

Problem Set IV

Econometrics I - FGV EPGE

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Problem 1 – (points: 3)

In the next questions, you will be asked to derive CLTs for the sample mean under different assumptions. You should decide what type of theorem to use in each case and *impose the assumptions you think you need*. You have less assumptions than you need on purpose. Justify your choices.

- a) Suppose X_t is an α -mixing weakly stationary process with $\mathbb{E}[X_t] = 0$ and covariance function $\gamma(h)$. Let $Y_t = X_t + 0.5X_{t-1}$. Derive a CLT for $\frac{1}{T} \sum_{t=2}^T Y_t$.
- b) Suppose that X_t is an i.i.d. scalar sequence with zero mean and finite fourth moment. If $Y_t \equiv X_t X_{t-1}$, derive a CLT for $\frac{1}{T} \sum_{t=2}^T Y_t$

Problem 2 – (points: 1)

During the derivation of the Augmented Dickey-Fuller test, we used a trick and invoked a result without proving it. You will prove it now. Consider p coefficients from an AR(p) process given by $\phi_1, \phi_2, \dots, \phi_p$. Consider the following $p \times p$ matrix B :

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

Let $\mathbf{y}_{t-1} = (y_{t-1}, y_{t-2}, \dots, y_{t-p})'$ be a $p \times 1$ vector. Show that:

$$B\mathbf{y}_{t-1} = \begin{bmatrix} y_{t-1} \\ \Delta y_{t-1} \\ \Delta y_{t-2} \\ \vdots \\ \Delta y_{t-p+1} \end{bmatrix}$$

Furthermore, let $\phi = (\phi_1, \phi_2, \dots, \phi_p)'$ and assume that there exist scalars $\rho \in \mathbb{R}$ and a vector $\beta = (\beta_1, \beta_2, \dots, \beta_{p-1})' \in \mathbb{R}^{p-1}$ such that:

$$\phi = \mathbf{B}' \begin{bmatrix} \rho \\ \beta \end{bmatrix}$$

Show that this transformation implies:

- (a) $\rho = \phi_1 + \phi_2 + \dots + \phi_p = \sum_{j=1}^p \phi_j$;
(b) An AR(p) process can be equivalently written as:

$$y_t = \rho y_{t-1} + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-2} + \dots + \beta_{p-1} \Delta y_{t-p+1} + \varepsilon_t$$

where ε_t is the innovation term.

Problem 3 – (points: 2)

Consider an AR-DL(1,2) model where y_t depends on its own lag and on current and lagged values of an exogenous variable x_t :

$$y_t = \alpha + \rho y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \varepsilon_t$$

where $|\rho| < 1$ ensures stationarity, and $\mathbb{E}[\varepsilon_t | y_{t-1}, y_{t-2}, \dots, x_t, x_{t-1}, \dots] = 0$.

- (a) **Impact Multiplier:** Suppose that at time $t = 0$, the system is in equilibrium with $y_t = \bar{y}$ for all $t \leq 0$ and $x_t = \bar{x}$ for all $t \leq 0$. At time $t = 1$, there is a one-time shock such that $x_1 = \bar{x} + 1$ (i.e., x increases by one unit), and then $x_t = \bar{x}$ for all $t \geq 2$.

Compute the *impact multiplier*, defined as the immediate effect on y_1 :

$$\frac{\partial y_1}{\partial x_1} = ?$$

- (b) **Dynamic Multipliers:** Continuing from part (a), compute the effect of this shock on y_t for $t = 2, 3, 4$. Show that:

$$y_2 - \bar{y} = \rho \beta_0 + \beta_1$$

$$y_3 - \bar{y} = \rho^2 \beta_0 + \rho \beta_1 + \beta_2$$

$$y_4 - \bar{y} = \rho^3 \beta_0 + \rho^2 \beta_1 + \rho \beta_2$$

What is the general pattern here?

- (c) **Long-Run Multiplier:** The *long-run multiplier* is defined as the cumulative effect of a permanent one-unit increase in x on the long-run equilibrium value of y .

Suppose instead that starting at $t = 1$, x increases permanently from \bar{x} to $\bar{x} + 1$ (i.e., $x_t = \bar{x} + 1$ for all $t \geq 1$). Let y^* denote the new long-run equilibrium value of y . Show that:

$$y^* = \frac{\alpha + (\beta_0 + \beta_1 + \beta_2)(\bar{x} + 1)}{1 - \rho}$$

and therefore the long-run multiplier is:

$$\frac{dy^*}{dx} = \frac{\beta_0 + \beta_1 + \beta_2}{1 - \rho}$$

Hint: In the new steady state, $y_t = y_{t-1} = y^*$ and $x_t = x_{t-1} = x_{t-2} = \bar{x} + 1$.

- (d) **General AR-DL(1,q) Case:** Consider the more general AR-DL(1,q) model:

$$y_t = \alpha + \rho y_{t-1} + \sum_{j=0}^q \beta_j x_{t-j} + \varepsilon_t$$

Using the same reasoning as in part (c), show that the long-run multiplier is:

$$\text{LRM} = \frac{\sum_{j=0}^q \beta_j}{1 - \rho}$$

Interpret this result: why does the autoregressive coefficient ρ amplify (if $\rho > 0$) or dampen (if $\rho < 0$) the long-run effect relative to the sum of the contemporaneous and lagged coefficients on x ?

Problem 4 – (points: 3)

Take the linear model $Y_t = X_t' \beta + e_t$ with $\mathbb{E}[Z_t e_t] = 0$ for some variable $Z_t \in \mathbb{R}^\ell$, where $Y_t \in \mathbb{R}$ and $X_t \in \mathbb{R}^k$. Consider the GMM estimator $\hat{\beta}$ of β and assume $\ell \geq k$. Let $J = T \hat{m}_T(\hat{\beta})' \hat{S}^{-1} \hat{m}_T(\hat{\beta})$ be the J -statistic for the test regarding overidentifying restrictions. S is the asymptotic variance of the moment conditions and \hat{S} is a consistent estimator.

We also let $\mathbf{X}_{T \times k} \equiv (X_1', \dots, X_T')'$ and $\mathbf{Z}_{T \times \ell} \equiv (Z_1', \dots, Z_T')'$. Finally, \mathbf{I}_ℓ is the $\ell \times \ell$ identity matrix. We will now show that $J \xrightarrow{d} \chi_{\ell-k}^2$ as $T \rightarrow \infty$ by demonstrating the following items:

- (a) Argue that we can write $S^{-1} = \mathbf{C}\mathbf{C}'$ and $S = \mathbf{C}'^{-1}\mathbf{C}^{-1}$ for some matrix \mathbf{C} .
- (b) Show that we can write $J = T (\mathbf{C}' \hat{m}_T(\hat{\beta}))' (\mathbf{C}' \hat{S} \mathbf{C})^{-1} \mathbf{C}' \hat{m}_T(\hat{\beta})$.

(c) Show that $\mathbf{C}'\hat{m}_T(\hat{\beta}) = \mathbf{A}_T\mathbf{C}'\hat{m}_T(\beta)$ where $\hat{m}_T(\beta) = \frac{1}{T} \sum_{t=1}^T Z_t e_t$ and

$$\mathbf{A}_T = \mathbf{I}_\ell - \mathbf{C}' \left(\frac{1}{T} \mathbf{Z}' \mathbf{X} \right) \left[\left(\frac{1}{T} \mathbf{X}' \mathbf{Z} \right) \hat{\mathbf{S}}^{-1} \left(\frac{1}{T} \mathbf{Z}' \mathbf{X} \right) \right]^{-1} \left(\frac{1}{T} \mathbf{X}' \mathbf{Z} \right) \hat{\mathbf{S}}^{-1} \mathbf{C}'^{-1}.$$

(d) Show that $\mathbf{A}_T \xrightarrow{p} \mathbf{I}_\ell - \mathbf{R}(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'$ where $\mathbf{R} = \mathbf{C}'\mathbb{E}[Z_t X_t']$.

(e) Show that $T^{1/2}\mathbf{C}'\hat{m}_T(\beta) \xrightarrow{d} u \sim N(0, \mathbf{I}_\ell)$.

(f) Show that $J \xrightarrow{d} u' (\mathbf{I}_\ell - \mathbf{R}(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}') u$.

(g) Show that $u' (\mathbf{I}_\ell - \mathbf{R}(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}') u \sim \chi_{\ell-k}^2$.

Hint: $\mathbf{I}_\ell - \mathbf{R}(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'$ is a projection matrix. What do we know about these matrices?

Problem 5 – (points: 2)

You want to estimate $\mu = \mathbb{E}[Y_i]$ under the assumption that $\mathbb{E}[X_i] = 0$, where Y_i and X_i are scalars and observed from a random sample $\{(Y_i, X_i)\}_{i=1}^n$. Find an efficient GMM estimator for μ . Why the information about X_i might help to improve the estimation of μ ?

Problem 6 – (points: 2)

In this question, we will explore the asymptotic distribution of the GMM estimator under model misspecification.

The observed data is $\{Y_i, X_i, Z_i\} \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^\ell$, $k > 1$ and $\ell > k > 1$, $i = 1, \dots, n$. The model assumed by the econometrician is $Y = X'\beta + e$ with $\mathbb{E}[Ze] = 0$.

(a) Given a weight matrix $W > 0$ write down the GMM estimator $\hat{\beta}$ for β .

(b) Suppose the model is misspecified. Specifically, assume that for some $\delta \neq 0$,

$$e = \frac{\delta}{\sqrt{n}} + u$$

$$\mathbb{E}[u | Z] = 0$$

with $\mu_Z = \mathbb{E}[Z] \neq 0$. Show that (13.32) implies that $\mathbb{E}[Ze] \neq 0$.

(c) Express $\sqrt{n}(\hat{\beta} - \beta)$ as a function of W , n , δ , and the variables (X_i, Z_i, u_i) .

- (d) Find the asymptotic distribution of $\sqrt{n}(\hat{\beta} - \beta)$ in this case and show that it has an asymptotic bias.
- (e) Is this misspecification bias eliminated if we use the optimal weight matrix? Justify your answer.
- (f) Will this misspecification affect consistency?