

Chapter 7: One sample tests

(Ott & Longnecker Sections: 5.4-5.7)

Part 1

<https://dzwang91.github.io/stat324/>



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- Sample mean test
- Z test
- T test

- ① State the null hypothesis H_0 and the alternative hypothesis H_A .
- ② Choose a significance level α . Typically 0.05, 0.01.
- ③ Choose a test statistic $T_n(X_1, \dots, X_n)$ and establish the rejection region.
- ④ Collect Data X_1, \dots, X_n . Two approaches:
 - compute the realization of the test statistic. If the test statistic is in the rejection region, we reject H_0 , otherwise we do not reject H_0 .
 - compute p-value. If p-value is greater than the given significance level, we do not reject H_0 . Otherwise, we reject H_0 .

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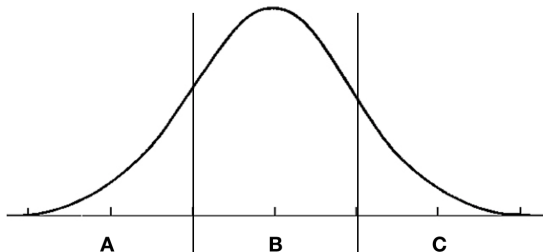
How do we choose the test statistic, rejection region and p-value?

One-tailed test (left tail)



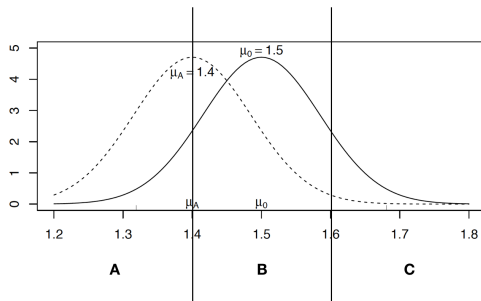
$$H_0 : \theta = \theta_0 \text{ v.s. } H_A : \theta < \theta_0$$

- If the following is the distribution of the test statistic given H_0 is true, then which part is the rejection region?



$$H_0 : \mu = 1.5 \text{ v.s. } H_A : \mu < 1.5$$

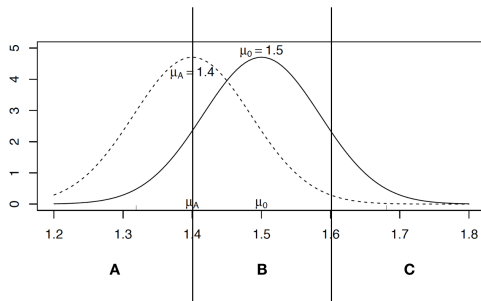
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Shade the probability that test statistic realizes to region A (or B or C) under $\mu = 1.5$ and $\mu = 1.4$.

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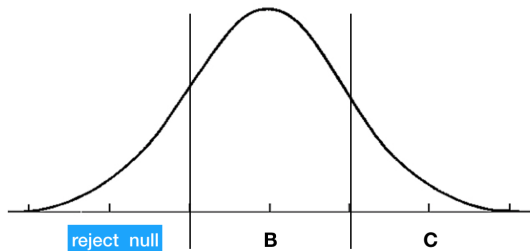
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One-tailed test (left tail)



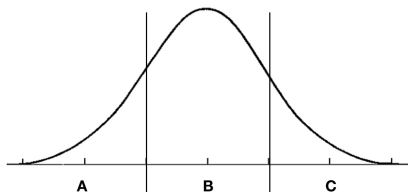
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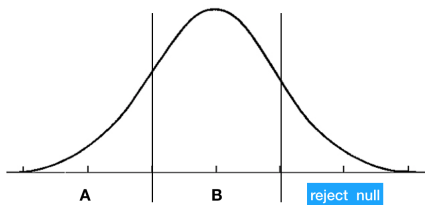


One-tailed test (right tail)



$$H_0 : \theta = \theta_0 \text{ v.s. } H_A : \theta > \theta_0$$

- If the following is the distribution of the test statistic given H_0 is true, then which part of A, B, C is the rejection region?





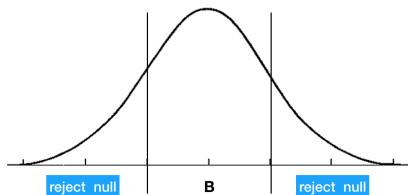
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- Rejection region is in both of the left and right tails of the test statistic.

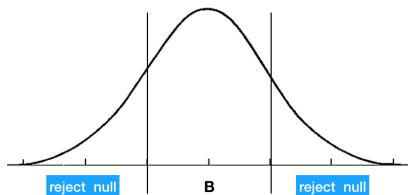


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- Rejection region is in both of the left and right tails of the test statistic.



- Conclusion: The rejection region corresponds to the alternative hypothesis.

**Population
mean**

$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu \neq \mu_0$$

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$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu > \mu_0$$

parametric tests

Sample mean test

Z test

T test

nonparametric test

Sign test

Sample mean test: use sampling distribution of \bar{X}

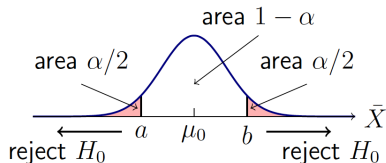
$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu \neq \mu_0$$

- We collect a sample of size n , X_1, \dots, X_n with known population variance σ^2 .
- Test statistic: sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$.
- Under H_0 , the sampling distribution of \bar{X} is approximately normal distributed with mean μ_0 and standard deviation σ/\sqrt{n} . (Why?)

How do we decide the rejection region?

- Use the distribution of the test statistic under H_0 to determine a rejection region that limits the type I error at significance level α :

$$\alpha = \mathbb{P}(\text{reject } H_0 \mid H_0 \text{ is true})$$

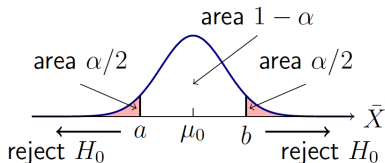


- Rejection region: $\bar{X} \geq b$ or $\bar{X} \leq a$

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- Rejection region: $\bar{X} \geq b$ or $\bar{X} \leq a$

How do we decide a and b ?

- Choice of b :

$$\frac{\alpha}{2} = \mathbb{P}(\bar{X} \geq b) = \mathbb{P}\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{b - \mu_0}{\sigma/\sqrt{n}}\right)$$

\Rightarrow

$$\frac{b - \mu_0}{\sigma/\sqrt{n}} = z_{\alpha/2}$$

\Rightarrow

$$b = \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- Choice of a :

$$\frac{\alpha}{2} = \mathbb{P}(\bar{X} \leq a) = \mathbb{P}\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq \frac{a - \mu_0}{\sigma/\sqrt{n}}\right)$$

\Rightarrow

$$\frac{a - \mu_0}{\sigma/\sqrt{n}} = -z_{\alpha/2}$$

\Rightarrow

$$a = \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$



How do we compute p-value?

- p-value is the probability of a test statistic realizing to a value that is **as or more extreme than** the one actually observed given the null hypothesis is true.
- If we denote the realization of \bar{X} is c , then p-value formula:

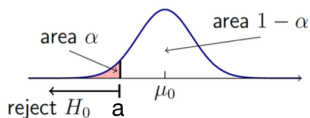
$$\text{p-value} = \mathbb{P}(\bar{X} \geq |c|) + \mathbb{P}(\bar{X} \leq -|c|) = 2\mathbb{P}(\bar{X} \geq |c|)$$

One-tailed sample mean test (left tail)



$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu < \mu_0$$

- Assume the population variance σ^2 is known.
- Rejection region: $\bar{X} \leq a$



One-tailed sample mean test (left tail)



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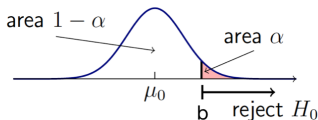
$$\text{p-value} = \mathbb{P}(\bar{X} \leq -|c|)$$

One-tailed sample mean test (right tail)



$$H_0 : \mu = \mu_0 \text{ v.s. } H_A : \mu > \mu_0$$

- Assume the population variance σ^2 is known.
- Rejection region: $\bar{X} \geq b$



- Choice of b :

$$\alpha = \mathbb{P}(\bar{X} \geq b) = \mathbb{P}\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{b - \mu_0}{\sigma/\sqrt{n}}\right)$$

\Rightarrow

$$\frac{b - \mu_0}{\sigma/\sqrt{n}} = z_\alpha$$

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- If we denote the realization of \bar{X} is c , then p-value formula:

$$\text{p-value} = \mathbb{P}(\bar{X} \geq |c|)$$

Z test: use sampling distribution of $\frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$

- Under $H_0 : \mu = \mu_0$,

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \approx N(0, 1)$$

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- When $H_A : \mu \neq \mu_0$, the rejection region is

$$Z > z_{\alpha/2} \text{ or } Z < -z_{\alpha/2}.$$

- When $H_A : \mu < \mu_0$, the rejection region is

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- When $H_A : \mu > \mu_0$, the rejection region is

$$Z > z_{\alpha}.$$

- **Conclusion:** Z test is equivalent to sample mean test.

Example



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- Step 1: $H_0 : \mu = 100$, $H_A : \mu \neq 100$.
- Step 2: Choose $\alpha = 0.05$

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- Step 1: $H_0 : \mu = 100$, $H_A : \mu \neq 100$.
- Step 2: Choose $\alpha = 0.05$
- Step 3: reject H_0 if $\bar{X} < a$ or $\bar{X} > b$ with

$$a = \mu_0 - z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 - 1.96 \frac{8}{10} = 98.432$$

$$b = \mu_0 + z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 + 1.96 \frac{8}{10} = 101.568$$

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$$b = \mu_0 + z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 + 1.96 \frac{8}{10} = 101.568$$

- Step 4: $\bar{X} = 102 > b$, therefore reject H_0 .

A quality control engineer finds that a sample of 100 light bulbs had an average life-time of 470 hours. Assuming a population standard deviation of $\sigma = 25$ hours, test whether the population mean is 480 hours vs. the alternative hypothesis $\mu < 480$ at a significance level of $\alpha = 0.05$.

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- Step 1: $H_0 : \mu = 480$, $H_A : \mu < 480$.
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$$\bar{X} < \mu_0 - z_{0.05} \frac{\sigma}{\sqrt{n}} = 480 - 1.645 \frac{25}{10} = 475.9$$

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- Step 1: $H_0 : \mu = 480$, $H_A : \mu < 480$.
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$$\bar{X} < \mu_0 - z_{0.05} \frac{\sigma}{\sqrt{n}} = 480 - 1.645 \frac{25}{10} = 475.9$$

- Step 4: $\bar{X} = 470 < 475.9$, therefore reject H_0 .

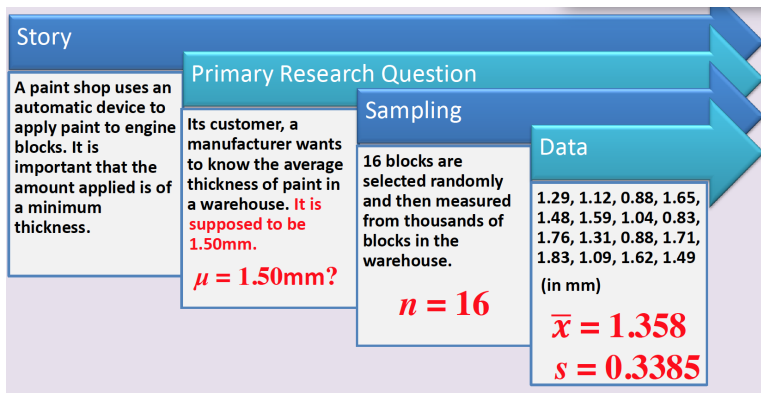
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T test

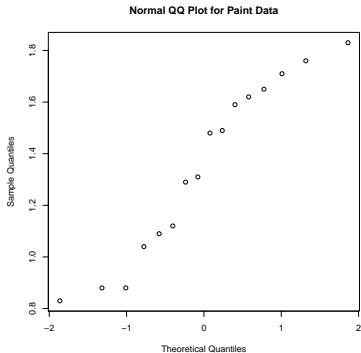
- We collect a sample X_1, \dots, X_n from a normal distribution, but with unknown variance σ^2 .
- Test statistic: $T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$, where s is the sample standard deviation
- Under $H_0 : \mu = \mu_0$, the sampling distribution of $\frac{\bar{X} - \mu_0}{s/\sqrt{n}}$ is a t-distribution with degrees of freedom $n - 1$.

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- Under $H_0 : \mu = \mu_0$, the sampling distribution of $\frac{\bar{X} - \mu_0}{s/\sqrt{n}}$ is a t-distribution with degrees of freedom $n - 1$.
- Rejection region:
 - When $H_A : \mu \neq \mu_0$: $T < -t_{\alpha/2, n-1}$ or $T > t_{\alpha/2, n-1}$.
 - When $H_A : \mu < \mu_0$: $T < -t_{\alpha, n-1}$.
 - When $H_A : \mu > \mu_0$: $T > t_{\alpha, n-1}$.



- Normality?

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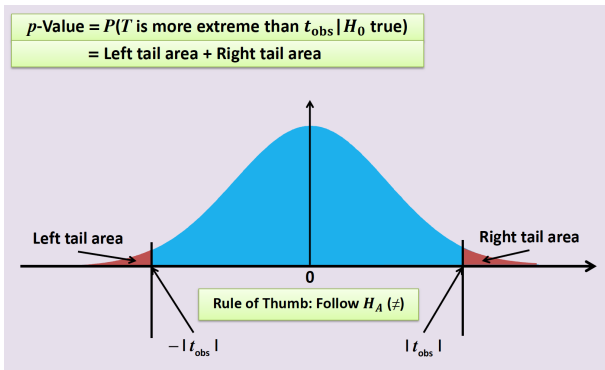
- Step 1:

$$H_0: \mu = 1.5, H_A: \mu \neq 1.5$$

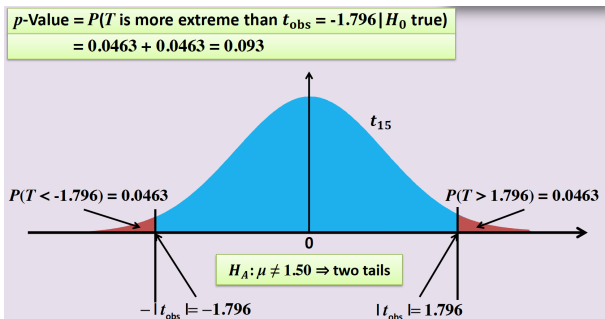
- Step 2: Choose $\alpha = 0.05$
- Step 3: Rejection region is $T < -t_{15,0.025}$ or $T > t_{15,0.025}$. From t table, $t_{15,0.025} = 2.13$.
- Step 4: $t_{obs} = \frac{1.348 - 1.50}{\frac{0.3385}{\sqrt{16}}} = -1.796$. t_{obs} does not fall in the rejection region, so we do not reject the null.

- For a two-sided test,

$$\text{p-value} = \mathbb{P}(T_{n-1} \geq |t_{\text{obs}}|) + \mathbb{P}(T_{n-1} \leq -|t_{\text{obs}}|)$$



Example



- If we choose $\alpha = 0.05$, then since $0.094 > 0.05$, we do not reject the null. However, if $\alpha = 0.1$, we reject the null.



We'll discuss how to calculate the sample size and power in hypothesis testing in the next lecture.