Markov chain approximation methods for continuous-time optimal control problems

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[1] The views expressed in this lecture are those of the authors and don't necessarily reflect the position of the Federal Reserve Bank of Cleveland or the Federal Reserve System.

Introduction

Optimization problems ubiquitous in economics.

This lecture considers Markov-chain approximation (MCA) methods:

- illustration and contrast with finite-difference methods.
- applies to income fluctuation problem with discrete-choice.

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This lecture considers Markov-chain approximation (MCA) methods:

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- applies to income fluctuation problem with discrete-choice.

Applications exist in literature but some benefits unexplored.

(see paper for further discussion)

Outline

Example 1: one-sector neoclassical growth model.

- Implicit finite-difference scheme of Achdou et al special case of MCA.

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General framework:

- Local consistency (moment-matching) conditions.
- Reduction to discrete-time discrete-state problem.

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General framework:

- Local consistency (moment-matching) conditions.
- Reduction to discrete-time discrete-state problem.

Example 2: Income-fluctuation problem with discrete-choice.

- Illustrates benefits of using alternatives to PFI.

(additional macrofinance application in paper)

Neoclassical growth model

Suppose farmer has preferences

$$U(c) = \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} u(c_t) dt\right] \tag{1}$$

and capital stock evolves according to

$$dk_t = [f(k_t) - \delta k_t - c_t]dt + \sigma(k_t) dZ_t, \tag{2}$$

for concave f, $\delta > 0$, BM Z and smooth σ vanishing outside of some $[\underline{k}, \overline{k}]$.

PROBLEM: maximize (1) s.t. (2) and $k_0 = k$.

Standard discrete-time approach

Given $k_0 \in [\underline{k}, \overline{k}]$ of capital, replace (2) with

$$k_{t+\Delta_t} = k_t + (f(k_t) - \delta k_t - c_t)\Delta_t + \sqrt{\Delta_t}\sigma(k_t)X_t$$

for $\Delta_t > 0$ and i.i.d. $(X_t)_{t=0}^{\infty}$ with mean zero assuming ± 1, and define

$$BV(k) = \max_{c \geq 0} \Delta_t u(c) + e^{-\rho \Delta_t} \mathbb{E}\left[V\left(k + (f(k) - \delta k - c)\Delta_t + \sqrt{\Delta_t}\sigma(k)X \right) \right] \tag{3}$$

Key points:

- Principle of Optimality $\implies$ value function unique solution to $BV = V$.
- T is contraction $\implies$ $\lim_{N\to\infty} B^N V_0 = V$ for any V_0.

MCA approach

MCA method approximates problem in fundamentally different way.

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- restricted to *finite* set $S := \{\underline{k}, \underline{k} + \Delta_k, \ldots, \overline{k}\}$.
- $\Delta k := k_{t+\Delta_t} - k_t$ supported on $\{-\Delta_k, 0, \Delta_k\}$.
- same *mean* and *variance* as original process.

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- same *mean* and *variance* as original process.

Example:

$$\begin{aligned} p(k, k \pm \Delta_k, c) &= \frac{\Delta_t}{\Delta_k^2}\left(\frac{\sigma^2(k)}{2} + \Delta_k[f(k) - c - \delta k]^{\pm}\right) \\ p(k, k, c) &= 1 - p(k, k - \Delta_k, c) - p(k, k + \Delta_k, c). \end{aligned} \tag{4}$$

Discrete-time dynamic programming arguments

Consider Bellman operator

$$\tilde{B}V(k) = \max_{c \geq 0} \Delta_t u(c) + e^{-\rho \Delta_t} \mathbb{E}\left[V(k')\right]. \tag{5}$$

As above $\tilde{B}$ is contraction so $\lim_{N \to \infty} \tilde{B}^N V_0 = V$ for any V_0.

(Approximately) same conditional moments:

- Mean: $\mathbb{E}[k_{t+\Delta_t} | k_t = k] = k + \Delta_t (f(k) - c - \delta k)$
- Variance: $\mathbb{E}[\mathrm{Var}(k_{t+\Delta_t}) | k_t = k] = \Delta_t \sigma(k)^2 + o(\Delta_t)$.

Comparison of above problems

Above laws of motion "close" to one another for small Δ_t.

$\implies$ fixed points B and $\tilde{B}$ of (3) and (5) "close" as $\Delta_t \to 0$.

Why is second formulation useful?

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Above laws of motion "close" to one another for small Δ_t.

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Why is second formulation useful? Key point:

Optimal control depends only on local payoffs.

Shape and *regularity* of value function irrelevant.

No non-linear root-finding

Maximization becomes

$$\max_{c\geq 0} u(c) + e^{-\rho\Delta_t}[f(k) - \delta k - c]^{+} V^F - [f(k) - \delta k - c]^{-} V^B + \frac{\sigma(k)^2}{2} V^C$$

where V^F, V^B and V^C = forward and backward and central differences.

Optimal c in closed-form for *any* V and f.

Figure 1 plots examples with

$$u(c) = 2c^{1/2}, f(k) = \max\left\{\sqrt{k}, 5\sqrt{k-5}\right\}, \rho = 1, \delta = 0.05, N = 1000.$$

Example

Converges with tol. 10^{-8} in < 0.03 seconds from initial $c(k) = f(k) - \delta k$.

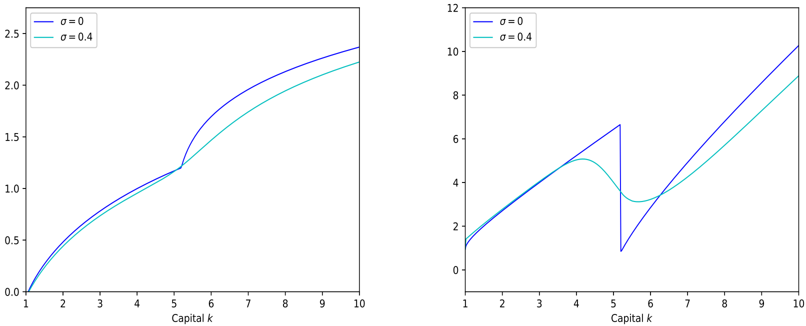

Figure 1: Value and policy functions

Intuition for computational benefits

Two properties make this fast:

1. Consumption update in closed-form.
2. Capital only moves up/down/stay $\implies$ updating matrix in PFI *sparse*.

Above features shared by other methods (endogenous grid, finite-differences).

Virtue of above: *generalizes more easily to richer environments*.

No need for complicated interpolation or clever choice of grids.

Formal analysis

Consider problems of form

$$V(X) = \max_{u \in \mathcal{C}} \mathbb{E}\left[\int_0^\tau e^{-\rho t} F(X_t, u_t)dt + I_{\tau<\infty} g(X_\tau)\right]$$

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dB_t$$

$$X_0 = X$$

where

- $X \subseteq \mathbb{R}^n$ denotes *state*; $B \in \mathbb{R}^n$ Brownian motion;
- $\mathcal{C}$ set of *admissible* controls in $\mathbb{R}^m$; and
- $\tau =$ (possibly stochastic) time when (X_t, t) exits some $U \subseteq \mathbb{R}^{n+1}$.

Basic idea: approximate above with simpler finite-state problem.

Local consistency

Definition

Suppose X solves $dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dZ_t$ and $(X^h)_{h>0}$ is family of Markov chains with finite state spaces $(S_h)_{h>0}$.

It is *locally consistent* if

$$\begin{aligned} \mathbb{E}[\Delta_n^h X] &= \Delta_n^h(X)\mu(X,u) + o(\Delta_t^h(X)) \\ \mathbb{E}[(\Delta_n^h X - \mathbb{E}[\Delta_n^h X])^2] &= \Delta_n^h(X)\sigma(X,u)\sigma(X,u)^t + o(\Delta_t^h(X)) \end{aligned} \tag{6}$$

where increments and times denoted

$$\Delta_n^h X = X_{n+1}^h - X_n^h \qquad\qquad t_n^h = \sum_{i=1}^{n} \Delta^h t(X_n^h, u_n^h).$$

Fundamental convergence result

Theorem (Kushner and Dupuis)

If $(X^h)_{h\geq 0}$ is locally consistent with X, then $X^h \to X$ in distribution as $h \to 0$. For all x we have $V^h(x) \to V(x)$ as $h \to 0$, where V^h is value function for finite-state problem.

Standard (discrete-time) arguments show V^h solves

$$V^h(x) = \max\left\{g(x), \max_u \Delta_t F(x,u) + e^{-\rho\Delta_t}\mathbb{E}^u[V^h(x')]\right\} \tag{7}$$

where x follows given Markov chain.

Solution methods well-understood for problem (7).

Comparison with finite-difference methods

Common method for PDE problems: *finite-differences* (Achdou et al (2020)).

Derives limit of Bellman equation (a PDE) and replaces derivatives with quotients.

Briefly, for any $t, h > 0$, Principle of Optimality gives

$$V(k,t) = \max_c \int_t^{t+h} e^{-\rho s} u(c(s))ds + V(k(t+h), t+h)$$

rearranging and using Ito's lemma gives

$$0 = \max_c e^{-\rho t} u(c) + \frac{\partial V}{\partial t} + [f(k) - c - \delta k] \frac{\partial V}{\partial k} + \frac{\sigma(k)^2}{2} \frac{\partial^2 V}{\partial k^2}.$$

Implicit finite-difference method

Write $V_j^n = V(k_j, t_n)$ for (k_j, t_n) in fixed grid.

Implicit method: given (V^n, c^n), V^{n+1} solves

$$\left(\frac{1}{\Delta_t} + \rho\right) V^{n+1} = \frac{V^n}{\Delta_t} + u(c^n) + [f(k) - c^n - \delta k]^+ (V^{n+1})^F$$
$$- [f(k) - c - \delta k]^- (V^{n+1})^B + \frac{\sigma(k)^2}{2} (V^{n+1})^C.$$

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Bellman equation of discrete problem:

$$\frac{(1 - e^{-\rho\Delta_t})}{\Delta_t} V = \max_{c \geq 0} u(c) + e^{-\rho\Delta_t} [f(k) - \delta k - c]^+ V^F$$
$$- e^{-\rho\Delta_t} [f(k) - \delta k - c]^- V^B + e^{-\rho\Delta_t} \frac{\sigma(k)^2}{2} V^C$$

IFD method special case of MCA

Fixed point may be written as solution to

$$0 =: \max_{c \in \Gamma(k)} u(c) + T_{\text{IFD}}(c; \Delta_t) V \tag{8}$$

for $T_{\text{IFD}}(c) : \mathbb{R}^N \to \mathbb{R}^N$. If $\Delta_t = \infty$ IFD algorithm becomes:

1. fix V_0 arbitrarily;
2. find c_0 by solving $\max_{c \in \Gamma(k)} u(c) + T(c) V_0$;
3. replace V_0 with V_1 s.t. $0 = u(c_0) + T(c_0) V_1$ and repeat until convergence.

This is policy function iteration “in disguise”.

Operators coincide in the limit.

IFD method special case of MCA

Lemma

Given Δ_t *and* (4) *write* $T(c;\Delta_t) = [e^{-\rho\Delta_t}P(c) - I]/\Delta_t$. *Then for all* c,

$$\lim_{\Delta_t \to 0} T(c;\Delta_t) = \lim_{\Delta_t \to \infty} T_{\text{IFD}}(c,\Delta_t) \in \mathbb{R}^{N\times N}.$$

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I.e. limit of implicit method as $\Delta_t \to \infty$ equivalent to:

1. particular choice of *Markov chain*; and
2. particular *method* for solving the resulting equation (PFI).

Policy function iteration sometimes not the best option.

Finite-dffierences vs MCA

Vast literatures for both techniques.

In my opinion, MCA deserves to be more widely known:

1. Intuitive task: *approx with Markov chain by matching moments*.
2. Discretized problem solved using techniques familiar to economists.
3. Avoids (technical) PDE theory.

For Point 2, you can use any technique you want for the discrete problem:

- e.g. VFI, PFI, modified PFI, etc.

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QUESTION: when to use each approach?

Pros and cons of implicit FDM and PFI

Previous lecture gave intuition for speed of implicit FDM:

1. FOCs in closed-form.
2. Transition matrix sparse $\implies$ updating step fast.
3. Typically converges in small number of iterations.

Point 2 breaks down in multiple dimensions and large state spaces:

- Transition matrix becomes less sparse.
- Approximating solution to $Ax = b$ slow for large A.

However, *no need to use PFI*. Often better to use modified PFI or alternatives.

Recap of PFI and modified PFI

State space S; timestep $\Delta_t \in \mathbb{R}^{|S|}$; transition probabilities $P : S^2 \times U \to [0,1]$.

DP equation of form

$$V(x) = \max_{u \in U} \Delta_t(x) f(x,u) + e^{-\rho \Delta_t(x)} \sum_{x' \in S} P(x,x',u) V(x').$$

Define $T(\hat{u}) := e^{-\rho \Delta_t} P(\hat{u}) - I$ and write Bellman equation as

$$0 = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + T(\hat{u}) v =: B(v) \tag{9}$$

PFI may be summarized:

- Given v_0 choose $\hat{u}(v_0)$ to solve $B(v_0) = F(\hat{u}(v_0)) + T(\hat{u}(v_0)) v_0$.
- Define $v_1 = -T(\hat{u}(v_0))^{-1} F(\hat{u}(v_0))$.
- Repeat above with v_1 in place of v_0 until convergence.

Modified PFI

Write $\hat{u}(v)$ for optimal control given v and abbreviate $\hat{u}_n := \hat{u}(v_n)$.

Updating law:

$$\begin{aligned} v_{n+1} &= -T(\hat{u}(v_n))^{-1} F(\hat{u}(v_n)) \\ &= v_n - T(\hat{u}(v_n))^{-1} B(v_n) \\ &= v_n + \sum_{j=0}^{\infty} (I + T(\hat{u}_n))^j B(v_n). \end{aligned} \tag{10}$$

Modified policy function iteration truncates sum in (10) at fixed k.

No solving of large linear system.

Durable and non-durable consumption

Illustrate with income fluctuation problem with discrete-choice.

Assume preferences over non-durable and durable consumption given by

$$U(c, D) := \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} u(c_t, D_t) dt\right] \tag{11}$$

for some u. Possible values of durable consumption

$$S_D := \{\underline{D}, \underline{D} + \Delta_D, \ldots, \underline{D} + N_D \Delta_D\}$$

for some $\underline{D}$, N_D and Δ_D.

(S_D primitive of problem, not chosen in discretization.)

Durable and non-durable consumption

Income: $y_t = e^{z_t}$ for AR(1) z. Price of durable good $= \overline{p}$.

- At $t \geq 0$ agent chooses whether to increase durable consumption.
- Opportunities to purchase arrive at rate $\lambda > 0$.
- As $\lambda \to \infty$ durable good changes instantaneously.

Law of motion for state variables

$$\begin{aligned} da_t &= [ra_t + y_t - c_t]dt - \overline{p}dD_t(q_t) \\ dz_t &= -\theta z_t dt + \sigma dZ_t \\ dD_t(q_t) &= dJ_t(q_t) \end{aligned} \tag{12}$$

$J =$ jump process with arrival rate λ; q_t indicates purchase of durable good.

Reduction to discrete problem

Two steps: approximate with finite-state problem; then solve that problem.

Define $S := S_a \times S_z \times S_D$, where

$$S_a := \{\underline{a} + \Delta_a, \dots, \overline{a} - \Delta_a\}$$
$$S_z := \{\underline{z} + \Delta_z, \dots, \overline{z} - \Delta_z\}$$

for some $\Delta_a, \Delta_y > 0$.

To ensure income remains on grid impose $\overline{p}\Delta_D = K\Delta_a$ for some $K \geq 1$.

Need to match conditional moments of process.

Transition probabilities

Wealth and income:

$$p(a \pm \Delta_a, z, D) = \frac{\Delta_t}{\Delta_a}[ra + y - c]^{\pm}.$$

Income:

$$p(a, z \pm \Delta_z, D) = \frac{\Delta_t}{\Delta_y^2}\left(\frac{\sigma^2}{2}1_{z \notin \{\underline{z}+\Delta_z, \overline{z}-\Delta_z\}} + \Delta_z[-\theta z]^{\pm}\right).$$

Durable good:

$$p(a - \overline{p}\Delta_D, z, D + \Delta_a) = \lambda 1_{q_t = (-\overline{p}\Delta_a, 0, \Delta_D)}\Delta_t$$

where

$$q_t \in \left\{(0,0,0), (-\overline{p}\Delta_a, 0, \Delta_D)\right\}.$$

Durable and non-durable consumption

Define

$$T(c,q;\Delta_t) = u(c,D) + \frac{1}{\Delta_t}\left(e^{-r\Delta_t}\mathbb{E}[V(a',z',D')] - V(a,z,D)\right).$$ details

Optimal policy for durable good

$$q := \overline{q}1_{V(a-K\Delta_a,z,D+\Delta_D)>V(a,z,D)}.$$

Durable and non-durable consumption

Define

$$T(c, q; \Delta_t) = u(c, D) + \frac{1}{\Delta_t}\left(e^{-r\Delta_t}\mathbb{E}[V(a', z', D')] - V(a, z, D)\right).$$

details

Optimal policy for durable good

$$q := \overline{q}1_{V(a-K\Delta_a, z, D+\Delta_D) > V(a,z,D)}.$$

Discrete choice poses no problems for policy functions:

- No need for interpolation.
- No clever choice of grids. Uniform and fixed throughout.

Numerical illustration

Parameters mainly from Fella (2014) (see paper for details).

Table 2 gives convergence times for MPFI with tolerance 10^{-5}.

	PFI	VFI	k = 10	k = 50	k = 100
(50, 10, 10)	0.382562	2.896908	0.349048	0.168027	0.159010
(100, 20, 10)	4.794657	21.105528	2.114040	0.608535	0.567014
(150, 30, 10)	16.679804	80.534670	9.213652	2.320960	1.716259
(200, 40, 10)	44.446097	253.265637	25.383114	7.502274	4.233294

Figure 2: Time until convergence for discrete choice problem

MPFI up to 10x faster than PFI for three-dimensional problem.

Durable consumption obviously discontinuous, but no problem for algorithm.

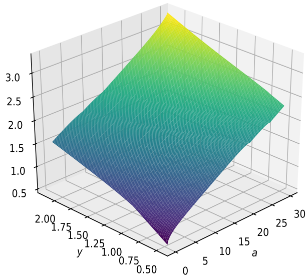

Figure 3: Non-durable and durable policy functions

Conclusion

Comprehensive theory for MCA methods developed by Kushner and Dupuis.

(authors dot all the i's and cross all the t's that I omitted here)

Goal of our paper: *make accessible to economists and illustrate some benefits*.

Main points:

- Implicit FDM of Achdou et al is PFI "in disguise".
- Modified PFI much faster than PFI when state space large.

Omitted from lecture: avoids headaches when state variables highly correlated.

Operator in discrete-choice case

We now define $\tilde{T}(c,q) = \lim_{\Delta_t \to 0} T(c,q;\Delta_t)$.

Simplification gives

$$\begin{aligned}
\tilde{T}(c,q) = u(c,D) &+ \frac{1}{\Delta_a}[ra+y-c]^+[V(a+\Delta_a,z,D) - V(a,z,D)] \\
&+ \frac{1}{\Delta_a}[ra+y-c]^-[V(a-\Delta_a,y,D) - V(a,y,D)] \\
&+ \frac{1}{\Delta_z^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^+\right)[V(a,z+\Delta_z,D) - V(a,z,D)] \\
&+ \frac{1}{\Delta_z^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^-\right)[V(a,z-\Delta_z,D) - V(a,z,D)] \\
&+ \lambda(V(a-q_a,z+q_z,D+q_D) - V(a,z,D)) - rV(a,z,D).
\end{aligned} \tag{13}$$

main