

Lecture 6: Estimation of ARMA Models

Raul Riva

FGV EPGE

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Intro

- So far, we took parameters as given when working with ARMA models;
- In practice, we need to *estimate* these parameters from data;
- There many ways to estimate ARMA models: maximum likelihood, method of moments, Kalman filter, etc;
- We will focus on MLE estimation;
- Usually, good software for ARMA estimation gives you several options;
- More than mastering math tricks and details, it is important to understand the *big picture*;

- It is always the MA part that will complicate things;
- A natural estimator for $AR(p)$ models is just the OLS estimator: regress y_t on y_{t-1}, \dots, y_{t-p} ;
- Mild conditions will guarantee consistency, asymptotic normality, bla, blah, blah...
- But for $ARMA(p, q)$ models, we cannot do that! We do not observe ε_t !!!

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- But for $ARMA(p, q)$ models, we cannot do that! We do not observe ε_t !!!
- MLE will require a *distributional assumption* for ε_t ;
- We will relax that later when we touch on “*quasi-MLE*”;
- We will start with *given* values of p and q and discuss model choice later;

Preliminaries

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- First step: characterize the joint distribution of the sample $\mathbf{y} = (y_1, \dots, y_T)'$;
- Denote this distribution by $f_{y_T, y_{T-1}, \dots, y_1}(\mathbf{y}; \Theta)$;

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- Recall: $f_{Y|X}(y, x) = f_{Y, X}(y, x) / f_X(x) \implies f_{Y, X}(y, x) = f_{Y|X}(y, x) f_X(x)$
- For any integer $k \geq 1$:

$$f_{y_T, y_{T-1}, \dots, y_1}(\mathbf{y}; \Theta) = f_{y_k, \dots, y_1}(y_k, \dots, y_1; \Theta) \cdot \prod_{t=k+1}^T f_{y_t|y_{t-1}, \dots, y_1}(y_t | y_{t-1}, \dots, y_1; \Theta)$$

Questions?

The AR(p) Case

The AR(p) Case

- Consider the AR(p) model below and let $\Theta = (c, \phi_1, \dots, \phi_p, \sigma^2)$:

$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2)$$

- Notice that $y_t | y_{t-1}, \dots, y_{t-p} \sim N(c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p}, \sigma^2)$. Therefore:

$$f_{y_t | y_{t-1}, \dots, y_1}(y_t | y_{t-1}, \dots, y_1; \Theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_t - c - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p})^2}{2\sigma^2}}$$

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- The likelihood of the first p observations, $f_{y_p, \dots, y_1}(y_p, \dots, y_1; \Theta)$, is more involved;
- Notice that the $p \times 1$ vector $\mathbf{y}_{1:p} = (y_1, \dots, y_p)'$ is multivariate normal;

$$\mathbf{y}_{1:p} \sim N(\boldsymbol{\mu}, \boldsymbol{\Omega}), \quad \boldsymbol{\mu} = \frac{c}{1 - \sum_{i=1}^p \phi_i} \mathbf{1}, \quad \boldsymbol{\Omega}_{ij} = \gamma(|i - j|) \quad \forall i, j \in \{1, \dots, p\}$$

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- The likelihood of the first p observations is given by:

$$f_{y_p, \dots, y_1}(y_p, \dots, y_1; \Theta) = (2\pi)^{-p/2} |\mathbf{\Omega}^{-1}|^{1/2} e^{-\frac{1}{2}(\mathbf{y}_{1:p} - \boldsymbol{\mu})' \mathbf{\Omega}^{-1} (\mathbf{y}_{1:p} - \boldsymbol{\mu})}$$

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- From here, we can write the full likelihood function:

$$\begin{aligned} f_{y_T, \dots, y_1}(\mathbf{y}; \Theta) &= f_{y_k, \dots, y_1}(y_k, \dots, y_1; \Theta) \cdot \prod_{t=k+1}^T f_{y_t | y_{t-1}, \dots, y_1}(y_t | y_{t-1}, \dots, y_1; \Theta) \\ &= (2\pi)^{-p/2} |\mathbf{\Omega}^{-1}|^{1/2} e^{-\frac{1}{2}(\mathbf{y}_{1:p} - \boldsymbol{\mu})' \mathbf{\Omega}^{-1}(\mathbf{y}_{1:p} - \boldsymbol{\mu})} \cdot \prod_{t=p+1}^T \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_t - c - \sum_{i=1}^p \phi_i y_{t-i})^2}{2\sigma^2}} \\ &= (2\pi)^{-T/2} \sigma^{-(T-p)} |\mathbf{\Omega}^{-1}|^{1/2} e^{-\frac{1}{2}(\mathbf{y}_{1:p} - \boldsymbol{\mu})' \mathbf{\Omega}^{-1}(\mathbf{y}_{1:p} - \boldsymbol{\mu})} \cdot e^{-\frac{1}{2\sigma^2} \sum_{t=p+1}^T (y_t - c - \sum_{i=1}^p \phi_i y_{t-i})^2} \end{aligned}$$

The Log-Likelihood Function

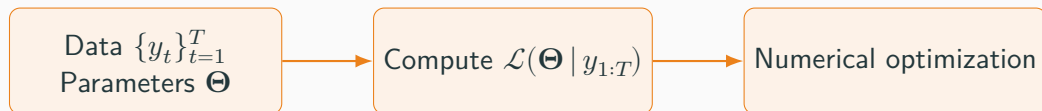
- We always optimize the log-likelihood function $\mathcal{L}(\Theta|\mathbf{y}) = \log \left(f_{y_T, \dots, y_1}(\mathbf{y}; \Theta) \right)$

$$\begin{aligned}\mathcal{L}(\Theta|\mathbf{y}) &= \log \left(f_{y_T, \dots, y_1}(\mathbf{y}; \Theta) \right) \\ &= -\frac{T}{2} \log(2\pi) \\ &\quad - (T - p) \log(\sigma) + \frac{1}{2} \log(|\mathbf{\Omega}^{-1}|) \\ &\quad - \frac{1}{2} (\mathbf{y}_{1:p} - \boldsymbol{\mu})' \mathbf{\Omega}^{-1} (\mathbf{y}_{1:p} - \boldsymbol{\mu}) - \frac{1}{2\sigma^2} \sum_{t=p+1}^T \left(y_t - c - \sum_{i=1}^p \phi_i y_{t-i} \right)^2\end{aligned}$$

- The blue part looks like the OLS objective function;
- The red part is “distorting” this objective function;

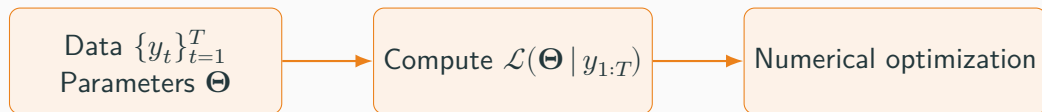
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- Notice that this requires inverting a $p \times p$ matrix any time we evaluate the function;



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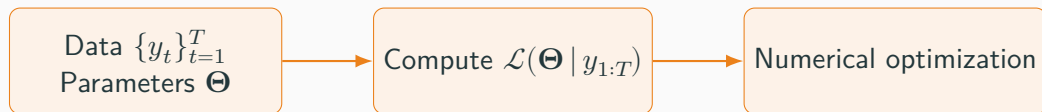
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- Wait a minute... what if $T \gg p$?
- In that case the main contribution to the log-likelihood function comes from the blue part;
- This suggests a simpler approach: *conditional* MLE;
- Assume that the first p observations are fixed (non-random);
- Approximate $\mathcal{L}(\Theta | \mathbf{y}_{1:T})$ by $\log \left(f_{y_{p+1}, \dots, y_T | y_{1:p}}(\mathbf{y}; \Theta) \right)$

The Numerical Shortcut for the AR(p) Case

- Recall that, up to a constant, we have:

$$\log \left(f_{y_{p+1}, \dots, y_T | y_{1:p}}(\mathbf{y}; \boldsymbol{\Theta}) \right) = - \sum_{t=p+1}^T \frac{(y_t - c - \sum_{i=1}^p \phi_i y_{t-i})^2}{2\sigma^2} - (T-p) \log(\sigma)$$

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- Estimators for c and ϕ_i 's are the same as the OLS from regressing y_t on y_{t-1}, \dots, y_{t-p} ;
- Super simple closed-form solutions! 😎
- The estimator for σ^2 is just the (biased) sample variance of the OLS residuals;

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- Super simple closed-form solutions! 😎
- The estimator for σ^2 is just the (biased) sample variance of the OLS residuals;
- If T is large, this is a very good approximation to the full MLE;
- $\mathcal{L}(\Theta | \mathbf{y})$ is efficiently computed using the Kalman filter – darker magic for the next year!

Questions?

The MA(q) Case

- Consider the MA(q) model below and let $\Theta = (\mu, \theta_1, \dots, \theta_q, \sigma^2)$:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2)$$

- There is no hope to get an “OLS”-type trick... we do not see the shocks...
- There are again two main approaches: full MLE and conditional MLE;
- We will focus on the conditional MLE approach;
- You can see the full MLE approach in Hamilton's book (Chapter 5);
- If T is large, the two approaches will give very similar results;
- Similar to the forecasting exercise in the last lecture!

The MA(q) Case

- The key observation is that $y_t | \varepsilon_{t-1}, \dots, \varepsilon_{t-q} \sim N(\mu + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \sigma^2)$;
- But how is that useful if we do not observe ε_t ?

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- The key observation is that $y_t | \varepsilon_{t-1}, \dots, \varepsilon_{t-q} \sim N(\mu + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \sigma^2)$;
- But how is that useful if we do not observe ε_t ?
- Let's assume that $\varepsilon_{-q+1} = \varepsilon_{-q+2} = \dots = \varepsilon_0 = \mathbb{E}[\varepsilon_t] = 0$;
- We can start a recursion, like in the forecasting case:

$$\varepsilon_1 = y_1 - \mu$$

$$\varepsilon_2 = y_2 - \mu - \theta_1 \varepsilon_1$$

$$\varepsilon_3 = y_3 - \mu - \theta_1 \varepsilon_2 - \theta_2 \varepsilon_1$$

$$\vdots$$

$$\varepsilon_t = y_t - \mu - \theta_1 \varepsilon_{t-1} - \dots - \theta_q \varepsilon_{t-q}$$

$$\vdots$$

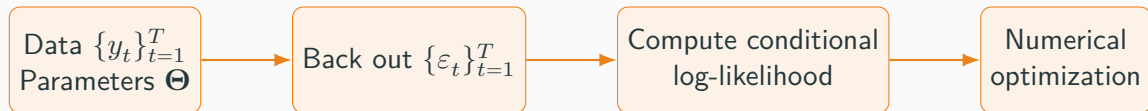
$$\varepsilon_T = y_T - \mu - \theta_1 \varepsilon_{T-1} - \dots - \theta_q \varepsilon_{T-q}$$

The Conditional Log-Likelihood Function

- From here, we can write the conditional log-likelihood function (up to a constant):

$$\begin{aligned}\log \left(f_{y_t, \dots, y_1 | \varepsilon_{-q+1} = \varepsilon_{-q+2} = \dots = \varepsilon_0 = 0}(\mathbf{y}; \Theta) \right) &= \sum_{t=q+1}^T \log \left(f_{y_t | \varepsilon_{t-1}, \dots, \varepsilon_{t-q}}(y_t | \varepsilon_{t-1}, \dots, \varepsilon_{t-q}; \Theta) \right) \\ &= - \sum_{t=1}^T \frac{(\varepsilon_t)^2}{2\sigma^2} - T \cdot \log(\sigma)\end{aligned}$$

- When there is an MA component, the logical flow is:



Questions?

The ARMA(p, q) Case

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- Consider a Gaussian ARMA(p, q) model and let $\Theta = (c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q, \sigma^2)$:

$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2)$$

- We can combine the two previous approaches;
- Given Θ , we will back out ε_t recursively;
- We also note that $y_t | y_{t-1}, \dots, y_1, \varepsilon_{t-1}, \dots, \varepsilon_{t-q} \sim N(c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p}, \sigma^2)$
- Then we are ready to use the conditioning trick once again!

The Recursion

- As we did with the $AR(p)$, assume y_1, \dots, y_p are fixed;
- Assume that $\varepsilon_p = \varepsilon_{p-1} = \dots = \varepsilon_{p-q+1} = 0$
- The first shock to be backed out is $\varepsilon_{p+1} = y_{p+1} - c - \sum_{i=1}^p \phi_i y_{p+1-i}$
- Then we get $\varepsilon_{p+2} = y_{p+2} - c - \sum_{i=1}^p \phi_i y_{p+2-i} - \theta_1 \varepsilon_{p+1}$
- And so on...
- You might be skeptical of “assuming values” for the shock... but usually p and q are small compared to T !
- You will almost never see $q > 10$ and $p > 20$ in practice!

The Conditional Log-Likelihood Function

- The conditional log-likelihood function, up to a constant, is given by:

$$\mathcal{L}(\Theta|\mathbf{y}) = \log \left(f_{y_t, \dots, y_1 | \varepsilon_{-q+1} = \varepsilon_{-q+2} = \dots = \varepsilon_0 = 0}(\mathbf{y}; \Theta) \right) = - \sum_{t=p+1}^T \frac{(\varepsilon_t)^2}{2\sigma^2} - (T-p) \log(\sigma)$$

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Regarding numerical optimization:

- Do we have guarantees the numerical method will converge to the global maximum? No.
- Is it much harder as we increase p and q ? Yes and no: increasing p is fine, but q is hell;
- Where to start the optimization? OLS estimates for ϕ are a good shot;
- What about θ ? Start with zeros or small values;
- Try several different starting points and make sure you get similar answers;

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- Ok great, we can estimate $\text{ARMA}(p, q)$ models;
- How to do inference?
- We will use standard MLE results;
- Important assumptions: a correctly specified model and Θ_0 must be an interior point;
- Recall that, if the model is correctly specified, then:

$$\sqrt{T}(\hat{\Theta} - \Theta_0) \xrightarrow{d} N(0, \mathcal{I}^{-1}(\Theta_0))$$

where $\mathcal{I}(\Theta)$ is the Fisher information matrix;

- Recall that, in this case, $\mathcal{I}(\Theta) = -\mathbb{E} \left[\frac{\partial^2 l_t(\Theta)}{\partial \Theta \partial \Theta'} \right] = \mathbb{E} \left[\frac{\partial l_t(\Theta|\mathbf{y})}{\partial \Theta} \frac{\partial l_t(\Theta|\mathbf{y})}{\partial \Theta'} \right]$, where $l_t(\Theta|\mathbf{y}) = \log \left(f_{y_t|\mathbf{y}_{t-1}}(y_t|\mathbf{y}_{t-1}; \Theta) \right)$;

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- Theory suggests two equally valid ways of estimating it. Let us define two objects:

1. The Hessian:

$$\mathcal{H}(\hat{\Theta}) \equiv \frac{1}{T-p} \cdot \frac{\partial^2 \mathcal{L}(\hat{\Theta}|\mathbf{y})}{\partial \Theta \partial \Theta'} = \frac{1}{T-p} \cdot \sum_{t=p+1}^T \frac{\partial^2 \log(f_{y_t|\mathbf{y}_{t-1}}(y_t|\mathbf{y}_{t-1}; \hat{\Theta}))}{\partial \Theta \partial \Theta'}$$

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2. The score function and its associated *outer product*:

$$\mathcal{S}(\hat{\Theta})_t \equiv \frac{\partial \log(f_{y_t|\mathbf{y}_{t-1}}(y_t|\mathbf{y}_{t-1}; \hat{\Theta}))}{\partial \Theta}; \quad \mathcal{O}(\hat{\Theta}) \equiv \frac{1}{T-p} \cdot \sum_{t=p+1}^T \mathcal{S}(\hat{\Theta})_t \mathcal{S}(\hat{\Theta})_t'$$

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- Then, we can estimate $\mathcal{J}(\Theta_0)$ by either $[-\mathcal{H}(\hat{\Theta})]$ or $[\mathcal{O}(\hat{\Theta})]$;

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- Then, we can estimate $\mathcal{J}(\Theta_0)$ by either $[-\mathcal{H}(\hat{\Theta})]$ or $[\mathcal{O}(\hat{\Theta})]$;
- (Adjust the starting point of the sum as needed, it doesn't matter asymptotically);

- What if ε_t is not Gaussian?
- The MLE will converge to a different Θ_0 ;
- This parameter is the *pseudo-true* value that minimizes the Kullback-Leibler divergence between the true model and the assumed Gaussian model;
- The idea, and the term *Quasi-MLE*, is due to White (Econometrica, 1982);

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- The asymptotic distribution is now:

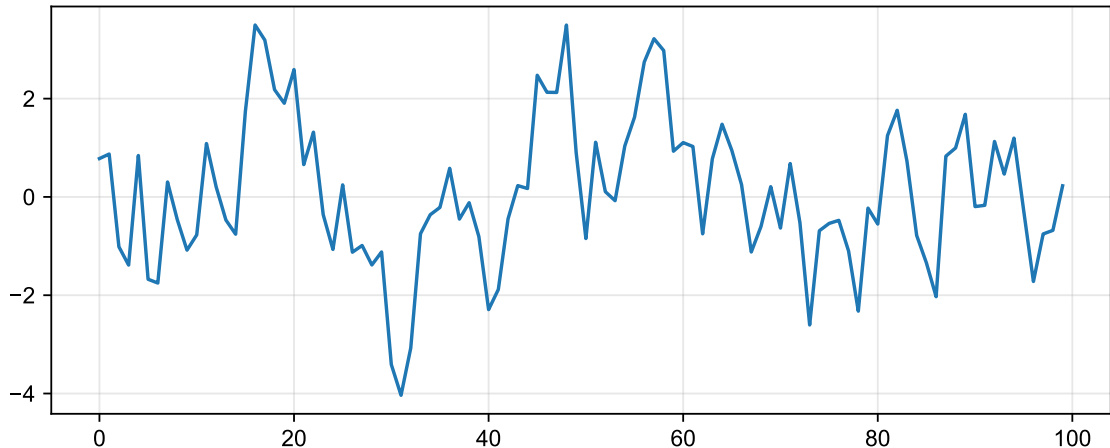
$$\sqrt{T}(\hat{\Theta} - \Theta_0) \xrightarrow{d} N\left(0, \underbrace{\mathcal{H}^{-1}(\Theta_0)\mathcal{J}(\Theta_0)\mathcal{H}^{-1}(\Theta_0)'}_{\text{the "sandwich" variance}}\right)$$

- The “bread” uses the Hessian and the “meat” uses the outer product of the score;
- The estimator for the sandwich is $\left[-\mathcal{H}^{-1}(\hat{\Theta})\mathcal{O}(\hat{\Theta})\mathcal{H}^{-1}(\hat{\Theta})'\right]$

Some Simulations

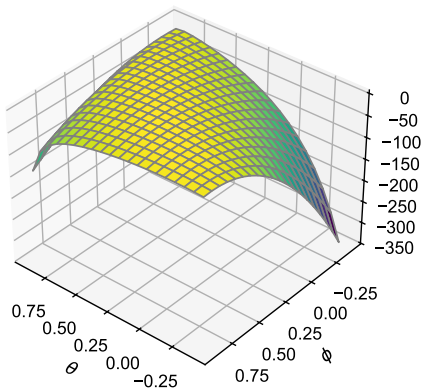
The ARMA(1,1) Case - Simulated Path ($\phi = 0.5, \theta = 0.2$)

Let's say we have an ARMA(1,1), with $\mu = 0$ and $\sigma^2 = 1$. Let's simulate one path:

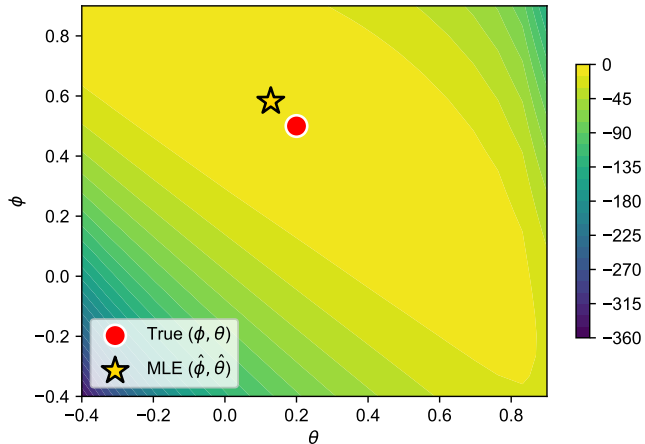


The Likelihood Surface ($T=100$)

3D Likelihood

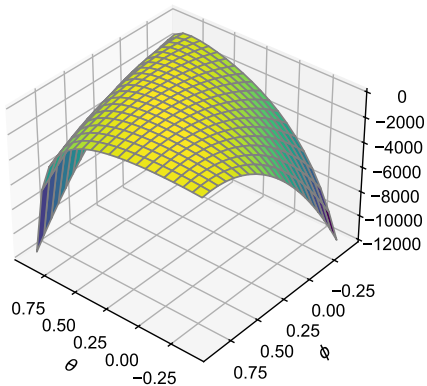


Contour: True vs MLE ($T=100$)

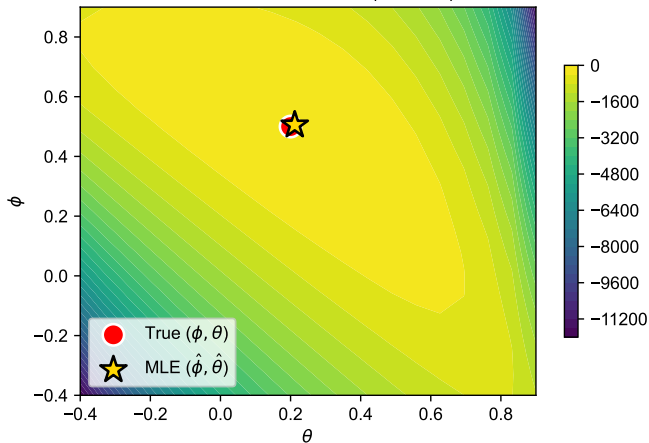


The Likelihood Surface ($T=5000$)

3D Likelihood

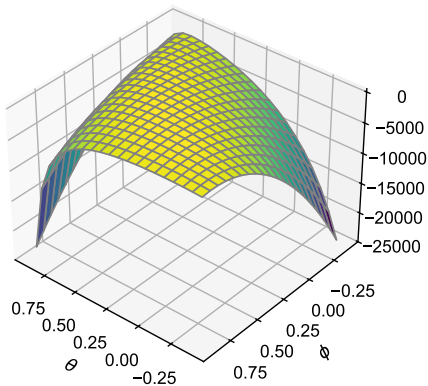


Contour: True vs MLE ($T=5000$)

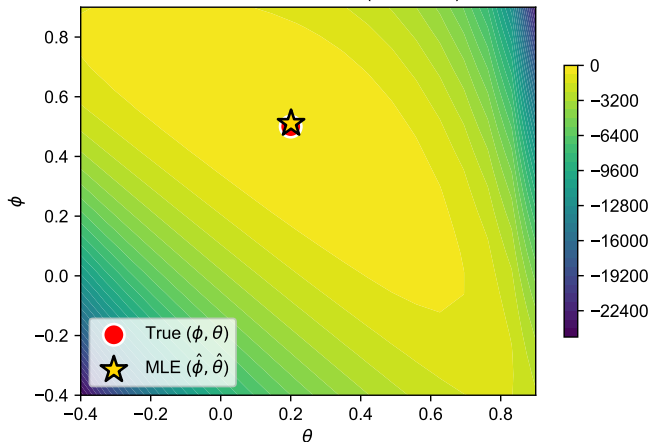


The Likelihood Surface ($T=10000$)

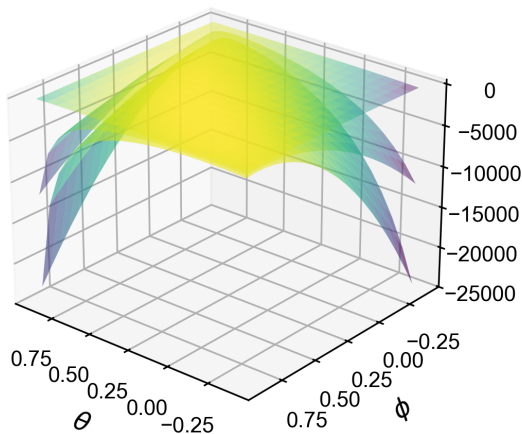
3D Likelihood



Contour: True vs MLE ($T=10000$)



The Likelihood Gets More Concentrated!



- $\uparrow T \Rightarrow$ tighter likelihood. Why?
- How is the Hessian at the optimum related to this?
- What is the connection with the asymptotic distribution of the MLE?

Questions?

How to choose p and q ?

The Model Selection Problem

- So far, we have assumed that p and q are known;
- In practice, we need to choose them from the data;
- Naive approach: maximize the log-likelihood over all possible (p, q) pairs;

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 - Adding parameters can never decrease the maximized log-likelihood;
 - Converges to a perfect fit *in sample* as $(p, q) \rightarrow \infty$ (overfitting);
- We need a formal criterion that **penalizes model complexity**;
- This leads to *information criteria*: balance fit vs. parsimony;

- The general form of information criteria is:

$$\text{IC} = -2 \cdot \mathcal{L}(\hat{\Theta}|\mathbf{y}) + \text{penalty}(k, T)$$

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- We choose the model that **minimizes** the IC;
- Different penalties lead to different criteria;
- The key trade-off: smaller penalty \implies more likely to select larger models;

The Main Information Criteria

Let $k = p + q + 2$ be the number of parameters in an $\text{ARMA}(p, q)$ model (including c and σ^2).

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Bayesian Information Criterion (BIC) or Schwarz Criterion (SIC):

$$\text{BIC} = -2 \cdot \mathcal{L}(\hat{\Theta}|\mathbf{y}) + k \log(T)$$

- It approximates the model with the highest posterior probability (assuming equal priors);
- It is **consistent**: selects the true model (if it is in the candidate set) with probability $\rightarrow 1$ as $T \rightarrow \infty$;

Comparing the Penalties

- Notice that for $T > 8$, we have $\log(T) > 2$, so BIC penalizes more heavily than AIC;

| Sample Size | AIC penalty | BIC penalty |
|-------------|-------------|-------------|
| $T = 50$ | $2k$ | $3.91k$ |
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- As $T \rightarrow \infty$: BIC penalty grows much faster than AIC;
- Implication: BIC tends to select **more parsimonious models** than AIC;

Step-by-step procedure:

1. Choose a maximum order p_{\max} and q_{\max} (often based on theory or exploratory analysis);
2. Estimate all ARMA(p, q) models for $p \in \{0, 1, \dots, p_{\max}\}$ and $q \in \{0, 1, \dots, q_{\max}\}$;
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How to Use Information Criteria in Practice

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Important notes:

- All models must be estimated on the **same sample** (same T);
- Start with reasonable p_{\max} and q_{\max} (e.g., 5-10 for quarterly data, 12-24 for monthly);
- If the selected model is at the boundary, consider increasing the maximum orders;

The End

- Chapter 5 from Hamilton's book for ARMA estimation;
- Chapter 28 from Hansen's book on model selection for MLE;