Advanced Macroeconomics II

Handout 6 - Speeding Up VFI

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February 20, 2023

Short recap

Prototypical DP problem:

$$V(z,k) = \max_{\{c,k'\}} u(c) + \beta E\left[V\left(z', k'\right)|z\right]$$
$$\text{s.t.} c + k' = f(z,k)$$
$$z' = h(z,\eta);\eta \text{ stochastic}$$

- We are looking for functions V, g^c, g^k: We cannot solve this.

We need to solve an approximate problem:

- Approximate continuous function: **Interpolation**
  - Requires "exact" solution of maximization problem: **Optimization**
  - Requires computing expectations: **Integration**

Why is VFI so costly?

$$V(z,k) = \max_{\{c,k'\}} u(c) + \beta E\left[V\left(z^{'},k^{'}\right)|z\right]$$

- Combination of maximization and expectations is lethal
- Optimizing requires *a lot* of function evaluations
  - Each evaluation requires expectations, interpolations and often derivatives

Key idea:

- Can we bypass the maximization step?
- Focus on the Euler equation

Carroll (2006)

Maximization requires satisfying FOC:

$$u^{'}(c) = \beta E\left[V_k\left(z^{'}, k^{'}\right)|z\right] \qquad c + k^{'} = f(z,k)$$

Usual approach:

- Fix (z,k) and solve for $\left(k^{'}, c\right)$
- Consumption is immediately given $k^{'}$: $c = f(z,k) - k^{'}$
- Problem is to try a bunch of $k^{'}$ to solve

$$u^{'}\left(f(z,k) - k^{'}\right) = \beta E\left[V_k\left(z^{'}, k^{'}\right)|z\right]$$

Carroll's approach:

- Fix $\left(k^{'}, z\right)$ and solve for k! Hence the endogenous grid name
- Problem is to solve:

$$f(z,k) = \underbrace{(u')^{-1}\left(\beta E\left[V_k\left(z^{'}, k^{'}\right)|z\right]\right) + k^{'}}$$

Standard algorithm

Algorithm 1: EGM: Standard Method

Function `EGM`(V,$\vec{k}$,$\vec{z}$,*parameters*)**:**

- **for** i=1:n_z **do**
  - **for** j=1:n_k **do**
    - $\mathrm{F(x)} = f(\vec{z}_i, x) - \vec{k}_j - (u')^{-1}\left(\beta E\left[V_k\left(z', \vec{k}_j\right)|\vec{z}_i\right]\right)$
    - # Find [k_min,k_max], check corners, further bracket zero
    - k_endo[j] = Roots(F,k_min,k_max)
    - $\mathrm{V\_endo[j]} = u(f(\vec{z}_i, k\_endo[j]) - \vec{k}_j) + \beta E\left[V\left(z', \vec{k}_j\right)|\vec{z}_i\right]$
  - # Interpolate value function to exogenous grid
  - V_new(i,:) = Interpolation(k_endo,V_endo,$\vec{k}$)
- return V_new

Change of variable - Know your states

- A change of variable makes things easier
- Define Y as total income, or cash on hand: $Y = f(z, k)$

$$
\begin{aligned}
V(z, Y) &= \max_{\{k'\}} u\left(Y - k'\right) + \beta E\left[V\left(z', Y'\right) | z\right] \\
\text{s.t.}\, Y' &= f\left(z', k'\right) \\
z' &= h(z, \eta)\,; \eta \text{ stochastic}
\end{aligned}
$$

- Control variable k' (partially) determines future state
- We still need to hang onto z as a state, why?

Change of variable - Know your states

Note that Y' is a function of k' and z' so we can write

$$\mathbb{V}\left(z, k'\right) = \beta E\left[V\left(Y'\left(z', k'\right), z'\right)|z\right]$$
$$\mathbb{V}_k\left(z, k'\right) = \beta E\left[V_Y\left(Y'\left(z', k'\right), z'\right)\frac{\partial Y\left(z', k'\right)}{\partial k}|z\right]$$

Now the problem is:

$$V\left(z, Y\right) = \max_{\left\{k'\right\}} u\left(Y - k'\right) + \mathbb{V}\left(z, k'\right)$$
$$\text{s.t.}\, Y' = f\left(z', k'\right)$$
$$z' = h\left(z, \eta\right); \eta \text{ stochastic}$$

Modified EGM

Algorithm 2: EGM: Change of State Method

Function `EGM`(V,$\vec{k}$,$\vec{z}$,*parameters*)**:**

Note: You already know Y for any (z,k), let $Y_{ij} = Y(\vec{z}_i, \vec{k}_j)$

for *i=1:*n_z **do**

for *j=1:*n_k **do**

$$\mathbb{V}_j = \beta E\left[V\left(Y(z', \vec{k}_j), z'\right)|\vec{z}_i\right]$$

$\vec{c}$_endo = $(u')^{-1}.(\mathbb{V}_k)$ (Note: Evaluating whole vector)

$\vec{Y}$_endo = c_endo + $\vec{k}$

$\vec{V}$_endo = $u(\vec{c}\_endo) + \mathbb{V}$

Interpolate value function to exogenous grid

V_new[i,:] = Interpolation($\vec{Y}$_endo, $\vec{V}$_endo, $\vec{Y}[i,:]$)

Change variable back to k. Note: $V_{ji} = V(Y(\vec{z}_i, \vec{k}_j), z_i) = V(\vec{z}_i, \vec{k}_j)$

Some comments

The method still has some flexibility

1. How to compute derivatives (you can get it from interpolation step)
2. How to compute expectations (no interpolation if z is discrete)
3. How to judge convergence (standard practice is to pass $\mathbb{V}$ along and judge convergence with it)
4. How to map to capital after convergence
    4.1 Interpolation of $V\left(Y_{endo}\right)$ to $V\left(Y_{exo}\right)$: Y_{exo} maps to k by construction
    4.2 Keep Y_{endo}. Solve for $\vec{k}(z)$ s.t. $Y_{endo}(z) = f\left(z, \vec{k}(z)\right)$

Labor Supply

Labor supply adds an equation

$$
\begin{aligned}
V(z,k) = \max_{\{c,k'\}} \; & u(c,\ell) + \beta E\left[V\left(z',k'\right)|z\right] \\
\text{s.t.}\, c + k' &= f(z,k,\ell) \\
z' &= h(z,\eta)\,;\eta \text{ stochastic}
\end{aligned}
$$

FOC:

$$u_c(c,\ell) = \beta E\left[V_k\left(z',k'\right)|z\right] \qquad -u_\ell(c,\ell) = f_\ell(z,k,\ell) \qquad c+k' = f(z,k,\ell)$$

Attempting EGM - Problems

Change of variable:

$$Y(z,k) = f(z,k,\ell(z,k)) = zk^{\alpha}\ell(z,k)^{1-\alpha} + (1-\delta)k$$

- Cannot define exogenous grid for Y. Grid depends on policy function.

Euler equation:

$$u_c(c,\ell(z,k)) = \beta E\left[V_k\left(z^{'},k^{'}\right)|z\right]$$

- In general, cannot invert this equation for c.
    - Special case for additively separable preferences: $u_c(c,\ell) = u_c(c)$

If only we knew $\ell(z,k)$ we could almost use EGM!

Barillas & Fernandez-Villaverde (2007)

Idea: Mix EGM and VFI in the spirit of Howard's policy function iteration

1. Fix a policy function for labor $\ell_0(z,k)$ (a good guess is $\ell_0(z,k) = \ell_{ss}$)
2. Conduct N steps of EGM given $\ell_0(z,k)$ (say $N = 10$)
   - EGM has to be modified to include labor.
   - Solve non-linear equation $u_c(c, \ell_0(z,k)) = \beta E\left[V_k\left(z', k'\right)|z\right]$
3. Conduct M steps of VFI (say $M = 1$) on the exogenous capital grid.
4. Replace $\ell(z,k)$ with the output of step 3 and conduct step 2.

Algorithm 3: EGM: Fixed labor supply

Function `EGM(`$V, \vec{k}, \vec{z}, \ell(z,k), k'(z,k)$, *parameters*`)`**:**

for *i=1:*n_z **do**

for *j=1:*n_k **do**

1. Solve for $\tilde{k}_{ij}$ s.t. $k'(\tilde{k}_{ij}) = \vec{k}_j$
2. Evaluate for labor from old policy: $\tilde{\ell}_{ij} = \ell(\vec{z}_i, \tilde{k}_{ij})$
3. Evaluate expectd value: $\mathbb{V} = \beta E\left[V\left(z', \vec{k}_j)\right) | \vec{z}_i\right]$
4. Recover consumption: $u_c\left(\tilde{c}_{ij}, \tilde{\ell}_{ij}\right) = \mathbb{V}_k$
5. Find endogenous capital $\hat{k}_{ij}$ s.t.: $\tilde{c}_{ij} + \vec{k}_j = f\left(\vec{z}_i, \hat{\mathrm{k}}_{\mathrm{ij}}, \tilde{\ell}_{ij}\right)$
6. Update value at endogenous grid: $V(\vec{z}_i, \hat{k}_{ij}) = u\left(\tilde{c}_{ij}, \tilde{\ell}_{ij}\right) + \mathbb{V}$

7. Interpolate to exogenous grid: V_new[i,:] = Interp($\hat{k}, V, \vec{k}$)

Algorithm 4: EGM: Fixed labor supply II

Function `EGM(`$V, \vec{k}, \vec{z}, \ell(z,k)$, *parameters*`)`:

for $i=1{:}n_z$ **do**

for $j=1{:}n_k$ **do**

1. Evaluate expectd value: $\mathbb{V} = \beta E\left[V\left(z', \vec{k}_j)\right) | \vec{z}_i\right]$
2. Define consumption given labor: $u_c\left(\hat{c}(\ell), \ell\right) = \mathbb{V}_k$
3. Find endogenous capital $\hat{k}_{ij}$ s.t.:

   $\hat{c}(\ell(\vec{z}_i, \hat{\mathsf{k}}_{\mathsf{ij}})) + \vec{k}_j = f\left(\vec{z}_i, \hat{\mathsf{k}}_{\mathsf{ij}}, \ell(\vec{z}_i, \hat{\mathsf{k}}_{\mathsf{ij}})\right)$

   (Requires root finding and interpolation of ℓ at guess $\hat{k}$)
4. Update value at endogenous grid:

   $V(\vec{z}_i, \hat{k}_{ij}) = u\left(\hat{c}(\ell(\vec{z}_i, \hat{k}_{ij})), \ell(\vec{z}_i, \hat{k}_{ij})\right) + \mathbb{V}$

5. Interpolate to exogenous grid: V_new[i,:] = Interp($\hat{k}, V, \vec{k}$)

Some comments

1. In step 1 you can get the inverse from an interpolation routine
   - One option is to use a root finder
   - If you are using your own routine (say Cubic Splines) you can code your own inverse function
2. In step 5 we can be more ambitious and also update labor.

   5'. Find endogenous capital $\hat{k}_{ij}$ s.t.: $\tilde{c}_{ij} + \vec{k}_j = f\left(\vec{z}_i, \hat{\mathrm{k}}_{\mathrm{ij}}, \ell\left(\vec{z}_i, \hat{\mathrm{k}}_{\mathrm{ij}}\right)\right)$

   - Doing this implies adding an interpolation step to the root finding
   - It also provides a better update of the value function

   6'. Update value at endogenous grid: $V(\vec{z}_i, \hat{k}_{ij}) = u\left(\tilde{c}_{ij}, \ell\left(\vec{z}_i, \hat{\mathrm{k}}_{\mathrm{ij}}\right)\right) + \mathbb{V}$
3. Step 4 can be simplified if utility is separable $u(c, \ell) = U(c) - H(\ell)$
   - In fact, all the algorithm gets easier
   - No need to carry around the policy function for labor

Algorithm 5: EGM: Endogenous labor supply with separable utility

Function `EGM(`V,$\vec{k}$, $\vec{z}$,*parameters*`)`:

for *i=1:*n_z **do**

  for *j=1:*n_k **do**

    1. Evaluate expectd value: $\mathbb{V} = \beta E\left[V\left(z', \vec{k}_j\right)\right) |\vec{z}_i\Big]$
    2. Recover consumption: $\tilde{c}_{ij} = (U')^{-1}(\mathbb{V}_k)$
    3. Find $(\hat{k}_{ij}, \tilde{\ell}_{ij})$ that solve FOC:
       3a. Define $\hat{k}(\ell)$ analytically from FOC: $\frac{-H'(\ell)}{U'(\tilde{c}_{ij})} = f_\ell(\vec{z}_i, \hat{\mathrm{k}}(\ell), \ell)$
       3b. Solve for $\tilde{\ell}_{ij}$ numerically s.t.: $\tilde{c}_{ij} + \vec{k}_j = f\left(\vec{z}_i, \hat{\mathrm{k}}(\tilde{\ell}_{\mathrm{ij}}), \tilde{\ell}_{ij}\right)$
       3c. Assign endogenous grid point $\hat{k}_{ij} = \hat{k}(\tilde{\ell}_{ij})$
    4. Update value at endogenous grid: $V(\vec{z}_i, \hat{k}_{ij}) = u\left(\tilde{c}_{ij}, \tilde{\ell}_{ij}\right) + \mathbb{V}$

  5. Interpolate to exogenous grid: V_new[i,:] = Interp($\hat{k}$, V, $\vec{k}$)

Envelope Condition Method

Maliar & Maliar (2013)

First order conditions:

$$u_c(c,\ell) = \beta E\left[V_k\left(z',k'\right)|z\right] \tag{1}$$

$$\frac{-u_\ell(c,\ell)}{u_c(c,\ell)} = f_\ell(z,k,\ell) \tag{2}$$

$$c + k' = f(z,k,\ell) \tag{3}$$

Envelope condition:

$$V_k(z,k) = u_c(c,\ell)\, f_k(z,k,\ell) \tag{4}$$

Combining (1) and (4) we get a recursive equation for value derivative:

$$V_k(z,k) = \beta f_k(z,k,\ell)\, E\left[V_k\left(z',k'\right)|z\right] \tag{5}$$

Key: EGM works by solving (1), (2) and (3). ECM solves (4), (2) and (3). Equation (5) lets us update V_k directly without computing V

ECM - Algorithm - Inelastic labor supply

Algorithm 6: ECM: Inelastic labor supply

Function ECM($V_k, \vec{k}, \vec{z}$,*parameters*):

- **for** i=1:n_z **do**
  - **for** j=1:n_k **do**
    1. Get consumption analytically: $c_{ij} = (u')^{-1}\left(\frac{V_k(\vec{z}_i, \vec{k}_j)}{f_k\left(\vec{z}_i, \vec{k}_j\right)}\right)$

       Note: We are using present k, not future k.
    2. Get k': $k'_{ij} = f\left(\vec{z}_i, \vec{k}_j\right) - c_{ij}$
    3. Update V_k: $V_k^{new}\left(\vec{z}_i, \vec{k}_j\right) = \beta f_k\left(\vec{z}_i, \vec{k}_j\right) E\left[V_k\left(z', k'_{ij}\right)|\vec{z}_i\right]$

       Note: This step requires interpolation inside expectation
- return V_k^{new};

Some comments

1. No optimization or root finding in any step!
2. Careful when updating derivatives directly
   - You lose the power of the contraction mapping theorem
   - Particularly you lose uniqueness ($V_k = 0$ is a fixed point)
   - Remember that your function is monotone, so $V_k > 0$!
3. Alternative is to update with V as we do always:

   0'. Get derivative of V at grid nodes $V_k\left(\vec{z}_i, \vec{k}_j\right)$

   3'. Update value function: $V^{new}(z, k) = u(c_{ij}) + \beta E\left[V\left(z', k'_{ij}\right)|z\right]$
      - Note that you still need to do the interpolation inside the expectation
4. Authors say they get better results with V_k
   - They reference another paper (Maliar & Maliar, 2012) that solves problems with 16 states using variants of the ECM

ECM - Labor supply

Algorithm 7: ECM: Endogenous labor supply, Separable utility

Function `ECM(`$V_k, \vec{k}, \vec{z}$,*parameters*`)`**:**

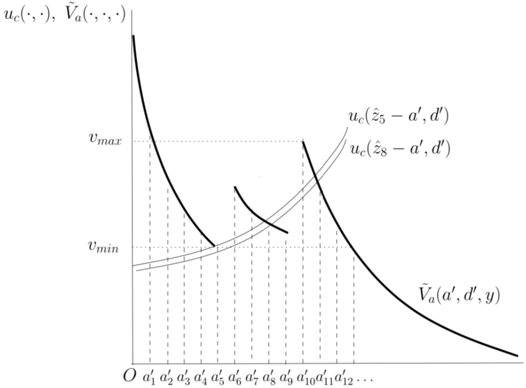
- **for** i=1:n_z **do**
  - **for** j=1:n_k **do**
    1. Get labor ℓ_{ij} numerically: $V_k(\vec{z}_i, \vec{k}_j) = \frac{H'\left(\ell_{ij}\right)}{f_\ell\left(\vec{z}_i, \vec{k}_j, \ell_{ij}\right)} f_k\left(\vec{z}_i, \vec{k}_j, \ell_{ij}\right)$

       Note: No interpolation or expectation
    2. Get consumption analytically: $c_{ij} = (U')^{-1}\left(\frac{H'\left(\ell_{ij}\right)}{f_\ell\left(\vec{z}_i, \vec{k}_j, \ell_{ij}\right)}\right)$
    3. Get k': $k'_{ij} = f\left(\vec{z}_i, \vec{k}_j, \ell_{ij}\right) - c_{ij}$
    3. Update V_k:
       $V_k^{new}\left(\vec{z}_i, \vec{k}_j\right) = \beta f_k\left(\vec{z}_i, \vec{k}_j, \ell_{ij}\right) E\left[V_k\left(z', k'_{ij}\right) | \vec{z}_i\right]$
- return V_k^{new};

Extensions

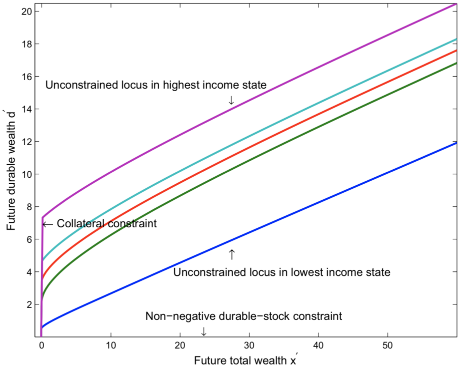
Non-Convex, Non-Smooth Problems - Fella (2014)

- Extend EGM to a problem with discrete state variable and continuous choices
- Discreteness is a problem because it generates kinks in the function
- Idea: EGM works away from the kinks!
- This is worth checking!



2 States+Borrowing Const. - Hintermaier & Koeniger (2010)

- Method for model with occasionally binding collateral constraints and non-separable utility in durable and non-durable consumption
- Good for applications with uninsurable income risk
- Idea: Solve the problem with a new state variable
  - x : Cash on hand or beginning of period wealth



Robustness

Initial conditions are tricky

- EGM and ECM put a lot of trust on our guess of V and V_k
- That trust is often misplaced during initial iterations
  - Initial guess won't capture curvature of the solution
- Always build a safety check
  - Your EGM or ECM might send you out of bounds
  - This often means negative savings
  - Under-estimate curvature at the bottom $\longrightarrow$ Over-estimate consumption
- If you are going out of bounds revert to traditional VFI