Chapter 7: One sample tests

(Ott & Longnecker Sections: 5.4-5.7)

Part 1

https://dzwang91.github.io/stat324/

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Review of hypothesis testing

1. State the null hypothesis H_0 and the alternative hypothesis H_A.
2. Choose a significance level α. Typically 0.05, 0.01.
3. Choose a test statistic $T_n(X_1, ..., X_n)$ and establish the rejection region.
4. Collect Data $X_1, ..., X_n$. Two approaches:
   - compute the realization of the test statistic. If the test statistic is in the rejection region, we reject H_0, otherwise we do not reject H_0.
   - compute p-value. If p-value is greater than the given significance level, we do not reject H_0. Otherwise, we reject H_0.

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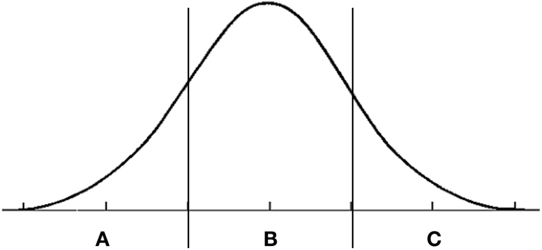
How do we choose the test statistic, rejection region and p-value?

One-tailed test (left tail)

$$H_0 : \theta = \theta_0 \text{ v.s. } H_A : \theta < \theta_0$$

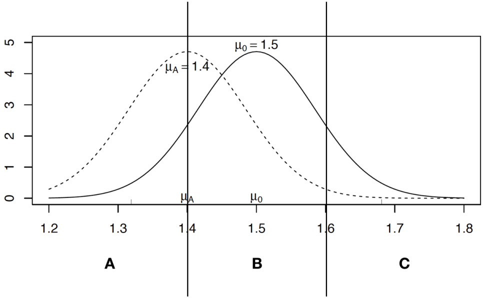
- If the following is the distribution of the test statistic given H_0 is true, then which part is the rejection region?



Example

$$H_0 : \mu = 1.5 \text{ v.s. } H_A : \mu < 1.5$$

- Consider a specific case of H_A: $\mu = 1.4$.



Shade the probability that test statistic realizes to region A (or B or C) under $\mu = 1.5$ and $\mu = 1.4$.

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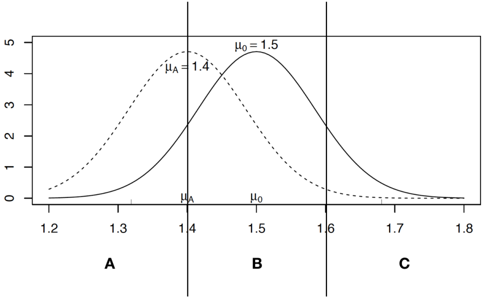

Shade the probability that test statistic realizes to region A (or B or C) under $\mu = 1.5$ and $\mu = 1.4$.

- Region A is the reject region.

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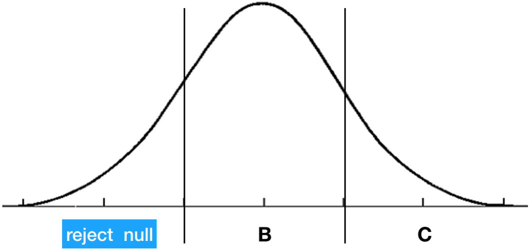

One-tailed test (right tail)

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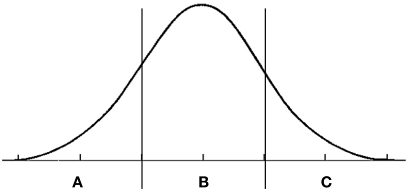

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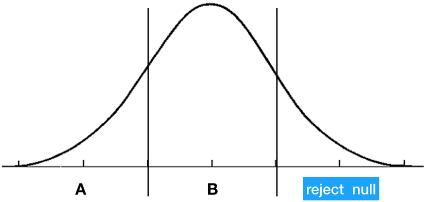

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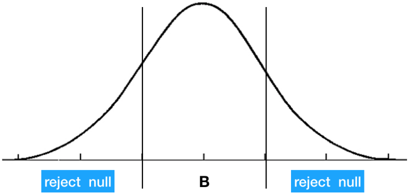
- Rejection region is in both of the left and right tails of the test statistic.



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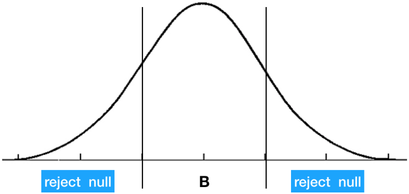


- Conclusion: The rejection region corresponds to the alternative hypothesis.

Outline

Population mean

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu \neq \mu_0$$

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu < \mu_0$$

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu > \mu_0$$

parametric tests

Sample mean test

Z test

T test

nonparametric test

Sign test

Sample mean test: use sampling distribution of $\bar{X}$

Two-tailed sample mean test

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu \neq \mu_0$$

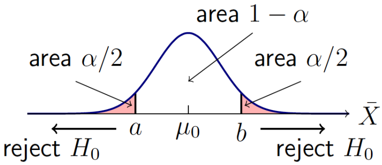
- We collect a sample of size n, $X_1, ..., X_n$ with known population variance σ^2.
- Test statistic: sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.
- Under H_0, the sampling distribution of $\bar{X}$ is approximately normal distributed with mean μ_0 and standard deviation $\sigma/\sqrt{n}$. (Why?)

Two-tailed sample mean test continued

How do we decide the rejection region?

- Use the distribution of the test statistic under H_0 to determine a rejection region that limits the type I error at significance level α:

$$\alpha = \mathbb{P}(\text{reject } H_0 \mid H_0 \text{ is true})$$



- Rejection region: $\bar{X} \geq b$ or $\bar{X} \leq a$

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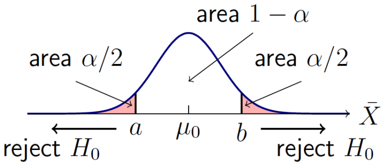


- Rejection region: $\bar{X} \geq b$ or $\bar{X} \leq a$

How do we decide a and b?

Two-tailed sample mean test continued

- Choice of b:

$$\frac{\alpha}{2} = \mathbb{P}(\bar{X} \geq b) = \mathbb{P}(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{b - \mu_0}{\sigma/\sqrt{n}})$$

$\Rightarrow$

$$\frac{b - \mu_0}{\sigma/\sqrt{n}} = z_{\alpha/2}$$

$\Rightarrow$

$$b = \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- Choice of a:

$$\frac{\alpha}{2} = \mathbb{P}(\bar{X} \leq a) = \mathbb{P}(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq \frac{a - \mu_0}{\sigma/\sqrt{n}})$$

$\Rightarrow$

$$\frac{a - \mu_0}{\sigma/\sqrt{n}} = -z_{\alpha/2}$$

$\Rightarrow$

$$a = \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Two-tailed sample mean test continued

How do we compute p-value?

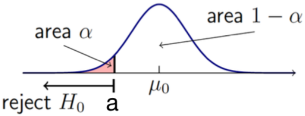
- p-value is the probability of a test statistic realizing to a value that is as or more extreme than the one actually observed given the null hypothesis is true.
- If we denote the realization of $\bar{X}$ is c, then p-value formula:

$$\text{p-value} = \mathbb{P}(\bar{X} \geq |c|) + \mathbb{P}(\bar{X} \leq -|c|) = 2\mathbb{P}(\bar{X} \geq |c|)$$

One-tailed sample mean test (left tail)

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu < \mu_0$$

- Assume the population variance σ^2 is known.
- Rejection region: $\bar{X} \leq a$



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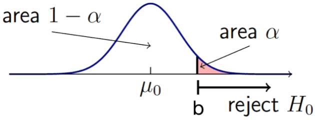
- Assume the population variance σ^2 is known.
- Rejection region: $\bar{X} \geq b$



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- If we denote the realization of $\bar{X}$ is c, then p-value formula:

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Z test: use sampling distribution of $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$

Z test

- Under $H_0 : \mu = \mu_0$,

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \approx N(0, 1)$$

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- When $H_A : \mu \neq \mu_0$, the rejection region is

$$Z > z_{\alpha/2} \text{ or } Z < -z_{\alpha/2}.$$

- When $H_A : \mu < \mu_0$, the rejection region is

$$Z < -z_{\alpha}.$$

- When $H_A : \mu > \mu_0$, the rejection region is

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- Conclusion: Z test is equivalent to sample mean test.

Example

A batch of 100 resistors have an average of 102 Ohms. Assuming a population standard deviation of 8 Ohms, test whether the population mean is 100 Ohms at a significance level $\alpha = 0.05$.

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- Step 1: $H_0 : \mu = 100$, $H_A : \mu \neq 100$.

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A batch of 100 resistors have an average of 102 Ohms. Assuming a population standard deviation of 8 Ohms, test whether the population mean is 100 Ohms at a significance level $\alpha = 0.05$.

- Step 1: $H_0 : \mu = 100$, $H_A : \mu \neq 100$.
- Step 2: Choose $\alpha = 0.05$

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- Step 1: $H_0 : \mu = 100$, $H_A : \mu \neq 100$.
- Step 2: Choose $\alpha = 0.05$
- Step 3: reject H_0 if $\bar{X} < a$ or $\bar{X} > b$ with

$$a = \mu_0 - z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 - 1.96 \frac{8}{10} = 98.432$$

$$b = \mu_0 + z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 + 1.96 \frac{8}{10} = 101.568$$

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$$b = \mu_0 + z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 + 1.96 \frac{8}{10} = 101.568$$

- Step 4: $\bar{X} = 102 > b$, therefore reject H_0.

Example

A quality control engineer finds that a sample of 100 light bulbs had an average life-time of 470 hours. Assuming a population standard deviation of $\sigma = 25$ hours, test whether the population mean is 480 hours vs. the alternative hypothesis $\mu < 480$ at a significance level of $\alpha = 0.05$.

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- Step 1: $H_0 : \mu = 480$, $H_A : \mu < 480$.
- Step 2: Choose $\alpha = 0.05$
- Step 3: reject H_0 if

$$\bar{X} < \mu_0 - z_{0.05}\frac{\sigma}{\sqrt{n}} = 480 - 1.645\frac{25}{10} = 475.9$$

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- Step 2: Choose $\alpha = 0.05$
- Step 3: reject H_0 if

$$\bar{X} < \mu_0 - z_{0.05}\frac{\sigma}{\sqrt{n}} = 480 - 1.645\frac{25}{10} = 475.9$$

- Step 4: $\bar{X} = 470 < 475.9$, therefore reject H_0.

What if the population standard deviation is unknown?

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T test

T test

- We collect a sample $X_1, ..., X_n$ from a normal distribution, but with unknown variance σ^2.
- Test statistic: $T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$, where s is the sample standard deviation
- Under $H_0 : \mu = \mu_0$, the sampling distribution of $\frac{\bar{X} - \mu_0}{s/\sqrt{n}}$ is a t-distribution with degrees of freedom $n - 1$.

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- Under $H_0 : \mu = \mu_0$, the sampling distribution of $\frac{\bar{X} - \mu_0}{s/\sqrt{n}}$ is a t-distribution with degrees of freedom $n - 1$.
- Rejection region:
  - When $H_A : \mu \neq \mu_0$: $T < -t_{\alpha/2, n-1}$ or $T > t_{\alpha/2, n-1}$.
  - When $H_A : \mu < \mu_0$: $T < -t_{\alpha, n-1}$.
  - When $H_A : \mu > \mu_0$: $T > t_{\alpha, n-1}$.

Example

Story

A paint shop uses an automatic device to apply paint to engine blocks. It is important that the amount applied is of a minimum thickness.

Primary Research Question

Its customer, a manufacturer wants to know the average thickness of paint in a warehouse. It is supposed to be 1.50mm.

$$\mu = 1.50\text{mm}?$$

Sampling

16 blocks are selected randomly and then measured from thousands of blocks in the warehouse.

$$n = 16$$

Data

1.29, 1.12, 0.88, 1.65, 1.48, 1.59, 1.04, 0.83, 1.76, 1.31, 0.88, 1.71, 1.83, 1.09, 1.62, 1.49

(in mm)

$$\bar{x} = 1.358$$

$$s = 0.3385$$

Example

- Normality?

Example

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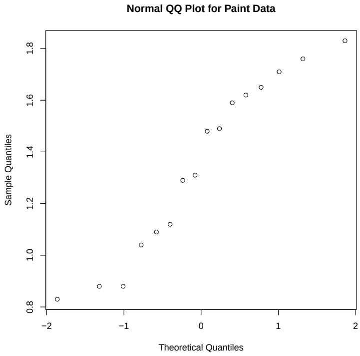


Example

- Step 1:

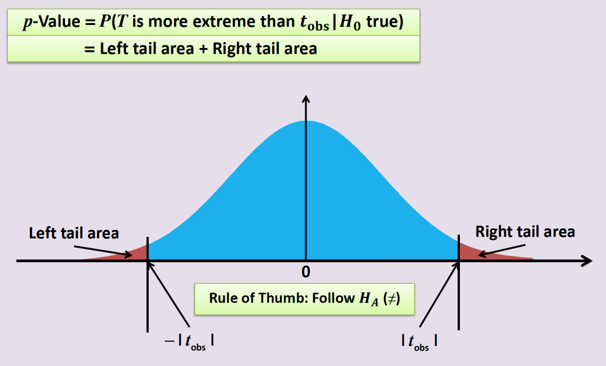
$$H_0\colon\ \mu = 1.5,\ H_A\colon\ \mu \neq 1.5$$

- Step 2: Choose $\alpha = 0.05$
- Step 3: Rejection region is $T < -t_{15,0.025}$ or $T > t_{15,0.025}$. From t table, $t_{15,0.025} = 2.13$.
- Step 4: $t_{obs} = \frac{1.348-1.50}{\frac{0.3385}{\sqrt{16}}} = -1.796$. t_{obs} does not fall in the rejection region, so we do not reject the null.

Hypothesis testing using p-value

- For a two-sided test,

$$\text{p-value} = \mathbb{P}(T_{n-1} \geq |t_{obs}|) + \mathbb{P}(T_{n-1} \leq -|t_{obs}|)$$



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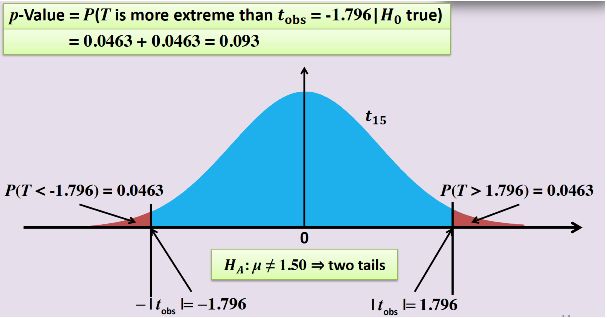

- If we choose $\alpha = 0.05$, then since $0.094 > 0.05$, we do not reject the null. However, if $\alpha = 0.1$, we reject the null.

What's the next?

We'll discuss how to calculate the sample size and power in hypothesis testing in the next lecture.