Lecture 6: Estimation of ARMA Models

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Intro

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- So far, we took parameters as given when working with ARMA models;
- In practice, we need to *estimate* these parameters from data;
- There many ways to estimate ARMA models: maximum likelihood, method of moments, Kalman filter, etc;
- We will focus on MLE estimation;
- Usually, good software for ARMA estimation gives you several options;
- More than mastering math tricks and details, it is important to understand the *big picture*;

A Preview

- It is always the MA part that will complicate things;
- A natural estimator for AR(p) models is just the OLS estimator: regress y_t on $y_{t-1}, \cdots, y_{t-p}$;
- Mild conditions will guarantee consistency, asymptotic normality, bla, blah, blah...
- But for ARMA(p, q) models, we cannot do that! We do not observe ε_t!!!

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- Mild conditions will guarantee consistency, asymptotic normality, bla, blah, blah...
- But for ARMA(p, q) models, we cannot do that! We do not observe ε_t!!!
- MLE will require a *distributional assumption* for ε_t;
- We will relax that later when we touch on "*quasi-MLE*";
- We will start with *given* values of p and q and discuss model choice later;

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- First step: characterize the joint distribution of the sample $\mathbf{y} = (y_1, \ldots, y_T)'$;
- Denote this distribution by $f_{y_T, y_{T-1}, \ldots, y_1}(\mathbf{y}; \mathbf{\Theta})$;

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- Recall: $f_{Y|X}(y, x) = f_{Y,X}(y, x)/f_X(x) \implies f_{Y,X}(y, x) = f_{Y|X}(y, x) f_X(x)$

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- For any integer $k \geq 1$:

$$\boxed{f_{y_T, y_{T-1}, \dots, y_1}(\mathbf{y}; \mathbf{\Theta}) = f_{y_k, \dots, y_1}(y_k, \dots, y_1; \mathbf{\Theta}) \cdot \prod_{t=k+1}^{T} f_{y_t | y_{t-1}, \dots, y_1}(y_t \mid y_{t-1}, \dots, y_1; \mathbf{\Theta})}$$

Questions?

The AR(p) Case

The AR(p) Case

- Consider the AR(p) model below and let $\boldsymbol{\Theta} = (c, \phi_1, \dots, \phi_p, \sigma^2)$:

$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2)$$

- Notice that $y_t | y_{t-1}, \dots, y_{t-p} \sim N(c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p}, \sigma^2)$. Therefore:

$$f_{y_t|y_{t-1},\dots,y_1}(y_t|y_{t-1}, \dots, y_1; \boldsymbol{\Theta}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_t - c - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p})^2}{2\sigma^2}}$$

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- The likelihood of the first p observations, $f_{y_p,\dots,y_1}(y_p,\dots,y_1;\boldsymbol{\Theta})$, is more involved;
- Notice that the $p \times 1$ vector $\mathbf{y}_{1:p} = (y_1, \dots, y_p)'$ is multivariate normal;

$$\mathbf{y}_{1:p} \sim N(\boldsymbol{\mu}, \boldsymbol{\Omega}), \quad \boldsymbol{\mu} = \frac{c}{1 - \sum_{i=1}^{p} \phi_i} \mathbf{1}, \quad \boldsymbol{\Omega}_{ij} = \gamma(|i-j|) \quad \forall i, j \in \{1, \dots, p\}$$

The AR(p) Case

- The likelihood of the first p observations is given by:

$$f_{y_p,\dots,y_1}(y_p,\dots,y_1;\mathbf{\Theta}) = (2\pi)^{-p/2}|\mathbf{\Omega}^{-1}|^{1/2}e^{-\frac{1}{2}(\mathbf{y}_{1:p}-\boldsymbol{\mu})'\mathbf{\Omega}^{-1}(\mathbf{y}_{1:p}-\boldsymbol{\mu})}$$

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- From here, we can write the full likelihood function:

$$\begin{aligned}
f_{y_T,\ldots,y_1}(\mathbf{y};\boldsymbol{\Theta}) &= f_{y_k,\ldots,y_1}(y_k,\ldots,y_1;\boldsymbol{\Theta}) \cdot \prod_{t=k+1}^{T} f_{y_t|y_{t-1},\ldots,y_1}(y_t \mid y_{t-1},\ldots,y_1;\boldsymbol{\Theta}) \\
&= (2\pi)^{-p/2}|\boldsymbol{\Omega}^{-1}|^{1/2}e^{-\frac{1}{2}(\mathbf{y}_{1:p}-\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\mathbf{y}_{1:p}-\boldsymbol{\mu})} \cdot \prod_{t=p+1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_t-c-\sum_{i=1}^{p}\phi_i y_{t-i})^2}{2\sigma^2}} \\
&= (2\pi)^{-T/2}\sigma^{-(T-p)}|\boldsymbol{\Omega}^{-1}|^{1/2}e^{-\frac{1}{2}(\mathbf{y}_{1:p}-\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\mathbf{y}_{1:p}-\boldsymbol{\mu})} \cdot e^{-\frac{1}{2\sigma^2}\sum\limits_{t=p+1}^{T}(y_t-c-\sum\limits_{i=1}^{p}\phi_i y_{t-i})^2}
\end{aligned}$$

The Log-Likelihood Function

- We always optimize the log-likelihood function $\mathcal{L}(\boldsymbol{\Theta}|\mathbf{y}) = \log\left(f_{y_T,\dots,y_1}(\mathbf{y};\boldsymbol{\Theta})\right)$

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\Theta}|\mathbf{y}) &= \log\left(f_{y_T,\dots,y_1}(\mathbf{y};\boldsymbol{\Theta})\right) \\
&= -\frac{T}{2}\log(2\pi) \\
&\qquad -(T-p)\log(\sigma) + \frac{1}{2}\log\left(|\boldsymbol{\Omega}^{-1}|\right) \\
&\qquad -\frac{1}{2}(\mathbf{y}_{1:p}-\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\mathbf{y}_{1:p}-\boldsymbol{\mu}) - \frac{1}{2\sigma^2}\sum_{t=p+1}^{T}(y_t - c - \sum_{i=1}^{p}\phi_i y_{t-i})^2
\end{aligned}
$$

- The blue part looks like the OLS objective function;
- The red part is “distorting” this objective function;

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- Full ML estimation requires optimizing this function w.r.t. $\boldsymbol{\Theta}$;
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Data $\{y_t\}_{t=1}^{T}$
Parameters $\boldsymbol{\Theta}$ → Compute $\mathcal{L}(\boldsymbol{\Theta} \mid y_{1:T})$ → Numerical optimization

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- In that case the main contribution to the log-likelihood function comes from the blue part;

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- Wait a minute... what if $T >> p$?
- In that case the main contribution to the log-likelihood function comes from the blue part;
- This suggests a simpler approach: *conditional* MLE;
- Assume that the first p observations are fixed (non-random);
- Approximate $\mathcal{L}(\boldsymbol{\Theta}|\mathbf{y}_{1:T})$ by $\log\left(f_{y_{p+1},\ldots,y_T|y_{1:p}}(\mathbf{y};\boldsymbol{\Theta})\right)$

The Numerical Shortcut for the AR(p) Case

- Recall that, up to a constant, we have:

$$\log\left(f_{y_{p+1},\dots,y_T|y_{1:p}}(\mathbf{y};\boldsymbol{\Theta})\right) = -\sum_{t=p+1}^{T} \frac{\left(y_t - c - \sum_{i=1}^{p}\phi_i y_{t-i}\right)^2}{2\sigma^2} - (T-p)\log(\sigma)$$

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- Estimators for c and ϕ_i's are the same as the OLS from regressing y_t on $y_{t-1},\ldots,y_{t-p}$;
- Super simple closed-form solutions! 😎
- The estimator for σ^2 is just the (biased) sample variance of the OLS residuals;

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- Estimators for c and ϕ_i's are the same as the OLS from regressing y_t on $y_{t-1},\ldots,y_{t-p}$;
- Super simple closed-form solutions! 😎
- The estimator for σ^2 is just the (biased) sample variance of the OLS residuals;
- If T is large, this is a very good approximation to the full MLE;
- $\mathcal{L}(\boldsymbol{\Theta}|\mathbf{y})$ is efficiently computed using the Kalman filter – darker magic for the next year!

Questions?

The MA(q) Case

- Consider the MA(q) model below and let $\boldsymbol{\Theta} = (\mu, \theta_1, \dots, \theta_q, \sigma^2)$:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2)$$

- There is no hope to get an "OLS"-type trick... we do not see the shocks...
- There are again two main approaches: full MLE and conditional MLE;
- We will focus on the conditional MLE approach;
- You can see the full MLE approach in Hamilton's book (Chapter 5);
- If T is large, the two approaches will give very similar results;
- Similar to the forecasting exercise in the last lecture!

The MA(q) Case

- The key observation is that $y_t|\varepsilon_{t-1},\dots,\varepsilon_{t-q} \sim N(\mu + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}, \sigma^2)$;
- But how is that useful if we do not observe ε_t?

The MA(q) Case

- The key observation is that $y_t|\varepsilon_{t-1}, \ldots, \varepsilon_{t-q} \sim N(\mu + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}, \sigma^2)$;
- But how is that useful if we do not observe ε_t?
- Let's assume that $\varepsilon_{-q+1} = \varepsilon_{-q+2} = \ldots = \varepsilon_0 = \mathbb{E}[\varepsilon_t] = 0$;
- We can start a recursion, like in the forecasting case:

$$
\begin{aligned}
\varepsilon_1 &= y_1 - \mu \\
\varepsilon_2 &= y_2 - \mu - \theta_1\varepsilon_1 \\
\varepsilon_3 &= y_3 - \mu - \theta_1\varepsilon_2 - \theta_2\varepsilon_1 \\
&\vdots \\
\varepsilon_t &= y_t - \mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} \\
&\vdots \\
\varepsilon_T &= y_T - \mu - \theta_1\varepsilon_{T-1} - \ldots - \theta_q\varepsilon_{T-q}
\end{aligned}
$$

The Conditional Log-Likelihood Function

- From here, we can write the conditional log-likelihood function:

$$\log\left(f_{y_t,\ldots,y_1|\varepsilon_{-q+1}=\varepsilon_{-q+2}=\ldots=\varepsilon_0=0}(\mathbf{y};\boldsymbol{\Theta})\right)=\sum_{t=q+1}^{T}\log\left(f_{y_t|\varepsilon_{t-1},\ldots,\varepsilon_{t-q}}(y_t|\varepsilon_{t-1},\ldots,\varepsilon_{t-q};\boldsymbol{\Theta})\right)$$
$$=-\sum_{t=1}^{T}\frac{(\varepsilon_t)^2}{2\sigma^2}-(T-q)\log(\sigma)$$

- When there is an MA component, the logical flow is:

Data $\{y_t\}_{t=1}^T$ Parameters $\boldsymbol{\Theta}$ → Back out $\{\varepsilon_t\}_{t=1}^T$ → Compute conditional log-likelihood → Numerical optimization

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- Consider a Guassian ARMA(p, q) model and let $\mathbf{\Theta} = (c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q, \sigma^2)$:

$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2)$$

- We can combine the two previous approaches;
- Given $\mathbf{\Theta}$, we will back out ε_t recursively;
- We also note that $y_t | y_{t-1}, \dots, y_1, \varepsilon_{t-1}, \dots, \varepsilon_{t-q} \sim N(c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p}, \sigma^2)$
- Then we are ready to use the conditioning trick once again!

The Recursion

- As we did with the AR(p), assume $y_1, \dots, y_p$ are fixed;
- Assume that $\varepsilon_p = \varepsilon_{p-1} = \dots = \varepsilon_{p-q+1} = 0$
- The first shock to be backed out is $\varepsilon_{p+1} = y_{p+1} - c - \sum_{i=1}^{p} \phi_i y_{p+1-i}$
- Then we get $\varepsilon_{p+2} = y_{p+2} - c - \sum_{i=1}^{p} \phi_i y_{p+2-i} - \theta_1 \varepsilon_{p+1}$
- And so on...
- You might be skeptical of "assuming values" for the shock... but usually p and q are small compared to T!
- You will almost never see $q > 10$ and $p > 20$ in practice!

The Conditional Log-Likelihood Function

- The conditional log-likelihood function, up to a constant, is given by:

$$\mathcal{L}(\boldsymbol{\Theta}|\mathbf{y}) = \log\left(f_{y_t,\ldots,y_1|\varepsilon_{-q+1}=\varepsilon_{-q+2}=\ldots=\varepsilon_0=0}(\mathbf{y};\boldsymbol{\Theta})\right) = -\sum_{t=p+1}^{T}\frac{(\varepsilon_t)^2}{2\sigma^2} - (T-p)\log(\sigma)$$

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Regarding numerical optimization:

- Do we have guarantees the numerical method will converge to the global maximum? No.
- Is it much harder as we increase p and q? Yes and no: increasing p is fine, but q is hell;
- Where to start the optimization? OLS estimates for ϕ are a good shot;
- What about θ? Start with zeros or small values;
- Try several different starting points and make sure you get similar answers;

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- Ok great, we can estimate ARMA(p, q) models;
- How to do inference?
- We will use standard MLE results;
- Important assumptions: a correctly specified model and $\mathbf{\Theta_0}$ must be an interior point;
- Recall that, if the model is correctly specified, then:

$$\sqrt{T}\left(\hat{\mathbf{\Theta}} - \mathbf{\Theta}_0\right) \xrightarrow{d} N(0, \mathcal{I}^{-1}(\mathbf{\Theta}_0))$$

  where $\mathcal{I}(\mathbf{\Theta})$ is the Fisher information matrix;
- Recall that, in this case, $\mathcal{I}(\mathbf{\Theta}) = -\mathbb{E}\left[\frac{\partial^2 l_t(\mathbf{\Theta})}{\partial \mathbf{\Theta} \partial \mathbf{\Theta}'}\right] = \mathbb{E}\left[\frac{\partial l_t(\mathbf{\Theta}|\mathbf{y})}{\partial \mathbf{\Theta}} \frac{\partial l_t(\mathbf{\Theta}|\mathbf{y})}{\partial \mathbf{\Theta}'}\right]$, where $l_t(\mathbf{\Theta}|\mathbf{y}) = \log\left(f_{y_t|\mathbf{y_{t-1}}}(y_t|\mathbf{y}_{t-1}; \mathbf{\Theta})\right)$;

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- Theory suggests two equally valid ways of estimating it. Let us define two objects:

1. The Hessian:

$$\mathcal{H}(\hat{\boldsymbol{\Theta}}) \equiv \frac{1}{T-p} \cdot \frac{\partial^2 \mathcal{L}(\hat{\boldsymbol{\Theta}}|\mathbf{y})}{\partial \boldsymbol{\Theta} \partial \boldsymbol{\Theta}'} = \frac{1}{T-p} \cdot \sum_{t=p+1}^{T} \frac{\partial^2 \log \left(f_{y_t|\mathbf{y_{t-1}}}(y_t|\mathbf{y}_{t-1}; \hat{\boldsymbol{\Theta}}) \right)}{\partial \boldsymbol{\Theta} \partial \boldsymbol{\Theta}'}$$

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2. The score function and its associated *outer product*:

$$\mathcal{S}(\hat{\boldsymbol{\Theta}})_t \equiv \frac{\partial \log\left(f_{y_t|\mathbf{y_{t-1}}}(y_t|\mathbf{y}_{t-1}; \hat{\boldsymbol{\Theta}})\right)}{\partial \boldsymbol{\Theta}}; \qquad \mathcal{O}(\hat{\boldsymbol{\Theta}}) \equiv \frac{1}{T-p} \cdot \sum_{t=p+1}^{T} \mathcal{S}(\hat{\boldsymbol{\Theta}})_t \mathcal{S}(\hat{\boldsymbol{\Theta}})_t'$$

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- Then, we can estimate $\mathcal{I}(\mathbf{\Theta}_0)$ by either $\left[-\mathcal{H}(\hat{\mathbf{\Theta}})\right]$ or $\left[\mathcal{O}(\hat{\mathbf{\Theta}})\right]$;

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- Then, we can estimate $\mathcal{I}(\mathbf{\Theta}_0)$ by either $\left[-\mathcal{H}(\hat{\mathbf{\Theta}})\right]$ or $\left[\mathcal{O}(\hat{\mathbf{\Theta}})\right]$;
- (Adjust the starting point of the sum as needed, it doesn't matter asymptotically);

Quasi-MLE

- What if ε_t is not Gaussian?
- The MLE will converge to a different $\mathbf{\Theta}_0$;
- This parameter is the *pseudo-true* value that minimizes the Kullback-Leibler divergence between the true model and the assumed Gaussian model;
- The idea, and the term *Quasi-MLE*, is due to White (Econometrica, 1982);

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- The idea, and the term *Quasi-MLE*, is due to White (Econometrica, 1982);
- The asymptotic distribution is now:

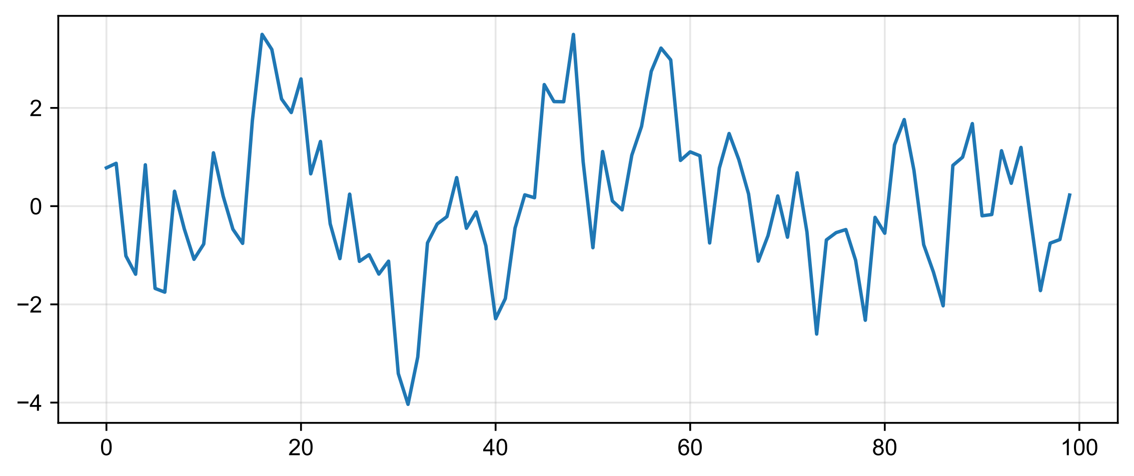
$$\sqrt{T}\left(\hat{\mathbf{\Theta}} - \mathbf{\Theta}_0\right) \xrightarrow{d} N\left(0, \underbrace{\mathcal{H}^{-1}(\mathbf{\Theta}_0)\mathcal{J}(\mathbf{\Theta}_0)\mathcal{H}^{-1}(\mathbf{\Theta}_0)'}_{\text{the "sandwich" variance}}\right)$$

- The "bread" uses the Hessian and the "meat" uses the outer product of the score;
- The estimator for the sandwich is $\left[-\mathcal{H}^{-1}(\hat{\mathbf{\Theta}})\mathcal{O}(\hat{\mathbf{\Theta}})\mathcal{H}^{-1}(\hat{\mathbf{\Theta}})'\right]$

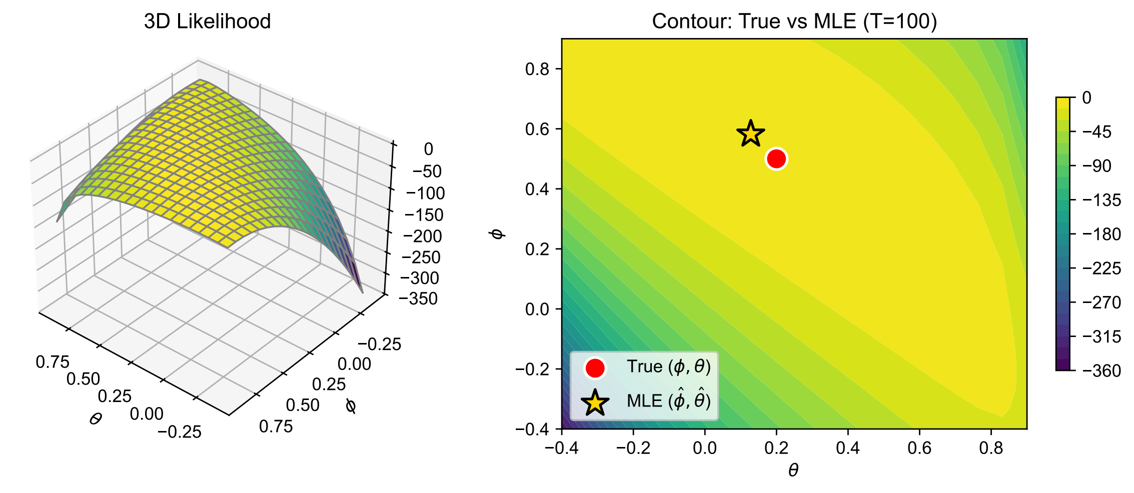
Some Simulations

The ARMA(1,1) Case - Simulated Path ($\phi = 0.5$, $\theta = 0.2$)

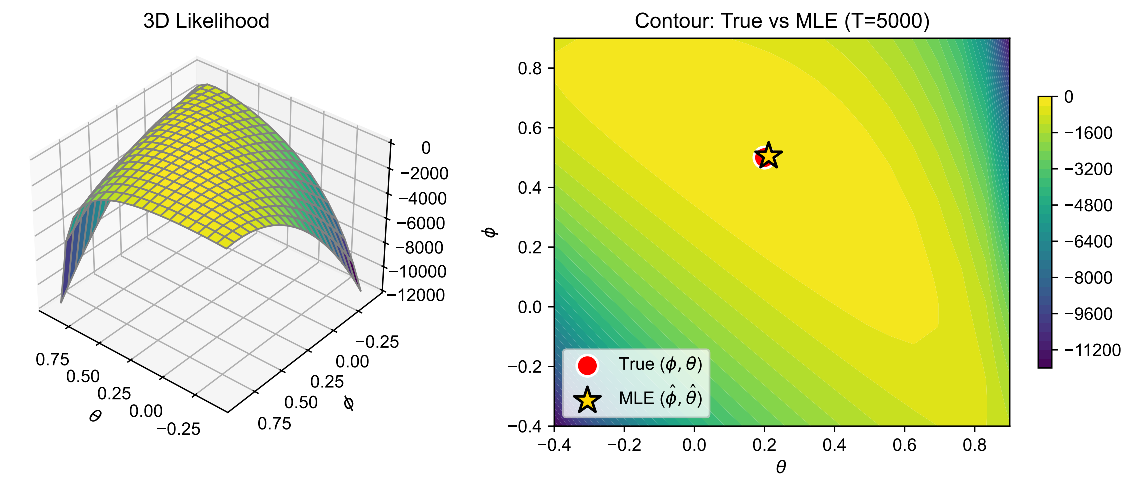
Let's say we have an ARMA(1,1), with $\mu = 0$ and $\sigma^2 = 1$. Let's simulate one path:



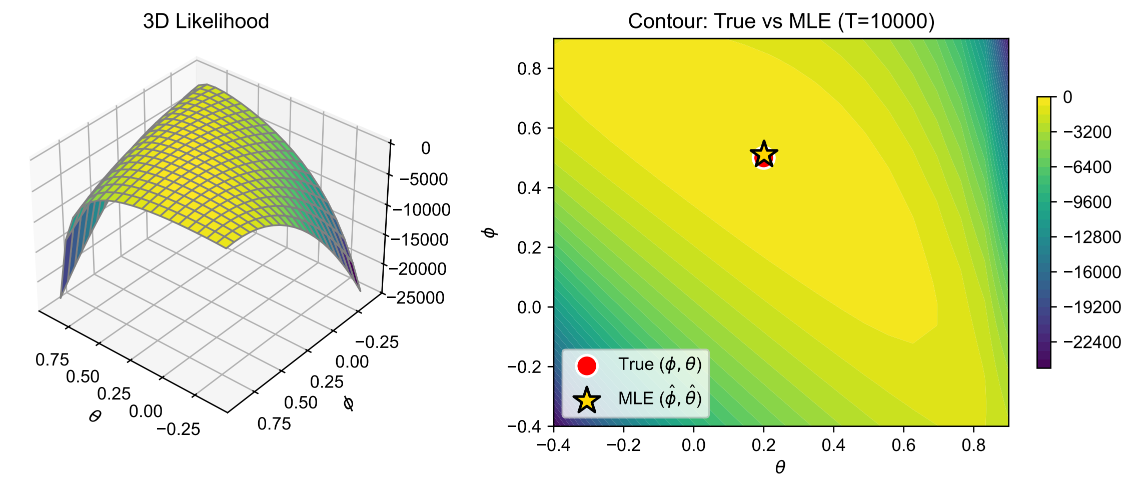
The Likelihood Surface (T=100)



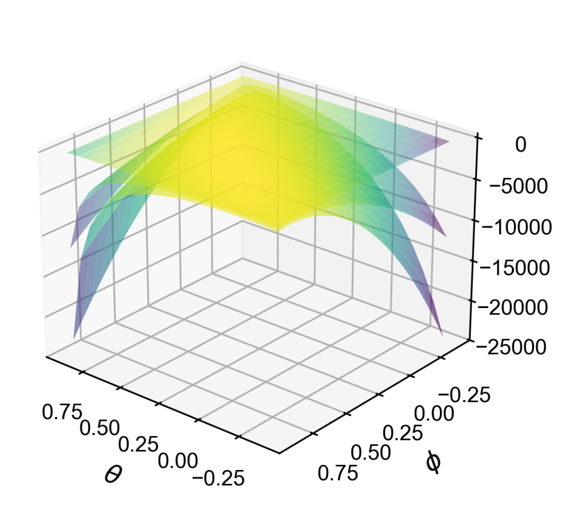
The Likelihood Surface (T=5000)



The Likelihood Surface (T=10000)



The Likelihood Gets More Concentrated!



- $\uparrow T \implies$ tighter likelihood. Why?
- How is the Hessian at the optimum related to this?
- What is the connection with the asymptotic distribution of the MLE?

Questions?

How to choose p and q?

The Model Selection Problem

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- In practice, we need to choose them from the data;
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- We need a formal criterion that **penalizes model complexity**;
- This leads to *information criteria*: balance fit vs. parsimony;

Information Criteria: General Framework

- The general form of information criteria is:

$$\boxed{\mathsf{IC} = -2 \cdot \mathcal{L}(\hat{\boldsymbol{\Theta}}|\mathbf{y}) + \mathsf{penalty}(k, T)}$$

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- We choose the model that **minimizes** the IC;
- Different penalties lead to different criteria;
- The key trade-off: smaller penalty $\implies$ more likely to select larger models;

The Main Information Criteria

Let $k = p + q + 2$ be the number of parameters in an ARMA(p, q) model (including c and σ^2).

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Bayesian Information Criterion (BIC) or **Schwarz Criterion (SIC)**:

$$\boxed{\text{BIC} = -2 \cdot \mathcal{L}(\hat{\boldsymbol{\Theta}}|\mathbf{y}) + k \log(T)}$$

- It approximates the model with the highest posterior probability (assuming equal priors);
- It is **consistent**: selects the true model (if it is in the candidate set) with probability $\to 1$ as $T \to \infty$;

Comparing the Penalties

- Notice that for $T > 8$, we have $\log(T) > 2$, so BIC penalizes more heavily than AIC;

Sample Size	AIC penalty	BIC penalty
$T = 50$	$2k$	$3.91k$
$T = 100$	$2k$	$4.61k$
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- As $T \to \infty$: BIC penalty grows much faster than AIC;
- Implication: BIC tends to select **more parsimonious models** than AIC;

How to Use Information Criteria in Practice

Step-by-step procedure:

1. Choose a maximum order $p_{\max}$ and $q_{\max}$ (often based on theory or exploratory analysis);
2. Estimate all ARMA(p, q) models for $p \in \{0, 1, \dots, p_{\max}\}$ and $q \in \{0, 1, \dots, q_{\max}\}$;
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Important notes:

- All models must be estimated on the **same sample** (same T);
- Start with reasonable $p_{\max}$ and $q_{\max}$ (e.g., 5-10 for quarterly data, 12-24 for monthly);
- If the selected model is at the boundary, consider increasing the maximum orders;

The End

References

- Chapter 5 from Hamilton's book for ARMA estimation;
- Chapter 28 from Hansen's book on model selection for MLE;