Chapter 11: Linear regression

(Section 2.1.1 and 3.1 of ISLR[1])

https://dzwang91.github.io/stat324/

[1] An Introduction to Statistical Learning with Applications in R

Outline

1. What is statistical learning
2. Pearson correlation coefficient
3. Simple linear regression
4. Estimating the coefficients
5. Assessing the accuracy of the coefficient estimates
6. Assessing the accuracy of the model
7. Simple linear regression in R

A motivating example: advertising data

- Suppose we are statistical consultants hired by a client to provide advice on how to improve sales of a particular product.
- The Advertising data set [2] consists of the sales of that product in 200 different markets, along with advertising budgets for the product in each of those markets for three different media: TV, radio, and newspaper.

[2] download the data set at http://www-bcf.usc.edu/~gareth/ISL/data.html

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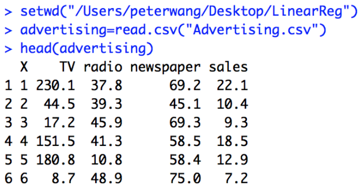

```
> setwd("/Users/peterwang/Desktop/LinearReg")
> advertising=read.csv("Advertising.csv")
> head(advertising)
  X    TV radio newspaper sales
1 1 230.1  37.8      69.2  22.1
2 2  44.5  39.3      45.1  10.4
3 3  17.2  45.9      69.3   9.3
4 4 151.5  41.3      58.5  18.5
5 5 180.8  10.8      58.4  12.9
6 6   8.7  48.9      75.0   7.2
```

Figure: Advertising data

[2] download the data set at http://www-bcf.usc.edu/~gareth/ISL/data.html

A motivating example: advertising data

- It is impossible for our client to directly increase sales of the product. But they can control the advertising expenditure in each of the three media.
- Therefore, if we determine that there is an association between advertising and sales, then we can instruct our client to adjust advertising budgets, thereby indirectly increasing sales.
- In other words, our goal is to develop an accurate model that can be used to predict sales on the basis of the three media budgets.

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We want to find a good function f such that

$$\text{Sales} \approx f(\text{TV, Radio, Newspaper})$$

Input variable and output variable

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- In general, suppose we have a quantitative response Y and p different predictors, $X_1, ..., X_p$. Then

$$Y = f(X_1, ..., X_p) + \epsilon$$

  - f is some fixed but unknown function of $X_1, ..., X_p$, and it's called regression function
  - f represents the systematic information that X provides about Y
  - ϵ is a random error term, independent of X and has mean 0

Example: income data

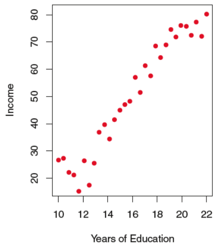

Figure: The red dots are the observed values of income (in tens of thousands of dollars) and years of education for 30 individuals.

Example: income data

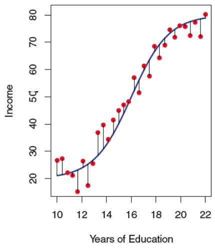

Figure: The blue curve represents the underlying relationship between income and years of education.

$$\text{Income} = f(\text{Years of education}) + \epsilon$$

Why estimate f: prediction

- In many situations, a set of input X are readily available, but the output Y cannot be easily obtained.

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- In many situations, a set of input X are readily available, but the output Y cannot be easily obtained.
- We can predict Y using
$$\hat{Y} = \hat{f}(X)$$

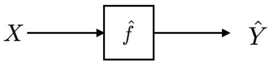

Figure: $\hat{f}$ is treated as a black box

  - $\hat{f}$ represents our estimate for f
  - $\hat{Y}$ represents the prediction of Y

The accuracy of prediction

- We use mean squared error (MSE) to measure the accuracy of the prediction:
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- MSE decomposition:
$$\begin{aligned}\text{MSE} &= \mathbb{E}(f(X) + \epsilon - \hat{f}(X))^2 \\ &= \mathbb{E}[(f(X) - \hat{f}(X))^2 + 2(f(X) - \hat{f}(X))\epsilon + \epsilon^2] \\ &= (f(X) - \hat{f}(X))^2 + 2(f(X) - \hat{f}(X))\mathbb{E}(\epsilon) + \mathbb{E}(\epsilon^2) \\ &= \underbrace{(f(X) - \hat{f}(X))^2}_{\text{Reducible}} + \underbrace{Var(\epsilon)}_{\textit{Irreducible}}\end{aligned}$$

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- Our goal for estimating f is to minimize the reducible error

Why estimate f: inference

- We are interested in
  - which predictors are associated with the response? Only a small fraction of the available predictors are substantially associated with Y, want to identify the important predictors
  - what is the relationship between the response and each predictor?
  - Can the relationship between Y and each predictor be adequately summarized using a linear equation, or is the relationship more complicated?
- Example: for the advertising data, we are interested in
  - which media contribute to sales?
  - how much increase in sales is associated with a given increase in TV advertising?

Next step of statistical learning

How can we estimate f?

Next step of statistical learning

How can we estimate f?

The most popular method: linear regression

Back to advertising data

What is the relationship between TV/Radio/newspaper advertising budgets and sales?

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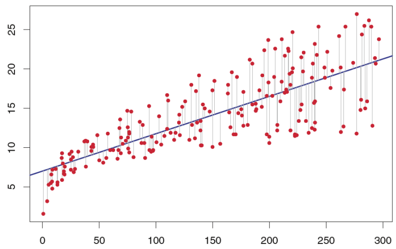

“All models are wrong, but some are useful”
George Box

- True regression functions are never linear!

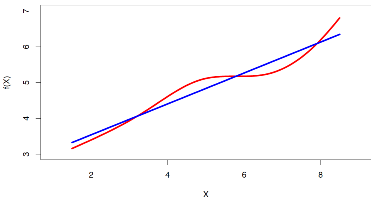

Figure: the red curve is the true regression function

- But linear regression is easy to implement and interpret!

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Motivation

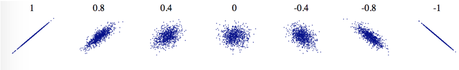
A natural question: for two random variables X and Y, how can we measure their association?

Pearson correlation coefficient

- For two random variables X and Y, Pearson correlation coefficient is defined as
$$\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y},$$
where
  - $cov(X,Y) = \mathbb{E}(X-\mu_X)(Y-\mu_Y)$, and μ_X and μ_Y are expectations of X and Y.
  - $\sigma_X = \sqrt{Var(X)}$: standard deviation of X
  - $\sigma_Y = \sqrt{Var(Y)}$: standard deviation of Y
- Range of $\rho_{X,Y}$:
$$-1 \leq \rho_{X,Y} \leq 1$$

Pearson correlation coefficient continued

- Visualization of $\rho_{X,Y}$:



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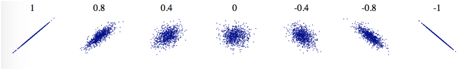

- When $\rho_{X,Y} = 1$, scatter is perfect straight line sloping up
- When $\rho_{X,Y} = -1$, scatter is perfect straight line sloping down
- When $\rho_{X,Y} = 0$, there is no linear association, then we call X and Y are uncorrelated.

Pearson correlation coefficient continued

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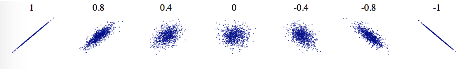

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Pearson correlation coefficient continued

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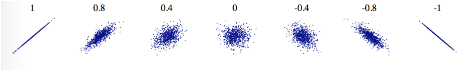

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- Conclusion: Pearson correlation coefficient measures the linear association
- When $\rho_{X,Y} > 0$, we say X and Y have a positive linear association
- When $\rho_{X,Y} < 0$, we say X and Y have a negative linear association

Sample Pearson correlation coefficient

- Given n pairs of data $(x_1, y_1), \ldots, (x_n, y_n)$, sample Pearson correlation coefficient is defined as

$$r_{xy} = \frac{S_{xy}}{S_x S_y} = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$
$$= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

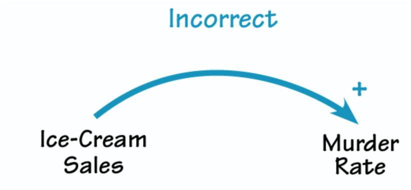
Correlation is not causation[3]

- A famous example: Ice cream sales is correlated with homicides in New York,

[3] Reading: Why correlation does not imply causation?

Correlation is not causation[3]

- A famous example: Ice cream sales is correlated with homicides in New York, but ice cream is not causing the death of people.


Incorrect

+

Ice-Cream Sales

Murder Rate

[3] Reading: Why correlation does not imply causation?

Correlation is not causation continued

- Why are ice cream sales and homicides correlated?

Correlation is not causation continued

- Why are ice cream sales and homicides correlated?
- There are some hidden factors which cause both of ice cream sales and homicides.

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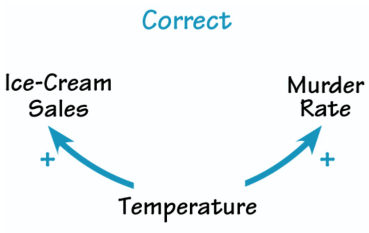

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Example

- Sir Francis Galton (1822-1911) was interested in how children resemble their parents. One simple measure of this is height.
- Galton measured the heights of father son pairs (in inches) at maturity.
- In the actual study, 1078 pairs were measured. For convenience, we will use a small subsample of $n = 14$ pairs:

Example continued

Family	Father's Height	Son's Height
1	71.3	68.9
2	65.5	67.5
3	65.9	65.4
4	68.6	68.2
5	71.4	71.5
6	68.4	67.6
7	65.0	65.0
8	66.3	67.0
9	68.0	65.3
10	67.3	65.5
11	67.0	69.8
12	69.3	70.9
13	70.1	68.9
14	66.9	70.2

- Goal: predict sons' height from father's height.

Example continued

- Which variable is input variable?

Example continued

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- A simple linear regression:
$$\text{Son's height} = \beta_0 + \beta_1 * \text{ Father's height} + \text{Random error}$$
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$$Y = \beta_0 + \beta_1 X + \epsilon$$
- Denote the height of son i by y_i, the height of father i by x_i, and the random error by ϵ_i, so that the model becomes:
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- Why do we add a random error term? The random error term picks up sources of variation in an individual son's height that are not due to his father's height (mother's genetics, environmental factors, etc.) and which cause the points to be "off line."

Intercept and slope

- β_0 is the intercept. It is the expected value of Y when $X = 0$.
- β_1 is the slope. It is the average increase of Y associated with a one-unit increase in X.
- Our goal: estimate the values of β_0 and β_1 from data. (what is the available (training) data?)

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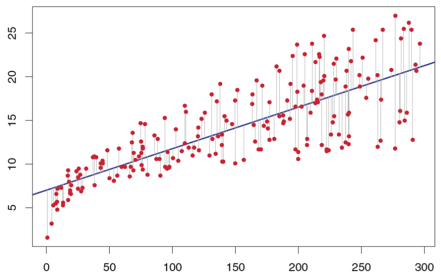
Simple linear regression

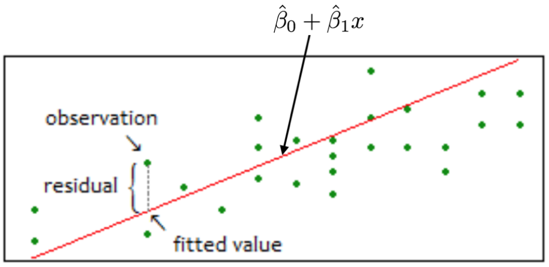
- We assume a model
$$Y = \beta_0 + \beta_1 X + \epsilon$$
- Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n observation pairs, each of which consists of a measurement of X and a measurement of Y.



Residual sum of squares

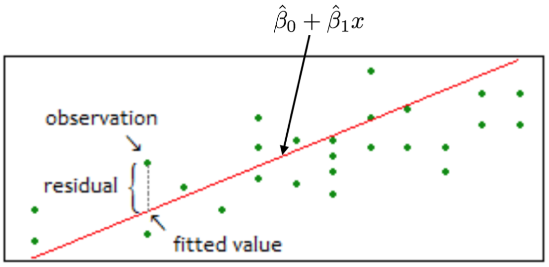
- Suppose $\hat{\beta}_0$ and $\hat{\beta}_1$ are estimates, then the estimated (fitted) value for given x_i is:
$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$
- The i-th residual: the difference between $\hat{y}_i$ and the observed y_i.
$$e_i = y_i - \hat{y}_i.$$
- Residual sum of squares (RSS):
$$\text{RSS} = e_1^2 + e_2^2 + \ldots + e_n^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2.$$
- RSS measures how well the line fits the data.

Ordinary least squares



- How can we decide the line?

Ordinary least squares



- How can we decide the line?
- Ordinary least squares (OLS):
$$\text{minimize RSS}(\hat{\beta}_0, \hat{\beta}_1) = \sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2$$

OLS estimator

- OLS estimator:
$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2},$$
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x},$$
where $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$.
- We call $\hat{\beta}_0 + \hat{\beta}_1 x$ the least squares/best fit/regression line.
- The residual sum of squares for the least squares line is also called the sum of squared errors (SSE). SSE is the smallest possible residual sum of squares in the universe of all possible lines.

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- Exercise: calculate $\hat{\beta}_0$ and $\hat{\beta}_1$ for the father and son data. ($\hat{\beta}_1 = 0.65$ and $\hat{\beta}_0 = 23.64$)

Back to Pearson correlation

- OLS estimator of slope:
$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{S_{xy}}{S_x^2} = \frac{S_{xy}}{S_x S_y}\frac{S_x S_y}{S_x^2} = r_{xy}\frac{S_y}{S_x}$$

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- We have
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- Conclusion: r_{xy} is the slope of the regression line for standardized data points.

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Unbiasedness

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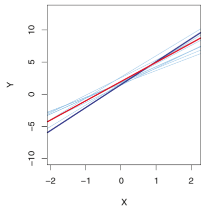

Figure: The simple linear regression is $Y = 2 + 3X + \epsilon$. The red line is the true regression function $2 + 3X$. The light blue lines are least squares lines for different sample. On average, the least squares lines are close to the true regression function.

Standard error of OLS estimator [4]

- Standard error of $\hat{\beta}_0$:

$$SE(\hat{\beta}_0) = \sqrt{Var(\hat{\beta}_0)} = \sqrt{\sigma^2[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}]}$$

where $\sigma^2 = Var(\epsilon)$.

- Standard error of $\hat{\beta}_1$:

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[4] Proofs are not required. See the extra slides in our course website.

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- When σ is unknown, we need to estimate the standard error. Replace σ^2 by its estimator $\frac{SSE}{n-2}$ where $SSE = \sum_{i=1}^n e_i^2$

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Confidence interval

- If
  1. The linear model is correct.
  2. The observations are independent.
  3. The variance around the true regression line is constant for all values of x.
  4. The random error around the true line is normal.

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- Therefore,
  $$\mathbb{P}\left(-t_{n-2,\alpha/2} \le \frac{\hat{\beta}_1 - \beta_1}{\widehat{SE}(\hat{\beta}_1)} \le t_{n-2,\alpha/2}\right) = 1 - \alpha$$

Confidence interval continued [5]

- $100(1-\alpha)\%$ confidence interval of β_1 is
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- Similarly, the $100(1-\alpha)\%$ confidence interval of β_0 is

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Hypothesis testing

- The most common hypothesis test in the simple linear regression is

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$$H_0: \beta_1 = 0 \text{ vs. } H_A: \beta_1 \neq 0,$$

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- Exercise: For the father and son data, $\hat{\sigma} = 1.78$, so $\widehat{SE(\hat{\beta}_1)} = 0.24$, and $t_{obs} = 2.70$. Comparing this to a t_{12}, the p-value is 0.0193. So we would reject at the 5% level, and conclude that father's height is related to son's height.

Outline

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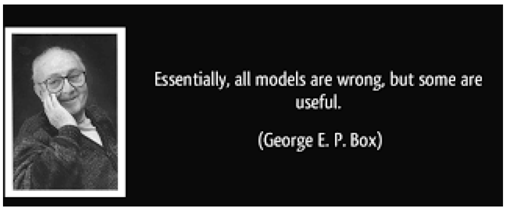
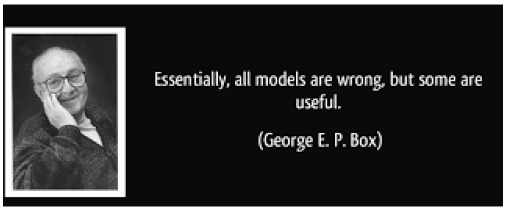


Essentially, all models are wrong, but some are useful.

(George E. P. Box)



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How good does the linear model fit the data?

R squared

- Total sum of squares:

$$SSTot = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

- Regression sum of squares:

$$SSReg = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$$

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- Sum of squares law:

$$\mathrm{SSTot} = \mathrm{SSReg} + \mathrm{SSE}$$

R squared continued

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- Exercise: for the father and son data, $R^2 = 0.38$. So we can say that about 38% of the variability in sons' heights can be explained by fathers' heights.

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Lab: simple linear regression

See R code at our course website.