Advanced Macroeconomics II

Handout 9 - Search Models II

Sergio Ocampo

Western University

December 7, 2020

Random Search

We want to endogenize search frictions:

- Arrival rate of job offers depends on search frictions
- Unemployed (and employed) workers search for vacancies
- Firms post vacancies to search for workers
- Relative number of vacancies per worker (market tightness) determine frictions

Random Search

We want to endogenize search frictions:

- Arrival rate of job offers depends on search frictions
- Unemployed (and employed) workers search for vacancies
- Firms post vacancies to search for workers
- Relative number of vacancies per worker (market tightness) determine frictions

We will pay for it in heterogeneity:

- All jobs are the same, only difference is employed/unemployed
- Wages determined by bargaining, but all firms are identical

Benchmark Model
Pissarides (1990)

Set up

- Time is continuous and agents have linear utility

$$\max E\left[\int_0^\infty e^{-\rho t} x_t dt\right]$$

  - If unemployed x_t is b_t and if employed it is w_t
  - Expectation is over unemployment spells

Set up

- Time is continuous and agents have linear utility

$$\max E\left[\int_0^\infty e^{-\rho t} x_t dt\right]$$

  - If unemployed x_t is b_t and if employed it is w_t
  - Expectation is over unemployment spells

- There are L agents in the economy
  - u is the unemployment rate and v is the vacancy rate
  - So uL is total number of unemployed and vL total vacancies

Set up

- Time is continuous and agents have linear utility

$$\max E\left[\int_0^{\infty} e^{-\rho t} x_t dt\right]$$

  - If unemployed x_t is b_t and if employed it is w_t
  - Expectation is over unemployment spells

- There are L agents in the economy
  - u is the unemployment rate and v is the vacancy rate
  - So uL is total number of unemployed and vL total vacancies

- Each instant (period) there are mL new jobs (matches)
  - Key: the number of matches depends on searchers and vacancies

Set up

- Time is continuous and agents have linear utility

$$\max E\left[\int_0^\infty e^{-\rho t} x_t dt\right]$$

  - If unemployed x_t is b_t and if employed it is w_t
  - Expectation is over unemployment spells
- There are L agents in the economy
  - u is the unemployment rate and v is the vacancy rate
  - So uL is total number of unemployed and vL total vacancies
- Each instant (period) there are mL new jobs (matches)
  - Key: the number of matches depends on searchers and vacancies
- Jobs (matches) end at an exogenous rate δ

Matching (functions)

$$mL = m\left(uL, vL\right)$$

Properties of $m\left(\cdot, \cdot\right)$:

1. Increasing in both arguments
2. Concave
3. Homogenous of degree 1 (CRS): $m = m\left(u, v\right)$

Matching (functions)

$$mL = m(uL, vL)$$

Properties of $m(\cdot, \cdot)$:

1. Increasing in both arguments
2. Concave
3. Homogenous of degree 1 (CRS): $m = m(u, v)$

Notes:

- Petrongolo and Pissarides (2001) justify the CRS assumption
    - Data from 30+ countries
- See Jolivet and Postel-Vinay (2012) for more references

Matching (functions) and tightness

- Define market tightness as $\theta \equiv {}^{v}\!/_{u}$

Matching (functions) and tightness

- Define market tightness as $\theta \equiv {}^{v}/_{u}$
- What is the vacancy matching rate?

$$q(\theta) \equiv \frac{mL}{vL} = m\left({}^{u}/_{v}, 1\right) = m\left({}^{1}/_{\theta}, 1\right)$$

  - We know that $q^{'}(\theta) \leq 0$, so more tightness reduces vacancies' matching rate
  - Every time interval Δ a fraction $q(\theta)\Delta$ of vacancies are filled
  - Mean duration of a vacancy is ${}^{1}/_{q(\theta)}$ (from matches following a Poisson process with arrival rate $q(\theta)$)

Matching (functions) and tightness

- Define market tightness as $\theta \equiv {}^{v}/_{u}$
- What is the vacancy matching rate?

$$q(\theta) \equiv \frac{mL}{vL} = m\left({}^{u}/_{v}, 1\right) = m\left({}^{1}/_{\theta}, 1\right)$$

  - We know that $q^{'}(\theta) \leq 0$, so more tightness reduces vacancies' matching rate
  - Every time interval Δ a fraction $q(\theta)\Delta$ of vacancies are filled
  - Mean duration of a vacancy is ${}^{1}/_{q(\theta)}$ (from matches following a Poisson process with arrival rate $q(\theta)$)
- What is the unemployed matching rate?

$$\frac{mL}{uL} = \theta q(\theta)$$

So tightness also determines unemployed's chances of employment

Matching (functions)

Urn Ball Matching

Matching (functions)

Urn Ball Matching

- Each unemployed sends one application at random
  - Probability a vacancy receives worker's application is $1/vL$
  - Probability a vacancy receives no application is $(1 - 1/vL)^{uL}$

Matching (functions)

Urn Ball Matching

- Each unemployed sends one application at random
  - Probability a vacancy receives worker's application is $^1/_{vL}$
  - Probability a vacancy receives no application is $(1 - {}^1/_{vL})^{uL}$
- If a vacancy receives more than one application, choose one at random

Matching (functions)

Urn Ball Matching

- Each unemployed sends one application at random
  - Probability a vacancy receives worker's application is $1/vL$
  - Probability a vacancy receives no application is $(1 - 1/vL)^{uL}$
- If a vacancy receives more than one application, choose one at random
- Number of matches is then:

$$M = \underbrace{vL}_{\text{No. Vacancies}} \underbrace{\left(1 - \left[1 - \frac{1}{vL}\right]^{uL}\right)}_{\text{1-Pr. No application}} \approx vL\left(1 + e^{\frac{u}{v}}\right) = vL\left(1 + e^{-\theta}\right)$$

where we use $e^x \approx 1 + x$ for small x and $x = 1/vL$ with large vL

Matching (functions)
Cobb-Douglas

Matching (functions)
Cobb-Douglas

$$m(u, v) = u^{\alpha} v^{1-\alpha}$$

- Problems in discrete time. No guarantee of $m(u, v) \leq \min\{u, v\}$, equivalently that $m(u, v) \leq 1$
  - Only defined in a range of tightness:

$$m(u, v) \leq u \longleftrightarrow \theta^{1-\alpha} \leq 1 \qquad m(u, v) \leq v \longleftrightarrow \theta^{-\alpha} \leq 1$$

Matching (functions)

Cobb-Douglas

$$m(u, v) = u^{\alpha} v^{1-\alpha}$$

- Problems in discrete time. No guarantee of $m(u, v) \leq \min\{u, v\}$, equivalently that $m(u, v) \leq 1$
  - Only defined in a range of tightness:

$$m(u, v) \leq u \longleftrightarrow \theta^{1-\alpha} \leq 1 \qquad m(u, v) \leq v \longleftrightarrow \theta^{-\alpha} \leq 1$$

Den Haan, Ramey and Watson (2001)

$$m(u, v) = \frac{uv}{(u^{\zeta} + v^{\zeta})^{\frac{1}{\zeta}}} \in (0, 1)$$

Labor flows and steady state

$$\dot{u} = \delta (1 - u) - \theta q (\theta) u$$

- Flow into unemployment of fired workers
- Flow out of unemployment of job finders

Labor flows and steady state

$$\dot{u} = \delta (1 - u) - \theta q (\theta) u$$

- Flow into unemployment of fired workers
- Flow out of unemployment of job finders
- Steady state:

$$u = \frac{\delta}{\delta + \theta q (\theta)}$$

- Steady state relationship forms the model's Beveridge curve

Firms (where vacancies come from)

- Firms produce using only labor and CRS techonlogy
  - CRS makes firm size immaterial, but other models make sense of it (Burdett & Mortense, 1998; Moscarini & Postel-Vinay, 2013)

Firms (where vacancies come from)

- Firms produce using only labor and CRS techonlogy
  - CRS makes firm size immaterial, but other models make sense of it (Burdett & Mortense, 1998; Moscarini & Postel-Vinay, 2013)
- Firms produce using the same technology: $y = zl$
  - l is either 0 or 1 depending on whether the vacancy is filled
  - z is common to all firms and subject to exogenous shocks

Firms (where vacancies come from)

- Firms produce using only labor and CRS techonlogy
  - CRS makes firm size immaterial, but other models make sense of it (Burdett & Mortense, 1998; Moscarini & Postel-Vinay, 2013)
- Firms produce using the same technology: $y = zl$
  - l is either 0 or 1 depending on whether the vacancy is filled
  - z is common to all firms and subject to exogenous shocks
- Posting a vacancy entails a cost $z\kappa$

Value functions: Vacancies

Value of a vacancy:

$$V = \frac{1}{1+\rho\Delta}\left[-z\kappa + \Delta q(\theta) J + (1 - \Delta q(\theta)) V\right]$$

As $\Delta \to 0$ we get:

$$\rho V = -z\kappa + q(\theta)(J - V)$$

Value functions: Vacancies

Value of a vacancy:

$$V = \frac{1}{1+\rho\Delta}\left[-z\kappa + \Delta q(\theta) J + (1-\Delta q(\theta)) V\right]$$

As $\Delta \to 0$ we get:

$$\rho V = -z\kappa + q(\theta)(J - V)$$

- There is free entry to the market so $V = 0$
- Expected profits from vacancy must cover the cost (κ)
- All firms (vacancies) make zero profits when combined.

$$J = \frac{z\kappa}{q(\theta)}$$

Value functions: Filled jobs

Value of a filled job position:

$$J = \frac{1}{1+\rho\Delta}\left[(z-w)\,\Delta + \Delta\delta\underbrace{V}_{=0} + (1-\Delta\delta)\,J\right]$$

$$\rho J = p - w - \delta J$$

Value functions: Filled jobs

Value of a filled job position:

$$J = \frac{1}{1+\rho\Delta}\left[(z-w)\,\Delta + \Delta\delta\underbrace{V}_{=0} + (1-\Delta\delta)\,J\right]$$

$$\rho J = p - w - \delta J$$

- Wages come from Nash bargaining... but they also satisfy:

$$(\rho+\delta)\,\frac{z\kappa}{q(\theta)} = z - w$$

So as w increases $q(\theta)$ must increase (θ decrease), which happens as fewer vacancies are posted. The value of a vacancy is lower due to higher w.

Value functions: Workers

$$\begin{aligned}\rho U &= b + \theta q(\theta)(W - U)\\ \rho W &= w + \delta(U - W)\end{aligned}$$

Value functions: Workers

$$\rho U = b + \theta q(\theta)(W - U)$$
$$\rho W = w + \delta(U - W)$$

We can further solve in terms of θ and w:

$$\rho U = \frac{(\rho+\delta) b + \theta q(\theta)}{\rho + \delta + \theta q(\theta)}$$
$$\rho W = \frac{\delta b + (\rho + \theta q(\theta)) w}{\rho + \delta\theta q(\theta)}$$

Although we will rarely use this, except for welfare analysis...

Wage determination

Total surplus of the match:

$$S = J - V + W - U$$

Wage determination

Total surplus of the match:

$$S = J - V + W - U$$

Nash Bargaining splits the surplus so that:

$$W - U = \beta S \qquad J - V = (1 - \beta)\, S$$

where β is the workers' bargaining power

Wage determination

Total surplus of the match:

$$S = J - V + W - U$$

Nash Bargaining splits the surplus so that:

$$W - U = \beta S \qquad J - V = (1 - \beta)\, S$$

where β is the workers' bargaining power

- In wage terms:

$$w = \rho U + \beta\,(z - \rho U)$$

  Wage is reservation wage (ρU) plus β of net surplus ($z - \rho U$)

Wage determination

Total surplus of the match:

$$S = J - V + W - U$$

Nash Bargaining splits the surplus so that:

$$W - U = \beta S \qquad J - V = (1 - \beta) S$$

where β is the workers' bargaining power

- In wage terms:

$$w = \rho U + \beta (z - \rho U)$$

  Wage is reservation wage (ρU) plus β of net surplus ($z - \rho U$)

- In terms of tightness:

$$w = (1 - \beta) b + \beta z (1 + \kappa \theta)$$

  Value to firm is output z plus forgone vacancy costs per worker: $z \kappa \frac{v}{u}$

Equilibrium (as a function of θ)

- To find an equilibrium we need to determine (θ, w, u, v)

Equilibrium (as a function of θ)

- To find an equilibrium we need to determine (θ, w, u, v)
- Turns out we can eliminate u using steady state relation:

$$u = \frac{\delta}{\delta + \theta q(\theta)}$$

Equilibrium (as a function of θ)

- To find an equilibrium we need to determine (θ, w, u, v)
- Turns out we can eliminate u using steady state relation:

$$u = \frac{\delta}{\delta + \theta q(\theta)}$$

- Then we only need (θ, w) which get from free entry + wage:

$$w = z\left(1 - (\rho + \delta)\frac{\kappa}{q(\theta)}\right)$$
$$w = (1 - \beta)b + \beta z(1 + \kappa\theta)$$

- Simple problem with a single crossing point.

Note about capital

- We can add capital in a partial equilibrium way
    - Generates dynamics, but not very interesting
    - See Pissarides' book
- Harder to do it in general equilibrium
    - Krusell, Mukoyama and Sahin (2010, ReStud) do it
    - Not a big effect of capital on model predictions

Shimer (2005)

The Shimer puzzle

Are DMP models capable of matching business cycle facts? Are business cycles better matched with productivity (z) shocks or job destruction shocks (δ)?

The Shimer puzzle

Are DMP models capable of matching business cycle facts? Are business cycles better matched with productivity (z) shocks or job destruction shocks (δ)?

- Short answer is that DMP fails unless $b \sim z$

The Shimer puzzle

Are DMP models capable of matching business cycle facts? Are business cycles better matched with productivity (z) shocks or job destruction shocks (δ)?

- Short answer is that DMP fails unless $b \sim z$

Large debate in the literature:

- Shimer (2005), Hall and Milgrom (2005), Hagerdorn and Manovskii (2008), Constain and Reiter (2008), Chorodow-Reich and Karabarbounis (2013), Hagerdorn, Karahan, Manovskii, Mitman (2014), many more

The Shimer puzzle

Unemployment and vacancies are twice as variable as productivity ($p = z$)

TABLE 1—SUMMARY STATISTICS, QUARTERLY U.S. DATA, 1951–2003

		u	v	v/u	f	s	p
Standard deviation		0.190	0.202	0.382	0.118	0.075	0.020
Quarterly autocorrelation		0.936	0.940	0.941	0.908	0.733	0.878
	u	1	−0.894	−0.971	−0.949	0.709	−0.408
	v	—	1	0.975	0.897	−0.684	0.364
Correlation matrix	v/u	—	—	1	0.948	−0.715	0.396
	f	—	—	—	1	−0.574	0.396
	s	—	—	—	—	1	−0.524
	p	—	—	—	—	—	1

Notes: Seasonally adjusted unemployment u is constructed by the BLS from the Current Population Survey (CPS). The seasonally adjusted help-wanted advertising index v is constructed by the Conference Board. The job-finding rate f and separation rate s are constructed from seasonally adjusted employment, unemployment, and mean unemployment duration, all computed by the BLS from the CPS, as explained in equations (1) and (2). u, v, f, and s are quarterly averages of monthly series. Average labor productivity p is seasonally adjusted real average output per person in the non-farm business sector, constructed by the Bureau of Labor Statistics (BLS) from the National Income and Product Accounts and the Current Employment Statistics. All variables are reported in logs as deviations from an HP trend with smoothing parameter 10^5.

Set up

- Standard DMP model (vacancy cost no longer scaled by z)

Set up

- Standard DMP model (vacancy cost no longer scaled by z)
- Aggregate shocks: (z, δ)
  - (z, δ) change randomly. Change follows Poisson with arrival rate λ
  - We draw new values for $\left(z', \delta'\right)$ from state-contingent distribution

Set up

- Standard DMP model (vacancy cost no longer scaled by z)
- Aggregate shocks: (z, δ)
  - (z, δ) change randomly. Change follows Poisson with arrival rate λ
  - We draw new values for $\left(z', \delta'\right)$ from state-contingent distribution
- Surplus of a match:

$$S_{z,\delta} = z - \left(b + \theta_{z,\delta} q\left(\theta_{z,\delta}\right) \beta S_{z,\delta}\right) - \delta S_{z,\delta} + \lambda\left(E\left[S_{z',\delta'}\right] - S_{z,\delta}\right)$$

- Free entry:

$$\kappa = q\left(\theta_{z,\delta}\right)(1-\beta) S_{z,\delta}$$

- Wage equation gives tightness:

$$\frac{\rho+\delta+\lambda}{q\left(\theta_{z,\delta}\right)} + \beta\theta_{z,\delta} = (1-\beta)\frac{z-b}{\kappa} + \lambda E\left[\frac{1}{q\left(\theta_{z',\delta'}\right)}\right]$$

Where do wages come from?

Values of workers

$$\rho U_{z,\delta} = b + \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\left[W_{z,\delta} - U_{z,\delta}\right] + \lambda\left[E\left[U_{z',\delta'}\right] - U_{z,\delta}\right]$$
$$\rho W_{z,\delta} = w_{z,\delta} - \delta\left[U_{z,\delta} - W_{z,\delta}\right] + \lambda\left[E\left[W_{z',\delta'}\right] - W_{z,\delta}\right]$$

Where do wages come from?

Values of workers

$$\rho U_{z,\delta} = b + \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\left[W_{z,\delta} - U_{z,\delta}\right] + \lambda\left[E\left[U_{z',\delta'}\right] - U_{z,\delta}\right]$$
$$\rho W_{z,\delta} = w_{z,\delta} - \delta\left[U_{z,\delta} - W_{z,\delta}\right] + \lambda\left[E\left[W_{z',\delta'}\right] - W_{z,\delta}\right]$$

Value of filled vacancy (already with free entry:

$$\rho J_{z,\delta} = z - w_{z,\delta} + \delta\left[0 - J_{z,\delta}\right] + \lambda\left[E\left[J_{z',\delta'}\right] - J_{z,\delta}\right]$$

Where do wages come from?

Values of workers

$$\rho U_{z,\delta} = b + \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\left[W_{z,\delta} - U_{z,\delta}\right] + \lambda\left[E\left[U_{z',\delta'}\right] - U_{z,\delta}\right]$$
$$\rho W_{z,\delta} = w_{z,\delta} - \delta\left[U_{z,\delta} - W_{z,\delta}\right] + \lambda\left[E\left[W_{z',\delta'}\right] - W_{z,\delta}\right]$$

Value of filled vacancy (already with free entry:

$$\rho J_{z,\delta} = z - w_{z,\delta} + \delta\left[0 - J_{z,\delta}\right] + \lambda\left[E\left[J_{z',\delta'}\right] - J_{z,\delta}\right]$$

We get surplus from the sum: $S_{z,\delta} = J_{z,\delta} + W_{z,\delta} - U_{z,\delta}$:

$$S_{z,\delta} = z - b - \delta S_{z,\delta} + \lambda\left[E\left[S_{z',\delta'}\right] - S_{z,\delta}\right] - \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\left[W_{z,\delta} - U_{z,\delta}\right]$$

Where do wages come from?

Total surplus:

$$S_{z,\delta} = z - b - \delta S_{z,\delta} + \lambda \left[E \left[S_{z',\delta'} \right] - S_{z,\delta} \right] - \theta_{z,\delta} q \left(\theta_{z,\delta} \right) \left[W_{z,\delta} - U_{z,\delta} \right]$$

Where do wages come from?

Total surplus:

$$S_{z,\delta} = z - b - \delta S_{z,\delta} + \lambda \left[E\left[S_{z',\delta'}\right] - S_{z,\delta}\right] - \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\left[W_{z,\delta} - U_{z,\delta}\right]$$

- From Nash bargaining we know that $S = \frac{W-U}{\beta} = \frac{J}{1-\beta}$, so:
  $$S_{z,\delta} = z - \left(b + \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\beta S_{z,\delta}\right) - \delta S_{z,\delta} + \lambda \left[E\left[S_{z',\delta'}\right] - S_{z,\delta}\right]$$

Where do wages come from?

Total surplus:

$$S_{z,\delta} = z - b - \delta S_{z,\delta} + \lambda \left[E\left[S_{z',\delta'}\right] - S_{z,\delta}\right] - \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\left[W_{z,\delta} - U_{z,\delta}\right]$$

- From Nash bargaining we know that $S = \frac{W-U}{\beta} = \frac{J}{1-\beta}$, so:
  $$S_{z,\delta} = z - \left(b + \theta_{z,\delta} q\left(\theta_{z,\delta}\right)\beta S_{z,\delta}\right) - \delta S_{z,\delta} + \lambda \left[E\left[S_{z',\delta'}\right] - S_{z,\delta}\right]$$
- From free entry we know that:
  $$S = \frac{J}{1-\beta} = \frac{\kappa}{q\left(\theta_{z,\delta}\right)(1-\beta)}$$

Where do wages come from?

Total surplus:

$$S_{z,\delta} = z - b - \delta S_{z,\delta} + \lambda \left[E\left[S_{z',\delta'} \right] - S_{z,\delta} \right] - \theta_{z,\delta} q\left(\theta_{z,\delta}\right) \left[W_{z,\delta} - U_{z,\delta} \right]$$

- From Nash bargaining we know that $S = \frac{W-U}{\beta} = \frac{J}{1-\beta}$, so:

  $$S_{z,\delta} = z - \left(b + \theta_{z,\delta} q\left(\theta_{z,\delta}\right) \beta S_{z,\delta} \right) - \delta S_{z,\delta} + \lambda \left[E\left[S_{z',\delta'} \right] - S_{z,\delta} \right]$$

- From free entry we know that:

  $$S = \frac{J}{1-\beta} = \frac{\kappa}{q\left(\theta_{z,\delta}\right)(1-\beta)}$$

- Replacing gives wages:

  $$\frac{\rho + \delta + \lambda}{q\left(\theta_{z,\delta}\right)} + \beta \theta_{z,\delta} = (1-\beta) \frac{z-b}{\kappa} + \lambda E\left[\frac{1}{q\left(\theta_{z',\delta'}\right)} \right]$$

Comparative statics

Assume there are no shocks, then $\lambda = 0$:

$$\frac{\rho + \delta}{q(\theta)} + \beta\theta = (1 - \beta)\frac{z - b}{\kappa}$$

Comparative statics

Assume there are no shocks, then $\lambda = 0$:

$$\frac{\rho + \delta}{q(\theta)} + \beta\theta = (1 - \beta)\frac{z - b}{\kappa}$$

We can get the elasticity of θ wrt to net labor productivity $z - b$:

$$\frac{\partial\theta}{\partial(z - b)}\frac{z - b}{\theta} = \frac{\rho + \delta + \beta\theta q(\theta)}{(\rho + \delta)(1 - \eta(\theta)) + +\beta\theta q(\theta)}$$

$\eta(\theta)$ is elasticity of job finding rate to tightness $\left(\eta(\theta) \equiv {}^{\partial\theta q(\theta)}/_{\partial\theta}{}^{1}/_{q(\theta)}\right)$

Comparative statics

Assume there are no shocks, then $\lambda = 0$:

$$\frac{\rho + \delta}{q(\theta)} + \beta\theta = (1 - \beta)\frac{z - b}{\kappa}$$

We can get the elasticity of θ wrt to net labor productivity $z - b$:

$$\frac{\partial\theta}{\partial(z - b)}\frac{z - b}{\theta} = \frac{\rho + \delta + \beta\theta q(\theta)}{(\rho + \delta)(1 - \eta(\theta)) + +\beta\theta q(\theta)}$$

$\eta(\theta)$ is elasticity of job finding rate to tightness $\left(\eta(\theta) \equiv {}^{\partial\theta q(\theta)}/_{\partial\theta}{}^{1}/_{q(\theta)}\right)$

- "This is large only if workers' bargaining power β is small and the elasticity η is close to one. But with reasonable parameter values, it is close to 1."
- "This implies that unless the value of leisure is close to labor productivity, the v-u ratio is likely to be unresponsive to changes in the labor productivity."
- That is the Shimer puzzle

How to calibrate the shocks

- Both random processes have a common driver, call it y

How to calibrate the shocks

- Both random processes have a common driver, call it y
- Assume y follows an Ornstein-Uhlenbeck process

$$dy = -\gamma y dt + \delta dB \qquad B \text{ a Brownian motion}$$

where B is a brownian motion satisfying:

How to calibrate the shocks

- Both random processes have a common driver, call it y
- Assume y follows an Ornstein-Uhlenbeck process

$$dy = -\gamma y dt + \delta dB \qquad B \text{ a Brownian motion}$$

where B is a brownian motion satisfying:

1. $B_0 = 0$
2. B_t is almost surely continuous (see notes in stochastic calculus)
3. B_t has independent increments
4. $B_{t+\Delta} - B_t \sim N(0, \Delta)$

How to calibrate the shocks

- Both random processes have a common driver, call it y
- Assume y follows an Ornstein-Uhlenbeck process

$$dy = -\gamma y dt + \delta dB \qquad B \text{ a Brownian motion}$$

  where B is a brownian motion satisfying:

  1. $B_0 = 0$
  2. B_t is almost surely continuous (see notes in stochastic calculus)
  3. B_t has independent increments
  4. $B_{t+\Delta} - B_t \sim N(0, \Delta)$

- We are going to have to simulate B to get to y to get to (z, δ)

How to simulate in continuous time?

We need to discretize y so that it is consistent with definition of OU process and shock arrival at rate λ

How to simulate in continuous time?

We need to discretize y so that it is consistent with definition of OU process and shock arrival at rate λ

1. Discretize y on a grid: $\vec{y} = [-n\Delta_y, -(n-1)\Delta_y, \ldots, 0, \ldots, n\Delta y]$
2. When λ hits process goes one step up or one step down:

$$y' = \begin{cases} y + \Delta_y & \text{with prob.} \frac{1}{2}\left(1 - \frac{y}{n\Delta_y}\right) \\ y - \Delta_y & \text{with prob.} \frac{1}{2}\left(1 + \frac{y}{n\Delta_y}\right) \end{cases}$$

   - Key: Prob of moving up decreases with level of y and viceversa
3. Set $\gamma = \frac{\lambda}{n}$, in practice we choose γ and let λ vary
4. Set $\sigma = \Delta_y \sqrt{\lambda}$

How to simulate in continuous time?

What do we do about the rest of the model?

How to simulate in continuous time?

What do we do about the rest of the model?

1. Define $z = b + e^{y}(z^{\star} - b)$ and $\delta = e^{y}\delta^{\star}$ and compute grids
2. Compute transition probabilities for y conditional on shock arrival
3. Solve for tightness using wage equation for each value of $y \in \vec{y}$
   - This involves solving a nonlinear set of equations once
4. Generate random time arrivals for shocks
   - Draws of a Poisson obtained as $t = {}^{-\log(x)}/_{\lambda}$, where $x \sim U(0,1)$
   - We get this by inverting the CDF of the Poisson!
   - Because Poisson arrivals are independent you can draw them all at once and use cumsum to get vector of arrival times
5. Start value of y in grid. Update at each arrival time.
   - Use uniform random draws and transition probability to choose move.
6. Evaluate z, δ, θ at values of y for each time.

How to simulate in continuous time?

One question left... how to get u out of the simulation?

How to simulate in continuous time?

One question left... how to get u out of the simulation?

$$
\begin{aligned}
\dot{u}(t) &= \delta(t)(1-u(t)) - \theta(t)\, q(\theta(t))\, u(t) \\
\dot{u}(t) &= \delta(t) - (\delta(t) + \theta(t)\, q(\theta(t)))\, u(t)
\end{aligned}
$$

How to simulate in continuous time?

One question left... how to get u out of the simulation?

$$\begin{aligned}
\dot{u}(t) &= \delta(t)(1-u(t)) - \theta(t)\,q(\theta(t))\,u(t) \\
\dot{u}(t) &= \delta(t) - (\delta(t) + \theta(t)\,q(\theta(t)))\,u(t)
\end{aligned}$$

This is a first order differential equation of the form:

$$\dot{x} = b(t) + a(t)\,x(t)$$

The solution of this equation is known:

$$x(t) = e^{\int_{t_0}^{t} a(\tau)d\tau} x_0 + e^{\int_{t_0}^{t} a(\tau)d\tau} \int_{t_0}^{t} e^{-\int_{t_0}^{s} a(\tau)d\tau} b(s)\,ds$$

How to simulate in continuous time?

- When a shock arrives θ adjusts instantaneously to its new level.
- Then in between shocks the value of θ is fixed.
- We can solve for the dynamics of u in intervals between shocks using:

$$
\begin{aligned}
x(t) &= e^{\int_{t_0}^{t} a d\tau} x_0 + e^{\int_{t_0}^{t} a d\tau} \int_{t_0}^{t} e^{-\int_{t_0}^{s} a d\tau} b ds \\
&= e^{a(t-t_0)} x_0 + e^{at} b \int_{t_0}^{t} e^{-as} ds \\
&= e^{a(t-t_0)} x_0 + e^{at} b \left[-\frac{e^{-as}}{a} \right]_{t_0}^{t} \\
&= e^{a(t-t_0)} \left(x_0 + \frac{b}{a} \right) - \frac{b}{a}
\end{aligned}
$$

- This gives a recursive way to update unemployment, given an initial condition u_0