Chapter 7 Summary: One sample tests

https://dzwang91.github.io/stat324/

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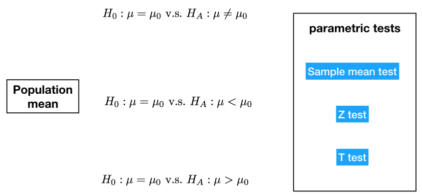
Hypothesis testing framework

1. State the null hypothesis H_0 and the alternative hypothesis H_A.
2. Choose a significance level α. Typically 0.05, 0.01.
3. Choose a test statistic $T_n(X_1, ..., X_n)$ and establish the rejection region.
4. Collect Data $X_1, ..., X_n$. Two approaches:
   - compute the realization of the test statistic. If the test statistic is in the rejection region, we reject H_0, otherwise we do not reject H_0.
   - compute p-value. If p-value is greater than the given significance level, we do not reject H_0. Otherwise, we reject H_0.

Tests for population mean



Two-tailed sample mean test

$$H_0: \mu = \mu_0 \text{ v.s. } H_A: \mu \neq \mu_0$$

- We collect a sample of size n, $X_1, ..., X_n$ with known population variance σ^2.
- Test statistic: sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.
- Under H_0, the sampling distribution of $\bar{X}$ is approximately normal distributed with mean μ_0 and standard deviation $\sigma/\sqrt{n}$.
- Rejection region: $\bar{X} \geq b$ or $\bar{X} \leq a$ where

$$b = \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad a = \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- If we denote the realization of $\bar{X}$ is c, then p-value formula:

$$\text{p-value} = \mathbb{P}(\bar{X} \geq |c|) + \mathbb{P}(\bar{X} \leq -|c|) = 2\mathbb{P}(\bar{X} \geq |c|)$$

One-tailed sample mean test (left tail)

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu < \mu_0$$

- Assume the population variance σ^2 is known.
- Rejection region: $\bar{X} \le a$ where

$$a = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$$

- If we denote the realization of $\bar{X}$ is c, then p-value formula:

$$\text{p-value} = \mathbb{P}(\bar{X} \le c)$$

One-tailed sample mean test (right tail)

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu > \mu_0$$

- Assume the population variance σ^2 is known.
- Rejection region: $\bar{X} \ge b$ where

$$b = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

- If we denote the realization of $\bar{X}$ is c, then p-value formula:

$$\text{p-value} = \mathbb{P}(\bar{X} \ge c)$$

Z test

- Under $H_0 : \mu = \mu_0$,

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \approx N(0, 1)$$

- When $H_A : \mu \ne \mu_0$, the rejection region is

$$Z > z_{\alpha/2} \text{ or } Z < -z_{\alpha/2}.$$

- When $H_A : \mu < \mu_0$, the rejection region is

$$Z < -z_\alpha.$$

- When $H_A : \mu > \mu_0$, the rejection region is

$$Z > z_\alpha.$$

T test

- We collect a sample $X_1, ..., X_n$ from a normal distribution, but with unknown variance σ^2.
- Test statistic: $T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$, where s is the sample standard deviation
- Under $H_0 : \mu = \mu_0$, the sampling distribution of $\frac{\bar{X} - \mu_0}{s/\sqrt{n}}$ is a t-distribution with degrees of freedom $n - 1$.
- Rejection region:
  - When $H_A : \mu \ne \mu_0$: $T < -t_{\alpha/2,n-1}$ or $T > t_{\alpha/2,n-1}$.
  - When $H_A : \mu < \mu_0$: $T < -t_{\alpha,n-1}$.
  - When $H_A : \mu > \mu_0$: $T > t_{\alpha,n-1}$.
- For a **two-sided** test,

$$\text{p-value} = \mathbb{P}(T_{n-1} \ge |t_{obs}|) + \mathbb{P}(T_{n-1} \le -|t_{obs}|)$$

Test for population median

Sign test

Sign test

Definition

For a random variable X, population median is defined as a parameter m such that

$$P(X \le m) = \frac{1}{2}$$

- Collect an i.i.d. sample $X_1, \ldots, X_n$ from some population with population median m.
- **Step 1**:
  1. Two-tailed test:
  $$H_0 : m = m_0 \text{ v.s. } m \neq m_0$$
  2. Left-tailed test:
  $$H_0 : m = m_0 \text{ v.s. } m < m_0$$
  3. Right-tailed test:
  $$H_0 : m = m_0 \text{ v.s. } m > m_0$$

Sign test

- **Step 2**: delete data points which are equal to m_0. Count remaining sample size n^*.
- **Step 3**: choose test statistic

  B = number of data values greater than m_0 in the remaining sample.
- Under $H_0 : m = m_0$, $B \sim \text{Binomial}(n^*, 0.5)$.
- **Step 4**: calculate p-value. Let b be the observation of B.
  - $H_A : m > m_0$:
  $$\text{p-value} = P(B \ge b \mid B \sim \text{Binomial}(n^*, 0.5))$$
  - $H_A : m < m_0$:
  $$\text{p-value} = P(B \le b \mid B \sim \text{Binomial}(n^*, 0.5))$$
  - $H_A : m \neq m_0$:
  $$\text{p-value} = 2 \times \min\{P(B \ge b), P(B \le b)\}.$$

Tests for population proportion

Z test

Tests for population proportion

- Let π be the population proportion. We want to test

$$\text{(right-tailed test)} \quad H_0 : \pi = \pi_0, \ v.s. \ H_A : \pi > \pi_0;$$

or

$$\text{(left-tailed test)} \quad H_0 : \pi = \pi_0, \ v.s. \ H_A : \pi < \pi_0;$$

or

$$\text{(two-tailed test)} \quad H_0 : \pi = \pi_0, \ v.s. \ H_A : \pi \neq \pi_0.$$

- We have data $X_1, ..., X_n \sim Ber(\pi)$.
- Under $H_0 : \pi = \pi_0$, the sample proportion $P = \frac{\sum_{i=1}^n X_i}{n}$ has

$$\mathbb{E}(P) = \pi_0$$

and

$$Var(P) = \frac{\pi_0(1-\pi_0)}{n}.$$

Tests for population proportion

- When $n\pi_0 > 5$ and $n(1-\pi_0) > 5$, the CLT holds:

$$Z = \frac{P - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}.$$

is approximately $N(0,1)$.
- **Test statistic**:

$$Z = \frac{P - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}.$$

- **P-value**: Let P_{obs} be the observed sample proportion.
  - $H_A : \pi > \pi_0$: calculate $z_{obs} = \frac{P_{obs} - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}$ and calculate $P(Z > z_{obs})$.
  - $H_A : \pi < \pi_0$: calculate $P(Z < z_{obs})$.
  - $H_A : \pi \neq \pi_0$: calculate $2 * P(Z > |z_{obs}|)$.

Power calculation

Power for two-tailed T test: see slides on Chapter 7, part 2

Sample size calculation

Sample size calculation for two-tailed Z test

- Goal: find the required sample size to achieve power at least $1-\beta$ for the specific simple alternative $H_A : \mu = \mu_1$
- The required sample size:

$$n = \left(\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_1 - \mu_0} \right)^2$$