Chapter 9: Comparing two paired populations

(Ott & Longnecker Sections: 6.4)

https://dzwang91.github.io/stat324/

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Outline

1. Paired t test
2. Paired sign test
3. Summary

Example

- Musculoskeletal disorders of the neck and shoulders are common in office workers because of repetitive tasks. Long periods of upper-arm elevation above 30 degrees have been shown to be related to disorders. It was thought that varying working conditions over the course of the day could alleviate some of these problems.
- "Can the diversification change the percentage of time where arm angle is below 30 degrees? "

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- "Can the diversification change the percentage of time where arm angle is below 30 degrees? "
- 8 office workers were randomly selected. They were observed for one work day under the standard conditions, and the percentage of time that their dominant upper-arm was below 30 degrees was recorded. The next day, these same individuals had their work diversified, and again were observed.

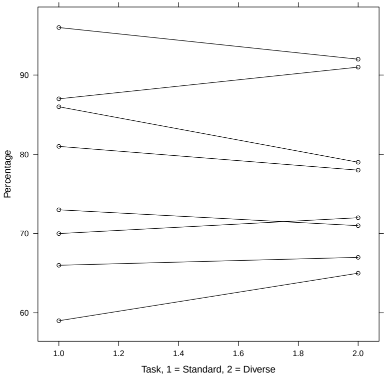
Example continued

- The results are:

Participant	1	2	3	4	5	6	7	8
Diverse	78	91	79	65	67	72	71	96
Standard	81	87	86	59	66	70	73	92

Example continued

- Dotplots connected for each individual:



Some go up, some go down, and some stay the same. It's unlikely that we will see much difference.

Example continued

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- Can we use two-sample t test or Welch t test?
  NO, because of the dependence. The data is clearly paired, since we are taking two measurements on each individual.

Paired t test

- The data consists of paired observations. Let:
  - $X_{1,i}$ = i-th data point from population 1
  - μ_1 = true mean of population 1
  - $X_{2,i}$ = i-th data point from population 2
  - μ_2 = true mean of population 2
  - $D_i = X_{1,i} - X_{2,i}$ = the difference for pair i
  - n = number of pairs
- We wish to test: $H_0 : \mu_1 - \mu_2 = \delta$ vs. $H_A : \mu_1 - \mu_2 \neq \delta$.

Paired t test continued

- If we can assume:
    - All of the *pairs* are indepdendent.
    - The differences follow a normal distribution. (how do we check?)

  Then the test statistic is:

$$t = \frac{\bar{D} - \delta}{\frac{S_D}{\sqrt{n}}}$$

  Where $\bar{D}$ is the mean of the differences and S_D is the sample standard deviation of the differences.

- Under H_0, test statistic is T distribution with $n-1$ degrees of freedom. (what is p-value and rejection region?)

Paired t test continued

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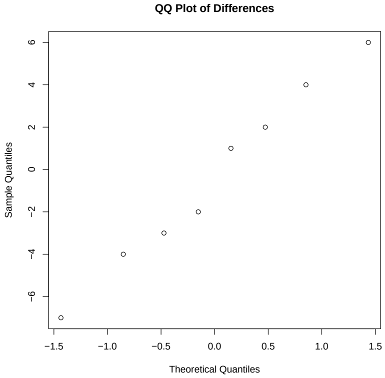
  Where $\bar{D}$ is the mean of the differences and S_D is the sample standard deviation of the differences.
- Under H_0, test statistic is T distribution with $n - 1$ degrees of freedom. (what is p-value and rejection region?)
- Paired t test is indeed one sample t test on the differences!

Back to example

- diverse - standard: -3, 4, -7, 6, 1, 2, -2, 4.

Back to example

- diverse - standard: -3, 4, -7, 6, 1, 2, -2, 4.
- check normality of the differences:



This looks quite straight, so we can assume normality.

Back to example

- For the example, $\bar{D} = 0.63$ and $S_D = 4.34$, so observation of test statistic is

$$t_{obs} = \frac{0.63}{\frac{4.34}{\sqrt{8}}} = 0.41.$$

Back to example

- For the example, $\bar{D} = 0.63$ and $S_D = 4.34$, so observation of test statistic is
$$t_{obs} = \frac{0.63}{\frac{4.34}{\sqrt{8}}} = 0.41.$$
- Compare to a T_7 distribution, p-value is about 0.702. Thus we would not reject. It seems that diversification of work has little effect on arm angle.

Paired t test in R

```
> rm(list=ls())
>
> #enter data#
> standard <- c(81, 87, 86, 59, 66, 70, 73, 92)
> diverse  <- c(78, 91, 79, 65, 67, 72, 71, 96)
>
> #reformat#
> all <- c(standard, diverse)
> task <- c(rep(1, 8), rep(2, 8))
> person <- c(1:8, 1:8)
```

Figure: make a dataset.

Paired t test in R

```
> #line graph#
> library(lattice)
>
> xyplot(all ~ task, group = person, type = 'o', col = 1, ylab = "Percentage",
xlab = "Task, 1 = Standard, 2 = Diverse")
```

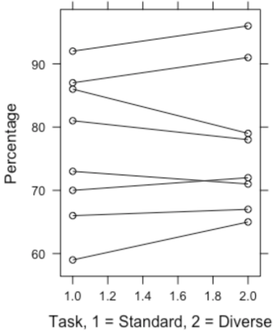


Figure: draw dotplot.

Paired t test in R

```
> #qq of diffs#
> diffs <- diverse - standard
> qqnorm(diffs, main = "QQ Plot of Differences")
```

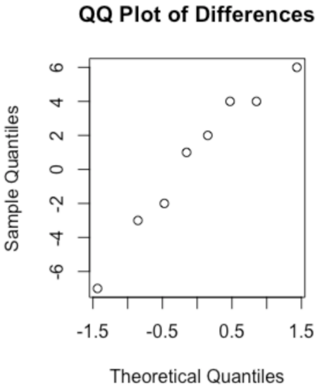


Figure: draw QQ plot.

Paired t test in R

```
> #Paired t test#
> #Output format is slightly different but these do the same thing#
> t.test(diverse, standard, paired = T)

        Paired t-test

data:  diverse and standard
t = 0.40728, df = 7, p-value = 0.696
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -3.003686  4.253686
sample estimates:
mean of the differences
                  0.625

> t.test(diffs)

        One Sample t-test

data:  diffs
t = 0.40728, df = 7, p-value = 0.696
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 -3.003686  4.253686
sample estimates:
mean of x
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```

Figure: paired t test.

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Paired sign test

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  - m_1 = true median of population 1
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  - n = number of pairs
  - σ_D^2 = true variance of the differences
- We wish to test: $H_0 : m_1 - m_2 = 0$ vs. $H_A : m_1 - m_2 \neq 0$.

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  - n = number of pairs
  - σ_D^2 = true variance of the differences
- We wish to test: $H_0 : m_1 - m_2 = 0$ vs. $H_A : m_1 - m_2 \neq 0$. Equivalently, let $m_D = m_1 - m_2$,

$$H_0 : m_D = 0, vs.\ H_A : m_D \neq 0$$

Paired sign test

- If $D_1, \ldots, D_n$ are an i.i.d. sample, then test statistic is

  $B =$ number of data values greater than 0 (Ignore values tied with 0).

- Note that if H_0 is true, then $B \sim \text{Binomial}(n^*, 0.5)$, where n^* is the number of data points not equal to 0.

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- Note that if H_0 is true, then $B \sim \text{Binomial}(n^*, 0.5)$, where n^* is the number of data points not equal to 0.
- P-value: Let b be the observation of B, then for two-sided alternative,

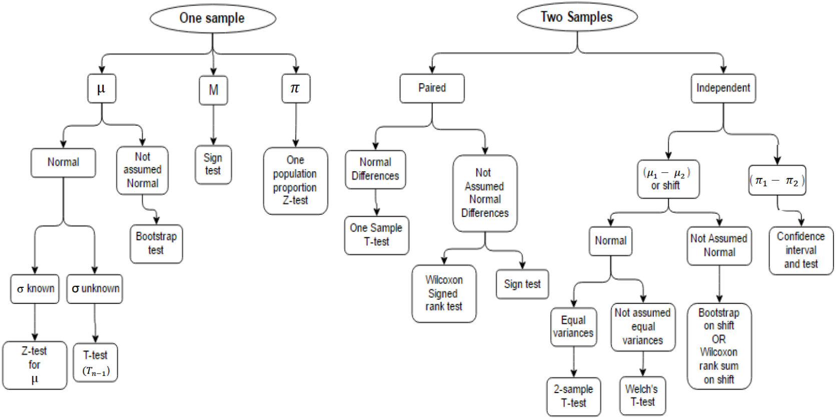
$$\text{p-value} = 2 \min\{P(B \geq b), P(B \leq b)\}.$$

Outline

Summary of paired tests

For paired populations, take difference to remove the dependence and then use one sample tests.

Summary of Chapter 7, 8 and 9



What's the next?

We'll discuss how to compare multiple independent populations after Midterm 2.