Chapter 7: One sample tests
(Ott & Longnecker Sections: 5.4-5.7, 5.9)

Part 2

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https://dzwang91.github.io/stat324/

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Overview

Population mean

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu \neq \mu_0$$

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu < \mu_0$$

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu > \mu_0$$

parametric tests

- Sample mean test
- Z test
- T test

Example: two-tailed T test

Story

A paint shop uses an automatic device to apply paint to engine blocks. It is important that the amount applied is of a minimum thickness.

Primary Research Question

Its customer, a manufacturer wants to know the average thickness of paint in a warehouse. It is supposed to be 1.50mm.

$$\mu = 1.50\text{mm}?$$

Sampling

16 blocks are selected randomly and then measured from thousands of blocks in the warehouse.

$$n = 16$$

Data

1.29, 1.12, 0.88, 1.65, 1.48, 1.59, 1.04, 0.83, 1.76, 1.31, 0.88, 1.71, 1.83, 1.09, 1.62, 1.49

(in mm)

$$\bar{x} = 1.358$$

$$s = 0.3385$$

Example: two-tailed T test

- Step 1:

$$H_0\colon \mu = 1.5,\ H_A\colon \mu \neq 1.5$$

- Step 2: Choose $\alpha = 0.05$
- Step 3: Rejection region is $T < -t_{15,0.025}$ or $T > t_{15,0.025}$. From t table, $t_{15,0.025} = 2.13$.
- Step 4: $t_{obs} = \frac{1.348-1.50}{\frac{0.3385}{\sqrt{16}}} = -1.796$. t_{obs} does not fall in the rejection region, so we do not reject the null.

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Is this test statistically good?

Outline

1. Power calculation

2. Sample size calculation

Power

Definition

The power of a test is the probability of rejecting H_0 given that the alternative hypothesis is true. That is

$$\text{Power} = 1 - \beta$$

where β is the Type II error.

- How can we calculate the power of a test?

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Definition

Simple hypothesis against a simple alternative:

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu = \mu_1,$$

where μ_0 and μ_1 are two constants.

Vignette one: power for two-tailed T test

$$H_0 : \mu = 1.5 \text{ v.s. } H_A : \mu = 1.4$$

- Power:

$$\begin{aligned} P(\text{Reject } H_0 | H_A \text{ is true}) &= P\left(|T| > 2.13 \middle| \mu = 1.4\right) \\ &= P\left(\left|\frac{\bar{X} - 1.5}{S/\sqrt{n}}\right| > 2.13 \middle| \mu = 1.4\right) = ? \end{aligned}$$

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- What is the distribution of $\frac{\bar{X}-1.5}{S/\sqrt{n}}$ when $\mu = 1.4$?

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- What is the distribution of $\frac{\bar{X}-1.5}{S/\sqrt{n}}$ when $\mu = 1.4$? NOT t distribution.

Power calculation continued

- Decomposition:

$$\frac{\bar{X} - 1.5}{S/\sqrt{n}} = \frac{\bar{X} - 1.4}{S/\sqrt{n}} + \frac{1.4 - 1.5}{S/\sqrt{n}}$$

Under $\mu = 1.4$, $\frac{\bar{X} - 1.4}{S/\sqrt{n}}$ is t_{n-1} distribution.

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$$\begin{aligned} Power &= P(|\frac{\bar{X} - 1.4}{S/\sqrt{n}} + \frac{1.4 - 1.5}{S/\sqrt{n}}| > 2.13|\mu = 1.4) \\ &= P(|t_{15} + \frac{1.4 - 1.5}{0.3385/\sqrt{16}}| > 2.13) = P(|t_{15} - 1.18| > 2.13) \\ &= P\left(T_{15} < -0.95\right) + P\left(T_{15} > 3.31\right) = 0.179 + 0.002 = 0.181 \end{aligned}$$

Power calculation continued

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- This power is low. This means that if the true population mean was $\mu = 1.4$, we would be very unlikely to reject the null based on a sample of size 16.

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What is the required sample size to achieve specific power?

Outline

1. Power calculation

2. Sample size calculation

Vignette two: sample size calculation for two-tailed Z test

- Two-tailed Z test:
  - $H_0 : \mu = \mu_0$ v.s. $H_A : \mu \neq \mu_0$,
  - Test statistic: $Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$
  - Rejection region to control Type I error at significance level α:

$$Z > z_{\alpha/2} \text{ or } Z < -z_{\alpha/2}.$$

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- Goal: find the required sample size to achieve power at least $1 - \beta$ for the specific simple alternative $H_A : \mu = \mu_1$

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$\Leftrightarrow$

$$\text{What is } n \text{ which satisfies } P(|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}| < z_{\alpha/2} | \mu = \mu_1) \leq \beta?$$

Vignette two: sample size calculation for two-tailed Z test

- Derivation:

$$P(|\frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}| < z_{\alpha/2} | \mu = \mu_1) \leq \beta$$

$$P(|Z + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}| < z_{\alpha/2}) \leq \beta$$

$$P(-z_{\alpha/2} < Z + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} < z_{\alpha/2}) \leq \beta$$

$$P(-z_{\alpha/2} - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} < Z < z_{\alpha/2} - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}) \leq \beta \qquad (\star)$$

Vignette two: sample size calculation for two-tailed Z test

- In order to make $(\star)$ hold, only need

$$P(Z < z_{\alpha/2} - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}) = \beta$$

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$$n = \left(\frac{\sigma\left(z_{\alpha/2} + z_\beta\right)}{\mu_1 - \mu_0}\right)^2$$

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- How does n change when α and β become small?

Vignette two: sample size calculation for two-tailed Z test

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$$n = \left(\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_1 - \mu_0}\right)^2$$

- How does n change when α and β become small?
- How does n change when the difference between μ_1 and μ_0 becomes small?

Back to example

- Determine the required sample size to have power 0.8 under $H_A : \mu = 1.4$

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- Determine the required sample size to have power 0.8 under $H_A : \mu = 1.4$
- We have $\beta = 1 - 0.8 = 0.2$, $z_{0.05/2} = 1.96$ and $z_{0.2} = 0.84$. Use $S = 0.3385$ as an estimator of σ. Then
$$n = (\frac{0.3385(1.96+0.84)}{1.5-1.4})^2 = 89.8, \text{ round up to } 90.$$

Take-home exercise

Find the required sample size for right-tailed Z test which achieves power at least $1 - \beta$ at significance level α for the specific simple alternative

$$H_A : \mu = \mu_1$$

What's the next?

We'll introduce sign test and discuss how to test population proportions in next lecture.