Chapter 5: Estimation

(Ott & Longnecker Sections: 4.12, 4.14 and 5.2)

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Part 2

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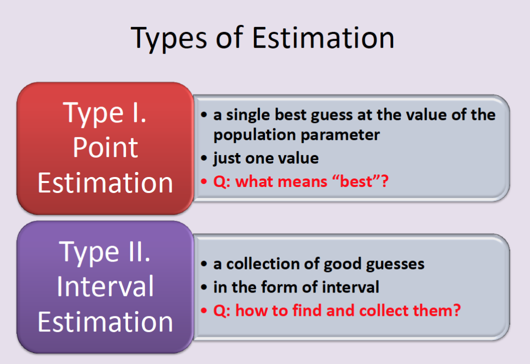
What do we study?

Key Concepts: estimator, statistic, estimate, bias, unbiased estimator, standard error, mean squared error, estimated standard error

Types of estimation



We'll focus on point estimation in this lecture.

Outline

Example

A car manufacturer uses an automatic device to apply paint to engine blocks. Since engine blocks get very hot, the paint must be heat-resistant, and it is important that the amount applied is of a certain minimum thickness. A warehouse contains thousands of blocks painted by the automatic device. The manufacturer wants to know the average amount of paint applied by the device, so 16 blocks are selected at random, and the paint thickness is measured in mm. The results are below: 1.29, 1.12, 0.88, 1.65, 1.48, 1.59, 1.04, 0.83, 1.76, 1.31, 0.88, 1.71, 1.83, 1.09, 1.62, 1.49

Sample mean

- Denote the population mean paint thickness of blocks by μ.
- What is μ?

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- What is μ?
- Define 16 i.i.d. RVs, X_1, X_2, ..., X_{16}. Then $\mathbb{E}(X_i) = \mu$.
- Our intuition serves us well here, the sample mean of these observations, which we define below, will be a good estimate of the population mean μ:

$$\hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

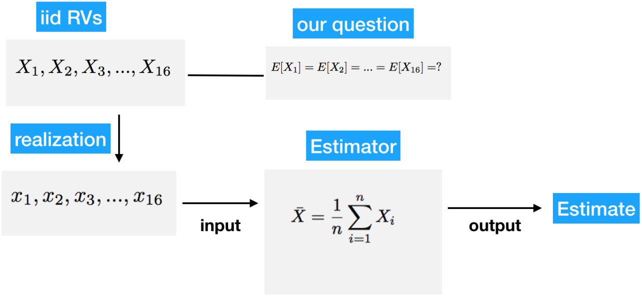
Outline

1. Sample mean
2. Estimator and estimate
3. Bias
4. Standard error
5. MSE
6. Sample variance and sample standard deviation
7. Estimated standard error of $\bar{X}$

Estimator and estimate

- The formula that describes how data from a sample would be used to compute a guess about a population parameter is called an estimator, or a statistic.
- The sample mean formula given above is an example of an estimator.
- The numerical value computed using the estimator once the data is collected is called an estimate.
- An estimator is a RV, and an estimate is a realization of that RV.

Estimator and estimate



A question

Why is sample mean a good estimator to the population mean μ?

Outline

Bias

Let $\hat{\theta}$ be an estimator to some unknown parameter θ.

- The bias of the estimator $\hat{\theta}$ is defined as:

$$\mathit{bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta.$$

- If the bias is equal to zero, the estimator $\hat{\theta}$ is called unbiased for θ.
- All other criterions being equal, smaller bias is better.

Outline

Standard error

- The variance of an estimator $\hat{\theta}$ is defined as

$$VAR(\hat{\theta}) = \mathbb{E}(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2.$$

- The square root of the variance is usually called the standard deviation or SD. However, when we are talking about estimating a parameter, we instead use the term standard error or SE, to remind us that this is the amount of error in estimation. Thus the square root of the variance of an estimator will be denoted $SE(\hat{\theta})$.
- All other criterions being equal, smaller standard error is better.

Outline

Mean squared error

- The mean squared error or MSE, of an estimator $\hat{\theta}$ is defined as:

$$MSE(\hat{\theta}) = VAR(\hat{\theta}) + \left(bias(\hat{\theta})\right)^2.$$

- All other things being equal, smaller MSE is better.
- MSE incorporates the information about both bias and variance.
- Bias-Variance tradeoff: sometimes one estimator is unbiased but has large variance, sometimes one estimator is biased but has low variance.
- Choose the one with the smallest MSE.

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$$VAR(\bar{X}) = VAR(\tfrac{X_1+X_2+...+X_n}{n}) = \tfrac{\sigma^2+\sigma^2+...+\sigma^2}{n^2} = \tfrac{\sigma^2}{n}.$$
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- We can see that $\bar{X}$ is unbiased for μ, since $E(\bar{X}) = \mu$.
- As n increases, the SE decreases. This is intuitive, since as we take a larger sample, we should do a better job of estimating.

Examples ($n = 3$)

iid RVs in the Sample	• X_1, X_2, X_3 with $E(X_i) = \mu$, $SD(X_i) = \sigma$
Goal	• estimate μ
Estimator 1 (unbiased): $\hat{\mu}_1 = \bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$	• $bias(\bar{X}) = \mu - \mu = 0$; $VAR(\bar{X}) = \frac{\sigma^2}{3}$ • $MSE(\bar{X}) = 0^2 + \frac{\sigma^2}{3} = \frac{\sigma^2}{3}$
Estimator 2 (unbiased): $\hat{\mu}_2 = \frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3$	• $bias(\hat{\mu}_2) = E(\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3) - \mu = \mu - \mu = 0$ • $VAR(\hat{\mu}_2) = VAR(\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3) = (\frac{1}{16} + \frac{1}{4} + \frac{1}{16})\ \sigma^2 = \frac{3\sigma^2}{8}$ • $MSE(\bar{X}) = 0^2 + \frac{3\sigma^2}{8} = \frac{3\sigma^2}{8} > MSE(\bar{X})$
Estimator 3 (biased): $\hat{\mu}_3 = X_1 + \frac{1}{2}X_2 - X_3$	• $bias(\hat{\mu}_3) = E(X_1 + \frac{1}{2}X_2 - X_3) - \mu = \frac{1}{2}\mu - \mu = -\frac{1}{2}\mu$ • $VAR(\hat{\mu}_3) = VAR(X_1 + \frac{1}{2}X_2 - X_3) = (1 + \frac{1}{4} + 1)\ \sigma^2 = \frac{9\sigma^2}{4}$ • $MSE(\bar{X}) = (-\frac{1}{2}\mu)^2 + \frac{9\sigma^2}{4} = \frac{\mu^2}{4} + \frac{9\sigma^2}{4}$

Outline

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- Use sample variance:

$$\hat{\sigma}^2 = S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X}$ is the sample mean.

- Sample variance is an estimator, and thus an RV. In this case S^2 is an estimator of σ^2.
- The reason we use $n-1$ in the denominator is that this makes the estimator S^2 unbiased for σ^2.

Sample standard deviation

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- Use sample standard deviation:

$$\hat{\sigma} = S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

- S is an estimator of σ, but it's **biased**.

Outline

Estimated standard error of $\bar{X}$

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- Estimated standard error of $\bar{X} = \frac{\hat{\sigma}}{\sqrt{n}} = \frac{S}{\sqrt{n}}$.
- For the paint data, we find $S = 0.3385$ mm, so estimated SE $= \frac{0.3385}{\sqrt{16}} = 0.085$ mm.

Conclusions

- The standard deviation is a property of the distribution of the random variable X, it measures the spread of the distribution.
- The standard error of $\bar{X}$ is a property of the estimator $\bar{X}$, it measures the accuracy of the estimator.
- The estimated standard error is an estimator of the standard error.

Yes, I know how confusing this is!!

What's the next?

In the next lecture, we'll discuss normality, the central limit theorem and confidence intervals.