\big([\theta(\overline{\sigma} - \sigma)]^+ + \sigma_\sigma \sigma(1-\gamma)\overline{k}_t\big) \\
p^{(2)}(y, x, \sigma - \Delta_\sigma) &= \frac{\Delta_t}{\Delta_\sigma}[\theta(\overline{\sigma} - \sigma)]^+.
\end{aligned}
$$

For the diffusion component we follow the non-local selection criteria

$$
\begin{aligned}
p^{(3)}(y, x \pm m_{12}\Delta_x, \sigma \pm m_{13}\Delta_\sigma) &= \overline{p}|z| \\
p^{(3)}(y, x \pm m_{22}\Delta_x, \sigma \pm m_{23}\Delta_\sigma) &= \overline{p}(1 - |z|)
\end{aligned}
$$

where the transitions and weights are given by Lemma 4.2. Using $W(X) = V(x,\sigma)e^{(1-\gamma)y}/(1-\gamma)$ and eliminating probabilities independent of controls, the maximization in the Bellman equation for the discretized problem becomes

$$
\begin{aligned}
&\max_{\overline{c},\overline{k}\geq 0} \Delta_t \frac{\rho \overline{c}^{1-\gamma}}{1-\gamma} + e^{-\rho\Delta_t}\left[p^{(2)}(y, x+\Delta_x, \sigma)\frac{\Delta_x V^{F,x}}{1-\gamma} + p^{(2)}(y, x, \sigma+\Delta_\sigma)\frac{\Delta_\sigma V^{F,\sigma}}{1-\gamma}\right] \\
&+ e^{-\rho\Delta_t}\Big[p^{(1)}(y+\Delta_y, x, \sigma)[e^{(1-\gamma)\Delta_y} - 1] + p^{(1)}(y-\Delta_y, x, \sigma)[e^{-(1-\gamma)\Delta_y} - 1]\Big]\frac{V}{1-\gamma}
\end{aligned}
$$

Dividing by $\Delta_t e^{-\rho\Delta_t}$, eliminating terms independent of the controls and abbreviating gives

$$
\begin{aligned}
0 = &\max_{\overline{c},\overline{k}\geq 0} e^{\rho\Delta_t}\frac{\rho\overline{c}^{1-\gamma}}{1-\gamma} + \frac{[e^{-(1-\gamma)\Delta_y} - 1]}{(1-\gamma)\Delta_y}V\overline{c} + (\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma\overline{k} \\
&+ \frac{1}{\Delta_y^2}\Big([e^{(1-\gamma)\Delta_y} - 1]\Big(\sigma^2\overline{k}^2/2 + \Delta_y(\Pi - r)\overline{k}\Big) + (1+\Delta_y)[e^{-(1-\gamma)\Delta_y} - 1]\sigma^2\overline{k}^2/2\Big)\frac{V}{1-\gamma}.
\end{aligned} \tag{46}
$$

We then have the following optimal policy functions. The proof and expressions for the constants are in Section B.2.

Lemma 4.3. *For $i \in \{E, H\}$ the policy functions for consumption are $\overline{c}_i = \rho_i^{1/\gamma} V_i^{-1/\gamma} e^{\rho\Delta_t/\gamma} E_c(\Delta_y)$ and the capital policy function for the expert is*

$$
\overline{k} = (E_1/E_2)\max\left\{\frac{\Pi - r}{\gamma\sigma^2} + \frac{\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}}{E_1 \gamma \sigma V}, 0\right\}
$$

where E_1, E_2 and E_c depend only upon Δ_y, Δ_t and ρ and tend to unity as $\Delta_y \to 0$.

4.4 Market clearing and consistency

Lemma 4.3 calculated policy functions for a discretized version of the problem of an agent facing given continuation values, risk-free rate, and law of motion of the wealth share. We wish to construct a map that iterates upon the continuation values of each agent, and so now impose two requirements: the market for bonds clears; and the law of motion of the aggregate state is consistent with individual policy functions. Using Lemma 4.3, the bond market-clearing condition becomes

$$(E_1/E_2)\max\left\{\frac{\Pi - r}{\gamma\sigma^2} + \frac{\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}}{E_1 \gamma \sigma V}, 0\right\} x \leq 1 \tag{47}$$

with equality if $r > \underline{r}$. Note the inequality in (47) may be strict if the storage technology is utilized in equilibrium. The left-hand side of (47) is decreasing in r so there are two cases to consider: if $\overline{k}(\underline{r})x \leq 1$ then $r = \underline{r}$; otherwise r solves $\overline{k}(r)x = 1$. Rearranging gives

$$r = \max\left\{\underline{r}, \Pi + \frac{\sigma}{E_1 V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}) - \frac{\gamma\sigma^2}{x}(E_1/E_2)^{-1}\right\}. \tag{48}$$

Substituting (48) into the expression for capital in Lemma 4.3 gives

$$\overline{k} = \max\left\{\min\left\{(E_1/E_2)\left(\frac{\Pi - \underline{r}}{\gamma\sigma^2} + \frac{E_1^{-1}}{\gamma\sigma V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\right), \frac{1}{x}\right\}, 0\right\}. \tag{49}$$

Expressions (48) and (49) give the risk-free rate and capital policy function consistent with bond market-clearing, given continuation values and law of motion for the wealth share. We now impose consistency between individual and aggregate laws of motion to obtain a single map defined on continuation values. Lemma B.1 gives the drift and diffusion terms for the wealth share in terms of the law of motion for wealth shares of the individual agents. Substitution of the above policy functions then gives the following.

Lemma 4.4. *If $V_E^{F,x} < 0$ everywhere then*

$$\sigma_x x = \sigma x(1-x)\max\left\{\min\left\{\frac{(E_1/E_2)[\Pi - \underline{r}]/\sigma^2 + E_2^{-1}\sigma_\sigma V_E^{F,\sigma}/[\sigma V_E]}{\gamma + (1-x)E_2^{-1}x[-V_E^{F,x}]/V_E}, \frac{1}{x}\right\}, 0\right\}, \tag{50}$$

and the drift in the wealth share is

$$\mu_x x = \left[\left(\rho_H^{1/\gamma}V_H^{-1/\gamma} - \rho_E^{1/\gamma}V_E^{-1/\gamma}\right)E_c(\Delta_y) + (\Pi - r)\overline{k} - \sigma^2\overline{k}^2 x\right]x(1-x)$$

where $\overline{k}$ is given by (49).

Proof. Equating the volatility of the wealth share with that implied by policy functions gives

$$\sigma_x x = \sigma x(1-x)\max\left\{\min\left\{(E_1/E_2)\frac{[\Pi - \underline{r}]}{\gamma\sigma^2} + \frac{E_2^{-1}}{\gamma\sigma V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}), \frac{1}{x}\right\}, 0\right\}.$$

In the absence of the maximum and minimum operators σ_x would solve

$$\sigma_x = \frac{(1-x)\big((E_1/E_2)[\Pi - \underline{r}]/\sigma + E_2^{-1}\sigma_\sigma V^{F,\sigma}/V\big)/\gamma}{1 + (1-x)E_2^{-1}x[-V^{F,x}]/[\gamma V]}$$

which gives the desired expression from case-by-case analysis. □

Given arbitrary continuation values and functions for the risk-free rate and drift and diffusion of the wealth share, Lemma 4.3 gives the optimal policy functions for the discretized problem. Expression (48) then gives the market-clearing interest rate as a function of the law of motion of wealth. Lemma 4.4 then imposes consistency of individual and aggregate laws of motion.

Prior to summarizing the equilibrium we show that one may proceed as in Section 2 and simplify our problem by considering the limit as $\overline{p}$ (and hence Δ_t) and Δ_y to zero. We subtract the value function from both sides of the discretized Bellman equation and divide by the timestep to obtain a functional equation of the form $0 = \max_{\overline{c},\overline{k}} \rho u(\overline{c}) + T(\overline{c},\overline{k};\overline{p},\Delta_y)V$ for some operator T. As with the growth model in Section 2, when $\overline{p}, \Delta_y \to 0$ we have convergence to a non-trivial operator.

Lemma 4.5. *For any $\overline{c}, \overline{k}$ we have $\lim_{\overline{p},\Delta_y \to 0} T(\overline{c},\overline{k};\overline{p},\Delta_y) = \overline{T}(\overline{c},\overline{k})$ where*

$$\begin{aligned}
\overline{T}(\overline{c},\overline{k})V = &-\Big(\rho - (1-\gamma)(r - \overline{c} + (\Pi - r)\overline{k} - \gamma\sigma^2\overline{k}^2/2)\Big)\frac{V}{1-\gamma} + \big([\mu_x x]^+ + \sigma_x x\sigma\overline{k}\big)\frac{V^{F,x}}{1-\gamma} \\
&+ \big([\theta(\overline{\sigma} - \sigma)]^+ + \sigma_\sigma\sigma\overline{k}\big)\frac{V^{F,\sigma}}{1-\gamma} + [\mu_x x]^-\frac{[-V^{B,x}]}{1-\gamma} + [\theta(\overline{\sigma} - \sigma)]^-\frac{[-V^{B,\sigma}]}{1-\gamma} \\
&+ \frac{(1-|z|)}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{12}\Delta_x, \sigma + m_{13}\Delta_\sigma) + V(x - m_{12}\Delta_x, \sigma - m_{13}\Delta_\sigma) - 2V) \\
&+ \frac{|z|}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{22}\Delta_x, \sigma + m_{23}\Delta_\sigma) + V(x - m_{22}\Delta_x, \sigma - m_{23}\Delta_\sigma) - 2V).
\end{aligned}$$

Further, if $\rho > (1-\gamma)\Big(r - \overline{c} + (\Pi - r)\overline{k} - \gamma\sigma^2\overline{k}^2/2\Big)$ then the above algorithm is globally convergent.

A proof of Lemma 4.5 is contained in Section B.2. We can now simplify the expressions in Lemma 4.3 and Lemma 4.4 by setting $E_1, E_2, E_c = 1$ and summarize them as follows.

Proposition 4.6. *The policy functions for consumption are $\overline{c}_i = \rho_i^{1/\gamma} V_i^{-1/\gamma}$ for $i \in \{E, H\}$ and the capital policy function for the expert is*

$$\overline{k} = \frac{1}{\gamma\sigma^2}\max\big\{\Pi - r + \sigma_x x\sigma V^{F,x}/V + \sigma_\sigma\sigma V^{F,\sigma}/V, 0\big\}$$

If $V_E^{F,x} < 0$ everywhere then the volatility of the wealth share satisfies

$$\sigma_x x = \sigma x(1-x)\max\left\{\min\left\{\frac{[\Pi - \underline{r}]/\sigma^2 + \sigma_\sigma V_E^{F,\sigma}/[\sigma V_E]}{\gamma + (1-x)x[-V_E^{F,x}]/V_E}, 1\right\}, 0\right\}. \tag{51}$$

For this value, the interest rate is

$$r = \max\left\{\underline{r}, \Pi + \frac{\sigma}{V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}) - \frac{\gamma\sigma^2}{x}\right\}.$$

Substituting in the above expression for capital then gives the drift in the wealth share is

$$\mu_x x = \left[\rho_H^{1/\gamma} V_H^{-1/\gamma} - \rho_E^{1/\gamma} V_E^{-1/\gamma} + (\Pi - r)\overline{k} - \sigma^2 \overline{k}^2 x\right] x(1-x).$$

4.5 Summary and example

We now summarize the algorithm for computing the competitive equilibrium. For the following write $S_{x\sigma} := S_x \times S_\sigma$ for S_x and S_σ given in (44).

Algorithm 4.7 (Individual problems)**.** *Given grid values for the risk-free rate and wealth share*

$$\{r(x,\sigma), \mu_x(x,\sigma), \sigma_x(x,\sigma)\}_{(x,\sigma)\in S_{x\sigma}},$$

for each $i \in \{E, H\}$ the value and policy functions are found as follows:

1. *Define consumption and expert investment using Proposition 4.6.*
2. *For these policy functions, solve $0 = \rho u(\overline{c}) + \overline{T}(\overline{c}, \overline{k})V$ for V, where $\overline{T}$ is given in Lemma 4.5.*
3. *Return to Step 1 and repeat until convergence.*

Algorithm 4.8 (Computation of competitive equilibrium)**.** *Given an initial guess*

$$V = (V_E(x,\sigma), V_H(x,\sigma))_{(x,\sigma)\in S_{x\sigma}}$$

for the value functions for each agent, an approximate equilibrium is found as follows:

1. *Given the value functions of the agents, calculate the risk-free rate and law of motion of the wealth share using Proposition 4.6.*
2. *Using the prices and law of motion found in Step 1, use Algorithm 4.7 to update V_E and V_H.*
3. *Return to Step 1 and repeat until convergence.*

Numerical illustration: we have calculated an example for the following parameters:

$$(\gamma = 0.5, \rho = (0.2, 0.1), \theta = 1, (\underline{\Sigma}, \overline{\Sigma}) = (0.1, 0.4), \overline{\sigma} = 0.25,$$
$$\sigma_\sigma = 0.2, \Pi_E = 0.065, \Pi_H = \underline{r} = 0, N = (160, 80), \overline{m} = 4).$$

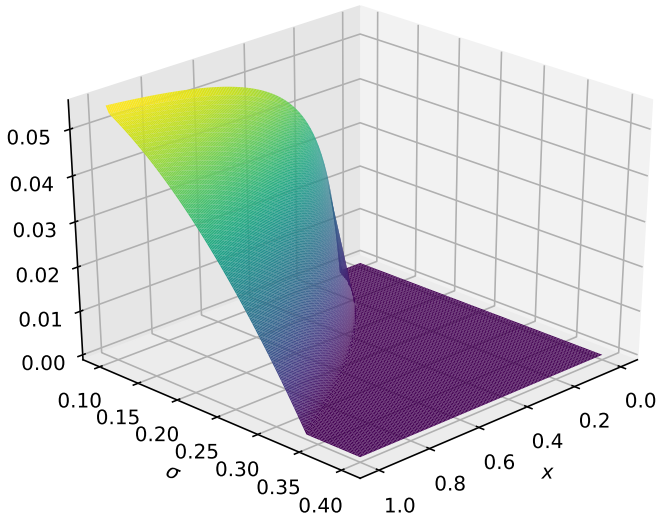

Figure 5: Interest rate

With a tolerance of 10^{-6} for both the consumer problem and the updating step for the aggregate law, it converges in less than 7 seconds beginning from an initial guess in which r, μ_x and σ_x are identically zero, which appears to make this an order of magnitude faster than the finite-difference scheme employed in D'Avernas and Vandeweyer (2019). Figure 5 gives the interest rate, Figure 6 gives the drift in the wealth share, and Figure 7 gives the volatility of the wealth share.

5 Conclusion

In this paper we explore several applications of the Markov-chain approximation (MCA) method of Harold Kushner and Paul Dupuis to optimal control problems in economics, illustrating some unutilized benefits. We first show that for certain choices of approximating chain the MCA method with policy function iteration coincides with a limiting version of the implicit finite difference scheme of Achdou et al. (2017). We then demonstrate the benefits of a more general specification by means of two examples. In the first, we use variations of modified policy function iteration to solve income fluctuation problems, both with and without discrete choices, while in the second, we show how the MCA method allows for a systematic solution method for problems with high correlation amongst state variables. In both cases the MCA is robust, easy to apply and results in an increase in speed

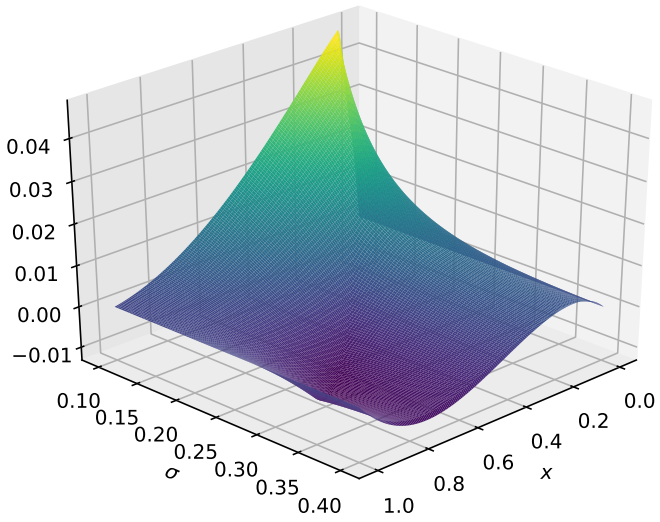

Figure 6: Drift in wealth share $\mu_x x$

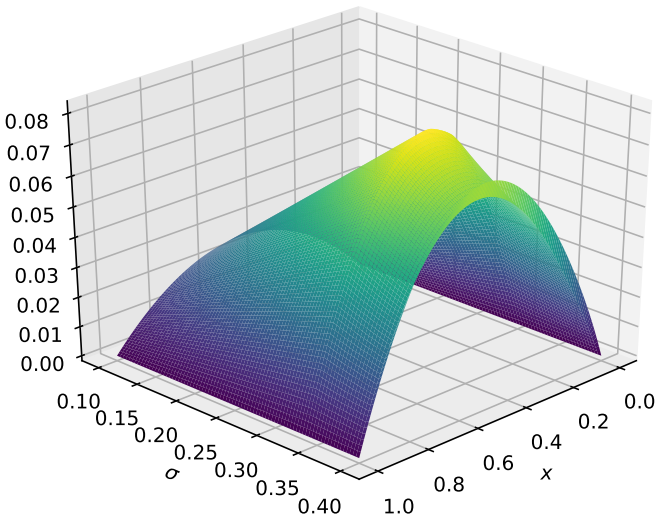

Figure 7: Drift in wealth share $\sigma_x x$

of more than an order of magnitude over finite-difference methods.

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A Linear-quadratic regulator problem

To verify the accuracy of the algorithms, this section records closed-form expressions for a class of linear-quadratic control problems. Lemma A.1 treats the standard case of an infinite-horizon linear-quadratic regulator problem, while Lemma A.3 treats the (slightly non-standard) case in which volatility is linear in the state variable. The former will be useful for calculating the benefits associated with modified policy function iteration while the latter will illustrate the applications of non-local transitions. Suppose that the objective to be maximized is

$$\int_0^\infty e^{-\rho t}\mathbb{E}[F(x_t,u_t)]dt$$

where for some symmetric positive definite matrices Q and R the flow payoffs are given by

$$F(x,u) = -\frac{1}{2}x^TQx - \frac{1}{2}u^TRu \tag{52}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$ are the state and control vectors, respectively, for some $n, q \geq 1$. Now suppose that for some$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q}$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ the law of motion for the state is

$$dx_t = [Ax_t + Bu_t]dt + \sigma(x)dZ_t$$

where $Z := (Z_t)_{t\geq 0}$ is m-dimensional Brownian motion. Write $\mu(x,u) = Ax + Bu$ for the drift as a function of the state and controls, and note that the Hamilton-Jacobi-Bellman equation is

$$\rho V(x) = \max_{u\in\mathbb{R}^q} F(x,u) + \sum_{i=1}^n \mu_i(x,u)V_i(x) + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}V_{ij}(x) \tag{53}$$

where $a_{ij}(x) = (\sigma(x)\sigma(x)^T)_{ij}$. If σ is constant then we obtain the following.

Lemma A.1. *The solution to* (53) *is* $V(x) = -x^TPx/2 - d$*, where* P *is a symmetric matrix that solves*

$$\rho P = Q + PA + A^TP - P^TBR^{-1}B^TP \tag{54}$$

the constant term is $d = [2\rho]^{-1}\text{trace}(\sigma\sigma^T P)$*, and the policy function is*

$$u = -R^{-1}B^T P x. \tag{55}$$

Corollary A.2. *The drift in the state variables* $(A - BR^{-1}B^T P)x$*.*

Note that P solves (54) if and only if it solves the undiscounted problem

$$0 = Q + P(A - \rho I/2) + (A - \rho I/2)^T P - P^T B R^{-1} B^T P$$

and so the analysis for the discounted case follows from results from the undiscounted case, with A replaced by $A - \rho I/2$. The standard linear-quadratic regulator problem analyzed in Lemma A.1 assumes constant volatility. To illustrate the flexibility of the method in treating some cases that appear difficult with finite-differences, we consider here an extension of the standard linear-quadratic framework in which the volatility is linear in the state variables. The flow payoff remains (52) for some symmetric positive definite matrices Q and R, but the law of motion for the state variables is now

$$dx_t = [Ax_t + Bu_t]dt + \sigma x_t dZ_t$$

where $\sigma \in \mathbb{R}^{n\times n}$ (and so $\sigma x_t \in \mathbb{R}^2$), and Z is scalar Brownian motion.

Lemma A.3. *If* σ *is a multiple of the identity then the value function is* $V(x) = -x^T P x/2$ *where* P *solves*

$$0 = Q + P(A + [\sigma^2 - \rho]I/2) + (A + [\sigma^2 - \rho]I/2)^T P - P^T B R^{-1} B^T P. \tag{56}$$

Lemma A.3 will be useful to illustrate how the FSMC method may be used to deal with problems for with the covariance matrix is degenerate, a case that appears difficult to analyze via other means.

We now use the closed-form expressions for value functions and policy functions in Lemma A.1 and Lemma A.3 to verify the accuracy of the Markov chain approximation method. There are many tests that we could conduct. We choose only a select few to illustrates the points highlighted in the main text and to provide confidence in the results recorded in the main text. We consider domains in $\mathbb{R}^3$ of the form $[0, M_0] \times [0, M_1] \times [0, M_2]$ for constants M_0, M_1 and M_2. Now define $\Delta_i = M_i/N_i$ for $i = 0, 1, 2$ and

$$\begin{aligned} S_0 &= \{\Delta_0, \ldots, M_0 - \Delta_0\} \\ S_1 &= \{\Delta_1, \ldots, M_1 - \Delta_1\} \\ S_2 &= \{\Delta_2, \ldots, M_2 - \Delta_2\}. \end{aligned} \tag{57}$$

We then let our grids be r $S = S_0 \times S_1 \times S_2$ and adopt the transition probabilities

$$\begin{aligned} p(x_0 \pm \Delta_0, x_1, x_2) &= \frac{\Delta_t}{\Delta_0^2}\left(\frac{\sigma_0^2}{2} + \Delta_0 (Ax + Bu)_0^{\pm}\right) \\ p(x_0, x_1 \pm \Delta_1, x_2) &= \frac{\Delta_t}{\Delta_1^2}\left(\frac{\sigma_1^2}{2} + \Delta_1 (Ax + Bu)_1^{\pm}\right) \\ p(x_0, x_1, x_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_1^2}\left(\frac{\sigma_1^2}{2} + \Delta_1 (Ax + Bu)_1^{\pm}\right). \end{aligned} \tag{58}$$

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	0.134048	1.458656	0.196963	0.082000	0.075110
(20, 20, 20)	1.928055	14.032890	1.725019	0.352964	0.219006
(30, 30, 30)	43.107914	76.018729	8.671721	2.081123	1.427002
(40, 40, 40)	264.771303	373.741105	38.424432	8.332970	4.794629
(50, 50, 50)	1164.369288	1145.196087	133.544498	32.746298	21.240257

Figure 8: Time until convergence

We wish to avoid case-by-case technicalities that distract us from the main goal and so consider problems with a single control. We normalize $R = 1$ and write $B = [b_0, b_1, b_2]^T$. The Riccati equation and policy function from Lemma A.1 become

$$\begin{aligned} \rho P &= Q + PA + A^T P - P^T B B^T P \\ u &= -B^T P x. \end{aligned} \tag{59}$$

We choose parameters for which the drift is always negative. Using (57) the maximization becomes gives

$$\max_{u \leq 0} \; -\frac{1}{2}u^2 + e^{-\rho \Delta_t}\big(b_0 V^{B0} + b_1 V^{B1} + b_2 V^{B2}\big)u$$

and so the optimal control is obvious

$$u = \min\{e^{-\rho \Delta_t}\big(b_0 V^{B0} + b_1 V^{B1} + b_2 V^{B2}\big), 0\}. \tag{60}$$

Notice that (57) and (60) are valid regardless of whether or not the timestep is state-dependent, or zero. We will choose our timestep to be as large as possible whilst ensuring the expressions for probabilities lie in the unit interval. If we restrict attention to controls such that $Ax + Bu \leq 0$ in each component the above probabilities will lie in the unit interval provided

$$1 \geq \Delta_t \left(\frac{\sigma_0^2}{\Delta_0^2} + \frac{\sigma_1^2}{\Delta_1^2} + \frac{\sigma_2^2}{\Delta_2^2} + \frac{(Ax + Bu)_0^-}{\Delta_0} + \frac{(Ax + Bu)_1^-}{\Delta_1} + \frac{(Ax + Bu)_2^-}{\Delta_2} \right).$$

To use the above to obtain an appropriate state-dependent timestep we require a bound on the control u. For this we will choose u to be $3 \times$ the true optimal policy function. Table 8 and Table 9 document the time until convergence for modified policy function iteration. Table 10 and Table 11 perform the analogous exercise for the generalized algorithm.

A.1 Degenerate covariance matrix

We now verify the accuracy of the Markov chains constructed in Section 4.3. In all of the following cases we set Q equal to the identity, omit drift and controls, set $(\sigma = 0.3, \rho = 0.15)$ and $M_1 = M_2 = 1$. Table 12

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	6.181670	6.181670	6.181670	6.181670	6.181670
(20, 20, 20)	2.531377	2.531377	2.531377	2.531377	2.531377
(30, 30, 30)	1.467515	1.467513	1.467515	1.467515	1.467515
(40, 40, 40)	0.999333	0.999330	0.999333	0.999333	0.999333
(50, 50, 50)	0.746817	0.746813	0.746816	0.746817	0.746817

Figure 9: Accuracy

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	0.111000	0.951169	0.110277	0.089105	0.099435
(20, 20, 20)	1.858931	9.828197	1.134998	0.302998	0.279003
(30, 30, 30)	26.876839	95.197244	10.406487	3.235998	1.526995
(40, 40, 40)	259.773637	408.568136	37.852798	10.821395	5.865596
(50, 50, 50)	1159.343633	1253.724452	129.647881	29.516229	17.856800

Figure 10: Generalize MPFI: Time until convergence

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	1.921812	1.921812	1.921812	1.921812	1.921812
(20, 20, 20)	1.117398	1.117398	1.117398	1.117398	1.117398
(30, 30, 30)	0.781886	0.781885	0.781886	0.781886	0.781886
(40, 40, 40)	0.600415	0.600412	0.600415	0.600415	0.600415
(50, 50, 50)	0.487141	0.487137	0.487141	0.487141	0.487141

Figure 11: Generalize MPFI: Accuracy

	m = 2	m = 4	m = 6	m = 8	m = 10
(50, 50)	8.854791	3.848835	3.576674	3.776008	4.239382
(100, 100)	7.616589	2.402118	1.857700	1.772860	1.855382
(150, 150)	7.202551	1.978822	1.385317	1.222015	1.202502
(200, 200)	7.042581	1.772902	1.143363	0.966804	0.910308
(250, 250)	6.904602	1.652381	1.009336	0.824529	0.745436
(300, 300)	6.841191	1.562504	0.927180	0.730902	0.647975

Figure 12: Accuracy with three points

	m = 2	m = 4	m = 6	m = 8	m = 10
(50, 50)	2.543416	0.571887	0.368057	0.428505	0.553616
(100, 100)	2.481971	0.486195	0.210015	0.166807	0.183523
(150, 150)	2.463159	0.462383	0.177174	0.115889	0.106008
(200, 200)	2.454088	0.453831	0.164089	0.096765	0.077508
(250, 250)	2.448759	0.448244	0.157307	0.087367	0.064290
(300, 300)	2.445255	0.444616	0.153302	0.081675	0.056718

Figure 13: Accuracy with five points

documents the L_1-norm of the percent difference between the true and computed value with three points, and Table 13 documents the same quantities for the case with five points.

B Macrofinance notes

B.1 Evolution of wealth shares

We now determine the law of motion for the wealth share x, by aggregating over the choices of experts and households. Lemma B.1 shows how the law of motion of the wealth share depends upon the law of motion of the wealth of the individual agents.

Lemma B.1. *Suppose the laws of motion of wealth for the two agents are*

$$\frac{da_E}{a_E} = \mu_E dt + \sigma_E dZ_t \qquad \frac{da_H}{a_H} = \mu_H dt + \sigma_H dZ_t \tag{61}$$

and that $x := N/(qK)$ denotes the wealth share of experts. Then the law of motion of $(x_t)_{t\geq 0}$ is given by

$$dx_t = \mu_x x_t dt + \sigma_x x_t dZ_t$$

where μ_x and σ_x are given by

$$\begin{aligned}\mu_x x &= \Big(\mu_E - \mu_H - (\sigma_E - \sigma_H)^2 x + \sigma_H[\sigma_H - \sigma_E]\Big)x(1-x)\\ \sigma_x x &= (\sigma_E - \sigma_H)x(1-x).\end{aligned}$$

Proof. Aggregating over (61) gives

$$\begin{aligned}\frac{dN_t}{N_t} &= \mu_E dt + \sigma_E dZ_t\\ d(q_t K_t) &= \mu_E N_t dt + \sigma_E N_t dZ_t + \mu_H(q_t K_t - N_t)dt + \sigma_H(q_t K_t - N_t)dZ_t\\ \frac{d(q_t K_t)}{q_t K_t} &= [\mu_E x_t + \mu_H(1-x_t)]dt + [\sigma_E x_t + \sigma_H(1-x_t)]dZ_t.\end{aligned} \tag{62}$$

Note that if $da_t/a_t = \mu_a dt + \sigma_a dZ_t$ and $db_t/b_t = \mu_b dt + \sigma_b dZ_t$, then $c_t = a_t/b_t$ satisfies $dc_t/c_t = (\mu_a - \mu_b + \sigma_b^2 - \sigma_a\sigma_b)dt + (\sigma_a - \sigma_b)dZ_t$. Applying to (62) gives

$$\begin{aligned}\frac{dx_t}{x_t} &= (\mu_E - [\mu_E x_t + \mu_H(1-x_t)] + [\sigma_E x_t + \sigma_H(1-x_t)]^2 - \sigma_E[\sigma_E x_t + \sigma_H(1-x_t)])dt\\ &\quad + (\sigma_E - [\sigma_E x_t + \sigma_H(1-x_t)])dZ_t\\ &= [\mu_E - \mu_H](1-x_t)dt + [\sigma_E x_t + \sigma_H(1-x_t)]^2 dt - \sigma_E^2 x_t dt\\ &\quad - \sigma_H\sigma_E(1-x_t)dt + [\sigma_E - \sigma_H](1-x_t)dZ_t\\ &= [\mu_E - \mu_H](1-x_t)dt + [\sigma_E^2 x_t^2 + 2\sigma_E x_t \sigma_H(1-x_t) + \sigma_H^2(1-x_t)^2]dt - \sigma_E^2 x_t dt\\ &\quad - \sigma_H\sigma_E(1-x_t)dt + [\sigma_E - \sigma_H](1-x_t)dZ_t.\end{aligned}$$

The drift becomes

$$\begin{aligned}\mu_x &= [\mu_E - \mu_H](1-x_t) + \sigma_E^2 x_t^2 + 2\sigma_E x_t \sigma_H(1-x_t) + \sigma_H^2(1-x_t)^2 - \sigma_E^2 x_t - \sigma_H\sigma_E(1-x_t)\\ &= [\mu_E - \mu_H](1-x_t) + \big(\sigma_E^2 - 2\sigma_E\sigma_H + \sigma_H^2\big)x_t(x_t-1) + \sigma_H^2[(1-x_t)^2 - x_t(x_t-1)] - \sigma_H\sigma_E(1-x_t)\\ &= [\mu_E - \mu_H](1-x_t) + (\sigma_E - \sigma_H)^2 x_t(x_t-1) + \sigma_H[\sigma_H - \sigma_E](1-x_t)\end{aligned}$$

as desired. □

B.2 Individual problems

Proof of Lemma 4.3. Eliminating the terms independent of capital and dividing by V gives

$$\begin{aligned}&\frac{1}{V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma\overline{k} + \frac{1}{\Delta_y(1-\gamma)}[e^{(1-\gamma)\Delta_y} - 1](\Pi - r)\overline{k}\\ &+ \frac{1}{\Delta_y^2(1-\gamma)}\Big(e^{(1-\gamma)\Delta_y} - 1 + (1+\Delta_y)[e^{-(1-\gamma)\Delta_y} - 1]\Big)\sigma^2\overline{k}^2/2.\end{aligned}$$

The first-order condition is then

$$0 = \frac{1}{V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma + \frac{1}{\Delta_y(1-\gamma)}[e^{(1-\gamma)\Delta_y} - 1](\Pi - r)$$
$$- \frac{1}{\gamma(1-\gamma)}\left(\frac{1}{\Delta_y}[1 - e^{-(1-\gamma)\Delta_y}] - \frac{1}{\Delta_y^2}[e^{(1-\gamma)\Delta_y} - 2 + e^{-(1-\gamma)\Delta_y}]\right)\sigma^2\gamma\overline{k}$$

which may be written $0 = (\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma/V + E_1(\Delta_y)(\Pi - r) - E_2(\Delta_y)\sigma^2\gamma\overline{k}$, where

$$E_1(\Delta_y) = \frac{1}{1-\gamma}[e^{(1-\gamma)\Delta_y} - 1]/\Delta_y$$
$$E_2(\Delta_y) = \frac{1}{\gamma(1-\gamma)}\Big([2 - e^{-(1-\gamma)\Delta_y} - e^{(1-\gamma)\Delta_y}]/\Delta_y^2 + [1 - e^{-(1-\gamma)\Delta_y}]/\Delta_y\Big)$$
$$E_c(\Delta_y) = \Big((1 - e^{-(1-\gamma)\Delta_y})/\Delta_y\Big)^{-1/\gamma}(1-\gamma)^{1/\gamma}$$

from which rearrangement gives the result. □

Proof of Lemma 4.5. Using $\Delta_t(\overline{p}) = \overline{p}\Delta_t(1) =: \overline{p}\overline{\Delta}_t$, dividing all terms in the Bellman equation by Δ_t gives

$$\begin{aligned}
0 = {} & \frac{1}{\Delta_t}(e^{-\rho\Delta_t}[1 - 2\overline{p}] - 1)\frac{V}{1-\gamma} + \frac{\rho\overline{c}^{1-\gamma}}{1-\gamma} + e^{-\rho\Delta_t}\frac{1}{\Delta_y}[e^{-(1-\gamma)\Delta_y} - 1]\frac{\overline{c}V}{1-\gamma} \\
& + \frac{e^{-\rho\Delta_t}}{\Delta_y^2}\Big[\Big(\sigma^2\overline{k}^2/2 + \Delta_y(\Pi - r)\overline{k}\Big)[e^{(1-\gamma)\Delta_y} - 1] + (1+\Delta_y)[e^{-(1-\gamma)\Delta_y} - 1]\sigma^2\overline{k}^2/2\Big]\frac{V}{1-\gamma} \\
& + e^{-\rho\Delta_t}\big(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}\big)\sigma\overline{k} + e^{-\rho\Delta_t}\frac{1}{\Delta_y}[e^{(1-\gamma)\Delta_y} - 1]\frac{rV}{1-\gamma} \\
& + \frac{e^{-\rho\Delta_t}}{1-\gamma}\big([\mu_x x]^+ V^{F,x} + [\mu_x x]^-[-V^{B,x}] + [\theta(\overline{\sigma} - \sigma)]^+ V^{F,\sigma} + [\theta(\overline{\sigma} - \sigma)]^-[-V^{B,\sigma}]\big) \\
& + e^{-\rho\Delta_t}\frac{(1 - |z|)}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{12}\Delta_x, \sigma + m_{13}\Delta_\sigma) + V(x - m_{12}\Delta_x, \sigma - m_{13}\Delta_\sigma)) \\
& + e^{-\rho\Delta_t}\frac{|z|}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{22}\Delta_x, \sigma + m_{23}\Delta_\sigma) + V(x - m_{22}\Delta_x, \sigma - m_{23}\Delta_\sigma)).
\end{aligned} \tag{63}$$

Note that as $\Delta_y \to 0$ we have

$$\frac{1}{\Delta_y^2}[e^{(1-\gamma)\Delta_y} - 2 + e^{-(1-\gamma)\Delta_y}] + \frac{1}{\Delta_y}[e^{-(1-\gamma)\Delta_y} - 1] \to (1-\gamma)^2 - (1-\gamma) = -\gamma(1-\gamma)$$

which gives the result upon substitution. The global convergence of the algorithm then follows from the fundamental results of policy function iteration recapiulated in Section 3.1. □