Imrohoroglu, Imrohoroglu, and Joines (1995) - A Life Cycle Analysis of Social Security

Brief description of replication of Imrohoroglu, Imrohoroglu, and Joines (1995). I make numerous notational changes. This is not intended as a full description of the model, so much as making it clear how the model is expressed in terms of the VFI Toolkit; for a full description of the model you should consult the original paper.

The model is a general equilibrium OLG model. The finite-horizon value function problem has one exogenous state (which takes two possible values, employed and unemployed), one endogenous state (assets), and 65 periods. The household value function problem is given by

$$
\begin{aligned}
V(k,z,j) = & \max_{c,k'} \frac{c^{1-\gamma}}{1-\gamma} + \beta s_j E_j[V(k',z',j+1)|z] \\
\text{subject to } & c + k' \leq (1+r(1-\tau_k))k + (1-\tau_s-\tau_u)wh\epsilon_j z\mathbb{I}_{(j<J_r)} + u(1-z)\mathbb{I}_{(j<J_r)} \dots \\
& \quad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\
& k' \geq 0
\end{aligned}
$$

There are $J = 65$ periods and $V(k,z,J+1) = 0$ for all k, & z. So household faces employment-status shocks (z) and solve a consumption-savings problem of choosing consumption c and next period assets k'. There are some basic taxes which are used to fund pensions SS that are received once retirement age is reached. When people die their assets are redistributed lump-sum accoss the living as Tr_{beq}. Households discount the future by pure discount factor β and conditional probability of survival s_j.

The earnings process z consists of two states, 'employed' which is when $z = 1$ and 'unemployed' which is when $z = 0$. While in principle it could be markov (and IIJ1995 describes it as such) the markov transition matrix is defined so that the actual shock is iid. (The rows of the markov transition matrix are all the same.) ϵ_j is a deterministic spline of earnings in terms of age and is used to generate the age profile of earnings.

The initial distribution of agents at birth is for them to have zero assets and the stationary distribution of shocks.

The government budget constraint consists of the following two (seperate) parts: the unemployment benefits tax, τ_u, pays for unemployment benefits u. The social security payroll tax, τ_s, pays for social security (pension) benefits SS.

The model has five general equilibrium constraints, the first is that the interest rate r equals the marginal product of capital minus the depreciation rate δ. The next two are fiscal: that the unemployment benefits tax balances unemployment benefits, and that the social security tax pays for social security pensions. The fifth is that the the (total across the population of the) lump-sum transfer of accidental bequests Tr_{beq} much equal the assets left behind by people on dying.

Notational differences from Imrohoroglu, Imrohoroglu, and Joines (1995), originals in parentheses: I refer to the replacement rate for social security (pension) benefits as b (θ) and the benefits themselves as SS (b). Age of retirement is Jr (j^*). Population growth rate n (ρ). Age-conditional survival probability is s_j (ψ_j). Exogenous shock –employment or unemployment– is z (s). In the Cobb-Douglas production function I use α as share of capital ($1-\alpha$) and A as the total factor productivity (B). Lump-sum transfers due to accidental bequests are Tr_beq (T).

For more details on the model see Imrohoroglu, Imrohoroglu, and Joines (1995).

Note that my codes treat tau_u as something to be determined in general equilibrium. In principle you could calculate it directly since ζ is known, and since the mean labor supply and the age of retirement are both known (due to exogenous labor supply). While both do depend on w, this could be canceled out. With an extension to endogenous labor τ_u would have to be treated as determined in general equilibrium.

For Figure 7 of IIJ1995, denoted here as Figure 7, I reproduce both the original (which graphs age-conditional pdfs) and also an 'alternative' which graphs age-conditional cdfs. This is because discretized pdfs are sensitive to the grids used (in terms of the mass at a given point, the y-axis of the graph; trivially, more points typically means less mass at each point), while plotting cdfs avoids this dependence. (An alternative to plotting cdfs is to plot non-parametric kernel etimators of the pdf.)

In Figure 2 I was unclear on how to calculate the interest rate that should be used to plot the 'planners' consumption profile. IIJ1995 also state that 'To remove the effects of social security on the capital stock and aggregate consumption, each of these profiles has been normalized to have the same aggregate consumption.' I have not done this as I was unsure if this just mean dividing each profile by it's own mean value or something more subtle that allowed for the cost of moving consumption between periods, in any case they look essentially the same in the replication as in the original.

Table 6 shows substantial differences between the replication and original for the model with medical shocks when the social security replacement rate is less than 0.3. As far as I can tell the reason for this in the replication results is that no equilibrium exists. The intuition for the lack of equilibrium has two parts. First, the medical expense shocks mean that agents want to enter retirement with much larger asset holdings (as insurance against the risk of medical expense shocks) and this means the equilibrium interest rate will have to be much lower, in fact it appears to require that the interest rate be negative. Second, there is a non-zero probability that an agent spends their working-life (periods 1 to 44) unemployed and then spends their retirement-life (periods 45 to 65) suffering a medical expense shock in every period. When the interest rate is low enough to be the kind of number required for general equilibrium this person (always unemployed, always sick) is unable to save enough during their working-life to cover all their medical expenses during retirement-life and so ends up with negative consumption and so has utility of negative-infinity. Notice that this is only going to happen when the social security is less than the medical expenses, which is the case for a social security replacement rate of less than 0.3, hence why these rows all show NaN; the Value function, not shown, is negative-infinity for all points in the state space. Notice that at age-1 there is a non-zero probability of ending up this person, and so everyone's period-1 value function ends up being negative-infinity.

Table 3, which reports the Welfare evaluations was 'tricky' to calculate. Notice that the lump sum L used to calculate the income-equivalent welfare evaluation enters the model in the exact same manner as the lump-sum transfers due to accidental bequests, Tr_beq. The numerical optimization algorithms had difficulty with this as the problem is to choose L to give the same welfare Ω at the same time as enforcing the general equilibrium conditions relating to r and Tr_beq. In the end I put a much larger weight (of 10, relative to 1 for the other conditions) on the condition relating to Tr_beq which helped the numerical algorithms distinguish the two. IIJ1995 does not explicitly state that they enforce the general equilibrium conditions relating to r and Tr_beq while doing the welfare evaluations but this seems the reasonable interpretation.

Table 6 I am not 100% sure of my results (simply as the magnitude of the differences from the results are much larger than in the rest of the paper) but I have checked my codes a number of times and they seem fine. I can rationalize the very negative average utility with medical shocks for low social security replacement rates as follows: there is a non-zero probability that someone is unemployed their entire working life and then suffers a medical shock every period of their retirement, this is obviously a terrible outcome and when $b < 0.3$ the pension income is insufficient to cover these medical shocks meaning that agents must all save large amounts to avoid reaching consumption of zero, meaning they get very meager consumption for their whole working life and hence very low (high negative numbers) utility.

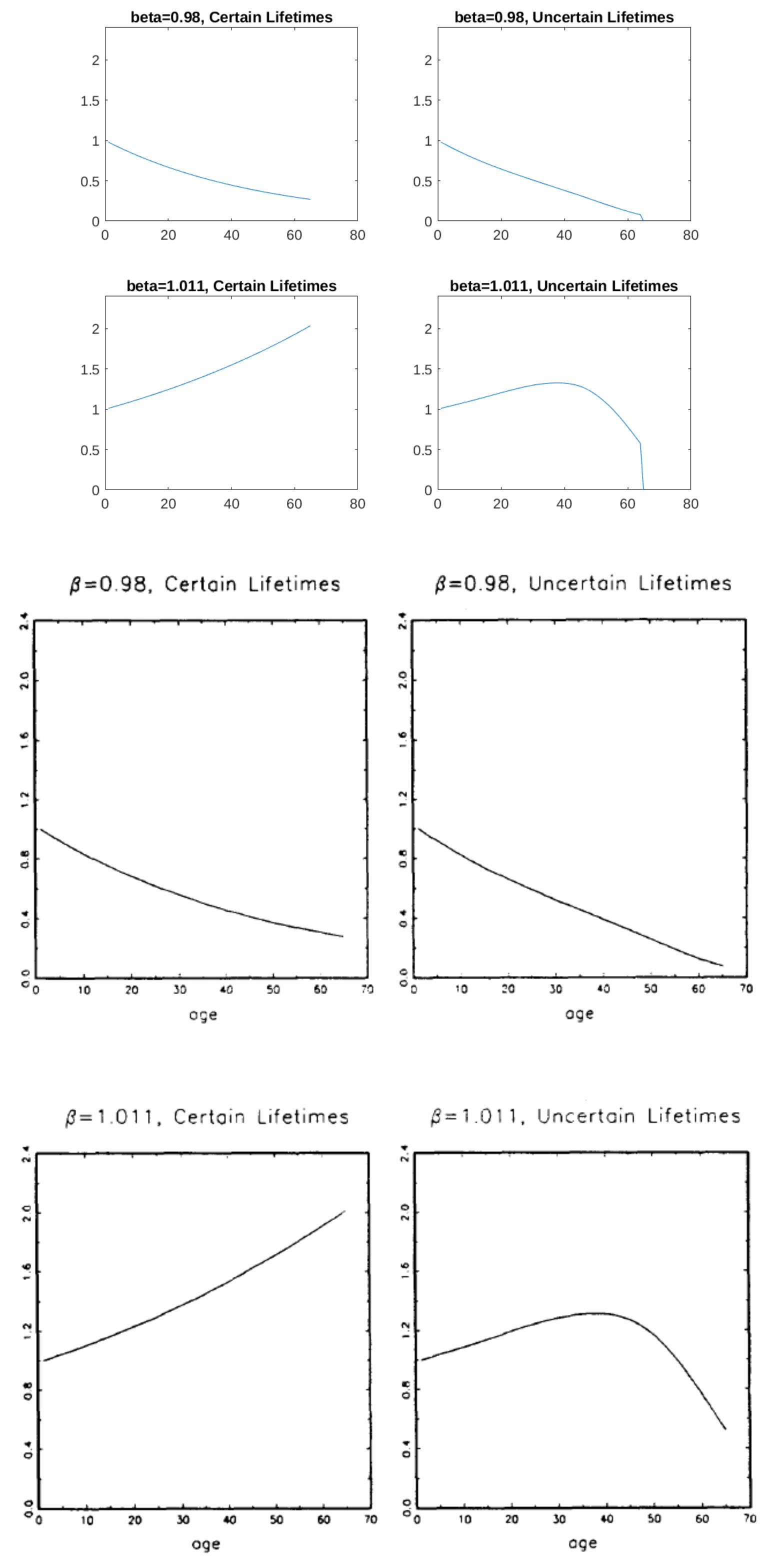


Figure 1: Figure 1 of Imrohoroglu, Imrohoroglu and Joines (1995)

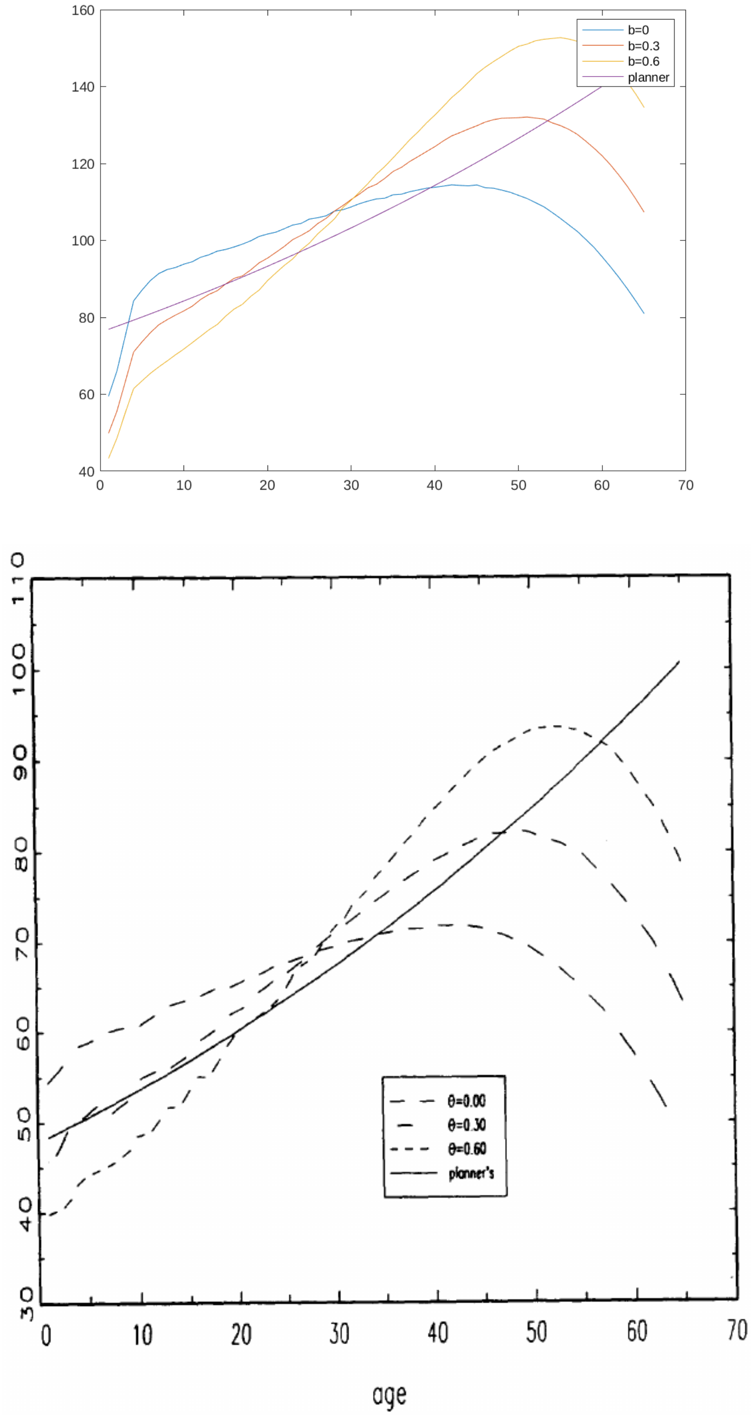
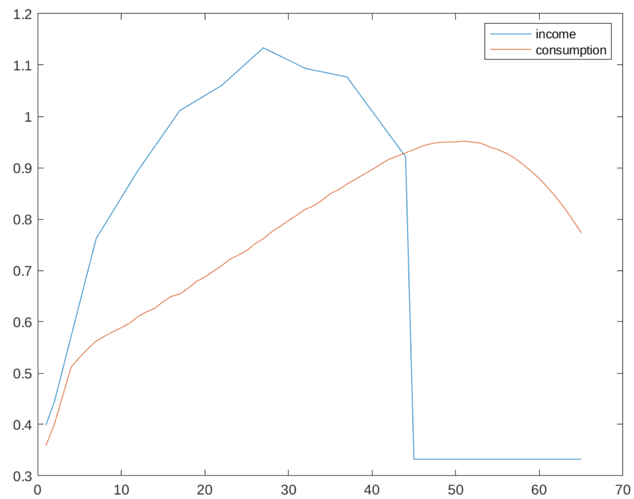


Figure 2: Figure 2 of Imrohoroglu, Imrohoroglu and Joines (1995)



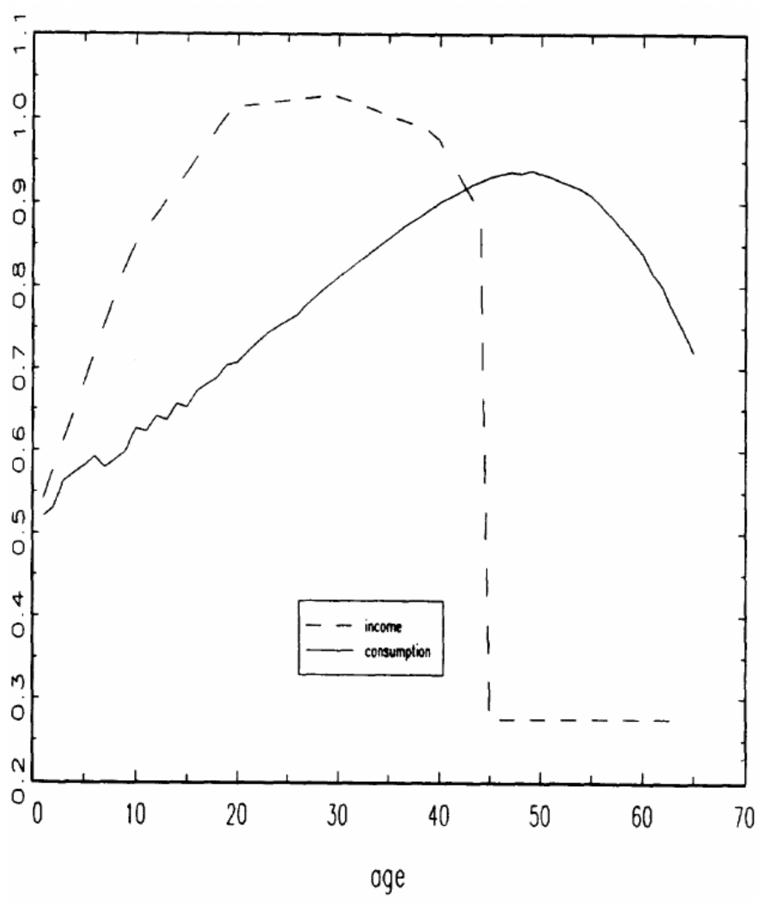


Figure 3: Figure 3 of Imrohoroglu, Imrohoroglu and Joines (1995)

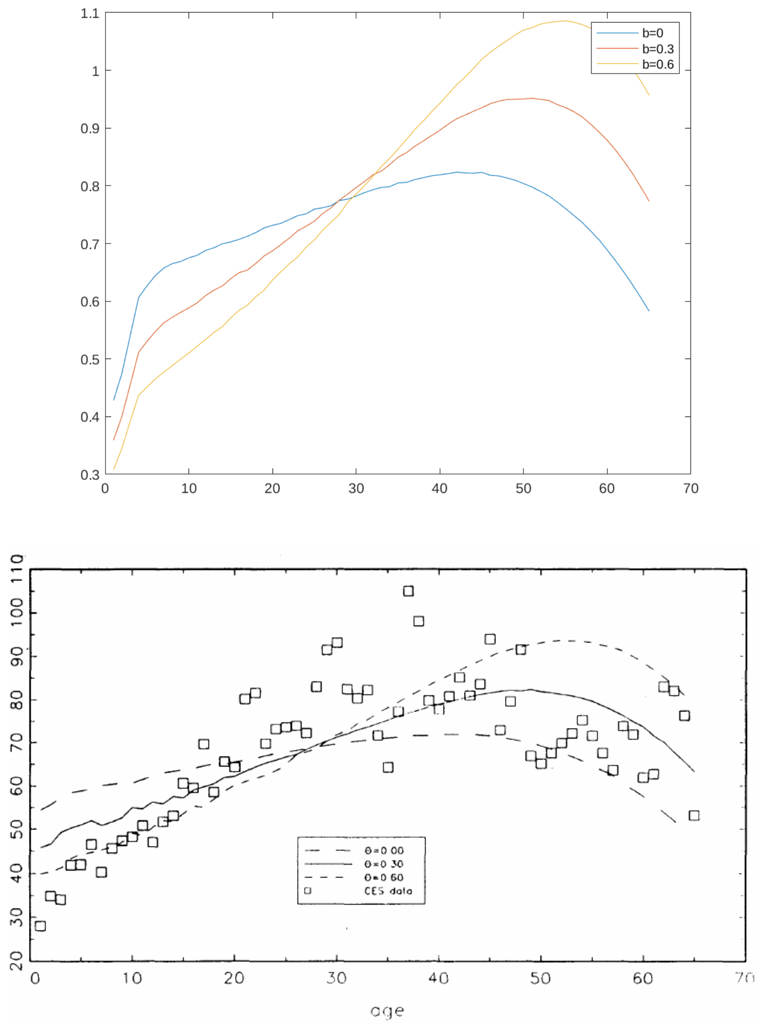


Figure 4: Figure 4 of Imrohoroglu, Imrohoroglu and Joines (1995)

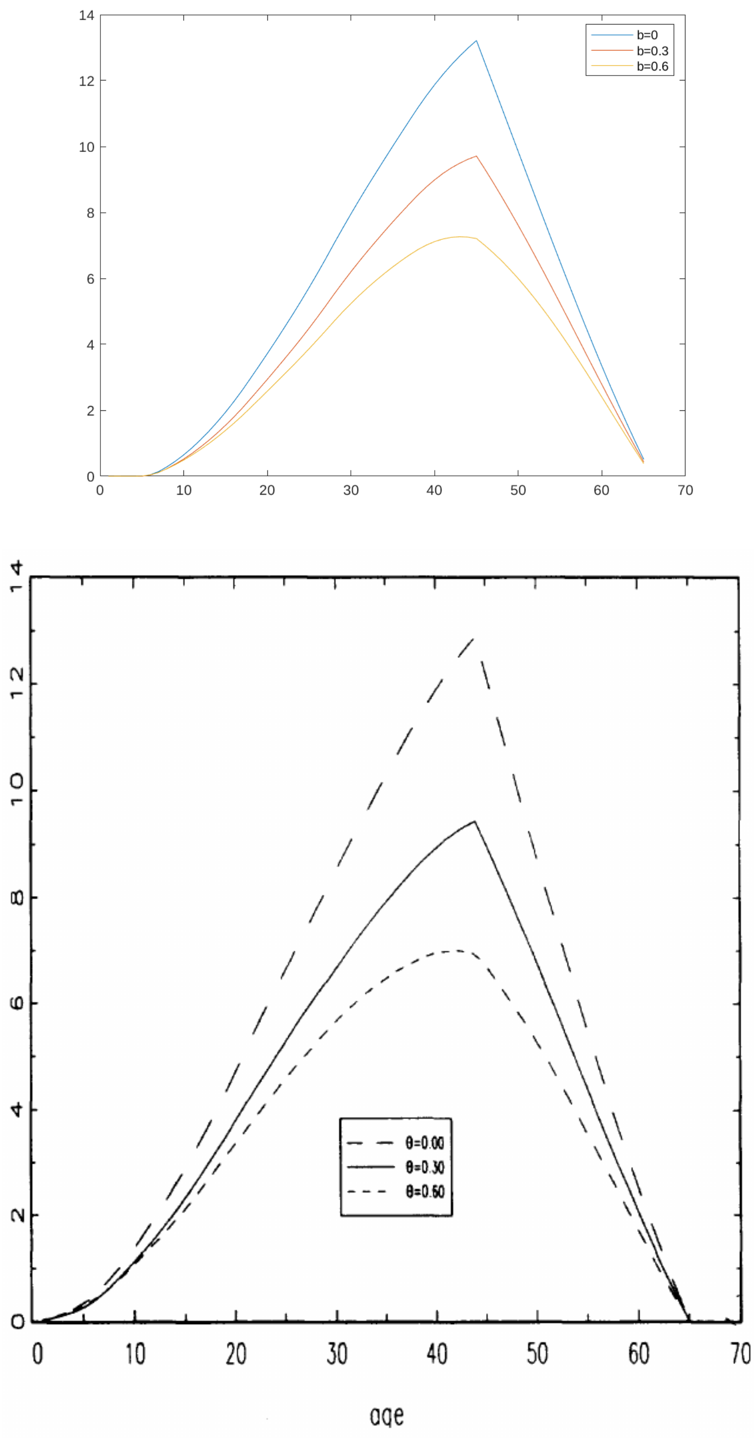


Figure 5: Figure 5 of Imrohoroglu, Imrohoroglu and Joines (1995)

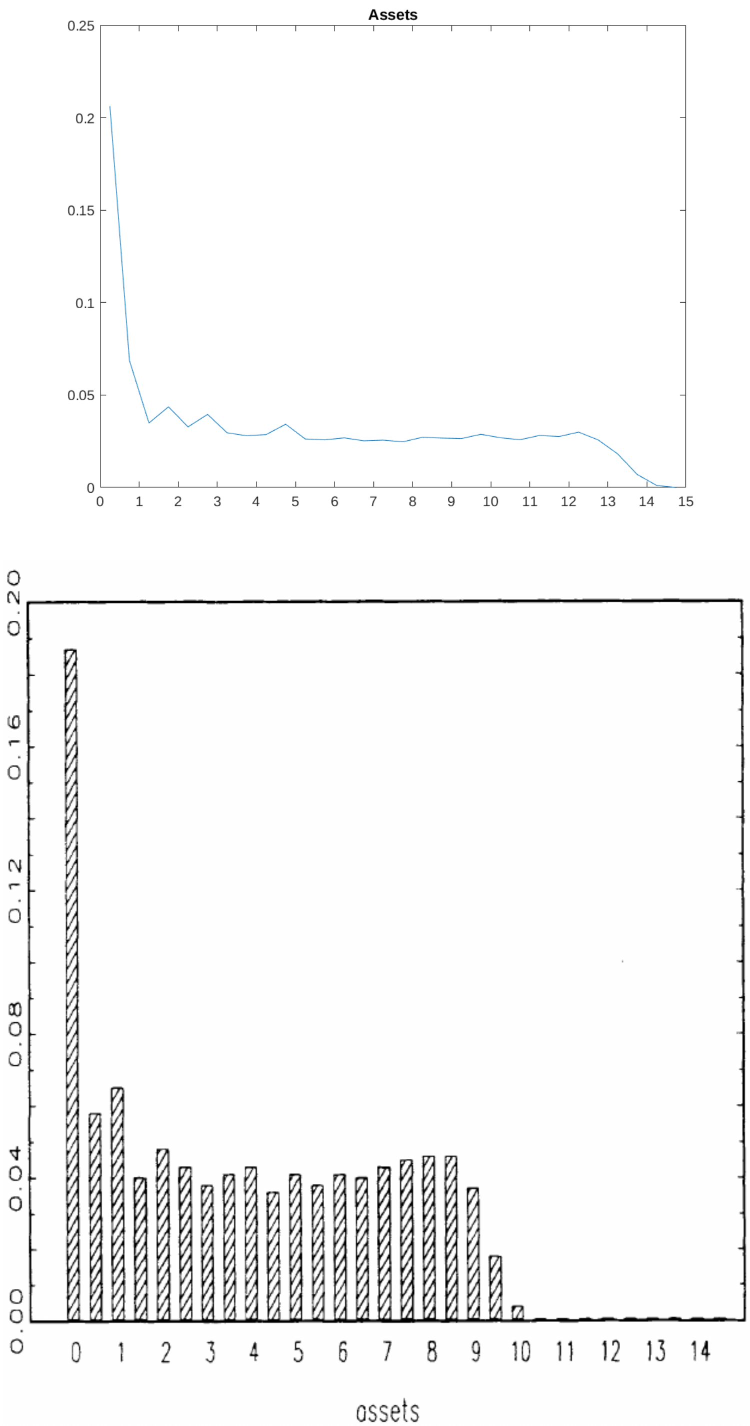


Figure 6: Figure 6 of Imrohoroglu, Imrohoroglu and Joines (1995)

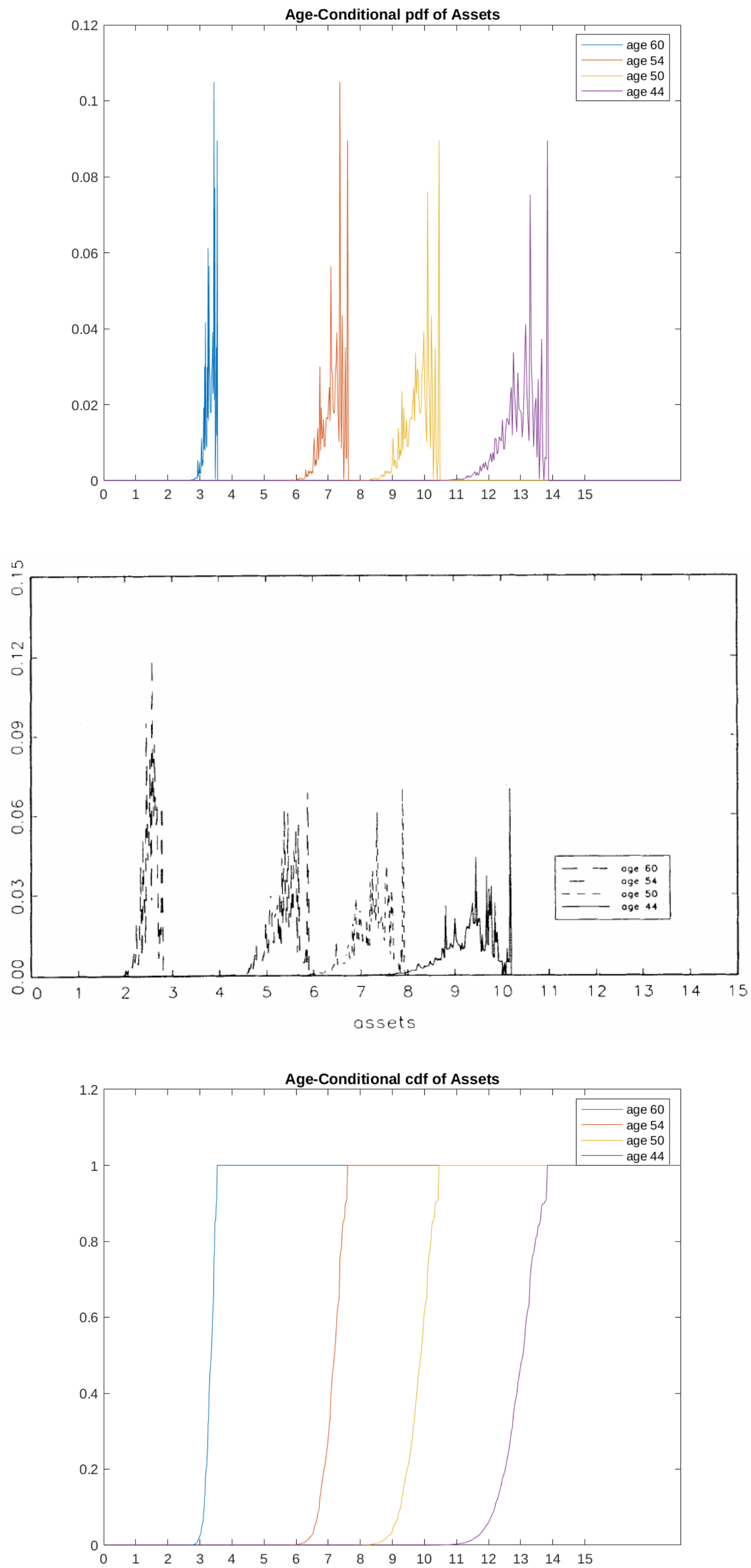


Figure 7: Figure 7 of Imrohoroglu, Imrohoroglu and Joines (1995)

Figure 8: Original Table 1 of Imrohoroglu, Imrohoroglu and Joines (1995)

Table 1. Population growth and lifetime uncertainty, $\beta = 1.011$, $\gamma = 2$

θ	Tax rate	Wage rate	Return to capital	Average consumption	Capital stock	Average income	Agerage utility
0.00	0.000	2.236	0.004	0.740	5.224	1.220	−97.859
0.10	0.020	2.161	0.009	0.742	4.751	1.179	−96.293
0.20	0.041	2.096	0.014	0.742	4.365	1.143	−95.476
0.30	0.061	2.038	0.019	0.741	4.060	1.114	$\boxed{-95.175}$
0.40	0.081	1.989	0.024	0.738	3.772	1.085	−95.339
0.50	0.102	1.947	0.028	0.735	3.553	1.062	−95.801
0.60	0.122	1.907	0.032	0.732	3.358	1.040	−96.548
1.00	0.203	1.781	0.046	0.716	2.773	0.971	−101.570

Table 1: Table 1 of Imrohoroglu, Imrohoroglu and Joines (1995)
Population growth and lifetime uncertainty, $\beta = 1.011$, $\gamma = 2$

b	Tax rate	Wage rate	Return to capital	Average consumption	Capital stock	Average income	Average utility
0.00	0.000	2.208	0.006	0.720	5.066	1.210	-99.979
0.10	0.021	2.133	0.011	0.722	4.602	1.169	-98.913
0.20	0.043	2.068	0.017	0.722	4.226	1.134	-98.588
0.30	0.064	2.012	0.021	0.720	3.913	1.103	-98.806
0.40	0.085	1.961	0.026	0.718	3.644	1.075	-99.442
0.50	0.107	1.916	0.031	0.715	3.414	1.050	-100.425
0.60	0.128	1.876	0.035	0.711	3.214	1.028	-101.688
1.00	0.214	1.744	0.051	0.694	2.613	0.954	-109.073

Note: b is the social security replacement rate (IIJ1995 call this θ).

Figure 9: Original Table 2 of Imrohoroglu, Imrohoroglu and Joines (1995)

Table 2. The role of population growth and lifetime uncertainty

	Zero population growth certain lifetimes			Population growth certain lifetimes			Population growth lifetime uncertainty		
θ	K/Q	r	Utility	K/Q	r	Utility	K/Q	r	Utility
0.00	5.734	−0.017	−150.91	5.163	−0.010	−147.57	4.282	0.004	−97.86
0.10	5.248	−0.011	−144.91	4.844	−0.006	−141.31	4.030	0.009	−96.29
0.20	4.766	−0.004	−142.14	4.488	0.000	−137.28	3.818	0.014	−95.48
0.30	4.371	0.002	$\boxed{-141.85}$	4.192	0.006	−135.18	3.644	0.019	$\boxed{-95.18}$
0.40	4.026	0.009	−143.53	3.943	0.011	$\boxed{-134.48}$	3.477	0.024	−95.34
0.50	3.739	0.016	−146.90	3.730	0.017	−134.88	3.346	0.028	−95.80
0.60	3.492	0.023	−151.86	3.530	0.022	−136.21	3.228	0.032	−96.55

Table 2: Table 2 of Imrohoroglu, Imrohoroglu and Joines (1995)
The role of population growth and lifetime uncertainty

	Zero population growth certain lifetimes			Population growth certain lifetimes			Population growth lifetime uncertainty		
b	K/Q	r	Utility	K/Q	r	Utility	K/Q	r	Utility
0.00	5.663	-0.016	-152.72	5.084	-0.009	-149.64	4.185	0.006	-99.98
0.10	5.100	-0.009	-147.44	4.684	-0.003	-143.83	3.935	0.011	-98.91
0.20	4.636	-0.002	-145.63	4.350	0.003	-140.67	3.727	0.017	-98.59
0.30	4.244	0.005	-146.33	4.061	0.009	-139.36	3.548	0.021	-98.81
0.40	3.910	0.012	-149.08	3.815	0.014	-139.46	3.390	0.026	-99.44
0.50	3.624	0.019	-153.66	3.600	0.020	-140.69	3.251	0.031	-100.43
0.60	3.376	0.027	-160.04	3.413	0.026	-142.90	3.128	0.035	-101.69

Note: b is the social security replacement rate (IIJ1995 call this θ).

Table 3: Table 3 of Imrohoroglu, Imrohoroglu and Joines (1995)
Welfare benefits of social security

b	0.10	0.20	0.30	0.40	0.50	0.60	1.0
κ	0.0013	0.0033	0.0061	0.0042	0.0001	-0.0011	-0.0359

Note: b is the social security replacement rate (IIJ1995 call this θ). κ is the welfare benefit of introducing social security relative to a situation of no social security, measured by equivalent variation and expressed as a fraction of aggregate income.

Figure 10: Original Table 3 of Imrohoroglu, Imrohoroglu and Joines (1995)

Table 3. The welfare benefits of social security

θ	0.10	0.20	.30	0.40	0.50	0.60	1.0
κ	0.0120	0.0184	0.0208	0.0195	0.0158	0.0100	−0.0268

Table 4: Table 4 of Imrohoroglu, Imrohoroglu and Joines (1995)
The role of intertemporal elasticity of substitution

	$1/\gamma = 0.67$ $(\gamma = 1.5)$		$1/\gamma = 0.50$ $(\gamma = 2.0)$		$1/\gamma = 0.25$ $(\gamma = 4.0)$	
b	K/Q	Utility	K/Q	Utility	K/Q	Utility
0.00	4.224	-170.54	4.185	-99.98	4.080	-66.35
0.10	4.022	-169.40	3.935	-98.91	3.681	-67.93
0.20	3.850	-168.76	3.727	-98.59	3.352	-72.72
0.30	3.701	-168.51	3.548	-98.81	3.079	-80.12
0.40	3.567	-168.54	3.390	-99.44	2.852	-89.83
0.50	3.447	-168.83	3.251	-100.43	2.660	-101.64
0.60	3.340	-169.31	3.128	-101.69	2.497	-115.80

Note: b is the social security replacement rate (IIJ1995 call this θ).

Figure 11: Original Table 4 of Imrohoroglu, Imrohoroglu and Joines (1995)

Table 4. The role of intertemporal elasticity of substitution

	$1/\gamma = 0.67$ $(\gamma = 1.5)$		$1/\gamma = 0.50$ $(\gamma = 2.0)$		$1/\gamma = 0.25$ $(\gamma = 4.0)$	
θ	K/Q	Utility	K/Q	Utility	K/Q	Utility
0.00	4.295	−168.69	4.282	−97.86	4.233	−59.29
0.10	4.100	−167.27	4.030	−96.29	3.813	$\boxed{-57.52}$
0.20	3.936	−166.34	3.818	−95.48	3.464	−58.52
0.30	3.795	−165.78	3.644	$\boxed{-95.18}$	3.186	−61.50
0.40	3.664	−165.54	3.477	−95.34	2.963	−65.93
0.50	3.539	$\boxed{-165.49}$	3.346	−95.80	2.758	−72.01
0.60	3.429	−165.66	3.228	−96.55	2.612	−78.51

Figure 12: Original Table 5 of Imrohoroglu, Imrohoroglu and Joines (1995)

Table 5. The role of the discount factor and productivity growth

	$\beta = 0.98, g = 0.0$			$\beta = 1.011, g = 0.022$			$\beta = 1.011, g = 0.0$		
θ	K/Q	r	Utility	K/Q	r	Utility	K/Q	r	Utility
0.00	3.177	0.033	$\boxed{-44.95}$	3.057	0.038	$\boxed{-61.69}$	4.282	0.004	−97.86
0.10	3.025	0.039	−45.78	2.965	0.041	−61.70	4.030	0.009	−96.29
0.20	2.898	0.044	−46.71	2.870	0.045	−61.83	3.818	0.014	−95.48
0.30	2.787	0.049	−47.74	2.795	0.049	−62.06	3.644	0.019	$\boxed{-95.18}$
0.40	2.692	0.054	−48.84	2.734	0.052	−62.37	3.477	0.024	−95.34
0.50	3.616	0.058	−49.95	2.671	0.055	−62.75	3.346	0.028	−95.80
0.60	2.530	0.062	−51.25	2.616	0.058	−63.17	3.228	0.032	−96.55
1.00	2.294	0.077	−56.83	2.428	0.068	−65.38	2.856	0.046	−101.57

Table 5: Table 5 of Imrohoroglu, Imrohoroglu and Joines (1995)
The role of the discount factor and productivity growth

	$\beta = 0.98$,$g = 0.0$			$\beta = 1.011$,$g = 0.022$			$\beta = 1.011$,$g = 0.0$		
b	K/Q	r	Utility	K/Q	r	Utility	K/Q	r	Utility
0.00	3.205	0.038	-47.25	3.425	0.025	-100.93	4.185	0.006	-99.98
0.10	3.024	0.042	-48.16	3.310	0.029	-101.62	3.935	0.011	-98.91
0.20	2.854	0.045	-49.17	3.205	0.032	-102.53	3.727	0.017	-98.59
0.30	2.689	0.049	-50.24	3.109	0.036	-103.62	3.548	0.021	-98.81
0.40	2.530	0.052	-51.38	3.022	0.039	-104.87	3.390	0.026	-99.44
0.50	2.369	0.055	-52.59	2.939	0.042	-106.30	3.251	0.031	-100.43
0.60	2.211	0.057	-53.87	2.861	0.046	-107.90	3.128	0.035	-101.69
1.00	1.539	0.066	-59.72	2.595	0.058	-115.84	2.740	0.051	-109.07

Note: b is the social security replacement rate (IIJ1995 call this θ).

Figure 13: Original Table 6 of Imrohoroglu, Imrohoroglu and Joines (1995)

Table 6. Risk of catastrophic illness in old age

	Prob. of illness 0.18 cost 25%		Prob. of illness 0.09 cost 35%		No catastrophic illness	
θ	K/Q	Utility	K/Q	Utility	K/Q	Utility
0.00	4.371	−99.20	4.370	−98.97	4.282	−97.86
0.10	4.111	−97.34	4.104	−97.12	4.030	−96.29
0.20	3.900	−96.44	3.881	−96.10	3.817	−95.48
0.30	3.709	$\boxed{-95.96}$	3.702	$\boxed{-95.76}$	3.644	$\boxed{-95.18}$
0.40	3.536	−95.97	3.528	−95.79	3.477	−95.34
0.50	3.393	−96.33	3.388	−96.17	3.346	−95.80
0.60	3.272	−96.98	3.265	−96.85	3.228	−96.55

Table 6: Table 6 of Imrohoroglu, Imrohoroglu and Joines (1995)
Risk of catastrophic illness in old age

	Prob. of illness 0.18 cost 25%		Prob. of illness 0.09 cost 35%		No catastrophic illness	
b	K/Q	Utility	K/Q	Utility	K/Q	Utility
0.00	5.286	-1433.22	5.286	-1433.22	4.185	-99.98
0.10	5.281	-350.95	5.281	-350.95	3.935	-98.91
0.20	5.257	-235.66	5.257	-235.66	3.727	-98.59
0.30	5.279	-211.01	5.279	-211.01	3.548	-98.81
0.40	5.225	-149.06	5.225	-149.06	3.390	-99.44
0.50	5.262	-156.83	5.262	-156.83	3.251	-100.43
0.60	4.957	-141.78	4.957	-141.78	3.128	-101.69

Note: b is the social security replacement rate (IIJ1995 call this θ).

References

Ayes Imrohoroglu, Selahattin Imrohoroglu, and Douglas Joines. A life cycle analysis of social security. Economic Theory, 6(1):83–114, 1995.