

Advanced Macroeconomics II

Handout 7 - Models with Distortions and GE

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Short recap

Prototypical DP problem:

$$\begin{aligned} V(z, k) &= \max_{\{c, k'\}} u(c) + \beta E \left[V(z', k') \mid z \right] \\ \text{s.t. } c + k' &= f(z, k) \\ z' &= h(z, \eta); \eta \text{ stochastic} \end{aligned}$$

- ▶ We are looking for functions $\mathbf{V}, \mathbf{g}^c, \mathbf{g}^k$.

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- ▶ We are looking for functions $\mathbf{V}, \mathbf{g}^c, \mathbf{g}^k$.

But that is not the actual problem we started with!

Macroeconomic model

- ▶ We had a representative agent choosing consumption (and labor) to solve:

$$\max_{\{c_t, \ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c, \ell) \quad \text{s.t.} \quad c_t + a_{t+1} = (1 + r_t) a_t + w_t \ell_t + \pi_t$$

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- ▶ We had a representative firm choosing capital and labor to solve:

$$\pi_t = \max_{\{k_t, \ell_t^d\}} f(z_t, k_t, \ell_t^d) - (r_t + \delta) k_t - w_t \ell_t^d$$

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- ▶ And we had prices that cleared markets:

$$\ell_t = \ell_t^d \quad a_t = k_t \quad c_t + a_{t+1} = f(z_t, k_t, \ell_t) + (1 + \delta) a_t$$

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- ▶ Map planner's solution to the competitive market allocation and prices
 - ▶ Planner solves for aggregate **quantities** $\{C, L, K\}$
 - ▶ We want to get individual quantities $\{c, \ell, a, k\}$ and prices $\{r, w\}$

$$c = C \quad \ell = L \quad k = a = K$$

$$r = f_k(z, K, L) - \delta \quad w = f_\ell(z, K, L)$$

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- ▶ **Key:** Planner's problem is a “simple” dynamic programming problem
 - ▶ We can solve it with the tools from the previous 6 lectures!

How to solve for the equilibrium directly?

- ▶ **Easy part:** Firm's problem is static
 - ▶ Solution depends on aggregate quantities
 - ▶ Solution gives us prices

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$$r = f_k(z, K, L) - \delta \quad w = f_\ell(z, K, L)$$

- ▶ **Hard part:** Consumer problem a dynamic programming problem
 - ▶ What are the states?
 - ▶ Consumer is a price taker: No clue about aggregate effect of choices
 - ▶ States must provide enough information to solve the problem
 - ▶ Consumer must know how states evolve

A DP problem for the consumer: (k, K)

$$V(k, \underbrace{z, K}_{\text{Agg. States}}) = \max_{\{c, \ell, k'\}} u(c, \ell) + \beta E \left[V(k', z', K') | z \right]$$

s.t. $c + k' = (1 + r)k + w\ell$

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 1. Where do prices come from?
 2. How to update aggregate states?

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Key: little k (the individual state) and big K (the aggregate state)

- ▶ In equilibrium they are the same, but the agent does not know it

A DP problem for the consumer: (k, K)

$$\begin{aligned} V(k, z, K) = & \max_{\{c, \ell, k'\}} u(c, \ell) + \beta E \left[V(k', z', K') \mid z \right] \\ \text{s.t.} \quad & c + k' = (1 + r)k + w\ell \\ & r = R(z, K, L) \\ & w = W(z, K, L) \\ & L = G_\ell(z, K) \\ & K' = G_k(z, K) \\ & z' = h(z, \eta), \text{ with } \eta \text{ stochastic} \end{aligned}$$

Key: Find functions R , W , G_ℓ and G_k .

- Given these you can solve consumer's problem

A recursive competitive equilibrium

An RCE is a set of a value function V , policy functions g_k and g_ℓ , updating functions G_k and G_ℓ and price functions R and W such that:

1. The value function V and policy functions g_k and g_ℓ solve the DP problem in previous slide
2. Pricing functions R and W satisfy the firm's first order conditions

$$R(z, K, L) = f_k(z, K, L) - \delta \quad W(z, K, L) = f_\ell(z, K, L)$$

3. Updating functions G_k and G_ℓ are consistent with individual optimization

$$G_k(z, K) = g_k(K, z, K) \quad G_\ell(z, K) = g_\ell(K, z, K)$$

Some comments

1. Market clearing:

The definition of RCE didn't include market clearing explicitly

- ▶ This is a device of the CRS technology of the firm
- ▶ At equilibrium prices demand for inputs is perfectly elastic
- ▶ Markets clear automatically
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3. Curse of dimensionality applies

- ▶ You have to solve the agent's problem off-equilibrium
- ▶ You need to know $g_k(k, z, K)$ for any combination of (k, K) , even though in equilibrium $k = K$

RCE algorithm

Algorithm 1: RCE Algorithm

input : Guess for updating functions (G_k, G_ℓ)

output: $V, g_k, g_\ell, G_k, G_\ell$

1. Solve the DP problem of the agent given G_k, G_ℓ :
 $(V, g_k, g_\ell) = T(V; G_k, G_\ell)$ (a fixed point problem) ;
 2. Update updating functions:
 $G_k(z, K) = g_k(K, z, K) \quad G_\ell(z, K) = g_\ell(K, z, K)$;
 3. Check convergence in updating functions ;
 4. Repeat (1)-(3) until convergence ;
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 - ▶ We no longer have the CMT... No reason for it to converge
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 - ▶ How to get it to converge?
 - ▶ Carefully...
 - ▶ The best strategy is the tortoise strategy: Slowly but surely
- 2' Dampened update of updating functions:

$$G_k^{n+1}(z, K) = \gamma g_k(K, z, K) + (1 - \gamma) G_k^n(z, K)$$

$$G_\ell^{n+1}(z, K) = \gamma g_\ell(K, z, K) + (1 - \gamma) G_\ell^n(z, K)$$

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- ▶ VFI is simplified with the RCE

$$-\frac{u_\ell(c, \ell)}{u_c(c, \ell)} = w \longrightarrow \ell(c; w)$$

Algorithm 2: EGM for RCE problem

Function EGM($V, \vec{k}, \vec{z}, \ell(z, k), k'(z, k), \text{parameters}$):

for $i=1:n_z$ *# productivity (aggregate state)* **do**

for $j=1:n_k$ *# capital (aggregate state)* **do**

 1. Evaluate prices: $r = R(\vec{z}_i, \vec{K}_j)$, $w = W(\vec{z}_i, \vec{K}_j)$

for $h=1:n_k$ *# capital (individual state)* **do**

 2. Expectd value: $\mathbb{V} = \beta E \left[V \left(\vec{k}_h, z', G_k(\vec{z}_i, \vec{K}_j) \right) \mid \vec{z}_i \right]$

 Requires interpolation on $K' = G_k(\vec{z}_i, \vec{K}_j)$

 3. Consumption from Euler: $u_c(\tilde{c}_{ijh}, \ell(\tilde{c}_{ijh}; w)) = \mathbb{V}_k$

 Analytical solution using $\ell(c; w)$ from FOC

 4. Endo. capital: $\hat{k}_{ijh} = \left(\tilde{c}_{ijh} + \vec{k}_h - w \ell(\tilde{c}_{ijh}; w) \right) / 1 + r$

 5. Update value at endogenous grid:

$\hat{V}(\hat{k}_{ijh}; \vec{z}_i, \vec{K}_j) = u(\tilde{c}_{ijh}, \ell(\tilde{c}_{ijh}; w)) + \mathbb{V}$

 6. Interpolate to exogenous grid: $V_new[:,i,j] = \text{Interp}(\hat{k}, \hat{V}, \vec{k})$

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 - ▶ Some cases we still need to change states to Y , or to Y_k

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Think of problems with agents that can invest, or manage businesses

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Think of problems with agents that can invest, or manage businesses

- ▶ No requirement that the grids on capital have to match: $\vec{K} \neq \vec{k}$
 - ▶ Often they are the same
- ▶ We do have to interpolate in taking expectations
 - ▶ EGM is fixing the future capital of the agent
 - ▶ The future capital of the economy is exogenous (to the agent)
 - ▶ The agent has to be “consistent” and use G_k to forecast K'

RCE applications

Many applications for RCE, but first:

- ▶ Check that your code works!
- ▶ The NGM's last gift to you...
Contrast RCE solution with Planner's DP problem

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Applications (all your heart's desire):

- ▶ Taxes (distortions in general)
- ▶ Multiple agents
- ▶ Externalities
- ▶ Business Cycle Accounting
- ▶ Non-stationary problems (transitions, life-cycle)

Application: Taxes/Wedges

Taxes (or wedges)

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- ▶ Distortionary taxes prevent us from using the planner's problem to solve for the market equilibrium
 - ▶ In fact that is the point! We want to know how to make the market equilibrium closer to the planner's solution
- ▶ Usual taxes:
 - ▶ Labor income taxes (possibly non-linear)
 - ▶ Capital income taxes or wealth taxes
 - ▶ Consumption taxes (dangerous!)

Taxes (or wedges) - Agent's problem

$$V(k, z, K; \tau) = \max_{\{c, \ell, k'\}} u(c, \ell) + \beta E \left[V(k', z', K'; \tau) \mid z \right]$$

$$\text{s.t.} \quad (1 + \tau_c) c + k' = (1 + (1 - \tau_k) r) k + (1 - \tau_\ell) w \ell + T$$

$$r = R(z, K, L)$$

$$w = W(z, K, L)$$

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$$K' = G_k(z, K)$$

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 - ▶ Agent takes taxes as given
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- ▶ What if you do care about the budget...
 1. Are you balancing the budget every period
 - ▶ Then you need to search for $\tau(z, K)$
 2. Are you allowing for deficit/surplus?
 - ▶ Where is Gov. getting/putting funds?
 - ▶ You have to figure out market clearing

Taxes as wedges

$$u_c(c, \ell) = \beta (1 + (1 - \tau_k)r) u_c(c', \ell')$$
$$-\frac{u_\ell(c, \ell)}{u_c(c, \ell)} = \left(\frac{1 - \tau_\ell}{1 + \tau_c} \right) w$$

- Taxes show up in the solution to the model as wedges in FOC

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- ▶ Taxes show up in the solution to the model as wedges in FOC
- ▶ You can rebate (lump-sum) the “tax revenue”
 - ▶ Taxes only affect combination, not level
- ▶ This is a powerful idea
 - ▶ Front and center in public economics
 - ▶ Core of equivalence results between models
 - ▶ Many ways of generating the same wedges
 - ▶ Implications for measurement: BCA

Non-linear taxes: Two options

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 - ▶ Benabou (2000-AER, 2002, ECMA)
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Non-linear taxes: Two options

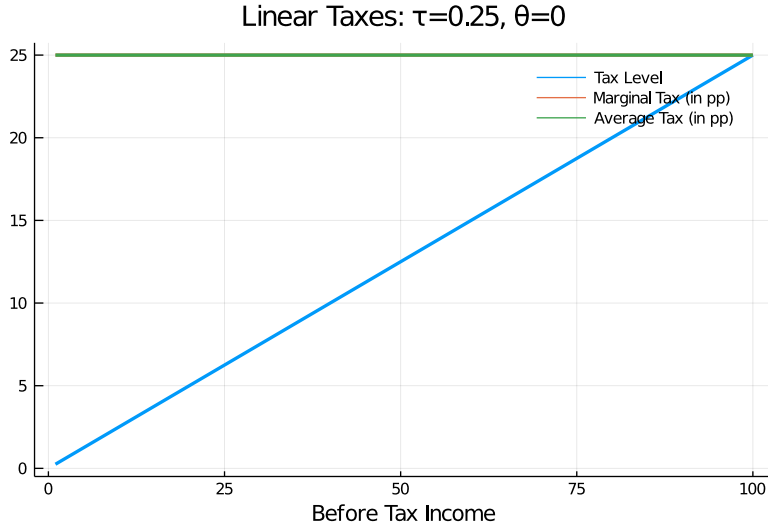
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2. Approximate tax system with smooth function
 - ▶ Benabou (2000-AER, 2002, ECMA)
 - ▶ Heathcote, Storesletten & Violante (2017-QJE)
 - ▶ If an agent has income y then after tax income is:

$$Y(y) = (1 - \tau) y^{1-\theta} + \underline{y} \quad T(y) = y - Y(y)$$

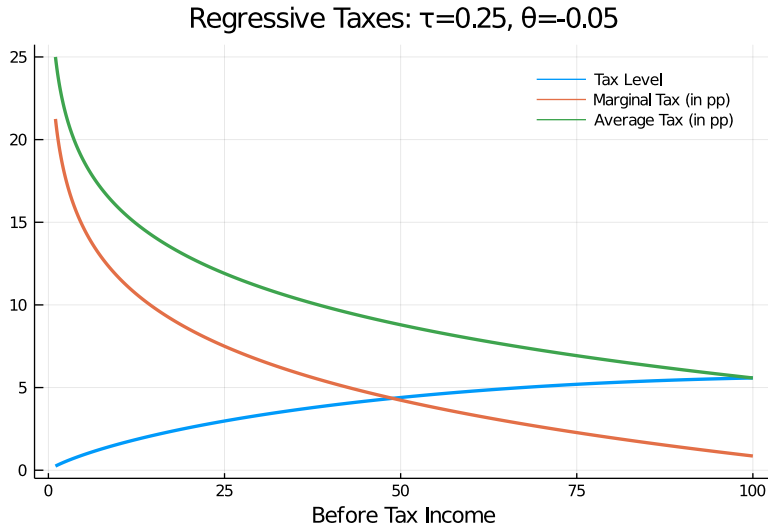
- ▶ Without transfers (\underline{y}) and zero progressivity ($\theta = 0$) we get tax rate τ
- ▶ Taxes are progressive (regressive) if ratio of marginal to average tax is larger (smaller) than 1

$$\frac{\text{mrg tax}}{\text{ave tax}} = \frac{1 - T'(y)}{1 - T(y)/y} = \frac{(1 - \theta)(1 - \tau)y^{-\theta}}{(1 - \tau)y^{-\theta} + \frac{\underline{y}}{y}} \leq (1 - \theta)$$

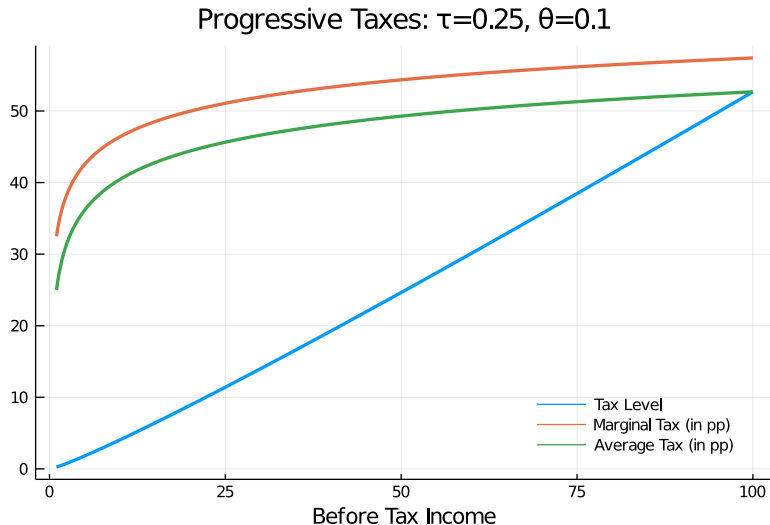
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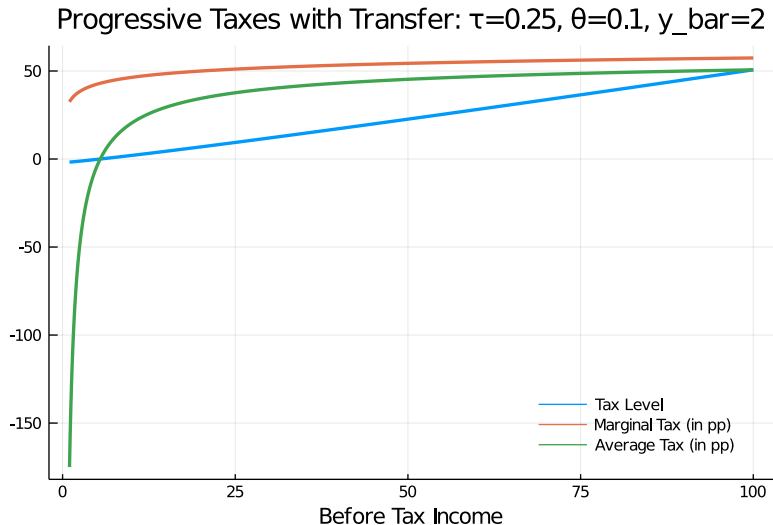
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Taxes (or wedges) - RCE

An RCE is a set of a value function V , policy functions g_k and g_ℓ , updating functions G_k and G_ℓ and price functions R and W such that, given taxes, transfers and expenditure $\{\tau_k, \tau_\ell, \tau_c, T, G\}$:

1. The $\{V, g_k, g_\ell\}$ solve the agent's DP problem
2. Pricing functions R and W satisfy the firm's first order conditions

$$R(z, K, L) = f_k(z, K, L) - \delta \quad W(z, K, L) = f_\ell(z, K, L)$$

3. Updating functions G_k and G_ℓ are consistent with agent optimization

$$G_k(z, K) = g_k(K, z, K) \quad G_\ell(z, K) = g_\ell(K, z, K)$$

4. Market clearing/Balanced budget:

$$G + c + K' = f(z, K, L) \quad \text{or: } G + T = \tau_c c + \tau_k R(z, K, L) K + \tau_w W(z, K, L) L$$

Taxes (or wedges) - Algorithm

Algorithm 3: RCE Algorithm with taxes/wedges

input : Guess for taxes/wedges $(G, T, \tau_k, \tau_c, \tau_\ell)$

output: $V, g_k, g_\ell, G_k, G_\ell, G, T, \tau_k, \tau_c, \tau_\ell$

1. Guess (G_k, G_ℓ) ;
 2. Solve the DP problem of the agent given $G_k, G_\ell, G, T, \tau_k, \tau_c, \tau_\ell$:
 $(V, g_k, g_\ell) = T(V; G_k, G_\ell, G, T, \tau_k, \tau_c, \tau_\ell)$ (a fixed point problem) ;
 3. Update updating functions:
 $G_k(z, K) = g_k(K, z, K) \quad G_\ell(z, K) = g_\ell(K, z, K)$;
 4. Check convergence in updating functions ;
 5. Repeat (2)-(4) until convergence ;
 6. Verify market clearing - Adjust taxes/transfer/spending ;
 7. Repeat (1)-(6) until market clears ;
-

Some comments

- ▶ General equilibrium == Outer loops
 - ▶ Outer loops are very expensive!
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$$G + T = \tau_c C + \tau_k rK + \tau_w wL$$

- ▶ Something has to be fixed
 - ▶ Sometimes it is taxes, sometimes it is expenditure
- ▶ Further complication: Dynamics
 - ▶ Taxes here are static, so is the budget
 - ▶ In general there can also be debt with deficit/surpluses
 - ▶ Change in market clearing ($K = k - D$), non-stationarity (transitions)

Application: Multiple Agents

Multiple agents - Model

- ▶ We already saw one of these:
 - ▶ Capitalist/Union model

Multiple agents - Model

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 - ▶ Capitalist/Union model
- ▶ Back then we cheated:
 - ▶ Union does not optimize... instead it fixes wages to avoid GE
- ▶ Lets try again

Multiple agents - Model

- ▶ There are three types of agents:
 - ▶ Capitalists
 - ▶ High-skilled workers
 - ▶ Low-skilled workers

Multiple agents - Model

- ▶ There are three types of agents:
 - ▶ Capitalists
 - ▶ High-skilled workers
 - ▶ Low-skilled workers
- ▶ Capitalists do not work but they own capital
- ▶ Workers are hand to mouth
- ▶ Production combines skill types with capital

Capitalists

$$V(k, z, K; w_l, w_h) = \max_{\{c, k'\}} u(c) + \beta E \left[V(k', z', K'; w'_l, w'_h) | z \right]$$

$$\text{s.t. } c + k' \leq \pi(z, k; w_l, w_h)$$

$$\pi(z, k; w_l, w_h) = \max_{\ell} f(z, k, \ell_l, \ell_h) - w_l \ell_l - w_h \ell_h + (1 - \delta) k$$

$$\log z' = \rho \log z + \eta; \quad \eta \sim N(0, \sigma_{\eta}^2)$$

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$$\log z' = \rho \log z + \eta; \quad \eta \sim N(0, \sigma_\eta^2)$$

- ▶ The production function is key
 - ▶ See Krusell, Ohanian, Rios-Rull & Violante (2000, ECMA)
- ▶ Capitalist needs to distinguish between k and K to predict wages

Workers

- ▶ The problem of the workers is symmetric and static:

$$\max u^i(w_i \ell, \ell) \quad \text{for } i = \{l, h\}$$

- ▶ Key here is the FOC given wages:

$$u_\ell^i(w_i \ell, \ell) = w_i u_c^i(w_i \ell, \ell)$$

- ▶ This condition gives closed form of $\ell_i(w_i)$ and $c_i(w_i)$

Market clearing - Labor

- ▶ From the profit maximization problem we get

$$w_l = f_l(z, K, \ell_l(w_l), \ell_h(w_h))$$

$$w_h = f_h(z, K, \ell_l(w_l), \ell_h(w_h))$$

- ▶ Solve for price functions that depend on aggregate states (z, K)
- ▶ Is it clear why these conditions imply market clearing?

Multiple agents - RCE

An RCE is a set of a value function V and policy function g_k for capitalists, updating function G_k and price functions W_L and W_H such that:

1. The value function V and policy functions g_k and g_ℓ solve the DP problem in previous slide
2. Pricing functions W_L, W_H satisfy the firm's first order conditions

$$W_L(z, K) = f_l(z, K, \ell_l(W_L(z, K)), \ell_h(W_H(z, K)))$$

$$W_H(z, K) = f_h(z, K, \ell_l(W_L(z, K)), \ell_h(W_H(z, K)))$$

3. Updating functions G_k and G_l are consistent with individual optimization

$$G_k(z, K) = g_k(K, z, K)$$

Application: Business Cycle Accounting

Business Cycle Accounting (CKM,2007)

- ▶ Main idea:
 - ▶ Use the model as a measurement device

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 - ▶ What shock or policy could have generated the observed data?

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- ▶ Main idea:
 - ▶ Use the model as a measurement device
- ▶ Change the question:
 - ▶ What are the effects of a shock or a policy?
 - ▶ What shock or policy could have generated the observed data?
- ▶ This is a crucial way in which we think about models
 - ▶ How to explain the world we have seen?
 - ▶ Which frictions or policies are most relevant?

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Method:

1. Use a “prototype” model with wedges
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Method:

1. Use a “prototype” model with wedges
 - ▶ The model can fit the data by construction by adjusting wedges
2. Analyze data with the model to recover wedges
 - ▶ Which wedges are important for the data?
3. Establish equivalence results between models and wedges
 - ▶ Some are obvious: wedges look like taxes
 - ▶ Some are not obvious: wedges can represent financial frictions

Application: Sovereign Default

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- ▶ Default models form a large literature on international econ
- ▶ Great example of dynamic programming:
 - ▶ Default option is inherently dynamic
- ▶ Great example of RCE:
 - ▶ Default and savings choice depend on prices!
 - ▶ Prices are endogenous... but taken as given

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- ▶ (Stochastic) Endowment economy
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- ▶ (Risk-neutral) Financial intermediary
 - ▶ Breaks even in expectation (wrt default)
- ▶ Default repercussion: Autarky
 - ▶ Output penalty during autarky
 - ▶ Autarky costly because of income fluctuation
 - ▶ Autarky ends with probability $\lambda \geq 0$

Sovereign default - Prices

Profits of intermediary:

$$\text{Pr} = qb' - \frac{1 - \delta}{1 + r}b' \longrightarrow \text{Pr} = 0$$

- ▶ Here δ is the probability of default
- ▶ δ is endogenous, in fact:

$$\delta = E_{s'} \left[g^D(s', b') | s \right] \quad \text{where } g^D(s', b') = \begin{cases} 1 & \text{if default} \\ 0 & \text{if no default} \end{cases}$$

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Free entry gives the zero profit condition:

$$q(s, b') = \begin{cases} \frac{1 - \sum_{s' \in S} \pi(s') g^D(s', b')}{R} & \text{if } b' < 0 \\ \frac{1}{R} & \text{if } b' \geq 0 \end{cases}$$

Sovereign default - DP

$$V^*(s, b) = \max_{d \in \{0,1\}} \{ (1 - d(s, b)) V(s, b) + d(s, b) V^A(s) \}$$

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$$V(s, b) = \max_{\{c, b'\}} \left\{ \frac{c(s, b)^{1-\sigma} - 1}{1 - \sigma} + \beta \sum_{s' \in S} \pi(s') V^*(s', b') \right\}$$

$$\text{s.t. } c(s, b) - q(s, b) b'(s, b) \leq y(s) + b$$

$$-B \leq b'(s, b) \quad [\text{B: borrowing limit}]$$

$$0 \leq c(s, b)$$

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$$V^A(s) = \frac{h(y(s))^{1-\sigma} - 1}{1 - \sigma} + \beta \sum_{s' \in S} \pi(s') (\lambda V^*(s', 0) + (1 - \lambda) V^A(s'))$$

Sovereign default - RCE

A Recursive Competitive Equilibrium is

1. Value functions V^* , V^A , V
2. Policy functions $g^c(s, b)$, $g^b(s, b)$, $g^D(s, b)$
3. Price functional given by $q(s, b) = \frac{(1 - \sum \pi(s') g^D(s', b'))}{R}$

Such that the value functions and policy functions solve the DP of previous slide taking q as given.