Advanced Macroeconomics II

Handout 8 - Heterogeneous Agent Models

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Heterogeneous agent models

- We want an economy with a continuum of agents
- Agents differ in their states:
  - Some are in debt, some have savings
  - Some have high income, some have low income
  - Some are young, some are old
  - Some own housing, some rent (some houses are larger than others)

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- Differences among agents are endogenous:
  - Arise from endogenous reaction to realization of shocks
- Agents interact in markets: Prices are endogenous
  - Labor market, capital market, housing market, etc
  - Demand and supply come from aggregating individual actions

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    - We get these by solving agents' DP problem (VFI+EGM)

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2. Distribution of agents across states
   - How many rich agents are there? How many high-income agents?
   - This is new... and time consuming... more on it later
3. Prices
   - Need to be consistent with market clearing
   - Demand and supply come from aggregating individual actions (1) with respect to the distribution (2)
   - Solution depends on market structure

Stationary Recursive Competitive Equilibrium

- We will focus on models without aggregate uncertainty
- All shocks are idiosyncratic and uncorrelated across agents
  - Agents' lives change period by period
  - Each agent gets different realization of shocks, causing change
    - A given agent can have high income one period and low the next
  - But there are no coordinated changes, there is always the same measure of agents in each states

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  - But there are no coordinated changes, there is always the same measure of agents in each states
- This means that aggregates do not change!
  - In particular prices are time invariant
  - More importantly: Distribution of agents is time invariant (stationary)

Hugget (1993) Economy

The individual problem (simplest problem)

- Agents live forever and choose consumption/savings
- Income (ϵ) is stochastic and follows a Markov process P:
  - This is an endowment economy, or a Lucas tree economy
  - This is the only shock in the economy
  - Realizations of the shock are independent across agents

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Agent's Problem:

$$V(\epsilon, a) = \max_{\{c, a'\}} u(c) + \beta E\left[V\left(\epsilon', a'\right) | \epsilon\right]$$

$$\text{s.t. } c + a' = (1+r)\,a + \epsilon \qquad a' \geq \underline{a}$$

Euler equation:

$$u'(c) = \beta E\left[V_a\left(\epsilon', a'\right) | \epsilon\right] = \beta(1+r)\,E\left[u'\left((1+r)\,a' + \epsilon' - g_a\left(\epsilon', a'\right)\right)\right]$$

The natural borrowing constraint

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The constraint must satisfy:

$$\underline{a} \geq {}^{-\epsilon_{min}}/_{r}$$

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Equilibrium: Integral over assets is zero

The distribution of agents

- Let $\overline{S}$ be the set of states and $[\underline{a}, \overline{a}]$ be the domain of assets
  - For admissible r, there exists $\overline{a} > 0$ such that $a \leq \overline{a}$ a.e.

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- Let $\mathcal{S}$ and $\mathcal{A}$ be σ-algebras over $\overline{S}$ and $[\underline{a}, \overline{a}]$ respectively
  - In practice we choose the Borel σ-algebras
- The distribution of agents is a function $\Gamma : \mathcal{S} \times A \to [0, 1]$
  - Γ is measurable with respect to $\mathcal{S}$ and $\mathcal{A}$
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  - Γ is measurable with respect to $\mathcal{S}$ and $\mathcal{A}$
  - Γ integrates to 1
- The distribution $\Gamma(S, A)$ answers the question:
  - What measure of agents have a state $s \in S \in \mathcal{S}$ and assets $a \in A \in \mathcal{A}$?

Updating the distribution

We can update the distribution by following the actions of agents

- Let $S \times A \in \mathcal{S} \times \mathcal{A}$ be a set in the σ-algebra
- We want to know if there are agents coming into the set $S \times A$

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1. From the Markov kernel of ϵ we also have the probability that $\epsilon^{'} \in S$:

$$\Pr\left(\epsilon^{'} \in S|\epsilon\right) = \int_{\epsilon^{'} \in S} P\left(\epsilon^{'}|\epsilon\right) d\epsilon$$

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2. Define an indicator function to know if $a^{'} \in A$ (for deterministic choices)

$$g\left(\epsilon, a, A\right) = \begin{cases} 1 & \text{if } a^{'}\left(\epsilon, z\right) \in A \\ 0 & \text{otw} \end{cases}$$

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- Both functions depend on the initial state of the agent (ϵ, a)

Updating the distribution

Updating function for all $(S, A) \in \mathcal{S} \times \mathcal{A}$:

$$\Gamma^{'}(S, A) = \int_{\overline{S}} \int_{\overline{A}} \underbrace{g(\epsilon, a, A) \cdot \Pr\left(\epsilon^{'} \in S | \epsilon\right)}_{\text{Markov Kernel:} Q\left(\epsilon^{'} a^{'} | \epsilon, a\right)} \cdot d\Gamma(\epsilon, a)$$

- Updating the distribution through **adjoint Markov operator of Q**
- Objective: fixed point of the adjoint operator (Γ such that $\Gamma^{'} = \Gamma$)
- If you are interested in the properties that guarantee a unique stationary distribution look at SLP or section 24 of the math camp notes

Updating the distribution

- When exogenous state is discrete updating is simpler (often the case)
- We can work with: $\mathcal{S} = \{\{\epsilon_1\}, \ldots, \{\epsilon_n\}, \ldots, \{\epsilon_N\}\}$ instead of σ-algebra

Updating function for all $(\epsilon, A) \in \overline{S} \times \mathcal{A}$:

$$\Gamma'\left(\epsilon', A\right) = \sum_{\overline{S}} \int_{\overline{A}} g\left(\epsilon, a, A\right) \cdot \underbrace{P\left(\epsilon' | \epsilon\right)}_{\text{Markov Transition Matrix}} \cdot d\Gamma\left(\epsilon, a\right)$$

Stationary RCE

A S-RCE is a set of a value function (V), policy function $\left(a^{'}\right)$, distribution (Γ), and price (r) such that:

1. Given r the value and policy functions solve the agent's problem
2. Given the policy function, Γ is a fixed point of the adjoint operator
3. Given the distribution and policy functions the market for bonds clears:

$$\underbrace{\int\int a^{'}(\epsilon, a) \cdot d\Gamma(\epsilon, a)}_{\text{Net Supply}} = 0 \longleftrightarrow \underbrace{\int\int c(\epsilon, a) \cdot d\Gamma(\epsilon, a)}_{\text{Demand for Goods}} = \underbrace{\int\int \epsilon \cdot d\Gamma(\epsilon, a)}_{\text{Supply of Goods}}$$

Algorithm: VFI with EGM

Algorithm 1: EGM for S-RCE problem

Function $\texttt{EGM}(V(\epsilon, a), \vec{a}, \vec{\epsilon}, r, parameters)$:

for $i=1:n_e$ *# Income state (exogenous, current period)* **do**

for $j=1:n_a$ *# Savings (endogenous, future period)* **do**

1. Expected value: $\mathbb{V} = \beta E\left[V\left(\epsilon', \vec{a}_j\right)|\vec{\epsilon}_i\right]$
2. Consumption from Euler: $u_c\left(\tilde{c}_{ij}\right)\right) = \mathbb{V}_a$
   Solve analytically for $\tilde{c}_{ij}$
3. Endo. assets: $\hat{a}_{ij} = \left(\tilde{c}_{ij} + \vec{a}_j - \vec{\epsilon}_i\right)/1 + r$
4. Update value at endogenous grid: $\hat{V}(\vec{\epsilon}_i, \hat{a}_{ij}) = u\left(\tilde{c}_{ij}\right)\right) + \mathbb{V}$

5. Check borrowing constraint before interaction:
   $a^{'}(\vec{\epsilon}_i, a) = \underline{a}$ for all $a \in [\underline{a}, \hat{a}(\vec{\epsilon}_i, \underline{a})]$
6. Interpolate to exogenous grid: V_new[i,:] = Interp($\hat{a}$, $\hat{V}$, $\vec{a}$)

Some comments

- This is the easiest DP problem we have had
  - No non-linear equations, no interpolation inside loop
- We have to check for the borrowing constraint! And adjust!
  - In EGM we find (endogenous) assets today that would give a given $a^{'}$
  - When $a^{'} = \underline{a}$ we get the assets for which agent is just at the constraint
  - The agent will be constrained below that point

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  - When $a^{'} = \underline{a}$ we get the assets for which agent is just at the constraint
  - The agent will be constrained below that point
- Adding labor choice does not complicate things much (see Lecture 7)
- You can also use the ECM of Maliar and Maliar, try it out!

Alternative: PFI with EGM

Algorithm 2: EGM for S-RCE problem

Function EGM($g_a(\epsilon, a), \vec{a}, \vec{\epsilon}, r, parameters$):

for *i=1:n_e* *# Income state (exogenous, current period)* **do**

for *j=1:n_a* *# Savings (endogenous, future period)* **do**

1. RHS of Euler:
   $RHS = \beta E\left[U\left((1+r)\vec{a}_j + w\vec{\epsilon}_i - g_a(\epsilon^{'}, \vec{a}_j)\right) | \vec{\epsilon}_i\right]$
2. Consumption from Euler: $u_c\left(\tilde{c}_{ij}\right) = RHS$
   Solve analytically for $\tilde{c}_{ij}$
3. Endo. assets: $\hat{a}_{ij} = \left(\tilde{c}_{ij} + \vec{a}_j - \vec{\epsilon}_i\right)/1 + r$

4. Check borrowing constraint: If $\hat{a}(\vec{\epsilon}_i, \underline{a}) > \underline{a}$ then agent chooses $a' = \underline{a}$ for $a \in [\underline{a}, \hat{a}(\vec{\epsilon}_i, \underline{a})]$
5. Interpolate consumption to exogenous grid and obtain implied a' from budget constraint.

Algorithm: The histogram method (Young, 2011)

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- **Inputs:** Transition matrix for exogenous states and policy function

Steps:

1. Get a fine grid for assets
2. Use (inverse) linear interpolation to map $a'(\epsilon, a)$ to the grid

Algorithm: The histogram method (Young, 2011)

Algorithm 3: Histogram Method - Transition matrix

Function `EGM`$(a^{'}(\epsilon, a), \vec{a}, \vec{\epsilon})$:

0. Generate fine grid for assets and interpolate policy function $a^{'}$ to it.

for *i=1:n$_e$* *# Income state* **do**

  for *j=1:n$_a$* *# Assets (on histogram grid)* **do**

1. Identify $h^\star$ s.t. $a^{'}(\vec{\epsilon}_i, \vec{a}_j) \in [\vec{a}_{h^\star}, \vec{a}_{h^\star+1}]$

   1.1. Save the index for future use: $h(\vec{\epsilon}_i, \vec{a}_j) = h^\star$

2. Define weight on lower grid node: $\omega = \frac{a^{'}(\vec{\epsilon}_i, \vec{a}_j) - \vec{a}_{h^\star}}{\vec{a}_{h^\star+1} - \vec{a}_{h^\star}}$

3. Assign weights to transition function for assets:

$$g(\vec{\epsilon}_i, \vec{a}_j, \vec{a}_{h^\star}) = \omega \qquad g(\vec{\epsilon}_i, \vec{a}_j, \vec{a}_{h^\star+1}) = 1 - \omega$$

4. All other entries for $(\vec{\epsilon}_i, \vec{a}_j)$ are zero so **g is sparse**.

5. Check that $g \in [0, 1]$ for all states.

Algorithm: The histogram method (Young, 2011)

- We now have a way to get the Transition matrix for assets on our grid
  - To get the index of assets on the grid we use inverse interpolation:

$$h^{\star} = \text{floor}\left(\left(\frac{a^{'}(\epsilon, a) - \min(\vec{a})}{\max(\vec{a}) - \min(\vec{a})}\right)^{\frac{1}{\theta_a}}(n_a - 1) + 1\right)$$

if $a^{'}(\epsilon, a) \in [\min(\vec{a}), \max(\vec{a})]$. If not then either $h^{\star} = 1$ or $h^{\star} = n_a$

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  if $a^{'}(\epsilon, a) \in [\min(\vec{a}), \max(\vec{a})]$. If not then either $h^{\star} = 1$ or $h^{\star} = n_a$

- We have at least three options to get the stationary distribution:
  1. Loops (avoid in Matlab at all cost)
  2. Matrices (Young, 2010)
  3. Smart matrices (Tan, 2020)

Algorithm: Stationary distribution with loops

Algorithm 4: Histogram Method - Updating distributions

Function `EGM`($\Gamma, g(\epsilon, a, a^{'}), P(\epsilon, \epsilon^{'})$):

$\Gamma^{'} = 0$ # Initialize new distribution $\Gamma^{'}$ for accumulation

for *i=1:n_e* # *Income state* **do**

  for *j=1:n_a* # *Assets* **do**

    for *k=1:n_e* # *Future income state* **do**

$$\Gamma^{'}(k, h(i,j)) = \Gamma^{'}(k, h(i,j)) + g(\vec{\epsilon}_i, \vec{a}_j) \cdot P(\vec{\epsilon}_i, \vec{\epsilon}_k) \cdot \Gamma(i,j)$$

$$\Gamma^{'}(k, h(i,j)+1) = \Gamma^{'}(k, h(i,j)+1) + (1 - g(\vec{\epsilon}_i, \vec{a}_j)) \cdot P(\vec{\epsilon}_i, \vec{\epsilon}_k) \cdot \Gamma(i,j)$$

Repeat until convergence

Algorithm: Stationary distribution with matrices

The state space is now effectively discrete, so we can list them (vectorize):

$$\vec{s} = \left[(\vec{\epsilon}_1, \vec{a}_1) \quad \dots \quad (\vec{\epsilon}_{n_\epsilon}, \vec{a}_1) \quad (\vec{\epsilon}_1, \vec{a}_2) \quad \dots \quad (\vec{\epsilon}_{n_\epsilon}, \vec{a}_2) \quad \dots \quad (\vec{\epsilon}_1, \vec{a}_{n_a}) \quad \dots \quad (\vec{\epsilon}_{n_\epsilon}, \vec{a}_{n_a}) \right]^T$$

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We can then get a transition matrix from $\vec{s}$ to $\vec{s}'$:

$$T = (P \otimes 1_{n_a \times n_a}) \odot G$$

where P is the transition matrix for the exogenous income process:

$$P = \begin{bmatrix} \Pr\left(\epsilon_1' | \epsilon_1\right) & \dots & \Pr\left(\epsilon_{n_e}' | \epsilon_1\right) \\ \vdots & \ddots & \\ \Pr\left(\epsilon_1' | \epsilon_{n_e}\right) & \dots & \Pr\left(\epsilon_{n_e}' | \epsilon_{n_e}\right) \end{bmatrix}$$

and $G = \underbrace{\left[g \left(\underbrace{\epsilon, a}_{s_i}, a' \right) \right]}_{\text{Size: } (n_\epsilon n_a) \times n_a} \otimes 1_{1 \times n_e}$ is the transition matrix for assets.

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- Armed with a transition matrix T we can either:
  - Iterate on $\Gamma' = T^T \Gamma$ many times
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- However the matrix G (and thus T) is very sparse, you can use that to your advantage!
- Despite sparseness these matrices are expensive to store and use
  - Lots of wasteful storage because of repeated versions of G and P

Algorithm: Smarter matrices (Tan, 2020)

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Key idea: Take the distribution forward one dimension at a time

1. Define $G_{n_\epsilon n_a \times n_\epsilon n_a}$ that maps $(\epsilon, a) \to \left(\epsilon, a^{'}\right)$ (see in next slide)
2. Start from a distribution Γ, an $n_\epsilon n_a$ vector
3. Update only the savings decision: $\hat{\Gamma} = G^{'}\Gamma$
4. Reshape $\hat{\Gamma}$ into a $n_\epsilon \times n_a$ matrix (This matrix is interpreted as $\left(\epsilon, a^{'}\right)$)
5. Update exogenous state: $\Gamma^{'} = P^{'}\hat{\Gamma}$ (result is a $n_\epsilon \times n_a$ matrix)
6. Reshape into new distribution $\Gamma^{'}$

Algorithm: The histogram method (Tan, 2020)

Algorithm 5: Histogram Method - Transition matrix

Function $\texttt{EGM}(a^{'}(\epsilon, a), \vec{a}, \vec{\epsilon})$:

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3. Assign weights to transition function for assets:

$$g(\vec{\epsilon}_i, \vec{a}_j, \vec{\epsilon}_i, \vec{a}_{h^\star}) = \omega \qquad g(\vec{\epsilon}_i, \vec{a}_j, \vec{\epsilon}_i, \vec{a}_{h^\star+1}) = 1 - \omega$$

The second ϵ is just a repeat of the same state

4. All other entries for $(\vec{\epsilon}_i, \vec{a}_j)$ are zero so **g is sparse**.

5. Check that $g \in [0, 1]$ for all states.

Algorithm: S-RCE

Algorithm 6: S-RCE Algorithm

input : Guess for price (r)
output: $V, a^{'}, \Gamma, r$

1. Solve the DP problem of the agent given r:
   $(V, a^{'}) = T(V; r)$ (a fixed point problem) ;
2. Find stationary distribution with histogram method ;
3. Check market clearing: $\sum_i \sum_j a^{'}(\vec{\epsilon}_i, \vec{a}_j) \cdot \Gamma(i, j)$;
4. Update prices to clear market
   Manually by tatonnement or with a Root finder ;
5. Repeat (1)-(4) until market clears ;

Some comments

- You can improve the code by using the insight of Howard's policy iteration algorithm
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  - Definitely use MacQueen-Porteus bounds if your are using VFI
- You don't need to iterate on the distribution until convergence!
  - Waste to iterate until convergence without the "true" policy function
- Alternative:

  2'. Iterate forward the distribution N times
  - After you iterate N times you check market clearing and update prices, then get new policy functions
  - Doing this means you have to change the convergence criteria

  4'. Repeat (1),(2'),(3),(4) until convergence in Γ and market clearing
  - You should also dampen the updating of the distribution (no need to update fully if you are not using the right policy function)

Aiyagari (1994) Economy

The Aiyagari economy

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- Production is just as in NGM:
  - Representative firm with CRS technology
  - This will greatly simplify market clearing!

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  - Representative firm with CRS technology
  - This will greatly simplify market clearing!
- Reinterpretation of the shock to individual agents
  - Shock is now labor efficiency instead of income endowment
  - This does not really change the problem

Individual problem

- Income depends on labor efficiency (ϵ) and the market wage (w)
  - ϵ is stochastic and follows a Markov process P
  - This is the only shock in the economy
  - Realizations of the shock are independent across agents

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Agent's Problem:

$$V(\epsilon, a) = \max_{\{c,a'\}} u(c) + \beta E\left[V\left(\epsilon', a'\right)|\epsilon\right]$$

$$\text{s.t. } c + a' = (1+r)\,a + w\epsilon \qquad a' \geq \underline{a}$$

Euler equation:

$$u'((1+r)\,a + w\epsilon) = \beta E\left[V_a\left(\epsilon', a'\right)|\epsilon\right]$$

Production and savings

Output is produced from capital and labor: $Y = F(K, L)$

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- **Key:** F has constant returns to scale
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Production and savings

Output is produced from capital and labor: $Y = F(K, L)$

- **Key:** F has constant returns to scale
  - Demand for inputs is perfectly elastic at equilibrium prices
  - The only way to clear the market is to make producers indifferent
- **Equilibrium prices**

$$r = F_k(K, L) - \delta \qquad w = F_L(K, L)$$

where the aggregate capital and labor come from distribution:

$$K = \int\int a \cdot d\Gamma(\epsilon, a) \qquad L = \int\int \epsilon \cdot d\Gamma(\epsilon, a)$$

But L supply is exogenous! So we can say: $L = \int \epsilon \Gamma_\epsilon(\epsilon)\, d\epsilon = 1$

Changes to RCE

Algorithm 7: S-RCE Algorithm

input : Guess for price (r)
output: $V, a^{'}, \Gamma, r$

1. Solve the DP problem of the agent given (r, w):
   $(V, a^{'}) = T(V; r)$ (a fixed point problem) ;
2. Find stationary distribution method (or update dist. N times) ;
3. Update prices to ensure market clearing:
   $K = \sum_i \sum_j a \cdot \Gamma(i, j) \quad \longrightarrow \quad r = F_k(K, 1) - \delta \quad w = F_L(K, 1)$;
   Dampen updating of prices if necessary ;
4. Repeat (1)-(3) until prices converge ;

OLG Aiyagari Economy

Life cycle: OLG

- Many applications in macro-labor require life cycle dynamics

Life cycle: OLG

- Many applications in macro-labor require life cycle dynamics
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Life cycle: OLG

- Many applications in macro-labor require life cycle dynamics
- That implies two changes:
  1. A new state: Age
  2. A new solution method for the agent's problem: Backward induction
- Age is a tricky state. It will increase the dimension of the problem. There is no way around it.
- Backward induction is a blessing! No need for value functions!
  - Main benefit: No more approximation of the derivative V_a!
  - Other benefit: No fixed point to solve agent's problem

Individual problem

Agent's Problem:

$$V^{h}\left(\epsilon,a\right)=\max_{\left\{c_{h},a_{h}^{'}\right\}}u\left(c_{h}\right)+\beta E\left[V^{h+1}\left(\epsilon^{'},a^{'}\right)|\epsilon\right]$$

$$\text{s.t. } c_{h}+a_{h}^{'}=\left(1+r\right)a+w\epsilon \qquad a_{h}^{'}\geq\underline{a}$$

- Age is a state (h), but we write it as an index

Individual problem

Agent's Problem:

$$V^h(\epsilon, a) = \max_{\{c_h, a_h^{'}\}} u(c_h) + \beta E\left[V^{h+1}\left(\epsilon^{'}, a^{'}\right)|\epsilon\right]$$

$$\text{s.t. } c_h + a_h^{'} = (1+r)\,a + w\epsilon \qquad a_h^{'} \geq \underline{a}$$

- Age is a state (h), but we write it as an index

Euler equation:

$$u^{'}(c_h(\epsilon, a)) = \beta E\left[V_a^{h+1}\left(\epsilon^{'}, a^{'}\right)|\epsilon\right]$$

- Pair with envelope condition to get rid of value function:

$$u^{'}(c_h(\epsilon, a)) = \beta E\left[u^{'}\left(c^{h+1}\left(\epsilon^{'}, a^{'}\right)\right)(1+r)\,|\epsilon\right]$$

OLG-RCE

A S-RCE is a set of a value function (V), policy function $\left(a^{'}\right)$, distribution (Γ), and prices (r, w) such that:

1. Given (r, w) the value and policy functions solve the agent's problem
   - Recall that both value and policy functions are indexed by age
2. Given policy functions, Γ is a fixed point of the adjoint operator
3. Given the distribution the input markets clear:

$$K = \sum_{h=1}^{H} \int \int a \cdot d\Gamma^{h}(\epsilon, a) \qquad L = \sum_{h=1}^{H} \int \int \epsilon \cdot d\Gamma^{h}(\epsilon, a)$$

$$r = F_{k}(K, L) - \delta \qquad w = F_{L}(K, L)$$

Algorithm: Backward induction

Algorithm 8: Backward Induction EGM for S-RCE problem

Function `EGM`($\vec{a}, \vec{\epsilon}, r, w, parameters$):

0. Last period we apply terminal conditions

$$a^H(\epsilon, a) = 0 \text{ and } c^H = (1+r)a + w\epsilon$$

for *h=H-1:1* *# Age, counting backwards* **do**

  for *i=1:n_e* *# Income state (exogenous, current period)* **do**

    for *j=1:n_a* *# Savings (endogenous, future period)* **do**

      1. Consumption from Euler (Solve analytically for $\tilde{c}_{ij}^h$):

$$u_c\left(\tilde{c}_{ij}^h\right) = \beta E\left[u_c\left(c^{h+1}(\epsilon', \vec{a}_j)\right) | \vec{\epsilon}_i\right]$$

      2. Endo. assets: $\hat{a}_{ij}^h = \left(\tilde{c}_{ij}^h + \vec{a}_j - w\vec{\epsilon}_i\right)/1 + r$

    3. Check for constraint binding as before

    4. Interpolate **policy functions** to exogenous grid:

$$c^h[\text{i,:}]=\text{Ip}(\hat{a}^h, \tilde{c}^h, \vec{a}) \text{ and } a^h[i,:] = (1+r)\vec{a} + w\vec{\epsilon}_i - c^h[i,:]$$

Some comments

- The solution is found in one iteration of the algorithm
  - No fixed point involved
- Cost comes with age! Another loop to go through

Some comments

- The solution is found in one iteration of the algorithm
  - No fixed point involved
- Cost comes with age! Another loop to go through
- After interpolation check for borrowing constraint / negative consumption
- The problem is quite flexible
  - Terminal conditions can involve bequests
  - The agent's problem can change with age (working/retirement)

Stationary distribution

- Finding the distribution is not conceptually different than before
- Practical difference: We need to index transition function g by age

Stationary distribution

- Finding the distribution is not conceptually different than before
- Practical difference: We need to index transition function g by age
- Implementation requires additional outer loop that goes through age
  - It is better to have this loop go backwards in age
  - Dist. of agents of age H maps into new-borns: $\Gamma^H(\epsilon, a) \longrightarrow \Gamma^1\left(\epsilon^{'}, a^{'}\right)$
  - Dist. of agents of age $H-1$ maps into age H: $\Gamma^{H-1}(\epsilon, a) \longrightarrow \Gamma^H\left(\epsilon^{'}, a^{'}\right)$

Algorithm: S-RCE

Algorithm 9: S-RCE Algorithm

input : Guess for price (r)
output: V, a', Γ, r

1. Solve the DP problem given (r, w) (Single iteration of Backwards Induction) ;
2. Find stationary distribution (or update dist. N times) ;
3. Update prices to ensure market clearing:
   $$K = \sum_h \sum_i \sum_j a \cdot \Gamma^h(i,j) \quad \longrightarrow \quad r = F_k(K,1) - \delta \quad w = F_L(K,1)$$
   Dampen updating of prices if necessary ;
4. Repeat (1)-(3) until prices converge ;

Final comments

Parallel programming and simulation

- Parallel programming is necessary for large-scale models
- Julia has built-in parallel options
  - Multithreading is the way to start
  - Multithreading distributes tasks across processors
  - Easiest to implement for loops
  - Loops are the bottleneck of the code
- Many results come from simulation
  - Need to simulate draws from the exogenous Markov Process
  - Check code for simulation of MP in Lecture 5
  - Simulation is parallelizable!
- Avoid simulation if possible, check Ocampo & Robison (2023)