Advanced Macroeconomics II

Handout 9 - Search Models I

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  - Focus on employment vs unemployment
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- Sub-type of "occupational choice models"
  - Focus on employment vs unemployment
  - Applies more broadly!
- **Key:** Type changes cannot happen at will
  - Search frictions explain inability to switch occupations at will
  - Technically: opportunities to switch arrive randomly

Partial equilibrium vs General equilibrium

- Are wages determined in equilibrium?

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  1. Draw wage offers from exogenous distribution (McCall, 1970)
  2. Draw wage offers from endogenous distribution (Burdett & Mortensen, 1998)
  3. Determine wages by bargaining (upon matching) (Pissarides, 1990)

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- Are search frictions determined in equilibrium?
  1. Offers follow an exogenous process (arrive with fixed probability)
  2. Offers are endogenous but one-sided (depend on search effort)
  3. Offers follow an endogenous process (depends on aggregate behavior)

Idiosyncratic and aggregate heterogeneity

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- Aggregate heterogeneity is harder (sometimes)
  - Can you aggregate the results of your model?
    - No need to track individual behavior
  - Can you solve the model block-recursively?
    - No need to know aggregates from the beginning

Three families of models

1. McCall models
2. DMP models
3. Directed/Competitive search models

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1. McCall models
2. DMP models
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Objectives:

1. Give you an overview of these models
2. Bare-bones application
3. Show you continuous and discrete time versions

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- Agents have linear utility and discount utility at rate β

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1. Employed (boring agents)
   - Agents are hand-to-mouth
   - Characterized by a wage w
   - Employment is permanent (no firing/quitting/on-the-job-search)

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1. Employed (boring agents)
    - Agents are hand-to-mouth
    - Characterized by a wage w
    - Employment is permanent (no firing/quitting/on-the-job-search)
2. Unemployed (slightly more interesting)
    - Agents are hand-to-mouth
    - All unemployed are equal:
        - All receive the same unemployment benefits $b > 0$
    - Wage offer received every period from some distribution

Aside: Interpretation issues

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  - Value of leisure? Value of home production? Value of unemployment insurance?

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- Large debate in the literature
  - Chorodow-Reich & Karabarbounis (2016) and Hall & Millgrom (2005) look at this in relation to the cyclicality of unemployment
  - Aguiar, Hurst & Karabarbounis (2013) Series of papers on time use data show what b can be

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- Linear utility is feature/bug of many search models (DMP/BurdettMortensen)

Benefits/Drawbacks

- Flexible framework for a variety of questions:
  - Effect of unemployment benefits/welfare programs // Effect of human capital on search (and uncertainty about human capital) // Effect of savings and access to credit for employment fluctuations

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  - Model real world policies and qualifying criteria for welfare programs

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- Flexible framework for a variety of questions:
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- Model can handle a lot of individual heterogeneity
  - Model real world policies and qualifying criteria for welfare programs
- Model is in partial equilibrium!
  - Wage offers and job offers are exogenous
  - Subject to Lucas Critique
  - Can be computationally (depends on VFI)

Basic McCall - Set up

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    "Over 40% of the employed workers who separate into unemployment ("EU" flow) return, after the jobless spell, to their last employer"

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    "Over 40% of the employed workers who separate into unemployment ("EU" flow) return, after the jobless spell, to their last employer"
- Notation:
  - $W(w)$ is the value of an employed agent with wage w
  - U is the value of an unemployed agent

Basic McCall - Bellman equations

Employed agents:

$$W(w) = w + \beta W(w) \longrightarrow W(w) = \frac{w}{1-\beta}$$

:

$$W(w) = \sum_{t=0}^{\infty} \beta^t w = w \sum_{t=0}^{\infty} \beta^t = \frac{w}{1-\beta}$$

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Note: Easy to extend to include expiration of benefits, life cycle, human capital and earning penalties, family or joint search

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- The solution of the model is an acceptance rule for jobs
  - We are looking for $w \in [0, \infty]$ such that $W(w) \geq U$
  - We know that W is increasing in wages, and that U is constant
  - Then there exists a unique w^R such that

$$W\left(w^R\right) = U$$

... and thus $W(w) > U$ for $w > w^R$

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- So acceptance rule takes the form of a reservation wage w^R

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$$\frac{w^R}{1-\beta} = b + \frac{\beta}{1-\beta}\int_0^\infty \max\left\{w^R, w\right\} dF(w)$$

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- Turns out, this equation is the fixed point of a contraction!

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$$w^R = T\left(w^R\right) = (1-\beta)\, b + \beta \int_0^\infty \max\left\{w^R, w\right\} dF(w)$$

- We can solve by iterating following the CMT!

Basic McCall - Reservation wage

We can also get a bit more with some integration:

$$\begin{aligned}\beta \int_0^{\infty} \max\left\{w^R, w\right\} dF\left(w\right) &= \beta w^R F\left(w^R\right) + \beta \int_{w^R}^{\infty} w dF\left(w\right) \\ &= \beta w^R + \beta \int_{w^R}^{\infty} \left(w - w^R\right) dF\left(w\right)\end{aligned}$$

Basic McCall - Reservation wage

Then use integration by parts (treating the ∞ with care):

$$\begin{aligned}
\lim_{\overline{w}\to\infty}\int_{w^R}^{\overline{w}}\left(w-w^R\right)dF\left(w\right) &= \lim_{\overline{w}\to\infty}\left[\left(w-w^R\right)F\left(w\right)\Big|_{w^R}^{\overline{w}}-\int_{w^R}^{\overline{w}}F\left(w\right)dw\right] \\
&= \lim_{\overline{w}\to\infty}\left[\left(\overline{w}-w^R\right)F\left(\overline{w}\right)-\int_{w^R}^{\overline{w}}F\left(w\right)dw\right] \\
&= \lim_{\overline{w}\to\infty}\left[\int_{w^R}^{\overline{w}}F\left(\overline{w}\right)-F\left(w\right)dw\right] \\
&= \int_{w^R}^{\infty}\left(1-F\left(w\right)\right)dw
\end{aligned}$$

Basic McCall - Reservation wage

Replacing back we get the standard representation:

$$w^R = b + \frac{\beta}{1-\beta} \int_{w^R}^{\infty} (1 - F(w))\, dw$$

- Given a functional form for F we can actually solve the integral
- Pareto distributions are popular for closed form solutions

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- You can do this by iterating on the contraction mapping
  - That is very slow
- Better to approach it as root finding:

$$G(x) = x - b - \frac{\beta}{1-\beta}\int_x^{\infty}(1 - F(w))\,dw$$

  - w^R is a root of this function

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- What happens if b changes?
- What happens if F changes? (application to mean preserving spreads)
- Implications for duration of unemployment
  - Let N be the periods of unemployment?
  - $\Pr(N=1) = 1 - F\left(w^R\right)$ (probability of an acceptable offer)
  - $\Pr(N=2) = F\left(w^R\right)\left(1 - F\left(w^R\right)\right)$ (probability of an unacceptable offer followed by an acceptable offer)
  - $\Pr(N=n) = \left(F\left(w^R\right)\right)^n\left(1 - F\left(w^R\right)\right)$ (key: job offers are iid)
  - Unemployment duration distributed geometrically with $\lambda = F\left(w^R\right)$
  - Mean unemployment duration is $1/(1-F(w^R))$

Heterogeneity - What about the distribution?

- In the limit all agents become employed
- Distribution of employed is easy:
  - Truncated distribution of job offers: $G(w) = \frac{F(w)}{1-F\left(w^R\right)}$

McCall Models

Continuous Time

Bellman equations

- New notation:
  - α: Odds that a worker gets an offer (or average number of offers per period)
  - Δ: length of a period
  - r: Interest rate for discounting $\left(\beta = \frac{1}{1+r}\right)$

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- Value of an employed agent:

$$U = \Delta b + \frac{\Delta \alpha}{1+\Delta r} \int_0^\infty \max\{U, W(w)\}\, dF(w) + \frac{1-\Delta\alpha}{1+\Delta r} U$$

$$rU = b + \alpha \int_0^\infty \max\{0, W(w) - U\}\, dF(w)$$

Reservation wage

- Just as before we want w^R such that $rU = rW\left(w^R\right) = w^R$

$$w^R = b + \frac{\alpha}{r}\int_0^{\infty} \max\left\{0, w - w^R\right\} dF\left(w\right)$$
$$w^R = b + \frac{\alpha}{r}\int_{w^R}^{\infty} w - w^R dF\left(w\right)$$
$$w^R = b + \frac{\alpha}{r}\int_{w^R}^{\infty} 1 - F\left(w\right) dw$$

- This is (basically) the same equation we had before

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- They are everywhere in continuous time
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- In this case the Poisson process refers to the arrival of job offers
- Let X_t be a random variable that indicates the number of job offers received by time t. Then $X_t \in \{0, 1, 2, \ldots\}$.
- X_t is a stationary Markov process
  - Probability that $X_t = m$ goes to $X_{t+h} = m + 1$ is independent of m and t
  - We only care about the increment in time and increment in offers

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- $P_0(t+\Delta)$ is the probability no offers occur until $t+\Delta$
  - Equivalent to no offers until t **and** no offers between t and $t+\Delta$
  - Probability of no offers between t and $t+\Delta$ is **independent** of history

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$$P_0(t+\Delta) = P_0(t)\,P_0(\Delta) = P_0(t)\,(1 - \Delta\alpha + o(\Delta))$$

- We can define the change in $P_0(t)$ with respect to time:

$$P_0'(t) = \frac{P_0(t+\Delta) - P_0(t)}{\Delta} = P_0(t)\left(-\alpha + \frac{o(\Delta)}{\Delta}\right) = -\alpha P_0(t)$$

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- We can solve this differential equation: $P_0(t) = e^{-\alpha t}$

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- The average unemployment duration is then:

$$D=\int_0^\infty \underbrace{t}_{\text{Duration}}\cdot\underbrace{\alpha\left(1-F\left(w^R\right)\right)}_{\text{Pr. Accepting an offer in } t}\cdot\underbrace{e^{-\alpha\left(1-F\left(w^R\right)\right)t}}_{\text{Pr. No offer until } t}dt=\frac{1}{\alpha\left(1-F\left(w^R\right)\right)}$$

McCall Models

Extension: Job Destruction

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$$W(w) = \Delta w + \frac{1 - \Delta\delta}{1 + \Delta r} W(w) + \frac{\Delta\delta}{1 + \Delta r} U$$
$$rW(w) = w + \delta (U - W(w))$$

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- Value of an unemployed (same as before):

$$rU = b + \alpha \int_0^\infty \max\left\{0, W(w) - U\right\} dF(w)$$

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- From the value of the employed we get: $w^R = rW\left(w^R\right) = rU$

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\begin{aligned}
r\left(W\left(w\right) - U\right) &= w - w^R + \delta\left(U - W\left(w\right)\right) \\
\left(r + \delta\right)\left(W\left(w\right) - U\right) &= w - w^R
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- Replacing on the value of the unemployed:

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\begin{aligned}
w^R &= b + \frac{\alpha}{r + \delta}\int_{w^R}^{\infty} w - w^R dF\left(w\right) \\
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- We now discount with a lower rate (because of separations)

Evolution of employment/unemployment

We now have flows in and out of states:

$$
\begin{aligned}
u_{t+\Delta} &= u_t - \Delta\alpha\left(1 - F\left(w^R\right)\right) + \Delta\delta e_t + o\left(\Delta\right) \\
\frac{u_{t+\Delta} - u_t}{\Delta} &= -\alpha\left(1 - F\left(w^R\right)\right)u_t + \delta\left(1 - u_t\right) + \frac{o\left(\Delta\right)}{\Delta} \\
\dot{u} &= -\alpha\left(1 - F\left(w^R\right)\right)u_t + \delta\left(1 - u_t\right)
\end{aligned}
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u_{t+\Delta} &= u_t - \Delta\alpha\left(1 - F\left(w^R\right)\right) + \Delta\delta e_t + o\left(\Delta\right) \\
\frac{u_{t+\Delta} - u_t}{\Delta} &= -\alpha\left(1 - F\left(w^R\right)\right) u_t + \delta\left(1 - u_t\right) + \frac{o\left(\Delta\right)}{\Delta} \\
\dot{u} &= -\alpha\left(1 - F\left(w^R\right)\right) u_t + \delta\left(1 - u_t\right)
\end{aligned}
$$

The steady state of unemployment is $\dot{u} = 0$, so:

$$
u_{ss} = \frac{\delta}{\delta + \alpha\left(1 - F\left(w^R\right)\right)}
$$

We can get comparative statics through total differential of reservation wage equation

More results: Dispersion of wages

Results from Hornstein, Krusell & Violante (2011)

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- Objective: Compute “mean to min” ratio of wages

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- Go back to the value of an unemployed:

$$rU = b + \alpha \int_{w^R}^{\infty} \left(W\left(w\right) - U\right) dF\left(w\right)$$

$$w^R = b + \frac{\alpha}{r+\delta} \int_{w^R}^{\infty} w - w^R dF\left(w\right)$$

$$w^R = b + \frac{\alpha\left(1 - F\left(w^R\right)\right)}{r+\delta} \int_{w^R}^{\infty} w - w^R d\frac{F\left(w\right)}{1 - F\left(w^R\right)}$$

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1. $jf = \alpha\left(1 - F\left(w^R\right)\right)$ is the job finding rate
2. $G\left(w\right) = \frac{F(w)}{1-F\left(w^R\right)}$ is the distribution of wages of the employed

More results: Dispersion of wages

Some algebra:

$$w^R = b + \frac{jf}{r+\delta}\int_{w^R}^{\infty} w - w^R dG(w)$$

$$w^R = b + \frac{jf}{r+\delta}\left(\int_{w^R}^{\infty} w dG(w) - w^R \int_{w^R}^{\infty} dG(w)\right)$$

$$w^R = b + \frac{jf}{r+\delta}\left(\overline{w} - w^R\right)$$

$$1 = b + \frac{jf}{r+\delta}\left(\frac{\overline{w}}{w^R} - 1\right)$$

the result:

$$\frac{\overline{w}}{w^R} = \frac{1 + \frac{jf}{r+\delta}}{r + \frac{jf}{r+\delta}}$$

McCall Models

Extension: Human capital

Model setup

Results based on Ljungqvist & Sargent (1998)

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- Study hysteresis (persistence) of unemployment in Europe

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Model setup

Results based on Ljungqvist & Sargent (1998)

- Study hysteresis (persistence) of unemployment in Europe
- Focus on scarring effects of unemployment
  - Depreciation of human capital
- Welfare benefits extend unemployment duration
  - Side effect: Long term unemployment due to low human capital
  - Better to stay on welfare than to work if human capital is too low

Model setup

- Discrete time and a continuum of workers with linear utility
  - Perpetual youth model with survival probability α

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- Variable search effort: $s > 0$
  - Effort has a cost $C(s)$ and changes probability of offers $\pi(s)$

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  - h increases while employed and decreases while unemployed

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  - h increases while employed and decreases while unemployed
- Unemployment income b
  - Benefits financed with income taxes τ
  - Optional: Benefits expire with probability p_b and go to $\underline{b}$

Assumptions on human capital

- Human capital changes in discrete amounts
  - Human capital lies on a grid:

$$\vec{h} = \left[\underline{h}, \underline{h} + \Delta, \ldots, \underline{h} + n\Delta, \ldots, \overline{h}\right]$$

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- Changes in human capital are capped by $\underline{h}$ and $\overline{h}$
- Newborns are unemployed with some h

Bellman equations

Employed agent:

$$W(w,h) = (1-\tau)\,wh + \alpha\beta\delta U\left(\max\{h-\psi_f\Delta, \underline{h}\}\right) \\ + \alpha\beta(1-\delta)\,W\left(w, \min\left\{h+\Delta, \overline{h}\right\}\right)$$

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Unemployed agent:

$$U(h) = \max_{s} \quad (1-\tau)\,b - C(s) + \alpha\beta\pi(s)\int_0^\infty \max\left\{W\left(w,h^{'}\right),U\left(h^{'}\right)\right\}dF(w) \\ + \alpha\beta(1-\pi(s))\,U\left(h^{'}\right)$$

$$\text{s.t. } h^{'} = \max\left\{h-\psi_u\Delta,\underline{h}\right\}$$

Government

- Let $G^U(h)$ be the distribution of h for the unemployed
- Let $G^W(w,h)$ be the joint distribution of h for the employed
  - We set it up such that: $\int_h dG^U(h) + \int_w \int_h dG^W(w,h) = 1$
  - The unemployment rate is $u = \int_h dG^U(h)$

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  - The unemployment rate is $u = \int_h dG^U(h)$
- Government solves for τ such that:

$$\int_h bdG^U(h) = \int_h \tau bdG^U(h) + \int_w \int_h \tau whdG^W(w,h)$$

$$(1-\tau)\,bu = \tau \int_w \int_h whdG^W(h)(w,h)$$

$$(1-\tau)\,bu = \tau E[wh](1-u)$$

Stationary RCE

A Stationary RCE is a list of policy functions for the unemployed $\left\{s^{\star}(h), w^{R}(h)\right\}$, value functions for the employed and unemployed $\{W(w, h), U(h)\}$, a stationary distribution for employed and unemployed agents $\left\{G^{U}, G^{W}\right\}$, and a tax rate $\{\tau\}$, such that:

1. Given taxes and the value function of the employed, policy functions are optimal for the unemployed, and the value function satisfies the Bellman equation
2. Given taxes and the value function of the unemployed, the value function of the employed satisfies the Bellman equation
3. Given policy functions the distributions are stationary
4. Given distributions tax rate balances the government budget

Algorithm

0. Discretize wages (you can also discretize the offer distribution F)
1. Choose a tax rate
2. Solve for the value functions using VFI
   2.1 In each VFI step solve the policy functions of the unemployed
3. Use histogram method to update the distributions. Two options:
   3.1 List all states in a vector

   $$\{(U, \underline{h}), \ldots, (U, \overline{h}), (W, w_1, \overline{h}), \ldots, (W, w_N, \underline{h}), \ldots, (W, w_1, \bar{h}), \ldots, (W, w_N, \overline{h})\}$$

   Create transition matrix.

   3.2 Use loops to update. This can be easier to code. Probably as fast.
4. Compute the government deficit
5. Update tax rate (say with golden-section or any root-finding)

McCall Models

Extension: Savings and Income Risk

Model setup

- Continuous time and a continuum of risk averse agents

$$E_0 \left[\int_0^\infty e^{-\rho t} \left(u\left(c_t\right) - e\left(s_t\right) \right) dt \right]$$

  - Utility is CRRA $u\left(c_t\right) = {}^{c_t^{1-\gamma}-1}/_{1-\gamma}$ and effort cost is $e\left(s_t\right) = {}^{\mu s^\eta}/_{\eta}$

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- Agents can save is risk-less asset:

$$\dot{a}=ra+\underbrace{y}_{\text{Income}}-c \qquad \text{s.t. } a\geq\underline{a}$$

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  - Job offers come from distribution $F\left(w\right)$

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  - Job offers come from distribution $F\left(w\right)$
- Jobs end endogenously (on-the-job-search) or exogenously at rate δ

Bellman equations

Employed agent:

$$\rho W(a,w) = \max_{\{c,s\}} \quad u(c) - e(s) + W_a(a,w)\underbrace{(ra + w - c)}_{\dot{a}} + \lambda s \int \max\{W(a,x) - W(a,w), 0\}\, dF(x) + \delta (U - W(a,w))$$

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Unemployed agent:

$$\rho U(a) = \max_{\{c,s\}} \quad u(c) - e(s) + U_a(a) \underbrace{(ra + b - c)}_{\dot{a}} + \lambda s \int \max\{W(a,x) - U(a), 0\} dF(x)$$

Optimality conditions

The consumption and search effort choices are immediate:

$$
\begin{aligned}
u^{'}\left(c^{U}\left(a\right)\right) &= U_{a}\left(a\right) \\
e^{'}\left(s^{U}\left(a\right)\right) &= \lambda \int \max\left\{W\left(a,x\right) - U\left(a\right), 0\right\} dF\left(x\right) \\
u^{'}\left(c^{W}\left(a,w\right)\right) &= W_{a}\left(a,w\right) \\
e^{'}\left(s^{W}\left(a,w\right)\right) &= \lambda \int \max\left\{W\left(a,x\right) - W\left(a,w\right), 0\right\} dF
\end{aligned}
$$

reservation wages are also simple:

$$
\begin{aligned}
w_{R}^{U}\left(a\right) &= b \\
w_{R}^{W}\left(a,w\right) &= w
\end{aligned}
$$

Immediate from job offer and separation rates being independent of (a, w)

Solution

- You can use finite difference or Markov chain approximation to solve this problem
- You can also use simple (but slower) value function iteration
- For the distribution use the KFE or histogram method
- Careful with how to manage draws from wages
- You can also pose the equivalent discrete time version and use all the tools we have
- Even without the numerical solution Jeremy shows that the model gives many closed form expressions (READ THE PAPER!)

Other Resources

Other resources

- Quantecon has a good set of notes on how to solve these models
  - Click here and check Lectures I-V
- Sargent & Junqvist - The book!

Applications

- UI: Chetty (2008, JPE), Schmieder et al (2013)
- Earnings losses after layoffs: Kolikowski (2013)
- Bankruptcy: Athreya & Simpson (2006), Chen (2012), Athreya et al (2014)
- Mortgage Crisis: Herkenhoff & Ohanian (2012)