STAT324: Introductory Applied Statistics for Engineers **Spring 2019**

Chapter 5 — Estimation

1. Distributions of Functions of RVs

   - Two RVs are said to be **independent** if the realization of one of them does not change the probability distribution of the other, and vice versa. If two RVs are not independent, then they are **dependent**.
   - Some rules of expectation and variance follow:
     (a) $E(c) = c$.
     (b) $E(c * X) = c * E(X)$.
     (c) $E(X + c) = E(X) + c$.
     (d) $E(X + Y) = E(X) + E(Y)$.
     (e) $VAR(c) = 0$.
     (f) $VAR(c * X) = c^2 VAR(X)$.
     (g) $VAR(X + c) = VAR(X)$.
     (h) If X and Y are independent, $VAR(X + Y) = VAR(X) + VAR(Y)$.
   - A sample of size n from a population is called a **simple random sample** if every possible sample of size n is equally likely to be drawn.
   - We say a sample is drawn **with replacement** if an element is replaced to the population before the next element is drawn. There is a chance the same element could be drawn more than once. Otherwise we say the sample is drawn **without replacement**, and every element can be drawn at most once.
   - A collection of RVs X_1, X_2, ..., X_n are said to be **independent and identically distributed**, or **iid**, if the following things are true:
     - They are all independent from one another. That is, the realization of any one of them does not change the probability distribution of any other one.
     - They all have exactly the same probability distribution.

2. Estimation

   - Sample mean: $\hat{\mu} = \bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$

- Sample variance of X: $\hat{\sigma}^2 = S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$
- Sample standard deviation of X: $\hat{\sigma} = S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$
- The formula that describes how data from a sample would be used to compute a guess about a population parameter is called an **estimatOR**, or a **statistic**. The numerical value computed once the data is collected is called an **estimATE**. An estimator is an RV, and an estimate is a realization of that RV.
- The **bias** in an estimator $\hat{\theta}$ is defined as:

$$bias(\hat{\theta}) = E(\hat{\theta}) - \theta.$$

  If the the bias is equal to zero, the estimator $\hat{\theta}$ is called **unbiased** for θ. All other things being equal, smaller bias is better.
- The variance of an estimator $\hat{\theta}$ is defined as $VAR(\hat{\theta})$. All other things being equal, smaller variance is better. The square root of the variance is usually called the standard deviation or SD. However, when we are talking about estimating a parameter, we instead use the term **standard error** or **SE**, to remind us that this is the amount of error in estimation. Thus the square root of the variance of an estimator will be denoted $SE(\hat{\theta})$.
- The **mean squared error**, or **MSE**, of an estimator $\hat{\theta}$ is defined as:

$$MSE(\hat{\theta}) = VAR(\hat{\theta}) + bias(\hat{\theta})^2.$$

  All other things being equal, smaller MSE is better.
- $E(\bar{X}) = E(\frac{X_1+X_2+...+X_n}{n}) = \frac{\mu+\mu+...+\mu}{n} = \mu.$
- $VAR(\bar{X}) = VAR(\frac{X_1+X_2+...+X_n}{n}) = \frac{\sigma^2+\sigma^2+...+\sigma^2}{n^2} = \frac{\sigma^2}{n}.$
- $SE(\bar{X}) = \sqrt{VAR(\bar{X})} = \frac{\sigma}{\sqrt{n}}.$
- Estimated standard error of $\bar{X}$: $\widehat{SE(\bar{X})} = \frac{\hat{\sigma}}{\sqrt{n}} = \frac{S}{\sqrt{n}}.$

3. A **normal quantile-quantile plot** or **normal QQ plot** can be used to evaluate normality. If the data appears to be drawn from a normally distributed population, the points in the plot will usually fall on a roughly straight line.

4. The Central Limit Theorem can be stated as follows. Let X_1, X_2, ..., X_n be a collection of iid RVs with $E(X_i) = \mu$ and $VAR(X_i) = \sigma^2$. For large enough n, the distribution of $\bar{X}$ will be approximately normal with $E(\bar{X}) = \mu$ and $VAR(\bar{X}) = \frac{\sigma^2}{n}$. That is, $\bar{X} \dot{\sim} N(\mu, \frac{\sigma^2}{n})$. The required size for n depends on the nature of the true distribution of X_i. The closer the distribution of X_i is to normal, the smaller n is required for the approximation to be good. Usually about $n = 30$ is sufficient.

5. Confidence Intervals

   - The interpretatoin for a confidence interval constructed for a population parameter θ, is that if you had theoretically taken many samples from the population, and created a different interval for each sample, $100(1-\alpha)\%$ of them would cover the true value of θ. This is usually shortened to saying we have $100(1-\alpha)\%$ **confidence** that the interval covers θ.
   - When using $\bar{X}$ to estimate μ, if the X_i are normal and σ is known, or n is large enough for the CLT to work, then a $100(1-\alpha)\%$ CI for μ is given by:

     $$\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}.$$

   - When using $\bar{X}$ to estimate μ, if the X_i are normal, σ is unknown, and the sample size is small, then a $100(1-\alpha)\%$ CI for μ is given by:

     $$\bar{X} \pm t_{(n-1,\alpha/2)} \frac{S}{\sqrt{n}}.$$

   - The general form for a CI often looks like:

     $$\text{estimate} \pm \text{multiplier} * \text{estimated SE(estimator)}$$

   - When intending to create a $100(1-\alpha)\%$ CI for μ, assuming normality and a large sample size, the n required to achive a half-width of no larger than H is given by:

     $$n = \frac{(z^2_{\alpha/2})(\sigma^2)}{H^2}.$$

6. Bootstrap Methods

   - When the sample does not look like it was drawn from a normal population, and the sample size is too small to use the CLT to approximate the sampling distribution of $t = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$, the **bootstrap** can be used to approximate the distribution of t. The steps are as follows:
     (1) Compute the estimate of the sample mean from the data sampled, $\bar{x}$.
     (2) Draw a simple random sample, with replacement, of size n, from the sample data. Call these observations x_1^*, x_2^*, ..., x_n^*. Often this means that the same data point will be repeated twice in the resampling.
     (3) Compute the mean and sd of the resampled data. Call these things $\bar{x}^*$ and s^*.
     (4) Compute the statistic $\hat{t} = \frac{\bar{x}^*-\bar{x}}{\frac{s^*}{\sqrt{n}}}$.
     (5) Repeat steps 2-4 a large number of times, and compute $\hat{t}$ from each one. This is an approximation to the sampling distribution of t.

   - Using the bootstrap, a $100(1-\alpha)\%$ CI for μ based on the approximate sampling distribution of t is given by:

     $$(\bar{x} - \hat{t}_{(\alpha/2)}\frac{s}{\sqrt{n}}, \bar{x} - \hat{t}_{(1-\alpha/2)}\frac{s}{\sqrt{n}}),$$

     where $\hat{t}_{(\alpha/2)}$ and $\hat{t}_{(1-\alpha/2)}$ are the $\alpha/2$ and $1-\alpha/2$ critical values of the approximate sampling distribution.

7. Estimation of a Population Proportion

   - If a sample can be considered a collection of iid RVs Y_i where the outcome of each is either zero or one, then we define the sample proportion:

     $$\text{Sample proportion: } \hat{\pi} = P = \frac{\sum_{i=1}^{n} Y_i}{n}.$$

   - $E(P) = \pi$, $VAR(P) = \frac{\pi(1-\pi)}{n}$, $SE(P) = \sqrt{\frac{\pi(1-\pi)}{n}}$.

   - So long as $n\pi > 5$ and $n(1-\pi) > 5$, the approximate distribution of P is:

     $$P \dot{\sim} N(\pi, \frac{\pi(1-\pi)}{n}).$$

   - So long as $n\pi > 5$ and $n(1-\pi) > 5$, an approximate $100(1-\alpha)\%$ CI for π would be of the form:

     $$P \pm z_{\alpha/2}\sqrt{\frac{P(1-P)}{n}}.$$