

Problem Set 2 - Proposed Solutions

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Problem 1. (a) Weak (covariance) stationarity of $y_t = \cos(t+\theta)$, where $\theta \sim \text{Unif}\{0, \pi/2, \pi, 3\pi/2\}$ and is time-invariant.

Mean. Using $\cos(t+\theta) \in \{\cos t, -\sin t, -\cos t, \sin t\}$ (from the formula $\cos(t+\theta) = \cos t \cos \theta - \sin t \sin \theta$) with equal probabilities,

$$\mathbb{E}[y_t] = \frac{1}{4}(\cos t - \sin t - \cos t + \sin t) = 0,$$

which does not depend on t .

Autocovariance. Since $\mathbb{E}[y_t] = 0$,

$$\gamma(\tau) = \text{Cov}(y_t, y_{t+\tau}) = \mathbb{E}[\cos(t+\theta)\cos(t+\tau+\theta)].$$

Using $\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$,

$$\gamma(\tau) = \frac{1}{2}\{\cos \tau + \mathbb{E}[\cos(2t+\tau+2\theta)]\}.$$

Here $2\theta \in \{0, \pi, 2\pi, 3\pi\}$ with equal probabilities, so

$$\mathbb{E}[\cos(2t+\tau+2\theta)] = \frac{1}{4}\{\cos \phi + \cos(\phi+\pi) + \cos(\phi+2\pi) + \cos(\phi+3\pi)\} = 0$$

for $\phi = 2t + \tau$. Hence

$$\gamma(\tau) = \frac{1}{2} \cos \tau,$$

which depends only on the lag τ (and not on t). Therefore $\{y_t\}$ is covariance (weakly) stationary with variance $\gamma(0) = \frac{1}{2}$.

(b) Not strictly stationary. Strict stationarity requires all finite-dimensional distributions to be invariant to time shifts. Consider the one-dimensional marginals at $t = 0$ and $t = \pi/4$:

$$y_0 \in \{1, 0, -1, 0\} \quad (\text{each with prob. } 1/4) \Rightarrow \mathbb{P}(y_0 = 0) = \frac{1}{2}, \quad \mathbb{P}(y_0 = \pm 1) = \frac{1}{4},$$

whereas

$$y_{\pi/4} \in \left\{ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\} \Rightarrow \mathbb{P}\left(y_{\pi/4} = \pm \frac{\sqrt{2}}{2}\right) = \frac{1}{2}.$$

These marginal distributions are different, hence the process is *not* strictly stationary.

(c) A different process:

$$z_t = \begin{cases} 1, & \forall t \quad \text{with probability } \frac{1}{2}, \\ 0, & \forall t \quad \text{with probability } \frac{1}{2}. \end{cases}$$

Let $Z \in \{0, 1\}$ denote the path value. Then $z_t \equiv Z$ for all t .

First two moments.

$$\mathbb{E}[z_t] = \mathbb{E}[Z] = \frac{1}{2}, \quad \mathbb{E}[z_t^2] = \mathbb{E}[Z^2] = \frac{1}{2},$$

$$\text{Var}(z_t) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Autocovariance. For any lag h ,

$$\gamma_z(h) = \text{Cov}(z_t, z_{t+h}) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

which depends only on h (in fact, is constant). Hence $\{z_t\}$ is weakly stationary.

Not mean-ergodic. For any realized path, either $z_t \equiv 1$ for all t or $z_t \equiv 0$ for all t . Thus the time average

$$\bar{z}_T = \frac{1}{T} \sum_{t=1}^T z_t = \begin{cases} 1, & \text{if } Z = 1, \\ 0, & \text{if } Z = 0, \end{cases}$$

for every T . Therefore \bar{z}_T does *not* converge (a.s. or in probability) to the ensemble mean $\mathbb{E}[z_t] = \frac{1}{2}$. The process is weakly stationary but not mean-ergodic (perfect dependence across time prevents a law of large numbers along the time axis).

Problem 2.

$$\text{AR}(2): \quad (1 - 1.1L + 0.18L^2)y_t = \varepsilon_t, \quad \varepsilon_t \sim \text{WN}(0, \sigma_\varepsilon^2).$$

Stationarity. The AR polynomial is $\Phi(z) = 1 - 1.1z + 0.18z^2$. Its roots are

$$0.18z^2 - 1.1z + 1 = 0 \implies z_1 = 5, \quad z_2 = \frac{10}{9},$$

both outside the unit circle, hence the process is stationary.

Mean. Since there is no constant term, the unique stationary mean is

$$\mu = \mathbb{E}[y_t] = 0.$$

Autocorrelations and autocovariances. Write the model as $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$ with $\phi_1 = 1.1$ and $\phi_2 = -0.18$. The Yule–Walker equations give

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} = \frac{55}{59}, \quad \rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad (k \geq 2).$$

Equivalently, since the characteristic equation for ρ_k is $\lambda^2 - \phi_1\lambda - \phi_2 = 0$ with roots $\lambda_1 = 0.9$ and $\lambda_2 = 0.2$,

$$\rho_k = A(0.9)^k + B(0.2)^k, \quad A = \frac{\rho_1 - \lambda_2}{\lambda_1 - \lambda_2} = \frac{432}{413}, \quad B = 1 - A = -\frac{19}{413}.$$

Hence

$$\rho_k = \frac{432}{413} (0.9)^k - \frac{19}{413} (0.2)^k, \quad k \geq 0, \quad (\rho_0 = 1).$$

Variance. With $\rho_2 = \phi_1 \rho_1 + \phi_2 = \frac{1247}{1475}$, the Yule–Walker equation $\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_\varepsilon^2$ yields

$$\gamma_0 = \frac{\sigma_\varepsilon^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2} = \frac{36875}{4674} \sigma_\varepsilon^2 \approx 7.8894 \sigma_\varepsilon^2.$$

Therefore, the whole autocovariance function is

$$\gamma_k = \gamma_0 \rho_k = \frac{36875}{4674} \sigma_\varepsilon^2 \left[\frac{432}{413} (0.9)^k - \frac{19}{413} (0.2)^k \right], \quad k = 0, 1, 2, \dots$$

Problem 3.

$$f(x) = a_p x^p + a_{p-1} x^{p-1} + \cdots + a_1 x + a_0, \quad a_p \neq 0, a_0 \neq 0,$$

and let λ be a root of f .

(a) If $\lambda = 0$, then $f(0) = a_0 = 0$, which contradicts $a_0 \neq 0$. Hence $\lambda \neq 0$.

(b) Define the “reversed-coefficients” polynomial

$$g(x) = a_p + a_{p-1}x + \cdots + a_1x^{p-1} + a_0x^p.$$

From $f(\lambda) = 0$ we have

$$\sum_{k=0}^p a_k \lambda^k = 0 \implies \sum_{k=0}^p a_k \lambda^{k-p} = 0 \implies \sum_{k=0}^p a_k \left(\frac{1}{\lambda}\right)^{p-k} = 0.$$

But the last expression equals $g(1/\lambda)$. Therefore $g(1/\lambda) = 0$, so $1/\lambda$ is a root of g . (Equivalently, $g(x) = x^p f(1/x)$.)

(c) If all roots of f satisfy $|\lambda_j| > 1$, then by part (b) the roots of g are $\{1/\lambda_j\}_{j=1}^p$ (counting multiplicities). Hence $|1/\lambda_j| < 1$ for all j , i.e., all roots of g lie inside the unit circle.

Problem 4. (a) For any constant c , $L^k c = c$ for all k . Hence

$$\Phi(L)c = \left(1 - \sum_{i=1}^p \phi_i\right)c.$$

Let $m \equiv \Psi(L)\mu$ (note m is a constant). Using $\Phi(L)\Psi(L) = 1$,

$$\mu = \Phi(L)[\Psi(L)\mu] = \Phi(L)m = \left(1 - \sum_{i=1}^p \phi_i\right)m,$$

which implies

$$\boxed{\Psi(L)\mu = \frac{\mu}{1 - \sum_{i=1}^p \phi_i}}$$

(the denominator is nonzero since a unit root would violate the assumption). Consequently,

$$y_t = \Psi(L)\mu + \Psi(L)\varepsilon_t = \frac{\mu}{1 - \sum_{i=1}^p \phi_i} + \Psi(L)\varepsilon_t.$$

(b) Opening the multiplication:

$$\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots - \phi_1 L \psi_0 - \phi_1 L^2 \psi_1 - \phi_1 L^3 \psi_2 - \cdots - \phi_2 L^2 \psi_0 - \phi_2 L^3 \psi_1 - \phi_2 L^4 \psi_2 - \cdots - \cdots = I.$$

Write as a polynomial in L :

$$\psi_0 + L(\psi_1 - \phi_1 \psi_0) + L^2(\psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0) + L^3(\psi_3 - \phi_1 \psi_2 - \phi_2 \psi_1 - \phi_3 \psi_0) + \cdots = I.$$

For those two operators to be equal, we need all the coefficients associated to L be equal to zero (as we would do with a polynomial):

$$\psi_k = \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2} + \cdots + \phi_p \psi_{k-p}, \quad \forall k \geq p,$$

$$\text{and} \quad \psi_0 = I.$$

(c) From (1),

$$\psi_k = \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2} + \cdots + \phi_p \psi_{k-p} \quad (k > p),$$

let us write the associated matrix:

$$F = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

As discussed in the TA session, we want to know the eigenvalues of this matrix so we can raise it to a power. This will lead us to the characteristic polynomial:

$$\det(F - \lambda I_p) = 0 \implies \lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_p = 0.$$

Hence the characteristic polynomial is

$$\chi(r) = r^p - \phi_1 r^{p-1} - \cdots - \phi_p.$$

Let $z = 1/r$. Multiplying the equation by z^p yields

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0,$$

which is precisely the AR polynomial $\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$. By assumption all roots z_j of Φ satisfy $|z_j| > 1$ (Otherwise, we know from lecture 4 that the inverse of the lag polynomial would not exist); therefore the roots of χ are $r_j = 1/z_j$ and satisfy $|r_j| < 1$. Thus all roots of the difference equation lie strictly inside the unit circle.

(d) Let r_1, \dots, r_q be the distinct roots of the characteristic polynomial $\chi(r) = r^p - \phi_1 r^{p-1} - \cdots - \phi_p$. From the previous result we know that $|r_j| < 1$ for all j (they are the inverses of the AR roots).

The general solution of the linear homogeneous recursion $\psi_k = \phi_1 \psi_{k-1} + \cdots + \phi_p \psi_{k-p}$ is

$$\psi_k = \sum_{j=1}^q P_j(k) r_j^k, \quad k \geq 0,$$

where P_j is a polynomial of degree $m_j - 1$ if r_j has multiplicity m_j . Hence there exist constants $C > 0$, $m = \max_j m_j$, and $0 < \rho < 1$ (take $\rho = \max_j |r_j|$) such that

$$|\psi_k| \leq C k^{m-1} \rho^k \quad \text{for all } k.$$

Therefore

$$\sum_{k=0}^{\infty} \psi_k^2 \leq C^2 \sum_{k=0}^{\infty} k^{2(m-1)} \rho^{2k} < \infty,$$

because a polynomial factor times a geometric term with ratio < 1 is summable. Consequently,

$$\sum_{i=0}^{\infty} \psi_i^2 < \infty.$$

Problem 5. (a)

- **IPCA** (Índice de Preços ao Consumidor Amplo) is Brazil's broad consumer-price index, produced by IBGE. It measures the change in prices of a consumer basket faced by households with monthly income between 1 and 40 minimum wages in large metropolitan areas. It is the *target* inflation measure for monetary policy (BCB's inflation targeting regime).
- **IGP-M** (Índice Geral de Preços – Mercado) is produced by FGV and is a composite index: roughly 60% IPA (wholesale/producer prices), 30% IPC (consumer prices), and 10% INCC (construction). Because of the high IPA weight, IGP-M is more sensitive to exchange rate, commodity, and wholesale price shocks. It is widely used in contract indexation (e.g., rents, utilities), but it is not the policy target.

(e) The data shows that IPCA is less volatile and more persistence than IGPM. This was the expected result, since IGPM is largely exposed to exchange rate fluctuations and other supply-side shocks.

Reproducible code (R)

```
# =====
# IPCA vs IGP-M: Descriptive Analysis
# =====

# ---- 0) Packages ----
library(here)
library(readr)
library(tidyverse)

# ---- 1) Load & clean data ----
csv_path <- here("problem_sets", "PS2", "ipca_igpm.csv")
df <- readr::read_csv(csv_path, show_col_types = FALSE)
names(df) <- tolower(names(df))
df$date <- as.Date(paste0(df$date, "-01"))
df$date <- as.Date(df$date)
df <- df[!is.na(df$igpm) & !is.na(df$ipca), ]

# ---- 2) Time-series plot (MoM) ----
df_long <- df |>
  tidyr::pivot_longer(ipca:igpm, names_to = "index", values_to = "mom") |>
  mutate(index = toupper(index))

p <- ggplot(df_long, aes(date, mom, color = index)) +
  geom_line() +
  labs(title = "Monthly MoM Inflation: IPCA vs IGP-M",
       x = NULL, y = "MoM (%)", color = NULL) +
```

```

theme_minimal(base_size = 12)
print(p)

# ---- 3) Summary stats & autocovariances ----
summarise_series <- function(x) {
  data.frame(
    mean      = mean(x, na.rm = TRUE),
    variance  = var(x, na.rm = TRUE),
    stringsAsFactors = FALSE
  )
}

as_cov <- function(x, L = 24) {
  ac <- acf(x, type = "covariance", plot = FALSE, na.action = na.omit, lag.max = L)
  data.frame(lag = as.integer(ac$lag[, , 1]),
             autocov = as.numeric(ac$acf[, , 1]))
}

ip_stats <- summarise_series(df$ipca); rownames(ip_stats) <- "IPCA"
ig_stats <- summarise_series(df$igpm); rownames(ig_stats) <- "IGP-M"
tab_stats <- rbind(ip_stats, ig_stats)

cat("\n==== Sample Mean & Variance (MoM, in % units) ====\n")
print(round(tab_stats, 4))

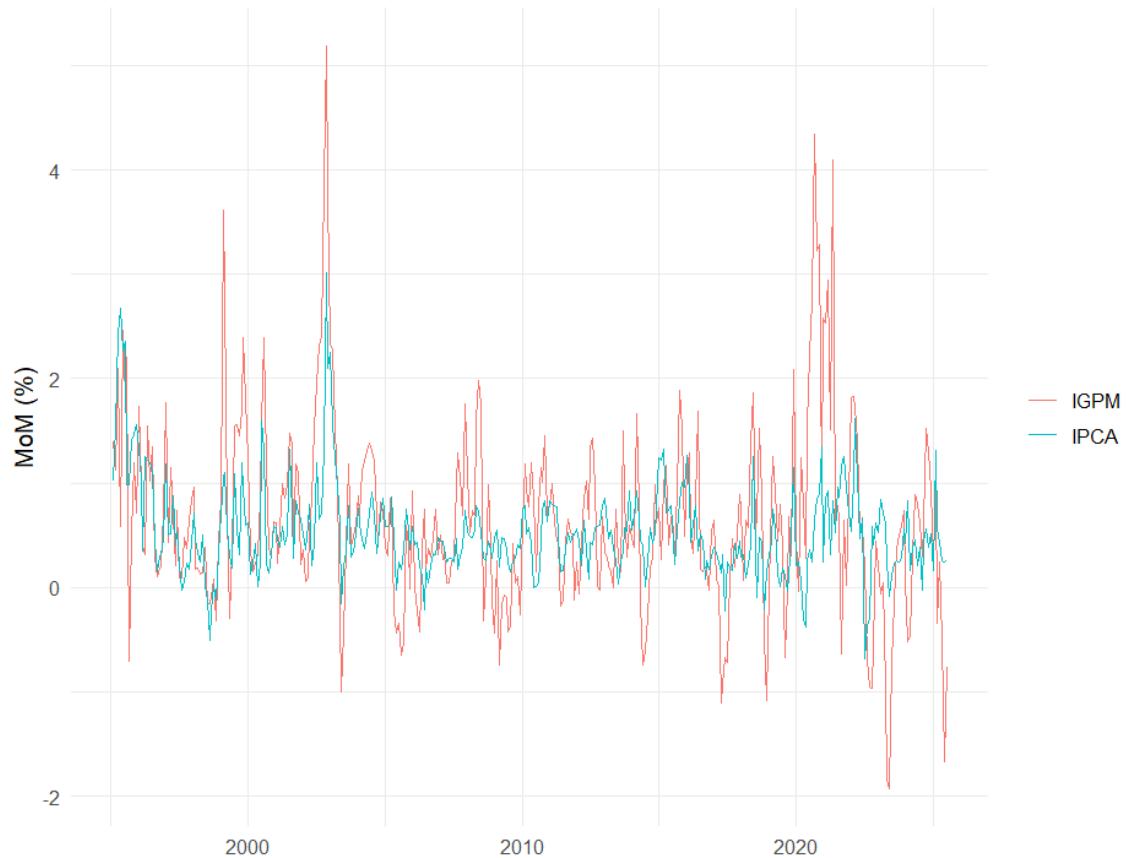
ip_cov <- as_cov(df$ipca, 24); ip_cov$series <- "IPCA"
ig_cov <- as_cov(df$igpm, 24); ig_cov$series <- "IGP-M"

cat("\n==== IPCA Autocovariances (lags 0..24) ====\n")
print(round(ip_cov[, c("lag", "autocov")], 5))
cat("\n==== IGP-M Autocovariances (lags 0..24) ====\n")
print(round(ig_cov[, c("lag", "autocov")], 5))

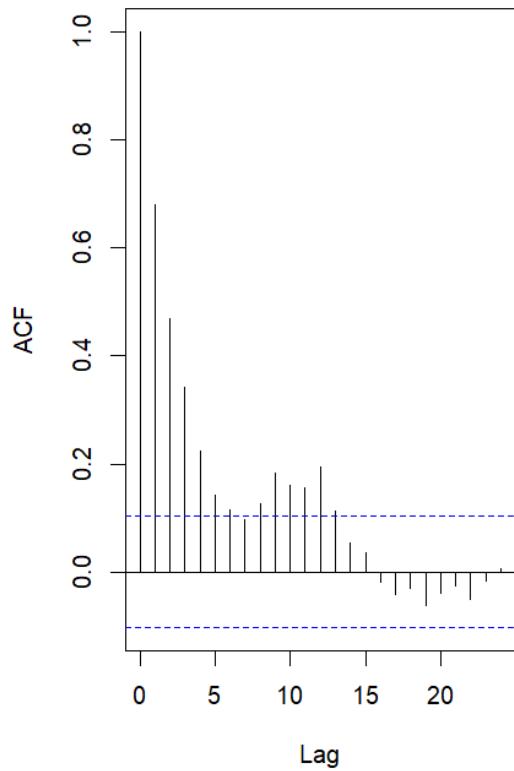
# ---- 4) ACF plots (persistence) ----
par(mfrow = c(1, 2))
acf(na.omit(df$ipca), lag.max = 24, main = "IPCA: ACF (0 24 )")
acf(na.omit(df$igpm), lag.max = 24, main = "IGP-M: ACF (0 24 )")
par(mfrow = c(1, 1))

```

Monthly MoM Inflation: IPCA vs IGP-M



IPCA: ACF (0–24)



IGP-M: ACF (0–24)

