

PS1 - Proposed Solutions

Taric Latif Padovani

November 9, 2025

Problem 1. 1. We are given a strictly (weakly) stationary $\{X_t\}$ with $[X_t] = 0$, covariance function $\gamma(h) = \text{Cov}(X_t, X_{t-h})$, and the process is α -mixing. Define

$$Y_t = X_t + 0.5 X_{t-1}, \quad t \in \mathbb{Z}.$$

Because Y_t is a finite linear combination of (X_t) , it is also weakly stationary and (under the standard fact that finite linear transforms preserve mixing with the same rate) α -mixing.

To apply a CLT for strongly/ α -mixing sequences (e.g. Ibragimov–Linnik), we *impose* the usual additional conditions:

- there exists $\delta > 0$ such that $|X_0|^{2+\delta} < \infty$ (hence also $|Y_0|^{2+\delta} < \infty$);
- the mixing coefficients satisfy $\sum_{h=1}^{\infty} \alpha(h)^{\delta/(2+\delta)} < \infty$.

Under these assumptions, the CLT for stationary α -mixing sequences gives

$$\sqrt{T}(\bar{Y}_T - Y_0) \xrightarrow{d} \mathcal{N}(0, \Omega_Y), \quad \bar{Y}_T := \frac{1}{T} \sum_{t=2}^T Y_t,$$

and since $Y_0 = X_0 + 0.5 X_{-1} = 0$, this is

$$\sqrt{T} \frac{1}{T} \sum_{t=2}^T Y_t \xrightarrow{d} \mathcal{N}(0, \Omega_Y).$$

(The fact that the sum starts at $t = 2$ instead of $t = 1$ is $o_p(1)$.)

It remains to identify the long-run variance. For $b_0 = 1$ and $b_1 = 0.5$ we can write

$$Y_t = \sum_{j=0}^1 b_j X_{t-j}.$$

Hence, for $h \in \mathbb{Z}$,

$$\gamma_Y(h) := \text{Cov}(Y_t, Y_{t-h}) = \sum_{j=0}^1 \sum_{k=0}^1 b_j b_k \gamma(h+j-k).$$

Plugging $b_0 = 1$, $b_1 = 0.5$,

$$\gamma_Y(h) = 1 \cdot 1 \cdot \gamma(h) + 1 \cdot 0.5 \cdot \gamma(h+1) + 0.5 \cdot 1 \cdot \gamma(h-1) + 0.5 \cdot 0.5 \cdot \gamma(h) = 1.25 \gamma(h) + 0.5 \gamma(h+1) + 0.5 \gamma(h-1).$$

The long-run variance is then

$$\Omega_Y = \sum_{h=-\infty}^{\infty} \gamma_Y(h) = \gamma_Y(0) + 2 \sum_{h=1}^{\infty} \gamma_Y(h),$$

with

$$\gamma_Y(0) = (Y_t) = 1.25 \gamma(0) + \gamma(1).$$

Therefore, under the mixing and moment assumptions we imposed,

$$\sqrt{T} \frac{1}{T} \sum_{t=2}^T Y_t \xrightarrow{d} \mathcal{N}(0, \Omega_Y), \quad \Omega_Y = \gamma_Y(0) + 2 \sum_{h=1}^{\infty} [1.25 \gamma(h) + 0.5 \gamma(h+1) + 0.5 \gamma(h-1)].$$

2. Now X_t is i.i.d., scalar, with $X_t = 0$ and $X_t^4 < \infty$. Define

$$Y_t := X_t X_{t-11}.$$

Because the X_t 's are i.i.d., (Y_t) is a *finite-dependence* (in fact, 11-dependence) process: for $h = 1, \dots, 10$ the sets of indices $\{t, t-11\}$ and $\{t-h, t-h-11\}$ are disjoint, so Y_t and Y_{t-h} are independent. Only at lag $h = 11$ is there overlap, but even there the covariance will turn out to be zero. A classical CLT for m -dependent sequences (e.g. Billingsley, or the standard m -dependent CLT) applies if $Y_t^2 < \infty$, which we *do* have because

$$Y_t^2 = (X_t^2 X_{t-11}^2) = (X_t^2) (X_{t-11}^2) = (X_t^2)^2 < \infty$$

(using independence and the given finite fourth moment).

First, the mean:

$$Y_t = (X_t) (X_{t-11}) = 0.$$

Second, the covariances. For $h = 1, \dots, 10$ we have independence $\Rightarrow (Y_t, Y_{t-h}) = 0$. For $h = 11$,

$$Y_t = X_t X_{t-11}, \quad Y_{t-11} = X_{t-11} X_{t-22},$$

so

$$(Y_t, Y_{t-11}) = [X_t X_{t-11}^2 X_{t-22}] - Y_t Y_{t-11} = X_t X_{t-11}^2 X_{t-22} = 0,$$

because $X_t = 0$. Thus *all* off-diagonal covariances are zero, and the long-run variance is just

$$\Omega = (Y_t) = (X_t^2 X_{t-11}^2) = (X_t^2)^2.$$

We sum from $t = 2$ to T in the statement, but for this process the first 11 terms are “incomplete”. Replacing $\frac{1}{T} \sum_{t=2}^T$ with $\frac{1}{T} \sum_{t=12}^T$ changes the average by $O(T^{-1})$, which is negligible at \sqrt{T} -rate. Hence, by the m -dependent CLT,

$$\sqrt{T} \frac{1}{T} \sum_{t=2}^T Y_t \xrightarrow{d} \mathcal{N}(0, (X_t^2)^2).$$

So the CLT we can state is

$$\sqrt{T} \frac{1}{T} \sum_{t=2}^T X_t X_{t-11} \xrightarrow{d} \mathcal{N}(0, (X_t^2)^2),$$

where we used the fact that (Y_t) is 11-dependent with zero mean and finite second moment, and we imposed no extra mixing assumptions beyond the i.i.d. structure.

Problem 2. We are given

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}, \quad \mathbf{y}_{t-1} = \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p} \end{bmatrix}.$$

1. Show that $\mathbf{B}\mathbf{y}_{t-1}$ has the desired form.

Compute the product row by row.

- First row of \mathbf{B} is $(1, 0, \dots, 0)$, so

$$(\text{row } 1) \cdot \mathbf{y}_{t-1} = 1 \cdot y_{t-1} = y_{t-1}.$$

- Second row is $(1, -1, 0, \dots, 0)$, so

$$(\text{row } 2) \cdot \mathbf{y}_{t-1} = y_{t-1} - y_{t-2} = \Delta y_{t-1}.$$

- Third row is $(0, 1, -1, 0, \dots, 0)$, so

$$(\text{row } 3) \cdot \mathbf{y}_{t-1} = y_{t-2} - y_{t-3} = \Delta y_{t-2}.$$

- Continuing this pattern, the k -th nonzero row (for $k = 2, \dots, p$) is zero everywhere except a 1 and a -1 in consecutive positions, giving

$$y_{t-(k-1)} - y_{t-k} = \Delta y_{t-k+1}.$$

Thus

$$\mathbf{B}\mathbf{y}_{t-1} = \begin{bmatrix} y_{t-1} \\ \Delta y_{t-1} \\ \Delta y_{t-2} \\ \vdots \\ \Delta y_{t-p+1} \end{bmatrix},$$

as required.

Now let

$$\phi = (\phi_1, \phi_2, \dots, \phi_p)' \in \mathbb{R}^p, \quad \begin{bmatrix} \rho \\ \beta \end{bmatrix} = \begin{bmatrix} \rho \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{bmatrix} \in \mathbb{R}^p,$$

and assume

$$\phi = \mathbf{B}' \begin{bmatrix} \rho \\ \beta \end{bmatrix}.$$

We need to work out the explicit form of \mathbf{B}' .

Since

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix},$$

its transpose is

$$\mathbf{B}' = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

So

$$\phi = \mathbf{B}' \begin{bmatrix} \rho \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{p-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} \rho \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{p-1} \end{bmatrix}.$$

Carrying out the multiplication componentwise:

- First component:

$$\phi_1 = \rho + \beta_1.$$

- Second component (use row $(0, -1, 1, 0, \dots)$):

$$\phi_2 = -\beta_1 + \beta_2.$$

- Third component:

$$\phi_3 = -\beta_2 + \beta_3.$$

- Continue this pattern up to

$$\phi_{p-1} = -\beta_{p-2} + \beta_{p-1},$$

and the last row of \mathbf{B}' is $(0, 0, \dots, 0, -1)$, so

$$\phi_p = -\beta_{p-1}.$$

So we have the system

$$\begin{aligned} \phi_1 &= \rho + \beta_1, \\ \phi_2 &= -\beta_1 + \beta_2, \\ \phi_3 &= -\beta_2 + \beta_3, \\ &\vdots \\ \phi_{p-1} &= -\beta_{p-2} + \beta_{p-1}, \\ \phi_p &= -\beta_{p-1}. \end{aligned}$$

(a) Show that $\rho = \sum_{j=1}^p \phi_j$.
Sum all p equations above:

$$\sum_{j=1}^p \phi_j = (\rho + \beta_1) + (-\beta_1 + \beta_2) + (-\beta_2 + \beta_3) + \cdots + (-\beta_{p-2} + \beta_{p-1}) + (-\beta_{p-1}).$$

This is a telescoping sum: every $+\beta_k$ is canceled by a $-\beta_k$ in the next term, and the final $-\beta_{p-1}$ cancels the last $+\beta_{p-1}$. What remains is

$$\sum_{j=1}^p \phi_j = \rho.$$

Hence

$$\boxed{\rho = \sum_{j=1}^p \phi_j.}$$

(b) Rewrite the AR(p) in ADF form.

Start from the usual AR(p):

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t = \phi' \mathbf{y}_{t-1} + \varepsilon_t.$$

By assumption, $\phi = \mathbf{B}' \begin{bmatrix} \rho \\ \beta \end{bmatrix}$, so

$$\phi' \mathbf{y}_{t-1} = (\mathbf{B}'[\rho; \beta])' \mathbf{y}_{t-1} = [\rho; \beta]' \mathbf{B} \mathbf{y}_{t-1}.$$

But from the first part we already proved

$$\mathbf{B} \mathbf{y}_{t-1} = \begin{bmatrix} y_{t-1} \\ \Delta y_{t-1} \\ \Delta y_{t-2} \\ \vdots \\ \Delta y_{t-p+1} \end{bmatrix}.$$

Therefore,

$$\phi' \mathbf{y}_{t-1} = [\rho; \beta_1; \dots; \beta_{p-1}]' \begin{bmatrix} y_{t-1} \\ \Delta y_{t-1} \\ \Delta y_{t-2} \\ \vdots \\ \Delta y_{t-p+1} \end{bmatrix} = \rho y_{t-1} + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-2} + \cdots + \beta_{p-1} \Delta y_{t-p+1}.$$

Substitute back into the AR(p) equation:

$$y_t = \rho y_{t-1} + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-2} + \cdots + \beta_{p-1} \Delta y_{t-p+1} + \varepsilon_t.$$

So we have shown

$$y_t = \rho y_{t-1} + \sum_{j=1}^{p-1} \beta_j \Delta y_{t-j} + \varepsilon_t,$$

which is precisely the ADF regression form, and by part (a) the “level” coefficient ρ is the sum of the original AR coefficients.

Problem 3. 1. Impact multiplier.

Initial steady state (for $t \leq 0$): $y_t = \bar{y}$ and $x_t = \bar{x}$. Plugging into the model,

$$\bar{y} = \alpha + \rho\bar{y} + (\beta_0 + \beta_1 + \beta_2)\bar{x} \implies \bar{y} = \frac{\alpha + (\beta_0 + \beta_1 + \beta_2)\bar{x}}{1 - \rho}.$$

At $t = 1$ we have $x_1 = \bar{x} + 1$, while $y_0 = \bar{y}$, $x_0 = \bar{x}$, $x_{-1} = \bar{x}$. Then

$$y_1 = \alpha + \rho\bar{y} + \beta_0(\bar{x} + 1) + \beta_1\bar{x} + \beta_2\bar{x} + \varepsilon_1.$$

Compare this with what y_1 would be without the shock (i.e. with $x_1 = \bar{x}$); the only term that changes is $\beta_0 x_1$. Hence

$$\frac{\partial y_1}{\partial x_1} = \beta_0.$$

2. Dynamic multipliers for $t = 2, 3, 4$.

From part (a),

$$y_1 = \bar{y} + \beta_0 + (\varepsilon_1 - \bar{\varepsilon}),$$

but since we are looking at deterministic effects of the x -shock, take $\varepsilon_t = 0$ throughout and work with deviations from \bar{y} .

At $t = 2$: $x_2 = \bar{x}$ (shock is over), $x_1 = \bar{x} + 1$, $x_0 = \bar{x}$, and $y_1 = \bar{y} + \beta_0$. Then

$$y_2 = \alpha + \rho y_1 + \beta_0 x_2 + \beta_1 x_1 + \beta_2 x_0 = \alpha + \rho(\bar{y} + \beta_0) + \beta_0 \bar{x} + \beta_1(\bar{x} + 1) + \beta_2 \bar{x}.$$

Subtract the steady state value $\bar{y} = \alpha + \rho\bar{y} + (\beta_0 + \beta_1 + \beta_2)\bar{x}$:

$$y_2 - \bar{y} = \rho\beta_0 + \beta_1.$$

At $t = 3$: now $x_3 = \bar{x}$, $x_2 = \bar{x}$, $x_1 = \bar{x} + 1$, and $y_2 = \bar{y} + \rho\beta_0 + \beta_1$. Then

$$y_3 = \alpha + \rho y_2 + \beta_0 \bar{x} + \beta_1 \bar{x} + \beta_2(\bar{x} + 1).$$

Subtracting \bar{y} again,

$$y_3 - \bar{y} = \rho(y_2 - \bar{y}) + \beta_2 = \rho(\rho\beta_0 + \beta_1) + \beta_2 = \rho^2\beta_0 + \rho\beta_1 + \beta_2.$$

At $t = 4$: all $x_4, x_3, x_2 = \bar{x}$. We have

$$y_4 = \alpha + \rho y_3 + \beta_0 \bar{x} + \beta_1 \bar{x} + \beta_2 \bar{x},$$

so

$$y_4 - \bar{y} = \rho(y_3 - \bar{y}) = \rho(\rho^2\beta_0 + \rho\beta_1 + \beta_2) = \rho^3\beta_0 + \rho^2\beta_1 + \rho\beta_2.$$

General pattern (for $t \geq 3$):

$$y_t - \bar{y} = \rho^{t-1}\beta_0 + \rho^{t-2}\beta_1 + \rho^{t-3}\beta_2.$$

That is, the shock to x_1 propagates through the AR part via powers of ρ , while the three β 's line up with lags of ρ .

3. Long-run multiplier for a permanent increase.

Now suppose $x_t = \bar{x} + 1$ for all $t \geq 1$. In the new steady state we have

$$y_t = y_{t-1} = y^*, \quad x_t = x_{t-1} = x_{t-2} = \bar{x} + 1.$$

Plug into the model:

$$y^* = \alpha + \rho y^* + \beta_0(\bar{x} + 1) + \beta_1(\bar{x} + 1) + \beta_2(\bar{x} + 1).$$

Collect terms:

$$y^* - \rho y^* = \alpha + (\beta_0 + \beta_1 + \beta_2)(\bar{x} + 1) \implies (1 - \rho)y^* = \alpha + (\beta_0 + \beta_1 + \beta_2)(\bar{x} + 1).$$

Hence

$$y^* = \frac{\alpha + (\beta_0 + \beta_1 + \beta_2)(\bar{x} + 1)}{1 - \rho}.$$

If we differentiate the new steady state with respect to the permanent level of x (i.e. treat “+1” as “+ dx ”), we get the long-run multiplier:

$$\frac{dy^*}{dx} = \frac{\beta_0 + \beta_1 + \beta_2}{1 - \rho}.$$

4. General AR-DL(1, q) case.

For

$$y_t = \alpha + \rho y_{t-1} + \sum_{j=0}^q \beta_j x_{t-j} + \varepsilon_t,$$

a permanent increase $x_t = \bar{x} + 1$ for all $t \geq 1$ implies in the new steady state

$$y^* = \alpha + \rho y^* + \sum_{j=0}^q \beta_j (\bar{x} + 1).$$

Thus

$$(1 - \rho)y^* = \alpha + \left(\sum_{j=0}^q \beta_j \right) (\bar{x} + 1), \implies y^* = \frac{\alpha + \left(\sum_{j=0}^q \beta_j \right) (\bar{x} + 1)}{1 - \rho}.$$

So the long-run effect of raising x by 1 permanently is

$$\text{LRM} = \frac{\sum_{j=0}^q \beta_j}{1 - \rho}.$$

Interpretation. The numerator $\sum_{j=0}^q \beta_j$ is the total contemporaneous + distributed-lag effect of x on y *before* accounting for feedback through y_{t-1} . Because current y feeds into future y through the AR term, every initial effect is propagated forward and summed as a geometric series with ratio ρ . If $\rho > 0$, these feedbacks reinforce the initial effect, magnifying it by $1/(1 - \rho)$. If $\rho < 0$, the feedback oscillates in sign and the net long-run effect is dampened relative to the positive-lag sum.

Problem 4. We have the linear model

$$Y_t = X_t' \beta + e_t, \quad [Z_t e_t] = 0, \quad Z_t \in \mathbb{R}^\ell, \quad X_t \in \mathbb{R}^k, \quad \ell \geq k.$$

Let

$$\hat{m}_T(\beta) = \frac{1}{T} \sum_{t=1}^T Z_t(Y_t - X_t' \beta), \quad \hat{S} \rightarrow_p S,$$

and the GMM estimator minimizes $\hat{m}_T(\beta)' \hat{S}^{-1} \hat{m}_T(\beta)$. The J -statistic is

$$J = T \hat{m}_T(\hat{\beta})' \hat{S}^{-1} \hat{m}_T(\hat{\beta}).$$

We prove step by step that $J \xrightarrow{d} \chi_{\ell-k}^2$.

1. Since S is the asymptotic variance of the moments, assume it is symmetric positive definite. Then we can take a Cholesky (or any square-root) factorization

$$S^{-1} = CC', \quad \text{with } C \text{ invertible.}$$

Equivalently,

$$S = C'^{-1} C^{-1},$$

because premultiplying and postmultiplying by the inverses gives $(C'^{-1} C^{-1})^{-1} = CC' = S^{-1}$.

2. Start from

$$J = T \hat{m}_T(\hat{\beta})' \hat{S}^{-1} \hat{m}_T(\hat{\beta}).$$

Insert $I = C'^{-1} C'$ on the left and $I = CC^{-1}$ on the right, and use $\hat{S}^{-1} \approx CC'$:

$$J = T (C' \hat{m}_T(\hat{\beta}))' (C' \hat{S} C)^{-1} (C' \hat{m}_T(\hat{\beta})).$$

This is the desired expression.

3. Write the sample moment as

$$\hat{m}_T(\beta) = \frac{1}{T} Z'(Y - X\beta), \quad \text{where } Z := (Z_1, \dots, Z_T)', \quad X := (X_1, \dots, X_T)', \quad Y := (Y_1, \dots, Y_T)'.$$

The linear GMM estimator $\hat{\beta}$ with weight \hat{S}^{-1} solves the sample FOC

$$\left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \hat{m}_T(\hat{\beta}) = 0.$$

Now write

$$\hat{m}_T(\hat{\beta}) = \hat{m}_T(\beta) - \left(\frac{1}{T} Z' X \right) (\hat{\beta} - \beta),$$

because $Y - X\hat{\beta} = (Y - X\beta) - X(\hat{\beta} - \beta)$. Plug this into the FOC:

$$\left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \left[\hat{m}_T(\beta) - \left(\frac{1}{T} Z' X \right) (\hat{\beta} - \beta) \right] = 0,$$

so

$$\left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \hat{m}_T(\beta) = \left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \left(\frac{1}{T} Z' X \right) (\hat{\beta} - \beta).$$

Hence

$$\hat{\beta} - \beta = \left[\left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \left(\frac{1}{T} Z' X \right) \right]^{-1} \left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \hat{m}_T(\beta).$$

Substitute this back into

$$\hat{m}_T(\hat{\beta}) = \hat{m}_T(\beta) - \left(\frac{1}{T} Z' X \right) (\hat{\beta} - \beta)$$

to get

$$\hat{m}_T(\hat{\beta}) = \hat{m}_T(\beta) - \left(\frac{1}{T} Z' X \right) \left[\left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \left(\frac{1}{T} Z' X \right) \right]^{-1} \left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \hat{m}_T(\beta).$$

Now premultiply by C' :

$$C' \hat{m}_T(\hat{\beta}) = \left\{ I_\ell - C' \left(\frac{1}{T} Z' X \right) \left[\left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \left(\frac{1}{T} Z' X \right) \right]^{-1} \left(\frac{1}{T} X' Z \right) \hat{S}^{-1} C'^{-1} \right\} C' \hat{m}_T(\beta).$$

Define

$$A_T = I_\ell - C' \left(\frac{1}{T} Z' X \right) \left[\left(\frac{1}{T} X' Z \right) \hat{S}^{-1} \left(\frac{1}{T} Z' X \right) \right]^{-1} \left(\frac{1}{T} X' Z \right) \hat{S}^{-1} C'^{-1},$$

so that

$$C' \hat{m}_T(\hat{\beta}) = A_T C' \hat{m}_T(\beta).$$

4. By LLN,

$$\frac{1}{T} Z' X \xrightarrow{p} [Z_t X_t'], \quad \frac{1}{T} X' Z \xrightarrow{p} [X_t Z_t'] = [Z_t X_t']', \quad \hat{S} \xrightarrow{p} S.$$

Hence

$$A_T \xrightarrow{p} I_\ell - C' [Z_t X_t'] ([X_t Z_t'] S^{-1} [Z_t X_t'])^{-1} [X_t Z_t'] S^{-1} C'^{-1}.$$

Define

$$R := C' [Z_t X_t'] \in \mathbb{R}^{\ell \times k}.$$

Note that

$$[X_t Z_t'] S^{-1} [Z_t X_t'] = ([Z_t X_t'])' S^{-1} ([Z_t X_t']) = R' R,$$

because $S^{-1} = C C'$ and $R = C' [Z_t X_t']$. Also,

$$[X_t Z_t'] S^{-1} C'^{-1} = ([Z_t X_t'])' C C' C'^{-1} = ([Z_t X_t'])' C = R'.$$

Therefore the probability limit is

$$A_T \xrightarrow{p} I_\ell - R(R' R)^{-1} R'.$$

This is the orthogonal projector onto the orthogonal complement of the column space of R .

5. By the multivariate CLT for i.i.d. (or suitable mixing) data,

$$T^{1/2} \hat{m}_T(\beta) = T^{1/2} \left(\frac{1}{T} \sum_{t=1}^T Z_t e_t \right) \xrightarrow{d} N(0, S).$$

Premultiplying by C' and using $S = C'^{-1} C^{-1}$,

$$T^{1/2} C' \hat{m}_T(\beta) \xrightarrow{d} N(0, C' S C) = N(0, C' C'^{-1} C^{-1} C) = N(0, I_\ell).$$

So define

$$u := \lim_d T^{1/2} C' \hat{m}_T(\beta) \sim N(0, I_\ell).$$

6. Recall from (b) and (c):

$$J = T(C'\hat{m}_T(\hat{\beta}))'(C'\hat{S}C)^{-1}(C'\hat{m}_T(\hat{\beta})).$$

But $C'\hat{S}C \xrightarrow{p} C'SC = I_\ell$, so

$$J = (T^{1/2}C'\hat{m}_T(\hat{\beta}))'(C'\hat{S}C)^{-1}(T^{1/2}C'\hat{m}_T(\hat{\beta})) \xrightarrow{d} (\lim T^{1/2}C'\hat{m}_T(\hat{\beta}))'(\lim C'\hat{S}C)^{-1}(\lim T^{1/2}C'\hat{m}_T(\hat{\beta})).$$

From (c) and (d),

$$T^{1/2}C'\hat{m}_T(\hat{\beta}) = A_T T^{1/2}C'\hat{m}_T(\beta) \xrightarrow{d} (I_\ell - R(R'R)^{-1}R')u.$$

Also, $(C'\hat{S}C)^{-1} \rightarrow I_\ell$. Hence

$$J \xrightarrow{d} u'(I_\ell - R(R'R)^{-1}R')u.$$

7. Finally, $P := I_\ell - R(R'R)^{-1}R'$ is the symmetric idempotent matrix that projects onto the orthogonal complement of $\text{span}(R)$. Its rank is

$$\text{rank}(P) = \ell - \text{rank}(R).$$

Since $R = C'[Z_t X_t']$ is $\ell \times k$ and we assume the usual rank condition for GMM/IV (i.e. $[Z_t X_t']$ has full column rank k), we have $\text{rank}(R) = k$. Thus

$$\text{rank}(P) = \ell - k.$$

If $u \sim N(0, I_\ell)$ and P is a symmetric idempotent matrix of rank r , then

$$u'Pu \sim \chi_r^2.$$

Here $r = \ell - k$, so

$$u'(I_\ell - R(R'R)^{-1}R')u \sim \chi_{\ell-k}^2.$$

Combining with (f) gives

$$J \xrightarrow{d} \chi_{\ell-k}^2.$$

Problem 5. We observe i.i.d. data $\{(Y_i, X_i)\}_{i=1}^n$ and want to estimate

$$\mu = [Y_i]$$

given the extra (true) moment

$$[X_i] = 0.$$

Consider the moment conditions

$$g_i(\mu) = \begin{bmatrix} Y_i - \mu \\ X_i \end{bmatrix}, \quad [g_i(\mu)] = 0.$$

Then the sample moments are

$$\hat{g}_n(\mu) = \begin{bmatrix} \bar{Y} - \mu \\ \bar{X} \end{bmatrix}, \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i, \quad \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i.$$

Efficient GMM. Let

$$S = \left(\begin{bmatrix} Y_i - \mu \\ X_i \end{bmatrix} \right) = \begin{bmatrix} \sigma_Y^2 & \sigma_{YX} \\ \sigma_{YX} & \sigma_X^2 \end{bmatrix},$$

where $\sigma_Y^2 = (Y_i)$, $\sigma_X^2 = (X_i)$, and $\sigma_{YX} = (Y_i, X_i)$. The efficient GMM weighting matrix is $W = S^{-1}$. The GMM estimator solves

$$\hat{\mu} = \arg \min_{\mu} \hat{g}_n(\mu)' W \hat{g}_n(\mu).$$

Write $W = (w_{jk})$. Then the FOC is

$$\frac{\partial}{\partial \mu} [(\bar{Y} - \mu, \bar{X}) W (\bar{Y} - \mu, \bar{X})'] = -2[w_{11}(\bar{Y} - \mu) + w_{12}\bar{X}] = 0,$$

so

$$w_{11}(\bar{Y} - \hat{\mu}) + w_{12}\bar{X} = 0 \implies \hat{\mu} = \bar{Y} + \frac{w_{12}}{w_{11}}\bar{X}.$$

Now compute $W = S^{-1}$. Since

$$S^{-1} = \frac{1}{\sigma_Y^2 \sigma_X^2 - \sigma_{YX}^2} \begin{bmatrix} \sigma_X^2 & -\sigma_{YX} \\ -\sigma_{YX} & \sigma_Y^2 \end{bmatrix},$$

we have

$$w_{11} = \frac{\sigma_X^2}{\sigma_Y^2 \sigma_X^2 - \sigma_{YX}^2}, \quad w_{12} = \frac{-\sigma_{YX}}{\sigma_Y^2 \sigma_X^2 - \sigma_{YX}^2}.$$

Hence

$$\frac{w_{12}}{w_{11}} = -\frac{\sigma_{YX}}{\sigma_X^2}.$$

So the efficient GMM estimator is

$$\boxed{\hat{\mu} = \bar{Y} - \frac{(Y_i, X_i)}{(X_i)} \bar{X}},$$

with the covariance and variance replaced by their sample analogs:

$$(Y_i, X_i) \approx \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X}), \quad (X_i) \approx \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2.$$

Why does X_i help? If X_i is correlated with Y_i , then Y_i contains a component that is “predictable” from X_i . Because we *know* the true mean of X_i is 0, we can remove from \bar{Y} the part that is due to a nonzero sample mean \bar{X} . The correction

$$-\frac{(Y, X)}{(X)} \bar{X}$$

is exactly the control-variate adjustment: it has mean zero but lowers the variance of the estimator. If X_i is uncorrelated with Y_i ($(Y, X) = 0$), the adjustment is zero and we are back to $\hat{\mu} = \bar{Y}$.

Problem 6. We observe $\{(Y_i, X_i, Z_i)\}_{i=1}^n$ and the econometrician posits the linear IV/GMM model

$$Y_i = X_i' \beta + e_i, \quad [Z_i e_i] = 0,$$

with $\ell = \dim Z_i > k = \dim X_i > 1$. We study what happens when this moment is (locally) misspecified.

1. **GMM estimator with a given weight matrix $W > 0$.**

Define the sample moment

$$\hat{m}_n(\beta) := \frac{1}{n} \sum_{i=1}^n Z_i(Y_i - X_i' \beta) = \frac{1}{n} Z'(Y - X\beta),$$

where $Z = (Z_1', \dots, Z_n')'$, $X = (X_1', \dots, X_n')'$, $Y = (Y_1, \dots, Y_n)'$. Given a fixed positive definite weight matrix W (of size $\ell \times \ell$), the GMM estimator is

$$\hat{\beta} = \arg \min_{\beta} \hat{m}_n(\beta)' W \hat{m}_n(\beta).$$

For the linear case this has the closed form

$$\boxed{\hat{\beta} = ((n^{-1} X' Z) W (n^{-1} Z' X))^{-1} (n^{-1} X' Z) W (n^{-1} Z' Y).}$$

2. **Misspecification implies $[Ze] \neq 0$.**

We are told the true error is

$$e_i = \frac{\delta}{\sqrt{n}} + u_i, \quad [u_i | Z_i] = 0,$$

and $\mu_Z := [Z_i] \neq 0$, with $\delta \neq 0$. Then

$$[Z_i e_i] = [Z_i (\delta/\sqrt{n} + u_i)] = \frac{\delta}{\sqrt{n}} [Z_i] + [Z_i u_i].$$

Because $[u_i | Z_i] = 0 \Rightarrow [Z_i u_i] = 0$, we get

$$[Z_i e_i] = \frac{\delta}{\sqrt{n}} \mu_Z \neq 0$$

when $\delta \neq 0$ and $\mu_Z \neq 0$. So the moment $[Ze] = 0$ is violated at order $1/\sqrt{n}$.

3. **Linear expansion of $\sqrt{n}(\hat{\beta} - \beta)$.**

Start from the sample FOC for linear GMM:

$$(n^{-1} X' Z) W \hat{m}_n(\hat{\beta}) = 0.$$

Write the sample moment at $\hat{\beta}$ as

$$\hat{m}_n(\hat{\beta}) = \hat{m}_n(\beta) - (n^{-1} Z' X)(\hat{\beta} - \beta),$$

because $Y - X\hat{\beta} = (Y - X\beta) - X(\hat{\beta} - \beta)$. Plugging in,

$$(n^{-1}X'Z)W[\hat{m}_n(\beta) - (n^{-1}Z'X)(\hat{\beta} - \beta)] = 0,$$

so

$$(n^{-1}X'Z)W\hat{m}_n(\beta) = (n^{-1}X'Z)W(n^{-1}Z'X)(\hat{\beta} - \beta).$$

Hence

$$\hat{\beta} - \beta = [(n^{-1}X'Z)W(n^{-1}Z'X)]^{-1}(n^{-1}X'Z)W\hat{m}_n(\beta).$$

Now substitute the misspecified error:

$$\hat{m}_n(\beta) = \frac{1}{n} \sum_{i=1}^n Z_i e_i = \frac{1}{n} \sum_{i=1}^n Z_i \left(\frac{\delta}{\sqrt{n}} + u_i \right) = \frac{\delta}{\sqrt{n}} \left(\frac{1}{n} \sum_{i=1}^n Z_i \right) + \frac{1}{n} \sum_{i=1}^n Z_i u_i.$$

Therefore

$$\sqrt{n}(\hat{\beta} - \beta) = [(n^{-1}X'Z)W(n^{-1}Z'X)]^{-1}(n^{-1}X'Z)W \left[\delta \left(\frac{1}{n} \sum_{i=1}^n Z_i \right) + \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n Z_i u_i \right) \right].$$

This is the requested expression in terms of W, n, δ and (X_i, Z_i, u_i) .

4. Asymptotic distribution and asymptotic bias.

Let

$$G := [Z_i X_i'] \quad (\ell \times k), \quad \mu_Z := [Z_i], \quad \Omega := (Z_i u_i) \quad (\ell \times \ell).$$

By LLN/CLT,

$$n^{-1}X'Z \xrightarrow{p} G', \quad n^{-1}Z'X \xrightarrow{p} G, \quad \frac{1}{n} \sum_{i=1}^n Z_i \xrightarrow{p} \mu_Z,$$

and

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n Z_i u_i \right) \xrightarrow{d} N(0, \Omega).$$

Define

$$Q := G'WG \quad (k \times k), \quad \text{nonsingular by relevance.}$$

Then, by continuous mapping,

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} Q^{-1}G'W(\delta\mu_Z + \eta), \quad \eta \sim N(0, \Omega).$$

So

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N \left(\underbrace{Q^{-1}G'W\delta\mu_Z}_{\text{asymptotic bias}}, Q^{-1}G'W\Omega W G Q^{-1} \right).$$

The mean is nonzero whenever $\delta \neq 0$ and $\mu_Z \neq 0$, so the limiting distribution is centered away from 0: this is the asymptotic bias induced by the local misspecification.

5. Does the optimal weight remove the bias?

The optimal weight is $W = \Omega^{-1}$. Plugging into the bias term,

$$\text{bias} = (G'\Omega^{-1}G)^{-1}G'\Omega^{-1}\delta\mu_Z.$$

This is generally *not* zero unless μ_Z happens to lie in the null space of $G'\Omega^{-1}$ (a non-generic case). So choosing the optimal weight minimizes asymptotic variance but does *not* remove the bias created by the drifting violation $[Ze] = (\delta/\sqrt{n})\mu_Z$.

6. Does this misspecification affect consistency?

Here the violation is *local*:

$$[Z_ie_i] = \frac{\delta}{\sqrt{n}}\mu_Z \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

So the population moment condition is satisfied in the limit. Therefore the GMM estimator still converges in probability to the true β :

$$\hat{\beta} \xrightarrow{p} \beta.$$

What changes is the *center* of the \sqrt{n} -distribution: it is no longer 0 but the nonzero vector in part (d). So: the misspecification does *not* break consistency, but it does create an asymptotic bias at the \sqrt{n} scale.