Advanced Macroeconomics II

Handout 7 - Models with Distortions and GE

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Short recap

Prototypical DP problem:

$$
\begin{aligned}
V(z,k) = \max_{\{c,k'\}} \; & u(c) + \beta E\left[V\left(z', k'\right)|z\right] \\
\text{s.t.}\, c + k' &= f(z,k) \\
z' &= h(z,\eta)\,;\eta \text{ stochastic}
\end{aligned}
$$

- We are looking for functions V, g^c, g^k.

But that is not the actual problem we started with!

Macroeconomic model

- We had a representative agent choosing consumption (and labor) to solve:

$$\max_{\{c_t,\ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c,\ell) \qquad \text{s.t. } c_t + a_{t+1} = (1+r_t)\, a_t + w_t \ell_t + \pi_t$$

- We had a representative firm choosing capital and labor to solve:

$$\pi_t = \max_{\left\{k_t,\ell_t^d\right\}} f\left(z_t, k_t, \ell_t^d\right) - (r_t+\delta)\, k_t - w_t \ell_t^d$$

- And we had prices that cleared markets:

$$\ell_t = \ell_t^d \qquad a_t = k_t \qquad c_t + a_{t+1} = f\left(z_t, k_t, \ell_t\right) + (1+\delta)\, a_t$$

Equilibrium vs Planner's problem

- FWT lets us solve planner's problem
- Map planner's solution to the competitive market allocation and prices
  - Planner solves for aggregate **quantities** $\{C, L, K\}$
  - We want to get individual quantities $\{c, \ell, a, k\}$ and prices $\{r, w\}$

$$c = C \qquad \ell = L \qquad k = a = K$$

$$r = f_k(z, K, L) - \delta \qquad w = f_\ell(z, K, L)$$

- **Key:** Planner's problem is a "simple" dynamic programming problem
  - We can solve it with the tools from the previous 6 lectures!

How to solve for the equilibrium directly?

- **Easy part:** Firm's problem is static
  - Solution depends on aggregate quantities
  - Solution gives us prices

$$r = f_k(z, K, L) - \delta \qquad w = f_\ell(z, K, L)$$

- **Hard part:** Consumer problem a dynamic programming problem
  - What are the states?
  - Consumer is a price taker: No clue about aggregate effect of choices
  - States must provide enough information to solve the problem
  - Consumer must know how states evolve

A DP problem for the consumer: (k, K)

$$V(k, \underbrace{z, K}_{\text{Agg. States}}) = \max_{\{c,\ell,k'\}} u(c,\ell) + \beta E\left[V\left(k', z', K'\right)|z\right]$$

$$\text{s.t.} \qquad c + k' = (1+r)\,k + w\ell$$

- This problem looks a lot like the one we have been working with
- But the problem is incomplete:
  1. Where do prices come from?
  2. How to update aggregate states?

Key: little k (the individual state) and big K (the aggregate state)

- In equilibrium they are the same, but the agent does not know it

A DP problem for the consumer: (k, K)

$$V(k,z,K) = \max_{\{c,\ell,k'\}} u(c,\ell) + \beta E\left[V\left(k',z',K'\right)|z\right]$$

$$\begin{aligned}
\text{s.t.} \qquad & c + k' = (1+r)\,k + w\ell \\
& r = R(z,K,L) \\
& w = W(z,K,L) \\
& L = G_\ell(z,K) \\
& K' = G_k(z,K) \\
& z' = h(z,\eta)\,,\ \text{with } \eta \text{ stochastic}
\end{aligned}$$

Key: Find functions R, W, G_ℓ and G_k.

- Given these you can solve consumer's problem

A recursive competitive equilibrium

An RCE is a set of a value function V, policy functions g_k and g_ℓ, updating functions G_k and G_ℓ and price functions R and W such that:

1. The value function V and policy functions g_k and g_ℓ solve the DP problem in previous slide
2. Pricing functions R and W satisfy the firm's first order conditions

$$R(z, K, L) = f_k(z, K, L) - \delta \qquad W(z, K, L) = f_\ell(z, K, L)$$

3. Updating functions G_k and G_l are consistent with individual optimization

$$G_k(z, K) = g_k(K, z, K) \qquad G_\ell(z, K) = g_\ell(K, z, K)$$

Some comments

1. The definition of RCE didn't include market clearing explicitly
    - This is a device of the CRS technology of the firm
    - At equilibrium prices demand for inputs is perfectly elastic
    - Markets clear automatically
    - Not the case in all models
2. Consistency only has to apply **in equilibrium**
    - As you converge to the equilibrium consistency does not have to hold
    - The agent's DP can be solved given any update functions
3. Curse of dimensionality applies
    - You have to solve the agent's problem off-equilibrium
    - You need to know $g_k(k, z, K)$ for any combination of (k, K), even though in equilibrium $k = K$

RCE algorithm

Algorithm 1: RCE Algorithm

input : Guess for updating functions (G_k, G_ℓ)
output: $V, g_k, g_\ell, G_k, G_\ell$

1. Solve the DP problem of the agent given G_k, G_ℓ:
   $(V, g_k, g_\ell) = T(V; G_k, G_\ell)$ (a fixed point problem) ;
2. Update updating functions:
   $G_k(z, K) = g_k(K, z, K) \qquad G_\ell(z, K) = g_\ell(K, z, K)$;
3. Check convergence in updating functions ;
4. Repeat (1)-(3) until convergence ;

Some comments

- Why would this converge?
  - We no longer have the CMT... No reason for it to converge
  - ***Eppur si muove***
- How to get it to converge?
  - Carefully...
  - The best strategy is the tortoise strategy: Slowly but surely

  2' Dampened update of updating functions:

$$G_k^{n+1}(z,K) = \gamma g_k(K,z,K) + (1-\gamma)\, G_k^n(z,K)$$
$$G_\ell^{n+1}(z,K) = \gamma g_\ell(K,z,K) + (1-\gamma)\, G_\ell^n(z,K)$$

- VFI is simplified with the RCE

$$-\frac{u_\ell(c,\ell)}{u_c(c,\ell)} = w \longrightarrow \ell(c;w)$$

Algorithm 2: EGM for RCE problem

Function $\texttt{EGM}(V, \vec{k}, \vec{z}, \ell(z,k), k'(z,k), parameters)$:

for *i=1:*n_z *# productivity (aggregate state)* **do**

for *j=1:*n_k *# capital (aggregate state)* **do**

1. Evaluate prices: $r = R(\vec{z}_i, \vec{K}_j), w = W(\vec{z}_i, \vec{K}_j)$

for *h=1:*n_k *# capital (individual state)* **do**

2. Expectd value: $\mathbb{V} = \beta E\left[V\left(\vec{k}_h, z', G_k(\vec{z}_i, \vec{K}_j)\right)|\vec{z}_i\right]$

Requires interpolation on $K' = G_k(\vec{z}_i, \vec{K}_j)$

3. Consumption from Euler: $u_c\left(\tilde{c}_{ijh}, \ell(\tilde{c}_{ijh}; w)\right) = \mathbb{V}_k$

Analytical solution using $\ell(c; w)$ from FOC

4. Endogenous $\hat{k}$: $\hat{k}_{ijh} = \left(\tilde{c}_{ijh} + \vec{k}_h - w\ell(\tilde{c}_{ijh}; w)\right)/1 + r$

5. $\hat{V}$ at end. grid: $\hat{V}(\hat{k}_{ijh}; \vec{z}_i, \vec{K}_j) = u\left(\tilde{c}_{ijh}, \ell(\tilde{c}_{ijh}; w)\right) + \mathbb{V}$

6. Interpolate to exogenous grid: V_new[:,i,j] = Interp$(\hat{k}, \hat{V}, \vec{k})$

Some comments

- In most cases we can now solve everything analytically
  - Big advantage of EGM in these problems
  - Some cases we still need to change states to Y, or to Y_k

$$c + k^{'} \leq Y_k + w\ell$$

  Think of problems with agents that can invest, or manage businesses

- No requirement that the grids on capital have to match: $\vec{K} \neq \vec{k}$
  - Often they are the same, but K only used for prices (usually smoother)
- We do have to interpolate in taking expectations
  - EGM is fixing the future capital of the agent
  - The future capital of the economy is exogenous (to the agent)
  - The agent has to be "consistent" and use G_k to forecast $K^{'}$

RCE applications

Many applications for RCE, but first:

- Check that your code works!
- The NGM's last gift to you ... Contrast RCE with Planner's DP problem

Applications (all your heart's desire):

- Taxes (distortions in general)
- Multiple agents
- Externalities
- Business Cycle Accounting
- Non-stationary problems (transitions, life-cycle)

Application: Taxes/Wedges

Taxes (or wedges)

- Classical question in economics: Effect of taxes
- Distortionary taxes prevent us from using the planner's problem to solve for the market equilibrium
  - In fact that is the point! We want to know how to make the market equilibrium closer to the planner's solution
- Usual taxes:
  - Labor income taxes (possibly non-linear)
  - Capital income taxes or wealth taxes
  - Consumption taxes (dangerous!)

Taxes (or wedges) - Agent's problem

$$
\begin{aligned}
V(k,z,K;\tau) = &\max_{\{c,\ell,k'\}} u(c,\ell) + \beta E\left[V\left(k',z',K';\tau\right)|z\right] \\
\text{s.t.} \quad & (1+\tau_c)c + k' = (1+(1-\tau_k)r)\,k + (1-\tau_\ell)w\ell + T \\
& r = R(z,K,L) \\
& w = W(z,K,L) \\
& L = G_\ell(z,K) \\
& K' = G_k(z,K) \\
& z' = h(z,\eta)\text{, with } \eta \text{ stochastic}
\end{aligned}
$$

Some comments

- Taxes do not need to be constant
  - You can have functions $\tau(z, K)$ (say for countercyclical policy)
  - You might need to find those functions in equilibrium (yet another loop!)
- This problem is independent of the government's budget
  - Agent takes taxes as given
  - These taxes need not balance the budget
  - This is important for interpretation as wedges (next slide)
- What if you do care about the budget...
  1. Are you balancing the budget every period? Need to search for $\tau(z, K)$
  2. Are you allowing for deficit/surplus?
     - Where is Gov. getting/putting funds? Figure out effect on market clearing

Taxes as wedges

$$u_c(c,\ell) = \beta\left(1+(1-\tau_k)r\right)u_c\left(c',\ell'\right)$$
$$-\frac{u_\ell(c,\ell)}{u_c(c,\ell)} = \left(\frac{1-\tau_\ell}{1+\tau_c}\right)w$$

- Taxes show up in the solution to the model as wedges in FOC
- You can rebate (lump-sum) the "tax revenue"
  - Taxes only affect combination, not level
- This is a powerful idea
  - Front and center in public economics
  - Equivalence results between models (many ways of getting same wedges)
  - Implications for measurement: Business Cycle Accounting (BCA)

Non-linear taxes: Two options

1. Map the tax system
   - Different tax brackets have different rates (e.g., exemption levels)

2. Approximate tax system with smooth function

   Benabou (2000-AER, 2002-ECMA) Heathcote, Storesletten & Violante (2017-QJE)

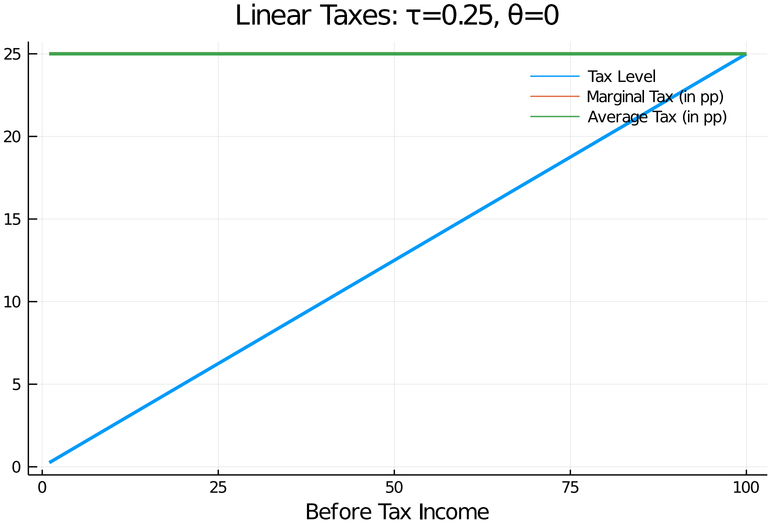
   - If an agent has income y then after tax income is:

$$Y(y) = (1-\tau)\, y^{1-\theta} + \underline{y} \qquad T(y) = y - Y(y)$$

   - Without transfers $\left(\underline{y}\right)$ and zero progressivity $(\theta = 0)$ we get tax rate τ
   - Taxes are progressive (regressive) if ratio of marginal to average tax is larger (smaller) than 1

$$\frac{\text{mrg tax}}{\text{ave tax}} = \frac{1 - T'(y)}{1 - {}^{T(y)}/_{y}} = \frac{(1-\theta)(1-\tau)\, y^{-\theta}}{(1-\tau)\, y^{-\theta} + \frac{\underline{y}}{y}} \leq (1-\theta)$$

Non-linear taxes



Taxes (or wedges) - RCE

An RCE is a set of a value function V, policy functions g_k and g_ℓ, updating functions G_k and G_ℓ and price functions R and W such that, given taxes, transfers and expenditure $\{\tau_k, \tau_\ell, \tau_c, T, G\}$:

1. The $\{V, g_k, g_\ell\}$ solve the agent's DP problem
2. Pricing functions R and W satisfy the firm's first order conditions

$$R(z, K, L) = f_k(z, K, L) - \delta \qquad W(z, K, L) = f_\ell(z, K, L)$$

3. Updating functions G_k and G_l are consistent with agent optimization

$$G_k(z, K) = g_k(K, z, K) \qquad G_\ell(z, K) = g_\ell(K, z, K)$$

4. Market clearing/Balanced budget:

$$G + c + K^{'} = f(z, K, L) \qquad \text{or: } G + T = \tau_c c + \tau_k R(z, K, L) K + \tau_w W(z, K, L) L$$

Taxes (or wedges) - Algorithm

Algorithm 3: RCE Algorithm with taxes/wedges

input : Guess for taxes/wedges $(G, T, \tau_k, \tau_c, \tau_\ell)$
output: $V, g_k, g_\ell, G_k, G_\ell, G, T, \tau_k, \tau_c, \tau_\ell$

1. Guess (G_k, G_ℓ) ;
2. Solve the DP problem of the agent given $G_k, G_\ell, G, T, \tau_k, \tau_c, \tau_\ell$:
   $(V, g_k, g_\ell) = T(V; G_k, G_\ell, G, T, \tau_k, \tau_c, \tau_\ell)$ (a fixed point problem) ;
3. Update updating functions:
   $G_k(z, K) = g_k(K, z, K) \qquad G_\ell(z, K) = g_\ell(K, z, K)$;
4. Check convergence in updating functions ;
5. Repeat (2)-(4) until convergence ;
6. Verify market clearing - Adjust taxes/transfer/spending ;
7. Repeat (1)-(6) until market clears ;

Some comments

- General equilibrium == Outer loops
  - Outer loops are very expensive!
  - You have to do everything over and over again
- Not all taxes/wedges can be free to choose

$$G + T = \tau_c c + \tau_k r K + \tau_w w L$$

  - Something has to be fixed
  - Sometimes it is taxes, sometimes it is expenditure
- Further complication: Dynamics
  - Taxes here are static, so is the budget
  - In general there can also be debt with deficit/surpluses
  - Change in market clearing ($K = k - D$), non-stationarity (transitions)

Application: Multiple Agents

Multiple agents - Model

- We already saw one of these:
  - Capitalist/Union model
- Back then we cheated:
  - Union does not optimize... instead it fixes wages to avoid GE
- Lets try again

Multiple agents - Model

- There are three types of agents:
  - Capitalists
  - High-skilled workers
  - Low-skilled workers
- Capitalists do not work but they own capital
- Workers are hand to mouth
- Production combines skill types with capital

Capitalists

$$
\begin{aligned}
V\left(k, z, K; w_l, w_h\right) = \max_{\left\{c, k'\right\}} \; & u\left(c\right) + \beta E\left[V\left(k', z', K'; w_l', w_h'\right) |z\right] \\
\text{s.t. } & c + k' \leq \pi\left(z, k; w_l, w_h\right) \\
& \pi\left(z, k; w_l, w_h\right) = \max_{\ell} f\left(z, k, \ell_l, \ell_h\right) - w_l \ell_l - w_h \ell_h + \left(1 - \delta\right) k \\
& \log z' = \rho \log z + \eta; \qquad \eta \sim N\left(0, \sigma_\eta^2\right)
\end{aligned}
$$

- The production function is key
  - See Krusell, Ohanian, Rios-Rull & Violante (2000,ECMA)
- Capitalist needs to distinguish between k and K to predict wages

Workers

- The problem of the workers is symmetric and static:

$$\max\ u^i\left(w_i\ell,\ell\right) \qquad \text{fot } i=\{l,h\}$$

- Key here is the FOC given wages:

$$u^i_\ell\left(w_i\ell,\ell\right)=w_i u^i_c\left(w_i\ell,\ell\right)$$

  - This condition gives closed form of $\ell_i\left(w_i\right)$ and $c_i\left(w_i\right)$

Market clearing - Labor

- From the profit maximization problem we get

$$w_l = f_l\left(z, K, \ell_l\left(w_l\right), \ell_h\left(w_h\right)\right)$$
$$w_h = f_h\left(z, K, \ell_l\left(w_l\right), \ell_h\left(w_h\right)\right)$$

  - Solve for price functions that depend on aggregate states (z, K)

- Is it clear why these conditions imply market clearing?

Multiple agents - RCE

An RCE is a set of a value function V and policy function g_k for capitalists, updating function G_k and price functions W_L and W_H such that:

1. The value function V and policy functions g_k and g_ℓ solve the DP problem in previous slide
2. Pricing functions W_L, W_H satisfy the firm's first order conditions

$$W_L(z,K) = f_l\left(z,K,\ell_l\left(W_L(z,K)\right),\ell_h\left(W_H(z,K)\right)\right)$$
$$W_H(z,K) = f_h\left(z,K,\ell_l\left(W_L(z,K)\right),\ell_h\left(W_H(z,K)\right)\right)$$

3. Updating functions G_k and G_l are consistent with individual optimization

$$G_k(z,K) = g_k(K,z,K)$$

Application: Business Cycle Accounting

Business Cycle Accounting (CKM,2007)

- Main idea:
  - Use the model as a measurement device
- Change the question:
  - What are the effects of a shock or a policy?
  - What shock or policy could have generated the observed data?
- This is a crucial way in which we think about models
  - How to explain the world we have seen?
  - Which frictions or policies are most relevant?

Business Cycle Accounting (CKM,2007)

Method:

1. Use a "prototype" model with wedges
   - The model can fit the data by construction by adjusting wedges
2. Analyze data with the model to recover wedges
   - Which wedges are important for the data?
3. Establish equivalence results between models and wedges
   - Some are obvious: wedges look like taxes
   - Some are not obvious: wedges can represent financial frictions

Application: Sovereign Default

Sovereign default

- Default models form a large literature on international econ
- Great example of dynamic programming:
  - Default option is inherently dynamic
- Great example of RCE:
  - Default and savings choice depend on prices!
  - Prices are endogenous... but taken as given

Basic model - Arellano (2008)

- (Stochastic) Endowment economy
  - Output follows an exogenous Markov process
- Benevolent government (planner) chooses:
  - Borrowing/savings and whether to default on debt
- (Risk-neutral) Financial intermediary
  - Breaks even in expectation (wrt default)
- Default repercussion: Autarky
  - Output penalty during autarky
  - Autarky costly because of income fluctuation
  - Autarky ends with probability $\lambda \geq 0$

Sovereign default - Prices

Profits of intermediary:

$$\Pr = qb^{'} - \frac{1-\delta}{1+r}b^{'} \longrightarrow \Pr = 0$$

- Here δ is the probability of default
- δ is endogenous, in fact:

$$\delta = E_{s'}\left[g^{D}\left(s^{'}, b^{'}\right)|s\right] \qquad \text{where } g^{D}\left(s^{'}, b^{'}\right) = \begin{cases} 1 & \text{if default} \\ 0 & \text{if no default} \end{cases}$$

Free entry gives the zero profit condition:

$$q\left(s, b'\right) = \begin{cases} \frac{1-\sum\limits_{s^{'}\in S}\pi(s')g^{D}(s',b')}{R} & \text{if } b' < 0 \\ \frac{1}{R} & \text{if } b' \geq 0 \end{cases}$$

Sovereign default - DP

$$V^{\star}(s,b) = \max_{d\in\{0,1\}} \left\{(1-d(s,b))\,V(s,b) + d(s,b)\,V^{A}(s)\right\}$$

$$\begin{aligned}
V(s,b) &= \max_{\{c,b'\}} \left\{ \frac{c(s,b)^{1-\sigma}-1}{1-\sigma} + \beta \sum_{s'\in S} \pi\left(s'\right) V^{\star}(s',b') \right\} \\
\text{s.t.} \quad & c(s,b) - q(s,b)\,b'(s,b) \leq y(s) + b \\
& -B \leq b'(s,b) \qquad \text{[B: borrowing limit]} \\
& 0 \leq c(s,b)
\end{aligned}$$

$$V^{A}(s) = \frac{h(y(s))^{1-\sigma}-1}{1-\sigma} + \beta \sum_{s'\in S} \pi(s')\left(\lambda V^{\star}(s',0) + (1-\lambda)\,V^{A}(s')\right)$$

Sovereign default - RCE

A **Recursive Competitive Equilibrium** is

1. Value functions $V^{\star}, V^{A}, V$
2. Policy functions $g^{c}(s,b), g^{b}(s,b), g^{D}(s,b)$
3. Price functional given by $q(s,b) = \frac{\left(1-\Sigma\pi(s')g^{D}(s',b')\right)}{R}$

Such that the value functions and policy functions solve the DP of previous slide taking q as given.